1a) Eigenvalues $\det(X \cdot I - A) = \det\begin{pmatrix} X & -4 \\ -1 & X \end{pmatrix} = X^2 - 4 = (X-2)(X+2)$

$\Longrightarrow \lambda(A) = \{-2, 2\}$

Eigenvectors

$V_{-2}(A) = \ker\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \ker\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\rangle$

$V_2(A) = \ker\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} = \ker\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$

$\Longrightarrow R = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -2 & \\ & 2 \end{pmatrix}$

$A = R \Lambda R^{-1}$

Transform initial condition:

$$W_0(x) = R^{-1} U_0(x) = \begin{cases} R^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x < 0 \\ R^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, & x > 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, & x < 0 \\ \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}, & x > 0 \end{cases}$$

Solving the decoupled Riemann problem gives

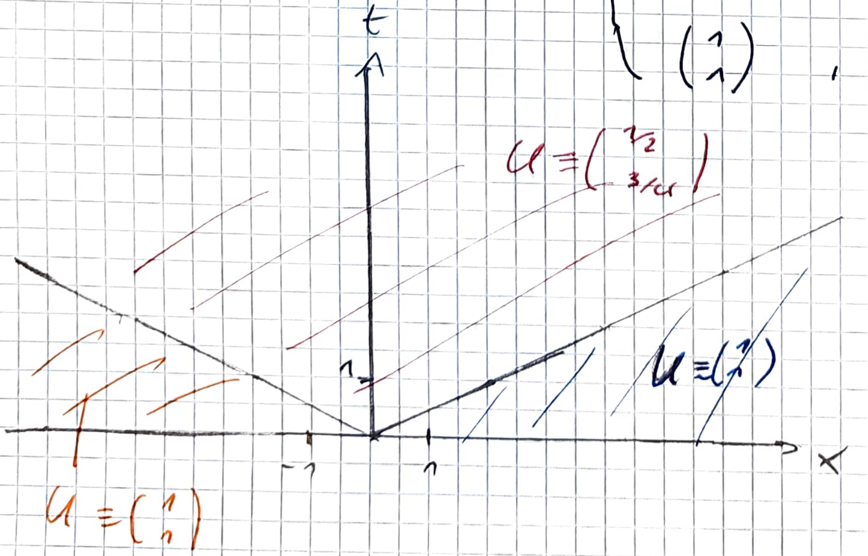
[from lecture notes: $W^p(x,t) = \begin{cases} W_L^p & \text{if } x < \lambda_p t \\ W_R^p & \text{if } x > \lambda_p t \end{cases}$]

$$W^1(x,t) = \begin{cases} \frac{1}{2}, & x < -2t \\ \frac{1}{4}, & x > -2t \end{cases}$$

$$W^2(x,t) = \begin{cases} \frac{1}{2}, & x < 2t \\ \frac{3}{4}, & x > 2t \end{cases}$$

$$\Rightarrow W(x,t) = \begin{cases} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, & x < -2t \\ \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}, & -2t < x < 2t \\ \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}, & x > 2t \end{cases}$$

$$\leadsto U(x,t) = R\,W(x,t) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x < -2t \\ \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix}, & -2t < x < 2t \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & x > 2t \end{cases}$$



1b) Eigenvalues: $\det(\lambda I - A) = \det\begin{pmatrix} \lambda-2 & \\ & \lambda-2 \end{pmatrix} = (\lambda-2)^2$

$\Rightarrow \lambda(A) = \{2\}$

Eigenspace: $V_2(A) = \ker\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

$\Rightarrow R = R^{-1} = I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix} = A$

→ No need for transformation. Solving the Riemann problem as before

$$u(x,t) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & x < 2t \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x > 2t \end{cases}$$