High order TVD schemes

Two ways to construct formally second order accurate schemes with the TVD property. The first one is the wave approach. The second one is the MUSCL approach.

1. The wave approach

start from $u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(f_{j+1/2} - f_{j-1/2}\right)$ (1)

Φ will be a monotone scheme, $\Phi = \Phi(u, v)$ and $\Phi(\uparrow, \downarrow)$

$\Phi_{j+1/2} = \Phi(u_j, u_{j+1})$ we set

$$(\Delta f)^+_{j+1/2} = f(u_{j+1}) - \Phi_{j+1/2}, \quad (\Delta f)^-_{j+1/2} = \Phi_{j+1/2} - f(u_j)$$

and $(\Delta f)^+_{j+1/2} + (\Delta f)^-_{j+1/2} = f(u_{j+1}) - f(u_j) := \Delta_{j+1/2} f$

we introduce the local CFL number with $\lambda = \Delta t/\Delta x$

$$\nu^+_{j+1/2} = \lambda \frac{(\Delta f)^+_{j+1/2}}{\Delta_{j+1/2} u} \quad \nu^-_{j+1/2} = \lambda \frac{(\Delta f)^-_{j+1/2}}{\Delta_{j+1/2} u}$$

thanks to the monotonicity of Φ, that

$$\nu^+_{j+1/2} \geq 0, \quad \nu^-_{j+1/2} \leq 0.$$

$\Big((\Delta f)^+_{j+1/2} = f(u_{j+1}) - \Phi_{j+1/2} = \Phi(u_{j+1}, u_{j+1}) - \Phi(u_j, u_{j+1}) > 0 \quad u_{j+1} > u_j$ (resp. < 0)

$\Delta_{j+1/2} u = u_{j+1} - u_j > 0$ (resp. < 0) $\Big)$

$\Big((\Delta f)^-_{j+1/2} = \Phi_{j+1/2} - f(u_j) = \Phi(u_j, u_{j+1}) - \Phi(u_j, u_j) < 0 \quad u_{j+1} > u_j \Big)$

Finally, $\nu^+_{j+1/2} + \nu^-_{j+1/2} = \lambda \frac{\Delta_{j+1/2} f}{\Delta_{j+1/2} u}$

Additional notations: for any sequence y_K.

$$\Delta_- y_K = y_K - y_{K-1} = \Delta_{K-1/2} y. \quad \Delta_+ y_K = y_{K+1} - y_K = \Delta_{K+1/2} y$$

- Case of a scalar problem

$f(u) = au. \quad a>0 \quad \nu = a\Delta t/\Delta x$

The Lax Wendroff scheme:

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{\nu^2}{2}\left[(u_{j+1}^n - u_j^n) - (u_j^n - u_{j-1}^n)\right]$$

rewrites as

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}(\Delta_{j+1/2}u + \Delta_{j-1/2}u) + \frac{\nu^2}{2}(\Delta_{j+1/2}u - \Delta_{j-1/2}u)$$

$$= u_j^n - \frac{\nu-\nu^2}{2}\Delta_{j+1/2}u - \frac{\nu+\nu^2}{2}\Delta_{j-1/2}u$$

$$= u_j^n - \nu\Delta_{j-1/2}u - \left(\frac{\nu(1-\nu)}{2}(\Delta_{j+1/2}u - \Delta_{j-1/2}u)\right)$$

$$= u_j^n - \nu\Delta_{j-1/2}u - \Delta_-\left[\frac{\nu(1-\nu)}{2}\Delta_{j+1/2}u\right]$$

This shows that, since $a>0$, the Lax-Wendroff scheme is obtained from the upwind scheme (which is monotone) and a perturbation. $-\Delta_-\left[\frac{\nu(1-\nu)}{2}\Delta_{j+1/2}u\right]$. In term of flux, the Lax Wendroff flux is obtained from the upwind flux with a modification

$$\hat{f}_{j+1/2}^{LW} = \hat{f}_{j+1/2}^{up} + \frac{\nu(1-\nu)}{2\lambda}\Delta_{j+1/2}u \qquad (7)$$

$$\left(u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(au_j - au_{j-1}) - \frac{\nu(1-\nu)}{2}\cdot\left(\Delta_{j+1/2}u - \Delta_{j-1/2}u\right)\cdot\frac{\Delta t}{\Delta x}\cdot\frac{1}{\Delta t/\Delta x}\right)$$

$$= u_j^n - \frac{\Delta t}{\Delta x}\left[\left(au_j + \frac{\nu(1-\nu)}{2\lambda}\Delta_{j+1/2}u\right) - \left(au_{j-1} + \frac{\nu(1-\nu)}{2\lambda}\Delta_{j-1/2}u\right)\right]$$

The Lax Wendroff is not TVD, but the upwind scheme is monotone. So we will consider the following flux,

$$\hat{f}_{j+1/2} = \hat{f}^{up}_{j+1/2} + \underline{\varphi(r_j)} \frac{\nu(1-\nu)}{2\lambda} \Delta_{j+1/2} u.$$

with $r_j = \frac{u_j - u_{j-1}}{u_{j+1} - u_j} = \frac{\Delta_{j-1/2} u}{\Delta_{j+1/2} u}$.

The question is how to choose φ so that the scheme is TVD under a CFL condition that is close to $0 \le \nu \le 1$ and then that the scheme is formally second order accurate.

And then that the scheme is formally second order accurate.

and then that the flux (7) in incremental form

We first write (1) with the flux (7) in incremental form

$$u_j^{n+1} = u_j^n - \lambda\left(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}\right)$$

$$= u_j^n - \lambda\Big(a u_j - a u_{j-1} \cancel{(a u_{j-1} - a u_{j-2})} + \frac{\nu(1-\nu)}{2\lambda} \Delta_{j+1/2} u \cdot \varphi(r_j)$$

$$- \frac{\nu(1-\nu)}{2\lambda} \Delta_{j-1/2} u \, \varphi(r_{j-1})\Big)$$

$$= u_j^n - \lambda a \Delta u_{j-1/2} - \frac{\nu(1-\nu)}{2} \Delta_{j+1/2} u \, \varphi(r_j) + \frac{\nu(1-\nu)}{2} \Delta_{j-1/2} u \, \varphi(r_{j-1})$$

$u_{j+1} - u_j$

Harten's Lemma.

$$= u_j^n + C_{j+1/2} \Delta_{j+1/2} u - D_{j-1/2} \Delta_{j-1/2} u$$

$$= u_j^n - \nu \Delta_{j-1/2} u - \frac{\nu(1-\nu)}{2} \left(\Delta_{j+1/2} u / \Delta_{j-1/2} u\right) \cdot \Delta_{j-1/2} u \, \varphi(r_j)$$

$$+ \frac{\nu(1-\nu)}{2} \Delta_{j-1/2} u \, \varphi(r_{j-1})$$

$$= u_j^n - \Delta_{j-1/2} u \left\{ \nu + \frac{\nu(1-\nu)}{2} \cdot \varphi(r_j) \frac{\Delta_{j+1/2} u}{\Delta_{j-1/2} u} - \frac{\nu(1-\nu)}{2} \varphi(r_{j-1}) \right\}$$

$\Rightarrow C_{j+1/2} = 0$. $\quad D_{j-1/2} = \nu + \frac{\nu(1-\nu)}{2} \frac{\Delta_{j+1/2}u\,\varphi(r_j) - \Delta_{j-1/2}u\,\varphi(r_{j-1})}{\Delta_{j-1/2}u}$

$$= \nu + \frac{\nu(1-\nu)}{2} \frac{\Delta_-\left(\varphi(r_j)\Delta_{j+1/2}u\right)}{\Delta_{j-1/2}u}$$

$$= \nu \left\{ 1 + \frac{1-\nu}{2}\left[\frac{\varphi(r_j)}{r_j} - \varphi(r_{j-1})\right]\right\} \geq 0 \text{ and } \leq 1$$

If we assume that there exists $\Phi \geq 0$ such that

$$\left| \frac{\varphi(r_j)}{r_j} - \varphi(r_j) \right| \leq \Phi. \overset{(9)}{\Rightarrow} -\Phi \leq \frac{\varphi(r_j)}{r_j} - \varphi(r_j) \leq \Phi$$

$$\Rightarrow \nu\left(1 - \frac{1-\nu}{2}\Phi\right) \leq D_{j-1/2} \leq \nu\left(1 + \frac{1-\nu}{2}\Phi\right)$$

and since we want $0 \leq \nu \leq 1$, we get $0 \leq D_{j-1/2} \leq 1$

if $\Phi \in [0, 2]$. We will aslo ask $\varphi(r) = 0$ if $r \leq 0$

so because of the form of r, we will have first order at extrema

Thus the condition (9) leads to

$$0 \leq \frac{\varphi(r)}{r} \leq 2. \quad 0 \leq \varphi(r) \leq 2$$

It also can be shown that if $\varphi(1) = 1$, the scheme is second order.

Then one can see that with these two requirements the flux must be a baryentric combination of the Lax-Wendroff and the Beam-Warming scheme:

$$u_j^{n+1} = u_j^n - \nu\Delta_{j-1/2}u - \Delta_-\left(\frac{\nu(1-\nu)}{2}\Delta_{j+1/2}u\right)$$

Lax-Wendroff corresponds to $\varphi(r) = 1$.

Beam-Warming corresponds to $\varphi(r) = r$.

$\Rightarrow$ so that $\varphi = (1-\theta(r))\varphi^{LW}_{(r)} + \theta(r)\varphi^{BW}_{(r)}$

$\varphi = 1 + \theta(r)(r-1)$

so that we chose

$$\varphi = (1-\theta(r))\,\varphi^{LW}(r) + \theta(r)\,\varphi^{BW}(r)$$

$$\varphi(r) = 1 + \theta(r)(r-1)$$

$\theta \in [0, 1]$

The nonlinear case.

$$\frac{u_j^{n+1}-u_j^n}{\delta t} + \frac{f(u_{j+1})-f(u_{j-1})}{2\delta x} = 0$$

Taylor expansion
modified equation

is $$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + \frac{\delta t}{2}\frac{\partial^2 u}{\partial t^2} = O(\Delta x^2)$$

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right) = -\frac{\partial}{\partial x}\left(a(u)u_t\right) = \frac{\partial}{\partial x}\left(a(u)\frac{\partial f}{\partial x}\right)$$

Lax-Wendroff scheme for nonlinear equation.

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f_{j+1}-f_{j-1}\right) + \frac{\lambda^2}{2}\left(a_{j+1/2}(f_{j+1}-f_j) - a_{j-1/2}(f_j-f_{j-1})\right)$$

where $a_{j+1/2} = a(u_j, u_{j+1})$

which is Lipschitz continuous and consistent with f_u:

$$a(u_j, u_{j+1}) = f_u(u_j) + O(\Delta_{j+1/2}u)$$

We look for schemes of the type

$$u_j^{n+1} = u_j^n - \lambda\left(\Phi_{j+1/2} - \Phi_{j-1/2}\right) - \lambda\Delta_-\left\{\varphi(r_j^+)\,\alpha^+_{j+1/2}(\Delta_{j+1/2}f)^+ - \varphi(r_{j+1}^-)\,\alpha^-_{j+1/2}(\Delta_{j+1/2}f)^-\right\}$$

with $\alpha^+_{j+1/2} = \frac{1}{2}\left(1-\nu^+_{j+1/2}\right)$ and $\alpha^-_{j+1/2} = \frac{1}{2}\left(1+\nu^-_{j+1/2}\right)$

and $$r_j^+ = \frac{\alpha^+_{j-1/2}(\Delta_{j-1/2}f)^+}{\alpha^+_{j+1/2}(\Delta_{j+1/2}f)^+} \qquad r_j^- = \frac{\alpha^-_{j+1/2}(\Delta_{j+1/2}f)^-}{\alpha^-_{j-1/2}(\Delta_{j-1/2}f)^-}$$

The MUSCL approach

1. Limiters.

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(\hat{f}_{j+1/2} - \hat{f}_{j-1/2}\right)$$

Φ being a monotone flux.

$$\hat{f}_{j+1/2} = \Phi\left(u^-_{j+1/2},\ u^+_{j+1/2}\right)$$

with

$$u^-_{j+1/2} = u_j + \frac{\Delta x}{2} S(u_{j-1}, u_j, u_{j+1}) = L(u_{j-1}, u_j, u_{j+1})$$

$$u^+_{j+1/2} = u_{j+1} - \frac{\Delta x}{2} S(u_j, u_{j+1}, u_{j+2}) = R(u_j, u_{j+1}, u_{j+2})$$

We introduce a few properties that we want to impose of the functionals R and L.

<u>Homogeneity (H)</u>. means that for any $\alpha \in \mathbb{R}$.

$$L(\alpha u, \alpha v, \alpha w) = \alpha L(u,v,w). \quad R(\alpha u, \alpha v, \alpha w) = \alpha R(u,v,w)$$

<u>Invariance by translation (T)</u>: for any $\alpha \in \mathbb{R}$:

$$L(u+\alpha, v+\alpha, w+\alpha) = \alpha + L(u,v,w) \quad R(u+\alpha, v+\alpha, w+\alpha) = \alpha + R(u,v,w)$$

<u>Symmetry (S)</u> We have $L(u,v,w) = R(w,v,u)$

We immediately get:

Proposition 3.1: There exists $\psi: \mathbb{R} \to \mathbb{R}$ such that

$$L(u,v,w) = v + \frac{1}{2}\psi\left(\frac{w-v}{v-u}\right)(v-u).$$

$$R(u,v,w) = v - \frac{1}{2}\psi\left(\frac{v-u}{w-v}\right)(w-v)$$

proof:

$$L(u,v,w) = v + L(u-v, 0, w-v) \quad \left(\text{Invariance by translation}\right)$$

$$= v + \underline{L\left(-1, 0, \frac{w-v}{-(u-v)}\right)} \cdot (v-u) \quad (\text{Homogeneity})$$

and we define ψ as $\quad \frac{1}{2}\psi\left(\frac{w-v}{v-u}\right)$

$$\psi(x) = 2L(-1, 0, x)$$

Then, using the symmetry, and the homogeneity.

$$R(u,v,w) = L(w, v, u)$$

$$= v + L(w-v, 0, u-v) = v + L\left(1, 0, \frac{u-v}{w-v}\right)(w-v)$$

$$= v - L\left(-1, 0, \frac{v-u}{w-v}\right)(w-v) = v - \frac{1}{2}\psi\left(\frac{v-u}{w-v}\right)(w-v)$$

Remark 3.2 (constraint on ψ). From now on, we make the following assumption on ψ:

If $r \le 0$, then $\psi(r) = 0$.

It means, in view of proposition 3.1 that if $u_i - u_{i-1}$ and $u_{i+1} - u_i$ are of opposite sign, i.e. u_i is out of the interval $[\min(u_{i-1}, u_{i+1}), \max(u_{i-1}, u_{i+1})]$, u_i is a local extremum, that $R(u_{i-1}, u_i, u_{i+1}) = L(u_{i-1}, u_i, u_{i+1}) = u_i$; the slope is 0.

We request another property: new extrema should not be created.

Monotonicity (M) $\quad \min(v,w) \le L(u,v,w) \le \max(v,w)$

$$\min(u,v) \le R(u,v,w) \le \max(u,v)$$

proposition 3.3.

The functional L and R satisfy the monotonicity property if for any $x \in \mathbb{R}$, $x \neq 0$,
$$0 \le \frac{\psi(x)}{x} \le 2$$

proof: we have
$$L(u,v,w) = v + \frac{1}{2}\psi\left(\frac{w-v}{v-u}\right)(v-u)$$
$$= v + \frac{1}{2}\psi\left(\frac{w-v}{v-u}\right)\cdot\frac{v-u}{w-v}(w-v)$$

So that setting
$$\theta = \frac{1}{2}\psi\left(\frac{w-v}{v-u}\right)\frac{v-u}{w-v}$$

we have
$$L(u,v,w) = (1-\theta)v + \theta w$$

This shows that the monotonicity is possible if and only if convex combination

$$\boxed{0 \le \frac{\psi(x)}{x} \le 2 \quad \text{when } x=0.}$$

$$\left(0 \le \theta \le 1,\ 0 \le 1-\theta \le 1\right)$$

The last requirement is that the convexity of data should be respected by the reconstruction. Namely, we ask that

$\boxed{\text{Convexity}}$: (C) if $u_j - u_{j-1} \le u_{j+1} - u_j$ then
$$u^+_{j-1/2} - u_{j-1} \le u_j - u^+_{j-1/2} \le u^-_{j+1/2} - u_j \le u_{j+1} - u^-_{j+1/2}$$

proposition 3.4.

If the (H). (T) (S) property holds true. then the (C) property is equivalent to

- If $x \geq 1$, $1 \leq x\psi(\frac{1}{x}) \leq \psi(x) \leq x$.

proof: We see that

$$u_{j+1} = u_j + \frac{u_{j+1}-u_j}{u_j-u_{j-1}}(u_j-u_{j-1})$$

$$= u_j + \theta(u_j-u_{j-1}) \qquad \theta := \frac{u_{j+1}-u_j}{u_j-u_{j-1}}$$

so that $u_j-u_{j-1} \leq u_{j+1}-u_j$ is equivalent to $u_j-u_{j-1} \leq \theta(u_j-u_{j-1})$, i.e.

$$(\theta-1)(u_j-u_{j-1}) \geq 0$$

Assuming that $\theta > 1$ is then equivalent to assuming that

$$u_j-u_{j-1} \geq 0$$

$$u^+_{j-1/2} = R(u_{j-1},u_j,u_{j+1}) = u_j - \frac{1}{2}\psi\left(\frac{u_j-u_{j-1}}{u_{j+1}-u_j}\right)(u_{j+1}-u_j)$$

$$= u_j - \frac{1}{2}\psi\left(\frac{u_j-u_{j-1}}{u_{j+1}-u_j}\right)\frac{u_{j+1}-u_j}{u_j-u_{j-1}}(u_j-u_{j-1})$$

$$u^-_{j+1/2} = L(u_{j-1},u_j,u_{j+1}) = u_j + \frac{1}{2}\psi\left(\frac{u_{j+1}-u_j}{u_j-u_{j-1}}\right)(u_j-u_{j-1})$$

~~we set θ =~~

the convexity (C) write:

$$u_j - \frac{1}{2}\theta\psi\left(\frac{1}{\theta}\right)(u_j-u_{j-1}) - u_{j-1} \leq u_j - \left(u_j - \frac{1}{2}\theta\psi\left(\frac{1}{\theta}\right)(u_j-u_{j-1})\right)$$

(15) $$\leq u_j + \frac{1}{2}\psi(\theta)(u_j-u_{j-1}) - u_j \leq u_{j+1}-u_j-\frac{1}{2}\psi(\theta)(u_j-u_{j-1})$$

$$\Rightarrow (u_j-u_{j-1})\left(1-\frac{\theta}{2}\psi\left(\frac{1}{\theta}\right)\right) \leq \frac{\theta}{2}\psi\left(\frac{1}{\theta}\right)(u_j-u_{j-1}) \leq \frac{1}{2}\psi(\theta)(u_j-u_{j-1})$$

$$\leq \left(\theta - \frac{1}{2}\psi(\theta)\right)(u_j-u_{j-1})$$

then since $y_j - y_{j-1} \geq 0$, we get

$$1 - \frac{\theta}{2}\psi\left(\frac{1}{\theta}\right) \leq \frac{\theta}{2}\psi\left(\frac{1}{\theta}\right) \leq \frac{1}{2}\psi(\theta) \leq \theta - \frac{1}{2}\psi(\theta) \quad (*)$$

that is $\quad 1 - \theta\psi\left(\frac{1}{\theta}\right) \leq 0,\ 0 \leq \theta - \psi(\theta),\ \theta\psi\left(\frac{1}{\theta}\right) \leq \psi(\theta)$

$$\Rightarrow \boxed{1 \leq \theta\psi\left(\frac{1}{\theta}\right) \leq \psi(\theta) \leq \theta}$$

- If now $\cancel{0 \leq \theta < 1}$ $(0 < \theta < 1)$, we have

$$0 < \frac{y_{j+1} - y_j}{y_j - y_{j-1}} < 1 \quad \text{and} \quad y_j - y_{j-1} \leq 0 \Rightarrow y_{j+1} - y_j \leq 0.$$

we get again

$$(y_j - y_{j-1})\left(1 - \frac{\theta}{2}\psi\left(\frac{1}{\theta}\right)\right) \leq \frac{\theta}{2}\psi\left(\frac{1}{\theta}\right)(y_j - y_{j-1}) \leq \frac{1}{2}\psi(\theta)(y_j - y_{j-1})$$
$$\leq \left(\theta - \frac{1}{2}\psi(\theta)\right)(y_j - y_{j-1})$$

after simplifying by $y_j - y_{j-1} \leq 0$ we get $\quad 1 \geq \theta\psi\left(\frac{1}{\theta}\right)$

$$1 - \frac{\theta}{2}\psi\left(\frac{1}{\theta}\right) \geq \frac{\theta}{2}\psi\left(\frac{1}{\theta}\right) \geq \frac{1}{2}\psi(\theta) \geq \theta - \frac{1}{2}\psi(\theta)$$

Setting $\mu = \frac{1}{\theta} > 1$, we get

$$1 - \frac{\psi(\mu)}{2\mu} \geq \frac{1}{2}\frac{\psi(\mu)}{\mu} \geq \frac{1}{2}\psi\left(\frac{1}{\mu}\right) \geq \frac{1}{\mu} - \frac{1}{2}\psi\left(\frac{1}{\mu}\right)$$

$$\Rightarrow \mu - \frac{1}{2}\psi(\mu) \geq \frac{1}{2}\psi(\mu) \geq \frac{\mu}{2}\psi\left(\frac{1}{\mu}\right) \geq 1 - \frac{\mu}{2}\psi\left(\frac{1}{\mu}\right) \quad (**)$$

$$\Rightarrow \boxed{\cancel{1 \leq \mu\psi\left(\frac{1}{\mu}\right) \leq \psi(\mu) \leq \mu}}$$

$$1 \leq \mu\psi\left(\frac{1}{\mu}\right) \leq \psi(\mu) \leq \mu.$$

proposition 3.5. If (H). (T). (S). (M) and (C) hold true, then:

- If $\theta \geq 1$, $1 \leq \theta \psi(\frac{1}{\theta}) \leq \psi(\theta) \leq \theta$. $(\cancel{\otimes})$
- If $\theta \in [0,1)$. $\theta \leq \psi(\theta) \leq 2\theta$.

proof: If (H). (T). (S). + (C) $\Rightarrow$ $(\cancel{\otimes})$

Assume that (H). (S). (T) and (M) hold true. for $\theta \in [0,1]$.

We have $0 \leq \frac{\psi(\theta)}{\theta} \leq 2 \Rightarrow 0 \leq \psi(\theta) \leq 2\theta$. page 8 So from the previous statement. $1 \leq \frac{1}{\theta}\psi(\theta) \Rightarrow \theta \leq \psi(\theta)$ page 10

$\Rightarrow$ $\theta \leq \psi(\theta) \leq 2\theta$.

Remark 3.6. We notice that the convexity property implies that

$\psi(1) = 1$.

Traditionally, the values $u^{\pm}_{j+1/2}$ are not defined from a function ψ but a function φ such that

$$u^{-}_{j+1/2} = u_j + \frac{1}{2}\varphi(r_j)(u_{j+1} - u_j) \quad u^{+}_{j-1/2} = u_j - \frac{1}{2}\varphi(\frac{1}{r_j})(u_j - u_{j-1})$$

where $r_j = \dfrac{u_j - u_{j-1}}{u_{j+1} - u_j}$ $\quad r_j = \frac{1}{\theta}$ $\Rightarrow$ $\varphi(r) = r\psi(\frac{1}{r})$.

Then we get.

Proposition 3.7. Under the assumptions of proposition 3.5, the function $\varphi(r) = r\psi(\frac{1}{r})$. We have

- If $0 \leq r \leq 1$, $r \leq \varphi(r) \leq 1$
- If $r \geq 1$, $1 \leq \varphi(r) \leq \min(2, r)$
- $\varphi(r) \leq r\varphi(\frac{1}{r})$ if $r \geq 1$.

proof.

For $0<r<1$, the convexity requirement is

$$1\le \frac{\psi(r)}{r} \le \psi\left(\frac{1}{r}\right) \le \frac{1}{r}$$

and $\varphi(r) = r\psi\left(\frac{1}{r}\right) \Rightarrow \varphi\left(\frac{1}{r}\right) = \frac{1}{r}\psi(r)$

$$\Rightarrow \quad r\le \psi(r) \le r\psi\left(\frac{1}{r}\right) \le 1$$

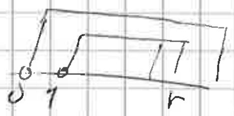
$$\Rightarrow r\le r\varphi\left(\frac{1}{r}\right) \le \varphi(r) \le 1$$

$\Rightarrow r\in[0,1)$. $r\le\varphi(r)\le 1$

for $r>1$ we have $r\psi\left(\frac{1}{r}\right)\le\psi(r)\le r$ from (*)

i.e. $\varphi(r)\le r\varphi\left(\frac{1}{r}\right)\le r$

we also have $\quad 1\le r\psi\left(\frac{1}{r}\right)\le r \Rightarrow 1\le\varphi(r)\le r.$



Last (M) request that for $r>1$

$$a<\psi\left(\frac{1}{r}\right)\le\frac{2}{r} \Rightarrow 0\le\varphi(r)\le 2$$

$$\Rightarrow \boxed{1\le\varphi(r)\le \min(2,r).}$$