Exercise set 3

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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In this exercise sheet we study the implicit central difference method (ICD), i.e., the central difference method with implicit euler method.

Given the conservation laws

$$\partial_t u + \partial_x F(u) = 0, \tag{1}$$

the ICD reads

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(F(u_{j+1}^{n+1}) - F(u_{j-1}^{n+1})\right), \tag{2}$$

where we consider a uniform point distribution $\{x_j\}_{j=1}^{N+1}$ over the interval $[a,b]$, where $x_1 = a$ and $x_{N+1} = b$ and $u_j \approx u(x_j)$, while the superscript n denotes the timestep t^n, hence, $u_j^n \approx u(t^n, x_j)$.

We consider, in particular, the linear transport equations, i.e., $F(u) = c \cdot u$, with $c \in \mathbb{R}$ the transport speed. This allows to collect all the $n+1$ terms on the left hand side and obtain a matrix M that we can invert to have the method ICD written as

$$\mathbf{u}^{n+1} = M^{-1}\mathbf{u}^n. \tag{3}$$

Moreover, we consider periodic boundary conditions, so that $u(t^n, a) = u_1^n = u_{N+1}^n = u(t^n, b)$, for every t^n.

Problem 3.1 Mass matrix (2pts)

1. Write explicitly the scheme for the linear transport equation and the mass matrix M and highlight which parameters it depends on.

   Solution

   The ICD method for the transport equation can be written as

   $$u_j^{n+1} = u_j^n - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) \tag{4}$$

   $$u_j^{n+1} + \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = u_j^n. \tag{5}$$

   Hence, if we define $\beta = \frac{c\Delta t}{2\Delta x}$ the mass matrix is on a general line j

   $$M_{j,j} = 1, \quad M_{j,j+1} = \beta, \quad M_{j,j-1} = -\beta, \qquad j = 2, \dots, N-1. \tag{6}$$

   For the first and last line (Nth line, we skip the point $N+1$ which is identical to the first because of the periodic boundary conditions) we have to apply the periodic boundary conditions. So,

   $$\begin{aligned} M_{1,1} = 1, \quad M_{1,2} = \beta, \quad M_{1,N} = -\beta \\ M_{N,N} = 1, \quad M_{N,1} = \beta, \quad M_{N,N-1} = -\beta. \end{aligned}$$

2. Find the condition under which the matrix M is diagonally dominant by rows and, hence, invertible.

Solution

The matrix M is diagonally dominant by row if

$$|M_{j,j}| > \sum_{k\neq j} |M_{j,k}|$$
$$1 > 2|\beta|.$$

This condition is verified when

$$2|\beta| = \frac{|c|\Delta t}{\Delta x} < 1. \tag{7}$$

This condition is the so–called CFL (Courant–Friedrichs–Levy) condition.

Problem 3.2 Energy stability (2pts)

Define the energy of the discrete solution $\mathbf{u}^n$ as

$$\mathcal{E}(\mathbf{u}^n) = \sum_{j=1}^{N} \frac{(u_j^n)^2}{2}. \tag{8}$$

Prove that the ICD is energy stable, i.e., $\mathcal{E}(\mathbf{u}^{n+1}) \leq \mathcal{E}(\mathbf{u}^n)$.

Solution

We recall the definition of the ICD scheme

$$u_j^{n+1} - u_j^n + \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = 0. \tag{9}$$

Now, we multiply by u_j^{n+1} the whole scheme and we obtain

$$u_j^{n+1}(u_j^{n+1}) - u_j^n) + \beta\left(u_j^{n+1}u_{j+1}^{n+1} - u_j^{n+1}u_{j-1}^{n+1}\right) = 0. \tag{10}$$

Using the equality $\alpha_1(\alpha_1-\alpha_2) = \frac{\alpha_1^2}{2} - \frac{\alpha_2^2}{2} + \frac{1}{2}(\alpha_1-\alpha_2)^2$, we can write

$$\frac{(u_j^{n+1})^2}{2} - \frac{(u_j^n)^2}{2} + \frac{1}{2}(u_j^{n+1} - u_j^n)^2 + \beta\left(u_j^{n+1}u_{j+1}^{n+1} - u_j^{n+1}u_{j-1}^{n+1}\right) = 0. \tag{11}$$

Summing over j and collecting the terms that sum up to the energy, we have

$$\mathcal{E}(\mathbf{u}^n) = \sum_j \frac{(u_j^n)^2}{2} = \sum_j \left(\frac{(u_j^{n+1})^2}{2} + \frac{1}{2}(u_j^{n+1} - u_j^n)^2 + \beta\left(u_j^{n+1}u_{j+1}^{n+1} - u_j^{n+1}u_{j-1}^{n+1}\right)\right) \tag{12}$$

$$= \mathcal{E}(\mathbf{u}^{n+1}) + \underbrace{\sum_j \frac{1}{2}(u_j^{n+1} - u_j^n)^2}_{\geq 0} + \beta \underbrace{\sum_j \left(u_j^{n+1}u_{j+1}^{n+1} - u_j^{n+1}u_{j-1}^{n+1}\right)}_{=0,\text{ telescopic sum}} \geq \mathcal{E}(\mathbf{u}^{n+1}). \tag{13}$$

Hence, the energy is decreasing in time, so the ICD scheme is energy stable.

Problem 3.3 Von Neumann Stability (2pts)

Prove that the ICD is also von Neumann stable. Suppose that u is a wave with wave number $\lambda \in \mathbb{N}^+$ and it grows exponentially in time, i. e.,

$$u(t,x) = e^{\alpha t} e^{i\lambda x}. \tag{14}$$

This is a solution for the transport equation with periodic boundary conditions on $[0, 2\pi]$ for $\alpha = 0$. If for the method $\alpha(\lambda)$ is positive for any λ, the scheme is unstable, if $\alpha(\lambda)$ is non–positive for all λ, the scheme is stable.

Prove that ICD is von Neumann stable.

Solution

Starting from the definition of the ICD scheme, we substitute the ansatz (14) and the discretizations $t^n = n\Delta t$ and $x_j = j\Delta x$, obtaining

$$u_j^{n+1} - u_j^n + \beta \left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = 0, \tag{15}$$

$$e^{\alpha t^{n+1}} e^{i\lambda x_j} - e^{\alpha t^n} e^{i\lambda x_j} + \beta \left(e^{\alpha t^{n+1}} e^{i\lambda x_{j+1}} - e^{\alpha t^{n+1}} e^{i\lambda x_{j-1}}\right) = 0, \tag{16}$$

$$e^{\alpha t^n} e^{i\lambda x_j} \left(e^{\alpha\Delta t} - 1 + e^{\alpha \Delta t}\beta \left(e^{i\lambda\Delta x} - e^{-i\lambda\Delta x}\right)\right) = 0, \tag{17}$$

$$e^{\alpha\Delta t} = \frac{1}{1 + i2\beta\sin(\lambda\Delta x)}. \tag{18}$$

Now, we can study the absolute value of the amplification factor $|e^{\alpha\Delta t}|$, which is $|1 + i2\beta\sin(\lambda\Delta x)|^{-1} = 1/\sqrt{1 + 4\beta^2\sin^2(\lambda\Delta x)} < 1$ for all possible $\lambda \neq 0$. Hence, the ICD is also von Neumann stable.

Problem 3.4 Implementation of the method (4pts)

Implement the ICD in MATLAB (or Python or Julia). Use already implemented functions for the inversion of the matrix.

Keep as input variables c the speed of the transport equation, N number of subintervals of the domain, CFL number ($c\Delta t \leq \text{CFL}\Delta x$), T the final time, a and b the domain extrema, u_0 the initial conditions.

At the end, test your code with the following parameters and plot the initial and final solutions on the same figure. $c = 1$, $N = 200$, $\text{CFL} = 0.75$, $T = 2$, $a = 0$, $b = 1$, $u_0 = \sin(2\pi x)$.

Submit the code.

Solution

```
% We implement the implicit central difference method for transport
% equation
% d_t u + a d_x u=0
% with periodic boundary conditions.
% The method reads
% u^{n+1}_j=u^n_j - c dt/2dx (u^{n+1}_{j+1}-u^{n+1}_{j-1})
plot_evolution=1;
close all

c=1;
CFL = 0.75; %dt/dx

N=200;
T=2;
```

```matlab

u0=@(x) sin(2*pi*x);
a=0;
b=1;


%%
[UU,xx,tt,XX,TT]=ICD(c,N,CFL,T,a,b,u0);
%for plotting purposes

mesh(XX,TT,UU)
xlabel('x')
ylabel('time')

figure()
plot(xx,UU(1,:), 'DisplayName','t=0')
hold on
plot(xx,UU(end,:), 'DisplayName','t=T')
xlabel('x')
legend()


%%

if(plot_evolution)
    for n=[1:ceil(numel(tt)/20):numel(tt),numel(tt)]
        figure(4)
        plot(xx,UU(n,:),...
            xx,u0(xx-c*tt(n)),'r--')
        legend('numerical','analytical','Location','SE')
        xlabel x
        ylabel u
        title(sprintf('upwind, t=%g',tt(n)))
        ylim([-1.5,1.5])
        disp(n)
        drawnow
        pause(.02)
    end
end


function [UU,xx,tt,XX,TT]=ICD(c,N,CFL,T,a,b,u0)
% Tools of the method

dx=(b-a)/N;
dt=dx/CFL/c;
Nt=ceil(T/dt);
dt=T/(Nt);

xx=linspace(a,b,N+1);
xx=xx(1:end-1);

tt=linspace(0,T,Nt+1);

[XX,TT]=meshgrid(xx,tt);


UU=zeros(size(XX));
UU(1,:)=u0(XX(1,:));
%Iteration matrix to be inverted is defined as
% M_ii=1, M_{i,i+1}=c dt/2dx, M_{i,i-1}=-c dt/2dx
M=eye(N) + c*dt/dx/2*(diag(ones(N-1,1),1) -diag(ones(N-1,1),-1));
M(N,1)=c*dt/dx/2;
M(1,N)=-c*dt/dx/2;
```

```

invM=inv(M);
un=reshape(UU(1,:),N,1);
for it=1:Nt
    UU(it+1,:)=invM*reshape(UU(it,:),N,1);
end
end
```

Listing 1: implicitCD.m

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