Exercise set 1

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
Lecturer: Dr. Philipp Öffner

Problem 1.1 Types of PDEs (2 pt)

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the solution of the PDE

$$a\partial_{xx}u + b\partial_{xy}u + c\partial_{yy}u + d\partial_x u + e\partial_y u + fu = g, \tag{1}$$

where $a, b, c, d, e, f, g \in \mathbb{R}$ are constant coefficients and let $T \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ be a nonsingular transformation of $\mathbb{R}^2$, i. e., for every $(x, y) \in \mathbb{R}^2$ we have that the Jacobian of the transformation is different from zero $\det(JT(x, y)) \neq 0$. Then the transformed equation is of the same type.

Problem 1.2 Method of characteristics (4 pt)

Consider the one-dimensional linear advection equation

$$\begin{aligned} \partial_t u(x,t) + a(x)\partial_x u(x,t) &= 0 && \text{for } (x,t) \in D \times [0,\infty) \\ u(x,0) &= \sin(x) && \text{for } x \in D, \end{aligned} \tag{2}$$

where $D \subseteq \mathbb{R}$ is the spatial domain. For each of the following advection coefficients $a(x)$ and domains D, justify the existence (or non-existence) of a solution to (1), plot the family of characteristics in the (x, t)-plane and write down the explicit solution.

a) $a(x) = \alpha x$, $\alpha \in \mathbb{R}$, and $D = \mathbb{R}$,

b) $a(x) = -1$, and $D = [0, \infty)$,

c) $a(x) = 1$, and $D = [0, \infty)$.

Problem 1.3 Linear advection equation (4 pt)

a) Consider the one-dimensional linear advection equation with variable coefficients

$$\begin{cases} \partial_t u(x,t) + a(x)\partial_x u(x,t) = 0 & \text{for } (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{3}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function. Let $a : \mathbb{R} \to \mathbb{R}$ be given by

$$a(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ -x, & \text{if } 0 < x \leq 1, \\ -1, & \text{if } x > 1. \end{cases} \tag{4}$$

Use the method of characteristics to find the solution $u(x, t)$, for $(x, t) \in \mathbb{R} \times [0, \infty)$, and a generic initial condition u_0.

b) Determine now the solution $u(x, t)$ of the one-dimensional linear conservation law

$$\begin{cases} \partial_t u(x,t) + \partial_x\big(a(x)u(x,t)\big) = 0 & \text{for } (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{5}$$

for $(x,t) \in \mathbb{R} \times [0,\infty)$, and a generic initial condition u_0.

Hint: Make use of the characteristics found in (a).

c) For the two solutions computed in (a) and (b), respectively, what can be said about $\int_{\mathbb{R}} |u(x,t)| \, dx$ and $\max_{x\in\mathbb{R}} |u(x,t)|$ as a function of time t?

d) Let $u(x,t)$ be a solution of the conservation law (5), where we now consider a general coefficient $a \in C^1$. Assume that $u(x,t)$ is C^1 and that $\lim_{|x|\to\infty} u(x,t) = 0$ for all $t \geq 0$ (you may assume $u(x,t)$ to be well-behaved at $x = \pm\infty$ to avoid technical difficulties). Show that $u(x,t)$ is conserved, in the sense that $\int_{\mathbb{R}} u(x,t) \, dx = \int_{\mathbb{R}} u_0(x) \, dx$ for all $t \geq 0$. Is this also true for the linear advection equation?

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Additional Informations

- Exercise classes will be held by Davide Torlo (Y27 K48, `davide.torlo@math.uzh.ch`) on Thursday form 10.00 to 12.00 in Y27 H26.
- There might be an additional exercise class.
- There will be a written or an oral exam at the end. It will be decided after the first two weeks of the course. The date of the written exam will be the 29th of June. Sufficient to participate in the exam are 50 % of the total points of **all** exercise sheets.
- There will be each week an exercise sheet on Thursday. You have to hand in to Davide Torlo at the beginning of the Thursday exercise class or in the mailbox in Y27 K floor or to Philipp Öffner at the end of Thursday lesson.
- For a consultation hour, please write an e-mail to davide.torlo@math.uzh.ch or philipp.oeffner@math.uzh.ch or ask after the lecture.