Exercise set 6

Numerical Methods for Hyperbolic Partial Differential Equations

IMATH, FS-2020

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Consider the following methods written in the conservation form, with the usual FD notation and $\lambda = \Delta t/\Delta x$:

$$u_j^{n+1} = u_j^n - \lambda \left(f^{num}(u_j^n, u_{j+1}^n) - f^{num}(u_{j-1}^n, u_j^n) \right), \tag{1}$$

- Lax–Friedrichs

$$f^{num}(u,v) = \frac{f(v)+f(u)}{2} - \frac{v-u}{2\lambda}; \tag{2}$$

- Rusanov (or Local Lax–Friedrichs)

$$f^{num}(u,v) = \frac{f(v)+f(u)}{2} - \max\left(|f'(u)|, |f'(v)|\right) \frac{v-u}{2}; \tag{3}$$

- Godunov

$$f^{num}(u,v) = \begin{cases} \min_{u\le\theta\le v} f(\theta) & \text{if } u \le v, \\ \max_{v\le\theta\le u} f(\theta) & \text{if } v \le u, \end{cases} \tag{4}$$

  which for fluxes f with a unique local minimum ω such that $f'(\omega) = 0$ and $f''(\omega) > 0$ and no local maxima (e.g. convex fluxes) can be written as

$$f^{num}(u,v) = \max\left(f\left(\max(u,\omega)\right), f\left(\min(v,\omega)\right)\right); \tag{5}$$

- Linearised Roe

$$f^{num}(u,v) = \begin{cases} f(u) & \text{if } \hat{A} \ge 0, \\ f(v) & \text{if } \hat{A} < 0, \end{cases} \tag{6}$$

  where

$$\hat{A} = \begin{cases} \frac{f(v)-f(u)}{v-u} & \text{if } u \ne v, \\ f'(u) & \text{if } u = v; \end{cases}$$

- Engquist–Osher

$$f^{num}(u,v) = \frac{f(v)+f(u)}{2} - \frac{1}{2}\int_u^v |f'(\theta)| d\theta, \tag{7}$$

  which for fluxes f with a unique local minimum ω such that $f'(\omega) = 0$ and $f''(\omega) > 0$ (e.g. convex fluxes) can be written as

$$f^{num}(u,v) = f^+(u) + f^-(v), \tag{8}$$

  where

$$\begin{cases} f^+(u) = f(\max(u,\omega)), \\ f^-(u) = f(\min(u,\omega)). \end{cases} \tag{9}$$

Problem 6.1 Code all of them (5pts)

Code all the methods in a unified way, where the input of the numerical flux are f, u, v and optional other inputs as λ or f' or ω.

Test the different methods on Burgers' equation for a shock test on the domain $[-2, 2]$ with Dirichlet BC for the Riemann problem

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases} \tag{10}$$

and then on

$$u_0(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases} \tag{11}$$

1. For each of the RPs and all the schemes, plot at the final time $T = 2$ the exact and the approximate solution with $N = 100$ cells.
2. What do you notice on the rarefaction wave? How does the Linearised Roe behave and why? (Heuristic reasons are enough)

Hint

Create a function `numericalFlux` with input a string with the name of the scheme, a symbolic function f of the flux, the left value u, the right value v and an extra input, which can be the value ω for GOdunov and Engquist–Osher, λ for Lax–Friedrichs or f' for Rusanov. This function codes the numerical fluxes defined before (consider the case where there exists a local minimum of the flux in 0 as for Burgers').
A second function `runScheme` takes as input the type of scheme and the specification of the problem: domain, final time, flux, its derivative, number of cells, CFL and so on.
Finally, a script runs the schemes with the different schemes.

Problem 6.2 Monotone schemes (3 pts)

A monotone scheme is such that if $u_j^n \leq v_j^n$ for all j, then $u_j^{n+1} \leq v_j^{n+1}$ for all j. Prove or disprove that the following schemes are monotone under some CFL conditions.

1. Godunov scheme (4),
2. Linerised Roe (6),
3. Emgquist–Osher (7).

Problem 6.3 Everything is upwind (2pts)

Show that if f is monotone and convex, then Godunov (4), Linearised Roe (6) and Engquist–Osher (7) are the upwind schemes.

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