Exercise set 5

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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Problem 5.1 Lax–Friedrichs scheme (5pts + 1 extra point)

We study in this exercise the Lax–Friedrichs scheme. For a conservation law

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \tag{1}$$

with $x \in [a,b]$ and $t \in [0,T]$, the method reads

$$u_j^{n+1} = \frac{u_{j-1}^n + u_{j+1}^n}{2} - \frac{\Delta t}{\Delta x} \frac{f(u_{j+1}^n) - f(u_{j-1}^n)}{2}, \tag{2}$$

with the usual FD notation.

1. Prove that the scheme is first order accurate: define the local truncation error as

   $$E = u(t^{n+1}, x_j) - u_j^{n+1}, \tag{3}$$

   supposing that $u(t^n, x_j) = u_j^n$ for all j and that $u(t,x)$ is regular enough, prove that $E = \mathcal{O}(\Delta t^2)$, with the usual CFL conditions $\Delta t = \lambda \Delta x$.
2. Find the conditions under which the scheme is von Neumann stable (use periodic boundary conditions and linear transport equation $f(u) = c \cdot u$).
3. Code the method for a linear transport equation with periodic boundary conditions. Pass as input c the speed of the transport equation, N number of subintervals of the domain, CFL number ($\Delta t := \text{CFL}\Delta x/|c|$), T the final time, a and b the domain extrema, u_0 the initial conditions.
4. Test the code with $c = 1$, $N = 200$, $\text{CFL} = 0.9$, $T = 1$, $a = -1$, $b = 1$, $u_0 = \cos(\pi x)$.
5. Check the numerical order of accuracy: for number of cells $N \in \{2^k | k = 1, \dots, 10\}$, run the Lax–Friedrichs scheme and compute the final $\mathbb{L}^2$ error with respect to the exact solution

   $$\|u(T) - u_{ex}(T)\|_2 := \sqrt{\Delta x \sum_{j=1}^{N} \left(u(T,x_j) - u_{ex}(T,x_j)\right)^2}. \tag{4}$$

   Plot the error vs Δx and a reference first order decay. Verify that they have the same rate of convergence.
6. (Extra 1 point) Run again the script for the order convergence with CFL=1. What do you observe and why?

Problem 5.2 Source terms (5pts + 2 extra pts)

Consider now the linear transport equation with a source term

$$\partial_t u(t,x) + c\partial_x u(t,x) = S(u(x,t)) \tag{5}$$

with initial condition $u_0(x)$.

1. Compute the exact solution of (5) for smooth initial data u_0 when the source is $S(u(t,x)) = s \cdot u(t,x)$ for $s \in \mathbb{R}$.
2. Write an explicit first order scheme based on the Lax–Friedrichs (2) to solve the linear equation with a source term. The update formula for u_j^{n+1} should depend only on u_{j-1}^n, u_j^n and u_{j+1}^n, i.e., the *footprint* of the stencil is 3 (there are infinitely many possibilities). Prove that it is first order accurate with Taylor expansions as in (3).
3. Create a new function that implements your method for the linear equation with the linear source term $S(u) = s \cdot u$ (modify the function of the previous exercise, adding also an extra input s).
4. Test it with $s = -0.5$ and $c = 1$, CFL $= 0.9$, $T = 1$, $a = -1$, $b = 1$, $u_0 = \cos(\pi x)$ and compute the numerical error decay as in (4). Is the accuracy of the scheme correct?
5. Consider the following semi–implicit scheme based on the Lax–Wendroff method

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{c^2\Delta t^2}{2\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + \Delta t \frac{S(u_j^{n+1}) + S(u_j^n)}{2} - \frac{c\Delta t^2}{4\Delta x}\left(S(u_{j+1}^n) - S(u_{j-1}^n)\right). \tag{6}$$

   What is its order of accuracy? Use the Taylor expansion.
6. (Extra 2 points) Implement the method (6) for the linear source $S(u) = s \cdot u$ and periodic boundary conditions and test the numerical accuracy of the scheme.

Submit the code for both exercises.

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