Numerical Methods for Hyperbolic PDEs
Homework 1

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Exercise 1

(1a)

We first find necessary conditions for a smooth solution u of the IVP, where we assume the domain is $(x,t) \in \mathcal{D} = \mathbb{R} \times \mathbb{R}_{\geq 0}$. Since the PDE is linear (so also quasi-linear) of first order, the integral surface

$$\mathcal{S} = \{x, t, u(x,t) : (x,t) \in \mathcal{D}\} \subset \mathbb{R}^3$$

is the union of characteristic curves for the solution u. The characteristic field of the equation is the constant vector field

$$\mathbf{F}(x,t,u) = \begin{pmatrix} 8/3 \\ 1 \\ 0 \end{pmatrix},$$

satisfying $\mathbf{F} \cdot \begin{pmatrix} -u_x \\ -u_t \\ 1 \end{pmatrix} = 0$. Thus, setting $\Phi = (x,t,u)$, the characteristic equations given by the ODE system $\dot{\Phi} = F(\Phi)$ are

$$\begin{cases} \dot{x}(s) = 8/3 \\ \dot{t}(s) = 1 \\ \dot{u}(s) = 0 \end{cases}$$

Solving this, we get

$$\begin{cases} x(s) = 8/3s + x_0 \\ t(s) = s + t_0 \\ u(s) = u(x(s), t(s)) = C, \end{cases}$$

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where x_0, t_0, C are constants. Inserting the initial values we get

$$t(0) = t_0 = 0$$
$$u(0) = C = u(x_0, 0) = \exp(x_0^2).$$

Using $u(x(s), s) = C$ for all $s \geq 0$ we obtain the necessary condition

$$u(x(s), s) = u(8/3s + x_0, s) = \exp(x_0^2),$$

for all $x_0 \in \mathbb{R}$ and $s \geq 0$. By a substitution, using that

$$\begin{aligned} \mathbb{R} &\to \mathbb{R} \\ s &\mapsto 8/3s + x_0 \end{aligned}$$

is bijective, it follows that the solution u must satisfy

$$u(x, t) = \exp\left((x - 8/3t)^2\right).$$

Now it can easily be checked that this is indeed a solution of the IVP.

(1b)

Since the equation is again quasi-linear of first order, we can use the same approach as in (1a) and the same domain $\mathcal{D}$. Assuming again a smooth solution ρ, we have the characteristic vector field

$$\mathbf{F}(x, t, \rho) = \begin{pmatrix} 2\rho \\ 1 \\ 0 \end{pmatrix},$$

leading to the characteristic equations

$$\begin{cases} \dot{x}(s) = 2\rho \\ \dot{t}(s) = 1 \\ \dot{\rho}(s) = 0 \end{cases}$$

Since $\rho(s) = \rho(x(s), t(s))$ must be constant we obtain

$$\begin{cases} x(s) = 2Cs + x_0 \\ t(s) = s + t_0 \\ \rho(s) = C = \rho(0) \end{cases}$$

Inserting the inital values we get $t(s) = s$ and

$$\rho(s) = C = \begin{cases} 3, & x_0 < 0 \\ 4, & x_0 \geq 0 \end{cases}$$

$$x(s) = \begin{cases} 6s + x_0, & x_0 < 0 \\ 8s + x_0 & x_0 \geq 0 \end{cases}$$

Note that the characteristic curves $\{(x(t), t)\}_{x_0 \in \mathbf{R}}$ for the IVP do not cover all of $\mathcal{D}$, so there cannot exist a smooth solution.

However, we can find a weak solution. The PDE is of the form

$$\rho_t + f(\rho)_x = 0,$$

where $f(z) = z^2$. This is a scalar conservation law as a Riemann problem. Using the Rankine-Hugoniot condition in chapter 3.2.1 of the lecture notes, we find that the shock speed satisfies

$$s(t) = \frac{f(4) - f(3)}{4 - 3} = 16 - 9 = 7.$$

A weak solution is therefore given by

$$u(x,t) = \begin{cases} 3, & x < 7t \\ 4, & x \geq 7t. \end{cases}$$

Exercise 2

Let $\mathcal{D} = [0,1] \times \mathbb{R}_{\geq 0}$. We see as in Exc. (1a), that the PDE has characteristic curves of the form

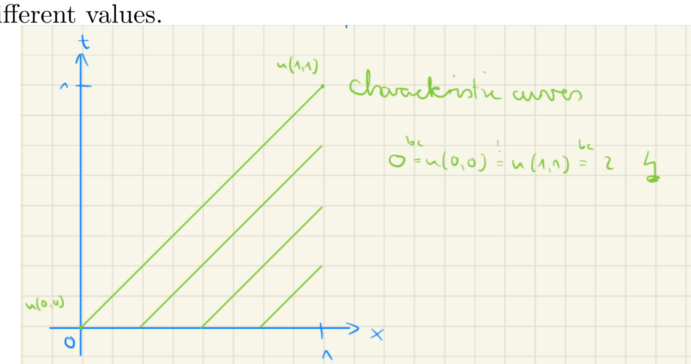
$$x(t) = t + x_0, \quad x_0 \in \mathbb{R}.$$

Therefore, a smooth solution u has the form $u(x,t) = u(x-t, 0) = \phi(x-t)$ for some function ϕ defined on $(-\infty, 1]$ which is smooth on $(-\infty, 1)$. Therefore, using the boundary conditions, we get

$$u(0,t) = \phi(-t) \stackrel{!}{=} 0, \quad t \geq 0$$
$$u(1,t) = \phi(1-t) \stackrel{!}{=} 2, \quad t \geq 0$$

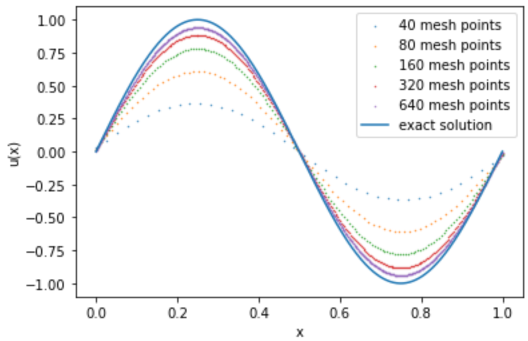
Thus, inserting $t = 0, t = 1$ respectively, we get $\phi(0) = 0$ and $\phi(0) = 2$, which is a contradiction. So no smooth solution can exist.

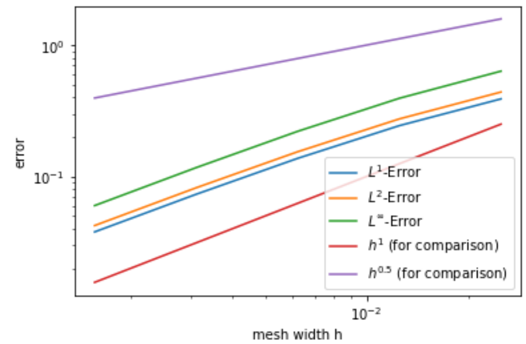
This makes sense physically: The value u of at $(x,t) = (0,0)$ gets transported along the line $x = t$ to the point $(1,1)$, but the boundary condition assumes different values.



Exercise 3

N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.389	-	0.439	-	0.633	-
80	0.244	0.676	0.274	0.680	0.394	0.686
160	0.137	0.826	0.154	0.829	0.221	0.835
320	0.073	0.909	0.082	0.912	0.117	0.919
640	0.038	0.952	0.042	0.955	0.060	0.960



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L1 average convergence rate: 0.8461225657687025
L2 average convergence rate: 0.8493698611798441
Linf average convergence rate: 0.8551629751356852
N=40
L1 Error at N=40: 0.38923302848668573
L2 Error  at N=40: 0.43903877844187666
Linf Error at N=40: 0.6330159194916749
N=80
L1 Error at N=80: 0.24362661560581
L2 Error  at N=80: 0.27411998636520424
Linf Error at N=80: 0.3935569426046138
L1 local convergence rate at N=80 : 0.675962382915764
L2 local convergence rate  at N=80: 0.6795408521992293
Linf local  convergence rate at N=80: 0.6856693917659438
N=160
L1 Error at N=160: 0.13740391449661243
L2 Error  at N=160: 0.1542774109075745
Linf Error at N=160: 0.22064638481508903
L1 local convergence rate at N=160 : 0.8262486470791478
L2 local convergence rate  at N=160: 0.8292806804609256
Linf local  convergence rate at N=160: 0.8348362805240694
N=320
L1 Error at N=320: 0.07318614062592668
L2 Error  at N=320: 0.08198966235056662
Linf Error at N=320: 0.11673426176222013
L1 local convergence rate at N=320 : 0.9087807315757385
L2 local convergence rate  at N=320: 0.9120129154835559
Linf local  convergence rate at N=320: 0.9185080522830033
N=640
L1 Error at N=640: 0.037828251432594016
L2 Error  at N=640: 0.0422828891918873
Linf Error at N=640: 0.060002689477895554
L1 local convergence rate at N=640 : 0.9521063779454269
L2 local convergence rate  at N=640: 0.9553680597832739
Linf local  convergence rate at N=640: 0.9601289847018769
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