Heron's Formula, Brahmagupta's Formula, and Two Heron-Like Formulae

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1 Introduction

1.1 Motivation

The most well-known formula for the area of a triangle is $A = \frac{1}{2}bh$ where b is the base length and h is the height of the triangle. This formula requires only two measurements and can be derived easily. On the other hand, Heron's formula specifies the area of a triangle in terms of the three side lengths, a, b, and c. Although it may seem less desirable to require three measurements rather than two, it is often inconvenient to first specify a base and then measure the height of a triangle, especially so when measuring angles is not feasible. Thus, Heron's formula can be quite useful.

Further, there is a nice generalization of Heron's formula, known as Brahmagupta's formula, which gives the area of a cyclic quadrilateral in terms of the four side lengths. This formula can be derived easily by using Heron's formula. Brahmagupta's formula is convenient to use because it works for any cyclic quadrilateral and requires only the side lengths to be measured. In addition, the formula bares a close resemblance to Heron's formula, and indeed by taking one side length of the quadrilateral to be zero, Brahmagupta's formula reduces to Heron's formula.

Although Brahmagupta's formula is convenient, its use is somewhat limited as it works only for cyclic quadrilaterals. There are many formulae for the area of a general convex quadrilateral, but most of these require five or six measurements to be taken (for example the four side lengths and two diagonal lengths). However, it turns out that Heron's formula can be used to derive two lesser-known formulae for the area of general convex quadrilateral, both of which require only three length measurements to be taken. In addition, these formulae are actually quite similar in form to Heron's formula after they have been further factorized.

This paper will feature proofs of Heron's formula, Brahmagupta's formula, and the two lesser-known 'Heron-like' formulae mentioned in the preceding paragraph. The proofs given will be in the style of classical euclidean geometry, although modern algebraic and/or trigonometric proofs exist. Before giving the proofs, some historical context will be given.

1.2 Historical Context

Heron of Alexandria (c. 100 AD) was a Greco-Egyptian mathematician and engineer who made numerous contributions to various mathematical and physical subjects. Also known as Hero of Alexandria, he was primarily concerned with practical applications of mathematics. His most significant contribution to the field of mathematics was his *Metrica*, which was written in three books. The first book primarily deals with the measurement of the area of triangles, squares, rectangles, trapezoids, various other regular polygons, circles, and surfaces of cylinders, cones, and spheres. This includes Heron's original derivation of his formula for the area of a triangle. Also of note in the first book is Heron's algorithm for approximating the square root of a non-square integer. It is a straightforward algorithm that works by successively averaging an under-approximation with an over-approximation, and is still commonly used today. In addition to his work in mathematics, Heron made many contributions to the field of mechanics such as his *Catoptrica* which deals with the elementary properties of mirrors, and also treats several problems concerning the construction of mirrors [1]. Interestingly enough, Brahmagupta's formula is not featured in *Metrica*, and, in fact, cyclic quadrilaterals are not explicitly discussed in any of Heron's works. This was largely the case in most of classical Greek mathematics.

In contrast to their Greek counterparts, Indian mathematicians were greatly interested in cyclic quadrilaterals and may have studied them as early as 800 BC. In fact, the earlier Indian mathematicians did not generally distinguish between cyclic and general quadrilaterals, although most of the properties stated hold only for cyclic ones. In his work *Correctly Established Doctrine of Brahma*, Brahmagupta (598-670 AD) states many interesting properties of cyclic quadrilaterals, including a formula for one's area. However, no proofs are given for any of these properties, and it is never explicitly stated that the quadrilateral must be cyclic. As a result, many mathematicians believed these results to be incorrect for several centuries. Eventually, proofs of the formula given by Brahmagupta for the area of a cyclic quadrilateral were found, and today it is commonly known as Brahmagupta's formula [2].

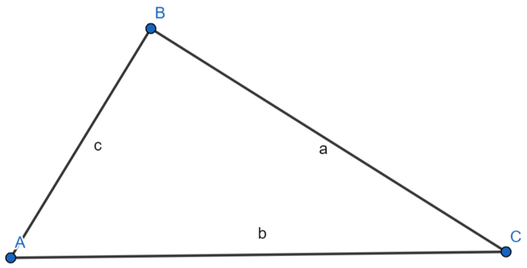
2 Heron's and Brahmagupta's Formulae

We begin this section by stating and proving Heron's formula. Note that it is not necessary to specify a base and then measure the height of the triangle to use this formula. The proof given is adapted from an outline given in exercise 6.11 on page 194 of [1].

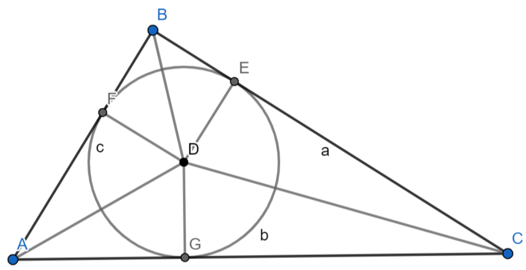
Theorem 1 (**Heron's Formula**)**.** *Given a triangle* ΔABC *with side lengths* $a, b,$ *and* c*, let* $s = \frac{1}{2}(a + b + c)$ *be the semi-perimeter. Then:*

$$area(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}.$$

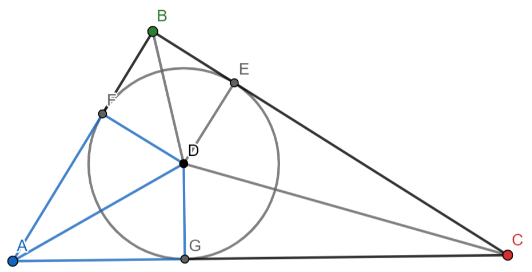
Proof. Given a triangle ΔABC, let a, b, and c denote the lengths of segments BC, AC, and AB respectively. In addition, let $s = \frac{1}{2}(a + b + c)$ denote the semi-perimeter of ΔABC.

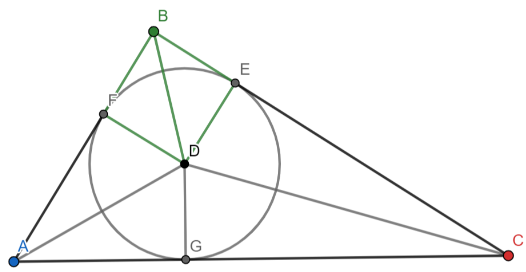


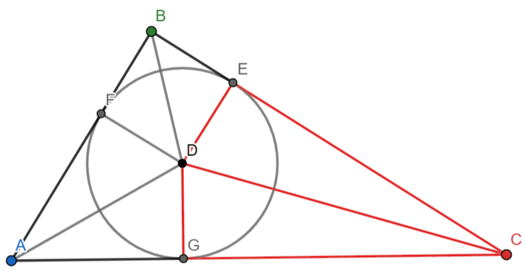
The internal angle bisectors of any triangle meet at a point called the incenter. Let D denote the incenter of ΔABC and construct segments AD, BD, and CD. In addition, connect perpendicular lines to D from each of segments BC, AB, and AC. Denote these lines as ED, FD, and GD respectively. The incenter of any triangle is the center of a circle (called the incircle) which is internally tangent to the triangle and passes through points E, F, and G.

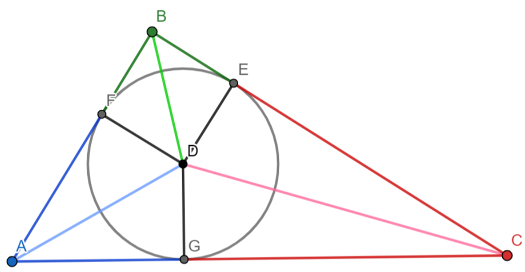


Now since segment AD is the internal bisector of $\angle A$, we have $\angle FAD = \angle GAD$. We also have $\angle AFD = \angle AGD$ since both are right by construction. Hence $\Delta AFD \sim \Delta AGD$ by AA similarity. In addition, segment AD is common to both triangles, and so $\Delta AFD \cong \Delta AGD$ (shown in blue below). Similarly, it holds that $\Delta BFD \cong \Delta BED$ (shown in green) and $\Delta CED \cong \Delta CGD$ (shown in red).





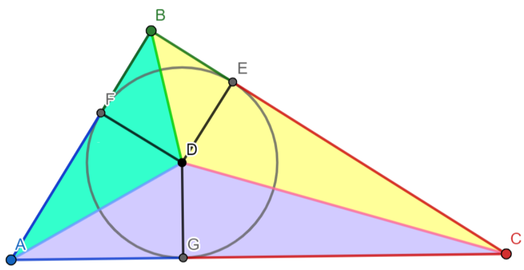




Hence we have $AF = AG$, $BF = BE$, and $CE = CG$. Therefore

$$\begin{aligned} 2s &= a + b + c \\ &= BC + AC + AB \\ &= BE + CE + CG + AG + AF + BF \\ &= 2BE + 2CG + 2AG \\ &= 2BE + 2(CG + AG) \\ &= 2BE + 2AC \\ \implies s &= BE + AC. \end{aligned} \tag{1}$$

Now using the traditional formula area $= \frac{1}{2}$(base)(height), we have area$(\Delta ABD) = \frac{1}{2}(AB)(FD)$, area$(\Delta BCD) = \frac{1}{2}(BC)(ED)$, and area$(\Delta ACD) = \frac{1}{2}(AC)(GD)$. The area of ΔABC is the sum of these three areas.



Note that $FD = ED = GD$ since all are radii of the incircle. Therefore

$$\begin{aligned} \text{area}(\Delta ABC) &= \frac{1}{2}(AB)(FD) + \frac{1}{2}(BC)(ED) + \frac{1}{2}(AC)(GD) \\ &= \frac{1}{2}(AB + BC + AC)GD \\ &= \frac{1}{2}(c + a + b)GD \\ &= s \cdot GD \\ &= (BE + AC)GD \end{aligned}$$

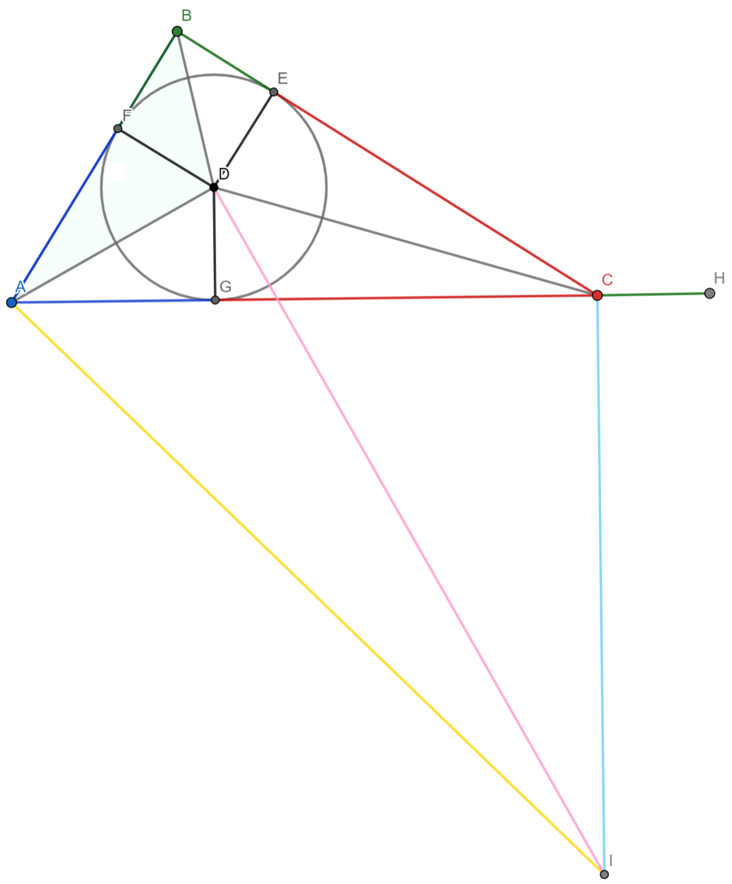
where the last equality used equation (1). Now extend AC to H such that $CH = BE$. Then we have

$$\begin{aligned} BE + AC &= CH + AC \\ &= AH, \end{aligned}$$

and so

$$\text{area}(\Delta ABC) = AH \cdot GD. \tag{2}$$

Next, construct a segment perpendicular to AD through point D (in pink) and construct a segment perpendicular to AH through point C (in light blue). Let point I be the point of intersection of these two lines, and construct segment AI (in yellow).



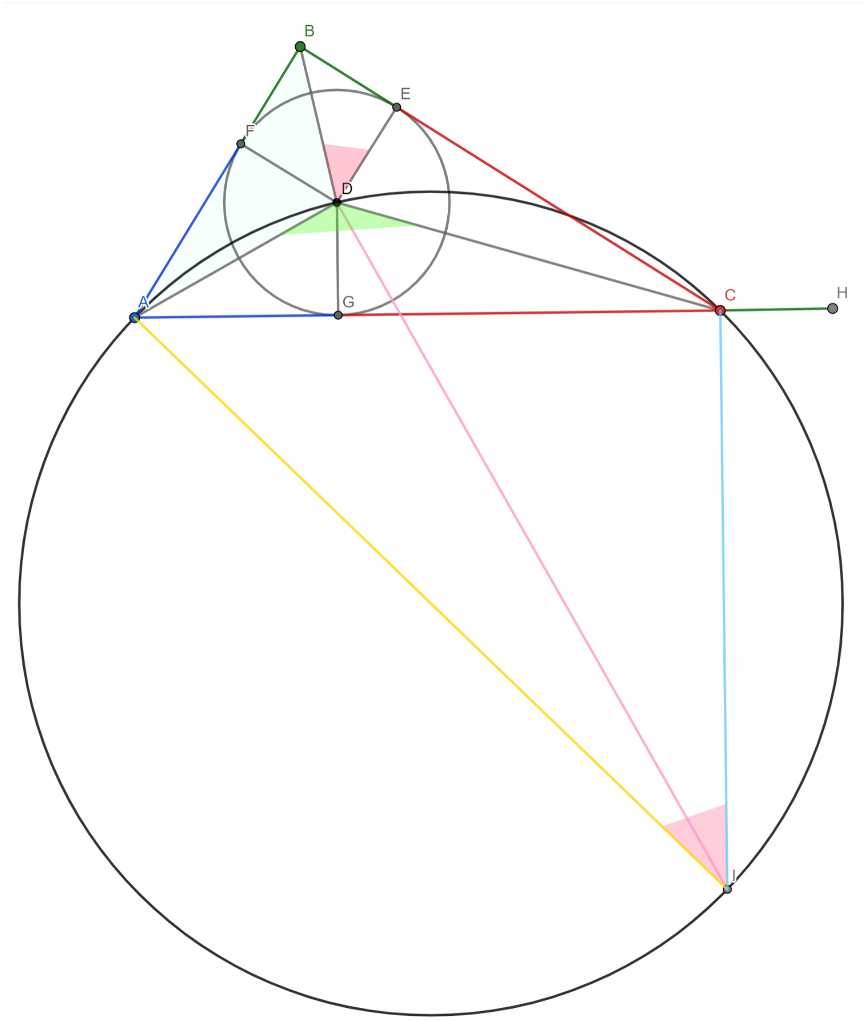
Note that $\angle ADI$ and $\angle ACI$ are both right by construction, and so we can infer from Thale's theorem that points D and C are both on the circumference of a circle with diameter AI. Hence the quadrilateral $ADCI$ is a cyclic quadrilateral. This implies that $\angle AIC$ is supplementary to $\angle ADC$.
By adding all the angles around point D, we have

$$360^\circ = \angle ADG + \angle ADF + \angle BDF + \angle BDE + \angle CDE + \angle CDG.$$

Dividing the above by 2 and recalling that $\Delta AFD \cong \Delta AGD$, $\Delta BFD \cong \Delta BED$, and $\Delta CED \cong \Delta CGD$ yields

$$\begin{aligned} 180^\circ &= \frac{1}{2}(2\angle ADG + 2\angle BDE + 2\angle CDG) \\ &= \angle ADG + \angle CDG + \angle BDE \\ &= \angle ADC + \angle BDE, \end{aligned}$$

and so $\angle BDE$ is supplementary to $\angle ADC$. Thus both $\angle AIC$ and $\angle BDE$ are supplementary to $\angle ADC$ and hence we have $\angle AIC = \angle BDE$.



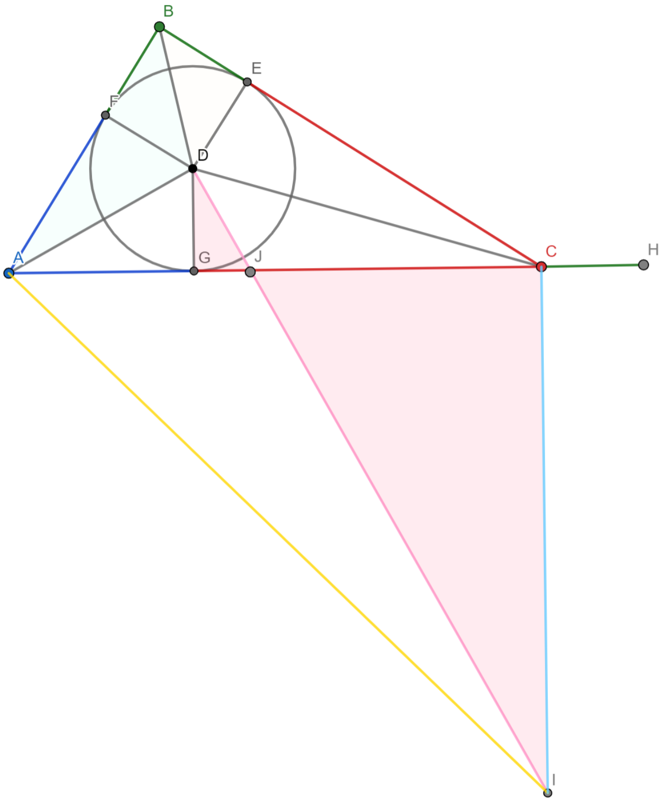
Now $\angle ACI = \angle BED$ since both are right by construction, and therefore $\Delta AIC \sim \Delta BDE$ by AA similarity. So we have

$$\frac{CI}{DE} = \frac{AC}{BE}.$$

Note that $DE = DG$ since both are radii of the incircle, and $BE = CH$ by construction. Therefore,

$$\frac{CI}{DG} = \frac{AC}{CH}. \tag{3}$$

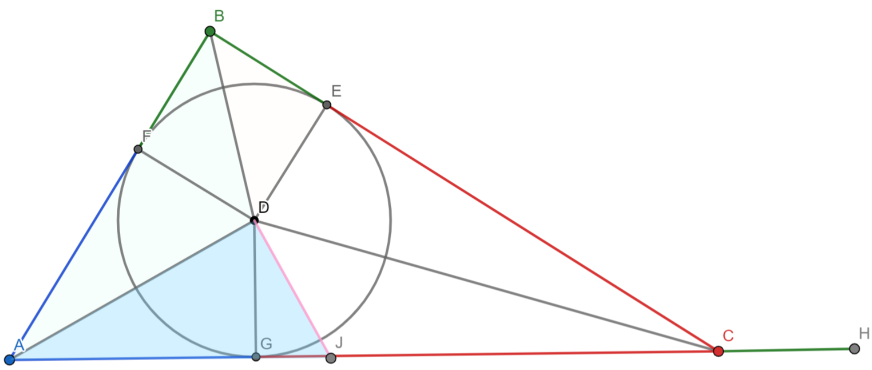
Let J be the point of intersection of segment DI with segment GC. Then $\angle GJD = \angle CJI$ since they are vertically opposite, and $\angle JDG = \angle JCI$ since both are right by construction. Thus $\Delta JIC \sim \Delta JGD$ by AA similarity.



Hence we have

$$
\begin{aligned}
& \frac{CJ}{GJ} = \frac{CI}{DG} \\
\implies & \frac{CJ}{GJ} = \frac{AC}{CH} && \text{by (3)} \\
\implies & \frac{CJ}{GJ} + 1 = \frac{AC}{CH} + 1 \\
\implies & \frac{CJ}{GJ} + \frac{GJ}{GJ} = \frac{AC}{CH} + \frac{CH}{CH} \\
\implies & \frac{CJ + GJ}{GJ} = \frac{AC + CH}{CH} \\
\implies & \frac{CG}{GJ} = \frac{AH}{CH} \\
\implies & CG \cdot CH = AH \cdot GJ. && (4)
\end{aligned}
$$

Note that ΔADJ is right by construction, and since GD is the altitude from its hypotenuse, we have $GJ \cdot GA = (GD)^2$ by the geometric mean.



Hence multiplying both sides of equation (4) by $AH \cdot GA$ yields

$$\begin{aligned}
AH \cdot CG \cdot CH \cdot GA &= (AH)^2(GJ \cdot GA) \\
&= (AH)^2(GD)^2 \\
&= (AH \cdot GD)^2 \\
\implies \sqrt{AH \cdot CG \cdot CH \cdot GA} &= AH \cdot GD,
\end{aligned}$$

and so by equation (2) we have

$$\text{area}(\Delta ABC) = \sqrt{AH \cdot CG \cdot CH \cdot GA}. \tag{5}$$

Now recall that $CH = BE$ by construction, and so equation (1) implies that $s = AH$. Hence $CH = AH - AC = s - b$. Finally, using that $\Delta AFD \cong \Delta AGD$, $\Delta BFD \cong \Delta BED$, and $\Delta CED \cong \Delta CGD$, we have

$$\begin{aligned}
CG &= AH - (AG + CH) \\
&= s - (AF + BE) \\
&= s - (AF + BF) \\
&= s - AB \\
&= s - c,
\end{aligned}$$

and

$$\begin{aligned}
GA &= AH - (CG + CH) \\
&= s - (CE + BE) \\
&= s - BC \\
&= s - a.
\end{aligned}$$

Thus inserting the above expressions for GA, CG, CH, and AH into equation (5) yields

$$\text{area}(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$$

as required.

□

Brahmagupta's formula is very similar in form to Heron's formula, and in fact by taking $d = 0$ it reduces to Heron's formula. Although the proof of Heron's formula was somewhat lengthy and involved several constructions, it can be used to prove Brahamagupta's formula, and the derivation is relatively quick in comparison. It requires a bit of algebraic manipulation, but by being careful with notation we can keep it fairly tidy. The proof given is from [3].

Theorem 2 (Brahmagupta's Formula). *Given a cyclic quadrilateral $ABCD$ with side lengths a, b, c, and d, let $s = \frac{1}{2}(a + b + c + d)$ be the semi-perimeter. Then:*

$$\textit{area}(ABCD) = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Proof. First, note that Heron's formula can be rewritten as

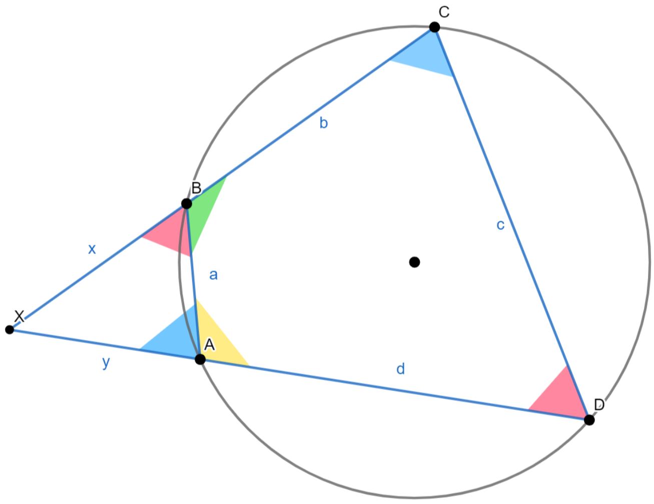
$$\begin{aligned}
\text{area}(\Delta ABC) &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{\frac{1}{16}(a+b+c)(b+c-a)(a+c-b)(a+b-c)} \\
&= \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}
\end{aligned} \tag{6}$$

since here $s = \frac{1}{2}(a + b + c)$. Similarly, Brahmagupta's formula can be rewritten as

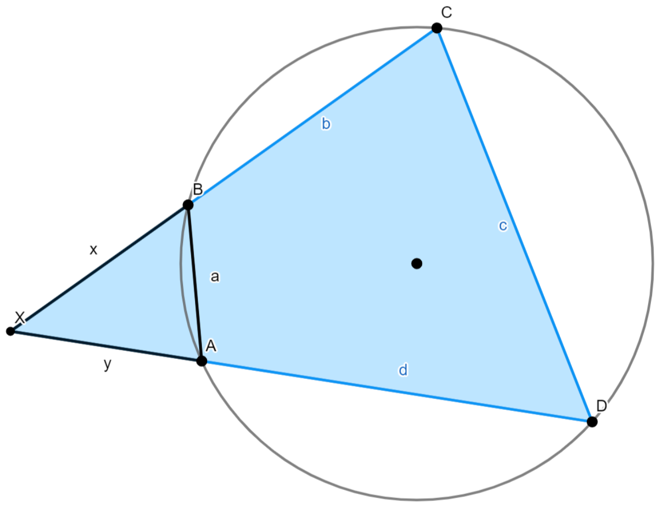
$$\text{area}(ABCD) = \frac{1}{4}\sqrt{(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)}.$$

Now, given a cyclic quadrilateral $ABCD$, let a, b, c, and d denote the lengths of segments AB, BC, CD, and DA respectively. Since $ABCD$ is cyclic, its opposite angles are supplementary. If, in addition, $ABCD$ has both of its opposite sides parallel, then its adjacent angles are all supplementary as well, and so it follows easily that all four of its interior angles are right. So in this case, $ABCD$ is a rectangle and we have area$(ABCD) = a \cdot b$, which is what Brahmagupta's formula reduces to if we take $a = c$ and $b = d$.

Henceforth we shall assume, without loss of generality, that AD and BC are not parallel and intersect at point X. Let $XB = x$ and $XA = y$. We know that $\angle ADC$ is supplementary to $\angle ABC$ since $ABCD$ is cyclic, and clearly $\angle ABX$ is supplementary to $\angle ABC$ as well. Hence $\angle ADC = \angle ABX$ (in red). By a similar argument, $\angle BCD = \angle BAX$ (in blue).



Therefore $\Delta DCX \sim \Delta ABX$ by AA similarity. Note that the area of $ABCD$ is the area of ΔDCX minus the area of ΔABX.



Let $S_1 = (b + x) + (d + y) + c$, $S_2 = (d + y) + c - (b + x)$, $S_3 = (b + x) + c - (d + y)$, and $S_4 = (b + x) + (d + y) - c$. Also let λ be the ratio of similarity between the sides of ΔDCX and ΔABX.

Then by using the version of Heron's formula given in equation (6) we have

$$\begin{aligned}
4 \cdot \text{area}(ABCD) &= 4\left[\text{area}(\Delta DCX) - \text{area}(\Delta ABX)\right] \\
&= \sqrt{S_1S_2S_3S_4} - \sqrt{(\lambda S_1)(\lambda S_2)(\lambda S_3)(\lambda S_4)} \\
\implies (4 \cdot \text{area}(ABCD))^2 &= \left(\sqrt{S_1S_2S_3S_4} - \lambda^2\sqrt{S_1S_2S_3S_4}\right)^2 \\
&= \left((1-\lambda^2)\sqrt{S_1S_2S_3S_4}\right)^2 \\
&= (1-2\lambda^2+\lambda^4)S_1S_2S_3S_4 \\
&= (1+2\lambda^2+\lambda^4-4\lambda^2)S_1S_2S_3S_4 \\
&= \left[(1+\lambda^2)^2-(2\lambda)^2\right]S_1S_2S_3S_4 \\
&= (1+\lambda^2-2\lambda)(1+\lambda^2+2\lambda)S_1S_2S_3S_4 \\
&= (1-\lambda)^2S_1S_4(1+\lambda)^2S_2S_3 \\
\implies \text{area}(ABCD) &= \frac{1}{4}\sqrt{(S_1-\lambda S_1)(S_2+\lambda S_2)(S_3+\lambda S_3)(S_4-\lambda S_4)}. \qquad (7)
\end{aligned}$$

Now each of λS_i for $i = 1, 2, 3, 4$ are terms corresponding to using equation (6) to calculate the area of ΔABX. Hence $\lambda S_1 = x + y + a$, $\lambda S_2 = x + a - y$, $\lambda S_3 = y + a - x$ and $\lambda S_4 = x + y - a$. Therefore, after simplifying we have

$$\begin{aligned}
S_1 - \lambda S_1 &= b + d + c - a \\
S_2 + \lambda S_2 &= d + c - b + a \\
S_3 + \lambda S_3 &= b + c - d + a \\
S_4 - \lambda S_4 &= b + d - c + a.
\end{aligned}$$

Rearranging the above four terms and inserting into equation (7) yields

$$\text{area}(ABCD) = \frac{1}{4}\sqrt{(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)}\ ,$$

which, as noted above, is equivalent to Brahmagupta's formula.

□

Heron's and Brahmagupta's formulae are both very convenient to use because they require only the side lengths of the polygon in question to calculate its area. This is nice because measuring a side length is generally quite straight-forward, whereas other measurements usually involve angles in some form, such as height, which requires a perpendicular line to be drawn.

3 'Heron-Like' Formulae

Although Brahmagupta's formula looks nice due to its similarity in form to Heron's formula, and is convenient since it requires only side lengths, its use is somewhat limited because it only applies to cyclic quadrilaterals. In this section we will prove two formulae that can be used to compute the area of a general convex quadrilateral. Aside from being more general than Brahmagupta's formula, these formulae also have the nice property that they each require only three length measurements. In addition, they can be further factorized in such a way that they appear quite 'Heron-like'. To state the two new formulae in a succinct way, we need a definition.

Definition 1 (Bimedian). *A bimedian is a line segment in a quadrilateral joining the midpoints of two opposite sides.*

With this definition, we then have the following theorem:

Theorem 3. *Given a convex quadrilateral $ABCD$, let p and q denote the lengths of its diagonals, and let m and n denote the lengths of its bimedians. Then:*

$$\textit{area}(ABCD) = \frac{1}{4}\sqrt{\left[(p+q)^2-4m^2\right]\left[4m^2-(p-q)^2\right]} \qquad (8)$$

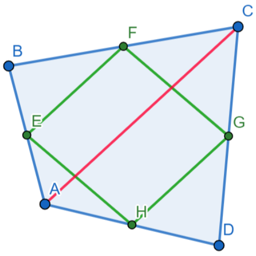
and

$$\textit{area}(ABCD) = \frac{1}{2}\sqrt{\left[(m+n)^2-q^2\right]\left[q^2-(m-n)^2\right]}. \qquad (9)$$

The proof of Theorem 3 will require the following lemma, first proved by French mathematician Pierre Varignon (1654-1722). The proof given is adapted from [4].

Lemma 1 (**Varignon's Theorem**). *The midpoints of the sides of any convex quadrilateral form a parallelogram. Further, the lengths of the sides of the paralleogram are half the length of the diagonals of the quadrilateral, and the area of the parallelogram is half the area of the quadrilateral.*

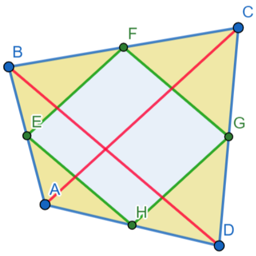
Proof. Given a convex quadrilateral $ABCD$, let E, F, G, and H be the midpoints of its sides. Construct segments EF, FG, GH, and HE. Also construct segment AC.



Note that EF joins the midpoints of sides AB and BC of ΔABC, and GH joins the midpoints of sides AD and DC of ΔADC. The third side of both ΔABC and ΔADC is AC and hence both EF and GH are parallel to and half the length of AC. Therefore the quadrilateral $EFGH$ has opposite sides EF and GH both parallel and equal in length, which implies that $EFGH$ is a parallelogram.

Now construct segment BD and let T_Q and T_P denote the areas of the quadrilateral $ABCD$ and the parallelogram $EFGH$ respectively. Then we have

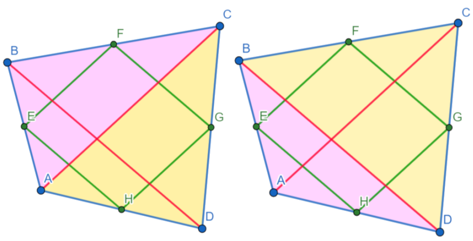
$$T_P = T_Q - \text{area}(\Delta AEH + \Delta BEF + \Delta CFG + \Delta DHG).$$



Since EF, FG, GH, and HE join the midpoints of sides AB, BC, CD, and DA, we have $\Delta ABD \sim \Delta AEH$, $\Delta BAC \sim \Delta BEF$, $\Delta CBD \sim \Delta CFG$, and $\Delta DAC \sim \Delta DHG$, all with a ratio of similarity of $\frac{1}{2}$ for the sides. This shows that the lengths of the sides of the paralleogram are half the length of the diagonals of the quadrilateral. In addition, $\text{area}(\Delta ABD) = 4 \cdot \text{area}(\Delta AEH)$ and similarly for the other three pairs of triangles. Therefore,

$$T_P = T_Q - \frac{1}{4}\text{area}(\Delta ABD + \Delta BAC + \Delta CBD + \Delta DAC).$$

Now note that $\text{area}(\Delta BAC + \Delta DAC) = \text{area}(ABCD) = \text{area}(\Delta ABD + \Delta CBD)$.



Hence we have

$$\begin{aligned} T_P &= T_Q - \frac{1}{4}(T_Q + T_Q) \\ &= \frac{1}{2}T_Q, \end{aligned}$$

and so the area of the paralleogram $EFGH$ is half the area of the quadrilateral $ABCD$.

□

Additionally, the following semi-factorized version of Heron's formula will be useful. Note its similarity in form to equations (8) and (9).

Lemma 2. *The area of a triangle ΔABC with side lengths a, b, and c is given by*

$$area(\Delta ABC) = \frac{1}{4}\sqrt{((a+b)^2 - c^2)(c^2 - (a-b)^2)}.$$

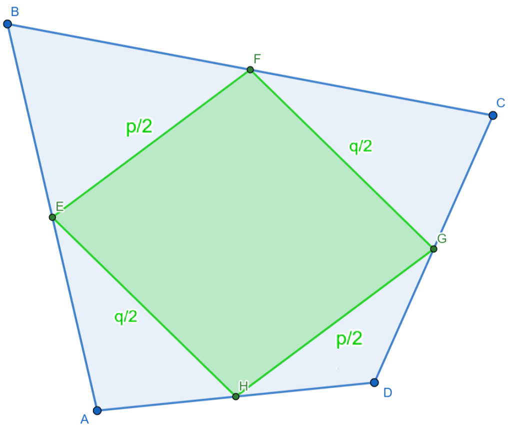
Proof. In the proof of Brahmagupta's formula, it was shown that equation (6) is equivalent to Heron's formula. Hence

$$\begin{aligned} \text{area}(\Delta ABC) &= \frac{1}{4}\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)} \\ &= \frac{1}{4}\sqrt{[(a+b+c)(a+b-c)]\,[(c+a-b)(c-(a-b))]} \\ &= \frac{1}{4}\sqrt{((a+b)^2-c^2)(c^2-(a-b)^2)}. \end{aligned}$$

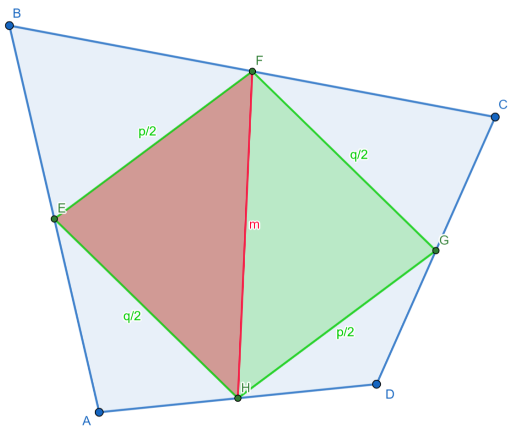
□

We now have all the tools needed to prove equations (8) and (9). This proof is originally from [5].

Proof of Theorem 3. Given a convex quadrilateral $ABCD$ with diagonal lengths p and q and bimedian lengths m and n, let E, F, G, and H be the midpoints of its sides. Construct segments EF, FG, GH, and HE. By Varignon's Theorem, $EF = HG = \frac{1}{2}p$ and $EH = FG = \frac{1}{2}q$. Also, $\text{area}(ABCD) = 2 \cdot \text{area}(EFGH)$.



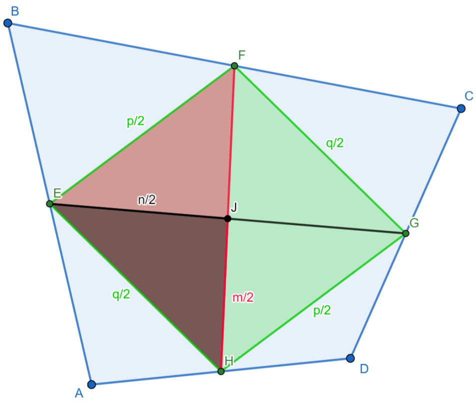
To derive equation (8), construct the bimedian FH with length m. The segment FH is also a diagonal of the parallelogram $EFGH$, and so divides $EFGH$ into two triangles with equal areas.



Hence using Lemma 2 yields

$$
\begin{aligned}
\text{area}(ABCD) &= 2 \cdot \text{area}(EFGH) \\
&= 4 \cdot \text{area}(\Delta FHE) \\
&= \frac{4}{4}\sqrt{\left[(\frac{1}{2}(p+q))^2 - m^2\right]\left[m^2 - (\frac{1}{2}(p-q))^2\right]} \\
&= \sqrt{\left[\frac{1}{4}(p+q)^2 - m^2\right]\left[m^2 - \frac{1}{4}(p-q)^2\right]} \\
&= \sqrt{\frac{1}{4}\left[(p+q)^2 - 4m^2\right] \cdot \frac{1}{4}\left[4m^2 - (p-q)^2\right]} \\
&= \frac{1}{4}\sqrt{\left[(p+q)^2 - 4m^2\right]\left[4m^2 - (p-q)^2\right]}.
\end{aligned}
$$

Now to derive equation (9), construct the second bimedian EG with length n, and let point J be the intersection of EG with FH. The segments EG and FH are the two diagonals of the parallelogram $EFGH$, and so they bisect each other, meaning that $EJ = \frac{1}{2}EG = \frac{1}{2}n$ and $HJ = \frac{1}{2}FH = \frac{1}{2}m$. This also implies that area$(\Delta FHE) = 2 \cdot$ area(ΔJHE).



Therefore by using Lemma 2 we have

$$
\begin{aligned}
\text{area}(ABCD) &= 2 \cdot \text{area}(EFGH) \\
&= 4 \cdot \text{area}(\Delta FHE) \\
&= 8 \cdot \text{area}(\Delta JHE) \\
&= \frac{8}{4}\sqrt{\left[(\frac{1}{2}(m+n))^2 - (\frac{1}{2}q)^2\right]\left[(\frac{1}{2}q)^2 - (\frac{1}{2}(m-n))^2\right]} \\
&= 2\sqrt{\frac{1}{4}\left[(m+n)^2 - q^2\right] \cdot \frac{1}{4}\left[q^2 - (m-n)^2\right]} \\
&= \frac{2}{4}\sqrt{\left[(m+n)^2 - q^2\right]\left[q^2 - (m-n)^2\right]} \\
&= \frac{1}{2}\sqrt{\left[(m+n)^2 - q^2\right]\left[q^2 - (m-n)^2\right]}.
\end{aligned}
$$

□

Corollary. *Given a convex quadrilateral $ABCD$, let p and q denote the lengths of its diagonals, and let m and n denote the lengths of its bimedians. Also let $s_1 = \frac{1}{2}(p+q+2m)$ and $s_2 = \frac{1}{2}(m+n+q)$. Then:*

$$\textit{area}(ABCD) = \sqrt{s_1(s_1-p)(s_1-q)(s_1-2m)} \tag{10}$$

and

$$\textit{area}(ABCD) = 2\sqrt{s_2(s_1-m)(s_1-n)(s_1-q)}. \tag{11}$$

Proof. Starting from equation (8), observe that

$$\begin{aligned}
\operatorname{area}(ABCD) &= \frac{1}{4}\sqrt{\left[(p+q)^2 - 4m^2\right]\left[4m^2 - (p-q)^2\right]} \\
&= \frac{1}{4}\sqrt{16\left[\frac{(p+q)^2}{4} - m^2\right]\left[m^2 - \frac{(p-q)^2}{4}\right]} \\
&= \frac{4}{4}\sqrt{\left[\left(\frac{p+q}{2} + m\right)\left(\frac{p+q}{2} - m\right)\right]\left[\left(m + \frac{p-q}{2}\right)\left(m - \frac{p-q}{2}\right)\right]} \\
&= \sqrt{s_1(s_1 - 2m)(s_1 - q)(s_1 - p)}.
\end{aligned}$$

Next, starting from equation (9) we have

$$\begin{aligned}
\operatorname{area}(ABCD) &= \frac{1}{2}\sqrt{\left[(m+n)^2 - q^2\right]\left[q^2 - (m-n)^2\right]} \\
&= \frac{1}{2}\sqrt{\left[(m+n) + q\right]\left[(m+n) - q\right] \cdot \left[q + (m-n)\right]\left[q - (m-n)\right]} \\
&= \frac{1}{2}\sqrt{16 \cdot \frac{m+n+q}{2} \cdot \frac{m+n-q}{2} \cdot \frac{q+m-n}{2} \cdot \frac{q-m+n}{2}} \\
&= 2\sqrt{s_2(s_2 - q)(s_2 - n)(s_2 - m)}.
\end{aligned}$$

□

Equations (10) and (11) make clear the similarity in form of Heron's formula to the formulae given in Theorem 3.

4 Conclusion

Geometric proofs of Heron's and Brahmagupta's formulae were given. Both of these famous formulae have been known for over a millennia. These formulae are convenient to use because they require only side length measurements; no use of angles is required.

In addition, two lesser-known and more recently discovered formulae for the area of a quadrilateral were given and proved using Heron's formula. Unlike Brahmagupta's formula, which applies only to cyclic quadrilaterals, the formulae in Theorem 3 work for general convex quadrilaterals. Further, they use only three length measurements.

Note that there is no analogue of Heron's and Brahmagupta's formulae for a general convex quadrilateral, in the sense that knowing the four side lengths of a general convex quadrilateral is not enough to determine its area. This is interesting because among the four lengths that a quadrilateral's two diagonals and two bimedians make up, only three are needed to determine its area using equation (8) or (9). The reason this makes sense is because constructing a bimedian requires knowing the midpoints of two sides of the quadrilateral, and so in practice one would still need to take more than three measurements. However, it should be noted that the formulae from Theorem 3 are still useful for the same reason that Heron's and Brahmagupta's formulae are useful; constructing and measuring diagonals and bimedians does not require the use of angles whatsoever.

References

[1] Howard Eves. *An Introduction to the History of Mathematics, 6th ed.*
Saunders College, 1990.
0-03-029558-0, 978-0-03-029558-4

[2] Eisso Atzema. *From Brahmagupta to Euler: on the formula for the area of a cyclic quadrilateral.*
BSHM Bulletin: Journal of the British Society for the History of Mathematics, Vol. 30, 2015.
`https://doi.org/10.1080/17498430.2014.942818`

[3] Albrecht Hess. *A Highway from Heron to Brahmagupta.*
Forum Geometricorum, Vol. 12, 2012.
`http://forumgeom.fau.edu/FG2012volume12/FG201215.pdf`

[4] Ellina Grigorieva. *Methods of Solving Complex Geometry Problems.*
Springer, 2013.
978-3-319-00704-5

[5] Martin Josefsson. *Heron-like formulas for quadrilaterals.*
The Mathematical Gazette, Vol. 100, 2016.
`https://doi.org/10.1017/mag.2016.118`