C1 Assignment 2 Report

1104630

January 7, 2016

1 Theory

1.1 DIRK Methods and Order of Convergence

In this assignment, we were interested in the experimental order of convergence and the error of approximated solutions to the initial value problem,

$$y'(t) = f(t, y(t)), \quad t_0 = t(0).$$

where we investigated two different functions f.

$$f(t,u) = 1 - \cos(t)(\cos(t) - 1) - u^2, \quad u_0 = 0,\ T = 2\pi \tag{1}$$
$$f(t,u) = 1 - \cos(t)(\cos(t) - e^{\lambda t}) - e^{-2\lambda t}u^2 + \lambda u, \quad u_0 = 0,\ T = 10,\ \lambda = -0.1 \tag{2}$$

Both these equations can be solved analytically which allowed us to calculate the error between the approximated solution and the exact solution.

We considered the following five diagonally implicit Runge-Kutta methods:

Forward Euler
: An explicit method. Using the Taylor expansion, it can be shown that the theoretical order of convergence for this method is 1.

Backward Euler
: An implicit method. Using the Taylor expansion, it can be shown that the theoretical order of convergence for this method is 1.

Crank-Nicholson
: An implicit method. Also known as Gauss-Legendre method, this method arises from considering the integral trapezoidal quadrature rule. The theoretical order of convergence for this method is 2.

3-step Heun
: An explicit method. The theoretical order of convergence for this method is 3.

2-step DIRK
: An implicit method. The theoretical order of convergence of this method is 2.

2 Results

2.1 Maximum Error vs. eoc

The following results tables give the maximum error and the experimental order of convergence for each of the models and DIRK schemes for $\tau = 0.1 \times 2^{-12}$.

Table 1: Equation 1

Scheme	Maximum Error	eoc
Forward Euler	1.20×10^{-5}	0.999956
Backward Euler	1.20×10^{-5}	0.999864
Crank-Nicholson	5.60×10^{-8}	1.99265
3-stage Heun	2.70×10^{-10}	-2.29754
2-stage DIRK	1.02×10^{-7}	1.996

Table 2: Equation 2

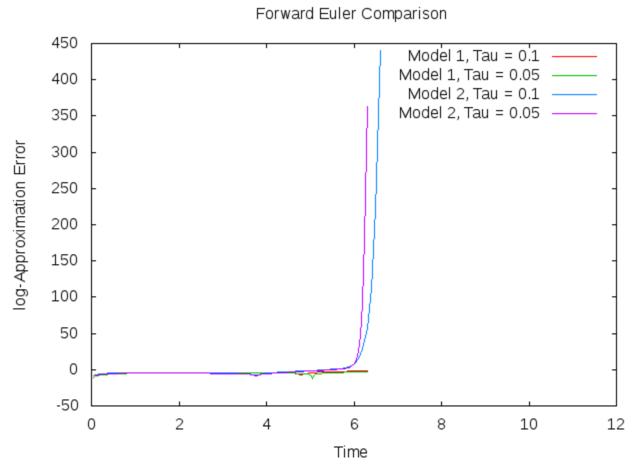
Scheme	Maximum Error	eoc
Forward Euler	1.60×10^{-3}	1.00479
Backward Euler	9.41×10^{-4}	1.00319
Crank-Nicholson	1.27×10^{-3}	1.00385
3-stage Heun	1.51×10^{-9}	-2.18845
2-stage DIRK	2.00×10^{-3}	1.00624

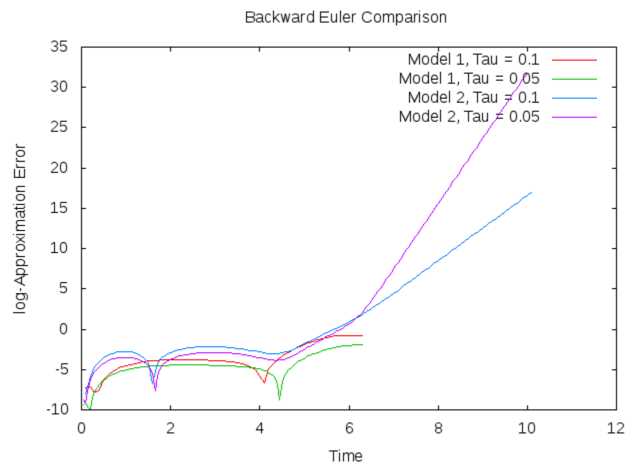
From the tables we see that eoc of Forward Euler and Backward Euler converge. From the values in the terminal, we also see that the 3-stage Heun converges initially but eventually the numerical errors causes the eoc to not converge.

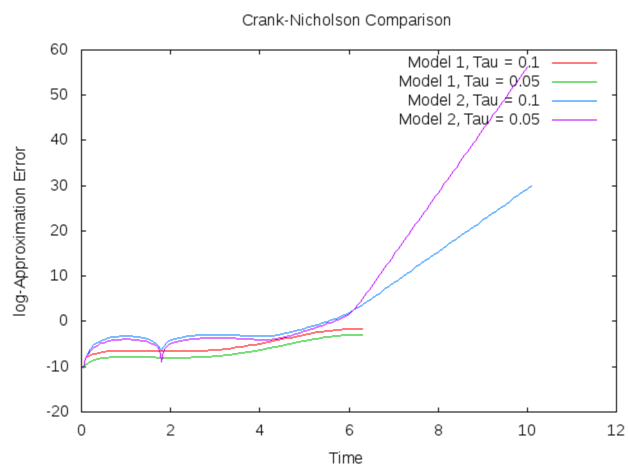
However, there are some unexpected results for Crank-Nicholson and 2-stage DIRK as the eoc for the two models do not converge to the same value. It is probable that the implementation for implicit methods is wrong.

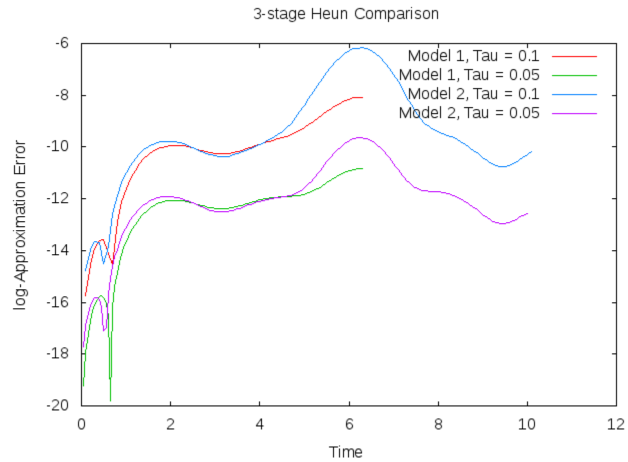
3 Approximation Error over Time

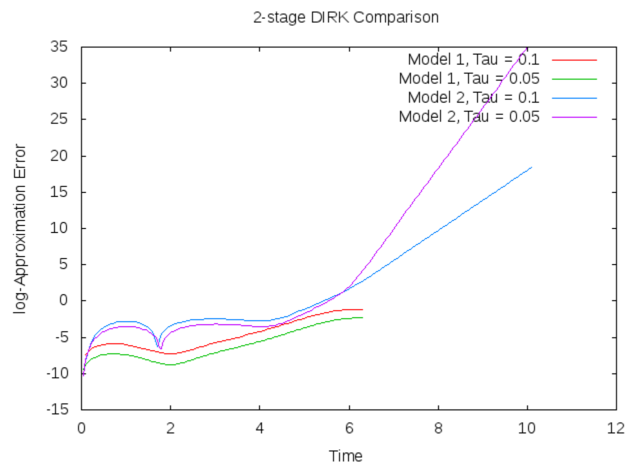
For each DIRK scheme, we graphed the log-approximation error of the approximate solution to the IVP against the time.


Forward Euler Comparison
450
400
350
300
250
200
150
100
50
0
-50
0
2
4
6
8
10
12
log-Approximation Error
Time
Model 1, Tau = 0.1
Model 1, Tau = 0.05
Model 2, Tau = 0.1
Model 2, Tau = 0.05


Backward Euler Comparison
35
30
25
20
15
10
5
0
-5
-10
0
2
4
6
8
10
12
log-Approximation Error
Time
Model 1, Tau = 0.1
Model 1, Tau = 0.05
Model 2, Tau = 0.1
Model 2, Tau = 0.05


Crank-Nicholson Comparison
60
50
40
30
20
10
0
-10
-20
0
2
4
6
8
10
12
log-Approximation Error
Time
Model 1, Tau = 0.1
Model 1, Tau = 0.05
Model 2, Tau = 0.1
Model 2, Tau = 0.05





We originally thought that the graphs would be increasing as time increased as the approximation and numerical errors should lead to increased error in the approximation of the solution. However, apart from the Forward Euler method, the graphs are quite erratic, with sharp decreases at certain time steps. This may be due to the fact that approximation error at a given time step could be cancelling out previous approximation errors. For example, as the exact solution for the second model is $e^{-\lambda t}\sin(t)$ which oscillates, so the changing sign of the derivative could cause the errors to cancel.

4 External Libraries

Although `odeint` was installed, there was not enough time to compare the library with the implementation.