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MAT827
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Sheet 2

Deadline: 13.03.2024, 12:00 PM

Exercise 1 (Points: 6)
Show that the Lax-Wendroff scheme is the second order scheme and Von Neumann stability analysis with periodic condition for the advection equation:

$$u_t + au_x = 0, a > 0.$$

Exercise 2 (Points: 3)
Show that the Lax-Wendroff scheme is a dispersive scheme by modified equation, for example, the advection equation:

$$u_t + au_x = 0, a > 0.$$

Exercise 3 (Points: 6)
Consider the following IBVP:

$$\begin{cases} u_t + 2u_x = 0, \quad x \in [-10, 10] \\ u(x,0) = \begin{cases} -1, x < 0 \\ 1, x > 0 \end{cases} \end{cases}$$

is subjected to the following boundary condition: $u[0] = u[1]$ and $u[N+1] = u[N]$.

(a) Find the exact solution.

(b) Implement the problem with upwind and Lax-Wendroff schemes with a uniform grid of meshes $N = 100$, when $t = 2$. Plot the numerical solutions with the exact solution. What do you observe?

Exercise 1 (Points: 6)
Show that the Lax-Wendroff scheme is the second order scheme and Von Neumann stability analysis with periodic condition for the advection equation:

$$u_t + au_x = 0, a > 0.$$

The Lax-Wendroff scheme is: $(\lambda = a\Delta t/\Delta x)$

$$\tilde{\mathbf{U}}_i^{n+1} = \tilde{\mathbf{U}}_i^n - \frac{\lambda}{2}(\tilde{\mathbf{U}}_{i+1}^n - \tilde{\mathbf{U}}_{i-1}^n) + \frac{\lambda^2}{2}\left(\tilde{U}_{i+1}^n - 2\tilde{\mathbf{U}}_i^n + \tilde{\mathbf{U}}_{i-1}^n\right).$$

Order: (we write $U_i^n = u(t_n, x_i)$ shorthand notation, $\tilde{U}_i^n$ for the numerical approx)

$$U_i^{n+1} = U_i^n + \Delta t (U_i^n)_t + \frac{\Delta t^2}{2}(U_i^n)_{tt} + \frac{\Delta t^3}{6}(U_i^n)_{ttt} + \mathcal{O}(\Delta t^4)$$

$$U_{i+1}^n = U_i^n + \Delta x (U_i^n)_x + \frac{\Delta x^2}{2}(U_i^n)_{xx} + \frac{\Delta x^3}{6}(U_i^n)_{xxx} + \mathcal{O}(\Delta x^4)$$

$$U_{i-1}^n = U_i^n - \Delta x (U_i^n)_x + \frac{\Delta x^2}{2}(U_i^n)_{xx} - \frac{\Delta x^3}{6}(U_i^n)_{xxx} + \mathcal{O}(\Delta x^4)$$

For local error assume $U_i^n = \tilde{U}_i^n \ \forall i$

$$\Rightarrow \underbrace{|\tilde{U}_i^{n+1} - U_i^{n+1}|}_{\text{local error}}$$

$$= \left| U_i^n - \frac{\lambda}{2}(U_{i+1}^n - U_{i-1}^n) + \frac{\lambda^2}{2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) - U_i^{n+1} \right|$$

$$= \Big| U_i^n - \frac{a\Delta t}{2\Delta x}\left(2\Delta x (U_i^n)_x + \mathcal{O}(\Delta x^3)\right) + \frac{a^2 \Delta t^2}{2\Delta x^2}\left(\Delta x^2 (U_i^n)_{xx} + \mathcal{O}(\Delta x^4)\right)$$
$$- U_i^n - \Delta t (U_i^n)_t - \frac{\Delta t^2}{2}(U_i^n)_{tt} + \mathcal{O}(\Delta t^3) \Big|$$

$$= \Big| \underbrace{-a\Delta t (U_i^n)_x - \Delta t (U_i^n)_t}_{= -\Delta t \left((U_i^n)_t + a (U_i^n)_x\right) = 0} + \mathcal{O}(\Delta t \Delta x^2) + \underbrace{\frac{a^2\Delta t^2}{2}(U_i^n)_{xx} - \frac{\Delta t^2}{2}(U_i^n)_{tt}}_{= \frac{\Delta t^2}{2}\left(a^2 (U_i^n)_{xx} - (U_i^n)_{tt}\right) \overset{u_{tt} = a^2 u_{xx}}{=} 0} + \mathcal{O}(\Delta t^2 \Delta x^2) + \mathcal{O}(\Delta t^3) \Big|$$

$$\left(\begin{aligned} u_t &= -a u_x \\ u_{tt} &= -a u_{xt} = (-a u_t)_x = a^2 u_{xx} \end{aligned} \right)$$

$$|\lambda| = \left|\frac{a\Delta t}{\Delta x}\right| \leq 1$$

$$= \mathcal{O}(\Delta t \Delta x^2) + \mathcal{O}(\Delta t^3) \overset{!}{=} \mathcal{O}(\Delta t^3)$$

stability + consistency $\rightarrow$ global error order 2

Von Neumann stability analysis: We assume periodic boundary conditions and expand U_j^n as Fourier series:

$$u(x,t) = \underbrace{\sum_k}_{\text{we sum over } \mathbb{Z}} \hat{u}_k(t)\, e^{-ikx} \implies U_j^n = u(\underbrace{x_j}_{j\Delta x}, t_n) = \sum_k \underbrace{\hat{u}_k(t_n)}_{=:\hat{U}_k^n} e^{-ikj\Delta x}$$

$$U_{j\pm 1}^n = u(x_{j\pm 1}, t_n) = \sum_k \hat{U}_k^n\, e^{-ik(j\pm 1)\Delta x}$$

Plugging this into the Lax-Wendroff scheme we get:

$$\sum_k \hat{U}_k^{n+1} e^{-ikj\Delta x} = \sum_k \hat{U}_k^n e^{-ikj\Delta x} - \frac{\lambda}{2}\left(\sum_k \hat{U}_k^n \left(e^{-ik(j+1)\Delta x} - e^{-ik(j-1)\Delta x}\right)\right)$$

$$+ \frac{\lambda^2}{2}\left(\sum_k \hat{U}_k^n \left(e^{-ik(j+1)\Delta x} - 2e^{-ikj\Delta x} + e^{-ik(j-1)\Delta x}\right)\right)$$

$$= \sum_k e^{-ikj\Delta x}\left(\hat{U}_k^n - \frac{\lambda}{2}\hat{U}_k^n\left(e^{-ik\Delta x} - e^{ik\Delta x}\right) + \frac{\lambda^2}{2}\hat{U}_k^n\left(e^{-ik\Delta x} - 2 + e^{ik\Delta x}\right)\right)$$

by projecting onto Fourier basis functions we get:

$$\hat{U}_k^{n+1} \stackrel{!}{=} \hat{U}_k^n \underbrace{\left(1 - \frac{\lambda}{2}\left(e^{-ik\Delta x} - e^{ik\Delta x}\right) + \frac{\lambda^2}{2}\left(e^{-ik\Delta x} + e^{ik\Delta x} - 2\right)\right)}_{=:A} \quad \forall k \in \mathbb{Z}$$

For stability we need $|A| \stackrel{!}{\leq} 1$. Using $\sin(k\Delta x) = \frac{1}{2i}\left(e^{-ik\Delta x} - e^{ik\Delta x}\right)$
and $\cos(k\Delta x) = \frac{1}{2}\left(e^{-ik\Delta x} + e^{ik\Delta x}\right)$ we get:

$$A = 1 - i\lambda \sin(\overbrace{k\Delta x}^{=:\omega}) + \lambda^2\left(\cos(k\Delta x) - 1\right) = 1 - 2i\lambda \sin\left(\tfrac{\omega}{2}\right)\cos\left(\tfrac{\omega}{2}\right) - 2\lambda^2 \sin^2\left(\tfrac{\omega}{2}\right)$$

$$\sin\left(2\tfrac{\omega}{2}\right) = 2\sin\left(\tfrac{\omega}{2}\right)\cos\left(\tfrac{\omega}{2}\right)$$

$$\cos\left(2\tfrac{\omega}{2}\right) = 1 - 2\sin^2\left(\tfrac{\omega}{2}\right)$$

$$\Longrightarrow |A|^2 = \left(1 - 2\lambda^2 \sin^2\left(\frac{\omega}{2}\right)\right)^2 + 4\lambda^2 \sin^2\left(\frac{\omega}{2}\right)\cos^2\left(\frac{\omega}{2}\right)$$

$$= 1 + 4\lambda^4 \sin^4\left(\frac{\omega}{2}\right) - 4\lambda^2 \sin^2\left(\frac{\omega}{2}\right) + 4\lambda^2 \sin^2\left(\frac{\omega}{2}\right)\cos^2\left(\frac{\omega}{2}\right) \overset{!}{\leq} 1$$

$$\Longleftrightarrow 0 \geq 4\lambda^2 \sin^2\left(\frac{\omega}{2}\right)\left(\lambda^2 \sin^2\left(\frac{\omega}{2}\right) \underbrace{-1 + \cos^2\left(\frac{\omega}{2}\right)}_{= -\sin^2\left(\frac{\omega}{2}\right)}\right)$$

$$= -4\lambda^2 \sin^2\left(\frac{\omega}{2}\right)\left(\lambda^2 - 1\right) \qquad \forall\, \omega = k\Delta x,\ k \in \mathbb{Z}$$

$$\Longrightarrow (\lambda^2 - 1) \leq 0 \Longrightarrow |\lambda| \leq 1$$

Also $|\lambda| \leq 1$ is sufficient as $|\lambda| \leq 1 \Rightarrow |A| \leq 1$

Exercise 2 (Points: 3)
Show that the Lax-Wendroff scheme is a dispersive scheme by modified equation, for example, the advection equation:

$$u_t + au_x = 0, a > 0.$$

We modify the order calculation from a) to include higher order terms:

$$|\tilde{U}_i^{n+1} - U_i^{n+1}| \text{ — local error}$$

$$= \left| U_i^n - \frac{\lambda}{2}(U_{i+1}^n - U_{i-1}^n) + \frac{\lambda^2}{2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) - U_i^{n+1} \right|$$

$$= \Big| U_i^n - \frac{a\Delta t}{2\Delta x}\left(2\Delta x (U_i^n)_x + \frac{\Delta x^3}{3}(U_i^n)_{xxx} + \mathcal{O}(\Delta x^5)\right)$$

$$+ \frac{a^2 \Delta t^2}{2\Delta x^2}\left(\Delta x^2 (U_i^n)_{xx} + \mathcal{O}(\Delta x^4)\right) - U_i^n - \Delta t (U_i^n)_t - \frac{\Delta t^2}{2}(U_i^n)_{tt} - \frac{\Delta t^3}{6}(U_i^n)_{ttt} + \mathcal{O}(\Delta t^3)\Big|$$

$$= \Big| \underbrace{-a\Delta t (U_i^n)_x - \Delta t (U_i^n)_t}_{= -\Delta t\left((U_i^n)_t + a(U_i^n)_x\right) = 0} - \frac{a}{6}\Delta t \Delta x^2 (U_i^n)_{xxx} + \underbrace{\frac{a^2\Delta t^2}{2}(U_i^n)_{xx} - \frac{\Delta t^2}{2}(U_i^n)_{tt}}_{= \frac{\Delta t^2}{2}\left(a^2 (U_i^n)_{xx} - (U_i^n)_{tt}\right) \overset{u_{tt} = a^2 u_{xx}}{=} 0} + \mathcal{O}(\Delta t^2 \Delta x^2) - \frac{\Delta t^3}{6}(U_i^n)_{ttt} + \mathcal{O}(\Delta t^4) \Big|$$

$$= \Delta t \Delta x^2 \frac{a}{6}(U_i^n)_{xxx} + \mathcal{O}(\Delta t^2 \Delta x^2) + \frac{\Delta t^3}{6}(U_i^n)_{ttt} + \mathcal{O}(\Delta t^4)$$

$$= \Delta t^3 \frac{a^3}{6}\left(\frac{1}{\lambda^2} - 1\right)(U_i^n)_{xxx} + \mathcal{O}(\Delta t^4)$$

$$\Delta x = \frac{a\Delta t}{\lambda}$$

$$\begin{pmatrix} u_t = -a u_x \\ u_{tt} = -a u_{xt} = (-a u_t)_x = a^2 u_{xx} \\ u_{ttt} = (a^2 u_t)_{xx} = -a^3 u_{xxx} \end{pmatrix}$$

We see that the leading error term depends on $(U_i^n)_{xxx}$. This means the scheme is dispersive as discussed in the lecture

Exercise 3 (Points: 6)
Consider the following IBVP:

$$\begin{cases} u_t + 2u_x = 0, \quad x \in [-10, 10] \\ \underset{u_0(x)}{u(x,0)} = \begin{cases} -1, x < 0 \\ 1, x > 0 \end{cases} \end{cases}$$

is subjected to the following boundary condition: $u[0] = u[1]$ and $u[N+1] = u[N]$.

(a) Find the exact solution.

(b) Implement the problem with upwind and Lax-Wendroff schemes with a uniform grid of meshes $N = 100$, when $t = 2$. Plot the numerical solutions with the exact solution. What do you observe?

a) As discussed in the lecture, the exact solution of such a linear advection equation is

$$u(x,t) = u_0(x-2t) = \begin{cases} -1 & , \; x < 2t \\ 1 & , \; x \geq 2t \end{cases}$$

The Lax-Wendroff scheme is: ($\lambda = a\Delta t/\Delta x$)

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\lambda}{2}(\mathbf{U}_{i+1}^n - \mathbf{U}_{i-1}^n) + \frac{\lambda^2}{2}\left(U_{i+1}^n - 2\mathbf{U}_i^n + \mathbf{U}_{i-1}^n\right).$$

$$\begin{pmatrix} U_1^{n+1} \\ \vdots \\ U_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1-\lambda^2 & \frac{\lambda}{2}(\lambda-1) & & 0 \\ \frac{\lambda}{2}(\lambda+1) & \ddots & \ddots & \\ & \ddots & \ddots & \frac{\lambda}{2}(\lambda-1) \\ 0 & & \frac{\lambda}{2}(\lambda+1) & 1-\lambda^2 \end{pmatrix} \begin{pmatrix} U_1^n \\ \vdots \\ U_N^n \end{pmatrix} + \begin{pmatrix} \frac{\lambda}{2}(\lambda+1)\, U_0^n \\ 0 \\ \vdots \\ 0 \\ \frac{\lambda}{2}(\lambda-1)\, U_{N+1}^n \end{pmatrix}$$

$U_0^n = U_1^n$
$U_{N+1}^n = U_N^n$

$$= \begin{pmatrix} 1-\frac{\lambda^2}{2}+\frac{\lambda}{2} & \frac{\lambda}{2}(\lambda-1) & & 0 \\ \frac{\lambda}{2}(\lambda+1) & 1-\lambda^2 & \ddots & \\ & \ddots & 1-\lambda^2 & \frac{\lambda}{2}(\lambda-1) \\ 0 & & \frac{\lambda}{2}(\lambda+1) & 1-\frac{\lambda^2}{2}-\frac{\lambda}{2} \end{pmatrix} \begin{pmatrix} U_1^n \\ \vdots \\ U_N^n \end{pmatrix}$$

upwind - scheme:

$$U^{n+1} := \begin{pmatrix} u_1^{n+1} \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1-\lambda & & & 0 \\ \lambda & \ddots & & \\ & \ddots & \ddots & \\ 0 & & \lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix} + \begin{pmatrix} \lambda u_0^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$u_0^n = u_1^n$

$$= \begin{pmatrix} 1 & & & 0 \\ \lambda & 1-\lambda & & \\ & \ddots & \ddots & \\ 0 & & \lambda & 1-\lambda \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ u_N^n \end{pmatrix}$$