Exercise set 9

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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Problem 9.1 E-schemes (2pts)

Consider a consistent monotone 3-point numerical flux $f^{num}(u,v)$. The scheme given by this flux is said to be an *E-scheme* if it satisfies

$$(u_{j+1}-u_j)(f^{num}(u_j,u_{j+1})-f(u)) \leq 0 \quad \forall u \in [u_j,u_{j+1}] \text{ or } \forall u \in [u_{j+1},u_j], \tag{1}$$

according to the sign of $(u_{j+1}-u_j)$.

1. Prove that a 3-point monotone scheme is an E-scheme.
2. Consider the Godunov flux

$$f_G^{num}(u,v) := \begin{cases} \min_{u_j \leq u \leq u_{j+1}} f(u) & u_j \leq u_{j+1}, \\ \max_{u_{j+1} \leq u \leq u_j} f(u) & u_{j+1} \leq u_j. \end{cases} \tag{2}$$

   Prove that every monotone 3-point scheme with numerical flux f^{num} under CFL≤ 1 verifies

$$\begin{cases} f^{num}(u_j,u_{j+1}) \leq f_G^{num}(u_j,u_{j+1}) & \text{if } u_j \leq u_{j+1}, \\ f^{num}(u_j,u_{j+1}) \geq f_G^{num}(u_j,u_{j+1}) & \text{if } u_{j+1} \leq u_j. \end{cases} \tag{3}$$

Problem 9.2 Lax Wendroff (3 pts)

Code the two step Lax Wendroff scheme, where

$$\begin{cases} u_j^{n+1} = u_j^n - \lambda \left(f(u_{j+1/2}^{n+1/2}) - f(u_{j-1/2}^{n+1/2}) \right), \\ u_{j+1/2}^{n+1/2} = \frac{u_j+u_{j+1}}{2} - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_j^n) \right). \end{cases} \tag{4}$$

1. Write it in the conservation formulation.
2. Code it in the usual code (check solution of Exercise Set 8).
3. Compare it with the classical Lax Wendroff scheme: check the accuracy order and the computational times:
   For $N \in \{2^k : k=1,\dots,10\}$ number of cells solve the advection equation $u_t+u_x=0$ with $u_0(x)=\cos(\pi x)$ on $[-2,2]$ with periodic boundary conditions. Compute the $\mathbb{L}^2$ error with respect the exact solution for both schemes and all N and the time needed to solve the scheme (see `tic, toc` in Matlab and package `time` in Python). Plot the error wrt N in an appropriate scale and the time and N in an appropriate scale. What can you say on the error? And on the time?

Problem 9.3 MUSCL (5 pts + 1 extra)

In this problem we code the MUSCL scheme with different choices of *switching function*. The scheme can be summarized in the following steps.

1. Given the solution u_j^n, reconstruct the solution inside the cell $[x_{j-1/2}, x_{j+1/2}]$ as linear function

$$u_j^n(x) = u_j^n + \sigma_j^n(x - x_j), \quad x \in [x_{j-1/2}, x_{j+1/2}]. \tag{5}$$

   Define $u_{j,\pm}^n := u_j^n(x_{j\pm1/2})$.

2. Using this reconstruction compute an intermediate step at time $t^{n+1/2}$ for the interface values

$$u_{j,\pm}^{n+1/2} = u_{j,\pm}^n - \frac{\lambda}{2}\left(f(u_{j,+}^n) - f(u_{j,-}^n)\right). \tag{6}$$

3. Apply the numerical flux on the intermediate states

$$u_j^{n+1} = u_j^n - \lambda\left(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}\right) \tag{7}$$

   where the numerical fluxes are defined as

$$F_{j+1/2}^{n+1/2} = f^{num}(u_{j,+}^{n+1/2}, u_{j+1,-}^{n+1/2}). \tag{8}$$

Check figure 1 for an idea of the steps of the scheme.

The left choice to define this scheme are the way we define the slope σ_j^n, and it will depend on u_{j-1}, u_j, u_{j+1} and the numerical fluxes f^{num}.

The slope can be defined in a general way as

$$\sigma_j^n(u_{j-1}^n, u_j^n, u_{j+1}^n) = \frac{u_{j+1}^n - u_j^n}{\Delta x}\phi(r_j^n), \quad r_j^n := \frac{u_j^n - u_{j-1}^n}{u_{j+1}^n - u_j^n}. \tag{9}$$

Here, follows some possible choices for the *switching* function.

Name	*Switching* function $\phi(r)$
Upwind	0
Lax–Wendroff	1
Beam–Warming	r
Minmod	$\text{minmod}(1, r)$
Superbee	$\max(0, \min(1, 2r), \min(2, r))$
MC	$\max\left(0, \min\left(\frac{1+r}{2}, 2, 2r\right)\right)$
van Leer	$\frac{r+\lvert r\rvert}{1+\lvert r\rvert}$

Table 1: The most popular flux limiters.

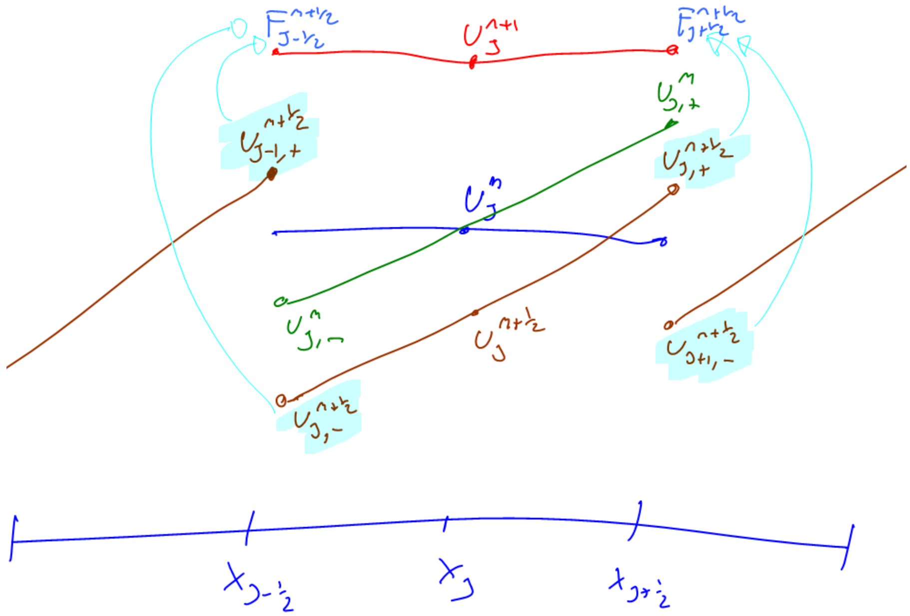

Figure 1: An amazing sketch of the MUSCL scheme. In order blue, green, brown, lightblue, red.

Note that

$$\operatorname{minmod}(a,b) = \begin{cases} \operatorname{sign}(a)\min(|a|,|b|) & \text{if } ab \geq 0, \\ 0 & \text{else.} \end{cases} \tag{10}$$

1. Which condition should the switching function and the numerical flux f^{num} verify in order to have a TVD and second order scheme on linear advection problems?
2. Code a function `switchingFunctions`, which code all the functions contained in Table 1 (inputs: a string with the name of the function and r, output the value of the function evaluated in r). In another script, plot all the functions in one figure on the interval $[0,5]$. Which of the functions does not verify the TVD conditions and which one does not verify the second order condition?
3. Code the MUSCL scheme as presented above, making use of the old function for numerical fluxes and of the `switchingFunctions`.
   Hint: build a function `computeSlope`, which compute the slopes of the intervals, check that the reconstruction that you obtain is TVD (a simple linear advection problem with few Ns is enough to check it).
4. Test the accuracy of the method on $\partial_t u + \partial_x u = 0$ with $u_0(x) = 1 + 0.2\cos(\pi x)$ on $[-2, 2]$ for all switching functions and Rusanov as numerical flux. Which limiter performs better? Which one does not reach second order of accuracy?

5. Test the TVD property of the scheme. For $\partial_t u + \partial_x u = 0$ with

$$u_0(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

on $[-2, 2]$ with all the limiters and Rusanov numerical flux compute the TV of the solution as a function of time on a simulation with $N = 100$ cells, $T = 1$ and CFL=0.9. Which limiter does not fulfill the TVD conditions?

6. Test the scheme with Burgers' equation. What is happening? Why? Can you find a switching function and a numerical flux that, under some conditions, give a TVD scheme?

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