Exercise set 3

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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In this exercise sheet we study the implicit central difference method (ICD), i.e., the central difference method with implicit euler method.

Given the conservation laws

$$\partial_t u + \partial_x F(u) = 0, \tag{1}$$

the ICD reads

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(F(u_{j+1}^{n+1}) - F(u_{j-1}^{n+1})\right), \tag{2}$$

where we consider a uniform point distribution $\{x_j\}_{j=1}^{N+1}$ over the interval $[a, b]$, where $x_1 = a$ and $x_{N+1} = b$ and $u_j \approx u(x_j)$, while the superscript n denotes the timestep t^n, hence, $u_j^n \approx u(t^n, x_j)$.

We consider, in particular, the linear transport equations, i.e., $F(u) = c \cdot u$, with $c \in \mathbb{R}$ the transport speed. This allows to collect all the $n+1$ terms on the left hand side and obtain a matrix M that we can invert to have the method ICD written as

$$\mathbf{u}^{n+1} = M^{-1}\mathbf{u}^n. \tag{3}$$

Moreover, we consider periodic boundary conditions, so that $u(t^n, a) = u_1^n = u_{N+1}^n = u(t^n, b)$, for every t^n.

Problem 3.1 Mass matrix (2pts)

- Write explicitly the scheme for the linear transport equation and the mass matrix M and highlight which parameters it depends on.
- Find the condition under which the matrix M is diagonally dominant by rows and, hence, invertible.

Problem 3.2 Energy stability(2pts)

Define the energy of the discrete solution $\mathbf{u}^n$ as

$$\mathcal{E}(\mathbf{u}^n) = \sum_{j=1}^{N} \frac{(u_j^n)^2}{2}. \tag{4}$$

Prove that the ICD is energy stable, i.e., $\mathcal{E}(\mathbf{u}^{n+1}) \leq \mathcal{E}(\mathbf{u}^n)$.

Problem 3.3 Von Neumann Stability (2pts)

Prove that the ICD is also von Neumann stable. Suppose that u is a wave with wave number $\lambda \in \mathbb{N}^+$ and it grows exponentially in time, i. e.,

$$u(t, x) = e^{\alpha t} e^{i\lambda x}. \tag{5}$$

This is a solution for the transport equation with periodic boundary conditions on $[0, 2\pi]$ for $\alpha = 0$. If for the method $\alpha(\lambda)$ is positive for any λ, the scheme is unstable, if $\alpha(\lambda)$ is non–positive for all λ, the scheme is stable.

Prove that ICD is von Neumann stable.

Problem 3.4 Implementation of the method (4pts)

Implement the ICD in MATLAB (or Python or Julia). Use already implemented functions for the inversion of the matrix.

Keep as input variables c the speed of the transport equation, N number of subintervals of the domain, CFL number ($c\Delta t \leq \text{CFL}\Delta x$), T the final time, a and b the domain extrema, u_0 the initial conditions.

At the end, test your code with the following parameters and plot the initial and final solutions on the same figure. $c = 1$, $N = 200$, $\text{CFL} = 0.75$, $T = 2$, $a = 0$, $b = 1$, $u_0 = \sin(2\pi x)$.

Submit the code.

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