Exercise set 7

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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Problem 7.1 Monotonicity preserving (2pts)

A monotonicity preserving scheme is a method that

$$\begin{cases} u_j^n \le u_{j-1}^n \Rightarrow u_j^{n+1} \le u_{j-1}^{n+1} & \forall n, j \\ u_j^n \ge u_{j-1}^n \Rightarrow u_j^{n+1} \ge u_{j-1}^{n+1} & \forall n, j. \end{cases} \tag{1}$$

Prove that if the scheme is a constant coefficient schemes with footprint $2K+1$, i.e.,

$$u_j^{n+1} = \sum_{l=-K}^{K} C_l u_{j+l}^n, \tag{2}$$

then it is monotonicity preserving if and only if $C_l \ge 0$ for all $l = -K, \dots, K$.

Solution

($\Leftarrow$)

Let us first prove that MP (monotonicity preserving) implies $C_\ell \ge 0$ $\forall \ell = -K, \dots, K$.

BY CONTRADICTION, SUPPOSE $\exists L \in [\![-K, K]\!]$: $C_L < 0$.

Take $U_j^0 = \begin{cases} 0 & j \le 0 \\ 1 & j \ge 1 \end{cases}$ $j \in \mathbb{Z}$. U^0 IS MONOTONE INCREASING. $U_{j+1}^0 - U_j^0 \ge 0$ $\forall j \in \mathbb{Z}$.

Then $U_{-L+1}^1 - U_{-L}^1 = \sum_{\ell=-K}^{K} C_\ell \underbrace{\left(U_{-L+1+\ell}^0 - U_{-L+\ell}^0\right)}_{\ne 0 \text{ ONLY FOR } \ell = L} = C_L (U_1^0 - U_0^0) = C_L < 0$

$\Rightarrow U^1$ IS DECREASING! CONTRADICTION $\Rightarrow C_\ell \ge 0$ $\forall \ell = -K, \dots, K$.

($\Rightarrow$) Let's prove that $C_\ell \ge 0$ $\forall \ell \Rightarrow$ MP.

CASE 1 WE KNOW THAT $U_{j+1}^n - U_j^n \ge 0$ $\forall j \in \mathbb{Z}$

$\Rightarrow U_{j+1}^{n+1} - U_j^{n+1} = \sum_{\ell=-K}^{K} \underbrace{C_\ell}_{\ge 0} \underbrace{\left(U_{j+1+\ell}^n - U_{j+\ell}^n\right)}_{\ge 0} \ge 0$ $\forall j \in \mathbb{Z}$.

CASE 2 WE KNOW THAT $U_{j+1}^n - U_j^n \le 0$ $\forall j \in \mathbb{Z}$

$U_{j+1}^{n+1} - U_j^{n+1} = \underbrace{\sum_{\ell=-K}^{K} \underbrace{C_\ell}_{\ge 0} \underbrace{\left(U_{j+1+\ell}^n - U_{j+\ell}^n\right)}_{\le 0}}_{\le 0} \le 0$ $\forall j \in \mathbb{Z}$ $\boxtimes$

Problem 7.2 Condition on viscous form to TVD (2pts)

Prove that a FD scheme written in the viscous form

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{1}{2}\left(Q_{j+1/2}\Delta u_{j+1/2}^n - Q_{j-1/2}\Delta u_{j-1/2}^n\right), \tag{3}$$

where $\lambda = \Delta t/\Delta x$, $\Delta u_{j+1/2} = u_{j+1} - u_j$ and $f_{j+1/2} = f(u_j) - f(u_{j-1})$, is TVD under the condition

$$\lambda \left| \frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}} \right| \le Q_{j+1/2} \le 1. \tag{4}$$

Hint Use the result of Harten's Lemma (Theorem 3.13 of the notes).

Solution

LET'S RECALL HARTEN'S LEMMA. GIVEN A SCHEME IN THE INCREMENTAL FORM $U_j^{n+1} = U_j^n + C_{j+1/2}^n \Delta U_{j+1/2}^n - D_{j-1/2}^n \Delta U_{j-1/2}^n$, IT IS TVD IF $C_{j+1/2}^n, D_{j+1/2}^n \ge 0$ $\forall j \in \mathbb{Z}$ AND $C_{j+1/2} + D_{j+1/2} \le 1$ $\forall j \in \mathbb{Z}$.

WE CAN WRITE (3) IN THE INCREMENTAL FORM SETTING

$C_{j+1/2}^n = -\frac{\lambda}{2} \frac{f(U_{j+1}^n) - f(U_j^n)}{U_{j+1}^n - U_j^n} + \frac{1}{2} Q_{j+1/2}$ and $D_{j-1/2}^n = \frac{\lambda}{2} \frac{f(U_j^n) - f(U_{j-1}^n)}{U_j^n - U_{j-1}^n} + \frac{1}{2} Q_{j-1/2}$

NOW, CONDITION (4) VERIFIES THAT $C_{j+1/2}^n = -\frac{\lambda}{2} \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n} + \frac{1}{2} Q_{j+1/2}^n \ge 0$

AND THAT $D_{j+1/2}^n = \frac{\lambda}{2} \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n} + \frac{1}{2} Q_{j+1/2}^n \ge 0$ $\forall j \in \mathbb{Z}$.

AGAIN, IF WE WRITE $C_{j+1/2}^n + D_{j+1/2}^n = \cancel{-\frac{\lambda}{2} \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n}} + \frac{1}{2} Q_{j+1/2}^n + \cancel{\frac{\lambda}{2} \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n}} + \frac{1}{2} Q_{j+1/2}^n$

$= Q_{j+1/2}^n \le 1$,

WE SEE THAT (4) VERIFIES ALSO THE SECOND HYPOTHESIS OF HARTEN'S LEMMA. $\boxtimes$

Problem 7.3 Recipe for TVD schemes (2pts)

The condition (4) gives us a recipe for building TVD schemes. Consider footprint-3 schemes, i.e., $Q_{j+1/2} = Q(u_j, u_{j+1})$, and using the ansatz that

$$Q(u,v) = q(\lambda a(u,v)), \quad a(u,v) = \begin{cases} \frac{f(u)-f(v)}{u-v} & \text{if } u \ne v, \\ f'(u) & \text{if } u = v, \end{cases} \tag{5}$$

deduce conditions on the function q, such that the FD scheme would be TVD.

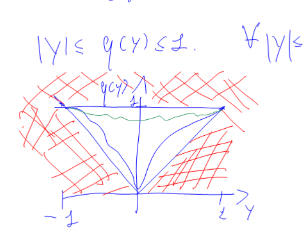
Solution

FROM (4) WE PROCEED SUBSTITUTING THE ANSATZ (5):

$\lambda \left| \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n} \right| \le Q_{j+1/2}^n \le 1$ (ASSUME $U_{j+1} \ne U_j$, OTHERWISE APPLY THE LIMIT FOR $U_j \to U_{j+1}$)

$\lambda \left| \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n} \right| \le q\left(\lambda \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n} \right) \le 1$. DEFINING $y := \lambda \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n}$, WE OBTAIN

$|y| \le q(y) \le 1$. $\forall |y| \le 1$.



ANY FUNCTION INSIDE THE CONSTRAINTS GIVES USE A TVD SCHEME.

Problem 7.4 TVD or not TVD (4pts)

1. Consider the Lax–Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{\lambda^2}{2}\left(a_{j+1/2}\Delta f_{j+1/2} - a_{j-1/2}\Delta f_{j-1/2}\right), \tag{6}$$

where $a_{j+1/2} = a(u_j, u_{j+1})$ using the definitions of previous exercises. Prove or disprove that the scheme is TVD.
Hint Use the criterion you found in Problem 7.3.

Solution

THE SCHEME IS NOT TVD. USING THE VISCOUS FORMULATION (3), WE CAN SET $Q_{j+1/2}^n$ TO BE SUCH THAT

$\lambda^2 a_{j+1/2}^n \Delta f_{j+1/2}^n = Q_{j+1/2}^n \Delta U_{j+1/2}^n \Leftrightarrow Q_{j+1/2}^n = \frac{\lambda^2 a_{j+1/2}^n \Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n} = \lambda^2 \left(\frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n} \right)^2$.

HENCE, $q(y) = y^2$ IN THIS CASE. WE KNOW THAT $y^2 \not\ge |y|$ $\forall |y| < 1$.

THE CONDITION WE FOUND BEFORE DOESN'T HOLD!

TO FINALLY PROVE THAT THE SCHEME IS NOT TVD, WE FIND A COUNTER EXAMPLE. TAKE $U_0(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$ FOR THE EQUATION $\partial_t u + \partial_x u = 0$.

$\begin{cases} U_j^0 = 0 & j \le 0 \quad j \in \mathbb{Z} \\ U_j^0 = 1 & j \ge 1 \quad j \in \mathbb{Z} \end{cases}$ $\quad U_j^1 = 0 \; j \le -1 \quad U_0^1 = 0 - \frac{\lambda}{2} + \frac{\lambda^2}{2}$

$\quad U_j^1 = 1 \; j \ge 2 \quad U_1^1 = 1 - \frac{\lambda}{2} - \frac{\lambda^2}{2}$

$TV(U^0) = 1$ $\quad TV(U^1) = \left| -\frac{\lambda}{2} + \frac{\lambda^2}{2} \right| + \left| 1 - \lambda^2 \right| + \left| \frac{\lambda}{2} + \frac{\lambda^2}{2} \right| \underset{\lambda<1}{=} \frac{\lambda}{2} - \cancel{\frac{\lambda^2}{2}} + 1 - \lambda^2 + \frac{\lambda}{2} + \cancel{\frac{\lambda^2}{2}}$

$= 1 + \lambda - \lambda^2 > 1$ IF $\lambda < 1$ $\boxtimes$

REMARK $q(1) = 1^2 = 1 = |1| = |\pm 1|$. THIS MEANS THAT FOR $\left| \lambda \frac{\Delta f_{j+1/2}^n}{\Delta U_{j+1/2}^n} \right| = 1$ $\forall j, n$ THE SCHEME WOULD BE TVD. (VERY RARE CONDITION).

2. Consider the Godunov method in its conservative form

$$u_j^{n+1} = u_j^n - \lambda\left(f^{num}(u_j^n, u_{j+1}^n) - f^{num}(u_{j-1}^n, u_j^n)\right) \tag{7}$$

where

$$f^{num}(u,v) = \begin{cases} \min_{u \le \theta \le v} f(\theta) & \text{if } u \le v, \\ \max_{v \le \theta \le u} f(\theta) & \text{if } v \le u, \end{cases}$$

prove or disprove that the scheme is TVD.
Hint Use directly Harten's Lemma conditions.

Solution

LET'S WRITE (7) IN THE INCREMENTAL FORM:

$U_j^{n+1} = U_j^n + C_{j+1/2}^n \Delta U_{j+1/2}^n - D_{j-1/2}^n \Delta U_{j-1/2}^n$ WITH

$C_{j+1/2}^n = \lambda \frac{f(U_j^n) - f^{num}(U_j^n, U_{j+1}^n)}{\Delta U_{j+1/2}}$ AND $D_{j-1/2}^n = \lambda \frac{f(U_j^n) - f^{num}(U_{j-1}^n, U_j^n)}{\Delta U_{j-1/2}^n}$.

CASE $U_j^n \le U_{j+1}^n$

$C_{j+1/2}^n = \underbrace{\lambda}_{\ge 0} \frac{\left[f(U_j^n) - \overbrace{\min_{U_j \le \theta \le U_{j+1}} f(\theta)}^{\le f(U_j^n)} \right] \ge 0}{\underbrace{U_{j+1}^n - U_j^n}_{\ge 0}} \ge 0$ $\quad D_{j+1/2}^n = \underbrace{\lambda}_{\ge 0} \frac{\left[f(U_{j+1}^n) - \overbrace{\min_{U_j \le \theta \le U_{j+1}} f(\theta)}^{\le f(U_{j+1}^n)} \right] \ge 0}{\underbrace{U_{j+1}^n - U_j^n}_{\ge 0}} \ge 0$

CASE $U_j^n \ge U_{j+1}^n$

$C_{j+1/2}^n = \lambda \frac{\left[f(U_j^n) - \overbrace{\max_{\le \theta \le} f(\theta)}^{\ge f(U_j^n)} \right] \le 0}{\underbrace{U_{j+1}^n - U_j^n}_{\le 0}} \ge 0$ $\quad D_{j+1/2}^n = \lambda \frac{\left[f(U_{j+1}^n) - \overbrace{\max_{\theta} f(\theta)}^{\ge f(U_{j+1}^n)} \right] \le 0}{\underbrace{U_{j+1}^n - U_j^n}_{\le 0}} \ge 0$

FINALLY

$C_{j+1/2}^n = \lambda \frac{f(U_j^n) - \max\min_{U_j \le \theta \le U_{j+1}} f(\theta)}{U_{j+1}^n - U_j^n} \le \lambda \max_{U_j \le \theta \le U_{j+1}} |f'(\theta)|$. SIMILARLY $D_{j+1/2}^n \le \lambda \max_{U_j \le \theta \le U_{j+1}} |f'(\theta)|$

$C_{j+1/2}^n + D_{j+1/2}^n \le 1$ IF $\lambda \max_{U_j \le \theta \le U_{j+1}} |f'(\theta)| \le \frac{1}{2}$ $\forall j \in \mathbb{Z}$. UNDER THIS CFL CONDITION IT IS TVD.

3. Test the Lax–Wendroff method for the linear transport equation $\partial_t u + \partial_x u = 0$ on $[-1, 1]$ with periodic boundary conditions and initial conditions $u_0 = \cos(\pi x)$. Plot the total variation of the scheme as a function of time, till $T = 1$, for CFL= 0.7 and CFL= 1. What do you observe? Why?
Hint You can use the solutions of the Exercise Set 5.

Solution

REMARK IN 7.4.1

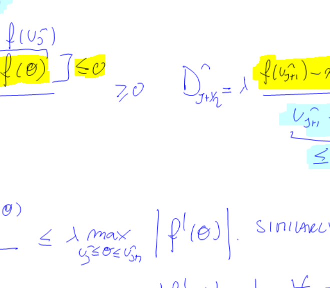


Figure 1: Total variation for Lax Wendroff scheme at different CFLs

```matlab
function [u,t,x, varargout] = lxw(c, N, cfl, T, a,b, u0, varargin)
%LAXFR   Solve the advection equation u_t + c u_x = 0 using the
%   upwind/downwind scheme. The solution is computed on the periodic domain
%   [-a, b) up to t=tMax.

    if nargin>7
        with_error=1;
    else
        with_error=0;
    end

  % Create the mesh
  x = linspace(a, b, N+1);
  dx = x(2)-x(1);
  x = [x(N), x];      % Mesh with ghost cells {x_0, ..., x_{N+1}}
  j = 2 : N+1;     % Index into the internal mesh  1,...,N

  % Set initial data and advection speed
  aMax = abs(c);
    if with_error
        u_exact=@(x,t) u0(x-c*t);
      uu_exact(1,:) = u_exact(x,0);
    end
    u(1,:) = u0(x);


  % Set the timestep and make sure that tMax/dt is an integer
  dt = cfl * dx/aMax;
  nt = ceil(T / dt)+1;
  t = [0:nt-1]*dt; % linspace(0, T, nt);
  %dt = t(2)-t(1);
  lambda = dt/dx;

  % Run the simulation
  for n = 2:nt
    % Update the solution
    u(n,j) = u(n-1,j)- lambda/2*c*(u(n-1,j+1)-u(n-1,j-1)) ...
            + c.^2*lambda^2/2*(u(n-1,j-1)-2*u(n-1,j)+u(n-1,j+1));

    % Set boundary conditions
    u(n,1) = u(n,N+1);
    u(n,N+2) = u(n,2);
        if with_error
            uu_exact(n,:) = u_exact(x,t(n));
        end
    end
    if with_error
      errL2end= norm(u(end,:)-uu_exact(end,:))*sqrt(dx);
        varargout={uu_exact,errL2end};
    else
        varargout={};
    end
  % Remove the ghost cell to the left
  x = x(2:end-1);
  u = u(:,2:end-1);
end
```

Listing 1: lxw.m

```matlab
% test lxf

c=1;
T=1;
a=-1;
b=1;
u0=@(x) cos(pi*x);

%% run lxw scheme at CFL 1 and 0.7
%
Nx = 40;
tfin = 1.;
cfl0 = 1;
with_error=1;
plot_evolution=1;
%
[ulxwCFL1,tCFL1,x,ex,err] = lxw(c, Nx, cfl0, tfin, a,b, u0, with_error);
cfl1=0.7;
[ulxwCFL07,tCFL07,x,ex,err] = lxw(c, Nx, cfl1, tfin, a,b, u0, with_error);

%% plot evolution
dx = x(2)-x(1);
u0 = @(x) cos(pi*x);
fig=figure(1);
%
figure(1)
subplot(211)
plot(x,ulxwCFL1(end,:),...
    x,u0(x-c*tfin),'r--')
legend('numerical','analytical','Location','SE')
xlabel x
ylabel u
title(sprintf('Lax-Wendroff, CFL=%g, t=%g',cfl0,tCFL1(end)))
ylim([-1.5,1.5])

subplot(212)
plot(x,ulxwCFL07(end,:),...
    x,u0(x-c*tfin),'r--')
legend('numerical','analytical','Location','SE')
xlabel x
ylabel u
title(sprintf('Lax-Wendroff, CFL=%g, t=%g',cfl1,tCFL07(end)))
ylim([-1.5,1.5])

saveas(fig, 'LxWtests.pdf')

%% plot TV instability for upwind

fig=figure
TVlxwCFL1=zeros(size(tCFL1));
ulxCFL1_1=zeros(size(ulxwCFL1));
ulxCFL1_1(:,1:end-1)=ulxwCFL1(:,2:end);
ulxCFL1_1(:,end)=ulxwCFL1(:,1);

TVlxwCFL07=zeros(size(tCFL07));
ulxwCFL07_1=zeros(size(ulxwCFL07));
ulxwCFL07_1(:,1:end-1)=ulxwCFL07(:,2:end);
ulxwCFL07_1(:,end)=ulxwCFL07(:,1);
for n=1:numel(tCFL1)
    TVlxwCFL1(n)= sum(abs(ulxwCFL1(n,:)-ulxCFL1_1(n,:)))*dx;
end
for n=1:numel(tCFL07)
    TVlxwCFL07(n)= sum(abs(ulxwCFL07(n,:)-ulxwCFL07_1(n,:)))*dx;
    %
end

semilogy(tCFL1,TVlxwCFL1,'DisplayName','LxW CFL=1')
hold on
semilogy(tCFL07,TVlxwCFL07,'DisplayName','LxW CFL=0.7')

xlabel('t','FontSize',16)
ylabel('TotalVariation','FontSize',16)
title('Total Variation','FontSize',16)
legend('FontSize',15,'Location','best')
hold off
saveas(fig,'TVlxw.pdf')
```

Listing 2: TVlxw.m

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Due date: Apr 30, 2020, h10.00 (use the upload tool of my.math.uzh.ch, see wiki.math.uzh.ch/public/student_upload_homework or if you have troubles send me an email).