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MAT827
Spring 2024
Institut für Mathematik
Universität Zürich

Sheet 9

Deadline: 15.05.2024, 12:00 PM

Exercise 1 (Points: 8)
Consider the following Riemann problem:

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0. \tag{1}$$

(a)

$$A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix},$$

and initial data

$$\boldsymbol{u}(x,0) = \begin{cases} (1,1)^\top, & x > 0, \\ (0,1)^\top, & x < 0. \end{cases}$$

(b)

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and initial data

$$\boldsymbol{u}(x,0) = \begin{cases} (0,1)^\top, & x > 0, \\ (1,0)^\top, & x < 0. \end{cases}$$

Sketch its solution in the (x,t) plane (specify all relevant data on your sketch).

Exercise 2 (Points: 7)
Design the Godunov scheme for the system in the previous exercise.

Exercise 1 (Points: 8)
Consider the following Riemann problem:

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = 0. \tag{1}$$

(a)

$$A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix},$$

and initial data

$$\boldsymbol{u}(x,0) = \begin{cases} (1,1)^\top, & x > 0, \\ (0,1)^\top, & x < 0. \end{cases}$$

(b)

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

and initial data

$$\boldsymbol{u}(x,0) = \begin{cases} (0,1)^\top, & x > 0, \\ (1,0)^\top, & x < 0. \end{cases}$$

Sketch its solution in the (x,t) plane (specify all relevant data on your sketch).

a) Eigenvalues: $\det(x \cdot I - A) = \det\begin{pmatrix} x & -4 \\ -1 & x \end{pmatrix} = x^2 - 4 = (x-2)(x+2)$

$\rightarrow \lambda(A) = \{-2, 2\}$

Eigenvectors: $V_{-2}(A) = \ker\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \ker\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\rangle$

$V_2(A) = \ker\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} = \ker\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$

$\rightarrow R = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$

Transform initial condition:

$$W_0(x) = \begin{cases} \frac{1}{4}\begin{pmatrix} 1 \\ 3 \end{pmatrix}, & x > 0 \\ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, & x < 0 \end{cases}$$

Solving the decoupled Rieman problem gives us:

$$W^p(x,t) = \begin{cases} W_L^p & \text{if } x < \lambda_p t \\ W_R^p & \text{if } x > \lambda_p t. \end{cases}$$

$$W^1(x,t) = \begin{cases} \frac{1}{4}, & x < -2t \\ \frac{1}{2}, & x > -2t \end{cases}$$

$$W^2(x,t) = \begin{cases} \frac{3}{4} & x < 2t \\ \frac{1}{2} & x > 2t \end{cases}$$

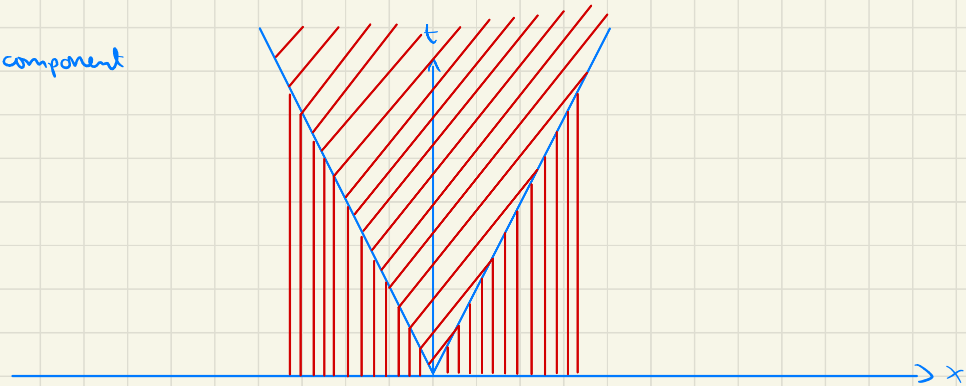
V/ sketch

$$\rightsquigarrow U(x,t) = RW(x,t) = \begin{pmatrix} 2(W^2 - W^1) \\ W^1 + W^2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & x < -2t \\ \begin{pmatrix} \frac{1}{2} \\ \frac{5}{4} \end{pmatrix} & -2t < x < 2t \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & x > 2t \end{cases}$$

1. component

$$\begin{cases} \begin{pmatrix}1\\1\end{pmatrix} & x<-2t \\ \begin{pmatrix}\frac{1}{2}\\ \frac{5}{4}\end{pmatrix} & -2t<x<2t \\ \begin{pmatrix}0\\1\end{pmatrix} & x>2t\end{cases}$$

2. component



b) Eigenvalues: $\det(x \cdot I - A) = \det\begin{pmatrix} x-2 & 0 \\ 0 & x-2 \end{pmatrix} = (x-2)^2$

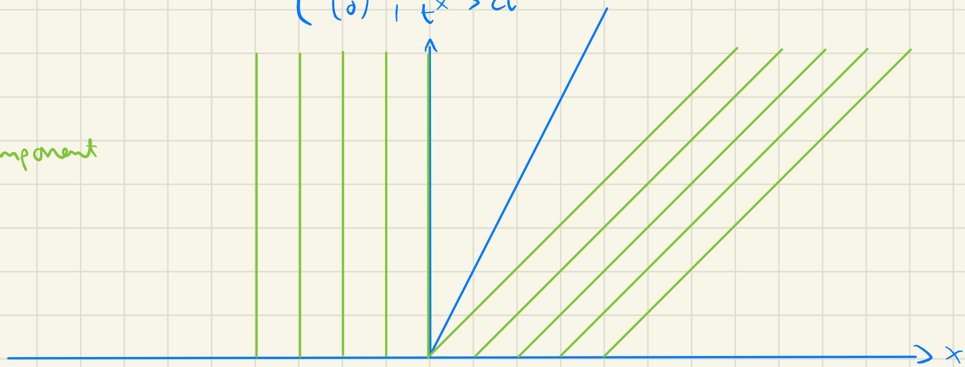
$\longrightarrow \lambda(A) = \{2\}$

Eigenvalues: $V_2(A) = \ker\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

$\longrightarrow$ no need for transformation

$$u(x,t) = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & x < 2t \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & x > 2t \end{cases}$$

1. component



2. component