

2. The finite volume scheme for the system from 1. is

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} (F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n)$$

with $F_{j+\frac{1}{2}}^n = F(U_j^n, U_{j+1}^n) := A U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}, 0)$,

where $U_{j+\frac{1}{2}}$ is the solution (exact or approximate) of the Riemann problem.

The Godunov flux uses the exact solution $U^* = U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}, 0)$.

In the transformed coordinates

$$W := R^{-1}U, \quad A = R \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} R^{-1}$$

the exact solution of the Riemann problem at the interface has a jump $W_R - W_L$.

Thus, in the original coordinates, the jump becomes

$$\begin{aligned} U_R - U_L &= R(W_R - W_L) \\ &= \sum_{p=1}^m (W_R^p - W_L^p) \Gamma_p \\ &= \sum_{p=1}^m \alpha_{j+\frac{1}{2}}^p \Gamma_p, \end{aligned}$$

where Γ_p are the eigenvectors of A .

Thus, since the Eigenvalues λ_i correspond to the wavespeed, we have

$$U^* = U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}, 0) = \sum_{p: \lambda_p < 0} \alpha_{j+\frac{1}{2}}^p \Gamma_p + U_j^n \quad (*)$$

Thus,

$$\begin{aligned} AU^* &= AU_j^n + \sum_{p: \lambda_p < 0} \alpha_{j+\frac{1}{2}}^p \underbrace{A \Gamma_p}_{= \lambda_p \Gamma_p} \\ &= AU_j^n + \sum_{p=1}^m \alpha_{j+\frac{1}{2}}^p \lambda_p^+ \Gamma_p. \end{aligned}$$

Similarly, we can represent $(\#)$ as

$$U^* = U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}, 0) = U_{j+1}^n - \sum_{p: \lambda_p > 0} \alpha_{j+\frac{1}{2}}^p \Gamma_p,$$

thus

$$AU^* = AU_{j+1}^n - \sum_{p=1}^m \alpha_{j+\frac{1}{2}}^p \lambda_p^+ \Gamma_p.$$

Averaging yields

$$\begin{aligned} AU^* &= \frac{1}{2} (AU_j^n + AU_{j+1}^n - \sum_{p=1}^m \alpha_{j+\frac{1}{2}}^p (\lambda_p^+ + \lambda_p^-) \Gamma_p) \\ &= \frac{1}{2} A(U_j^n + U_{j+1}^n) - \frac{1}{2} \sum_p |\lambda_p| \alpha_{j+\frac{1}{2}}^p \Gamma_p. \end{aligned}$$

The wave strength $\alpha_{j+\frac{1}{2}}^p$ is equal to $W_{j+1}^n - U_j^n$, thus

$$\begin{aligned} AU^* &= \frac{1}{2} A(U_j^n + U_{j+1}^n) - \frac{1}{2} R |A| (U_{j+1}^n - U_j^n) = \frac{1}{2} A(U_j^n + U_{j+1}^n) - \frac{1}{2} R |A| R^{-1} (U_{j+1}^n - U_j^n) \\ &=: \bar{F}_{j+\frac{1}{2}}^n, \text{ which gives the Godunov scheme.} \end{aligned}$$