Numerical Methods for Hyperbolic PDEs
Homework 3

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Exercise 1

1 (a)

Note that all given solution candidates are constant (thus smooth) on the domains

$$\Omega^- = \{(x,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0} | x < t\}$$
$$\Omega^+ = \{(x,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0} | x > t\}$$

respectively, and that the shock wave (the shared boundary of Ω^+ and Ω^-) is parameterized by the curve $x = \sigma(t) := t$, giving rise to a shock speed $s(t) = 1$. Denote U^- and U^+ the constant values of the solutions on Ω^- and Ω^+ respectively. With this notation, a solution candidate is a weak solution if and only if the Rankine-Hugoniot condition is fulfilled, in this case (for Burger's equation)

$$s(t) = \frac{(U^+)^2 - (U^-)^2}{2(U^+ - U^-)} \overset{!}{=} 1$$

We check this in all cases:

(i) $s(t) = \frac{1^2-0^2}{2(1-0)} = \frac{1}{2} \neq 1$

(ii) $s(t) = \frac{2^2-0^2}{2(2-0)} = 1$

(iii) $s(t) = \frac{0^2-2^2}{2(0-2)} = 1$

Thus, (i) is not a weak solution, but (ii) and (iii) are.

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1 (b)

Recall that the characteristic curves of Burger's equation are of the form

$$x(t) = u_0(x_0)t + x_0.$$

A shock/discontinuity forms when two characteristic curves originating from different initial values cross. More exactly, the solution has a discontinuity at the point (x, t) if and only if for every neighborhood of that point we can find an intersection (lying in that neighborhood) of characteristic curves originating from different initial values. So let $x_0, \tilde{x}_0 \in \mathbb{R}$, such that $u_0(x_0) \neq u_0(\tilde{x}_0)$ and let $x(t)$ and $\tilde{x}(t)$ be the corresponding characteristic curves. Then

$$\begin{aligned}
& x(t^*) = \tilde{x}(t^*) \\
\iff & u_0(x_0)t^* + x_0 = u_0(\tilde{x}_0)t^* + \tilde{x}_0 \\
\iff & \frac{u_0(x_0) - u_0(\tilde{x}_0)}{x_0 - \tilde{x}_0} t^* = -1.
\end{aligned}$$

Using the mean value theorem and that t is non-negative, if u_0 is differentiable, this implies the existence of an x_0^* between x_0 and $\tilde{x_0}$, such that

$$t^* = \frac{-1}{u_0'(x_0^*)} \text{ and } u_0'(x_0^*) < 0. \tag{1}$$

From now on, we assume that u_0 is differentiable and its derivative takes on a minimum. Now on the other hand, if there exists such a pair (t^*, x_0^*) satisfying Eq. (1), we take t^* and $u'(x_0^*)$ to be minimal. Then, by the mean value theorem it must hold for all $x_0 \neq \tilde{x}_0 \in \mathbb{R}$ that

$$\frac{u_0(x_0) - u_0(\tilde{x}_0)}{x_0 - \tilde{x}_0} \geq u_0'(x_0^*).$$

Since $u_0'(x_0^*) < 0$, we can take a sequence x_n with $x_n \neq x_0^*$ and limit x_0^* such that

$$0 > \frac{u_0(x_n) - u_0(x_0^*)}{x_n - x_0^*} \geq u_0'(x_0^*).$$

Define t_n, such that

$$\frac{u_0(x_n) - u_0(x_0^*)}{x_n - x_0^*} t_n = -1.$$

It holds that $t_n \geq t^*$ and $(x_n, t_n) \to (x_0^*, t^*)$ and at each point (x_n, t_n) we have crossing characteristics originating from different[1] initial values x_n and x_0^*, thus (x_0^*, t^*) is a discontinuity point.

We have shown that the moment of shock formation is at

$$t_{\min} = \frac{-1}{\min_{x \in \mathbb{R}} u_0'(x)}, \tag{2}$$

[1]they are different for n big enough since $u_0'(x_0^*) \neq 0$

In the case $u_0 = \sin(\pi x) + \frac{1}{2}$, we calculate:

$$u_0'(x) = \pi \cos(\pi x)$$

which has minimal value $-\pi$. Thus, Eq. (2) gives

$$t_{\min} = \frac{-1}{-\pi} = \frac{1}{\pi}.$$

Exercise 2

Let $U_l, U_r \in \mathbb{R}$. We know from the lecture (Equation 3.29 and 3.23 in the lecture notes) that the unique weak entropy solution for the Riemann problem

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u(x,0) = U_l, \quad x < 0 \\ u(x,0) = U_r, \quad x > 0, \end{cases}$$

are given by a shock wave if $U_l > U_r$, i.e.

$$u(x,t) = \begin{cases} U_l, & x < st \\ U_r, & x > st, \end{cases} \tag{3}$$

where by the Rankine-Hugoniot condition

$$s = \frac{(U_r)^2 - (U_l)^2}{2(U_r - U_l)},$$

and are given by a rarefaction wave if $U_l \leq U_r$, i.e.

$$u(x,t) = \begin{cases} U_l, & x \leq U_l t \\ \frac{x}{t}, & U_l t < x \leq U_r t \\ U_r, & x > U_r t, \end{cases} \tag{4}$$

(a) Since $1 > 0 > -1$, we have two shock waves forming with speeds

$$s_1 = \frac{(0)^2 - (1)^2}{2(0-1)} = \frac{1}{2}$$

and

$$s_2 = \frac{(-1)^2 - (0)^2}{2(-1-0)} = -\frac{1}{2}$$

The shock waves collide only at t=2, so for $0 < t < 2$ we have the weak entropy solution, given by translating and combining Eq. (3)

$$u(x,t) = \begin{cases} 1, & x < \frac{1}{2}t - 1 \\ 0, & \frac{1}{2}t - 1 < x < -\frac{1}{2}t + 1 \\ -1, & x > -\frac{1}{2}t + 1. \end{cases}$$

But for $t \geq 2$ we face the Riemann problem

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u(x,2) = 1, \quad x < 0 \\ u(x,2) = -1, \quad x > 0, \end{cases}$$

giving rise to a shock speed

$$s_3 = \frac{(-1)^2 - (1)^2}{2(-1-1)} = 0.$$

The solution is thus

$$u(x,t) = \begin{cases} 1, \quad x < \frac{1}{2}t - 1, & 0 \leq t < 2 \\ 0, \quad \frac{1}{2}t - 1 < x < -\frac{1}{2}t + 1, & 0 \leq t < 2 \\ -1, \quad x > -\frac{1}{2}t + 1, & 0 \leq t < 2 \\ 1, \quad x < 0, & t \geq 2 \\ -1, \quad x > 0, & t \geq 2 \end{cases}$$

(b) This time, since $-1 > 0 > 1$, rarefaction waves originate at $x_0 = -1$ and $\tilde{x}_0 = 1$ instead of shock waves. Using translated versions of Eq. (4) and combining, we get the solution

$$u(x,t) = \begin{cases} -1, \quad x \leq -t - 1 \\ \frac{x+1}{t}, \quad -t - 1 < x \leq -1 \\ 0, \quad -1 < x \leq 1, \\ \frac{x-1}{t}, \quad 1 < x \leq t + 1 \\ 1, \quad x > t + 1, \end{cases}$$

for all $t > 0$. This is the desired solution since the rarefaction waves don't intersect.