Exercise set 4

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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Problem 4.1 Linear upwind and downwind schemes (6pts)

We study in this exercise the upwind and downwind methods for linear transport equation

$$\partial_t u(t,x) + c\partial_x u(t,x) = 0, \tag{1}$$

with $c \in \mathbb{R}$ $x \in [a,b]$ and $t \in [0,T]$.

The two methods consider a first order finite difference discretization which follows the flux or goes in the opposite verse. We can define the upwind scheme as

$$u_j^{n+1} = u_j^n - \frac{\Delta t c^+}{\Delta x}\left(u_j^n - u_{j-1}^n\right) - \frac{\Delta t c^-}{\Delta x}\left(u_{j+1}^n - u_j^n\right), \tag{2}$$

while the downwind scheme is defined as

$$u_j^{n+1} = u_j^n - \frac{\Delta t c^-}{\Delta x}\left(u_j^n - u_{j-1}^n\right) - \frac{\Delta t c^+}{\Delta x}\left(u_{j+1}^n - u_j^n\right), \tag{3}$$

where $c^+ = \max(c,0)$ and $c^- = \min(c,0)$.

1. Prove that the previous scheme can also be written in the more general form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{c(u_{j+1}^n - u_{j-1}^n)}{2\Delta x} = \frac{\sigma |c| (u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{2\Delta x}, \quad \text{with } \sigma = \begin{cases} +1, & \text{(upwind)}, \\ -1, & \text{(downwind)}. \end{cases} \tag{4}$$

2. Compute the von Neumann analysis of the downwind scheme (3) for $c > 0$ and say if the scheme is stable or not.

3. Prove that the upwind scheme is total variation diminishing (use periodic boundary conditions), i.e.,

$$TV(\mathbf{u}^{n+1}) \leq TV(\mathbf{u}^n), \quad \text{with } TV(\mathbf{u}) = \sum_j \Delta x |u_{j+1} - u_j|. \tag{5}$$

4. Code both schemes in the form (4) with periodic boundary conditions, in a function where inputs are c the speed of the transport equation, N number of subintervals of the domain, CFL number ($\Delta t := \text{CFL}\Delta x / |c|$), T the final time, a and b the domain extrema, u_0 the initial conditions.

5. Use the coefficients $c = 1$, $N = 200$, $\text{CFL} = 0.9$, $T = 1$, $a = -1$, $b = 1$, $u_0 = \cos(\pi x)$ to test the schemes. Plot the total variation of the numerical solutions with respect to time in an appropriate scale. What do you observe?

6. Test the convergence order of the upwind scheme (2). For number of cells $N \in \{2^k | k = 1, \dots, 10\}$, run the upwind scheme and compute the final $\mathbb{L}^2$ error with respect to the exact solution

$$\|u(T) - u_{ex}(T)\|_2 := \sqrt{\Delta x \sum_{j=1}^N \left(u(T,x_j) - u_{ex}(T,x_j)\right)^2}. \tag{6}$$

   Plot the error vs Δx and a reference first order decay. Verify that they have the same rate of convergence.

Problem 4.2 Conservative upwind (4pts)

For the nonlinear equations, the definition of upwind scheme is not unique as there is no unique weak solution. Consider the Burgers' equation in two forms: the conservative

$$\partial_t u(t,x) + \partial_x \left(\frac{u(t,x)^2}{2} \right) = 0 \tag{7}$$

and the quasi linear form

$$\partial_t u(t,x) + u(t,x)\partial_x u(x,t) = 0. \tag{8}$$

We consider only nonnegative functions u for this exercise, hence, the wind blows from left to right. The two formulations lead to two different upwind schemes. Consider the conservative upwind scheme

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \frac{(u_j^n)^2 - (u_{j-1}^n)^2}{2} \tag{9}$$

and the quasi linear upwind given by

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} u_j^n \left(u_j^n - u_{j-1}^n \right). \tag{10}$$

1. Prove that (9) is conservative and that (10) is not conservative.
2. Given the Riemann problem

   $$u_0(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0, \end{cases} \tag{11}$$

   write the exact solution for the conservation law (7). Find the solution that the method (10), derived by the quasi linear form, would give.

   Hint: take a space discretization and evolve for one time step the solution for *interesting* points with the scheme (10). What do you observe? How will the method evolve for the all the time steps?
3. Code the two methods with Dirichlet boundary conditions on the left and outflow boundary conditions on the right (no conditions). Test the methods with the Riemann problem (11) and domain $[-1, 1]$, CFL=0.75, $T = 1$. Plot the two solutions and the exact one at final time $T = 1$.
4. Test the codes again first changing the Riemann problem into

   $$u_0(x) = \begin{cases} 1.5 & \text{if } x < 0, \\ 0.3 & \text{if } x \geq 0. \end{cases} \tag{12}$$

   What is the exact solution? Plot again the final solutions of the numerical methods and compare them with the exact one. And what happens if you choose the CFL=1.5?

Submit the code for both exercises.

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