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5. Iterative methods for the resolution of linear systems.

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- Iterative methods.
- Jacobi iterative method.
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5.1. Jacobi iterative method.

Iterative methods

Definition

In Numerical Methods, an *iterative method* is a mathematical procedure that uses an initial value to generate a sequence of improving approximate solutions for a class of problems, in which the n-th approximation is derived from the previous ones.

We are going to consider iterative methods for the following problem: finding $\overline{x}$ such that $F(\overline{x}) = b$, for $F : \mathbb{R}^n \to \mathbb{R}^n$ and $b \in \mathbb{R}^n$. An iterative method here is the construction of a function G and an initial value x_0 such that:

$$\lim_{k \to \infty} G^k(x_0) = \overline{x}.$$

Jacobi iterative method (i)

Let $A \in \mathbb{R}^{n\times n}$. We define the matrices $L_A, D_A, U_A \in \mathbb{R}^{n\times n}$ given by:

$$L_A(i,j) = \begin{cases} A(i,j), & i > j \\ 0 & i \leq j, \end{cases},$$

$$D_A(i,j) = \begin{cases} A(i,j), & i = j \\ 0 & i \neq j, \end{cases},$$

$$U_A(i,j) = \begin{cases} A(i,j), & i < j \\ 0 & i \geq j, \end{cases},$$

Then, $A = L_A + D_A + U_A$.

Remark

This has nothing to do with the LUP decomposition. Do not mistake different methods, even if they partially share notation.

Jacobi iterative method (ii)

We are going to solve $Ax = b$, for $b \in \mathbb{R}^n$ and A a non-singular matrix. This is equivalent to:

$$D_A x = b - L_A x - U_A x.$$

If A has a diagonal with non-null entries, this is equivalent to:

$$x = D_A^{-1}(b - L_A x - U_A x).$$

With this in mind, we may define Jacobi iterative method:

Definition

Let $A \in \mathbb{R}^{n \times n}$ a non-singular matrix with a diagonal with non-null entries and $b \in \mathbb{R}^n$. Then, Jacobi iterative method consists on defining a sequence given by $x_0 \in \mathbb{R}^n$ and

$$x_{k+1} = D_A^{-1}(b - L_A x_k - U_A x_k).$$

Jacobi method's convergence

Theorem (Jacobi Method's Convergence Theorem)

Let $(\mathbb{R}^n, \|\cdot\|)$ *be a normed space and let us consider* $\mathbb{R}^{n\times n}$ *with the matrix induced norm. Let* $A \in \mathbb{R}^{n\times n}$ *be a matrix with a diagonal with non-null entries,* $b \in \mathbb{R}^n$ *and* $\overline{x} = A^{-1}b$*. Then, if* x_k *is a sequence defined with the Jacobi method:*

$$\|x_k - \overline{x}\| \leq \|D_A^{-1}(L_A + U_A)\|^k \|x_0 - \overline{x}\|. \tag{†}$$

In particular, if $\|D_A^{-1}(L_A + U_A)\| < 1$*,* $x_k \to \overline{x}$*.*

Remark

When programming a computer to apply Jacobi's method, if $\|D_A^{-1}(L_A + U_A)\| < 1$, we can initiate in $x_0 = 0$.

Exercise

Prove that the hypothesis are satisfied if A is a matrix with a dominant diagonal.

Proof of Jacobi method's convergence

Clearly, it suffices to prove (†), as in that case, it is straightforward to see that $x_k \to \overline{x}$ if $\|D_A^{-1}(L_A + U_A)\| < 1$. The proof of (†) is done by induction. The base case, $k = 0$ is trivial. Let us supposed it is true for k and prove it for $k + 1$. Since $A\overline{x} = b$, then:

$$\overline{x} = D_A^{-1}(b - L_A\overline{x} - U_A\overline{x}).$$

Consequently, using that $\|\cdot\|$ is a matrix induced norm and the inductive hypothesis:

$$\begin{aligned}
\|x_{k+1} - \overline{x}\| &= \|D_A^{-1}(b - L_A x_k - U_A x_k) - D_A^{-1}(b - L_A\overline{x} - U_A\overline{x})\| \\
&= \|-D_A^{-1}(L_A + U_A)(x_k - \overline{x})\| \\
&\leq \|D_A^{-1}(L_A + U_A)\|\|x_k - \overline{x}\| \\
&\leq \|D_A^{-1}(L_A + U_A)\|^{k+1}\|x_0 - \overline{x}\|. \square
\end{aligned}$$

Jacobi and Frobenius norm

As we have seen in Exercise 3.10, the Frobenius norm is larger than the norm induced by the Euclidean norm. In particular, if for a given matrix A we have that $\|D_A^{-1}(L_A + U_A)\|_F < 1$, then Jacobi method converges to the solution of the linear system.

An example in which Jacobi method does not converge

Let us consider:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

and $b = 0$. Then, for all $x \in \mathbb{R}^2$:

$$D_A^{-1}(b - L_A - U_A)(x) = (-2x(2), -2x(1)).$$

Thus, if $x_0 = (1, 1)$, $x_k = ((-1)^k 2^k, (-1)^k 2^k)$, so the Jacobi method diverges in this specific example.

Remark

The non-convergence case illustrated in this example is very common whenever the terms of the diagonal are small in comparison with the other terms of the matrix. Obviously, the non-convergence may depend on the initial value (in particular, if we start on the solution, it converges).

Some remarks on Jacobi's method

- Unlike LUP or Cholesky, it is used to obtain approximations of the solutions of linear systems.
- Getting a good approximation can be with appropriate matrix faster than LUP or Cholesky, as the convergence is exponentially fast. This is true specially in high dimensions.