Dr. Yongle Liu
MAT827
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Institut für Mathematik
Universität Zürich

Sheet 7

Deadline: 01.05.2024, 12:00 PM

Exercise 1 (Points: 3, 2, 2)

(a) Prove that a finite difference (FD) scheme written in the viscous form

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{1}{2}\left(Q_{j+1/2}\Delta u_{j+1/2}^n - Q_{j-1/2}\Delta u_{j-1/2}^n\right), \tag{1}$$

where $\lambda = \Delta t/\Delta x$, $\Delta u_{j+1/2} = u_{j+1} - u_j$ and $\Delta f_{j+1/2} = f(u_j) - f(u_{j-1})$, is TVD under the condition

$$\lambda \left|\frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}}\right| \le Q_{j+1/2} \le 1. \tag{2}$$

Hint Use the result of Harten's Lemma.

(b) The condition (2) gives us a recipe for building TVD schemes. Consider footprint-3 schemes, i.e., $Q_{j+1/2} = Q(u_j, u_{j+1})$, and using the ansatz that

$$Q(u,v) = q(\lambda a(u,v)), \quad a(u,v) = \begin{cases} \frac{f(u)-f(v)}{u-v} & \text{if } u \neq v, \\ f'(u) & \text{if } u = v, \end{cases} \tag{3}$$

deduce conditions on the function q, such that the FD scheme would be TVD.

(c) Consider the Lax–Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{\lambda^2}{2}\left(a_{j+1/2}\Delta f_{j+1/2} - a_{j-1/2}\Delta f_{j-1/2}\right), \tag{4}$$

where $a_{j+1/2} = a(u_j, u_{j+1})$ is using the definitions of Exercise 1-(b). Prove or disprove that the scheme is TVD.

Hint Use the criterion you found in Exercise 1-(b).

Exercise 2 (Points: 4, 4)

For scalar conservation law

$$u_t + f(u)_x = 0,$$

we obtain the semi-discrete scheme in the finite volume framework:

$$\frac{d\bar{u}_j}{dt} - \frac{1}{\Delta x}(\hat{F}_{j+1/2}^n - \hat{F}_{j-1/2}^n) = 0,$$

where $\hat{F}_{j+1/2}^n = \hat{F}(u_{j+1/2}^-, u_{j+1/2}^+)$ is the numerical flux.

Now consider the following IBVP:

$$\begin{cases} u_t + u_x = 0, \quad x \in [-5, 5] \\ u(x,0) = \begin{cases} -1, x < 0 \\ 1, x > 0 \end{cases} \end{cases} \tag{5}$$

is subjected to the following boundary condition: $u[0] = u[1]$ and $u[N+1] = u[N]$.

(a) Implement Lax-Wenddroff and Beam-Warming schemes for (5) when $t = 1$. Please plot these schemes and the exact solution together with a proper mesh size, say $N = 100$, what do you observe?

(b) Consider the Lax-Friedrichs scheme:

$$\hat{F}^n_{j+1/2}(u^-_{j+1/2}, u^+_{j+1/2}) = \frac{1}{2}(f(u^-_{j+1/2}) + f(u^+_{j+1/2})) - \frac{\Delta x}{2\Delta t}(u^+_{j+1/2} - u^-_{j+1/2}),$$

for (5) when $t = 1$. Here

$$\begin{aligned}
u^-_{j+1/2} &= p_j(x_{j+1/2}), \quad x_{j-1/2} \le x \le x_{j+1/2}, \\
u^+_{j+1/2} &= p_{j+1}(x_{j+1/2}), \quad x_{j+1/2} \le x \le x_{j+3/2}, \\
p_j(x) &= \bar{u}_j + \sigma_j(x - x_j), \\
p_{j+1}(x) &= \bar{u}_{j+1} + \sigma_{j+1}(x - x_{j+1}),
\end{aligned}$$

and σ_j is computed by minmod limiter. Please plot the numerical solution and the exact solution together with a proper mesh size, say $N = 100$. What do you observe?

Exercise 1 (Points: 3, 2, 2)

(a) Prove that a finite difference (FD) scheme written in the viscous form

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{1}{2}\left(Q_{j+1/2}\Delta u_{j+1/2}^n - Q_{j-1/2}\Delta u_{j-1/2}^n\right), \qquad (1)$$

where $\lambda = \Delta t/\Delta x$, $\Delta u_{j+1/2} = u_{j+1} - u_j$ and $\Delta f_{j+1/2} = f(u_{j+1}) - f(u_j)$, is TVD under the condition

$$\lambda\left|\frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}}\right| \le Q_{j+1/2} \le 1. \qquad (2)$$

Hint Use the result of Harten's Lemma.

We write the FD scheme in incremental form:

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{1}{2}\left(Q_{j+1/2}\Delta u_{j+1/2}^n - Q_{j-1/2}\Delta u_{j-1/2}^n\right)$$

$\Delta f_{j+\frac{1}{2}} + \Delta f_{j-\frac{1}{2}} = f(u^n_{j+1}) - f(u^n_{j-1})$

$$\overset{!}{=} u_j^n + \underbrace{\frac{1}{2}\left(Q_{j+\frac{1}{2}} - \lambda\frac{\Delta f^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right)}_{=:\, C^n_{j+\frac{1}{2}}}\Delta u^n_{j+\frac{1}{2}} - \underbrace{\frac{1}{2}\left(Q_{j-\frac{1}{2}} + \lambda\frac{\Delta f^n_{j-\frac{1}{2}}}{\Delta u^n_{j-\frac{1}{2}}}\right)}_{=:\, D^n_{j-\frac{1}{2}}}\Delta u^n_{j-\frac{1}{2}}$$

We have $C^n_{j+\frac{1}{2}}, D^n_{j-\frac{1}{2}} \geq 0$ as $\lambda\left|\frac{\Delta f^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right| \leq Q^n_{j+\frac{1}{2}}$ and

$\lambda\left|\frac{\Delta f^n_{j-\frac{1}{2}}}{\Delta u^n_{j-\frac{1}{2}}}\right| \leq Q^n_{j-\frac{1}{2}}$

$$C^n_{j+\frac{1}{2}} + D^n_{j+\frac{1}{2}} = \frac{1}{2}\left(Q^n_{j+\frac{1}{2}} - \lambda\frac{\Delta f^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}} + Q^n_{j+\frac{1}{2}} + \lambda\frac{\Delta f^n_{j+\frac{1}{2}}}{\Delta u^n_{j+\frac{1}{2}}}\right) = Q^n_{j+\frac{1}{2}} \leq 1$$

$\xRightarrow{\text{Harten's Lemma}}$ FD is TVD

(b) The condition (2) gives us a recipe for building TVD schemes. Consider footprint-3 schemes, i.e., $Q_{j+1/2} = Q(u_j, u_{j+1})$, and using the ansatz that

$$Q(u,v) = q(\lambda a(u,v)), \quad a(u,v) = \begin{cases} \frac{f(u)-f(v)}{u-v} & \text{if } u \neq v, \\ f'(u) & \text{if } u = v, \end{cases} \tag{3}$$

deduce conditions on the function q, such that the FD scheme would be TVD.

We need $\lambda \left| \frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}} \right| \leq Q_{j+1/2} \leq 1.$

$$\lambda \left| \frac{\Delta f_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}} \right| \overset{!}{\leq} Q_{j+\frac{1}{2}} = q\left(\lambda \frac{f(u_j)-f(u_{j+1})}{u_j - u_{j+1}}\right) = q\left(\lambda \frac{\Delta f_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}}\right) \overset{!}{\leq} 1$$

This means we need to choose q s.t. $|x| \leq q(x) \leq 1$

$\forall x = \lambda \frac{\Delta f_{j+\frac{1}{2}}}{\Delta u_{j+\frac{1}{2}}}$

(c) Consider the Lax–Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{\lambda^2}{2}\left(a_{j+1/2}\Delta f_{j+1/2} - a_{j-1/2}\Delta f_{j-1/2}\right), \qquad (4)$$

where $a_{j+1/2} = a(u_j, u_{j+1})$ is using the definitions of Exercise 1-(b). Prove or disprove that the scheme is TVD.

Hint Use the criterion you found in Exercise 1-(b).

Comparing (1) and (4) we get

$$\lambda^2 a_{j+\frac12} \frac{\Delta f^n_{j+\frac12}}{\Delta u^n_{j+\frac12}} = \lambda^2 a^2_{j+\frac12} = Q_{j+\frac12} = Q(u_j, u_{j+1}) = q(\lambda a_{j+\frac12})$$

b) implies that Lax-Wendroff is TVD if

$$\forall j: \quad \lambda\left|\frac{\Delta f_{j+\frac12}}{\Delta u_{j+\frac12}}\right| \le \lambda^2 \left(\frac{\Delta f^n_{j+\frac12}}{\Delta u^n_{j+\frac12}}\right)^2 \le 1$$

↯ as $x^2 \le |x|$ for $|x| \le 1$

Now prove that LW is not TVD

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{\lambda^2}{2}\left(a_{j+1/2}\Delta f_{j+1/2} - a_{j-1/2}\Delta f_{j-1/2}\right)$$

Look at $u_t + u_x = 0 \to a_{j+\frac12} = 1 \ \forall j, \ f = id$

$$\leadsto u^{n+1}_j = u^n_j - \frac{\lambda}{2}\left(u^n_{j+1} - u^n_{j-1}\right) + \frac{\lambda^2}{2}\left(u^n_{j+1} - 2u^n_j + u^n_{j-1}\right)$$

let $n=0$ and $u^0_j = \begin{cases} 0, & j<0 \\ 1, & j \ge 0 \end{cases}$

$$= \left(-\frac{\lambda}{2} + \frac{\lambda^2}{2}\right)u^n_{j+1} + \left(1-\lambda^2\right)u^n_j + \left(\frac{\lambda}{2} + \frac{\lambda^2}{2}\right)u^n_{j-1}$$

$$\Rightarrow u^1_j = \begin{cases} 0, & j \le -2 \\ -\frac{\lambda}{2} + \frac{\lambda^2}{2}, & j = -1 \\ 1 - \frac{\lambda}{2} - \frac{\lambda^2}{2}, & j = 0 \\ 1, & j > 1 \end{cases}$$

choose $0 \le \lambda \le 1$

$$\sum_j \left|u^1_{j+1} - u^1_j\right| = \left|-\frac{\lambda}{2} + \frac{\lambda^2}{2}\right| + \left|1-\lambda^2\right| + \left|\frac{\lambda}{2} + \frac{\lambda^2}{2}\right| = \frac12 \overbrace{\left|\lambda^2 - \lambda\right|}^{\substack{0\le\lambda\le1 \\ = \lambda-\lambda^2}} + \frac12\left|\lambda^2+\lambda\right| + \left|1-\lambda^2\right| = 1 + \lambda - \lambda^2$$

$$> 1 = \sum_j \left|u^n_{j+1} - u^n_j\right|$$

$\to$ not TVD