Numerical Methods for Hyperbolic PDEs[1]

Prof. Adrian Ruf, FS 2021
13.08.2021, 30 minutes

Summary In a seminar room in HG, I am sitting at a table with the professor and the assistant, who writes the official protocol. I have sufficient paper in front of me to take notes, which I have to leave there after the exam.

The professor has in front of him the catalogue of possible exam questions he mentioned in the lecture and chooses questions from it. As he said in the lecture, all questions were either related to answers I had given before or followed one chapter of the lecture in a sensible way.

Most important: No questions on the exercises, nor on code, nor on tedious calculations and proofs from the lecture. Go through the results in the script, remember basic ideas of proofs, be able to present all topics in a plausible way and you will be well prepared :)

Make sure to know what you want to start with, choose something not too loose but not too concrete. (I did not good in this, but it was not a big issue.)

Content First order FVM (some flux, monotone schemes, TVD property, Harten's Lemma), Second Order FVM (some slope, limiter formulation, TVD region), Higher Order FVM (WENO), Entropy Solutions (discrete L^1/doubling of variables)

Protocol

First Order FVM

Prof: Is there something particular you would like to start with?

Me: The general formulation of finite volume methods. (I explain the usual update form.)

Prof: Could you please specify one particular flux function that we saw?

Me: I write down the Roe flux, explaining how the Roe average is connected to solving the linear transport equation for the Riemann problem given by the cell averages. I comment on that the scheme is not monotone and thus might fail to resolve rarefaction waves in nonlinear equations.

Prof: Now you chose the one and only non-monotone flux that we saw (apparently he did not expect this). What stability results do we have for monotone methods?

Me: L^∞, L^p, which is derived from the discrete entropy inequality, TVD property, Lipschitz continuity

[1] BSc/MSc Mathematics (I am from MSc), Core Subjects

in time.

Prof: What is the discrete entropy inequality?

Me: I write it down and mention that it involves the Kruzkhov entropy and how to get L^p results from it.

Prof: And what is the TVD that you mention?

Me: I give the discrete definition.

Prof: And can you give an idea on how to prove this for monotone schemes? (He explicitly wants no written formulas, just key words.)

Me: We use Harten's Lemma to show this property, involving coefficients C and D in the incremental FV form that have to be nonnegative and their sum has to be ≤ 1.

Prof: What do we need to show the conditions on C and D for monotone schemes?

Me: First, the general CFL criterion is (I write it down). For monotone schemes, we also have CFL* (I write down $\partial_a F \geq 0$, ...)

Prof: Finally, what is special for these hyperbolic schemes/equations that we don't have for others (e.g. parabolic)? Related to local changes and global effects? (It took me a bit to understand the question.)

Me: We have a finite travel speed of information.

Second Order Schemes

Prof: There is one aspect that we want to improve over the schemes we just talked about. What is it and how can we do this?

Me: They are only first order accurate, which refers to the truncation error. If we consider the REA frame, we just talked about fluxes needed in E, and we can improve the reconstruction to piece wise linear or polynomial.

Prof: Can explain how second order schemes work and give an example?

Me: I explain the piece wise linear form of p, that it is centered around U_j for conservation, and write down the minmod limiter, saying that it is TVD in contrast to other slopes that only consider one local gradient.

Prof: What is the minmod function, say at values -1, -2?

Me: Same signs, so non-zero, and minimal absolute value, so it is -1.

Prof: What is the TVD region, and the corresponding limiter formulation?

Me: I state both.

Higher Order Schemes

Prof: And what if we want to go even higher? Can you explain the WENO scheme?

Me: First, the formulation focuses on edge value reconstruction because that is what we need in the flux function. From approximation theory, we have this relation for the r-left-shift reconstruction: (I write down and explain the formula for $V^r_{j+1/2}$), where the coefficients are known from Shu's table. However, this is only k-th accurate for $k+1$ points in the stencil, so there is another result on linear combinations (write down the formula for $V^{\tilde{r}}_{j+1/2}$, which is of order $2k-1$. In the WENO scheme, we follow these steps: (I explain and write down $d,\ w,\ \alpha,\ \beta$, where I only give an idea of β and no formula).

Entropy Solutions

Prof: Now we talked a lot about numerical schemes. In the continuous case, what results do we have for entropy solutions?

Me: Uniqueness, L^1 and L^∞ bounds, ...

Prof: Can you give an idea of how to prove the L^1-bound?

Me: I state the result, mention that we prove local stability first and state this, explain and write down the doubling of variables technique by Kruzkhov involving the entropy inequality, explain that we will integrate the resulting relation over the domain A and use Green's theorem, where the terms with M will drop from the inequality and the local L^1 stability follows.

Prof: Thanks a lot, time is over.

Final Remarks In the protocol, I left out that sometimes it took the professor some more small questions to get me to the result he wanted to hear, but that was absolutely no problem. In general, both he and the assistant were very friendly, and the exam was exactly as described by the professor in the lecture.

Expected mark: 6
Received mark: 6