Minimal Surfaces

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1 Introduction

1.1 Geodesics and Plateau's Problem

Consider two fixed points p and q on a surface S in $\mathbb{R}^3$. Recall that the shortest path joining p to q on S will always be a geodesic. Hence finding a geodesic on S that joins p to q is equivalent to finding a curve γ on S which has the fixed points p and q as its start and end, and such that γ has the least length among all such curves.

The analogous problem in one higher dimension is to find a surface S in $\mathbb{R}^3$ which has a fixed curve γ as its boundary, and such that S has the least surface area among all such surfaces. This is known as *Plateau's problem*. In [1], it is shown that if S has the least surface area among all surfaces with a given fixed boundary curve γ, then it must be that H, the mean curvature of S, is equal to zero at all points on S.

1.2 Minimal Surfaces

The above discussion motivates the study of surfaces which have mean curvature equal to zero at all points:

Definition 1 (Minimal Surface). A minimal surface is a surface S whose mean curvature H is equal to zero at all points $p \in S$.

By the above remarks, if a surface S is a solution for Plateau's problem, then S must be a minimal surface. Note that any plane is clearly a minimal surface, and so a plane is considered to be a 'trivial' minimal surface. Any other minimal surface is considered to be nontrivial.

To mathematically interpret a minimal surface, recall that H is defined to be the arithmetic mean of the principal curvatures κ_1 and κ_2 at each point on S. Hence a minimal surface S must have $\kappa_1 = -\kappa_2$ at each point. Assuming S is not locally a plane (so that $\kappa_1 \neq 0$), this means $\kappa_1 \neq \kappa_2$, and so any principal vectors corresponding to κ_1 and κ_2 will be orthogonal. Since the principal curvatures at a point $p \in S$ give the maximum and minimum values of the normal curvature κ_n of any curve on S passing through p, and since the principal vectors are the tangent vectors of the curves giving these maximum and minimum values, it follows that a minimal surface is one such that at each point the curvature in one direction is matched with an equal and opposite curvature in the orthogonal direction. Hence on any minimal surface, each point that is not locally a plane will locally resemble a saddle surface.

On the other hand, a minimal surface can be physically interpreted as the surface of soap film bounded by wire frames. Consider the surface of a soap film created by dipping a simple closed curve γ made out of a wire frame into liquid soap. This surface will have energy in the form of surface tension, and this surface tension is proportional to the surface area of the soap film. Physical laws dictate that the surface of the soap film will settle in a shape that minimizes its energy, and hence will take on the shape of a surface that is of least possible surface area among all surfaces with boundary γ. Thus the surface of the soap film will be a solution for Plateau's problem, and so must be a minimal surface.

This paper will feature various necessary and sufficient conditions for certain classes of surfaces to be minimal, with examples of such surfaces being given throughout. The classes of surfaces which will be discussed are surfaces of revolution, ruled surfaces, and surfaces which are the graph of a smooth function.

2 Conditions for Minimal Surfaces

2.1 Preliminaries

Let S be a smooth surface with parameterization σ, and let the 1st and 2nd fundamental forms corresponding to σ be

$$\mathcal{F}_I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{and} \quad \mathcal{F}_{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

respectively. Since a minimal surface is defined in terms of its mean curvature H, the following proposition will come in handy during our study of minimal surfaces:

Proposition 1. The mean curvature of S at each point is given by

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

For proof, see Corollary 8.1.3 in [1]. □

As an immediate consequence of Proposition 1, we have the following characterization of minimal surfaces:

Corollary 1. The surface S is a minimal surface if and only if

$$LG + NE = 2MF$$

at each point of S.

Proof. By Proposition 1, we have

$$0 = H = \frac{LG - 2MF + NE}{2(EG - F^2)} \iff LG + NE = 2MF.$$

By definition, S is a minimal surface if and only if $H = 0$ at each point on S. □

2.2 Surfaces of Revolution

Recall that a surface of revolution S is one that is obtained by rotating a smooth plane curve γ, called the profile curve, about a straight line in that plane. By applying an isometry of $\mathbb{R}^3$, one can always assume the plane to be the xz-plane and the straight line to be the z-axis. Then if the profile curve is $\gamma(u) = (f(u), 0, g(u))$, a parameterization of S is given by $\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$. This parameterization is regular (so that S is a smooth surface) if $f(u) > 0$ at all points, i.e. if the profile curve γ never crosses the z-axis.

Example 1. Consider the surface of revolution S obtained by rotating the profile curve $\gamma(u) = (a\cosh\frac{u}{a}, 0, u)$, where a is a nonzero constant, about the z-axis. Such a surface is known as a *catenoid*.

Taking $a = 1$ to simplify calculations and using the parametization of S given by $\sigma(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$, we have

$$\begin{aligned} \sigma_u &= (\sinh u \cos v, \sinh u \sin v, 1), & \sigma_v &= (-\cosh u \sin v, \cosh u \cos v, 0), \\ \sigma_{uu} &= (\cosh u \cos v, \cosh u \sin v, 0), & \sigma_{vv} &= (-\cosh u \cos v, -\cosh u \sin v, 0), \\ \sigma_{uv} &= (-\sinh u \sin v, \sinh u \cos v, 0), \end{aligned}$$

and so

$$\begin{aligned} \vec{N_\sigma} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} &= \frac{(-\cosh u \cos v, -\cosh u \sin v, \sinh u \cosh u)}{\cosh^2 u} \\ &= (-\operatorname{sech} u \cos v, -\operatorname{sech} u \sin v, \tanh u). \end{aligned}$$

Thus the coefficients of the 1st and 2nd fundamental forms corresponding to σ are

$$E = \sigma_u \cdot \sigma_u = \cosh^2 u, \quad F = \sigma_u \cdot \sigma_v = 0, \quad G = \sigma_v \cdot \sigma_v = \cosh^2 u,$$

$$L = \vec{N_\sigma} \cdot \sigma_{uu} = -1, \quad M = \vec{N_\sigma} \cdot \sigma_{uv} = 0, \quad N = \vec{N_\sigma} \cdot \sigma_{vv} = 1.$$

Hence at each point of the surface S we have

$$LG + NE = -\cosh^2 u + \cosh^2 u = 0 = 2MF.$$

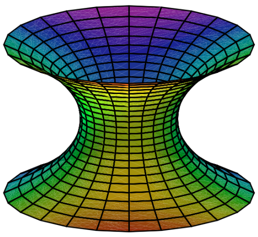
By Corollary 1, it follows that the catenoid S is a minimal surface. ■

Example 1 shows that a catenoid is one example of a surface of revolution which is also a minimal surface. In fact, apart from the plane, catenoids are the only minimal surfaces of revolution:

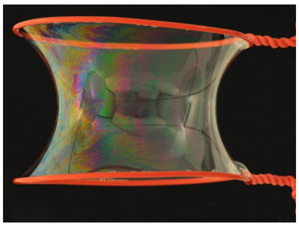
Theorem 1. Any minimal surface of revolution is either an open subset of a catenoid or an open subset of a plane.

For proof, see Proposition 12.2.2 in [1]. □

A catenoid can be physically constructed as a soap film by dipping an appropriately shaped wire frame into liquid soap:



A catenoid.



A soap film as a catenoid.

It should be noted that catenoids are historically significant, as they were the first nontrivial minimal surfaces in $\mathbb{R}^3$ to be discovered. They were first described (and proved to be minimal) by Leonhard Euler in 1744 [3].

2.3 Ruled Surfaces

Recall that a ruled surface can be described as the surface swept out by a moving straight line. Mathematically, a ruled surface S consists of a curve $\gamma(v)$ which describes the direction in which the line is moving, together with a vector $\delta(v)$, known as the ruling, which describes the direction in which the line is pointing at each point. A parameterization of S is then given by $\sigma(u, v) = \gamma(v) + u\delta(v)$. This parameterization is regular (so that S is a smooth surface) if $\dot{\gamma} \times \delta \neq \vec{0}$ at all points, i.e. if γ is never tangent to the ruling, meaning that the straight line is never moving in the same direction that it is pointing.

Example 2. Consider the ruled surface S swept out by a straight line that rotates at constant speed about an axis perpendicular to the line while simultaneously moving at constant speed along said axis. Such a surface is known as a *helicoid*. By applying an isometry of $\mathbb{R}^3$, one can always assume the line to be parallel to the xy-plane and the axis to be the z-axis.

If ω is the angular velocity of the rotating line and α is the its speed along the z-axis, then a curve describing the direction in which the line is moving is $\gamma(v) = (0, 0, \alpha v)$, and a vector which describes the direction in which the line is pointing is $\delta(v) = (\cos(\omega v), \sin(\omega v), 0)$. Using the parameterization of S given by $\sigma(u, v) = \gamma(v) + u\delta(v) = (u\cos(\omega v), u\sin(\omega v), \alpha v)$, we have

$$\begin{aligned}
\sigma_u &= (\cos(\omega v), \sin(\omega v), 0), & \sigma_v &= (-\omega u \sin(\omega v), \omega u \cos(\omega v), \alpha), \\
\sigma_{uu} &= (0, 0, 0), & \sigma_{vv} &= (-\omega^2 u \cos(\omega v), -\omega^2 u \sin(\omega v), 0), \\
\sigma_{uv} &= (-\omega \sin(\omega v), \omega \cos(\omega v), 0), & &
\end{aligned}$$

and so

$$\vec{N_\sigma} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(\alpha \sin(\omega v), -\alpha \cos(\omega v), \omega u)}{\sqrt{\alpha^2 + (\omega u)^2}}.$$

Thus the coefficients of the 1st and 2nd fundamental forms corresponding to σ are

$$E = \sigma_u \cdot \sigma_u = 1, \quad F = \sigma_u \cdot \sigma_v = 0, \quad G = \sigma_v \cdot \sigma_v = (\omega u)^2 + \alpha^2,$$

$$L = \vec{N_\sigma} \cdot \sigma_{uu} = 0, \quad M = \vec{N_\sigma} \cdot \sigma_{uv} = \frac{-\omega\alpha}{\sqrt{\alpha^2+(\omega u)^2}}, \quad N = \vec{N_\sigma} \cdot \sigma_{vv} = 0.$$

Hence at each point of the surface S we have

$$LG + NE = 0 = 2MF.$$

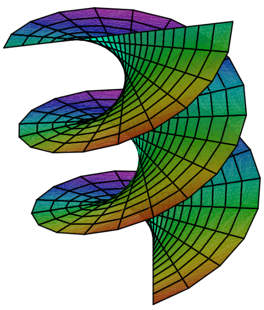
By Corollary 1, it follows that the helicoid S is a minimal surface. ■

Example 2 shows that a helicoid is one example of a ruled surface which is also a minimal surface. In fact, by a result analogous to Theorem 1, helicoids are the only minimal ruled surfaces, apart from the plane:

Theorem 2. Any minimal ruled surface is either an open subset of a helicoid or an open subset of a plane.

For proof, see Proposition 12.2.4 in [1]. □

A helicoid can be physically constructed as a soap film by dipping an appropriately shaped wire frame into liquid soap:



A helicoid.

A soap film as a helicoid.

It should be noted that helicoids are historically significant, as they were the second nontrivial minimal surfaces in $\mathbb{R}^3$ to be discovered, the first being catenoids. Helicoids were first described (and proved to be minimal) by Jean Baptiste Meusnier in 1776 [3].

2.4 Graphs of Functions

Recall that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, then the graph of f is a surface S in $\mathbb{R}^3$ consisting of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$. A parameterization for S is given by $\sigma(u, v) = (u, v, f(u, v))$, which will always be regular so long as f is smooth.

Theorem 3. The graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is a minimal surface if and only if

$$f_{xx}(1 + f_y^2) + f_{yy}(1 + f_x^2) = 2 f_x f_y f_{xy}. \qquad (1)$$

Proof. Let S be the graph of the smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Using the parameterization of S given by $\sigma(u, v) = (u, v, f(u, v))$, we have

$$\begin{aligned} \sigma_u &= (1, 0, f_u), & \sigma_v &= (0, 1, f_v), \\ \sigma_{uu} &= (0, 0, f_{uu}), & \sigma_{vv} &= (0, 0, f_{vv}), \\ \sigma_{uv} &= (0, 0, f_{uv}), \end{aligned}$$

and so

$$\vec{N_\sigma} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(-f_u, -f_v, 1)}{\sqrt{f_u^2 + f_v^2 + 1}}.$$

Thus the coefficients of the 1st and 2nd fundamental forms corresponding to σ are

$$E = \sigma_u \cdot \sigma_u = 1 + f_u^2, \qquad F = \sigma_u \cdot \sigma_v = f_u f_v, \qquad G = \sigma_v \cdot \sigma_v = 1 + f_v^2,$$

$$L = \vec{N_\sigma} \cdot \sigma_{uu} = \tfrac{f_{uu}}{\sqrt{f_u^2+f_v^2+1}}, \quad M = \vec{N_\sigma} \cdot \sigma_{uv} = \tfrac{f_{uv}}{\sqrt{f_u^2+f_v^2+1}}, \quad N = \vec{N_\sigma} \cdot \sigma_{vv} = \tfrac{f_{vv}}{\sqrt{f_u^2+f_v^2+1}}.$$

Note that f_u corresponds to f_x and f_v corresponds to f_y, since we are using the parameterization $\sigma(u, v) = (u, v, f(u, v))$. Hence, by Corollary 1, S is a minimal surface if and only if

$$\frac{f_{xx}}{\sqrt{f_x^2 + f_y^2 + 1}}(1 + f_y^2) + \frac{f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}}(1 + f_x^2) = 2\frac{f_{xy}}{\sqrt{f_x^2 + f_y^2 + 1}} f_x f_y$$

$$\iff f_{xx}(1 + f_y^2) + f_{yy}(1 + f_x^2) = 2 f_{xy} f_x f_y,$$

as required. □

Equation (1) in Theorem 3 is a complicated partial differential equation, and solving for f in general is by no means an easy task. However, by imposing certain restrictions on the form of f, an explicit solution may be found in some cases:

Example 3. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is of the form $f(x, y) = g(x) + h(y)$, where $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are both smooth functions. Then equation (1) becomes

$$g''(x)[1 + (h'(y))^2] + h''(y)[1 + (g'(x))^2] = 0,$$

and rearranging yields

$$-\frac{g''(x)}{1 + (g'(x))^2} = \frac{h''(y)}{1 + (h'(y))^2}.$$

Since the right-hand-side of the above depends only on x, whereas the left-hand-side depends only on y, it follows that both sides must be constant. Hence for some $\lambda \in \mathbb{R}$ we have

$$-\frac{g''(x)}{1 + (g'(x))^2} = \lambda = \frac{h''(y)}{1 + (h'(y))^2}. \qquad (2)$$

Let $u(x) = g'(x)$ so that $u'(x) = g''(x)$. Then we have

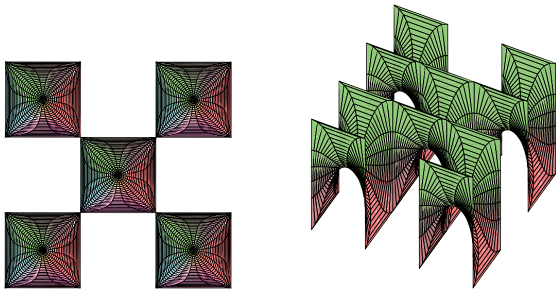
$$\begin{aligned} -\frac{u'(x)}{1 + u^2(x)} &= \lambda \\ \implies \int \frac{u'(x)}{1 + u^2(x)} dx &= -\lambda \int dx \\ \implies \arctan(u(x)) &= -\lambda x + C \\ \implies u(x) &= \tan(-\lambda x + C) \\ \implies g'(x) &= \tan(-\lambda x + C). \end{aligned}$$

If we set $\lambda = 1$ and $C = 0$ so that $g'(x) = \tan(-x) = -\tan x$, then integrating once more gives $g(x) = \ln(\cos x)$. Following the same exact procedure for the y-side of equation (2) will yield $h(y) = -\ln(\cos y)$. Hence

$$f(x, y) = g(x) + h(y) = \ln\left(\frac{\cos x}{\cos y}\right)$$

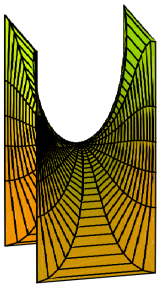
is a solution for equation (1). Note that $f(x, y)$ is not defined on all of $\mathbb{R}^2$, but restricting the domain to $-\frac{\pi}{2} < x, y < \frac{\pi}{2}$ yields a smooth function. It follows that the surface S with parametization $\sigma(u, v) = (u, v, \ln\left(\frac{\cos u}{\cos v}\right))$, where $-\frac{\pi}{2} < u, v < \frac{\pi}{2}$, is a minimal surface. ■

By using a theorem called the Schwarz Reflection Principle, it turns out that the domain of the surface S found in Example 3 can be extended to be a checkerboard domain [2]. This amounts to fitting pieces of the surface S together both horizontally and vertically. For this reason, the surface S, first discovered by Heinrich Sherck in 1834, is known as *Sherck's doubly periodic surface*. This surface is historically significant, as it was the third nontrivial minimal surface in $\mathbb{R}^3$ to be discovered, the first two being catenoids and helicoids. [3]

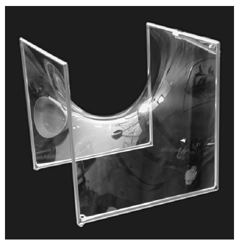


Sherck's doubly periodic surface.

One 'tile' of Sherck's doubly periodic surface can be physically constructed as a soap film by dipping an appropriately shaped wire frame into liquid soap:



A tile of Sherck's doubly periodic surface.



A soap film as a tile of Sherck's doubly periodic surface.

3 Conclusion

Minimal surfaces are defined to be surfaces which have mean curvature equal to zero at all points. They are significant because any solution for Plateau's problem is necessarily a minimal surface. In addition, the surface of a soap film bounded by a wire frame will always be a minimal surface. Any plane is a minimal surface, and so is considered to be a 'trivial' minimal surface.

Two examples of nontrivial minimal surfaces are catenoids and helicoids. Catenoids are significant for being the only nontrivial minimal surfaces of revolution, and for being the first nontrivial minimal surfaces to be discovered. Analogously, helicoids are significant for being the only nontrivial minimal ruled surfaces, and for being the second nontrivial minimal surfaces to be discovered.

The graph of a smooth function is a minimal surface if and only if the function satisfies the partial differential equation (1). This equation can be used to derive Sherck's doubly periodic surface, which is significant for being the third nontrivial minimal surface to be discovered.

Catenoids, helicoids, and Sherck's doubly periodic surface can all be physically constructed as soap films by dipping an appropriately shaped wire frame into liquid soap.

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