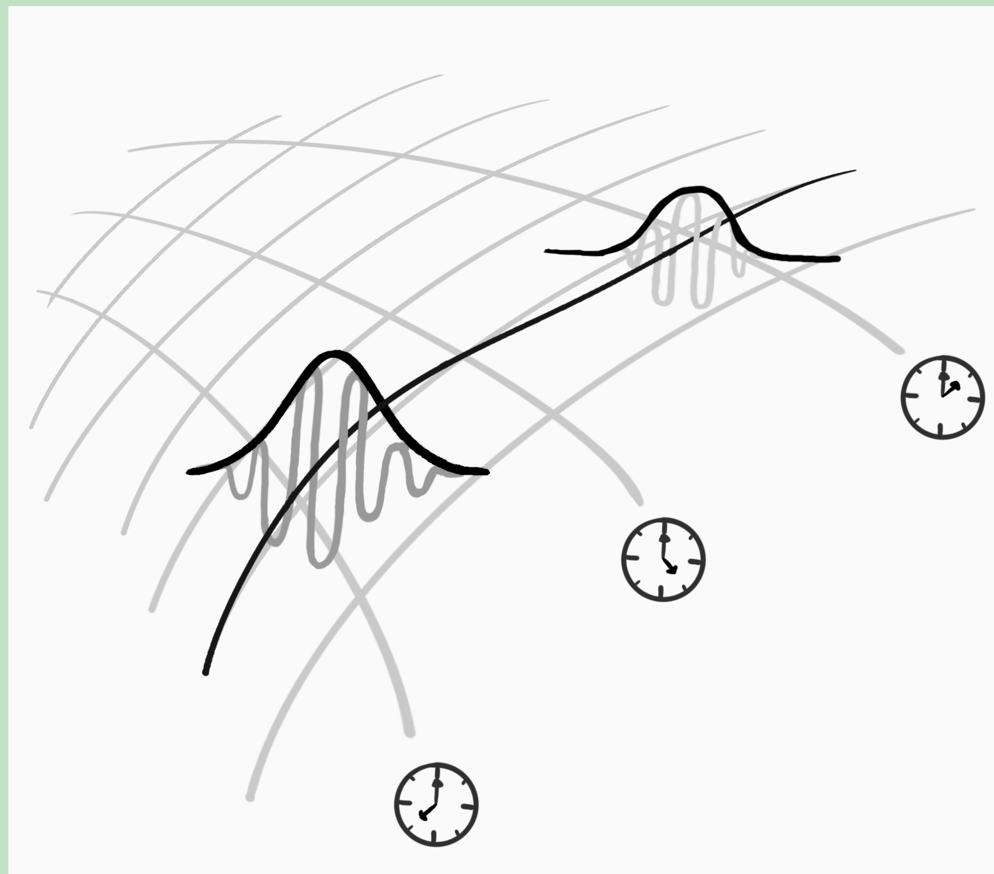
Geometry and Proper Time of a Relativistic Quantum Clock

Joseph Balsells with Martin Bojowald

The Pennsylvania State University

Institute for Gravitation and the Cosmos

In Einstein's theory of gravity, the proper time elapsed on a clock is identified with the geometric path length traversed by the clock in a metric geometry. When quantum effects become relevant, but not dominant, the classical theory of motion can be corrected by including quantum perturbations. We present a quasi-classical description of the relativistic quantum particle which retains the same formal structure as the classical theory. That is, we describe how the non-classical degrees of freedom of a quantum state evolve in an extension of the classical metric geometry.



Step 1: Hamiltonization of Geodesics

In classical mechanics, particle dynamics are determined by an action principle applied to an appropriate functional. In relativity theory, a suitable action is the arc length of worldlines, called the proper time $S[x^a, \dot{x}^a] = -mc\int_0^1 \sqrt{-g_{ab}(x)\dot{x}^a\dot{x}^b}\,d\tau$

We prefer to remove the square root by working with the equivalent einbein action

$$S\left[x^a, e, \dot{x}^a, \dot{e}\right] = -\int_0^1 \left(\frac{g_{ab}\dot{x}^a\dot{x}^b}{2e} - \frac{em^2c^2}{2}\right) d\tau.$$

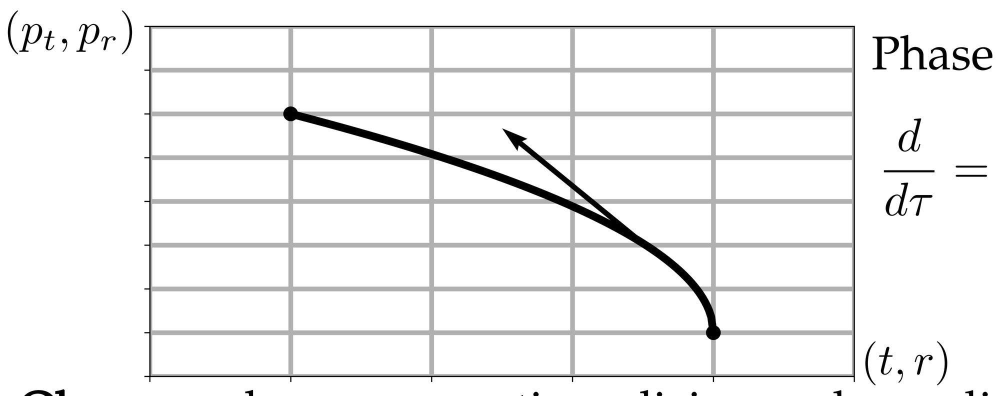
The Lagrangian is singular and may be converted to a constrained Hamiltonian action

$$S\left[x^a, e, p_a, p_e, \lambda\right] = \int_0^1 d\tau \left[p_a\dot{x}^a + p_e\dot{e} - H - \lambda\Phi\right].$$

Eliminate gauge freedom of the einbein field to obtain the proper time Hamiltonian

$$H = -\frac{1}{2m}\left(g^{ab}(x)p_ap_b + m^2c^2\right) \approx \frac{p^2}{2m}.$$

Generator of time translation with respect to proper time: $\frac{d}{d\tau} = \{\cdot, H\}$



Phase portrait tangent vector

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{\partial}{\partial t} + \frac{dr}{d\tau}\frac{\partial}{\partial r} + \frac{dp_t}{d\tau}\frac{\partial}{\partial p_t} + \frac{dp_r}{d\tau}\frac{\partial}{\partial p_r}$$

Observer chooses spacetime slicing and coordinates $x^a_{\text{obs}} = (\tau_{\text{obs}}, \boldsymbol{x}_{\text{obs}})$

Clock frame coordinates $x^a_{\text{clock}} = (\tau_{\text{clock}}, \boldsymbol{x}_{\text{clock}})$

Time dilation from the 0-component of the tangent vector.

$$\frac{d\tau_{\text{obs}}}{d\tau_{\text{clock}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{d\tau_{\text{obs}}}{d\tau_{\text{clock}}} = \frac{1}{\sqrt{1-\frac{r_s}{r}}}$$

$$\frac{d\tau_{\text{obs}}}{d\tau_{\text{clock}}} = \frac{1}{\sqrt{1-\frac{r_s}{r} - \frac{1}{1-\frac{r_s}{r}}\left(\frac{dr}{d\tau}\right)^2}}$$

Doppler and gravitational time dilation combined in general

$$\frac{d\tau_{\text{obs}}}{d\tau_{\text{clock}}} = \frac{1}{\sqrt{-g_{ab}\frac{dx^a_{\text{obs}}}{d\tau_{\text{obs}}}\frac{dx^b_{\text{obs}}}{d\tau_{\text{obs}}}}}$$

Step 2: Canonical quantization

Having established gauge conditions in Step 1, the problem of canonical quantization of the gauge theory is reduced to quantization of a non-constrained Hamiltonian theory. We formally obtain an operator algebra and Hamiltonian

$$[\hat{x}^a, \hat{p}_b] = i\hbar\delta^a_b \qquad \hat{H} = -\frac{1}{2m}\left(g^{ab}(\hat{x})\hat{p}_a\hat{p}_b + m^2c^2\right) + \text{ h.c.}$$

$$\hat{p}_b \mapsto \frac{\hbar}{i}\partial_b$$

This algebra may be represented on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$.

Reproduces the single-particle sector of quantum field theory [1].

Step 3: Geometrization

The inner product $\langle\cdot,\cdot\rangle$ defines a Kähler structure on $\mathcal{H}$ [2]. For any $\Phi, \Psi \in \mathcal{H}$ the inner product is decomposed into real and imaginary parts

$$\langle\Phi,\Psi\rangle = \frac{1}{2\hbar}G(\Phi,\Psi) + \frac{i}{2\hbar}\Omega(\Phi,\Psi).$$

Ω is a symplectic structure on $\mathcal{H}$. The symplectic structure allows to define **Hamiltonian vector fields**. The quantum state is a functional on the operator algebra. Any operator $\hat{F}$ on $\mathcal{H}$ defines a function on $\mathcal{H}$ by $F(\Psi) = \langle\Psi, \hat{F}\Psi\rangle$. The Hamiltonian vector field of F is

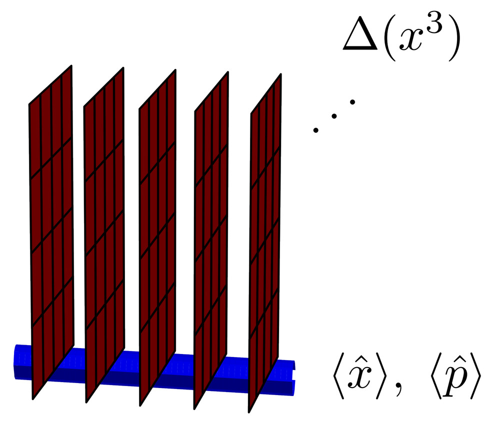
$$X_F(\Psi) = \frac{1}{i\hbar}\hat{F}\Psi.$$

Poisson structure. For two functions $g_{\hat{A}} = \langle\Psi, \hat{A}\Psi\rangle$ and $g_{\hat{B}} = \langle\Psi, \hat{B}\Psi\rangle$, the symplectic structure defines the Poisson bracket

$$\{g_{\hat{A}}, g_{\hat{B}}\} = \Omega(X_A, X_B) = \frac{1}{i\hbar}\langle[\hat{A},\hat{B}]\rangle.$$

The fundamental operators $(\hat{x}^a, \hat{p}_a)$ on $\mathcal{H}$ define a fiber bundle structure on $\mathcal{H}$. Fiber coordinates parametrize the quantum state by its correlation and fluctuation statistics

$$\begin{array}{ccccccc} & & \Delta(x^2) & \Delta(xp) & \Delta(p^2) & & \\ & \Delta(x^3) & \Delta(x^2p) & \Delta(xp^2) & \Delta(p^3) & & \\ \iddots & & & \vdots & & & \ddots \end{array}$$



Projection identifies all states, for which $\langle\Psi, \hat{x}\Psi\rangle = \langle\Phi, \hat{x}\Phi\rangle$, $\langle\Psi, \hat{p}\Psi\rangle = \langle\Phi, \hat{p}\Phi\rangle$

Dynamics from $H_Q(x, p, \Delta(x^a p_b)) \equiv \left\langle \hat{H}(x + (\hat{x} - x), p + (\hat{p} - p))\right\rangle$

$$= \sum_{c=0}^{\infty}\sum_{d=0}^{\infty} \frac{1}{c!d!}\frac{\partial^{c+d}H(x,p)}{\partial^c x^a \partial^d p_b}\,\Delta(x^a p_b).$$

Canonical variables s and p_s with $\{s, p_s\} = 1$ by the nonlinear mapping:

$$\Delta\left(x^2\right) = s^2 \qquad \Delta(xp) = sp_s \qquad \Delta\left(p^2\right) = p_s^2 + \frac{U}{s^2}.$$

Results: Quantum-perturbed Geodesics

Quantization of the radial degree of freedom to second order results in a quadratic effective quantum Hamiltonian

$$H_{\mathcal{Q}}(t, r, s, q_\hbar, p_t, p_r, p_s, \hbar/2) = -\frac{1}{2m}\left(g^{\bar{a}\bar{b}}_{\mathcal{Q}}p_{\bar{a}}p_{\bar{b}} + m^2c^2\right)$$

Quantum effective metric

$$\mathrm{g}^{\bar{a}\bar{b}}_{\mathcal{Q}} := \begin{pmatrix} g^{tt} + \frac{s^2}{2}\partial_r^2 g^{tt} & g^{tr} + \frac{s^2}{2}\partial_r^2 g^{tr} & s\partial_r g^{tr} & 0 \\ g^{tr} + \frac{s^2}{2}\partial_r^2 g^{tr} & g^{rr} + \frac{s^2}{2}\partial_r^2 g^{rr} & s\partial_r g^{rr} & 0 \\ s\partial_r g^{tr} & s\partial_r g^{rr} & g^{rr} & 0 \\ 0 & 0 & 0 & \frac{g^{rr}}{s^2} \end{pmatrix}.$$

Quantum proper time as arc-length in quantum extended metric geometry

$$\tau_{\mathcal{Q}} := \int \sqrt{-g^{\mathcal{Q}}_{\bar{a}\bar{b}}dx^{\bar{a}}dx^{\bar{b}}}$$

Corrections to gravitational time-dilation in Schwarzschild spacetime. The time shown by a clock infalling from finite position relative to the geodesic coordinate clock at infinity is

$$\frac{d\tau_{\text{obs}}}{d\tau_{\text{clock}}} = \gamma_c + \gamma_q \qquad \gamma_c = \frac{1}{\sqrt{1-\frac{r_s}{r}}} \qquad \gamma_q = \frac{1}{2}\frac{s^2}{r^2}\frac{\frac{r_s}{r}}{\left(1-\frac{r_s}{r}\right)^{5/2}} + \frac{1}{2}\left(\frac{\hbar/2}{mc\cdot s}\right)^2\sqrt{1-\frac{r_s}{r}}$$

These corrections scale with the ratio of the clock's Compton wavelength to its wave packet's spatial extent in agreement with the quantum time dilation effect obtained by Smith and Ahmadi [3].

References

[1] D. M. Gitman and I. V. Tyutin, Quantization of Fields with Constraints, Springer Series in Nuclear and Particle Physics (Springer Berlin, Heidelberg, 1990), ISBN 978-3-642-83938-2.
[2] M. Bojowald and A. Skirzewski, Rev. Math. Phys. 18, 713 (2006).
[3] A. R. H. Smith and M. Ahmadi, Nature Communications 11, 5360 (2020), ISSN 2041-1723

National Science Foundation

PennState Eberly College of Science