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MAT827
Spring 2024
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Sheet 7

Deadline: 01.05.2024, 12:00 PM

Exercise 1 (Points: 3, 2, 2)

(a) Prove that a finite difference (FD) scheme written in the viscous form

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{1}{2}\left(Q_{j+1/2}\Delta u_{j+1/2}^n - Q_{j-1/2}\Delta u_{j-1/2}^n\right), \tag{1}$$

where $\lambda = \Delta t/\Delta x$, $\Delta u_{j+1/2} = u_{j+1} - u_j$ and $\Delta f_{j+1/2} = f(u_j) - f(u_{j-1})$, is TVD under the condition

$$\lambda \left|\frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}}\right| \le Q_{j+1/2} \le 1. \tag{2}$$

Hint Use the result of Harten's Lemma.

(b) The condition (2) gives us a recipe for building TVD schemes. Consider footprint-3 schemes, i.e., $Q_{j+1/2} = Q(u_j, u_{j+1})$, and using the ansatz that

$$Q(u,v) = q(\lambda a(u,v)), \quad a(u,v) = \begin{cases} \frac{f(u)-f(v)}{u-v} & \text{if } u \neq v, \\ f'(u) & \text{if } u = v, \end{cases} \tag{3}$$

deduce conditions on the function q, such that the FD scheme would be TVD.

(c) Consider the Lax–Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{\lambda^2}{2}\left(a_{j+1/2}\Delta f_{j+1/2} - a_{j-1/2}\Delta f_{j-1/2}\right), \tag{4}$$

where $a_{j+1/2} = a(u_j, u_{j+1})$ is using the definitions of Exercise 1-(b). Prove or disprove that the scheme is TVD.
Hint Use the criterion you found in Exercise 1-(b).

Exercise 2 (Points: 4, 4)

For scalar conservation law

$$u_t + f(u)_x = 0,$$

we obtain the semi-discrete scheme in the finite volume framework:

$$\frac{d\bar{u}_j}{dt} - \frac{1}{\Delta x}(\hat{F}_{j+1/2}^n - \hat{F}_{j-1/2}^n) = 0,$$

where $\hat{F}_{j+1/2}^n = \hat{F}(u_{j+1/2}^-, u_{j+1/2}^+)$ is the numerical flux.
Now consider the following IBVP:

$$\begin{cases} u_t + u_x = 0, \quad x \in [-5, 5] \\ u(x,0) = \begin{cases} -1, x < 0 \\ 1, x > 0 \end{cases} \end{cases} \tag{5}$$

is subjected to the following boundary condition: $u[0] = u[1]$ and $u[N+1] = u[N]$.

(a) Implement Lax-Wenddroff and Beam-Warming schemes for (5) when $t = 1$. Please plot these schemes and the exact solution together with a proper mesh size, say $N = 100$, what do you observe?

(b) Consider the Lax-Friedrichs scheme:

$$\hat{F}^n_{j+1/2}(u^-_{j+1/2}, u^+_{j+1/2}) = \frac{1}{2}(f(u^-_{j+1/2}) + f(u^+_{j+1/2})) - \frac{\Delta x}{2\Delta t}(u^+_{j+1/2} - u^-_{j+1/2}),$$

for (5) when $t = 1$. Here

$$\begin{aligned} u^-_{j+1/2} &= p_j(x_{j+1/2}), \quad x_{j-1/2} \le x \le x_{j+1/2}, \\ u^+_{j+1/2} &= p_{j+1}(x_{j+1/2}), \quad x_{j+1/2} \le x \le x_{j+3/2}, \\ p_j(x) &= \bar{u}_j + \sigma_j(x - x_j), \\ p_{j+1}(x) &= \bar{u}_{j+1} + \sigma_{j+1}(x - x_{j+1}), \end{aligned}$$

and σ_j is computed by minmod limiter. Please plot the numerical solution and the exact solution together with a proper mesh size, say $N = 100$. What do you observe?