Numerical Methods for Hyperbolic PDEs
Homework 6

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Exercise 1

See extra pdf.

Exercise 2

(a)

For $t = \frac{0.5}{\pi}$ we observe order 1 convergence in all norms. For $t = \frac{1.5}{\pi}$ we only observe this order 1 convergence in the L^1-norm, while the error seems to plateau in the other two norms. We observe that the reason for this is an oscillation at the shock.

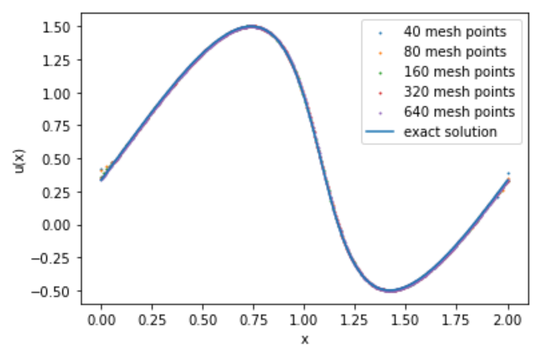

Figure 1: Numerical solutions with the the Lax-Wendroff scheme at time $t = \frac{0.5}{\pi}$

[*]Immatriculation Nr. 23-942-030
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N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.015	-	0.026	-	0.089	-
80	0.009	0.7077	0.016	0.6677	0.075	0.256
160	0.004	1.3039	0.008	1.0574	0.036	1.0458
320	0.001	1.4053	0.004	1.0458	0.017	1.0546
640	0.001	0.9924	0.002	1.0424	0.008	1.0449

Table 1: Errors and convergence rates for the Lax-Wendroff scheme at time $t = \frac{0.5}{\pi}$

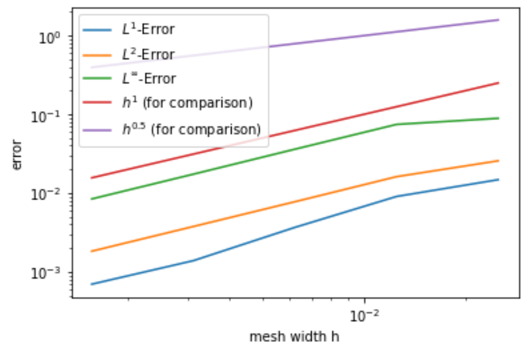

Figure 2: Error of numerical solution with the the Lax-Wendroff scheme at time $t = \frac{0.5}{\pi}$

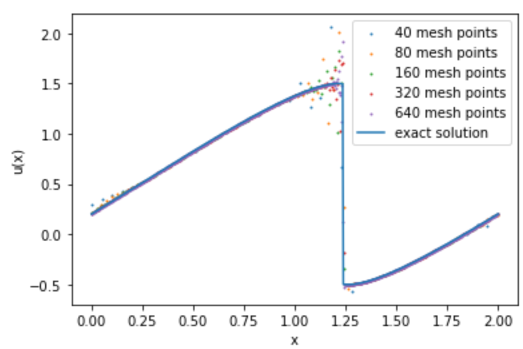

Figure 3: Numerical solutions with the the Lax-Wendroff scheme at time $t = \frac{1.5}{\pi}$

N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.12	-	0.235	-	0.832	-
80	0.067	0.837	0.165	0.5126	0.761	0.1282
160	0.028	1.2616	0.077	1.097	0.486	0.6464
320	0.016	0.8345	0.056	0.4515	0.47	0.0478
640	0.009	0.8275	0.051	0.1487	0.621	-0.4015

Table 2: Errors and convergence rates for the Lax-Wendroff scheme at time $t = \frac{1.5}{\pi}$

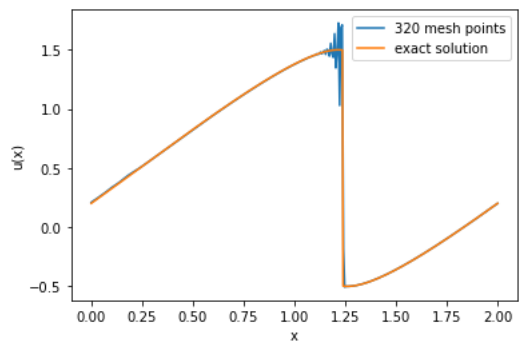

Figure 4: Numerical solution with the the Lax-Wendroff scheme at time $t = \frac{1.5}{\pi}$, 320 mesh points

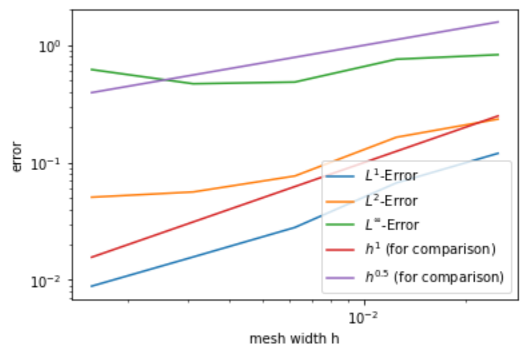

Figure 5: Error of numerical solution with the the Lax-Wendroff scheme at time $t = \frac{1.5}{\pi}$

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(b)

When only considering the errors on $[0, 0.75]$ at $t = \frac{1.5}{\pi}$ we observe order 1 convergence in all norms. This supports the hypothesis that the bad convergence at $t = \frac{1.5}{\pi}$ is caused by the oscillations around the shock.When only considering the errors on $[0, 0.75]$ at $t = \frac{1.5}{\pi}$ we observe order 1 convergence in all norms. This supports the hypothesis that the bad convergence at $t = \frac{1.5}{\pi}$ is caused by the oscillations around the shock.

N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.018	-	0.033	-	0.097	-
80	0.008	1.1322	0.015	1.1649	0.044	1.1477
160	0.004	0.9035	0.007	1.0064	0.02	1.0984
320	0.002	0.8962	0.004	1.013	0.01	1.072
640	0.001	1.0161	0.002	1.0085	0.005	1.0179

Table 3: Errors and convergence rates on $[0, 0.75]$ for the Lax-Wendroff scheme at time $t = \frac{1.5}{\pi}$

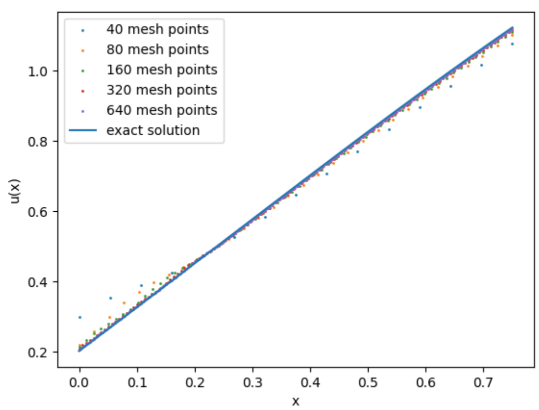

Figure 6: Numerical solutions on $[0, 0.75]$ of with the the Lax-Wendroff scheme at time $t = \frac{1.5}{\pi}$

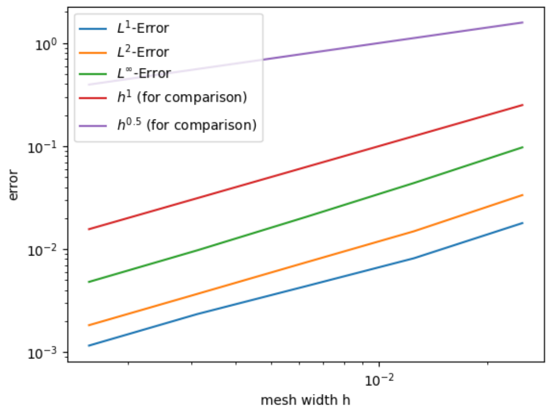

Figure 7: Error on $[0, 0.75]$ of numerical solution with the the Lax-Wendroff scheme at time $t = \frac{1.5}{\pi}$

(c)

Plotting the numerical versus exact solution with $N = 100$ when $t = \frac{1.5}{\pi}$ shows the oscillations in the Lax-Wendroff scheme before the shock.

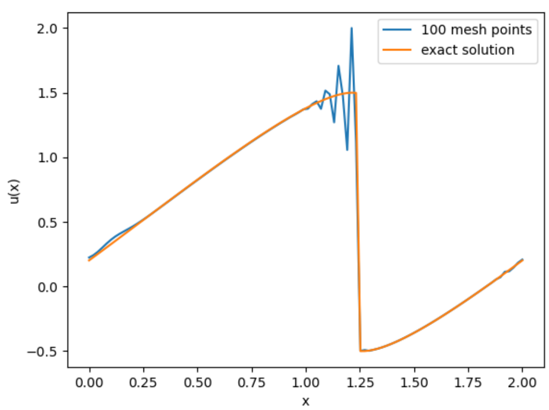

Figure 8: Numerical and exact solution with the the Lax-Wendroff scheme at time $t = \frac{1.5}{\pi}$, 100 mesh points

Exercise 3

We sketch the original proof of Lax and Richtmyer. Let $\mathcal{B}$ be a Banach space consisting of functions, containing the spacial solution of a well posed, linear initial value problem for each $0 \leq t \leq T$:

$$\begin{cases} u_t = Au(t) \\ u(0) = u_0 \end{cases}$$

where $A : \mathcal{B} \to \mathcal{B}$ is a (densely defined) linear differential operator. Note that we consider the time only as a parametrization of the solution, which is a family $(u(t))_t$ of functions in $\mathcal{B}$.

Denote the Banach space norm by $\|\cdot\|$. We call u a *genuiune solution* if u lies in the domain of A and

$$\left\| \frac{u(t+h) - u(t)}{h} - Au(t) \right\| \to 0, \quad \text{uniformly in t, } 0 \leq t \leq T.$$

, Define the operator

$$E_0(t)u_0 := u(t),$$

which is well defined since the problem is assumed well posed. Also, the well-posedness implies that there exists a constant K such that $\|E_0(t)u_0\| \leq K\|u_0\|$, i.e. they are uniformly bounded for $0 \leq t \leq T$. Note that the domain of E_0 are the unique initial values for the genuine solutions, which are assumed to be dense. We assume that we can extend E_0 continuously to the whole space $\mathcal{B}$, leading to the notion of weak solutions. We call this extension then E.

Consider now a linear numerical scheme $C_{\Delta t}$, seen as a *linear* operator on $\mathcal{B}$, where

$$U^{n+1} = C_{\Delta t} U^n.$$

We call the numerical scheme consistent, if for all genuine solutions u

$$\lim_{\Delta t \to 0} \left\| \frac{1}{\Delta t} \Big(C_{\Delta t} u(t) - u(t) \Big) - Au(t) \right\| = 0, \quad \text{uniformly in t}, 0 \leq t \leq T$$

.

We call the numerical scheme *convergent*, if for any sequence $\Delta_j t$, n_j, such that $n_j \Delta_j t \to t$ and for any initial data u_0, we have

$$\left\| \Big(C_{\Delta_j t} \Big)^{n_j} u_0 - E(t)u_0 \right\| \to 0, \quad 0 \leq t \leq T$$

Finally, we call the scheme *stable*, if the family of operators

$$\Big(C_{\Delta_j t} \Big)^n, \quad j \in \mathbb{N}, \quad 0 \leq n\Delta_j t \leq T \forall j$$

is uniformly bounded.

Convergent $\implies$ stable

Assume the scheme is convergent. Assume by contradiction, that there exists a sequence $n_j, \Delta_j t$ with $n_j \Delta_j t \le T$ and u_0 such that

$$\|C^{n_j}(\Delta_j t)u_0\| \to \infty, \quad (j \to \infty).$$

By passing to a subsequence, we can assume that there exists $n_j \Delta_j t$ converges to some t, $0 \le t \le T$. Since the scheme is convergent, we must have

$$C^{n_j}(\Delta_j t)u_0 \to E(t)u_0 \quad \text{in } \mathcal{B},$$

which gives a contradiction.

Stable and consistent $\implies$ Convergent

Assume the scheme is stable and consistent. Let

$$u(t) := E(t)u_0,$$

being a genuine solution. Let $\epsilon > 0$. By consistency, we can choose Δt small enough so that

$$\left\| \frac{1}{\Delta t}\Big(C_{\Delta t}u(t) - u(t)\Big) - Au(t) \right\| < \frac{\epsilon}{2}.$$

By the definition of a genuine solution, we can choose Δt also small enough so that

$$\left\| \frac{E(\Delta t)u(t) - u(t)}{\Delta t} - Au(t) \right\| < \frac{\epsilon}{2}.$$

Thus, by the triangle inequality,

$$\|C(\Delta t)u(t) - E(\Delta t)u(t)\| \le \epsilon \Delta t,$$

for Δt small enough. Define

$$\begin{aligned}
\psi_j &= \left[\Big(C(\Delta_j t)\Big)^{n_j} - E\Big(n_j \Delta_j t\Big)\right] u_0 \\
&= \sum_{k=0}^{n_j - 1} (C(\Delta_j t)^k \Big(C(\Delta_j t) - E(\Delta_j t)\Big) E((n_j - 1 - k)\Delta_j t) u_0.
\end{aligned}$$

Now we can estimate, for small enough $\Delta_j t$

$$\|\psi_j\| < K \sum_{k=0}^{n_j - 1} \epsilon \Delta_j t = K \epsilon n_j \Delta_j t < K \epsilon T.$$

Since ϵ was arbitrary, we obtain

$$\|\psi_j\| \to 0, \quad \text{as } \Delta_j t \to 0.$$

Suppose now that $n_j\Delta_j t \to t$ as $j \to \infty$, $0 < t < T$. For simplicity, assume $s := n_j\Delta_j t - t \geq 0$, so that we can write

$$\Big[E(n_j\Delta_j t) - E(t)\Big]u_0 = (E(s) - id)E(t)u_0.$$

(The case "≤ 0" is similar and gives the same results.) Then, since the operators $E(t)$ are uniformly bounded by well-posedness, there exists a constant $K_E > 0$ such that

$$\Big\|\Big[E(n_j\Delta_j t) - E(t)\Big]u_0\Big\| < K_E\|(E(s) - id)u_0\|,$$

converging to 0 as $j \to \infty$ since the solution is genuine. Therefore, since $\|\psi_j\| \to 0$, as $\Delta_j t \to 0$, we obtain

$$\Big\|\Big[C(\Delta_j t)^{n_j} - E(t)\Big]u_0\Big\| \to 0, \quad j \to \infty,$$

establishing convergence of the scheme. Note that if we assume that the genuine solutions are only dense in $\mathcal{B}$, we can extend this argument by the uniform boundedness principle.