Exercise set 2

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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Problem 2.1 Discontinuities (3pts)

Consider the Cauchy problem

$$\begin{aligned} u_t + f(u)_x &= 0 \qquad && \text{for } (x,t) \in \mathbb{R}\times\mathbb{R}_+ \\ u(x,0) &= u_0(x) && \text{for } x \in \mathbb{R}. \end{aligned} \tag{1}$$

a) Let $f(u) = u^2/2$, ie. let (1) be the Burgers' equation. Assume u_0 is smooth and that u_0' is negative at some point. Show that the solution of (1) will generate a discontinuity at time

$$T_b = \frac{-1}{\min_x u_0'(x)}.$$

Solution

For the first point we follow a more intuitive approach, while in the second we follow a more analytic one.

Take two points x_0, y_0, such that $x_0 < y_0$ and $u_0(x_0) > u_0(y_0)$. We know that the two characteristics will meet

$$\begin{cases} x_{x_0}(t) = x_0 + u_0(x_0)t \\ x_{y_0}(t) = y_0 + u_0(y_0)t \end{cases} \tag{2}$$

at $x_{x_0}(t) = x_{y_0}(t)$, hence when

$$\tilde{t} = \frac{-(x_0 - y_0)}{u_0(x_0) - u_0(y_0)}. \tag{3}$$

If we take all the possible points, we have that the breaking time will be at

$$T_b = \min_{x\in\mathbb{R}} \frac{-(x_0 - y_0)}{u_0(x_0) - u_0(y_0)} \leq \min_{x\in\mathbb{R}} \frac{-1}{u_0'(x)} = \frac{-1}{\min_{x\in\mathbb{R}} u_0'(x)}, \tag{4}$$

where the first inequality is clear, because the function u_0 is smooth and it is actually an equality. To prove it one can take two sequences of points that converge to the minimum point value of the derivative.

b) Generalize the result from a) with an arbitrary smooth convex flux f ($f''(x) > 0$ for all x).

Solution

Claim. *A discontinuity will happen at time*

$$T_b = \frac{-1}{\min_{x_0} f''(u_0(x))u'(x_0)}.$$

Proof of claim. We know that the characteristic curves are given by

$$x_{x_0}(t) = x_0 + f'(u_0(x_0))t, \tag{5}$$

and in this case the solution is given as

$$u(x_{x_0}(t), t) = u_0(x_0).$$

We will find a point $(x_{x_0}(T_b), T_b)$ such such that

$$\lim_{\tau \to t_B} |u_x(x_{x_0}(T_b), T_b)| = \infty,$$

which indeed will imply the discontinuity. By the chain rule, we have

$$u_x(x,t) = u_0'(x) \frac{\mathrm{d}x_0}{\mathrm{d}x},$$

and by differentiating (5) and some algebraic manipulation, we have

$$\frac{\mathrm{d}x_0}{\mathrm{d}x} = \frac{1}{1 + f''(u_0(x_0)) t u_0'(x_0)},$$

so

$$u_x(x,t) = \frac{u_0'(x)}{1 + f''(u_0(x_0)) t u_0'(x_0)},$$

and clearly, as $t \to T_b = \frac{-1}{\min_{x_0} f''(u_0(x)) u_0'(x_0)}$ then $|u_x(x,t)| \to \infty$ (Exercise for reader: Where did we use convexity of f and negativity of u_0'?). Conversely, every point where $|u_x(x,t)| \to \infty$ must be such that $1 + f''(u_0(x_0)) t u'(x_0) \to 0$ (since u_0' is bounded pointwise), so every discontinuity point must be of the form

$$t = \frac{-1}{f''(u_0(x)) u_0'(x_0)},$$

and T_b is the minimum of all those points. $\square$

Problem 2.2 Riemann problem (7pts)

Consider the Riemann problem for the scalar conservation law with discontinuous initial data:

$$u_t + f(u)_x = 0 \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{6}$$

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0; \\ u_r & \text{if } x > 0 \end{cases} \tag{7}$$

with constants u_l and u_r.

a) Assume $u_l > u_r$. Using the *Rankine-Hugoniot Conditions*, derive the expression for *shock* solution of (6) with initial conditions (7). Prove that it is a weak solution.

Solution

The shock solution is given as

$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x \geq st \end{cases},$$

where s is given by the Rankine-Hugoniot condition

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

We will show that u is indeed a weak solution. Let $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+, \mathbb{R})$ be a test function, then

$$\int_{\mathbb{R}}\int^{\infty} u\phi_t + f(u)\phi_x \,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} u_0(x)\phi(x,0)\,\mathrm{d}x = \underbrace{\int_{x<st} u\phi_t + f(u)\phi_x \,\mathrm{d}t\,\mathrm{d}x}_{=:A} \tag{8}$$

$$+ \overbrace{\int_{x\geq st} u\phi_t + f(u)\phi_x \,\mathrm{d}t\,\mathrm{d}x}^{=:B} + \int_{\mathbb{R}} u_0(x)\phi(x,0)\,\mathrm{d}x. \tag{9}$$

Since u is a smooth solution on $\{(x,t) \mid x < st\}$, we know that

$$u_t + f(u)_x = 0$$

(actually, this follows trivially since u is constant here). We have, by adding 0 (ie. $u_t + f(u)_x$):

$$u\phi_x + f(u)\phi_t = u\phi_t + f(u)\phi_x + (\overbrace{u_t + f(u)_x}^{=0})\phi = (u\phi_t + u_t\phi) + (f(u)\phi_x + f(u)_x\phi) = (u\phi)_t + (f(u)\phi)_x.$$

Choose an open set Ω such that $\operatorname{supp}\phi \subset \Omega$. Let Γ_- denote the boundary of the set $\{(x,t) \mid x < st\} \cap \Omega$, see Figure 1. Using the Green's formula

$$A = \int_{x<st} (u\phi)_t + (f(u)\phi)_x \,\mathrm{d}t\,\mathrm{d}x = \int_{\Omega\cap\{(x,t)|x<st\}} (u\phi)_t + (f(u)\phi)_x = \int_{\Gamma_-} -u\phi\,\mathrm{d}x + \int_{\Gamma_-} f(u)\phi\,\mathrm{d}t$$

and likewise we get that

$$B = \int_{\Gamma_+} -u\phi\,\mathrm{d}x + \int_{\Gamma_+} f(u)\phi\,\mathrm{d}t$$

where Γ_+ is the boundary of $\{(x,t) \mid x \geq st\}\cap\Omega$. We divide Γ_- into three sets $D_1 = \Gamma_- \cap \{(st,t) \mid t \in \mathbb{R}^+\}$ (the straight line $x = st$ that overlaps with Γ_-), $D_2 = \Gamma_- \cap (\mathbb{R}\times\{0\})$ (the x-axis that overlaps with Γ_-), and $D_c = \Gamma_- \setminus (D_1 \cup D_2)$. By choice of Ω, we know that $\operatorname{supp}\phi$ is completely contained in Ω, so ϕ vanishes on Γ_- outside of the x-axis, so ϕ vanishes on D_c and we are left with the integral over D_1 and D_2. For D_1 we choose the parametrization $x = st$ and get (notice that on Ω_- we have $u = u_l$)

$$\int_{D_1} -u\phi\,\mathrm{d}x + \int_{D_1} f(u)\phi\,\mathrm{d}t = \int_0^\infty -u_l\phi(st,t)s\,\mathrm{d}t + \int_0^\infty f(u_l)\phi(st,t)\,\mathrm{d}t$$

and for D_2 we insert the parametrization $(t, 0)$ (so the x-derivative of the parametrization vanishes):

$$\int_{D_2} -u\phi\,\mathrm{d}x + \int_{D_2} f(u)\phi\,\mathrm{d}t = \int_{-\infty}^0 -u_l\phi(x,0)\,\mathrm{d}x + \int_0^0 f(u_l)\phi(x,0)0\,\mathrm{d}t$$

so

$$\int_{\Gamma_-} -u\phi\,\mathrm{d}x + \int_{\Gamma_-} f(u)\phi\,\mathrm{d}t = -\int_{\mathbb{R}^-} \phi(x,0)u_l\,\mathrm{d}x - \int_{\mathbb{R}^+} u_l\phi(st,t)s\,\mathrm{d}t + \int_{\mathbb{R}^+} f(u_l)\phi(st,t)\,\mathrm{d}t$$

and similarly we get (remember the orientation of Γ_+)

$$\int_{\Gamma_+} -u\phi\,\mathrm{d}x + \int_{\Gamma_+} f(u)\phi\,\mathrm{d}t = -\int_{\mathbb{R}^+} \phi(x,0)u_r\,\mathrm{d}x + \int_{\mathbb{R}^+} u_r\phi(st,t)s\,\mathrm{d}t - \int_{\mathbb{R}^+} f(u_r)\phi(st,t)\,\mathrm{d}t.$$

Now, one gets

$$A + B = -\int_{\mathbb{R}} u_0(x)\phi(x,0)\,\mathrm{d}x + \int_{\mathbb{R}^+} s(u_r - u_l)\phi(st,t)\,\mathrm{d}t + \int_{\mathbb{R}^+} (f(u_l) - f(u_r))\phi(st,t)\,\mathrm{d}t$$

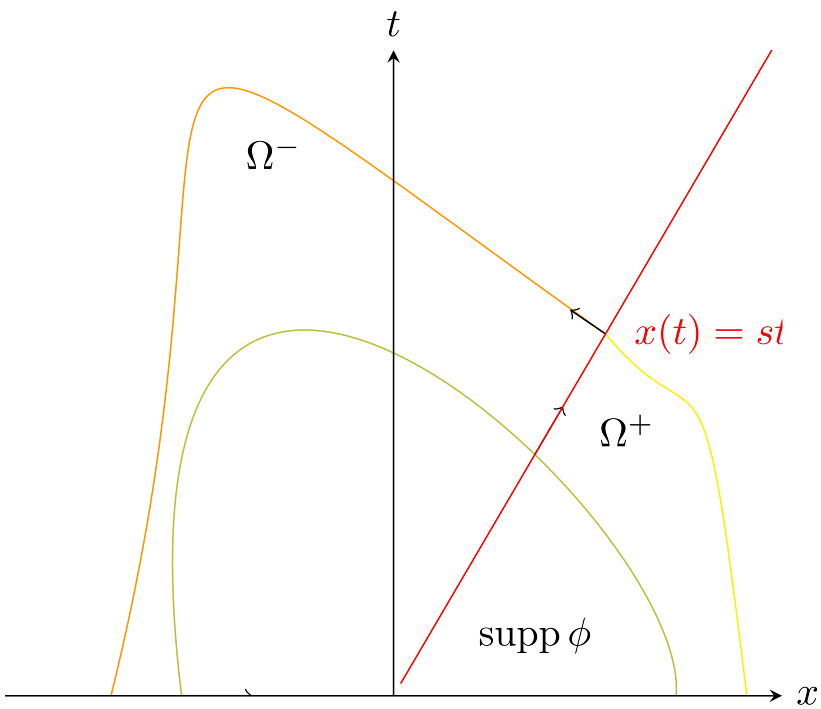

Figure 1: The domain for the integration for weak convergence. The arrows marks the orientation for the domain Ω^-.

as s is chosen such that

$$s(u_l - u_r) = \frac{f(u_l) - f(u_r)}{u_l - u_r},$$

we get

$$\int_{\mathbb{R}} \int^{\infty} u\phi_x + f(u)\phi_t \,\mathrm{d}t \,\mathrm{d}x + \int_{\mathbb{R}} u_0(x)\phi(x, 0) \,\mathrm{d}x = 0.$$

b) If f is a convex function, show that if $u_l < u_r$ then there exist more than one weak solution.

Solution

For parameters s_1, s_2 and α we construct the solution

$$u(x,t) = \begin{cases} u_l & x < s_1 t \\ \alpha & x \in [s_1 t, s_2 t] \\ u_r & \text{otherwise} \end{cases}.$$

We require that s_1 and s_2 obey the Rankine-Hugonoit conditions

$$s_1 = \frac{f(u_l) - f(\alpha)}{u_l - \alpha} \qquad s_2 = \frac{f(\alpha) - f(u_r)}{\alpha - u_r}$$

for each $\alpha \in (u_l, u_r)$ we see that we get a choice for s_1 and s_2, hence infinitely many solutions. One can do a similar computation as in the last exercise to see that these indeed will be weak solutions.

c) Consider (6) with the flux $f(u) = u(1-u)/2$, and with the following initial conditions:

$$u_0(x) \quad = \quad \begin{cases} 0, & \text{if } x < 0 \\ 2, & \text{if } x > 0, \end{cases} \tag{10}$$

$$u_0(x) \quad = \quad \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } 0 < x < 1; \\ 1, & \text{if } x > 1. \end{cases} \tag{11}$$

Use the *Rankine-Hugoniot Conditions* to calculate the *shock* solutions for both of the above initial conditions.

Solution

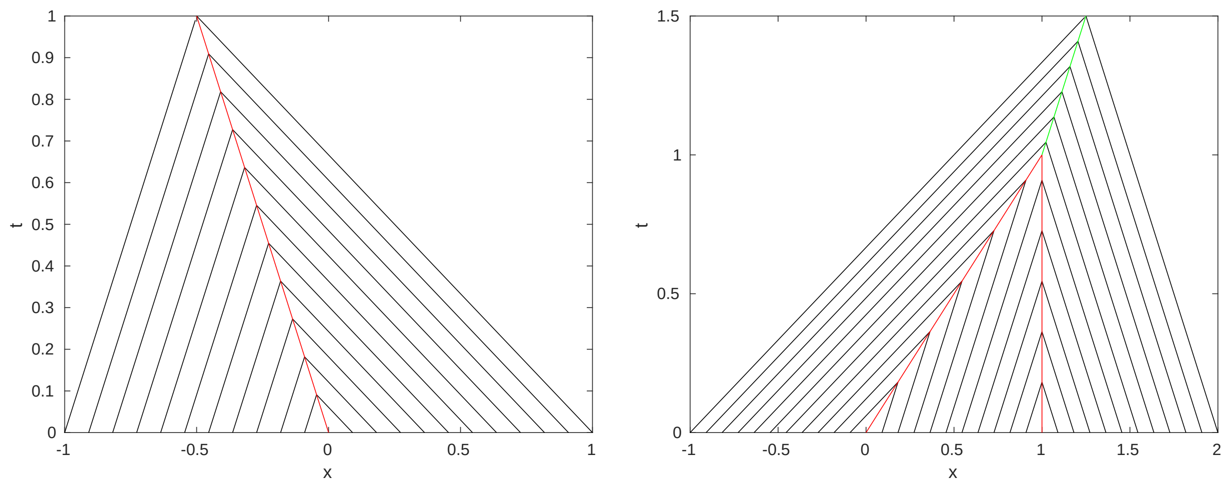

Figure 2: Riemann problems for (8) (left) and (9) (right)

For (8) we get a shock solution

$$u(x,t) = \begin{cases} 0 & x < st \\ 2 & x \geq st \end{cases}$$

where

$$s = \frac{f(2) - f(0)}{2} = -1/2.$$

For (9) we have two shocks with shock speeds calculated by the Rankine-Hugonoit condition. The shock speeds are $s_1 = 1$ and $s_2 = 0$, and the shock solution is

$$u(x,t) = \begin{cases} -1 & x < t \\ 0 & t < x < 1 \\ 1 & x > 1 \end{cases}.$$

when $t = 1$ we get a new Riemann problem with initial condition

$$u(x,1) = \begin{cases} -1 & x < 1 \\ 1 & x \geq 1 \end{cases}.$$

which again has the solution

$$u(x,1) = \begin{cases} -1 & x < 1 + 1/2t \\ 1 & x \geq 1 + 1/2t \end{cases}.$$

See pictures 2.

d) Consider (6) with the flux $f(u) = u^2/2$ (Burgers' equation), and following initial conditions:

$$u_0(x) \quad = \quad \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Derive an expression of the entropy satisfying solution.

Solution

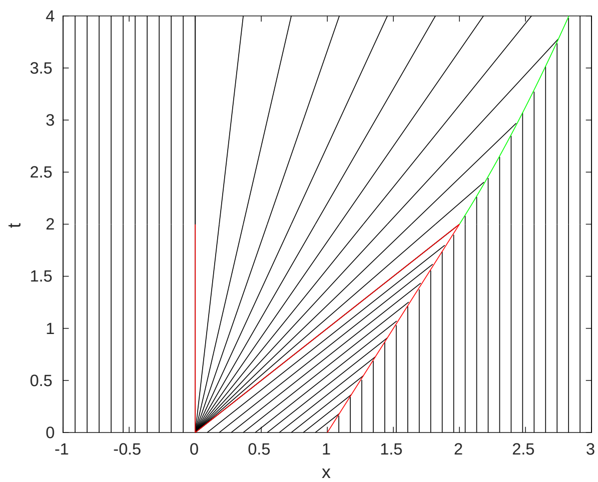

Figure 3: Solution for (10)

At first we have a shock and a rarefaction wave, giving the solution

$$u(x,t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & t < x < 1 + t/2 \\ 0 & \text{otherwise.} \end{cases}$$

at $t = 2$ the two solutions meet and we are left with a Rarefaction wave and a shock. The solution will be of the form

$$u(x,t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < \sigma(t) \\ 0 & \text{otherwise} \end{cases}$$

Here σ will obey the Rankine-Hugonoit conditions

$$\sigma'(t) = 1/2(\sigma(t)/t - 0) = \frac{1}{2}\sigma(t)/t$$

and initial condition $\sigma(2) = 2$. This ODE has the solution

$$\sigma(t) = (2t)^{1/2},$$

so the solution after $t = 2$ will be

$$u(x,t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < (2t)^{1/2} \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2.3 Entropy conditions and Lax entropy (5pts - optional)

Consider the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) &= u_0(x) && \text{for } x \in \mathbb{R}. \end{aligned} \tag{13}$$

a) Let $\Gamma = \{(\sigma(t), t) \mid t > 0\}$ be a smooth curve in $\mathbb{R} \times \mathbb{R}^+$ with $s(t) = \sigma'(t)$, and let u be the weak solution of (11), that is piecewise-smooth with discontinuities at Γ. Also assume $f, \eta, q \in C^2(\mathbb{R})$ and f and η are strictly convex and F is an entropy flux satisfying $q' = \eta' f'$. Prove that if u satisfies the entropy inequality,

$$\eta(u)_t + q(u)_x \leq 0$$

then, across the curve Γ, u satisfies

$$f'(u_l) > s(t) > f'(u_r).$$

Here $u_l \neq u_r$ are the traces of u from respectively the left and right sides of the curve Γ.

Solution

Let $\phi \in C_c^\infty(\mathbb{R} \times (0,\infty), \mathbb{R}^+)$ be a test function, and let (η, q) be a smooth entropy pair. Let V be an open set containing the support of ϕ, and divide V into V_l and V_r, being to the left and right of Γ respectively. The entropy inequality states that

$$\int_0^\infty \int_{\mathbb{R}} \eta(u)\phi_t + q(u)\phi_x \, \mathrm{d}x \, \mathrm{d}t \geq 0.$$

We may divide the integral into each of the parts V_l and V_r. On V_l the solution is smooth, so we produce

$$\int_{V_l} \eta(u)\phi_t + q(u)\phi_x \, \mathrm{d}V = \int_{V_l} \eta(u)\phi_t + q(u)\phi_x + \overbrace{(\eta'(u)u_t + \underbrace{\eta'(u)f'(u)}_{=q'(u)} u_x)}^{=0} \phi \, \mathrm{d}V = \int_{V_l} (\eta(u)\phi)_t + (q(u)\phi)_x \, \mathrm{d}V,$$

which, by using the Green's formula and noting that ϕ vanishes on $\partial V_l \cap \Gamma^c$, this turns into

$$\int_{V_l} (\eta(u)\phi)_t + (q(u)\phi)_x \, \mathrm{d}V = \int_\Gamma -\eta(u_l)\phi \, \mathrm{d}x + \int_\Gamma (q(u_l)\phi \, \mathrm{d}t$$

and by a similar computation for V_r we get (note the change of sign! This is because of the orientation)

$$\int_{V_r} \eta(u)\phi_t + q(u)\phi_x \, \mathrm{d}V = \int_\Gamma \eta(u_r)\phi \, \mathrm{d}x - \int_\Gamma (q(u_r)\phi \, \mathrm{d}t.$$

Inserting the parametrization $r(t) = (\sigma(t), t)$ for Γ, we get

$$0 \leq \int_0^\infty \int_{\mathbb{R}} \eta(u)\phi_t + q(u)\phi_x \, \mathrm{d}x \, \mathrm{d}t = \int_0^\infty \phi(\sigma(t), t) \left((\eta(u_r) - \eta(u_l)) \, s(t) + (q(u_l) - q(u_r)) \right) \mathrm{d}t.$$

If the above holds true for any test function, then

$$((\eta(u_r) - \eta(u_l))\, s(t) + (q(u_l) - q(u_r)) \geq 0.$$

Set

$$E(u) = (\eta(u_r) - \eta(u))\, \tilde{s}(u) + (q(u) - q(u_r))\,.$$

where

$$\tilde{s}(u) = \frac{f(u_r) - f(u)}{u_r - u}.$$

With a small computation, we find that

$$\begin{aligned}
E'(u) &= (\eta(u_r) - \eta(u))\, \tilde{s}'(u) - \eta'(u)\tilde{s}(t) + q'(u) \\
&= (\eta(u_r) - \eta(u)) \left(\frac{(f(u_r) - f(u)) - f'(u)(u_r - u)}{(u_r - u)^2} \right) - \eta'(u) \frac{f(u_r) - f(u)}{u_r - u} + q'(u) \\
&= (\eta(u_r) - \eta(u)) \left(\frac{(f(u_r) - f(u)) - f'(u)(u_r - u)}{(u_r - u)^2} \right) - \eta'(u) \frac{f(u_r) - f(u)}{u_r - u} + \underbrace{\eta'(u) f'(u)}_{=q'} \\
&= (\eta(u_r) - \eta(u)) \left(\frac{(f(u_r) - f(u)) - f'(u)(u_r - u)}{(u_r - u)^2} \right) - \eta'(u) \frac{f(u_r) - f(u) - f'(u)(u_r - u)}{u_r - u} \\
&= \frac{1}{(u_r - u)^2} \underbrace{(f(u_r) - f(u)) - f'(u)(u_r - u)}_{>0 \text{ if } u \neq u_r}) \left(\overbrace{\eta(u_r) - \eta(u) - \eta'(u)(u_r - u)}^{>0 \text{ if } u \neq u_r} \right)
\end{aligned}$$

where we used the convexity of f and η to see that

$$\eta(u_r) - \nu(u) - \eta'(u)(u_l - u) = \eta''(c)(u_r - u)^2 > 0 \qquad c \in (u_r, u)$$

and likewise for f. Hence $E'(u) > 0$ for $u \neq u_r$, which implies that E is strictly increasing. But $E(u_r) = 0$, and the entropy requirement states that $E(u_l) \geq 0$, so the only possibility is that $u_r < u_l$ (equality is ruled out by the discontinuity). Furthermore, we know that f' is a strictly increasing function, so using the mean value theorem, we see

$$s(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = f'(\xi) \qquad \xi \in (u_l, u_r).$$

Since f' is strictly increasing, we have

$$f'(u_l) > f'(\xi) > f'(u_r)$$

which give us

$$f'(u_l) > s(t) > f'(u_r)$$

and we are done.

b) Use this result to show that the non-unique shock solutions derived in Problem 2.2b) are not entropy solutions.

Solution

This is immediate since they do not obey the Lax entropy condition.

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