Exercise set 2

Numerical Methods for Hyperbolic Partial Differential Equations
IMATH, FS-2020
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Problem 2.1 Discontinuities (3pts)

Consider the Cauchy problem

$$\begin{aligned} u_t + f(u)_x &= 0 \qquad && \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) &= u_0(x) && \text{for } x \in \mathbb{R}. \end{aligned} \tag{1}$$

a) Let $f(u) = u^2/2$, ie. let (1) be the Burgers' equation. Assume u_0 is smooth and that u_0' is negative at some point. Show that the solution of (1) will generate a discontinuity at time

$$T_b = \frac{-1}{\min_x u_0'(x)}.$$

b) Generalize the result from a) with an arbitrary smooth convex flux f ($f''(x) > 0$ for all x).

Problem 2.2 Riemann problem (7pts)

Consider the Riemann problem for the scalar conservation law with discontinuous initial data:

$$u_t + f(u)_x = 0 \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{2}$$

$$u(x,0) = \begin{cases} u_l & \text{if } x < 0; \\ u_r & \text{if } x > 0 \end{cases} \tag{3}$$

with constants u_l and u_r.

a) Assume $u_l > u_r$. Using the *Rankine-Hugoniot Conditions*, derive the expression for *shock* solution of (6) with initial conditions (7). Prove that it is a weak solution.

b) If f is a convex function, show that if $u_l < u_r$ then there exist more than one weak solution.

c) Consider (6) with the flux $f(u) = u(1-u)/2$, and with the following initial conditions:

$$u_0(x) = \begin{cases} 0, & \text{if } x < 0 \\ 2, & \text{if } x > 0, \end{cases} \tag{4}$$

$$u_0(x) = \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } 0 < x < 1; \\ 1, & \text{if } x > 1. \end{cases} \tag{5}$$

Use the *Rankine-Hugoniot Conditions* to calculate the *shock* solutions for both of the above initial conditions.

d) Consider (6) with the flux $f(u) = u^2/2$ (Burgers' equation), and following initial conditions:

$$u_0(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Derive an expression of the entropy satisfying solution.

Problem 2.3 Entropy conditions and Lax entropy (5pts - optional)

Consider the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0 \qquad &\text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) &= u_0(x) \qquad &\text{for } x \in \mathbb{R}. \end{aligned} \tag{7}$$

a) Let $\Gamma = \{(\sigma(t), t) \mid t > 0\}$ be a smooth curve in $\mathbb{R} \times \mathbb{R}^+$ with $s(t) = \sigma'(t)$, and let u be the weak solution of (11), that is piecewise-smooth with discontinuities at Γ. Also assume $f, \eta, q \in C^2(\mathbb{R})$ and f and η are strictly convex and F is an entropy flux satisfying $q' = \eta' f'$. Prove that if u satisfies the entropy inequality,
$$\eta(u)_t + q(u)_x \leq 0$$
then, across the curve Γ, u satisfies
$$f'(u_l) > s(t) > f'(u_r).$$
Here $u_l \neq u_r$ are the traces of u from respectively the left and right sides of the curve Γ.

b) Use this result to show that the non-unique shock solutions derived in Problem 2.2b) are not entropy solutions.

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