Exercise set 6

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
Lecturer: Dr. Philipp Öffner
Teaching Assistant: Davide Torlo

Consider the following methods written in the conservation form, with the usual FD notation and $\lambda = \Delta t/\Delta x$:

$$u_j^{n+1} = u_j^n - \lambda \left(f^{num}(u_j^n, u_{j+1}^n) - f^{num}(u_{j-1}^n, u_j^n) \right), \tag{1}$$

- Lax–Friedrichs

$$f^{num}(u,v) = \frac{f(v)+f(u)}{2} - \frac{v-u}{2\lambda}; \tag{2}$$

- Rusanov (or Local Lax–Friedrichs)

$$f^{num}(u,v) = \frac{f(v)+f(u)}{2} - \max\left(|f'(u)|, |f'(v)|\right) \frac{v-u}{2}; \tag{3}$$

- Godunov

$$f^{num}(u,v) = \begin{cases} \min_{u\leq\theta\leq v} f(\theta) & \text{if } u \leq v, \\ \max_{v\leq\theta\leq u} f(\theta) & \text{if } v \leq u, \end{cases} \tag{4}$$

  which for fluxes f with a unique local minimum ω such that $f'(\omega) = 0$ and $f''(\omega) > 0$ and no local maxima (e.g. convex fluxes) can be written as

$$f^{num}(u,v) = \max\left(f\left(\max(u,\omega)\right), f\left(\min(v,\omega)\right)\right); \tag{5}$$

- Linearised Roe

$$f^{num}(u,v) = \begin{cases} f(u) & \text{if } \hat{A} \geq 0, \\ f(v) & \text{if } \hat{A} < 0, \end{cases} \tag{6}$$

  where

$$\hat{A} = \begin{cases} \frac{f(v)-f(u)}{v-u} & \text{if } u \neq v, \\ f'(u) & \text{if } u = v; \end{cases}$$

- Engquist–Osher

$$f^{num}(u,v) = \frac{f(v)+f(u)}{2} - \frac{1}{2}\int_u^v |f'(\theta)| d\theta, \tag{7}$$

  which for fluxes f with a unique local minimum ω such that $f'(\omega) = 0$ and $f''(\omega) > 0$ (e.g. convex fluxes) can be written as

$$f^{num}(u,v) = f^+(u) + f^-(v), \tag{8}$$

  where

$$\begin{cases} f^+(u) = f(\max(u,\omega)), \\ f^-(u) = f(\min(u,\omega)). \end{cases} \tag{9}$$

Problem 6.1 Code all of them (5pts)

Code all the methods in a unified way, where the input of the numerical flux are f, u, v and optional other inputs as λ or f' or ω.

Test the different methods on Burgers' equation for a shock test on the domain $[-2, 2]$ with Dirichlet BC for the Riemann problem

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases} \tag{10}$$

and then on

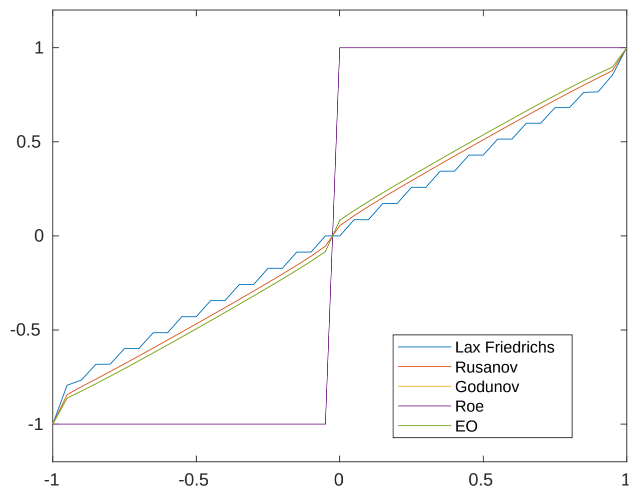
$$u_0(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases} \tag{11}$$

1. For each of the RPs and all the schemes, plot at the final time $T = 2$ the exact and the approximate solution with $N = 100$ cells.
2. What do you notice on the rarefaction wave? How does the Linearised Roe behave and why? (Heuristic reasons are enough)

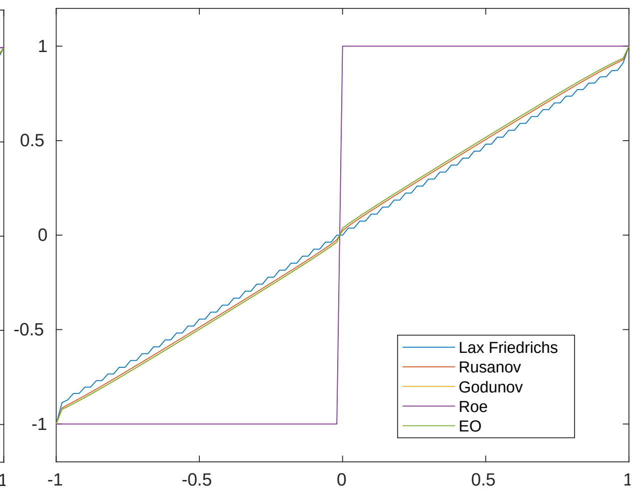
Hint

Create a function `numericalFlux` with input a string with the name of the scheme, a symbolic function f of the flux, the left value u, the right value v and an extra input, which can be the value ω for GOdunov and Engquist–Osher, λ for Lax–Friedrichs or f' for Rusanov. This function codes the numerical fluxes defined before (consider the case where there exists a local minimum of the flux in 0 as for Burgers').
A second function `runScheme` takes as input the type of scheme and the specification of the problem: domain, final time, flux, its derivative, number of cells, CFL and so on.
Finally, a script runs the schemes with the different schemes.

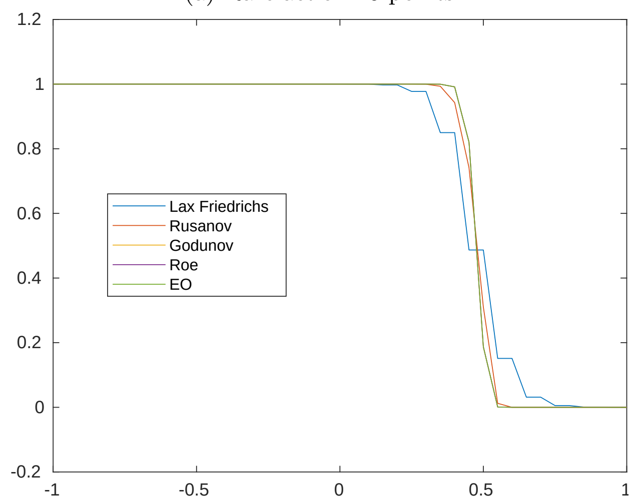
Solution



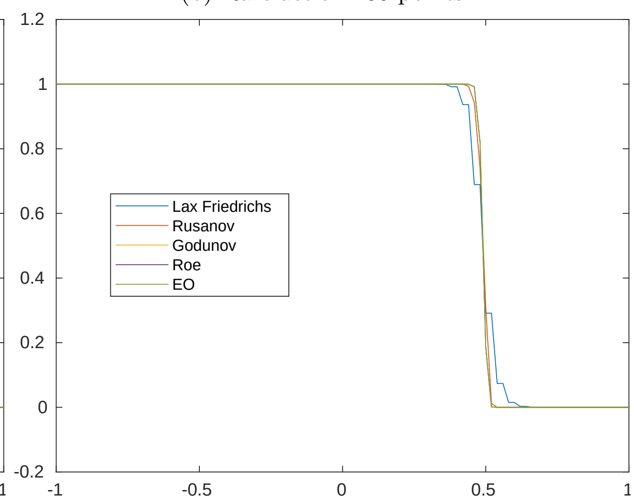
(a) Rarefaction 40 points



(b) Rarefaction 100 points



(c) Shock 40 points



(d) Shock 100 points

Figure 1: Comparison of different schemes on shock and rarefaction tests

We see in the rarefaction simulations Figures 1a and 1b that the linerized Roe does not move from the initial condition. This is due to the fact that the Riemann problem is such that $f(u_L) = f(u_R)$ and hence the for all the fluxes $\hat{A} = \frac{f(u)-f(v)}{u-v} = 0$ for all the possible points of the mesh. Hence, the discontinuity does not move.

This scheme is clearly violating the entropy relation, even though it is converging towards a weak solution, which is not the physical one.

```matlab
function [fNum] = numericalFlux(scheme,f,u,v,varargin)
% encode in this function different numerical fluxes
switch scheme
    case "Lax Friedrichs"
        % varargin should be lambda=dt/dx
        lam=varargin{1};
        fNum=(f(u)+f(v))/2-(v-u)/lam/2;
    case "Rusanov"
        % varargin should be f'
        df=varargin{1};
        fNum=(f(u)+f(v))/2-max(abs(df(u)),abs(df(v))).*(v-u)/2;
    case "Godunov"
        % varargin should be omega the unique local minimum of f
        omega=varargin{1}*ones(size(u));
        fNum=max(f(max(u,omega)),f(min(v,omega)));
    case "Roe"
        % varargin should be empty
        idxs=u==v;
        idx=logical(1-idxs);
        fNum=f(u);
        A=(f(u(idx))-f(v(idx)))./(u(idx)-v(idx));
        fNum(idx)=f(u(idx)).*(A>=0)+f(v(idx)).*(A<0);
    case "EO" %Engquist--Osher
        % varargin should be omega the unique local minimum of f
        omega=varargin{1}*ones(size(u));
        fNum=f(max(u,omega))+ f(min(v,omega));
end


end
```

Listing 1: numericalFlux.m

```matlab
function [u,x,t] = runScheme(scheme, f, df, N, CFL, T, a, b, u0, BC)

x=linspace(a,b,N+1);
dx=x(2)-x(1);
maxdfu=max(df(u0(x)));
dt=dx*CFL/maxdfu;
Nt=ceil(T/dt);
dt=T/Nt;
t=linspace(0,T,Nt+1);

if BC=="periodic"
    j=1:N;
    jR=[2:N,1];
    jL=[N, 1:N-1];
    u=zeros(Nt+1,N);
    x=x(j);
elseif BC=="dirichlet"
    j=[2:N];
    jL=[1:N-1];
    jR=[3:N+1];
    x=x(1:N+1);
    u=zeros(Nt+1,N+1);
    u(:,1)=u0(x(1));
    u(:,N+1)=u0(x(N+1));
```

```
end

u(1,:)=u0(x);
extra=defineExtraScheme(scheme,dt/dx,df);
for kt=2:Nt+1
    u(kt,j)=u(kt-1,j) - dt/dx*(...
        +numericalFlux(scheme,f,u(kt-1,j),u(kt-1,jR),extra)...
        -numericalFlux(scheme,f,u(kt-1,jL),u(kt-1,j),extra));
end

end


function extra=defineExtraScheme(scheme,lam,df)
switch scheme
    case "Lax Friedrichs"
        extra=lam;
    case "Rusanov"
        extra=df;
    case {"Godunov","EO"}
        extra=0; %only Burgers
    case "Roe"
        extra={};
end

end
```

Listing 2: runScheme.m

```
f=@(u) u.^2/2;
df=@(u) u;
T=1;
a=-1;
b=1;
uL=0;%1;%-1
uR=1;%0;%1
u0=@(x) uL*(x<0)+uR*(x>=0); %cos(pi*x);
BC="dirichlet";


%% run lxf scheme
scheme="Roe";%"Rusanov";%"Godunov";%"Roe";%Rusanov";%"EO";%"Lax Friedrichs"

Nx = 40;
CFL = 0.75;

[u,x,t]= runScheme(scheme, f, df, Nx, CFL, T, a, b, u0, BC);

%% plot evolution
plot_evolution=1;

dx = x(2)-x(1);
fig=figure(1);
if(plot_evolution)
    for n=[1:ceil(numel(t)/20):numel(t),numel(t)]
        %
        figure(1)
```

```
        plot(x,u(n,:))
        legend('numerical','Location','SE')
        xlabel x
        ylabel u
        title(sprintf('Lax-Friedrichs, CFL=%g, t=%g',CFL,t(n)))
        ylim([-1.5,1.5])


        %
        drawnow
        pause(.02)
    end
end

%%
schemes=["Lax Friedrichs";"Rusanov";"Godunov";"Roe";"EO"];
Nx = 100;
CFL = 0.75;

clear u x t

for k=1:length(schemes)
    scheme=schemes(k);
    [u{k},x{k},t{k}]= runScheme(scheme, f, df, Nx, CFL, T, a, b, u0, BC);
end

for k=1:length(schemes)
    scheme=schemes(k);
    plot(x{k},u{k}(end,:),'DisplayName',scheme)
    hold on
end
legend('Location','best')

ylim([min(u{1},[],'all')-0.2, max(u{1},[],'all')+0.2])
```

Listing 3: test.m

Problem 6.2 Monotone schemes (3 pts)

A monotone scheme is such that if $u_j^n \leq v_j^n$ for all j, then $u_j^{n+1} \leq v_j^{n+1}$ for all j. Prove or disprove that the following schemes are monotone under some CFL conditions.

1. Godunov

$$f^{num}(u,v) = \begin{cases} \min_{u\leq\theta\leq v} f(\theta) & \text{if } u \leq v, \\ \max_{v\leq\theta\leq u} f(\theta) & \text{if } v \leq u, \end{cases} \tag{12}$$

Solution

We can use the theorem proved in class for which a scheme is monotone iff, for any a, b, c in the solution domain,

$$\frac{\partial f^{num}(a,b)}{\partial u} \geq 0 \tag{13}$$

$$\frac{\partial f^{num}(a,b)}{\partial v} \leq 0 \tag{14}$$

under the condition

$$\left|\frac{\partial f^{num}(b,c)}{\partial u}\right| + \left|\frac{\partial f^{num}(a,b)}{\partial v}\right| \leq \frac{\Delta x}{\Delta t}. \tag{15}$$

To proceed with condition (13), we check the case $a \leq b$ which becomes

$$\frac{\partial f^{num}(u,v)}{\partial u} = \partial_u \min_{u\leq\theta\leq v} f(\theta) = \lim_{\varepsilon\to 0} \frac{\min_{u+\varepsilon\leq\theta\leq v} f(\theta) - \min_{u\leq\theta\leq v} f(\theta)}{\varepsilon} = \tag{16}$$

$$= \begin{cases} \lim_{\varepsilon\to 0^+} \frac{\min_{u+\varepsilon\leq\theta\leq v} f(\theta)-\min_{u\leq\theta\leq v} f(\theta)}{\varepsilon} \geq 0, \\ \lim_{\varepsilon\to 0^-} \frac{\min_{u+\varepsilon\leq\theta\leq v} f(\theta)-\min_{u\leq\theta\leq v} f(\theta)}{\varepsilon} \geq 0, \end{cases} \tag{17}$$

where the final inequality can be shown comparing the set where the minimum is computed using the correct sign of ε. For the case $b \leq a$ and the relation (14) the computations are analogous.

Clearly, also the condition (15) is verified providing CFL conditions

$$2\max_x |f'(u^n(x))| \leq \frac{\Delta x}{\Delta t}. \tag{18}$$

2. Linearised Roe

$$f^{num}(u,v) = \begin{cases} f(u) & \text{if } \hat{A} \geq 0, \\ f(v) & \text{if } \hat{A} < 0, \end{cases} \tag{19}$$

where

$$\hat{A} = \begin{cases} \frac{f(v)-f(u)}{v-u} & \text{if } u \neq v, \\ f'(u) & \text{if } u = v; \end{cases}$$

Solution

One can check the condition (13) to start.

$$\frac{\partial f^{num}(a,b)}{\partial u} = \begin{cases} f'(u) & \text{if } \hat{A} \geq 0 \\ 0 & \text{if } \hat{A} < 0 \end{cases} = \begin{cases} f'(u) & \text{if } \frac{f(u)-f(v)}{u-v} \geq 0, \\ 0 & \text{if } \hat{A} < 0. \end{cases} \tag{20}$$

We can imagine a situation where this is negative. A simple example could be for for Burgers' equation $f = u^2/2$, where $u = -1$ and $v = 2$. Clearly $\hat{A} = 1 \geq 0$, but $f'(u) = u = -1$. Hence, the scheme is not monotone.

3. Engquist–Osher

$$f^{num}(u,v) = \frac{f(v)+f(u)}{2} - \frac{1}{2}\int_u^v |f'(\theta)| d\theta. \tag{21}$$

Solution

Using the fundamental theorem of calculus we can compute

$$\frac{\partial f^{num}(a,b)}{\partial u} = \frac{f'(a)}{2} + \frac{1}{2}|f'(a)| = \max(f'(a),0)) \geq 0,$$
$$\frac{\partial f^{num}(a,b)}{\partial v} = \frac{f'(b)}{2} - \frac{1}{2}|f'(a)| = \min(f'(b),0)) \leq 0.$$

Finally, we show that

$$\left|\frac{\partial f^{num}(b,c)}{\partial u}\right| + \left|\frac{\partial f^{num}(a,b)}{\partial v}\right| = |\max(f'(b),0)| + |\min(f'(a),0)| = |f'(b)| \leq \frac{\Delta x}{\Delta t}, \tag{22}$$

which is verified under usual CFL conditions.

Problem 6.3 Everything is upwind (2pts)

Show that if f is monotone and convex, then Godunov (4), Linearised Roe (6) and Engquist–Osher (7) are the upwind schemes.

Solution

Let us recall that the upwind scheme can be written as

$$f^{num}(u,v) = \begin{cases} f(u) & \text{if } f' > 0, \\ f(v) & \text{if } f' \leq 0, \end{cases} \tag{23}$$

where we can write f' without ambiguity since the flux is monotone ($f' > 0$ or $f' < 0$ everywhere).

1. Godunov

$$f^{num}(u,v) = \begin{cases} \min_{u \leq \theta \leq v} f(\theta) & \text{if } u \leq v, \\ \max_{v \leq \theta \leq u} f(\theta) & \text{if } v \leq u, \end{cases} = \begin{cases} \begin{cases} f(u) & \text{if } u \leq v, \\ f(u) & \text{if } v \leq u, \end{cases} & \text{if } f' > 0, \\ \begin{cases} f(v) & \text{if } u \leq v, \\ f(v) & \text{if } v \leq u, \end{cases} & \text{if } f' \leq 0, \end{cases}$$

which is the upwind scheme.

2. Linearised Roe

$$f^{num}(u,v) = \begin{cases} f(u) & \text{if } \hat{A} > 0, \\ f(v) & \text{if } \hat{A} \leq 0, \end{cases} = \begin{cases} f(u) & \text{if } f' > 0, \\ f(v) & \text{if } f' \leq 0. \end{cases}$$

3. Engquist–Osher

$$f^{num}(u,v) = \frac{f(v)+f(u)}{2} - \frac{1}{2}\int_u^v |f'(\theta)| d\theta = \begin{cases} \frac{f(v)+f(u)}{2} - \frac{1}{2}\int_u^v f'(\theta)d\theta & \text{if } f' > 0, \\ \frac{f(v)+f(u)}{2} + \frac{1}{2}\int_u^v f'(\theta)d\theta & \text{if } f' > 0, \end{cases} =$$
$$= \begin{cases} \frac{f(v)+f(u)}{2} - \frac{f(v)-f(u)}{2} & \text{if } f' > 0, \\ \frac{f(v)+f(u)}{2} + \frac{f(v)-f(u)}{2} & \text{if } f' > 0, \end{cases} = \begin{cases} f(u) & \text{if } f' > 0, \\ f(v) & \text{if } f' > 0. \end{cases}$$

Organiser: Davide Torlo, Office: home (davide.torlo@math.uzh.ch)

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