

Exercise 1 (Points: 6)

Show that the Lax-Wendroff scheme is the second order scheme and Von Neumann stability analysis with periodic condition for the advection equation:

$$u_t + au_x = 0, a > 0.$$

Exercise 2 (Points: 3)

Show that the Lax-Wendroff scheme is a dispersive scheme by modified equation, for example, the advection equation:

$$u_t + au_x = 0, a > 0.$$

Exercise 3 (Points: 6)

Consider the following IBVP:

$$\begin{cases} u_t + 2u_x = 0, & x \in [-10, 10] \\ u(x, 0) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases} \end{cases}$$

is subjected to the following boundary condition: $u[0] = u[1]$ and $u[N+1] = u[N]$.

- (a) Find the exact solution.
- (b) Implement the problem with upwind and Lax-Wendroff schemes with a uniform grid of meshes $N = 100$, when $t = 2$. Plot the numerical solutions with the exact solution. What do you observe?

Exercise 1 (Points: 6)

Show that the Lax-Wendroff scheme is the second order scheme and Von Neumann stability analysis with periodic condition for the advection equation:

$$u_t + au_x = 0, a > 0.$$

The Lax-Wendroff scheme is: ($\lambda = a\Delta t/\Delta x$)

$$\tilde{U}_i^{n+1} = \tilde{U}_i^n - \frac{\lambda}{2}(\tilde{U}_{i+1}^n - \tilde{U}_{i-1}^n) + \frac{\lambda^2}{2} \left(\tilde{U}_{i+1}^n - 2\tilde{U}_i^n + \tilde{U}_{i-1}^n \right).$$

Order : (we write $U_i^n = u(t_n, x_i)$ shorthand notation, \tilde{U}_i^n for the numerical approx)

$$U_i^{n+1} = U_i^n + \Delta t(U_i^n)_t + \frac{\Delta t^2}{2}(U_i^n)_{tt} + \mathcal{O}(\Delta t^3)$$

$$U_{i,m}^n = U_i^n + \Delta x(U_i^n)_x + \frac{\Delta x^2}{2}(U_i^n)_{xx} + \frac{\Delta x^3}{6}(U_i^n)_{xxx} + \mathcal{O}(\Delta x^4)$$

$$U_{i-1}^n = U_i^n - \Delta x(U_i^n)_x + \frac{\Delta x^2}{2}(U_i^n)_{xx} - \frac{\Delta x^3}{6}(U_i^n)_{xxx} + \mathcal{O}(\Delta x^4)$$

For local error assume $U_i^n = \tilde{U}_i^n \forall i$

$$\Rightarrow |\tilde{U}_i^{n+1} - U_i^{n+1}| \quad \begin{matrix} \text{local} \\ \text{error} \end{matrix}$$

$$= |U_i^n - \frac{\lambda}{2}(U_{i,m}^n - U_{i-1}^n) + \frac{\lambda^2}{2}(U_{i,m}^n - 2U_i^n + U_{i-1}^n) - U_i^{n+1}|$$

$$= |U_i^n - \frac{a\Delta t}{2\Delta x} (2\Delta x(U_i^n)_x + \mathcal{O}(\Delta x^3)) + \frac{a^2 \Delta t^2}{2\Delta x^2} (\Delta x^2(U_i^n)_{xx} + \mathcal{O}(\Delta x^4))|$$

$$- |U_i^n - \Delta t(U_i^n)_t - \frac{\Delta t^2}{2}(U_i^n)_{tt} + \mathcal{O}(\Delta t^3)|$$

$$\left| \underbrace{-a\Delta t(U_i^n)_x - \Delta t(U_i^n)_t + \mathcal{O}(\Delta t\Delta x^2)}_{= -\Delta t((U_i^n)_t + a(U_i^n)_x)} + \underbrace{\frac{a^2 \Delta t^2}{2}(U_i^n)_{xx} - \frac{\Delta t^2}{2}(U_i^n)_{tt} + \mathcal{O}(\Delta t^2\Delta x^2)}_{\substack{= \frac{\Delta t^2}{2}(a^2(U_i^n)_{xx} - (U_i^n)_{tt}) \\ = 0}} \right|$$

$$= \mathcal{O}(\Delta t\Delta x^2)$$

$$\left(\begin{array}{l} u_t = -au_x \\ u_{tt} = -au_{xt} = (-au_t)_x = a^2u_{xx} \end{array} \right)$$

Von Neumann stability analysis: We assume periodic boundary conditions
 we sum over \mathbb{Z}
 and expand U_j^n as Fourier series:

$$u(x,t) = \sum_k \hat{u}_k(t) e^{-ikx} \Rightarrow U_j^n = u(x_j, t_n) = \sum_k \underbrace{\hat{u}_k(t_n)}_{=: \hat{U}_k^n} e^{-ikj\alpha x}$$

$$U_{j\pm 1}^n = u(x_{j\pm 1}, t_n) = \sum_k \hat{U}_k^n e^{-ik(j\pm 1)\alpha x}$$

Plugging this into the Lax-Wendroff scheme we get:

$$\begin{aligned} \sum_k \hat{U}_k^{n+1} e^{-ikj\alpha x} &= \sum_k \hat{U}_k^n e^{-ikj\alpha x} - \frac{\lambda}{2} \left(\sum_k \hat{U}_k^n (e^{-ik(j+1)\alpha x} - e^{-ik(j-1)\alpha x}) \right) \\ &\quad + \frac{\lambda^2}{2} \left(\sum_k \hat{U}_k^n (e^{-ik(j+1)\alpha x} - 2e^{-ikj\alpha x} + e^{-ik(j-1)\alpha x}) \right) \\ &= \sum_k e^{-ikj\alpha x} \left(\hat{U}_k^n - \frac{\lambda}{2} \hat{U}_k^n (e^{-ik\alpha x} - e^{ik\alpha x}) + \frac{\lambda^2}{2} \hat{U}_k^n (e^{-ik\alpha x} - 2 + e^{ik\alpha x}) \right) \end{aligned}$$

by projecting onto Fourier basis functions we get:

$$\hat{U}_k^{n+1} = \underbrace{\hat{U}_k^n \left(1 - \frac{\lambda}{2} (e^{-ik\alpha x} - e^{ik\alpha x}) + \frac{\lambda^2}{2} (e^{-ik\alpha x} + e^{ik\alpha x} - 2) \right)}_{=: A} \quad \forall k \in \mathbb{Z}$$

For stability we need $|A| \leq 1$. Using $\sin(k\alpha x) = \frac{1}{2i} (e^{-ik\alpha x} - e^{ik\alpha x})$
 and $\cos(k\alpha x) = \frac{1}{2} (e^{-ik\alpha x} + e^{ik\alpha x})$ we get:

$$\begin{aligned} A &= 1 - i\lambda \sin(k\alpha x) + \lambda^2 (\cos(k\alpha x) - 1) = 1 - 2i\lambda \sin\left(\frac{k\omega}{2}\right) \cos\left(\frac{k\omega}{2}\right) - 2\lambda^2 \sin^2\left(\frac{k\omega}{2}\right) \\ &\stackrel{:= \omega}{=} \\ \sin\left(2\frac{\omega}{2}\right) &= 2\sin\left(\frac{\omega}{2}\right) \cos\left(\frac{\omega}{2}\right) \\ \cos\left(2\frac{\omega}{2}\right) &= 1 - 2\sin^2\left(\frac{\omega}{2}\right) \end{aligned}$$

$$\Rightarrow |A|^2 = \left(\lambda - 2\lambda^2 \sin^2\left(\frac{\omega}{2}\right)\right)^2 + 4\lambda^2 \sin^2\left(\frac{\omega}{2}\right) \cos^2\left(\frac{\omega}{2}\right)$$

$$= \lambda + 4\lambda^4 \sin^4\left(\frac{\omega}{2}\right) - 4\lambda^2 \sin^2\left(\frac{\omega}{2}\right) + 4\lambda^2 \sin^2\left(\frac{\omega}{2}\right) \cos^2\left(\frac{\omega}{2}\right) \stackrel{!}{\leq} 1$$

$$\Leftrightarrow 0 \geq 4\lambda^2 \sin^2\left(\frac{\omega}{2}\right) \left(\lambda^2 \sin^2\left(\frac{\omega}{2}\right) \underbrace{- \lambda + \cos^2\left(\frac{\omega}{2}\right)}_{= -\sin^2\left(\frac{\omega}{2}\right)} \right)$$

$$= -4\lambda^2 \sin^2\left(\frac{\omega}{2}\right) (\lambda^2 - 1) \quad \forall \omega = k\alpha x, k \in \mathbb{Z}$$

$$\Rightarrow (\lambda^2 - 1) \leq 0 \Rightarrow |\lambda| \leq 1$$

Also $|\lambda| \leq 1$ is sufficient as $|\lambda| \leq 1 \Rightarrow |A| \leq 1$

Exercise 2 (Points: 3)

Show that the Lax-Wendroff scheme is a dispersive scheme by modified equation, for example, the advection equation:

$$u_t + au_x = 0, a > 0.$$

We modify the order calculation from a) to include higher order terms:

$$\begin{aligned}
 & |\tilde{U}_i^{n+1} - U_i^{n+1}| - \text{local error} \\
 &= |U_i^n - \frac{\lambda}{2}(U_{i+1}^n - U_{i-1}^n) + \frac{\lambda^2}{2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) - U_i^{n+1}| \\
 &= \left| U_i^n - \frac{a\Delta t}{2\Delta x} (2\Delta x(U_i^n)_x + \frac{\Delta x^3}{3}(U_i^n)_{xxx} + O(\Delta x^5)) \right. \\
 &\quad \left. + \frac{a^2 \Delta t^2}{2\Delta x^2} (\Delta x^2(U_i^n)_{xx} + O(\Delta x^4)) - U_i^n - \Delta t(U_i^n)_t - \frac{\Delta t^2}{2}(U_i^n)_{tt} + O(\Delta t^3) \right| \\
 &= \left| \underbrace{-a\Delta t(U_i^n)_x - \Delta t(U_i^n)_t - \frac{a}{6}\Delta t\Delta x^2(U_i^n)_{xxx}}_{=0} + \underbrace{\frac{a^2 \Delta t^2}{2}(U_i^n)_{xx} - \frac{\Delta t^2}{2}(U_i^n)_{tt} + O(\Delta t^2\Delta x^2)}_{\substack{= \frac{\Delta t^2}{2} (a^2(U_i^n)_{xx} - (U_i^n)_{tt}) \\ = 0}} \right| \\
 &= \Delta t\Delta x^2 \frac{a}{6}(U_i^n)_{xxx} + O(\Delta t^2\Delta x^2)
 \end{aligned}$$

We see that the leading error term depends on $(U_i^n)_{xxx}$. This means the scheme is dispersive as discussed in the lecture.

Exercise 3 (Points: 6)

Consider the following IVP:

$$\begin{cases} u_t + 2u_x = 0, & x \in [-10, 10] \\ u(x, 0) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases} \end{cases}$$

is subjected to the following boundary condition: $u[0] = u[1]$ and $u[N+1] = u[N]$.

- (a) Find the exact solution.
- (b) Implement the problem with upwind and Lax-Wendroff schemes with a uniform grid of meshes $N = 100$, when $t = 2$. Plot the numerical solutions with the exact solution. What do you observe?

a) As discussed in the lecture, the exact solution of such a linear advection equation is

$$u(x, t) = u_0(x - 2t) = \begin{cases} -1, & x < 2t \\ 1, & x \geq 2t \end{cases}$$

The Lax-Wendroff scheme is: ($\lambda = a\Delta t / \Delta x$)

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\lambda}{2} (\mathbf{U}_{i+1}^n - \mathbf{U}_{i-1}^n) + \frac{\lambda^2}{2} \left(\mathbf{U}_{i+1}^n - 2\mathbf{U}_i^n + \mathbf{U}_{i-1}^n \right).$$

$$\begin{pmatrix} \mathbf{U}_1^{n+1} \\ \vdots \\ \mathbf{U}_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1 - \lambda^2 & \frac{\lambda}{2}(\lambda+1) & 0 & & \\ \frac{\lambda}{2}(\lambda+1) & 1 - \lambda^2 & \frac{\lambda}{2}(\lambda-1) & 0 & \\ 0 & \frac{\lambda}{2}(\lambda+1) & 1 - \lambda^2 & \frac{\lambda}{2}(\lambda-1) & \\ & & 0 & 1 - \lambda^2 & \frac{\lambda}{2}(\lambda+1) \\ & & & \frac{\lambda}{2}(\lambda-1) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_1^n \\ \vdots \\ \mathbf{U}_N^n \end{pmatrix} + \begin{pmatrix} \frac{\lambda}{2}(\lambda+1) \mathbf{U}_0^n \\ 0 \\ \vdots \\ 0 \\ \frac{\lambda}{2}(\lambda-1) \mathbf{U}_{N+1}^n \end{pmatrix}$$

$$\begin{matrix} \mathbf{U}_0^n = \mathbf{U}_1^n \\ \mathbf{U}_{N+1}^n = \mathbf{U}_N^n \end{matrix}$$

$$= \begin{pmatrix} 1 - \frac{\lambda^2}{2} + \frac{\lambda}{2} & \frac{\lambda}{2}(\lambda+1) & 0 & & \\ \frac{\lambda}{2}(\lambda+1) & 1 - \lambda^2 & & & \\ 0 & & 1 - \lambda^2 & \frac{\lambda}{2}(\lambda-1) & \\ & & \frac{\lambda}{2}(\lambda+1) & 1 - \frac{\lambda^2}{2} - \frac{\lambda}{2} & \end{pmatrix} \begin{pmatrix} \mathbf{U}_1^n \\ \vdots \\ \mathbf{U}_N^n \end{pmatrix}$$

Wurzel-scheme:

$$U^{nn} := \begin{pmatrix} u_{11} & & \\ & \ddots & \\ & & u_{nn} \end{pmatrix} = \begin{pmatrix} \lambda & & & \\ & \ddots & & \\ & & \lambda & \\ & & & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} \lambda u_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$U_0^{nn} = U_n^{nn} = \begin{pmatrix} \lambda & & & \\ & \ddots & & \\ & & \lambda & -\lambda \\ & & & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$