Numerical Methods for Hyperbolic PDEs
Homework 2

James King*, Robert Ziegler†

March 13, 2024

Exercise 1

Let $U_i^n := u(t_n, x_i)$ be the exact solution at (t_n, x_i), assuming to exist, and let $\tilde{U}_i^n$ the numerical approximation. By Taylor expansion we know that

$$\begin{aligned} U_i^{n+1} &= U_i^n + \Delta t (U_i^n)_t + \frac{\Delta t^2}{2}(U_i^n)_{tt} + \mathcal{O}(\Delta t^3) \\ U_{i+1}^n &= U_i^n + \Delta x (U_i^n)_x + \frac{\Delta x^2}{2}(U_i^n)_{xx} + \frac{\Delta x^3}{6}(U_i^n)_{xxx} + \mathcal{O}(\Delta x^4) \\ U_{i-1}^n &= U_i^n - \Delta x (U_i^n)_x + \frac{\Delta x^2}{2}(U_i^n)_{xx} - \frac{\Delta x^3}{6}(U_i^n)_{xxx} + \mathcal{O}(\Delta x^4). \end{aligned}$$

We want to calculate the local error, so we assume $U_i^n = \tilde{U}_i^n$ for all i. Then the one-step error is

$$\begin{aligned} &\tilde{U}_i^{n+1} - U_i^{n+1} \\ &= U_i^n - \frac{\lambda}{2}(U_{i+1}^n - U_{i-1}^n) + \frac{\lambda^2}{2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) - U_i^{n+1} \\ &= U_i^n - \frac{a\Delta t}{2\Delta x}\Big(2\Delta x (U_i^n)_x + \mathcal{O}(\Delta x^3)\Big) + \frac{a^2\Delta t^2}{2\Delta x^2}\Big(\Delta x^2 (U_i^n)_{xx} + \mathcal{O}(\Delta x^4)\Big) \\ &\qquad - U_i^n - \Delta t (U_i^n)_t - \frac{\Delta t^2}{2}(U_i^n)_{tt} + \mathcal{O}(\Delta t^3) \\ &= \underbrace{-a\Delta t (U_i^n)_x - \Delta t (U_i^n)_t}_{= -\Delta t \left((U_i^n)_t + a(U_i^n)_x\right) = 0} + \mathcal{O}(\Delta t \Delta x^2) + \underbrace{\frac{a^2\Delta t^2}{2}(U_i^n)_{xx} - \frac{\Delta t^2}{2}(U_i^n)_{tt}}_{= \frac{\Delta t^2}{2}\left(a^2 (U_i^n)_{xx} - (U_i^n)_{tt}\right) = 0} + \mathcal{O}(\Delta t^2 \Delta x^2 + \Delta t^3) \\ &= \mathcal{O}(\Delta t^3), \end{aligned}$$

*Immatriculation Nr. 23-942-030
†Immatriculation Nr. 18-550-558

where we used in the last equality that $\mathcal{O}(\Delta t\Delta x^2 + \Delta t^2\Delta x^2 + \Delta t^3) = \mathcal{O}(\Delta t^3)$, since $\lambda = a\frac{\Delta t}{\Delta x} = const.$ and that

$$u_{tt} = -au_{xt} = (-au_t)_x = (a^2u)_{xx}. \tag{1}$$

Thus, it follows for the local truncation error τ_n

$$\tau_n = \frac{1}{\Delta t}(\tilde{U}_i^n - U_i^n) = \mathcal{O}(\Delta t^2).$$

Therefore, the method is of second order.

We perform the von-Neumann stability analysis: We assume periodic boundary conditions and expand U_j^n as Fourier series. Write $u(x,t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t)e^{-ikx}$ and let $\hat{U}_k^n = \hat{u}_k(t_n)$. Then

$$\begin{aligned}
U_j^n = u(x_j, t_n) &= \sum_k \hat{U}_k^n e^{-ikj\Delta x} \\
U_{j+1}^n = u(x_{j+1}, t_n) &= \sum_k \hat{U}_k^n e^{-ik(j+1)\Delta x} \\
U_{j-1}^n = u(x_{j-1}, t_n) &= \sum_k \hat{U}_k^n e^{-ik(j-1)\Delta x}.
\end{aligned}$$

Plugging this into the Lax-Wendroff scheme, we get

$$\begin{aligned}
U_j^{n+1} &= \sum_k \hat{U}_k^{n+1} e^{-ikj\Delta x} \\
&= \sum_k \hat{U}_k^n e^{-ikj\Delta x} - \frac{\lambda}{2}\Big(\sum_k \hat{U}_k^n (e^{-ik(j+1)\Delta x} - e^{-ik(j-1)\Delta x})\Big) \\
&\quad + \frac{\lambda^2}{2}\Big(\sum_k \hat{U}_k^n (e^{-ik(j+1)\Delta x} - 2e^{-ikj\Delta x} + e^{-ik(j-1)\Delta x})\Big) \\
&= \sum_k e^{-ikj\Delta x}\Big(\hat{U}_k^n - \frac{\lambda}{2}\hat{U}_k^n(e^{-ik\Delta x} - e^{ik\Delta x}) + \frac{\lambda^2}{2}\hat{U}_k^n(e^{-ik\Delta x} - 2 + e^{ik\Delta x})\Big).
\end{aligned}$$

By projecting onto Fourier basis functions we obtain, for all $k \in \mathbb{Z}$,

$$\hat{U}_k^{n+1} = \hat{U}_k^n \underbrace{\Big(1 - \frac{\lambda}{2}(e^{-ik\Delta x} - e^{ik\Delta x}) + \frac{\lambda^2}{2}(e^{-ik\Delta x} + e^{ik\Delta x} - 2)\Big)}_{:=A_k}.$$

For stability, we need $|A_k| \leq 1$ for all k. Using

$$\sin(k\Delta x) = \frac{1}{2i}(e^{-ik\Delta x} - e^{ik\Delta x})$$

and

$$\cos(k\Delta x) = \frac{1}{2}(e^{-ik\Delta x} + e^{ik\Delta x}),$$

we get with substituting $\omega_k := k\Delta x$:

$$A_k = 1 - i\lambda\sin(\omega_k) + \lambda^2(\cos(\omega_k) - 1) = 1 - 2i\lambda\sin(\frac{\omega_k}{2})\cos(\frac{\omega_k}{2}),$$

where we used the identities

$$\sin(2x) = 2\sin(x)\cos(x)$$

and

$$\cos(2x) = 1 - \sin^2(x).$$

Therefore,

$$\begin{aligned}
|A_k|^2 &= (1 - 2\lambda^2\sin^2(\frac{\omega_k}{2}))^2 + 4\lambda^2\sin^2(\frac{\omega_k}{2})\cos^2(\frac{\omega_k}{2}) \\
&= 1 + 4\lambda^4\sin^4(\frac{\omega_k}{2}) - 4\lambda^2\sin^2(\frac{\omega_k}{2}) + 4\lambda^2\sin^2(\frac{\omega_k}{2})\cos^2(\frac{\omega_k}{2}) \\
&\overset{!}{\leq} 1
\end{aligned}$$

Rewriting this gives

$$\begin{aligned}
0 &\geq 4\lambda^2\sin^2(\frac{\omega_k}{2})\Big(\lambda^2\sin^2(\frac{\omega_k}{2})\underbrace{-1 + \cos^2(\frac{\omega_k}{2})}_{=-\sin^2(\frac{\omega_k}{2})}\Big) \\
&= 4\lambda^2\sin^4(\frac{\omega_k}{2})(\lambda^2 - 1).
\end{aligned}$$

This is equivalent to $|\lambda| \leq 1$. Thus, $|\lambda| \leq 1$ is sufficient as this implies $|A_k| \leq 1$ for all k.

Exercise 2

We modify the order calculation from Exercise 1 to include higher order terms:

$$
\begin{aligned}
&\tilde{U}_i^{n+1} - U_i^{n+1} \\
&= U_i^n - \frac{\lambda}{2}(U_{i+1}^n - U_{i-1}^n) + \frac{\lambda^2}{2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) - U_i^{n+1} \\
&= U_i^n - \frac{a\Delta t}{2\Delta x}\Big(2\Delta x (U_i^n)_x + \frac{\Delta x^2}{3}(U_i^n)_{xxx} + \mathcal{O}(\Delta x^5)\Big) + \frac{a^2\Delta t^2}{2\Delta x^2}\Big(\Delta x^2 (U_i^n)_{xx} + \mathcal{O}(\Delta x^4)\Big) \\
&\quad - U_i^n - \Delta t (U_i^n)_t - \frac{\Delta t^2}{2}(U_i^n)_{tt} - \frac{\Delta t^3}{6}(U_i^n)_{ttt} + \mathcal{O}(\Delta t^4) \\
&= \underbrace{-a\Delta t (U_i^n)_x - \Delta t (U_i^n)_t}_{=-\Delta t\left((U_i^n)_t + a(U_i^n)_x\right)=0} - \frac{a}{6}\Delta t \Delta x^2 (U_i^n)_{xxx} + \underbrace{\frac{a^2\Delta t^2}{2}(U_i^n)_{xx} - \frac{\Delta t^2}{2}(U_i^n)_{tt}}_{=\frac{\Delta t^2}{2}\left(a^2(U_i^n)_{xx} - (U_i^n)_{tt}\right)=0} \\
&\quad - \frac{\Delta t^3}{6}(U_i^n)_{ttt} + \mathcal{O}(\Delta t^2 \Delta x^2 + \Delta t^4) \\
&= -\frac{a}{6}\Delta t \Delta x^2 (U_i^n)_{xxx} - \frac{\Delta t^3}{6}(U_i^n)_{ttt} + \mathcal{O}(\Delta t^2\Delta x^2 + \Delta t^4) \qquad [\Delta x = a\frac{\Delta t}{\lambda}] \\
&= \Delta t^3 \frac{a^3}{6}(1 - \frac{1}{\lambda^2})(U_i^n)_{xxx} + \mathcal{O}(\Delta t^4),
\end{aligned}
$$

where we used in the last equality that $u_{ttt} = -a^3 u_{xxx}$ which we can derive by Eq. (1), using

$$u_{ttt} = (a^2 u_t)_{xx} = -a^3 u_{xxx}.$$

We see that the leading error term depends on $(U_i^n)_{xxx}$. This means the scheme is dispersive as discussed in the lecture.

Exercise 3

As discussed in the lecture, the solution of the linear advection equation is

$$u(x,t) = u_0(x - 2t) = \begin{cases} -1, & x < 2t \\ 1, & x \geq 2t. \end{cases}$$

As we see in the animation, the solution Lax-Wendroff scheme spreads out over time in a wavy pattern at the discontinuity of the exact solution, whereas in the upwind scheme the numerical solution smoothes out over time. This makes sense since the LW-scheme is dispersion dominant and the upwind scheme is diffusion dominant.

Implementation of the Lax-Wendroff scheme:

$$\begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1-\lambda^2 & \frac{\lambda}{2}(\lambda-1) & & 0 \\ \frac{\lambda}{2}(\lambda+1) & \ddots & \ddots & \\ & \ddots & \ddots & \frac{\lambda}{2}(\lambda-1) \\ 0 & & \frac{\lambda}{2}(\lambda+1) & 1-\lambda^2 \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix} + \begin{pmatrix} \frac{\lambda}{2}(\lambda+1)\, u_0^n \\ 0 \\ \vdots \\ 0 \\ \frac{\lambda}{2}(\lambda-1)\, u_{N+1}^n \end{pmatrix}$$

$u_0^n = u_1^n$
$u_{N+1}^n = u_N^n$

$$= \begin{pmatrix} 1-\frac{\lambda^2}{2}+\frac{\lambda}{2} & \frac{\lambda}{2}(\lambda-1) & & 0 \\ \frac{\lambda}{2}(\lambda+1) & 1-\lambda^2 & \ddots & \\ & \ddots & 1-\lambda^2 & \frac{\lambda}{2}(\lambda-1) \\ 0 & & \frac{\lambda}{2}(\lambda+1) & 1-\frac{\lambda^2}{2}-\frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix}$$

Implementation of the upwind scheme:

$$u^{n+1} := \begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1-\lambda & & & 0 \\ \lambda & \ddots & & \\ & \ddots & \ddots & \\ 0 & & \lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix} + \begin{pmatrix} \lambda u_0^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$u_0^n = u_1^n$

$$= \begin{pmatrix} 1 & & & 0 \\ \lambda & 1-\lambda & & \\ & \ddots & \ddots & \\ 0 & & \lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix}$$