# Exercise set 7

**Numerical Methods for Hyperbolic Partial Differential Equations**
IMATH, FS-2020
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## Problem 7.1 Monotonicity preserving (2pts)

A monotonicity preserving scheme is a method that

$$\begin{cases} u_j^n \leq u_{j-1}^n \Rightarrow u_j^{n+1} \leq u_{j-1}^{n+1} & \forall n, j \\ u_j^n \geq u_{j-1}^n \Rightarrow u_j^{n+1} \geq u_{j-1}^{n+1} & \forall n, j. \end{cases} \tag{1}$$

Prove that if the scheme is a constant coefficient schemes with footprint $2K+1$, i.e.,

$$u_j^{n+1} = \sum_{l=-K}^{K} C_l u_{j+l}^n, \tag{2}$$

then it is monotonicity preserving if and only if $C_l \geq 0$ for all $l = -K, \dots, K$.

## Problem 7.2 Condition on viscous form to TVD (2pts)

Prove that a FD scheme written in the viscous form

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{1}{2}\left(Q_{j+1/2}\Delta u_{j+1/2}^n - Q_{j-1/2}\Delta u_{j-1/2}^n\right), \tag{3}$$

where $\lambda = \Delta t/\Delta x$, $\Delta u_{j+1/2} = u_{j+1} - u_j$ and $\Delta f_{j+1/2} = f(u_j) - f(u_{j-1})$, is TVD under the condition

$$\lambda \left| \frac{\Delta f_{j+1/2}}{\Delta u_{j+1/2}} \right| \leq Q_{j+1/2} \leq 1. \tag{4}$$

**Hint** Use the result of Harten's Lemma (Theorem 3.13 of the notes).

## Problem 7.3 Recipe for TVD schemes (2pts)

The condition (4) gives us a recipe for building TVD schemes. Consider footprint-3 schemes, i.e., $Q_{j+1/2} = Q(u_j, u_{j+1})$, and using the ansatz that

$$Q(u,v) = q(\lambda a(u,v)), \quad a(u,v) = \begin{cases} \frac{f(u)-f(v)}{u-v} & \text{if } u \neq v, \\ f'(u) & \text{if } u = v, \end{cases} \tag{5}$$

deduce conditions on the function $q$, such that the FD scheme would be TVD.

## Problem 7.4 TVD or not TVD (4pts)

1. Consider the Lax–Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(f(u_{j+1}^n) - f(u_{j-1}^n)\right) + \frac{\lambda^2}{2}\left(a_{j+1/2}\Delta f_{j+1/2} - a_{j-1/2}\Delta f_{j-1/2}\right), \tag{6}$$

   where $a_{j+1/2} = a(u_j, u_{j+1})$ using the definitions of previous exercises. Prove or disprove that the scheme is TVD.
   **Hint** Use the criterion you found in Problem 7.3.

2. Consider the Godunov method in its conservative form

$$u_j^{n+1} = u_j^n - \lambda\left(f^{num}(u_j^n, u_{j+1}^n) - f^{num}(u_{j-1}^n, u_j^n)\right) \tag{7}$$

   where

$$f^{num}(u,v) = \begin{cases} \min_{u\leq\theta\leq v} f(\theta) & \text{if } u \leq v, \\ \max_{v\leq\theta\leq u} f(\theta) & \text{if } v \leq u, \end{cases}$$

   prove or disprove that the scheme is TVD.
   **Hint** Use directly Harten's Lemma conditions.

3. Test the Lax–Wendroff method for the linear transport equation $\partial_t u + \partial_x u = 0$ on $[-1, 1]$ with periodic boundary conditions and initial conditions $u_0 = \cos(\pi x)$. Plot the total variation of the scheme as a function of time, till $T = 1$, for CFL$= 0.7$ and CFL$= 1$. What do you observe? Why?
   **Hint** You can use the solutions of the Exercise Set 5.

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