Exercise set 1

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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Teaching Assistant: Davide Torlo

Problem 1.1 Types of PDEs (2 pt)

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be the solution of the PDE

$$a\partial_{xx}u + b\partial_{xy}u + c\partial_{yy}u + d\partial_x u + e\partial_y u + fu = g, \tag{1}$$

where $a, b, c, d, e, f, g \in \mathbb{R}$ are constant coefficients and let $T \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ be a nonsingular transformation of $\mathbb{R}^2$, i. e., for every $(x, y) \in \mathbb{R}^2$ we have that the Jacobian of the transformation is different from zero $\det(JT(x, y)) \neq 0$. Then the transformed equation is of the same type.

Let $(\tilde{x}, \tilde{y}) = T(x, y)$ and $\tilde{u}(\tilde{x}, \tilde{y}) := u(T^{-1}(\tilde{x}, \tilde{y}))$ be the solution of the transformed PDE.

We can write the PDE (1) in following form

$$\begin{pmatrix} \partial_x & \partial_y \end{pmatrix} \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} u + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} u + fu = g, \tag{2}$$

and we see that the discriminant of the PDE is $\Delta = -4\det(A) = -4(ac - b^2/4)$, where

$$A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

If we apply the transformation to the variables x, y, we see that

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \partial_x\tilde{x}\partial_{\tilde{x}} + \partial_x\tilde{y}\partial_{\tilde{y}} \\ \partial_y\tilde{x}\partial_{\tilde{x}} + \partial_y\tilde{y}\partial_{\tilde{y}} \end{pmatrix} = \begin{pmatrix} \partial_x\tilde{x} & \partial_x\tilde{y} \\ \partial_y\tilde{x} & \partial_y\tilde{y} \end{pmatrix} \begin{pmatrix} \partial_{\tilde{x}} \\ \partial_{\tilde{y}} \end{pmatrix} = JT \begin{pmatrix} \partial_{\tilde{x}} \\ \partial_{\tilde{y}} \end{pmatrix}. \tag{3}$$

Applying the transformation of the variables to (2), we obtain

$$\begin{pmatrix} \partial_{\tilde{x}} & \partial_{\tilde{y}} \end{pmatrix} JT^T \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} JT \begin{pmatrix} \partial_{\tilde{x}} \\ \partial_{\tilde{y}} \end{pmatrix} \tilde{u} + \begin{pmatrix} d & e \end{pmatrix} JT \begin{pmatrix} \partial_{\tilde{x}} \\ \partial_{\tilde{y}} \end{pmatrix} \tilde{u} + f\tilde{u} = g, \tag{4}$$

hence, here the discriminant is

$$\tilde{\Delta} = -4\det(JT^T AJT) = -4\det(JT^T)\det(A)\det(JT). \tag{5}$$

Since $\det(JT) \neq 0$ it's either positive everywhere or negative everywhere and the determinant of its transposition is equal to itself. So, the sign of Δ is equal to the sign of $\tilde{\Delta}$ and if one is zero also the other one is zero.

Problem 1.2 Method of characteristics (4 pt)

Consider the one-dimensional linear advection equation

$$\begin{aligned} \partial_t u(x, t) + a(x)\partial_x u(x, t) &= 0 \qquad && \text{for } (x, t) \in D \times [0, \infty) \\ u(x, 0) &= \sin(x) && \text{for } x \in D, \end{aligned} \tag{6}$$

where $D \subseteq \mathbb{R}$ is the spatial domain. For each of the following advection coefficients $a(x)$ and domains D, justify the existence (or non-existence) of a solution to (1), plot the family of characteristics in the (x, t)-plane and write down the explicit solution.

a) $a(x) = \alpha x$, $\alpha \in \mathbb{R}$, and $D = \mathbb{R}$,

b) $a(x) = -1$, and $D = [0, \infty)$,

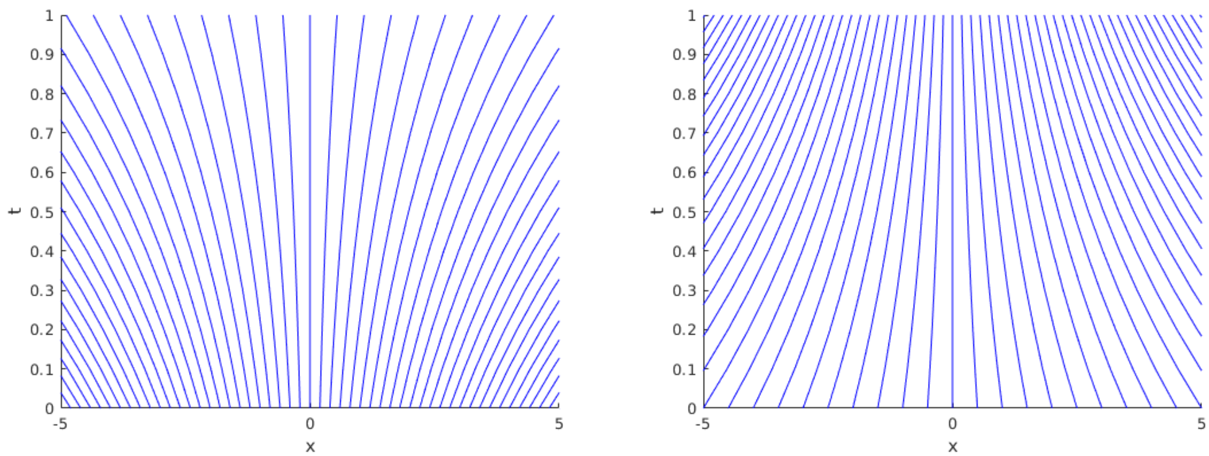
c) $a(x) = 1$, and $D = [0, \infty)$.

Solution (a): By definition, the characteristic curve starting at $x_0 \in \mathbb{R}$, $x(t)$, is that which verifies the ODE

$$\frac{dx_{x_0}}{dt} = a(x_{x_0}(t)) \tag{7}$$

$$x_{x_0}(0) = x_0. \tag{8}$$

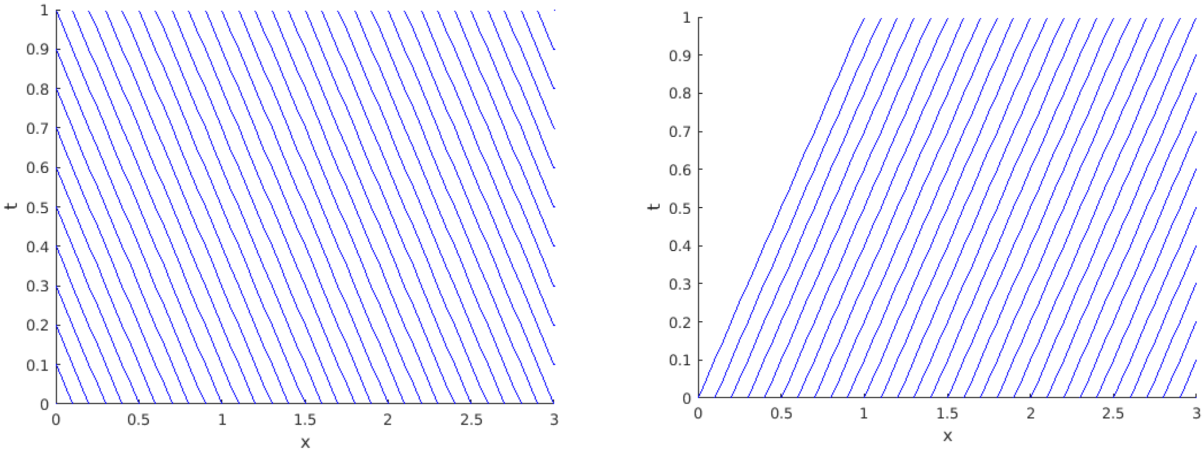
With $a(x) = \alpha x$, we thus find that $x_{x_0}(t) = x_0 e^{\alpha t}$. The characteristics are shown in figure 1. Note that tracing back the characterstic starting at a given (x, t) leads to the initial point $x_0 = x e^{-\alpha t}$. Given initial data $u_0(x) = \sin(x)$, it follows that $u(x, t) = \sin\left(x e^{-\alpha t}\right)$ is the solution obtained by the method of characteristics.



(a) $a = x$ (b) $a = -x$

Figure 1: Characteristic for $a(x) = \alpha x$.

Solution (b): In this case, the characteristics are given by $x_{x_0}(t) = x_0 - t$. The characteristics are plotted in figure 2a. The solution obtained by applying the method of characteristics is given by $u(x, t) = \sin(x + t)$. This defines $u(x, t)$ for arbitrary $(x, t) \in D \times [0, \infty)$.



(a) $a = -1$

(b) $a = +1$

Figure 2: Characteristic for $a(x) = \text{const}$.

Solution (c): In this last case, the characteristics are given by $x_{x_0}(t) = x_0 + t$. The characteristics are plotted in figure 2b. Applying the method of characteristics, we obtain $u(x,t) = \sin(x-t)$. However, *in contrast to (b)*, we note that this does not define $u(x,t)$ for arbitrary $(x,t) \in D \times [0,\infty)$, since the initial data has only been defined on $D = [0,\infty)$! We observe that the 'initial point' $x_0 = x - t$ for given $(x,t) \in D \times [0,\infty)$ will only lie in the domain D, provided that $x \geq t$. To determine the solution for $x < t$, we need to additionally supply a boundary condition $u(0,t) = g(t)$ along the left boundary $\{0\} \times [0,\infty)$. For any choice of $g(t)$, we will obtain a unique solution $u(x,t)$ by the method of characteristics. Therefore, $u(x,t)$ is not uniquely defined by the initial data in this case.

Problem 1.3 Linear advection equation (4 pt)

a) Consider the one-dimensional linear advection equation with variable coefficients

$$\begin{cases} \partial_t u(x,t) + a(x)\partial_x u(x,t) &= 0 \qquad \text{for } (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) &= u_0(x) \qquad \text{for } x \in \mathbb{R} \end{cases} \tag{9}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a given function. Let $a : \mathbb{R} \to \mathbb{R}$ be given by

$$a(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ -x, & \text{if } 0 < x \leq 1, \\ -1, & \text{if } x > 1. \end{cases} \tag{10}$$

Use the method of characteristics to find the solution $u(x,t)$, for $(x,t) \in \mathbb{R} \times [0,\infty)$, and a generic initial condition u_0.

Solution: Since the coefficient $a(x)$ is defined piecewise for $x < 0$, $0 \leq x \leq 1$, and $x > 1$, we need to consider the characteristics in these regions separately. A visual representation of the characteristics is given in Figure 3.

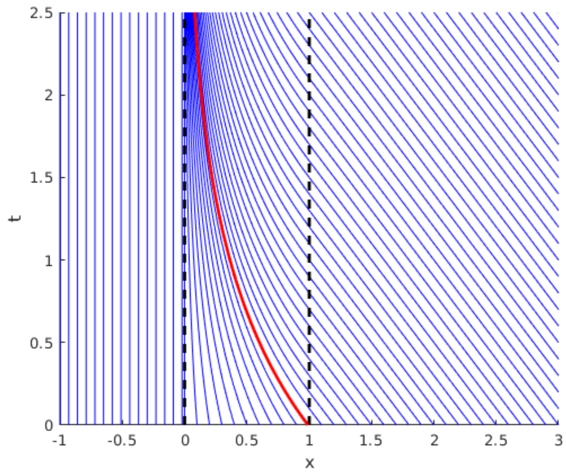


Figure 3: Characteristics for $a(x)$ defined by 10.

Region I ($x < 0$): Here, the characteristics are simply given by $x(t) = x_0$.

Region II ($0 \leq x \leq 1$): The characteristic equation is $\dot{x} = -x$, with solution $x(t) = \hat{x}\exp(-(t-\hat{t}))$, where $(\hat{x}, \hat{t}$ is the first point at which the characteristic enters this region (which may be at $\hat{t} \neq 0$) (cp. 3).

Region III ($x > 1$): The characteristic equation is $\dot{x} = -1$, with solution $x(t) = x_0 - t$. All of these characteristics originate in a point $(x_0, 0)$ with $x_0 > 1$, so that no characteristics from regions I, and II can enter region III.

To apply the method of characteristics to determine the solution $u(x,t)$ for a given (x,t), we must trace back along characteristics, until we arrive at a point $(x_0, 0)$ at the initial time.

- For $x < 0$, we clearly have $x_0 = x < 0$ for all points in region I.
- Similarly, for $x > 1$, tracing back along characteristics does not lead out of region III. Therefore, we find from $x(t) = x_0 - t$ that $x_0 = x + t$.
- Finally, for (x,t) in region II, we have $x(t) = \hat{x}\exp(-(t-\hat{t}))$, where $(\hat{x}, \hat{t})$ lies on the boundary of the region. The only possible boundary points are of the form $(\hat{x}, 0)$ or $(1, \hat{t})$, for some $\hat{x} \in [0,1]$, and $\hat{t} \geq 0$. As is clear from figure 3, this distinction divides region II into two subregions, with the characteristic emanating from $(1,0)$ corresponding to the boundary.
  - Thus, any (x,t) with $x \leq \exp(-t)$, can be traced back to $x_0 = x\exp(t)$.
  - If $x \geq \exp(-t)$, then we first trace back along the characteristic $x(t) = \hat{x}\exp(-(t-\hat{t}))$, until we hit $\hat{x} = 1$. From $x = \exp(-(t-\hat{t}))$, we now find that we hit the boundary at time $\hat{t} = t + \log(x) \leq t$. Tracing further back along the characteristic leads us into region III, so that (from the second bullet point above) $x_0 = \hat{x} + \hat{t} = 1 + t + \log(x)$.

By the method of characteristics, it follows that

$$u(x,t) = \begin{cases} u_0(x), & \text{if } x < 0, \\ u_0(x\exp(t)), & \text{if } 0 \leq x \leq e^{-t}, \\ u_0(1 + t + \log(x)), & \text{if } e^{-t} \leq x \leq 1, \\ u_0(x+t), & \text{if } x \geq 1. \end{cases}$$

b) Determine now the solution $u(x,t)$ of the one-dimensional linear conservation law

$$\begin{cases} \partial_t u(x,t) + \partial_x\big(a(x)u(x,t)\big) &= 0 \qquad \text{for } (x,t) \in \mathbb{R}\times[0,\infty) \\ u(x,0) &= u_0(x) \qquad \text{for } x \in \mathbb{R} \end{cases} \tag{11}$$

for $(x,t) \in \mathbb{R}\times[0,\infty)$, and a generic initial condition u_0.

Hint: Make use of the characteristics found in (a).

Solution: Note that at all points, where $a(x)$ is smooth, we can write (11) as an inhomogeneous advection equation

$$\partial_t u(x,t) + a(x)\partial_x u(x,t) = -a'(x)u(x,t).$$

This implies that along the characteristics $x_{x_0}(t)$ computed in part (a), we have

$$\frac{d}{dt}u(x_{x_0}(t),t) = -a'(x_{x_0}(t))u(x_{x_0},t).$$

In regions I and III (cp. part (a)), we have $-a'(x) = 0$. In region II, we have $-a'(x) = 1$. Since no characteristics leave region II, this implies that $u(x,t)$ will coincide with the solution obtained in (a), for all (x,t) in regions I and III.

On the other hand, the characteristics enter region II either at $\hat{t} = 0$ (if $0 \le x_0 \le 1$, or at $\hat{t} = x_0 - 1$ (if $x_0 > 1$). After entering region II, these characteristics solve the following ODE

$$\frac{d}{dt}u(x_{x_0}(t),t) = u(x_{x_0}(t),t),$$

with initial condition

$$u(x_{x_0}(\hat{t}),\hat{t}) = \begin{cases} u_0(xe^t), & \text{if } x \le e^{-t} \qquad [\Rightarrow \hat{t} = 0], \\ u_0(1+t+\log(x)), & \text{if } e^{-t} \le x \le 1 \ [\Rightarrow \hat{t} = t + \log(x)]. \end{cases}$$

In square brackets on the right, we have also recalled the appropriate value of $\hat{t}$ which has been obtained in the previous part, Problem 1.2(a). Solving the ODEs now leads to the solutions

$$u(x,t) = u_0(xe^t)e^t, \quad \text{and} \quad u_0(1+t+\log(x))e^{t-\hat{t}} = \frac{u_0(1+t+\log(x))}{x},$$

for the two cases. Recalling also the (unchanged) results in region I and III, from Problem 1.2(a), we finally find

$$u(x,t) = \begin{cases} u_0(x), & \text{if } x < 0, \\ u_0(xe^t)e^t, & \text{if } 0 \le x \le e^{-t}, \\ \dfrac{u_0(1+t+\log(x))}{x}, & \text{if } e^{-t} \le x \le 1, \\ u_0(x+t), & \text{if } x \ge 1. \end{cases}$$

c) For the two solutions computed in (a) and (b), respectively, what can be said about $\int_{\mathbb{R}} |u(x,t)|\,dx$ and $\max_{x\in\mathbb{R}} |u(x,t)|$ as a function of time t?

Solution: Clearly, the solution that we found for the advection equation ($u_t + au_x = 0$) satisfies $\max_x |u(x,t)| = \max_x |u_0(x)|$. On the other hand, the solution found for the conservation law $u_t + (au)_x$ in (b) has branches $u_0(xe^t)e^t$ ($\le \max_x |u_0(x)|e^t$), and $u_0(1+t+\log(x))/x$ in $[e^{-t},1]$, which also verifies $u_0(1+t+\log(x))/x \le \max_x |u_0(x)|e^t$.

Therefore, it achieves $\max_x |u(x,t)| \le e^t \max_x |u_0(x)|$. This bound is sharp: it is easy to find a $u_0(x)$ for which equality holds; e.g. $u_0 \equiv 1$ (cf. Problem 1.3).

In general, the quantity $\int_{\mathbb{R}} |u(x,t)|\,dx$ might decrease *or increase* for the advection equation with generic coefficient $a(x)$. For the present choice of $a(x)$, $\int_{\mathbb{R}} |u(x,t)|\,dx$ is found to be monotonically decreasing, simply by plugging in the solution found in 1.2.a):

$$\begin{aligned}\int_{\mathbb{R}} |u(x,t)|\,dx &= \int_{-\infty}^{0} |u(x,t)|\,dx + \int_{0}^{e^{-t}} |u(x,t)|\,dx + \int_{e^{-t}}^{1} |u(x,t)|\,dx + \int_{1}^{\infty} |u(x,t)|\,dx \\ &= \int_{-\infty}^{0} |u_0(x)|\,dx + \int_{0}^{e^{-t}} |u_0(xe^{t})|\,dx + \int_{e^{-t}}^{1} |u_0(1+t+\log(x))|\,dx + \int_{1}^{\infty} |u_0(x+t)|\,dx.\end{aligned}$$

Making the change of variables $y(x) = xe^t$ for the second term, $z(x) = 1 + t + \log(x)$ for the third, and $s(x) = x + t$ for the last, we obtain:

$$\int_{\mathbb{R}} |u(x,t)|\,dx = \int_{-\infty}^{0} |u_0(x)|\,dx + \int_{0}^{1} |u_0(y)|e^{-t}\,dy + \int_{1}^{1+t} |u_0(z)|e^{z-1-t}\,dz + \int_{1+t}^{\infty} |u_0(s)|\,ds.$$

And after the observation that for $t \geq 0$, we have that $e^{-t} \leq 1$, and that $e^{z-1-t} \leq 1$ if $z \in [1, 1+t]$, we conclude

$$\int_{\mathbb{R}} |u(x,t)|\,dx \leq \int_{\mathbb{R}} |u_0(x)|\,dx.$$

On the other hand, for the conservation law, we find

$$\begin{aligned}\int_{\mathbb{R}} |u(x,t)|\,dx &= \int_{-\infty}^{0} |u(x,t)|\,dx + \int_{0}^{e^{-t}} |u(x,t)|\,dx + \int_{e^{-t}}^{1} |u(x,t)|\,dx + \int_{1}^{\infty} |u(x,t)|\,dx \\ &= \int_{-\infty}^{0} |u_0(x)|\,dx + \int_{0}^{e^{-t}} |u_0(xe^{t})|e^{t}\,dx + \int_{e^{-t}}^{1} |u_0(1+t+\log(x))|\,\frac{dx}{x} + \int_{1}^{\infty} |u_0(x+t)|\,dx.\end{aligned}$$

Making the change of variables $y = xe^t$ for the second term, and $y = 1 + t + \log(x)$ for the third term, we see that these two contributions sum precisely to $\int_0^{1+t} |u_0(x)|\,dx$, so that

$$\int_{\mathbb{R}} |u(x,t)|\,dx = \int_{-\infty}^{0} |u_0(x)|\,dx + \int_{0}^{1+t} |u_0(x)|\,dx + \int_{1}^{\infty} |u_0(x+t)|\,dx = \int_{\mathbb{R}} |u_0(x)|\,dx.$$

Thus, the linear conservation law conserves the L^1 norm of $u(x,t)$ over time.

d) Let $u(x,t)$ be a solution of the conservation law (11), where we now consider a general coefficient $a \in C^1$. Assume that $u(x,t)$ is C^1 and that $\lim_{|x|\to\infty} u(x,t) = 0$ for all $t \geq 0$ (you may assume $u(x,t)$ to be well-behaved at $x = \pm\infty$ to avoid technical difficulties). Show that $u(x,t)$ is conserved, in the sense that $\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_0(x)\,dx$ for all $t \geq 0$. Is this also true for the linear advection equation?

Solution: We won't state any precise requirements on $u(x,t)$, and proceed by mostly formal calculations (you may come up with suitable conditions on $u(x,t)$): For the linear conservation law, we find

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} \partial_t u(x,t)\,dx = \int_{\mathbb{R}} -\partial_x(a(x)u(x,t))\,dx = [a(x)u(x,t)]_{x=-\infty}^{\infty} = 0,$$

where the last inequality follows provided that $u(x,t)$ decays to zero at infinity. Thus, $\int_{\mathbb{R}} u(x,t)\,dx$ is conserved in time.

On the other hand, for the advection equation, we find

$$\frac{d}{dt}\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} \partial_t u(x,t)\,dx = -\int_{\mathbb{R}} a(x)\partial_x u(x,t)\,dx = \int_{\mathbb{R}} (\partial_x a(x))u(x,t)\,dx,$$

which is non-zero in general, unless $a(x) = \text{const.}$.

Remark: Note that if $u(x,t)$ is advected by $a(x)$, then so is $|u(x,t)|$. That is, $|u(x,t)|$ is also solution of the advection equation (up to minor issues with smoothness). We therefore expect that

$$\int_{\mathbb{R}} |u(x,t)|\, dx \quad \text{is} \quad \begin{cases} \text{monotonically increasing, if } \partial_x a(x) \geq 0, \\ \text{monotonically decreasing, if } \partial_x a(x) \leq 0, \end{cases}$$

This was observed for the explicit solution computed in part (c), where $\partial_x a(x) \leq 0$.

Organiser: Davide Torlo, Y27 K48 (`davide.torlo@math.uzh.ch`)

Additional Informations

- Exercise classes will be held by Davide Torlo (Y27 K48, `davide.torlo@math.uzh.ch`) on Thursday form 10.00 to 12.00 in Y27 H26.
- There might be an additional exercise class.
- There will be a written or an oral exam at the end. It will be decided after the first two weeks of the course. The date of the written exam will be the 29th of June. Sufficient to participate in the exam are 50 % of the total points of **all** exercise sheets.
- There will be each week an exercise sheet on Thursday. You have to hand in to Davide Torlo at the beginning of the Thursday exercise class or in the mailbox in Y27 K floor or to Philipp Öffner at the end of Thursday lesson.
- For a consultation hour, please write an e-mail to davide.torlo@math.uzh.ch or philipp.oeffner@math.uzh.ch or ask after the lecture.