Universität Zürich
Institut für Mathematik
MAT827
Spring Semester 2024

Dr. Yongle Liu

Sheet 1: Due 12:00 PM, March 6

1. (6 Points) Solve the following initial value problems using Method of Characteristics:

(a) $$\begin{cases} u_t + \frac{8}{3}u_x = 0 \\ u(x,0) = \exp(x^2) \end{cases}$$

(b) $$\begin{cases} \rho_t + 2\rho\rho_x = 0 \\ \rho(x,0) = \rho_0(x) = \begin{cases} 3, & x < 0 \\ 4, & x \geq 0 \end{cases} \end{cases}$$

2. (3 points) Show that the PDE $u_t + u_x = 0$, $x \in [0,1]$, $t \in [0,\infty)$, has no smooth solutions satisfying the boundary condition $u(0,t) = 0$, $u(1,t) = 2$. Explain this physically. (Hint: draw several characteristic curves. This exercise makes the point that the boundary condition for transport equations has to be given carefully.)

3. (6 points) Implement the upwind scheme for the following IBVP

$$\begin{cases} u_t + 2u_x = 0, & x \in [0,1] \\ u(x,0) = \sin(2\pi x) \end{cases},$$

which is subjected to the periodic boundary condition. Use a sequence of uniform grids with meshes $N = 40, 80, 160, 320, 640$. Select Δt according to the CFL condition. Plot the computed solutions together with the exact one at times $t = 2$. Check the experimental convergence rates in the L^1-, L^2- and L^∞-norms and report the results in the following table:

Table 1: Errors and rates of the point values of u, $t = 2$.

N	L^1	rate	L^2	rate	L^∞	rate
40	15,6	-	2,8	-	0,6	-
80	19,5	–	2,5	0,5	0,4	1
160	22		2		0,2	
320	23,4		1,5		0,12	
640	24,1		1,1		0,06	

1. (6 Points) Solve the following initial value problems using Method of Characteristics:

(a) $\begin{cases} u_t + \frac{8}{3} u_x = 0 \\ u(x,0) = \exp(x^2) \end{cases}$ (b) $\begin{cases} \rho_t + \overbrace{2\rho\rho_x}^{=(\rho^2)_x} = 0 \\ \rho(x,0) = \rho_0(x) = \begin{cases} 3, & x < 0 \\ 4, & x \geq 0 \end{cases} \end{cases}$

a) $a = \frac{8}{3}$

if $x(t)$ is a characteristic curve

$$u_t + a u_x \overset{!}{=} 0 = \frac{d}{dt} u(x(t),t) = \partial_t u(x(t),t) + \partial_x u(x(t),t)\,\dot{x} = u_t + \dot{x} u_x \rightsquigarrow \begin{cases} \dot{x}(t) = a \\ x(0) = x_0 \end{cases} \rightsquigarrow x(t) = at + x_0$$

solution is constant along this curve

$$\Longrightarrow u(x(t),t) = u(x(0),0) = \exp(x_0^2) = \exp\left((x(t) - at)^2\right)$$

b)

if $x(t)$ is a characteristic curve

$$\rho_t + 2\rho\rho_x \overset{!}{=} 0 = \frac{d}{dt} \rho(x(t),t) = \partial_t \rho(x(t),t) + \partial_x \rho(x(t),t)\,\dot{x} = \rho_t + \dot{x}\rho_x \rightsquigarrow \begin{cases} \dot{x}(t) = \rho(x(t),t) \\ x(0) = x_0 \end{cases}$$

$$\rho(x(t),t) = \rho_0(x_0) \Longrightarrow \begin{cases} \dot{x}(t) = \rho_0(x_0) \\ x(0) = x_0 \end{cases} \rightsquigarrow x(t) = \rho_0(x_0)t + x_0$$

ρ constant along $x(t)$

The shock speed given by the Rankine-Hugoniot condition is

$$\frac{\rho_r^2 - \rho_l^2}{2(\rho_r - \rho_l)} = \frac{1}{2}(\underset{4}{\rho_r} + \underset{3}{\rho_l}) = 3{,}5$$

Therefore a weak solution is $\rho(x,t) = \begin{cases} 3 & x < 3{,}5t \\ 4 & x \geq 3{,}5t \end{cases}$

2. (3 points) Show that the PDE $u_t + u_x = 0, \quad x \in [0,1],\ t \in [0,\infty)$, has no smooth solutions satisfying the boundary condition $u(0,t) = 0$, $u(1,t) = 2$. Explain this physically. (Hint: draw several characteristic curves. This exercise makes the point that the boundary condition for transport equations has to be given carefully.)

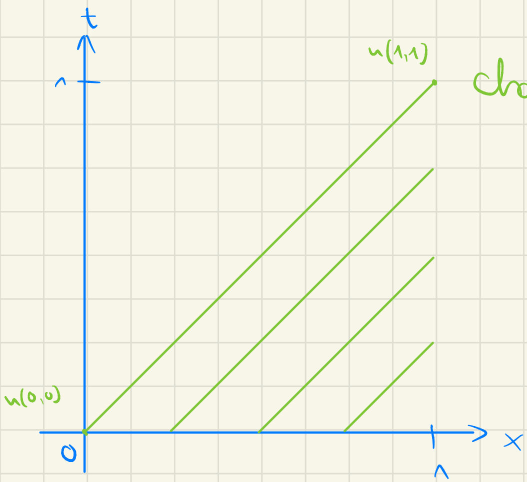
$a = 1$

As shown in the previous exercise $x(t) = at + x_0$ with $a = 1$ describes the characteristic curves of the PDE.

Let u be a smooth sol. of the PDE $\overset{\text{linear advection}}{\rightsquigarrow}$ it has the form $u(x,t) = \phi(x-t)$ for ϕ smooth.

$$u(0,t) = \phi(-t) \overset{!}{=} 0 \quad \forall t \geq 0 \implies \phi(0) = 0 \quad (t=0)$$
$$u(1,t) = \phi(1-t) \overset{!}{=} 2 \quad \forall t \geq 0 \implies \phi(0) = 2 \quad (t=1) \quad ↯$$

So there can't be such a ϕ



characteristic curves

$$0 \overset{bc}{=} u(0,0) \overset{!}{=} u(1,1) \overset{bc}{=} 2 \quad ↯$$

3. (6 points) Implement the upwind scheme for the following IBVP

$$\begin{cases} u_t + 2u_x = 0, & x \in [0,1] \\ u(x,0) = \sin(2\pi x) \end{cases},$$

which is subjected to the periodic boundary condition. Use a sequence of uniform grids with meshes $N = 40, 80, 160, 320, 640$. Select Δt according to the CFL condition. Plot the computed solutions together with the exact one at times $t = 2$. Check the experimental convergence rates in the L^1-, L^2- and L^∞-norms and report the results in the following table:

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640						

exact sol: $\sin(2\pi(x-2t))$

Upwind scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + 2\,\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \quad \rightsquigarrow \quad u_j^{n+1} = u_j^n - \overbrace{2\frac{\Delta t}{\Delta x}}^{=:\lambda}\left(u_j^n - u_{j-1}^n\right)$$

from periodic bc

$$U^{n+1} := \begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1-\lambda & & & \lambda \\ \lambda & \ddots & & \\ & \ddots & \ddots & \\ & & \lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix} =: M(\lambda)\, U^n$$

CFL: $\lambda = 2\frac{\Delta t}{\Delta x} = 2\Delta t N \overset{!}{\leq} 1$

$\Longrightarrow \Delta t \leq \frac{1}{2N}$