Exercise 1 (Points: 2, 2)

(a) Show that the Lax-Friedrichs scheme is monotone provided the CFL condition.

(b) Show that the Lax-Friedrichs scheme is a TVD scheme.

Lax-Friedrich:

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

with $F_{j+\frac{1}{2}}^n = F^{LxF}(U_j^n, U_{j+1}^n) = \frac{f(U_j^n) + f(U_{j+1}^n)}{2} - \frac{\Delta x}{2\Delta t}\left(U_{j+1}^n - U_j^n\right)$

a) We prove that L-F is monotone by using Lemma 4.8.:

$$\frac{\partial}{\partial a} F^{LxF}(a,b) = \frac{1}{2}\left(f'(a) + \frac{\Delta x}{\Delta t}\right) > 0 \rightarrow a \mapsto F^{LxF}(a,b) \text{ non-decreasing}$$

CFL condition:

$$\max_j |f'(U_j^n)| \leq \frac{\Delta x}{2\Delta t}$$

$$\frac{\partial}{\partial b} F^{LxF}(a,b) = \frac{1}{2}\left(f'(b) - \frac{\Delta x}{\Delta t}\right) < 0 \rightarrow b \mapsto F^{LxF}(a,b) \text{ non-increasing}$$

$\overset{\text{Lemma 4.8}}{\Longrightarrow}$ L-F is monotone

b) As $F^{LxF}(U_j^n, U_j^n) = \frac{f(U_j^n) + f(U_j^n)}{2} - 0 = f(U_j^n)$

L-F is consistent. As it is also monotone as shown in a), TVD follows from Harten's Lemma (4.13). (L-F is a three point finite volume scheme)