Exercises Numerical Method I 2023/24

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3 Matrix Analysis

Exercise 3.1. Define formally the addition of matrices and multiplication by a scalar in $\mathbb{R}^{n\times m}$.

Exercise 3.2. Let $n \geq 1$. Let us define the *identity matrix* $I \in \mathbb{R}^{n\times n}$ given by $I(i,i) = 1$ for all $i = 1, \dots n$, and $I(i,j) = 0$ if $i \neq j$. Show that,

- $AI = A$ for all $A \in \mathbb{R}^{m\times n}$.
- $IB = B$ for all $B \in \mathbb{R}^{n\times m}$.

Exercise 3.3. Show that the multiplication of matrices is associative, i.e., show that for all $A \in R^{n\times m}$, $B \in \mathbb{R}^{m\times \ell}$ and $C \in \mathbb{R}^{\ell\times k}$:

$$(AB)C = A(BC).$$

Exercise 3.4. Show that for all $A \in \mathbb{R}^{n\times m}$, and $B, C \in \mathbb{R}^{m\times \ell}$:

$$A(B + C) = AB + AC.$$

Exercise 3.5. Show that following properties are satisfied:

1. The identity matrix I is a permutation matrix.

2. Let $P, Q \in \mathbb{R}^{n\times n}$ be permutation matrices. Then $PQ \in \mathbb{R}^{n\times n}$ is a permutation matrix.

3. Let $P \in \mathbb{R}^{n\times n}$ be a permutation matrix. Then P' is a permutation matrix. Moreover, $PP' = P'P = I$; that is, P is invertible and $P' = P^{-1}$.

Exercise 3.6. Show that following properties are satisfied:

1. The identity matrix I is a diagonal matrix.

2. $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix if and only if it is upper triangular and lower triangular.

3. If $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix, L' is an upper triangular matrix.

4. If $U \in \mathbb{R}^{n\times n}$ is an upper triangular matrix, U' is a lower triangular matrix.

5. If $L_1, L_2 \in \mathbb{R}^{n\times n}$ are lower triangular matrices, so are $L_1 + L_2$ and L_1L_2.

6. If $U_1, U_2 \in \mathbb{R}^{n\times n}$ are upper triangular matrices, so are $U_1 + U_2$ and U_1U_2.

Exercise 3.7. Let $A \in \mathbb{R}^{n\times n}$ and Z be the matrix satisfying

$$Z(i,j) = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute ZAZ'.

Exercise 3.8. Let $\|\cdot\|$ be a matrix norm defined in $\mathbb{R}^{n\times n}$. Show that the following properties are satisfied:

1. $\|I\| \geq 1$, for I the identity matrix.

2. If $A \in \mathbb{R}^{n\times n}$ is a non-singular matrix, $\|A^{-1}\| \geq \|A\|^{-1}$.

3. If $A \in \mathbb{R}^{n\times n}$, $\|A^n\| \le \|A\|^n$ for all $n \in \mathbb{N}$.

Exercise 3.9. Show that for all $A \in \mathbb{R}^{n\times n}$:

$$\|A\|_F = \sqrt{\mathrm{tr}(AA')} = \sqrt{\mathrm{tr}(A'A)}.$$

Remember that for any matrix $B \in \mathbb{R}^{n\times n}$, we have that $\mathrm{tr}(B) = \sum_{i=1}^n B(i,i)$; that is, the trace of a matrix is the sum of the terms in the diagonal.

Exercise 3.10. Show that for all $A \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$:

$$\|Av\|_2 \le \|A\|_F \|v\|_2.$$

Moreover, give an example in which $\|\cdot\|_F$ is not the matrix norm induced by $\|\cdot\|_2$.

Exercise 3.11. Let us consider $(\mathbb{R}^n, \|\cdot\|_1)$. The objective of this exercise is to show that its induced matrix norm is given by:

$$F(A) = \max_{j=1,\dots,n} \|A(:,j)\|_1.$$

1. Let $A \in \mathbb{R}^{n\times n}$. Show that $\|Av\|_1 \le F(A)$ for all $v \in \mathbb{R}^n$ such that $\|v\|_1 = 1$.
2. Let $A \in \mathbb{R}^{n\times n}$ and j_A be such that $F(A) = \|A(:,j_A)\|_1$. Show that $\|Ae_{j_A}\|_1 = F(A)$. Here e_{j_A} is the vector such that $e_{j_A}(j_A) = 1$ and $e_{j_A}(j) = 0$ if $j \ne j_A$.
3. Show that its induced matrix norm is given by F.

Exercise 3.12. Let us consider $(\mathbb{R}^n, \|\cdot\|_\infty)$. The objective of this exercise is to show that its induced matrix norm is given by:

$$F(A) = \max_{i=1,\dots,n} \|A(i,:)\|_1$$

1. Let $A \in \mathbb{R}^{n\times n}$. Show that $\|Av\|_\infty \le F(A)$ for all $v \in \mathbb{R}^n$ such that $\|v\|_\infty = 1$.

2. Let $A \in \mathbb{R}^{n\times n}$ and i_A be such that $F(A) = \|A(i_A,:)\|_1$. Let us define v as follows: $v(j) = 1$ if $A(i_A, j) \geq 0$ and $v(j) = -1$ otherwise. Show that $\|Av\|_\infty = F(A)$.

3. Show that its induced matrix norm is given by F.

Exercise 3.13. Show that

$$F(A) = \sum_{i=1}^{n}\sum_{j=1}^{n} |A(i,j)|$$

is a matrix norm in $\mathbb{R}^{n\times n}$.

Exercise 3.14. Challenge. Let $f \in C^1(\mathbb{R},\mathbb{R})$. Then, show that for all $x \in \mathbb{R}$ the absolute condition number of f on x is given by $|f'(x)|$. In addition, if $f(x) \neq 0$, show that the relative condition number of f on x is given by $\frac{|xf'(x)|}{|f(x)|}$.

4 Direct methods for the resolution of linear systems

Exercise 4.1. Let $L \in \mathbb{R}^{n\times n}$ be a lower triangular matrix.

1. Show that $\det(L) = L(1,1)\cdots L(n,n)$.
2. Show that L is non-singular if and only if $L(i,i) \neq 0$ for all $i \neq 0$.

Exercise 4.2. Let $L_1, L_2 \in \mathbb{R}^{n\times n}$ be lower triangular matrices. Show that:

$$(L_1L_2)(i,i) = L_1(i,i)L_2(i,i), \quad i = 1,\dots,n.$$

Exercise 4.3. Let $L \in \mathbb{R}^{n\times n}$ be a lower non-singular triangular matrices.

1. Prove that L^{-1} is also a lower triangular matrix. With that purpose, let us consider $i,j \in \{1,\dots,n\}$ such that $j > i$. We have to show that $L^{-1}(i,j) = 0$. We recall that the inverse formula is given by:

   $$L^{-1}(i,j) = (\det L)^{-1}(-1)^{i+j}\det(L([1{:}j-1, j+1{:}n],[1{:}i-1, i+1{:}n])).$$

   In particular, we define $\tilde{L} = L([1{:}j-1, j+1{:}n],[1{:}i-1, i+1{:}n])$. Then,

   (a) Compute $\tilde{L}(a,b)$ for $a,b = 1,\dots,n$.
   Hint: you may want to consider separately the following four cases:
   - $a \leq j-1$, $b \leq i-1$,
   - $a \geq j$, $b \leq i-1$,
   - $a \leq j-1$, $b \geq i$,
   - $a \geq j$, $b \geq i$.

   (b) Show that $\tilde{L}$ is lower triangular.

   (c) Show that $\tilde{L}(i,i) = 0$.

   (d) Conclude that $L^{-1}(i,j) = 0$.

2. Show, with the Exercise 4.2, that $(L^{-1})(i,i) = \frac{1}{L(i,i)}$.

Exercise 4.4. State the analogous results of Exercises 4.1, 4.2 and 4.3 for upper triangular matrices. Prove them using that the adjoint matrix of an upper triangular matrix is a lower triangular matrix.

Exercise 4.5. Obtain the LUP decomposition of the following matrices:

1. $A_1 = \begin{pmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 6 \end{pmatrix}$,

2. $A_2 = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 5 & 3 \\ 2 & 4 & -6 \end{pmatrix}$,

3. $A_3 = \begin{pmatrix} -5 & 2 & -1 \\ 1 & 0 & 3 \\ 3 & 1 & 6 \end{pmatrix}$.

Exercise 4.6. Solve the following systems using the LUP decomposition obtained in Exercise 4.5:

1.
$$\begin{aligned} x(1) + x(2) + 4x(4) &= 5, \\ 2x(1) - x(2) + 5x(3) &= -6, \\ 5x(1) + 2x(2) + x(3) + 2x(4) &= 3, \\ -3x(1) + 2x(3) + 6x(4) &= 4. \end{aligned}$$

2.
$$\begin{aligned} x(1) + 3x(2) + 2x(3) &= 5, \\ x(1) + 5x(2) + 3x(3) &= 10, \\ 2x(1) + 4x(2) - 6x(3) &= -4. \end{aligned}$$

3.

$$\begin{aligned} -5x(1) + 2x(2) - x(3) &= 2, \\ x(1) + 3x(3) &= 2, \\ 3x(1) + x(2) + 6x(3) &= 2. \end{aligned}$$

Exercise 4.7. LU decomposition. Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Then, we are going to show that there is a LU decomposition in which all the terms of the diagonal of L are 1 if and only if $\det(A(1{:}i, 1{:}i)) \neq 0$ for all $i = 1, \ldots, n$.

1. Let us first show that if $\det(A(1{:}k, 1{:}k)) = 0$ for some k, then A does not admit a LU decomposition in which all the terms of the diagonal of L are 1. We are going to prove this result by contradiction; that is, assuming that such L and U matrices exist. In this item we use the notation $L_k = L(1{:}k, 1{:}k)$ and $U_k = U(1{:}k, 1{:}k)$:

   (a) Using that A is non-singular, show that $k < n$.

   (b) Show that:
   $$L_k U_k = A(1{:}k, 1{:}k). \tag{†}$$

   (c) Show that $\det(L_k) = 1$ and that $\det(L) = 1$.

   (d) Using (†), show that $\det(U_k) = 0$ and use this to obtain that $\det(U) = 0$.

   (e) Using $LU = A$, show that $\det(U) \neq 0$.

   (f) Obtain a contradiction.

2. Let us now show that if $\det(A(1{:}k, 1{:}k)) \neq 0$ for all $k = 1, \ldots, n$ there is a LU decomposition in which all the terms of the diagonal of L are 1. For that, we are going to adapt the proof of the LUP decomposition and show by induction that for all $k = 1, \ldots, n$ there are:

   - $L_k \in \mathbb{R}^{n\times n}$ a lower triangular matrix such that $L_k(i,i) = 1$ for all $i = 1, \ldots, k$.
   - $U_k \in \mathbb{R}^{n\times n}$ an upper triangular matrix such that $U_k(i,i) \neq 0$ for all $i = 1, \ldots, k$.

   such that:
   $$(L_k U_k)(:, 1{:}k) = A(:, 1{:}k). \tag{‡}$$

   With that purpose:

(a) Obtain U_1. Ensure that $U_1(1,1) \neq 0$.

(b) Obtain L_1 and prove that (‡) is satisfied for $k = 1$.

(c) From now on we are going to suppose that the result is satisfied for k and prove it for $k+1$ (here $k = 1, \dots, n-1$). Obtain U_{k+1} and prove that $U_{k+1}(i,i) \neq 0$ for all $i = 1, \dots, k$.

(d) Let $\tilde{L}_{k+1} \in \mathbb{R}^{(k+1)\times(k+1)}$ be given by:

$$\tilde{L}_{k+1}(i,j) = \begin{cases} L_k(i,j) & j \leq k, \\ 0 & i = 1, \dots, k; \quad j = k+1, \\ 1 & (i,j) = (k+1, k+1). \end{cases}$$

and $\tilde{U}_{k+1} = U(1{:}k+1, 1{:}k+1)$. Show that:

$$\tilde{L}_{k+1}\tilde{U}_{k+1} = A(1{:}k+1, 1{:}k+1).$$

(e) Prove that $U_{k+1}(k+1, k+1) \neq 0$.

(f) Obtain L_{k+1}. Show that $L_{k+1}(i,i) = 1$ for all $i = 1, \dots, k+1$

(g) Prove that (‡) is satisfied for $k+1$.

Exercise 4.8. Let $A \in \mathbb{R}^{n\times n}$ for $n \geq 2$ be a matrix such that $|A(i,i)| > \sum_{j\neq i} |A(i,j)|$ for all $i = 1, \dots, n$ (a matrix with a *strictly dominant diagonal*).

1. Show that $\det(A) \neq 0$. For that:

   (a) Prove that the result holds for $n = 2$.

   (b) Suppose that the result is true for n and prove it for $n+1$. For that, let B be given by:

   $$B(i,j) = \begin{cases} A(i,1) & i = 1, \dots, n+1, \\ A(i,j) - \frac{A(1,j)}{A(1,1)} A(i,1) & \text{otherwise.} \end{cases}$$

      i. Prove that $A(1,1) \neq 0$ (which, in particular, implies that B is well-defined).

      ii. Prove that $\det(A) = \det(B)$.

      iii. Prove that $B(1,j) = 0$ for $j = 2, \dots, n+1$.

iv. Prove that $\det(B) = A(1,1)\det(B(2{:}n+1, 2{:}n+1))$.

v. Prove that $B(2{:}n+1, 2{:}n+1)$ is a matrix of strictly dominant diagonal.

vi. Prove that $\det(B(2{:}n+1, 2{:}n+1)) \neq 0$.

vii. Prove that $\det(A) \neq 0$.

2. Prove that $\det(A(1{:}k, 1{:}k)) \neq 0$ for all $k = 1, \ldots, n$.

3. Using Exercise 4.7, show that A admits a LU decomposition in which $L(i,i) = 1$ for all $i = 1, \ldots, n$.

Exercise 4.9. Let $A \in \mathbb{R}^{n \times n}$ for $n \geq 2$ be a matrix such that $A(i,i) > \sum_{j \neq i} |A(i,j)|$ for all $i = 1, \ldots, n$ (a matrix with a *strictly dominant and positive diagonal*).

1. Show that $\det(A) > 0$. For that:

   (a) Prove that the result holds for $n = 2$.

   (b) Suppose that the result is true for n and prove it for $n+1$. For that, let B be given by:

$$B(i,j) = \begin{cases} A(i,1) & i = 1, \ldots, n+1, \\ A(i,j) - \frac{A(1,j)}{A(1,1)} A(i,1) & \text{otherwise.} \end{cases}$$

   i. Show that $A(1,1) > 0$ (which, in particular, implies that B is well-defined).

   ii. Prove that $\det(A) = \det(B)$.

   iii. Prove that $B(1,j) = 0$ for $j = 2, \ldots, n+1$.

   iv. Prove that $\det(B) = A(1,1)\det(B(2{:}n+1, 2{:}n+1))$.

   v. Prove that $B(2{:}n+1, 2{:}n+1)$ is a matrix of strictly dominant and positive diagonal.

   vi. Prove that $\det(B(2{:}n+1, 2{:}n+1)) > 0$.

   vii. Prove that $\det(A) > 0$.

2. Prove that $\det(A(1{:}k, 1{:}k)) > 0$ for all $k = 1, \ldots, n$.

3. Prove that if A is also symmetric, it admits a Cholesky decomposition.

Exercise 4.10. Show that the following matrices are positive definite and obtain their Cholesky decomposition:

1. $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$,

2. $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 13 & 23 \\ 4 & 23 & 77 \end{pmatrix}$,

3. $\begin{pmatrix} 1 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{pmatrix}$.

Exercise 4.11. Solve these systems using the Cholesky decomposition obtained in Exercise 4.10:

1.
$$\begin{aligned} x(1)+x(2)+x(3) &= 3, \\ x(1)+2x(2)+2x(3) &= 5, \\ x(1)+2x(2)+3x(3) &= 6. \end{aligned}$$

2.
$$\begin{aligned} x(1)+2x(2)+4x(3) &= 1, \\ 2x(1)+13x(2)+23x(3) &= 2, \\ 4x(1)+23x(2)+7x(3) &= 4. \end{aligned}$$

3.
$$\begin{aligned} x(1)-2x(2)-x(3) &= 3, \\ -2x(1)+5x(2)+2x(3) &= -7, \\ -x(1)+2x(2)+2x(3) &= -3. \end{aligned}$$

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5 Iterative methods for the resolution of linear systems

Exercise 5.1. Using Frobenius norm, show that we can obtain the solution of the following systems with the Jacobi method. In addition, compute the first two iterations, starting in $x_0 = 0$:

1.
$$\begin{aligned} 10x(1) + x(2) + x(3) &= 8, \\ x(1) - 5x(2) + x(3) &= 5, \\ 2x(1) - 2x(2) + 5x(3) &= -1. \end{aligned}$$

2.
$$\begin{aligned} 2x(1) + x(2) &= 1, \\ x(1) - 2x(2) &= 3. \end{aligned}$$

3.
$$\begin{aligned} 2x(1) - x(2) &= 1, \\ x(1) + 2x(2) &= 3. \end{aligned}$$

Exercise 5.2. Let $A \in \mathbb{R}^{n\times n}$ for $n \geq 2$ be a matrix such that $|A(i,i)| > \sum_{j\neq i} |A(i,j)|$ for all $i = 1, \dots, n$ (a matrix with a *strictly dominant diagonal*).

1. Show that $A(i,i) \neq 0$ for all $i = 1, \dots, n$.
2. Prove that if $\|\cdot\|$ is the induced matrix norm in $(\mathbb{R}^n, \|\cdot\|_\infty)$, then:
$$\|D_A^{-1}(L_A + U_A)\| < 1. \tag{†}$$
Hint: use the formula of the induced norm obtained in Exercise 3.12.
Remark: this result shows that the Jacobi method is convergent for matrices with a strictly dominant diagonal. Let us now use (†) to obtain A is non-singular, a result also proved in Exercise 4.8 but with a different method.

3. Show that if $Ax = 0$, then
$$x = -D_A^{-1}(L_A + U_A)x. \tag{‡}$$

4. Prove, using (†) and (‡), that if $Ax = 0$, then $x = 0$.

5. Explain why A is non-singular.

Exercise 5.3. Gauss-Seidel iterative method. In this exercise we are going to study the Gauss-Seidel iterative method. Let $A \in \mathbb{R}^n$ a non-singular matrix with a diagonal with non-null entries. For that we consider a normed space $(\mathbb{R}^n, \|\cdot\|)$ and we consider $\mathbb{R}^{n\times n}$ with its induced matrix norm. The Gauss-Seidel iterative method is very similar to the Jacobi method, but, the iterative formula is given by:

$$x_{k+1} = D_A^{-1}(b - L_A x_{k+1} - U_A x_k) \tag{†}$$

1. Write the equation satisfied by $x_{k+1}(i)$ for all $i = 1, \dots, n$. In particular, you must show that these $x_{k+1}(i)$ can be directly obtained with $x_k(j)$ for $j > i$ and with $x_{k+1}(j)$ for $j < i$.

2. Show that if x_{k+1} satisfies (†), then it satisfies:
$$x_{k+1} = (L_A + D_A)^{-1}(b - U_A x_k).$$

3. In this item we are going to prove that if $\overline{x} = A^{-1}b$ and if $(x_k)_{k\geq 0}$ is a sequence satisfying (†):
$$\|x_k - \overline{x}\| \leq \|(L_A + D_A)^{-1}U_A\|^k \|x_0 - \overline{x}\| \tag{‡}$$
For that:
   (a) Show that (‡) is satisfied for $k = 0$.
   (b) Prove that $\overline{x}$ satisfies:
   $$\overline{x} = (L_A + D_A)^{-1}(b - U_A\overline{x})$$
   (c) Prove that:
   $$x_{k+1} - \overline{x} = -(L_A + D_A)^{-1}U_A(x_k - \overline{x}).$$

(d) Prove that:

$$\|x_{k+1} - \overline{x}\| \leq \|(L_A + D_A)^{-1} U_A\| \|x_k - \overline{x}\|$$

(e) Conclude with an inductive argument.

4. State a sufficient condition for the convergence of the Gauss-Seidel method.

5. Obtain the first iteration with the Gauss-Seidel method of Item 1 of Exercise 5.1 starting in $x_0 = 0$.

6. **Challenge:** show that if $A \in \mathbb{R}^{n\times n}$ for $n \geq 2$ be a matrix such that $|A(i,i)| > \sum_{j\neq i} |A(i,j)|$ for all $i = 1, \ldots, n$ (a matrix with a *strictly dominant diagonal*), then the Gauss-Seidel method is convergence.

6 Least Square Problem

Exercise 6.1. Minimize, using normal equations, the following functional:
$J(x) = \|Ax - b\|_2^2$, for

1. $$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

2. $$A = \begin{pmatrix} -1 & 1 \\ 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 5 \\ 20 \end{pmatrix}$$

3. $$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix}.$$

In particular, give the vector x which minimizes J, and also provide the value $J(x)$.

Exercise 6.2. Optional exercise for which Matlab is required. Minimize, using the QU decomposition with Householder reflections, the following functional $J(x) = \|Ax-b\|_2^2$:

1. $$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

2. $$A = \begin{pmatrix} -1 & 1 \\ 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 5 \\ 20 \end{pmatrix}$$

3.
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix}.$$

In particular, give the vector x which minimizes J, and also provide the value $J(x)$.

Exercise 6.3. Linear interpolation. Let $n \geq 2$. Let us consider the problem of linear interpolation. In particular, let us consider the dataset $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ for $x_1, \ldots, x_n$ different real numbers and $y_1, \ldots, y_n$ any real numbers. Let us consider the set of functions $f_a(x) = a_0 + a_1 x$. We want to find the value a where the following minimum is obtained:
$$\min_{a \in \mathbb{R}^2} \sum_{i=1}^{n} |f_a(x_i) - y_i|^2 \tag{†}$$
For that:

1. Show that the problem (†) is equivalent to:
$$\min_{a \in \mathbb{R}^2} \left\| \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\|_2^2 \tag{‡}$$

2. Solve (‡) using normal equations (it suffices to find the values a_0 and a_1).

Exercise 6.4. Uniqueness of QU decomposition.

1. Let $Q_1, Q_2 \in \mathbb{R}^{n \times n}$ be orthogonal matrices and $U_1, U_2 \in \mathbb{R}^{n \times n}$ be upper triangular matrices satisfying
$$U_1(i,i) > 0, \quad U_2(i,i) > 0, \quad \forall i = 1, \ldots, n$$
and such that:
$$Q_1 U_1 = Q_2 U_2.$$
Then,

(a) Show that:

$$Q_2' Q_1 = U_2 U_1^{-1}$$

(b) Let $Q \in \mathbb{R}^{n\times n}$ be an upper triangular orthogonal matrix. Show that Q is a diagonal matrix whose entries are 1 are -1. **Hint:** use that $Q^{-1} = Q'$ to show that Q^{-1} is, at the same time, upper and lower triangular.

(c) Show that:

$$U_2 U_1^{-1} = I_n.$$

(d) Show that:

$$U_2 = U_1.$$

(e) Show that:

$$Q_1 = Q_2.$$

2. Let us consider $A = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $U = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Obtain all the orthogonal matrices Q such that $QU = A$.

3. **Challenge.** Let $A \in \mathbb{R}^{n\times m}$ with $n > m$ be a matrix with linearly independent columns. Using any method that provides the QU decomposition for non-singular square matrices in $\mathbb{R}^{n\times m}$, show that the QU decomposition is not unique even if we add the restriction that $U(i,i) > 0$ for all $i = 1, \ldots, m$.

Exercise 6.5. QU decomposition and Gram-Schmidt. Let $A \in \mathbb{R}^{n\times m}$ with $n \geq m$ a matrix with linearly independent columns. In this exercise we are going to use the Gram-Schmidt orthonormalization algorithm to obtain a QU decomposition of A.

1. Let us consider the vectors given by:

$$\begin{cases} u_1 = A(:,1), & v_1 = \frac{v_1}{\|v_1\|_2}, \\ u_2 = A(:,2) - (A(:,2)\cdot v_1)v_1, & v_2 = \frac{u_2}{\|u_2\|_2}, \\ \qquad\vdots & \qquad\vdots \\ u_m = A(:,m) - \sum_{j=1}^{m-1}(A(:,m)\cdot v_j)v_j, & v_m = \frac{u_m}{\|u_m\|_2}. \end{cases} \tag{$*$}$$

Prove that $u_i \neq 0$ for all $i = 1, \ldots, m$.

2. Show, by induction on m, that $\{v_1, \dots, v_m\}$ is an orthonormal set in $\mathbb{R}^n$.

3. Show that:
$$A(:,j) = \sum_{k=1}^{j} (A(:,j) \cdot v_k) v_k \tag{†}$$
Hint: show that $\|v_j\|_2 = A(:,j) \cdot v_j$ by multiplying the expression in $(*)$ by v_j. Alternatively, you may use properties of the orthonormal set proved in the Lineal Algebra course.

4. Provide an algorithm to obtain some vectors $\{v_{m+1}, \dots, v_n\}$ so that $\{v_1, \dots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$. **Hint:** you may use Gram-Schmidt and vectors from the canonical basis.

5. Let us consider Q the matrix defined by $Q(i,j) = v_j(i)$; that is, the matrix whose columns are given by the elements of $\{v_1, \dots, v_n\}$. Show that Q is orthogonal.

6. Prove that for any matrix $B_1 \in \mathbb{R}^{n \times m}$ and $B_2 \in \mathbb{R}^{m \times \ell}$, we have that:
$$(B_1 B_2)(:,j) = \sum_{k=1}^{m} B_2(k,j) B_1(:,k). \tag{‡}$$

7. Let us consider U the matrix given by
$$U(i,j) = \begin{cases} A(:,j) \cdot v_i & i \le j, \\ 0 & i > j \end{cases}$$
Prove, using (†) and (‡), that $QU = A$.

Exercise 6.6. Challenge. QU decomposition has been obtained with reflections in a plane, but it can also be obtained with rotations, as rotation matrices are orthogonal matrix. Develop and analogue theory using rotations instead of reflections, and provide several examples.

7 Time complexity of Numerical Methods

Exercise 7.1. Let a_n, b_n be essentially positive sequences and c_n be any sequence. Prove that:

1. $c_n = \mathcal{O}(|c_n|)$.
2. $\mathcal{O}(a_n) = \mathcal{O}(\lambda a_n), \quad \forall \lambda > 0$.
3. If $c_n = \mathcal{O}(b_n)$ and $b_n = \mathcal{O}(a_n)$, then $c_n = \mathcal{O}(a_n)$.

Exercise 7.2. Prove that the Θ notation is an equivalence relation in the set of essentially positive sequences.

Exercise 7.3. Let a_n, b_n be essentially positive sequences. Prove that:

1. $\Theta(a_n)\Theta(b_n) = \Theta(a_n b_n)$
2. $\Theta(a_n) + \Theta(b_n) = \Theta(a_n + b_n)$

Exercise 7.4. Are the following statements true or false? Why?

1. If $a_n = \mathcal{O}(n)$, then $\lim_{n\to\infty} \frac{a_n}{n^2} = 0$.
2. Let a_n, b_n be essentially positive sequences. Then, either $a_n = \mathcal{O}(b_n)$ or $b_n = \mathcal{O}(a_n)$.
3. Let a_n be an essentially positive sequence. Then, $2a_n = \Theta(a_n)$.
4. Let a_n, b_n be essentially positive sequences. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $b_n = \Theta(a_n)$.
5. Let a_n, b_n be essentially positive sequences. If $b_n = \mathcal{O}(a_n)$, then $e^{b_n} = \mathcal{O}(e^{a_n})$.

Exercise 7.5. Let $A \in \mathbb{R}^{n\times n}$. Let us consider as elementary step the following ones:

- Consulting an entry of A.
- Obtain the maximum of two elements.
- Additions.
- Taking the absolute value.

Show that the computational complexity of obtaining the matrix norm induced by the supreme norm is $\Theta(n^2)$. For that:

1. Using the equivalence in Exercise 3.12, show that it can be done in $\mathcal{O}(n^2)$ steps.
2. Get a contradiction if for any algorithm that gets the norm of any matrix does not consult all the entries of all the matrices.
3. Conclude.

Exercise 7.6. Let $L, U \in \mathbb{R}^{n\times n}$ be lower and upper non-singular triangular matrices respectively such that $L(i,i) = 1$ for all $i = 1, \dots, n$, and $b \in \mathbb{R}^n$. Let us consider as elementary steps consulting a term, additions, multiplications and divisions of real numbers.

1. Show that solving the systems $LUx = b$ requires $\mathcal{O}(n^2)$ elementary steps. For that:
   (a) Show that solving the systems $Ly = b$ requires $\mathcal{O}(n^2)$ elementary steps.
   (b) Show that solving the systems $Ux = y$ requires $\mathcal{O}(n^2)$ elementary steps.
   (c) Conclude.
2. Let us now provide a lower bound for a number of steps required for an algorithm to solve $LUx = b$. For that:
   (a) Compute the number entries of L that are not given by the restrictions.
   (b) Let us consider $L = I$, $U = I$ and $b = (1, \dots, 1)'$. Obtain the solution of $LUx = b$.
   (c) Let us suppose that we replace an entry of L that is not given by the restrictions by 1. Obtain the solution of $LUx = b$.

(d) Get a contradiction if an algorithm does not consult all the entries of L that are not obtained by the restrictions.

(e) Get a lower bound for a number of steps required for an algorithm to solve $LUx = b$.

3. Show that solving the systems $LUx = b$ requires $\Theta(n^2)$ elementary steps.

Exercise 7.7. Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Let us consider as elementary steps additions, subtractions, multiplications, divisions, computing the absolute value and taking the maximum of two numbers. Let us consider that multiplying by a permutation matrix is $\mathcal{O}(n^2)$, as explained throughout the course.

1. Show that there is $C > 0$ such that the obtention of P_1, P_1A, U_1, and L_1, and U_1 requires al most Cn^2 elementary steps.
2. Show that there is $C > 0$ independent of n or k such that the obtention of U_{k+1}, P_{k+1}, $P_{k+1}A$ and L_{k+1} is at most Cn^2. For that, we show that we already know: U_k, P_k, P_kA and L_k. To do this:
   (a) Get a bound for the number of steps necessary to compute $\tilde{U}_{k+1}$.
   (b) Get a bound for the number of steps necessary to compute $\tilde{P}_{k+1}$.
   (c) Get a bound for the number of steps necessary to compute P_{k+1}, $P_{k+1}A$ and U_{k+1}.
   (d) Get a bound for the number of steps necessary to compute Ł$_{k+1}$.
   (e) Conclude.
3. Conclude that the LUP decomposition requires at most $\mathcal{O}(n^3)$ elementary steps.

Exercise 7.8. Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix.

1. Let us suppose that we compute the determinant of A with the Leibniz formula seen in your Linear Algebra course; that is, with

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}.$$

   How many multiplications are required if we are applying blindly that formula?

2. Let us suppose that obtaining the LUP decomposition of A requires $\mathcal{O}(n^3)$ steps (as shown in Exercise 7.7). Provide an algorithm that computes the determinant of A in $\mathcal{O}(n^3)$ steps (with an upper bound in the number of steps that belongs to $\mathcal{O}(n^3)$). The elementary steps are the same ones as in Exercise 7.7.