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MAT827
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Sheet 5

Deadline: 17.04.2024, 12:00 PM

Exercise 1 (Points: 2.5, 2.5)
Consider the nonlinear hyperbolic problem

$$u_t + f(u)_x = 0.$$

(a) If the flux function $f(u)$ is strictly convex and $f(u)$ has a single minimum at the point ω and no local maximum, the Godunov flux is given the following

$$\widehat{f}^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}) = \begin{cases} \min\limits_{u^n_j \le \theta \le u^n_{j+1}} f(\theta) & \text{if } u^n_j \le u^n_{j+1}, \\ \max\limits_{u^n_{j+1} \le \theta \le u^n_j} f(\theta) & \text{if } u^n_j > u^n_{j+1}, \end{cases}$$

which can be simplified to

$$\widehat{f}^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}) = \max(f(\max(u^n_j, \omega)), f(\min(u^n_{j+1}, \omega))). \tag{1}$$

If the flux function $f(u)$ is strictly concave and $f(u)$ has a single maximum at the point ω and no local minimum, please derive a similar formula as (1).

(b) If the flux function $f(u)$ has a single minimum at a point ω, show that the Engquist-Osher flux is given the following

$$\widehat{f}^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}) = \frac{f(u^n_j) + f(u^n_{j+1})}{2} - \frac{1}{2}\int_{u^n_j}^{u^n_{j+1}} |f'(\theta)| \mathrm{d}\theta,$$

which can be written as

$$\widehat{f}^n_{j+\frac{1}{2}} = f(\max(u^n_j, \omega)) + f(\min(u^n_{j+1}, \omega)) - f(\omega).$$

Exercise 2 (Points: 5, 5)
Consider Burgers' equation

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0, & x \in [0, 2] \\ u_0(x, 0) = \sin(\pi x) + \frac{1}{2} \end{cases}, \tag{2}$$

which is subject to the periodic boundary condition.

(a) Implement linearized Roe, Lax-Friedrichs, Rusanov and Engquist-Osher schemes for (2) when $t = \frac{0.5}{\pi}$. Please give the experimental convergence rates in the L^1-, L^2- and L^∞-norms using a sequence of uniform grids with meshes $N = 40, 80, 160, 320, 640$ and also plot the numerical solutions together with the exact solution.

(b) Implement linearized Roe, Lax-Friedrichs, Rusanov and Engquist-Osher schemes for (2) when $t = \frac{1.5}{\pi}$. Please plot the numerical solutions together with the exact solution. What do you observe?