Numerical Methods for Hyperbolic PDEs
Homework 5

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Exercise 1

Please see separate pdf.

Exercise 2

Note that we have a scalar conservation law

$$u_t + f(u)_x = 0$$

with $f(u) = \frac{u^2}{2}$.

We implement the finite volume scheme

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

with the fluxes $F_{j+1/2}$ are determined by the schemes in the following. Δx is given by the mesh size, and at each time step we calculate Δt according to the CFL-condition (as discussed in the exercise class):

$$\max_j \left|f'(U_j^n)\right| \frac{\Delta t}{\Delta x} = \max_j \left|U_j^n\right| \frac{\Delta t}{\Delta x} \leq \frac{1}{2}.$$

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Linearized Roe scheme

For the *linearized Roe* scheme we use

$$F^n_{j+1/2} = F^{Roe}(U^n_j, U^n_{j+1}) \tag{1}$$

$$= \begin{cases} f(U^n_j), & \hat{A}_{j+1/2} \geq 0 \\ f(U^n_{j+1}), & \hat{A}_{j+1/2} < 0 \end{cases} \tag{2}$$

$$\begin{cases} \frac{(U^n_j)^2}{2}, & \hat{A}_{j+1/2} \geq 0 \\ \frac{(U^n_{j+1})^2}{2}, & \hat{A}_{j+1/2} < 0 \end{cases} \tag{3}$$

where $\hat{A}_{j+1/2}$ is given by the Roe average

$$\hat{A}_{j+1/2} := \begin{cases} \frac{f(U^n_{j+1}) - f(U^n_j)}{U^n_{j+1} - U^n_j}, & U^n_{j+1} \neq U^n_j \\ f'(U^n_j), & U^n_{j+1} = U^n_j \end{cases} \tag{4}$$

$$= \frac{U^n_{j+1} + U^n_j}{2}. \tag{5}$$

Lax-Friedrichs scheme

For the *Lax-Friedrichs* scheme we use

$$F^n_{j+1/2} = F^{LxF}(U^n_j, U^n_{j+1}) \tag{6}$$

$$= \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{\Delta x}{2\Delta t}(U^n_{j+1} - U^n_j) \tag{7}$$

$$= \frac{(U^n_j)^2 + (U^n_{j+1})^2}{4} - \frac{\Delta x}{2\Delta t}(U^n_{j+1} - U^n_j) \tag{8}$$

Rusanov scheme

For the *Rusanov* scheme we use

$$F^n_{j+1/2} = F^{Rus}(U^n_j, U^n_{j+1}) \tag{9}$$

$$= \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{\max(\left|f'(U^n_j)\right|, \left|f'(U^n_{j+1})\right|)}{2}(U^n_{j+1} - U^n_j) \tag{10}$$

$$= \frac{f(U^n_j) + f(U^n_{j+1})}{2} - \frac{\max(\left|U^n_j\right|, \left|U^n_{j+1}\right|)}{2}(U^n_{j+1} - U^n_j) \tag{11}$$

Engquist-Osher scheme

As we showed in the first exercise, the *Engquist-Osher* flux is given by

$$\begin{aligned}
F^n_{j+1/2} &= F^{EO}(U^n_j, U^n_{j+1}) && (12)\\
&= f(\max(U^n_j, 0)) + f(\min(U^n_{j+1}, 0)) - f(0) && (13)\\
&= \frac{\max(U^n_j, 0)^2 + \min(U^n_{j+1}, 0)^2}{2} && (14)
\end{aligned}$$

since f has a unique minimum at 0.

(a)

N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.0447	-	0.0433	-	0.0987	-
80	0.0227	0.9742	0.0224	0.949	0.0569	0.7938
160	0.0116	0.9718	0.0118	0.9273	0.0301	0.9177
320	0.006	0.9454	0.0062	0.9331	0.0154	0.971
640	0.0035	0.7905	0.0035	0.8294	0.008	0.9402

Table 1: Errors and convergence rates for the linearized Roe scheme at time $t = \frac{0.5}{\pi}$

N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.146	-	0.1252	-	0.1712	-
80	0.0784	0.8962	0.0661	0.9212	0.0893	0.9383
160	0.0402	0.9638	0.0343	0.9454	0.0475	0.9124
320	0.0205	0.9718	0.0179	0.9422	0.0265	0.8405
640	0.0105	0.9719	0.0095	0.9118	0.0154	0.7813

Table 2: Errors and convergence rates for the Lax-Friedrichs scheme at time $t = \frac{0.5}{\pi}$

N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.0878	-	0.0782	-	0.1295	-
80	0.0472	0.8944	0.0435	0.8457	0.0905	0.518
160	0.0244	0.951	0.023	0.9212	0.0481	0.9125
320	0.0127	0.9462	0.0119	0.9487	0.0238	1.0142
640	0.0067	0.9269	0.0062	0.9385	0.0113	1.0744

Table 3: Errors and convergence rates for the Rusanov scheme at time $t = \frac{0.5}{\pi}$

N	L^1-Error	rate	L^2-Error	rate	L^∞-Error	rate
40	0.0445	-	0.0431	-	0.0987	-
80	0.0226	0.9777	0.0224	0.9459	0.0569	0.7938
160	0.0116	0.9665	0.0118	0.9264	0.0301	0.9177
320	0.006	0.9433	0.0062	0.9307	0.0154	0.971
640	0.0035	0.7893	0.0035	0.8293	0.008	0.9402

Table 4: Errors and convergence rates for the Engquist-Osher scheme at time $t = \frac{0.5}{\pi}$

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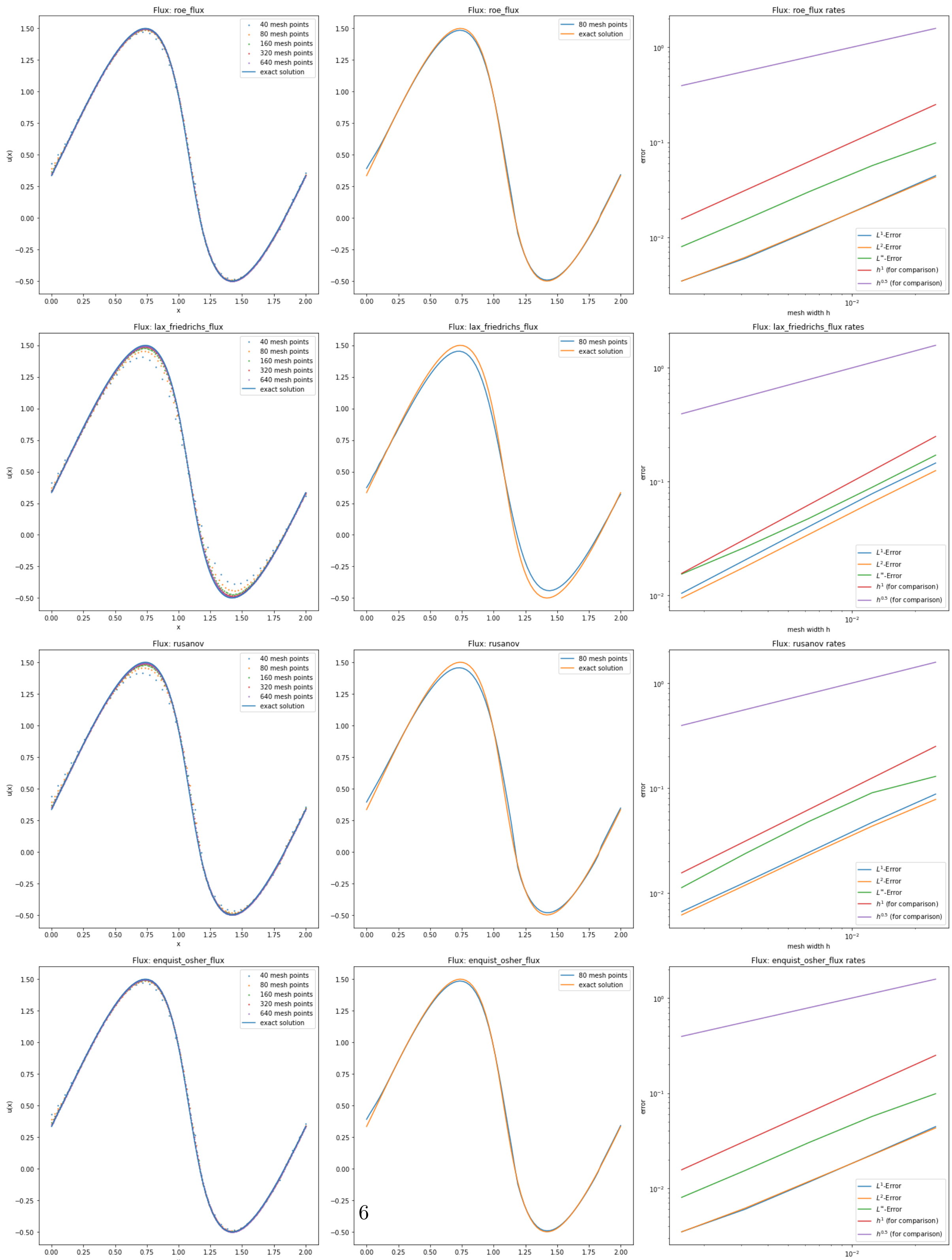

Figure 1: Numerical solutions at time $t = \frac{0.5}{\pi}$

(b)

We observe the diffusivity of the Lax-Friedrich scheme (especially around shocks) which leads to it being the worst out of the four schemes. We can confirm that the Rusanov scheme does indeed improve on the Lax-Friedrich scheme. The Enquist-Osher and linearised Roe schemes perform better. On smooth data, we see all schemes converge in all the observed norms with order around 1, but we don't see very strong evidence for first order convergence in the Lax-Friedrich scheme in L^∞ and the linearized Roe and Enquist-Osher scheme in L^1-norm, the rates seem to be slightly worse. Generally, we see the convergence rate worsen at the mesh size with $N = 640$. However, the rates could stabilize again for even smaller mesh sizes.

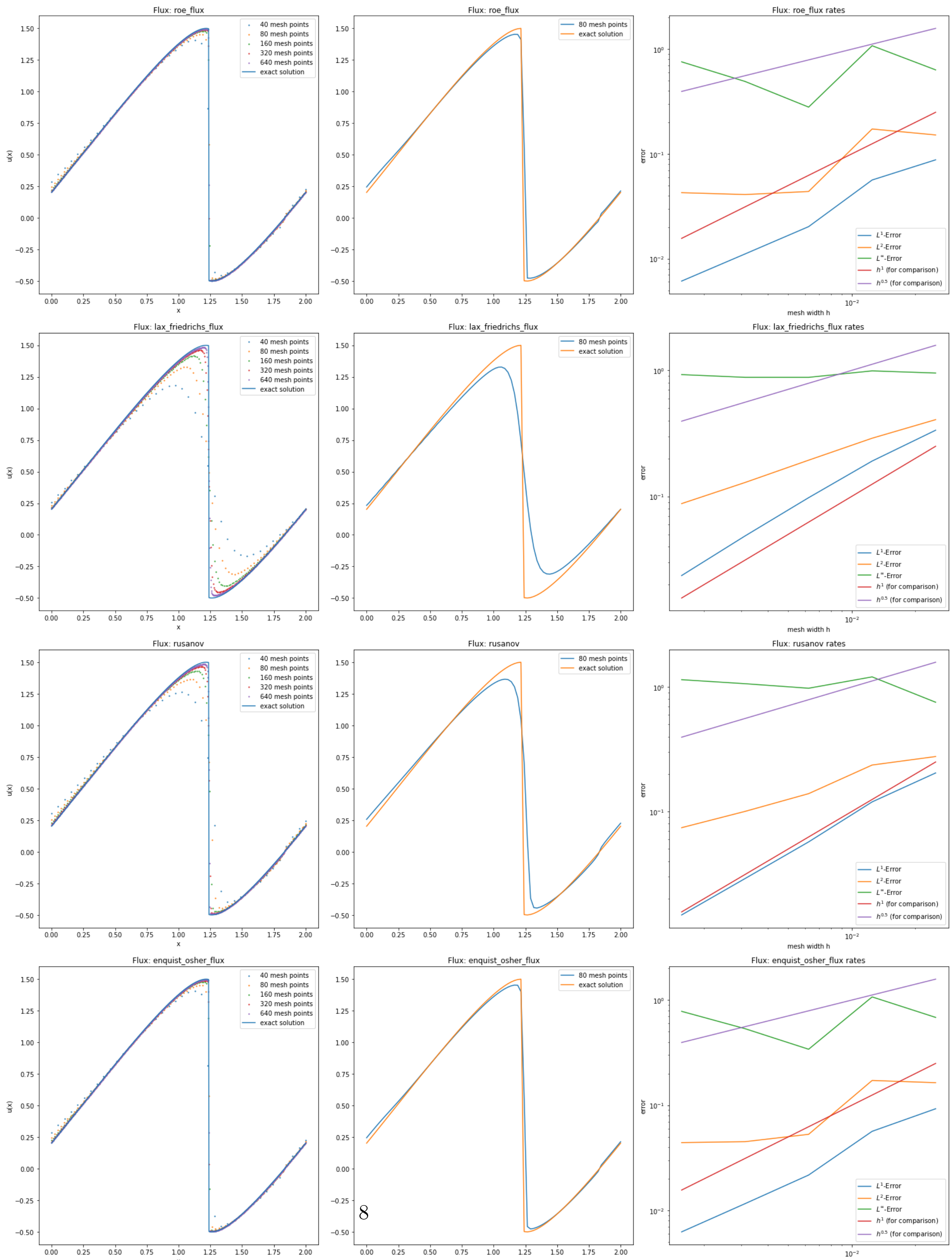

Figure 2: Numerical solutions at time $t = \frac{1.5}{\pi}$