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MAT827
Spring 2024
Institut für Mathematik
Universität Zürich

# Sheet 5

Deadline: 17.04.2024, 12:00 PM

**Exercise 1 (Points: 2.5, 2.5)**
Consider the nonlinear hyperbolic problem

$$u_t + f(u)_x = 0.$$

(a) If the flux function $f(u)$ is strictly convex and $f(u)$ has a single minimum at the point $\omega$ and no local maximum, the Godunov flux is given the following

$$\widehat{f}^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}) = \begin{cases} \min\limits_{u^n_j \le \theta \le u^n_{j+1}} f(\theta) & \text{if } u^n_j \le u^n_{j+1}, \\ \max\limits_{u^n_{j+1} \le \theta \le u^n_j} f(\theta) & \text{if } u^n_j > u^n_{j+1}, \end{cases}$$

which can be simplified to

$$\widehat{f}^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}) = \max(f(\max(u^n_j, \omega)), f(\min(u^n_{j+1}, \omega))). \tag{1}$$

If the flux function $f(u)$ is strictly concave and $f(u)$ has a single maximum at the point $\omega$ and no local minimum, please derive a similar formula as (1).

(b) If the flux function $f(u)$ has a single minimum at a point $\omega$, show that the Engquist-Osher flux is given the following

$$\widehat{f}^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}) = \frac{f(u^n_j) + f(u^n_{j+1})}{2} - \frac{1}{2}\int_{u^n_j}^{u^n_{j+1}} |f'(\theta)| \mathrm{d}\theta,$$

which can be written as

$$\widehat{f}^n_{j+\frac{1}{2}} = f(\max(u^n_j, \omega)) + f(\min(u^n_{j+1}, \omega)) - f(\omega).$$

**Exercise 2 (Points: 5, 5)**
Consider Burgers' equation

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, & x \in [0, 2] \\ u_0(x, 0) = \sin(\pi x) + \frac{1}{2} \end{cases}, \tag{2}$$

which is subject to the periodic boundary condition.

(a) Implement linearized Roe, Lax-Friedrichs, Rusanov and Engquist-Osher schemes for (2) when $t = \frac{0.5}{\pi}$. Please give the experimental convergence rates in the $L^1$-, $L^2$- and $L^\infty$-norms using a sequence of uniform grids with meshes $N = 40, 80, 160, 320, 640$ and also plot the numerical solutions together with the exact solution.

(b) Implement linearized Roe, Lax-Friedrichs, Rusanov and Engquist-Osher schemes for (2) when $t = \frac{1.5}{\pi}$. Please plot the numerical solutions together with the exact solution. What do you observe?

**Exercise 1 (Points: 2.5, 2.5)**
Consider the nonlinear hyperbolic problem

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(a) If the flux function $f(u)$ is strictly convex and $f(u)$ has a single minimum at the point $\omega$ and no local maximum, the Godunov flux is given the following

$$\widehat{f}^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}) = \begin{cases} \min\limits_{u^n_j \le \theta \le u^n_{j+1}} f(\theta) & \text{if } u^n_j \le u^n_{j+1}, \\ \max\limits_{u^n_{j+1} \le \theta \le u^n_j} f(\theta) & \text{if } u^n_j > u^n_{j+1}, \end{cases}$$

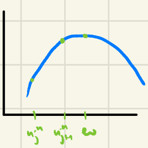
which can be simplified to

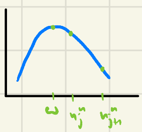
$$\widehat{f}^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}) = \max(f(\max(u^n_j, \omega)), f(\min(u^n_{j+1}, \omega))). \qquad (1)$$

If the flux function $f(u)$ is strictly concave and $f(u)$ has a single maximum at the point $\omega$ and no local minimum, please derive a similar formula as (1).

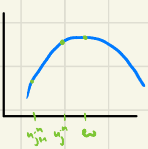
First let's consider the case $u_j^n \le u_{j+1}^n$. As f is strictly concave with no local minimum f takes on its minimum in $[u_j^n, u_{j+1}^n]$ on the boundary, so $\min\limits_{u_j^n \le \theta \le u_{j+1}^n} f(\theta) = \min(f(u_j^n), f(u_{j+1}^n))$

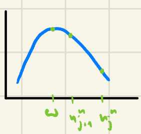
More precisely we have $\min(f(u_j^n), f(u_{j+1}^n)) = \begin{cases} f(u_j^n), & \text{if } \omega \ge u_{j+1}^n \\ f(u_{j+1}^n), & \text{if } \omega \le u_j^n \\ \min(f(u_j^n), f(u_{j+1}^n)), & \text{if } \omega \in (u_j^n, u_{j+1}^n) \end{cases}$

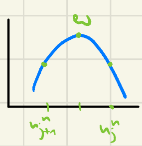



Now in the case $u_j^n > u_{j+1}^n$ we have $\max\limits_{u_{j+1}^n \le \theta \le u_j^n} f(\theta) = \begin{cases} f(u_j^n), & \omega \ge u_j^n \\ f(u_{j+1}^n), & \omega \le u_{j+1}^n \\ f(\omega), & \omega \in (u_{j+1}^n, u_j^n) \end{cases}$ for the same reason




We propose $\hat{f}^n_{j+\frac{1}{2}} = \min(f(\min(u_j^n, \omega)), f(\max(u_{j+1}^n, \omega)))$ and go through all the cases:

$f(\omega)$ max $(*)$

① $u_j^n \le u_{j+1}^n$:
- ⓐ $\omega \ge u_{j+1}^n \rightsquigarrow \hat{f}^n_{j+\frac{1}{2}} = \min(f(u_j^n), f(\omega)) \overset{(*)}{=} f(u_j^n)$ ✓
- ⓑ $\omega \le u_j^n \rightsquigarrow \hat{f}^n_{j+\frac{1}{2}} = \min(f(\omega), f(u_{j+1}^n)) \overset{(*)}{=} f(u_{j+1}^n)$ ✓
- ⓒ $\omega \in (u_j^n, u_{j+1}^n) \rightsquigarrow \hat{f}^n_{j+\frac{1}{2}} = \min(f(u_j^n), f(u_{j+1}^n))$ ✓

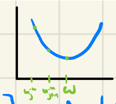
② $u_j^n > u_{j+1}^n$
- ⓐ $\omega \ge u_j^n \rightsquigarrow \hat{f}^n_{j+\frac{1}{2}} = \min(f(u_j^n), f(\omega)) \overset{(*)}{=} f(u_j^n)$ ✓
- ⓑ $\omega \le u_{j+1}^n \rightsquigarrow \hat{f}^n_{j+\frac{1}{2}} = \min(f(\omega), f(u_{j+1}^n)) \overset{(*)}{=} f(u_{j+1}^n)$ ✓
- ⓒ $\omega \in (u_{j+1}^n, u_j^n) \rightsquigarrow \hat{f}^n_{j+\frac{1}{2}} = \min(f(\omega), f(\omega)) = f(\omega)$ ✓

(b) If the flux function $f(u)$ has a single minimum at a point $\omega$, show that the Engquist-Osher flux is given the following

$$\widehat{f}^n_{j+\frac{1}{2}} = f(u_j^n, u_{j+1}^n) = \frac{f(u_j^n) + f(u_{j+1}^n)}{2} - \frac{1}{2}\int_{u_j^n}^{u_{j+1}^n} |f'(\theta)| \mathrm{d}\theta,$$

which can be written as

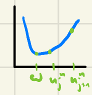
$$\widehat{f}^n_{j+\frac{1}{2}} = f(\max(u_j^n, \omega)) + f(\min(u_{j+1}^n, \omega)) - f(\omega).$$



Let us first consider $u_j^n \le u_{j+1}^n$.

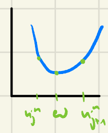
If $\omega \ge u_{j+1}^n$, then $f$ is decreasing on $[u_j^n, u_{j+1}^n]$ and $|f'(\theta)| = -f'(\theta)$ for $\theta \in [u_j^n, u_{j+1}^n]$

$$\rightsquigarrow \quad \frac{f(u_j^n) + f(u_{j+1}^n)}{2} + \frac{1}{2}\int_{u_j^n}^{u_{j+1}^n} f'(\theta)\, d\theta = f(u_{j+1}^n) = f(\omega) + f(u_{j+1}^n) - f(\omega) \quad \checkmark$$




If $\omega \le u_j^n$, then $f$ is increasing on $[u_j^n, u_{j+1}^n]$ and $|f'(\theta)| = f'(\theta)$ for $\theta \in [u_j^n, u_{j+1}^n]$

$$\rightsquigarrow \quad \frac{f(u_j^n) + f(u_{j+1}^n)}{2} - \frac{1}{2}\int_{u_j^n}^{u_{j+1}^n} f'(\theta)\, d\theta = f(u_j^n) = f(u_j^n) + f(\omega) - f(\omega) \quad \checkmark$$

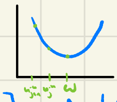



If $\omega \in (u_j^n, u_{j+1}^n)$ then $f$ is decreasing on $[u_j^n, \omega]$ and increasing on $[\omega, u_{j+1}^n]$, so

$|f'(\theta)| = -f'(\theta)$ for $\theta \in [u_j^n, \omega]$ and $|f'(\theta)| = f'(\theta)$ for $\theta \in [\omega, u_{j+1}^n]$.

$$\rightsquigarrow \quad \frac{f(u_j^n) + f(u_{j+1}^n)}{2} + \frac{1}{2}\int_{u_j^n}^{\omega} f'(\theta)\, d\theta - \frac{1}{2}\int_{\omega}^{u_{j+1}^n} f'(\theta)\, d\theta = f(\omega) = f(\omega) + f(\omega) - f(\omega)$$

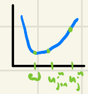
$$\widehat{f}^n_{j+\frac{1}{2}} = f(\max(u_j^n,\omega)) + f(\min(u_{j+1}^n,\omega)) - f(\omega).$$



Now for $u_{j+1}^n < u_j^n$

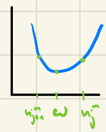
If $\omega \geq u_j^n$, then $f$ is decreasing on $[u_{j+1}^n, u_j^n]$ and $|f'(\theta)| = -f'(\theta)$ for $\theta \in [u_{j+1}^n, u_j^n]$

$$\rightsquigarrow \frac{f(u_j^n)+f(u_{j+1}^n)}{2} + \frac{1}{2}\int_{u_j^n}^{u_{j+1}^n} f'(\theta)\,d\theta = f(u_{j+1}^n) = f(\omega) + f(u_{j+1}^n) - f(\omega) \checkmark$$



If $\omega \leq u_{j+1}^n$, then $f$ is increasing on $[u_{j+1}^n, u_j^n]$ and $|f'(\theta)| = f'(\theta)$ for $\theta \in [u_{j+1}^n, u_j^n]$

$$\rightsquigarrow \frac{f(u_j^n)+f(u_{j+1}^n)}{2} - \frac{1}{2}\int_{u_j^n}^{u_{j+1}^n} f'(\theta)\,d\theta = f(u_j^n) = f(u_j^n) + f(\omega) - f(\omega) \checkmark$$



If $\omega \in (u_{j+1}^n, u_j^n)$ then $f$ is decreasing on $[u_{j+1}^n, \omega]$ and increasing on $[\omega, u_j^n]$, so

$|f'(\theta)| = -f'(\theta)$ for $\theta \in [u_{j+1}^n, \omega]$ and $|f'(\theta)| = f'(\theta)$ for $\theta \in [\omega, u_j^n]$.

$$\rightsquigarrow \frac{f(u_j^n)+f(u_{j+1}^n)}{2} - \frac{1}{2}\int_{u_j^n}^{\omega} f'(\theta)\,d\theta + \frac{1}{2}\int_{\omega}^{u_{j+1}^n} f'(\theta)\,d\theta = f(u_j^n) + f(u_{j+1}^n) - f(\omega) \checkmark$$