Numerical Methods for Hyperbolic PDEs
Homework 7

James King*, Robert Ziegler†

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Exercise 1

See extra pdf.

Exercise 2

Since this is simply the linear transport equation, the exact solution is

$$u(t) = u_0(x-t) = \begin{cases} -1, & x < t \\ 1, & x > 0 \end{cases}$$

(a)

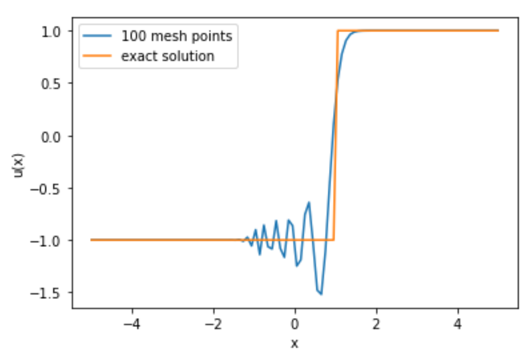

Figure 1: Numerical solutions with the the Lax-Wendroff scheme at time $t = 1$

*Immatriculation Nr. 23-942-030
†Immatriculation Nr. 18-550-558

We observe that Lax-Wendroff has oscillations before the shock. Beam-Warming has slightly less problems with oscillations and they occur after the shock.

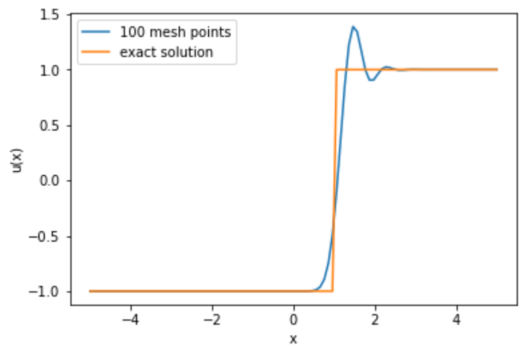

Figure 2: Numerical solutions with the the Beam-Warming scheme at time $t = 1$

(b)

Note that we use the explicit Euler rule to approximate the ODE, resulting in the standard first-order monotone finite volume scheme form

$$\bar{U}_j^{n+1} = \bar{U}_j^n - \frac{\Delta t}{\Delta x}\Big(F_{j+1/2}^n - F_{j-1/2}^n\Big).$$

For $\bar{U}_0$ we explicitly calculate

$$\bar{U}(x,0) = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x) = \begin{cases} -1, & x < -\frac{\Delta x}{2} \\ \frac{2x}{\Delta x}, & -\frac{\Delta x}{2} < x < \frac{\Delta x}{2} \\ 1, & x > \frac{\Delta x}{2} \end{cases}$$

We note that the numerical solution with the Lax-Friedrichs scheme is very diffusive and not at all very accurate. At 100 mesh points the graph seems to develop very small plateaus. But this seems to be an artifact of the choice of the number of mesh points to be even, since at odd number of mesh points as for example 101, this effect goes away.

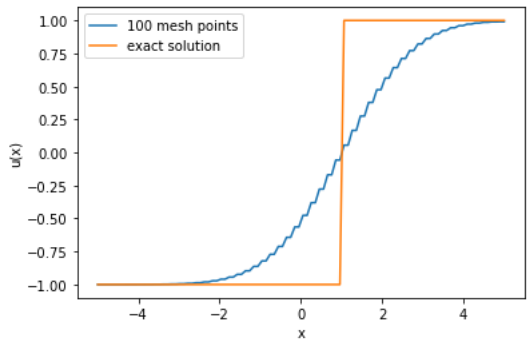

Figure 3: Numerical solutions with the the Lax-Friedrichs scheme at time $t = 1$, 100 mesh points

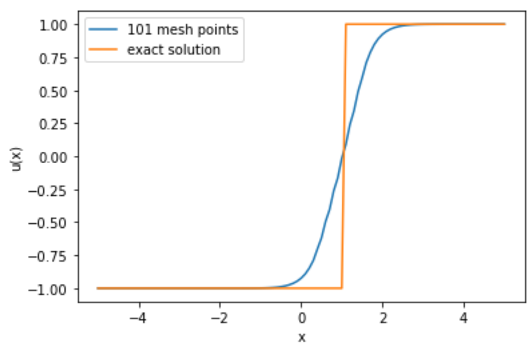

Figure 4: Numerical solutions with the the Lax-Friedrichs scheme at time $t = 1$, 101 mesh points