Numerical Methods for Hyperbolic PDEs
Homework 4

James King[*], Robert Ziegler[†]

April 5, 2024

Homework 3, Exercise 3

We derive the general form of the characteristic curves. Let $x_0 \in \mathbb{R}$. Then, if a smooth solution exists, a characteristic curve $x(t)$ with $x(0) = x_0$ satisfies

$$\begin{aligned}\frac{d}{dt}u(x(t),t) &\overset{!}{=} 0\\ &= u_t(x(t),t) + u_x(x(t),t)x'(t)\\ &\overset{!}{=} u_t(x(t),t) + u(x(t),t)u_x(x(t),t)\end{aligned}$$

Thus,

$$\begin{cases} x'(t) = u(x(t),t) = u_0(x_0) \\ x(0) = x_0 \end{cases}$$

which has the unique solution

$$x(t) = u_0(x_0)t + x_0.$$

Therefore, since a smooth solution is constant along the characteristic curves, u must satisfy for all $t > 0$

$$u(u_0(x_0)t + x_0, t) = u_0(x_0).$$

In our case, this gives

$$u(\sin(\pi x_0)t + \frac{1}{2}t + x_0, t) = \sin(\pi x_0) + \frac{1}{2} \tag{1}$$

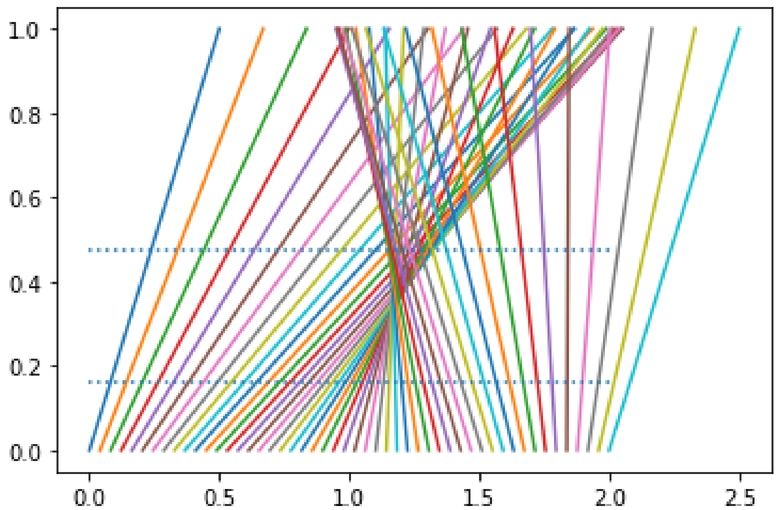
.

[*]Immatriculation Nr. 23-942-030
[†]Immatriculation Nr. 18-550-558

We want to substitute $x = \sin(\pi x_0)t + \frac{1}{2}t + x_0$. Then we recover x_0 from the tuple (x, t) for $t \neq 0$ by solving

$$\sin(\pi x_0)t + \frac{1}{2}t + x_0 - x \stackrel{!}{=} 0.$$

We solve for x_0 with the Newton-Method and insert it into Eq. (1). To choose suitable initial values for the Newton method, we plot the characteristic curves:



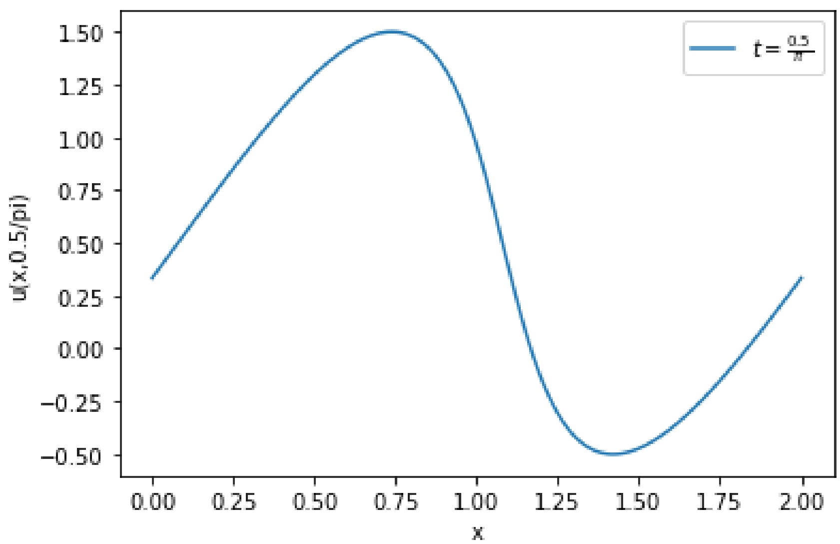
At time $t = \frac{1.5}{\pi}$, by the hint in the exercise class, the shock is located at

$$x_s = 1 + \frac{1}{2} \cdot \frac{1.5}{\pi}.$$

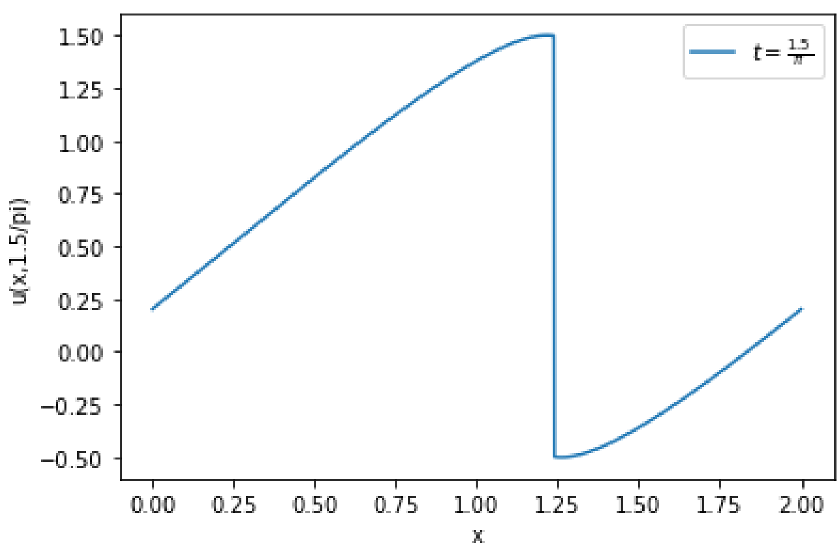
With the plot and the shock location, we use the following initial values for the Newton-Method, which are sufficient for convergence to the correct solution:

$$x_0^{\text{init}} = \begin{cases} x, & t = \frac{0.5}{\pi} \\ 0, & t = \frac{1.5}{\pi}, x < 0.2 \\ 0.5 & t = \frac{1.5}{\pi}, 0.2 \geq x < x_s \\ 1.5, & t = \frac{1.5}{\pi}, x \geq x_s \end{cases}$$

We obtain the following plots

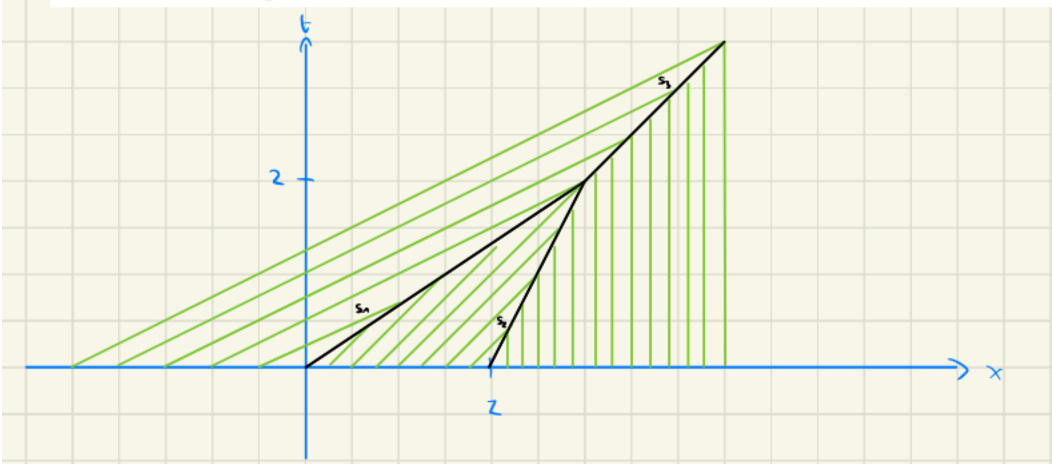


1.50
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
u(x,0.5/pi)
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x
$t = \frac{0.5}{\pi}$



1.50
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
u(x,1.5/pi)
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x
$t = \frac{1.5}{\pi}$

Homework 4, Exercise 1



$$\text{RH-cond}: \quad \frac{u_r^2 - u_l^2}{2(u_r - u_l)} = \frac{(u_r - u_l)(u_r + u_l)}{2(u_r - u_l)} = \frac{u_r + u_l}{2}$$

$$s_1 = \frac{2+1}{2} = \frac{3}{2}, \quad s_2 = \frac{1+0}{2} = \frac{1}{2}, \quad s_3 = \frac{2+0}{2} = 1$$

$$u(x,t) = \begin{cases} 2, & x < \frac{3}{2}t, & 0 \le t < 2 \\ 1, & \frac{3}{2}t < x < \frac{1}{2}t + 2, & 0 \le t < 2 \\ 0, & x > \frac{1}{2}t + 2, & 0 \le t < 2 \\ 2, & x < t + 1, & t \ge 2 \\ 0, & x > t + 1, & t \ge 2 \end{cases}$$

Homework 4, Exercise 2

Note that we have a scalar conservation law

$$u_t + f(u)_x = 0$$

with $f(u) = 2u$. Therefore, the CFL condition is

$$\max_j |f'(U_j^n)| \frac{\Delta t}{\Delta x} = 2\frac{\Delta t}{\Delta x} \le \frac{1}{2},$$

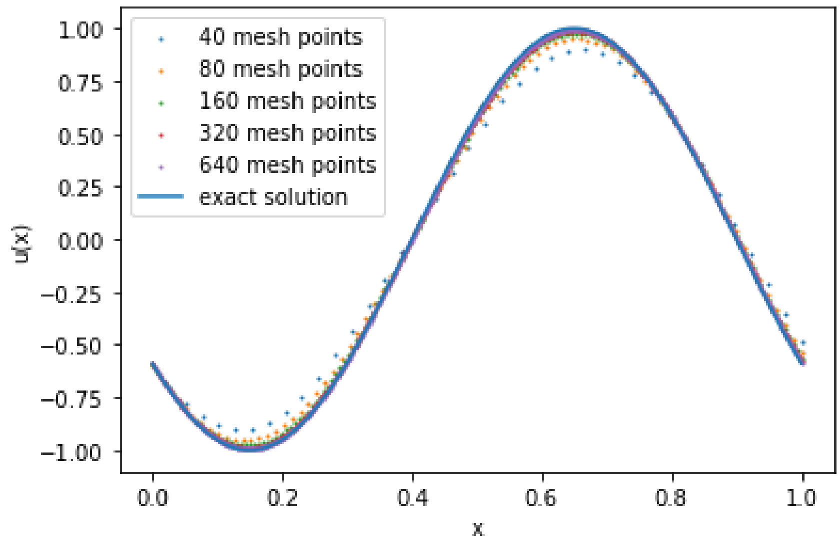
therefore we can choose $\Delta t = \frac{1}{4}\Delta x$. The Godunov flux is

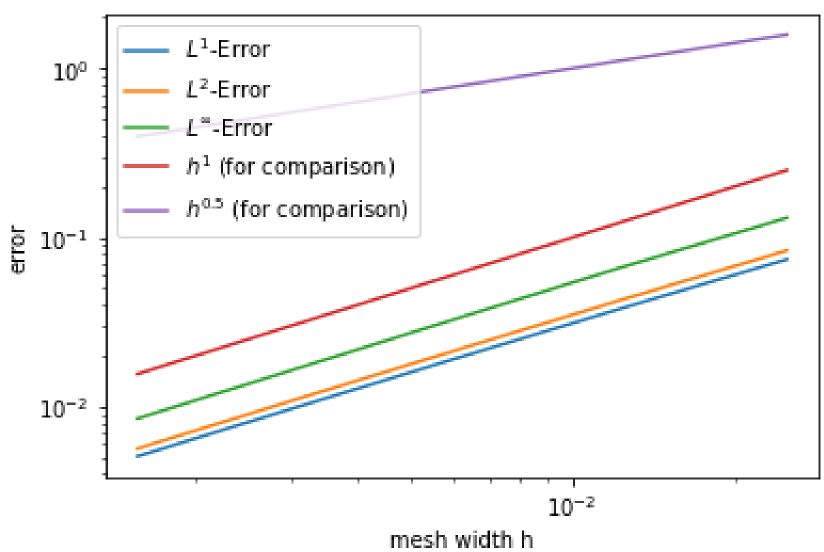
$$
\begin{aligned}
F_{j+\frac{1}{2}} &= \begin{cases} \min_{U_j^n \le \theta \le U_{j+1}^n} f(\theta), & U_j^n \le U_{j+1}^n \\ \max_{U_{j+1}^n \le \theta \le U_j^n} f(\theta), & U_j^n \ge U_{j+1}^n \end{cases} \\
&= f(U_j^n),
\end{aligned}
$$

since f is strictly increasing. As we showed in the first exercise sheet, the exact solution is

$$u(x,t) = \sin(2\pi(x - 2t)).$$

We get the following plots and convergence rates:





N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.0743	-	0.0838	-	0.1308	-
80	0.0386	0.9439	0.0434	0.9501	0.0675	0.9538
160	0.0198	0.964	0.0222	0.9692	0.034	0.989
320	0.0101	0.9754	0.0112	0.9807	0.017	0.9976
640	0.0051	0.9832	0.0057	0.9877	0.0085	0.999

Homework 4, Exercise 3

Note that we have a scalar conservation law

$$u_t + f(u)_x = 0$$

with $f(u) = \frac{u^2}{2}$. Therefore, the CFL condition is

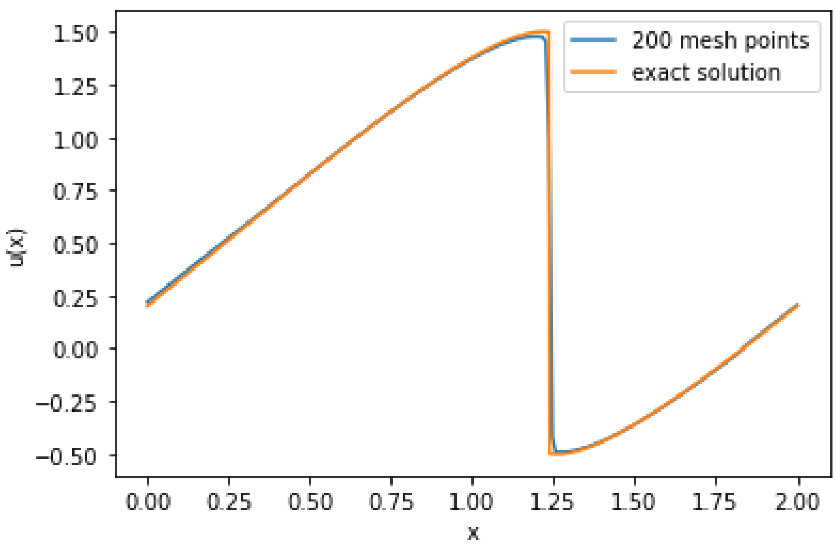
$$\max_j \left| f'(U_j^n) \right| \frac{\Delta t}{\Delta x} = \max_j \left| U_j^n \right| \frac{\Delta t}{\Delta x} \leq \frac{1}{2}.$$

Since the exact solution is bounded by the maximum of the initial values and we assume that numerical solution is close to the exact solution, we assume the bound $\max_j \left| U_j^n \right| \leq 2$ for all n. Therefore, we can choose $\Delta t = \frac{1}{4}\Delta x$. The Godunov flux is

$$\begin{aligned}
F_{j+\frac{1}{2}} &= \begin{cases} \min_{U_j^n \leq \theta \leq U_{j+1}^n} f(\theta), & U_j^n \leq U_{j+1}^n \\ \max_{U_{j+1}^n \leq \theta \leq U_j^n} f(\theta), & U_j^n \geq U_{j+1}^n \end{cases} \\
&= \begin{cases} f(U_j^n), & 0 \leq U_j^n \leq U_{j+1}^n \\ f(U_{j+1}^n), & U_j^n \leq U_{j+1}^n \leq 0 \\ 0, & U_j^n \leq 0 \leq U_{j+1}^n \\ f(U_j^n), & 0 \leq U_{j+1}^n \leq U_j^n \\ f(U_{j+1}^n), & U_{j+1}^n \leq U_j^n \leq 0 \\ f(\max\{\left|U_j^n\right|, \left|U_{j+1}^n\right|\}), & U_{j+1}^n \leq 0 \leq U_j^n \end{cases} \\
&= \max\{f(\max(U_j^n, 0)), f(\min(U_{j+1}^n, 0))\},
\end{aligned}$$

where we can easily verify the last equality case by case. As in homework sheet 3, exercise 3, we obtain the exact solution with a Newton method.

We get the following plot for a mesh size of 200:


200 mesh points
exact solution
u(x)
x
1.50
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00