2. The finite volume scheme for the system from 1. is

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right)$$

with $F_{j+\frac{1}{2}}^n = F(U_j^n, U_{j+1}^n) := A\, U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}, 0)$,

where $U_{j+\frac{1}{2}}$ is the solution (exact or approximate) of the Riemann problem.

The Godunov flux uses the exact solution $U^* = U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}, 0)$.

In the transformed coordinates

$$W := R^{-1} U, \qquad A = R \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} R^{-1}$$

the exact solution of the Riemann problem at the interface has a jump $W_R - W_L$.

Thus, in the original coordinates, the jump becomes

$$\begin{aligned} U_R - U_L &= R(W_R - W_L) \\ &= \sum_{p=1}^m \underbrace{(W_R^p - W_L^p)}_{:=\alpha^p} r_p \\ &= \sum_{p=1}^m \alpha^p r_p, \end{aligned}$$

where r_p are the eigenvectors of A.

Thus, since the Eigenvalues λ_i correspond to the wavespeed, we have

$$U^* = U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}, 0) = \sum_{p:\lambda_p<0} \alpha_{j+\frac{1}{2}}^p r_p + U_j^n \qquad (\star)$$

Thus,

$$\begin{aligned} AU^* &= AU_j^n + \sum_{p:\lambda_p<0} \alpha_{j+\frac{1}{2}}^p \underbrace{Ar_p}_{=\lambda_p r_p} \\ &= AU_j^n + \sum_{p=1}^m \alpha_{j+\frac{1}{2}}^p \lambda_p^- r_p . \end{aligned}$$

Similarly, we can represent $(\star)$ as

$$U^* = U_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}, 0) = U_{j+1} - \sum_{p:\lambda_p\geq 0} \alpha_{j+\frac{1}{2}}^p r_p ,$$

thus

$$AU^* = AU_{j+1} - \sum_{p=1}^m \alpha_{j+\frac{1}{2}}^p \lambda_p^+ r_p .$$

Averaging yields

$$\begin{aligned} AU^* &= \frac{1}{2}\left(AU_j^n + AU_{j+1}^n - \sum_{p=1}^m \alpha_{j+\frac{1}{2}}^p (\lambda_p^+ - \lambda_p^-) r_p\right) \\ &= \frac{1}{2} A(U_j^n + U_{j+1}^n) - \frac{1}{2}\sum_p |\lambda_p| \alpha_{j+\frac{1}{2}}^p r_p . \end{aligned}$$

The wave strength $\alpha_{j+\frac{1}{2}}^p$ is equal to $W_{j+1}^n - W_j^n$, thus

$$AU^* = \frac{1}{2}A(U_j^n + U_{j+1}^n) - \frac{1}{2} R|\Lambda|(W_{j+1}^n - W_j^n) = \frac{1}{2}A(U_j^n + U_{j+1}^n) - \frac{1}{2}R|\Lambda|R^{-1}(U_{j+1}^n - U_j^n)$$

$\stackrel{!}{=}: F_{j+\frac{1}{2}}$, which gives the Godunov scheme.

In a), we have

$$A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}, \quad R^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -2 & \\ & 2 \end{pmatrix}, \quad |\Lambda| = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}$$

$$R|\Lambda|R^{-1} = 2RR^{-1} = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix} = 2 \cdot I$$

Thus, the Godunov flux is

$$F_{j+\frac{1}{2}}^{n} = AU^* = \frac{1}{2}\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}(U_j^n + U_{j+1}^n) - (U_{j+1}^n - U_j^n) = \frac{1}{2}\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}U_j^n + \frac{1}{2}\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}U_{j+1}^n$$

b) $A = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \Lambda = |\Lambda| = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix},$

$$R|\Lambda|R^{-1} = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix} = 2 \cdot I$$

Thus, the Godunov flux is

$$F_{j+\frac{1}{2}}^{n} = \frac{1}{2}\begin{pmatrix} 2 & \\ & 2 \end{pmatrix}(U_j^n + U_{j+1}^n) - \frac{1}{2}R|\Lambda|R^{-1}(U_{j+1}^n - U_j^n)$$

$$= (U_j^n + U_{j+1}^n) - (U_{j+1}^n - U_j^n)$$

$$= 2U_j^n$$

The Godunov scheme then is

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right),$$

where Δt satisfies the CFL condition

$$\Delta t \leq \frac{\Delta x}{2\lambda_{max}} = \frac{\Delta x}{4},$$

where λ_{max} is the spectral radius of A, which is 2 for both a) and b)