CSE 101- Winter '18
Discussion Section
Week 2

Topics

- Topological ordering
- Strongly connected components
- Binary search
- Introduction to Divide and Conquer algorithms

Topological ordering

- Given a directed graph $G = (V,E)$ with $|V|=n$, assign labels $1,\ldots,n$ to $v_i \in V$ s.t. if v has label k, all vertices reachable from v have labels $> k$

Topological ordering

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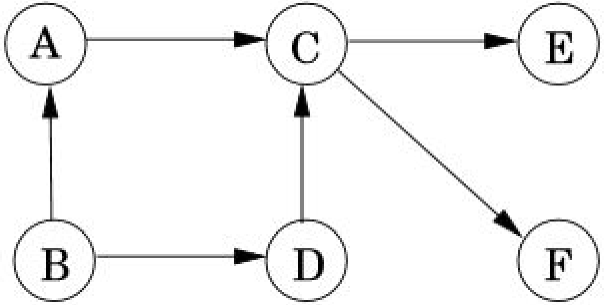
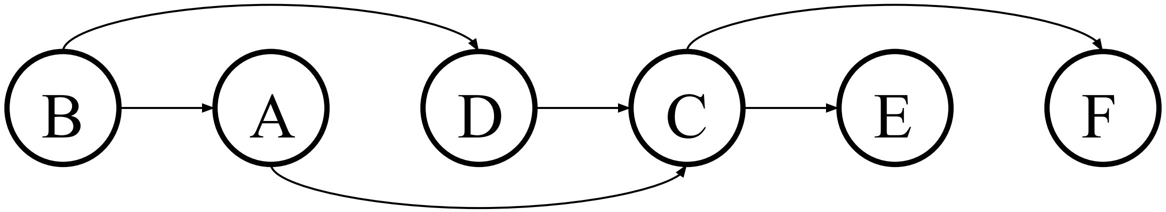
- Pictorially



=>

(only forward edges if vertices arranged in increasing order of labels)



Topological ordering

- If G has a directed cycle => no topological ordering
  - Why?

- **Theorem** – Every directed graph without a directed cycle (DAG) has a topological ordering

- Revised problem: Given a directed **acyclic** graph G = (V,E) with |V|=n, assign labels 1,...,n to $v_i \in V$ s.t. if v has label k, all vertices reachable from v have labels > k

Topological ordering – inductive approach

- How would you approach this problem?

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  - Find a vertex which you know can be labelled 1
  - What are the properties of such a “starting” vertex?

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  - What are the properties of such a “starting” vertex?

- Claim: A DAG G always has some vertex with indegree = 0
  - Take an arbitrary vertex v. If v doesn’t have indegree = 0, traverse any incoming edge to reach a predecessor of v. If this vertex doesn’t have indegree = 0, traverse any incoming edge to reach a predecessor, etc.
  - Eventually, this process will either Identify a vertex with indegree = 0, or else reach a vertex that has been reached previously (a contradiction, given that G is acyclic)

Topological ordering – inductive approach

- How would you approach this problem?
  - Find a vertex which you know can be labelled 1
  - What are the properties of such a “starting” vertex?
- Inductive (or recursive) approach
  - Find a vertex v with indegree(v) = 0; give it lowest available label;
  - Delete v (and incident edges, update degrees of remaining edges)
  - Repeat
- (Bonus) – can you do the same, beginning with an “ending” vertex?

Topological ordering – using DFS

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- Why?

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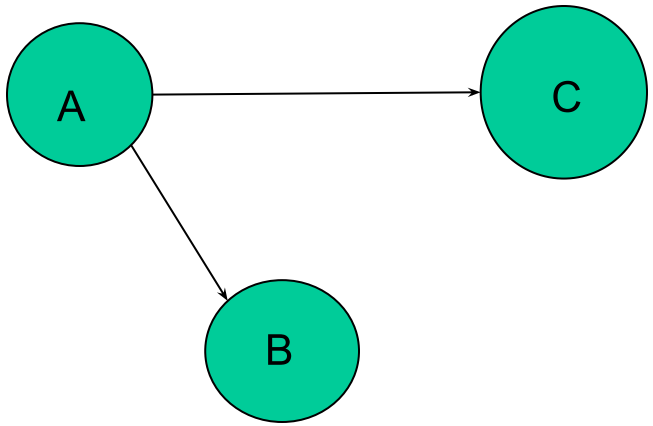
- Obvious solution:
  - Run DFS, then perform tasks in decreasing order of post numbers
  - Linear running time!

Topological ordering – example



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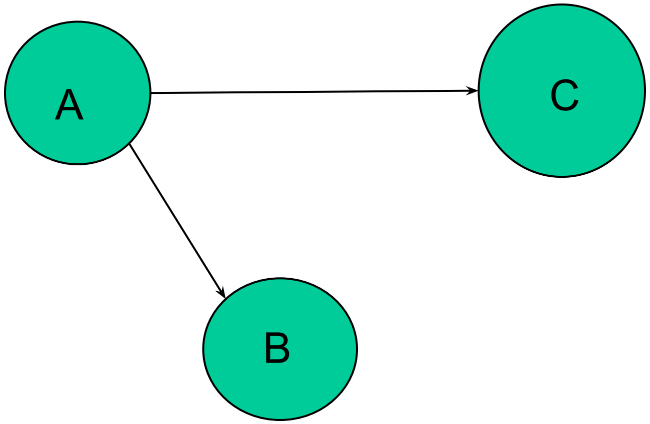

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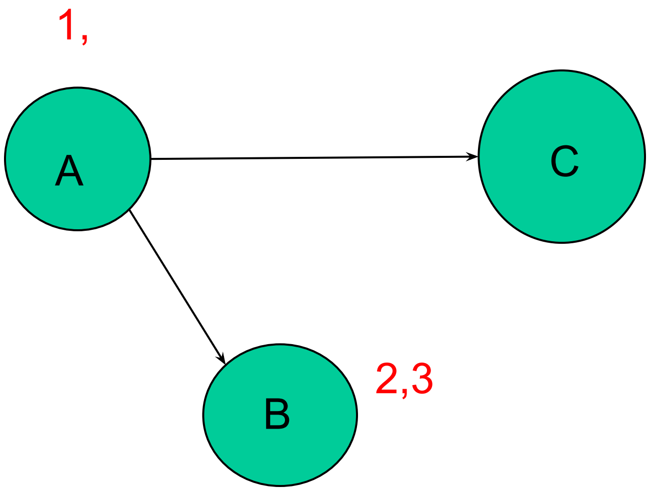

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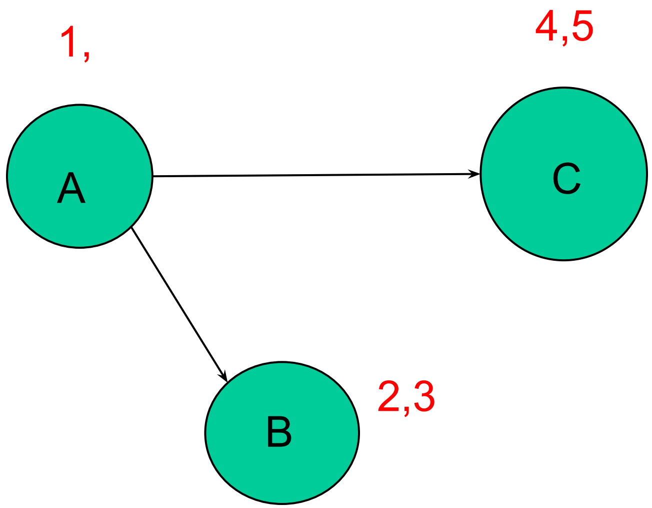

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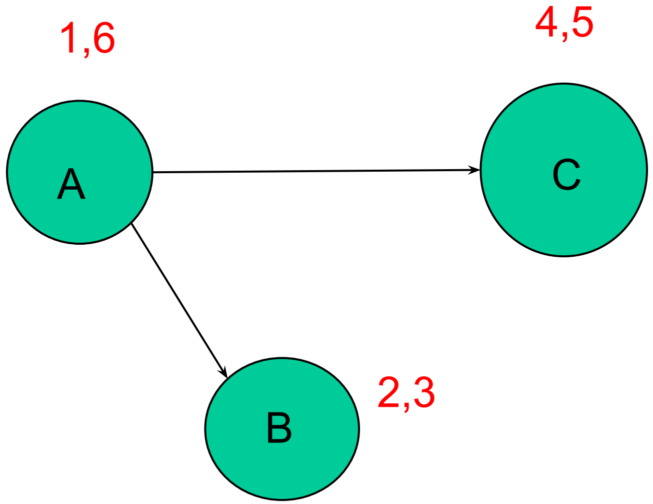

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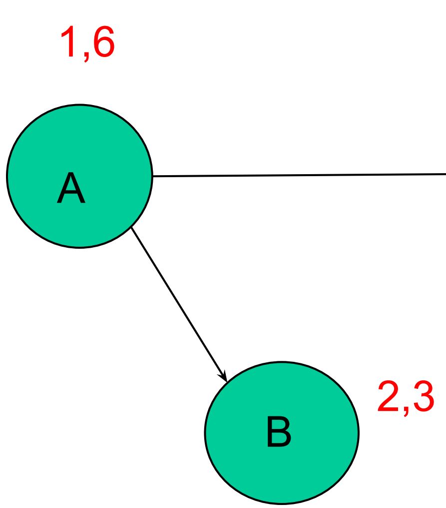
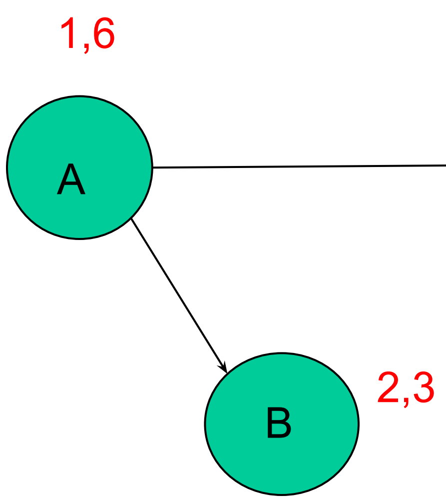


Vertices explored in alphabetical order

Topological order: A-C-B

Topological ordering – example



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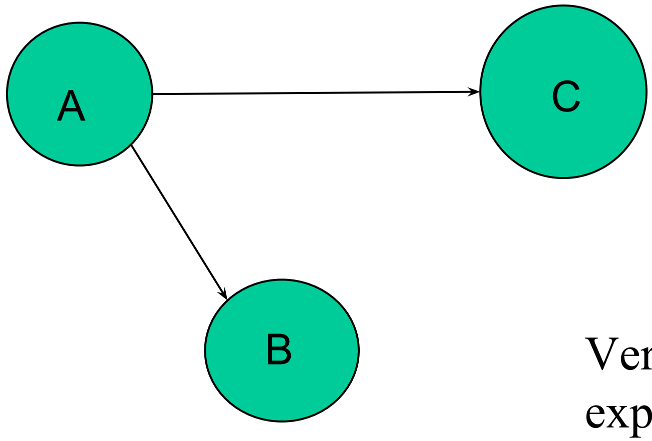

Vertices explored in order: B-C-A

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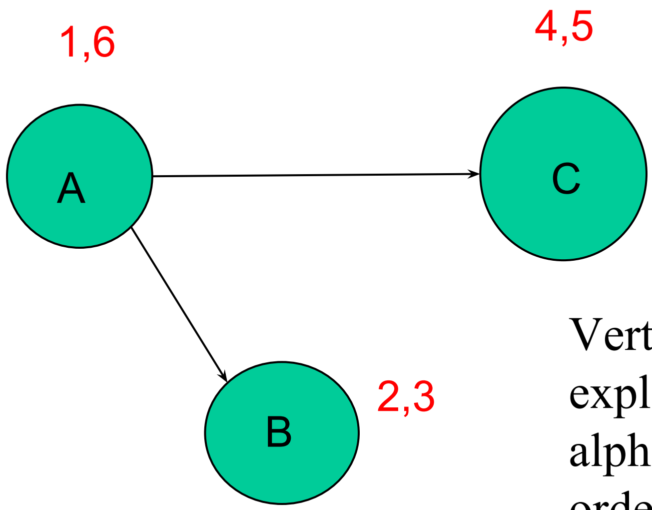

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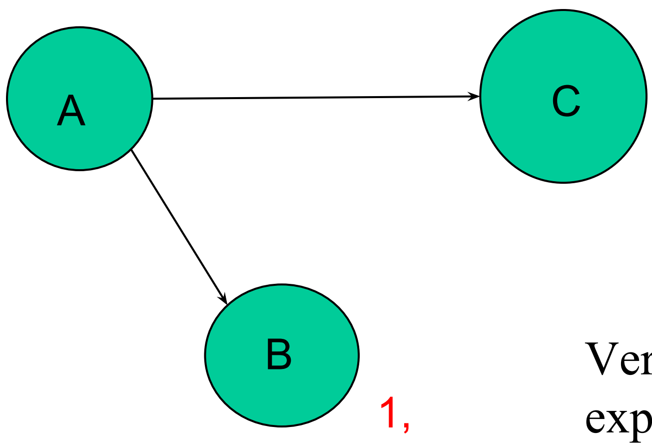

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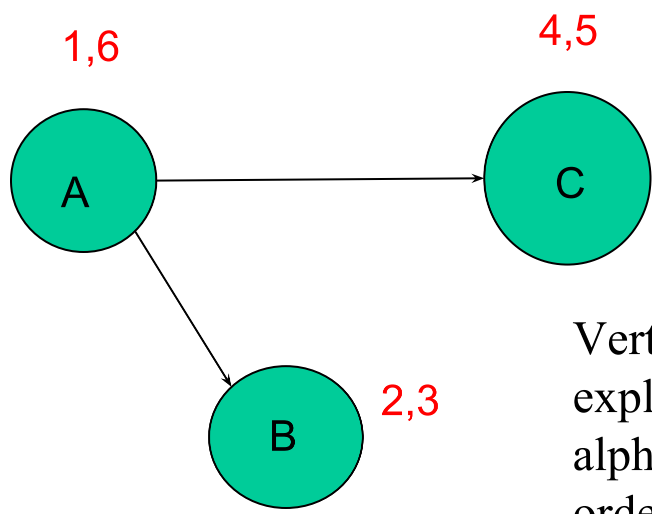

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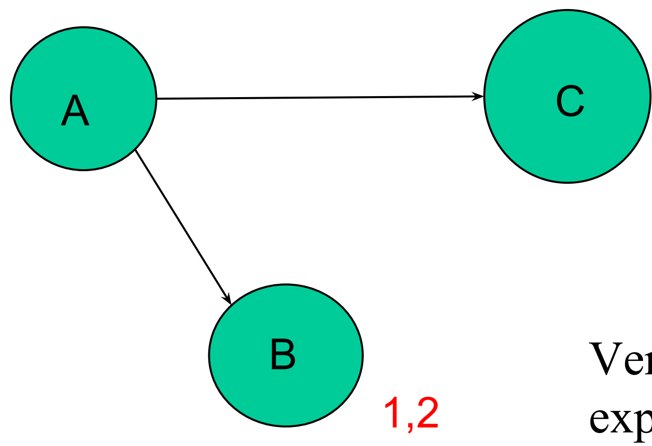

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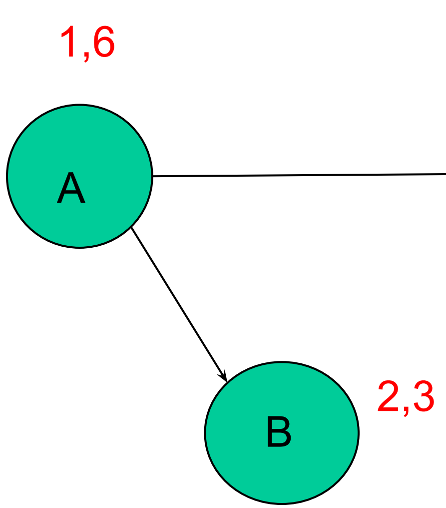

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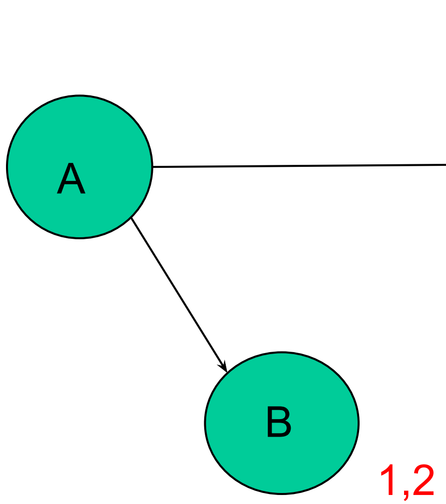

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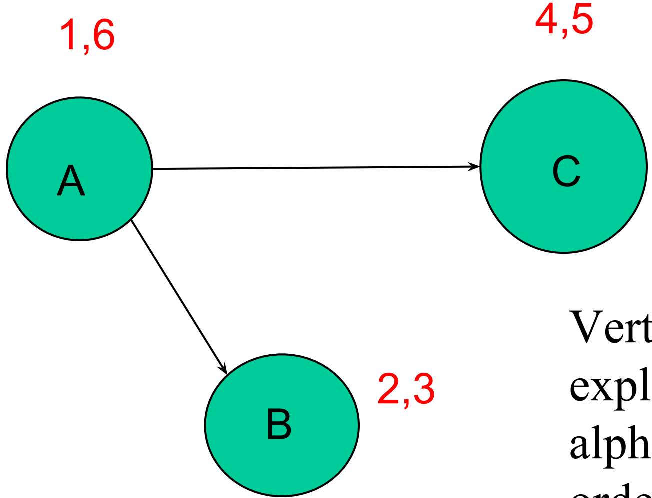

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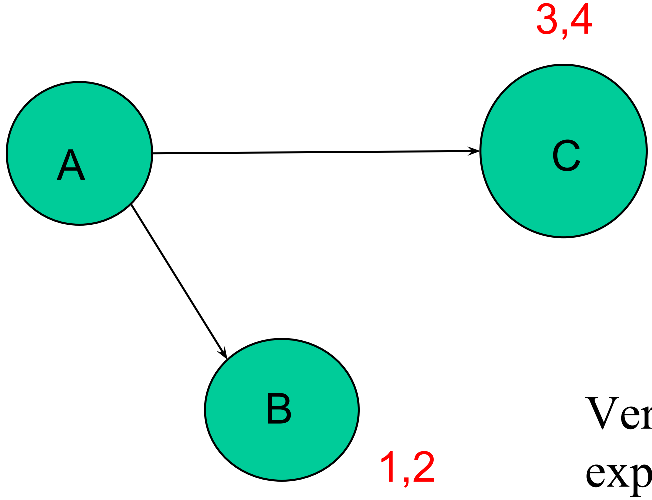

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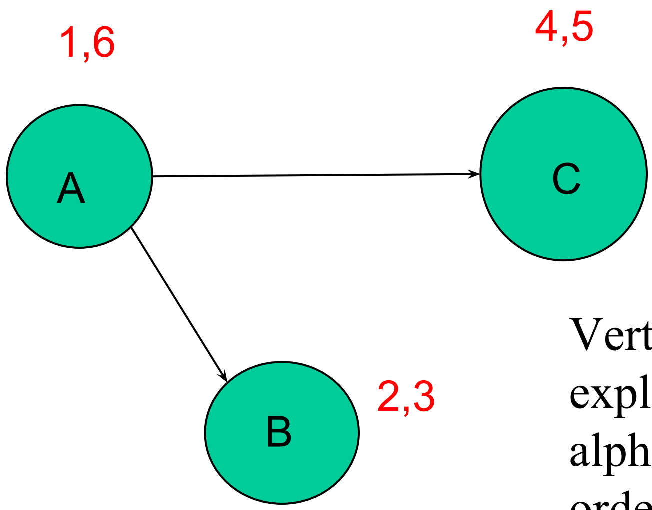

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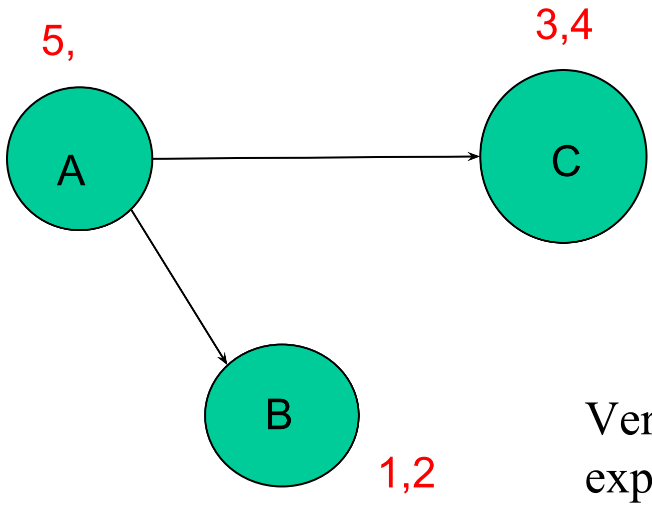

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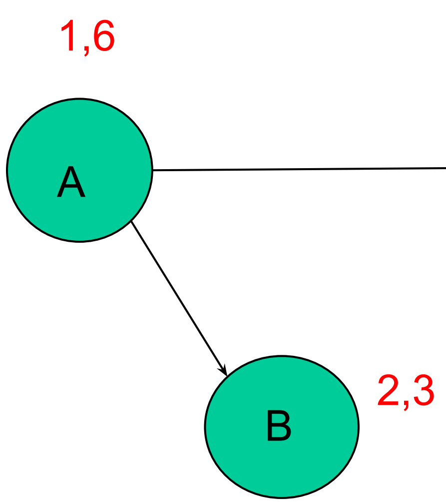

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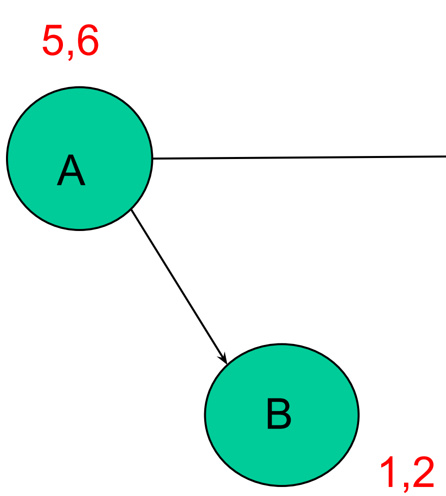

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Strongly Connected Components

- Two vertices u and v of a directed graph are connected if there is a path from u to v and a path from v to u.

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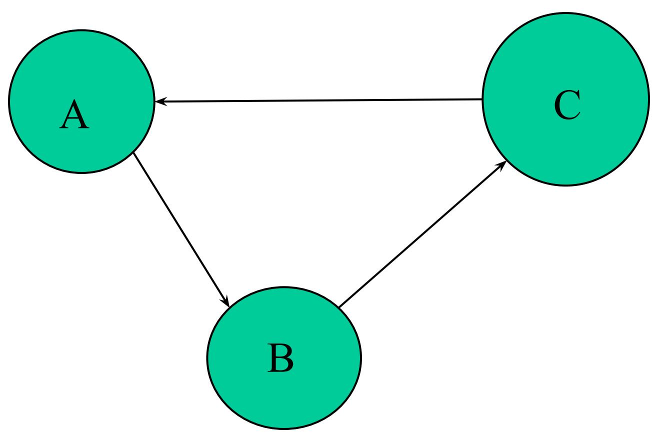
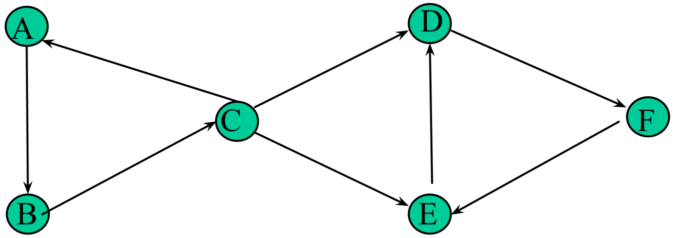
- Two vertices u and v of a directed graph are connected if there is a path from u to v and a path from v to u.
- This definition leads to a partition of a directed graph into disjoint sets of vertices -> strongly connected components



Meta graph

- Meta Graph – The graph obtained by shrinking each strongly connected component down to a single meta-node and drawing an edge from one meta-node to another if there is an edge between their respective components.
- Meta Graph is a DAG. (Why?)
- In other words, every directed graph is a dag of its strongly connected components.



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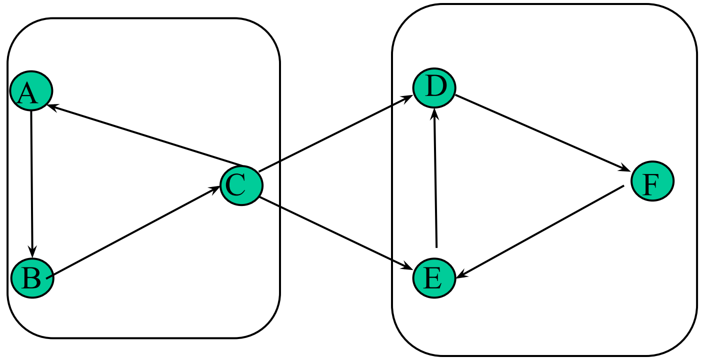

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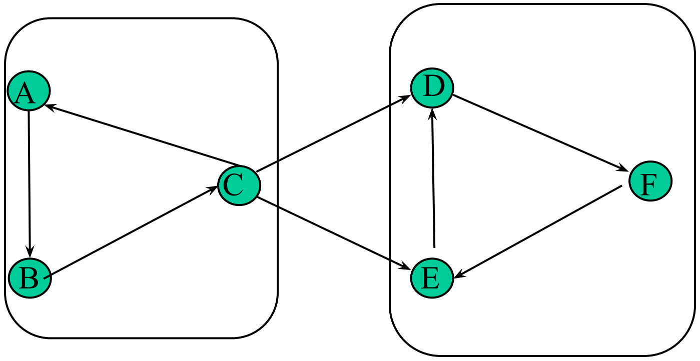

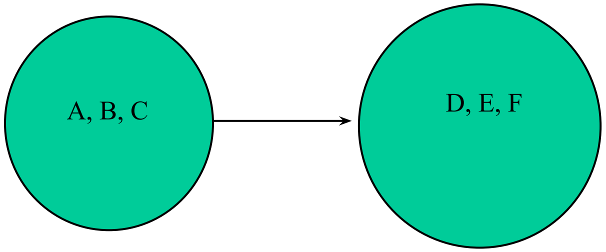

Binary search

- Looking up a word in the dictionary

- Problem: Find x in a sorted array A[1…n]

- Algorithm:
  - Compare x with middle element of array A
  - Recursively find x in left subarray **or** right subarray

Binary search pseudocode

```
Binary_search(A[], key, first, last)

   if (first > last) return not_found

   mid = (first + last)/2

   if (key == A[mid]) return mid

   if (key < A[mid])
      return Binary_search(A[],key,first,mid-1)

   if (key > A[mid])
      return Binary_search(A[],key,mid+1,last)
```

Divide and Conquer

- Divide – the problem (instance) into one or more subproblems

- Conquer – each subproblem recursively

- Combine – separate solutions

Binary search pseudocode

```
Binary_search(A[], key, first, last)

   if (first > last) return not_found //base case

   mid = (first + last)/2

   if (key == A[mid]) return mid

   //divide & conquer
   if (key < A[mid])
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```

Binary search revisited

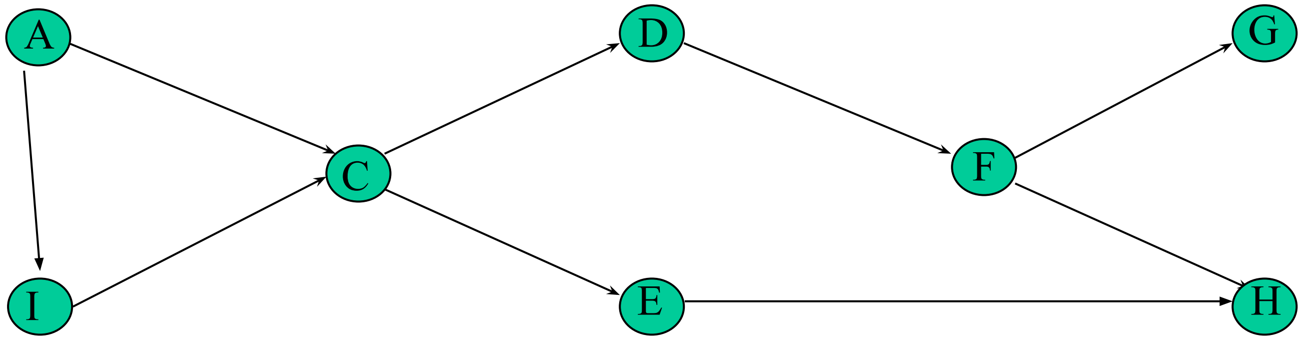
- Divide: compare x with middle element
- Conquer: recurse in <u>one</u> subarray
- Combine: trivial

- <u>Running time</u>
  - $T(n) = T(n/2) + O(1)$
  - **Master theorem**: $T(n) \leq a \cdot T(n/b) + O(n^d)$
    1. $T(n) = O(n^d)$ if $a < b^d$
    2. $T(n) = O(n^d \log n)$ if $a = b^d$
    3. $T(n) = O(n^{\log_b a})$ if $a > b^d$

  - Using master theorem, $T(n) = O(\log n)$ (Case 2)

Solved Exercise

- Run the DFS-based topological ordering algorithm on the following graph. Whenever you have a choice of vertices to explore, always pick the one that is alphabetically first.



- a) Indicate the pre and post numbers of the nodes.
- Solution

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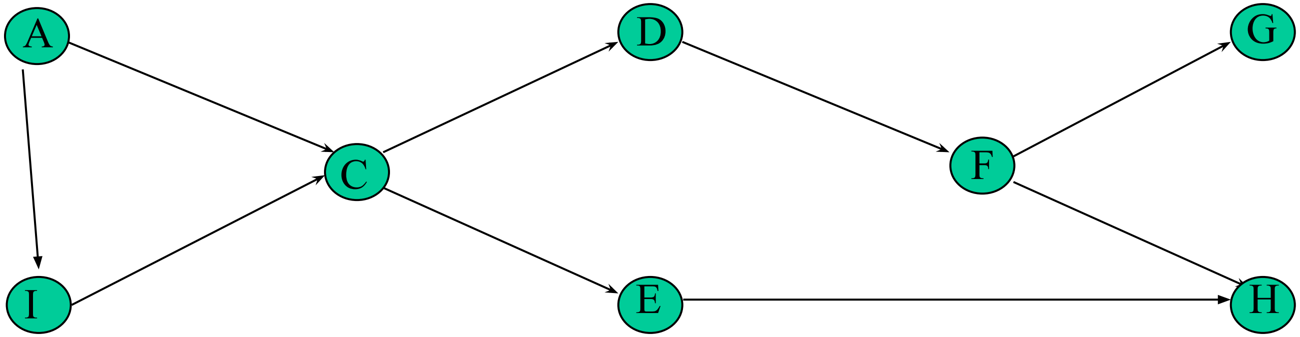

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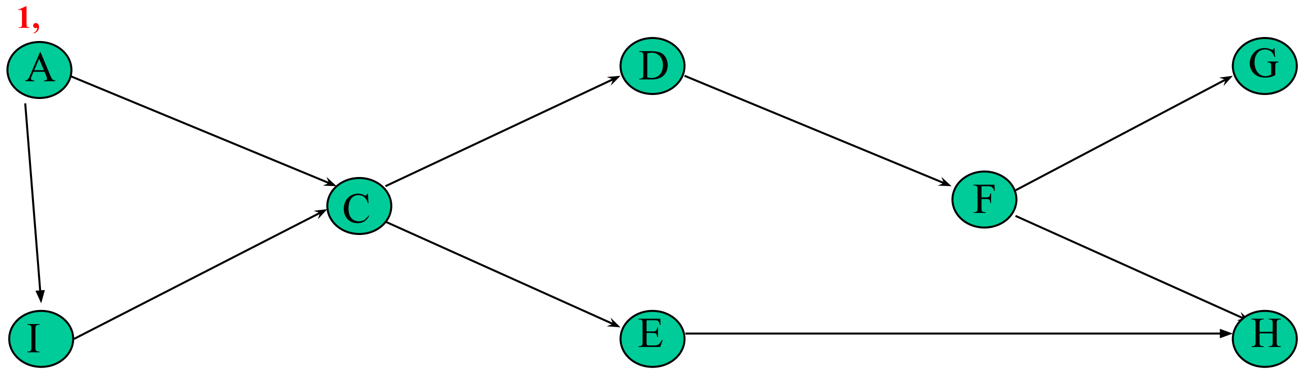

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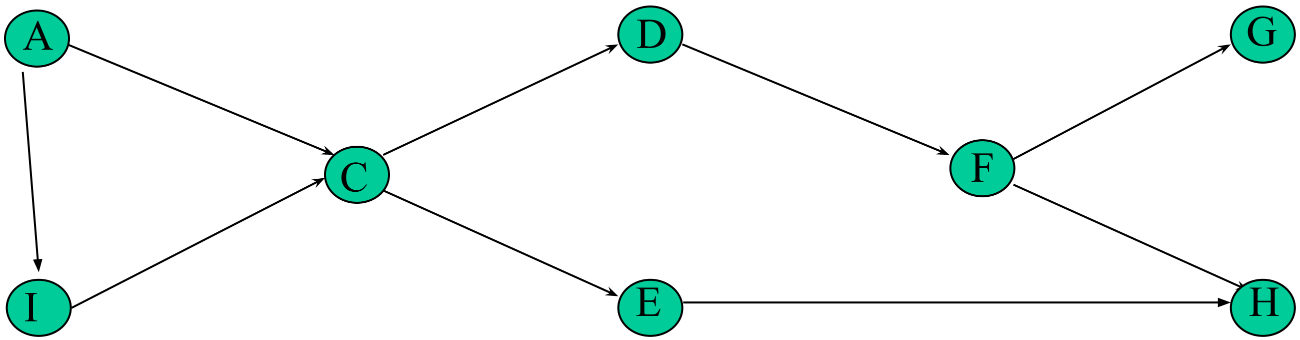

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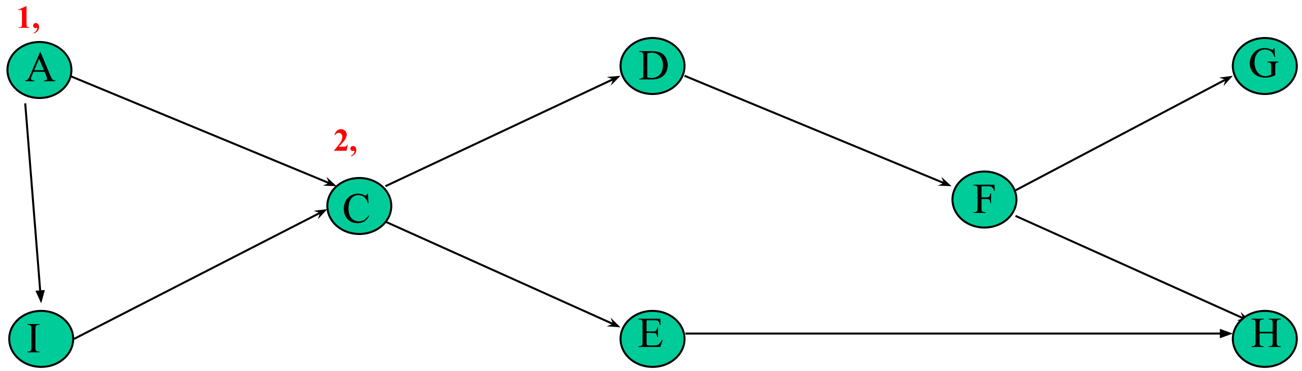

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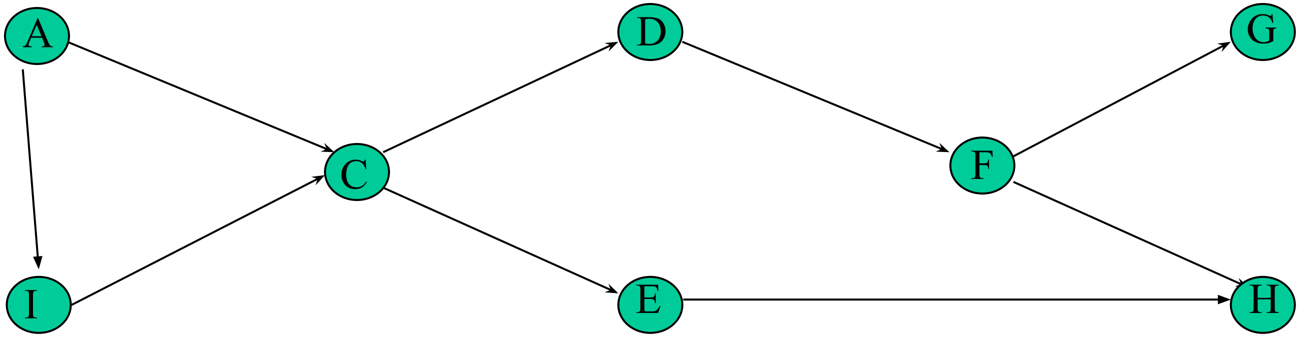

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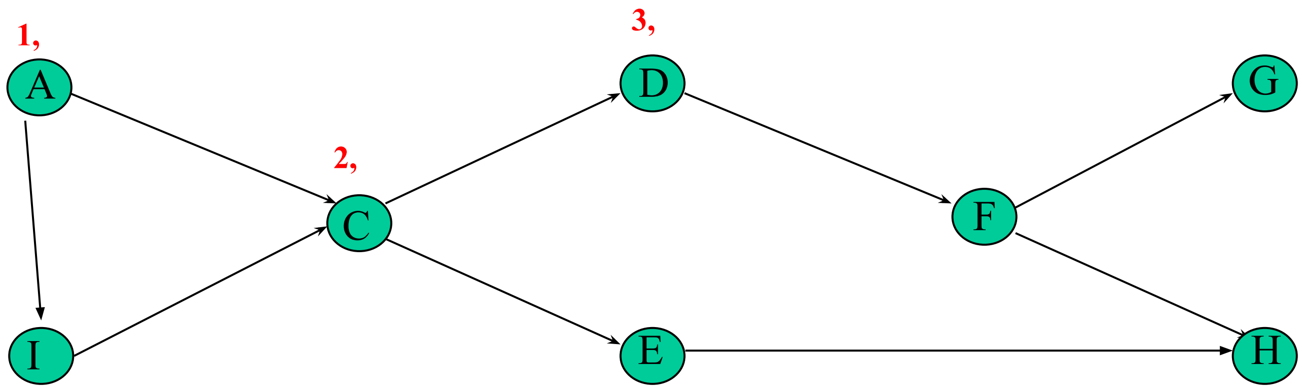

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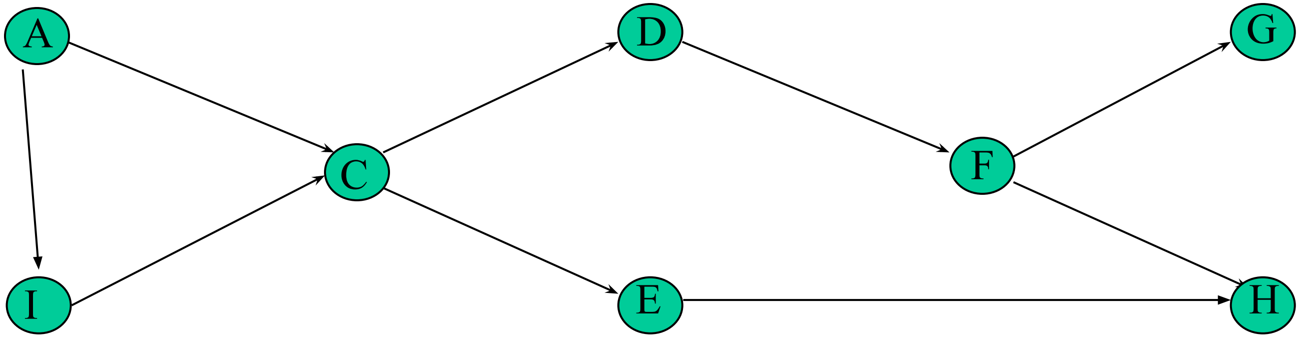

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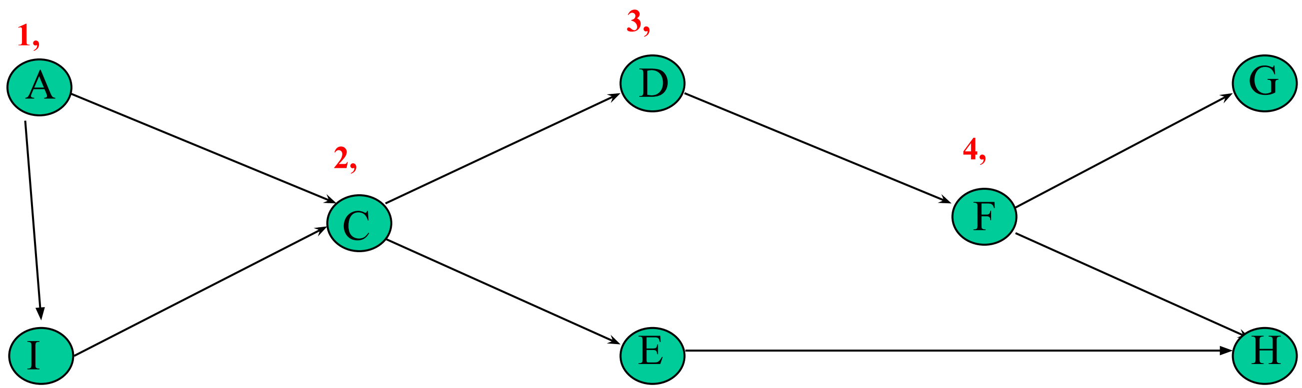

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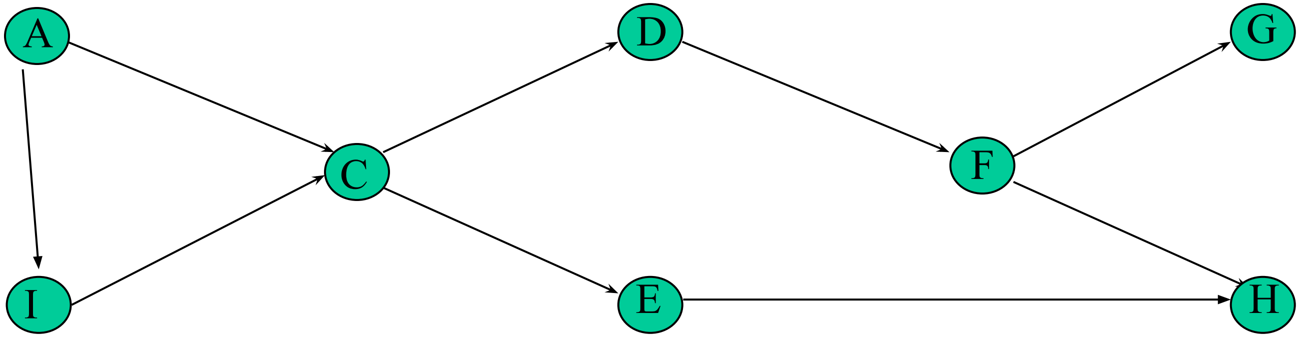

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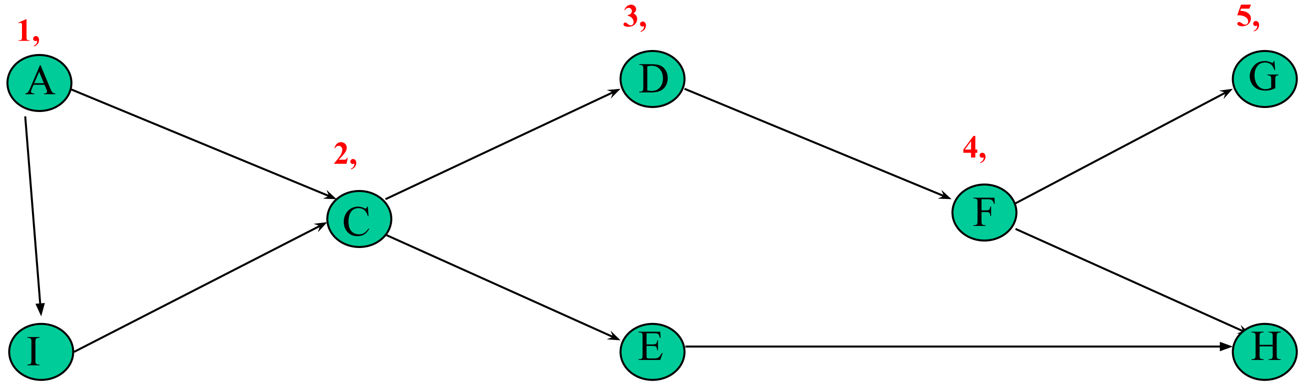

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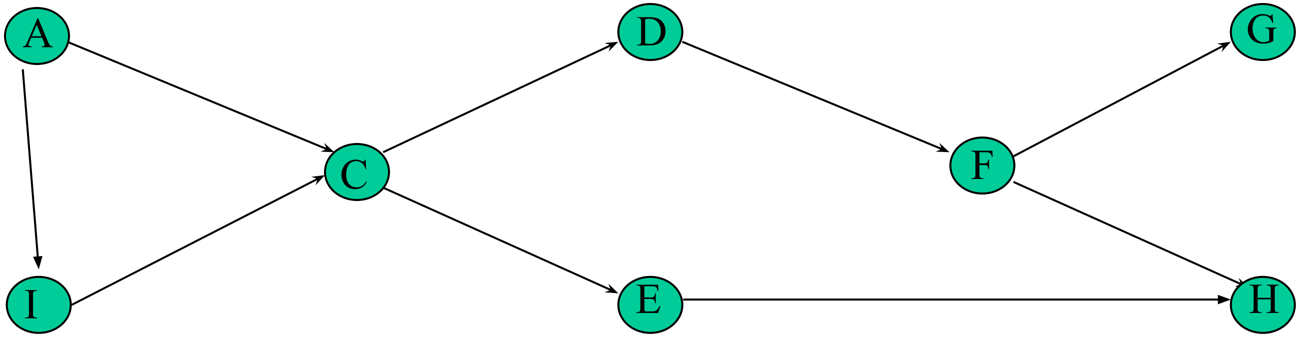

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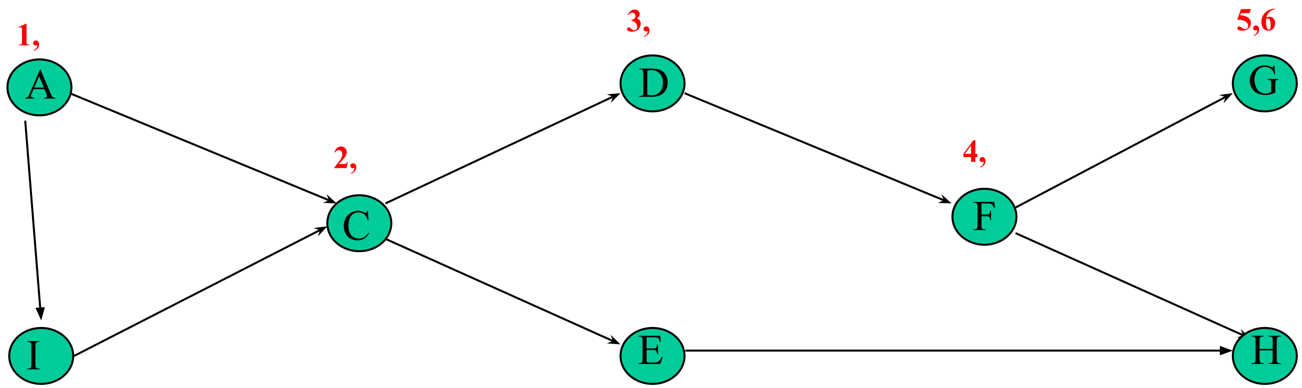

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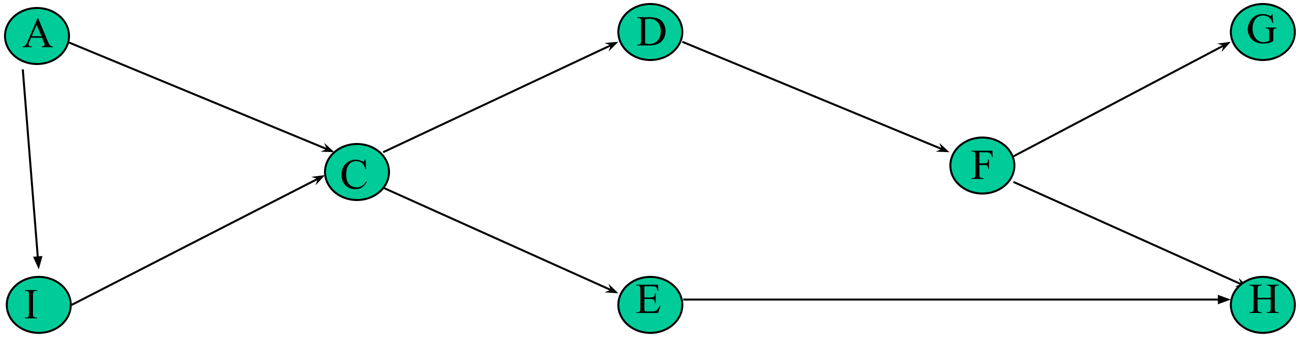

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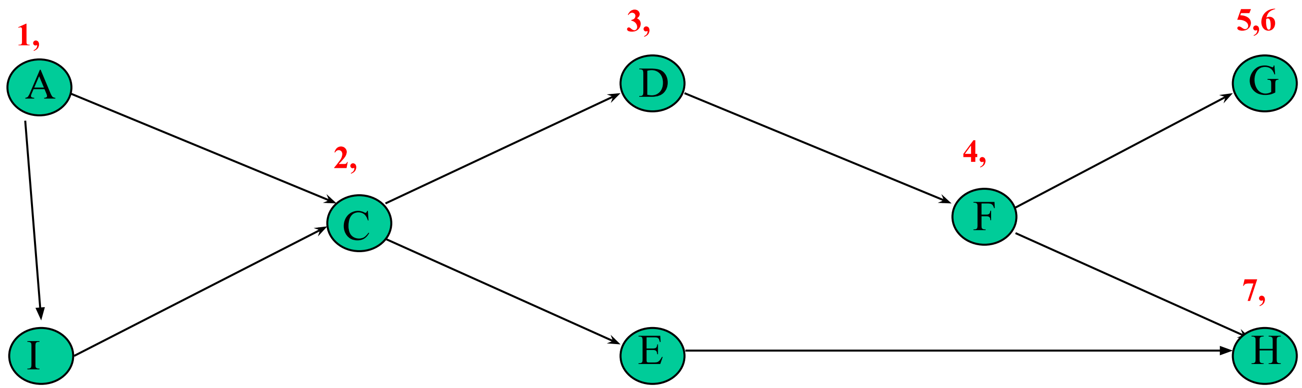

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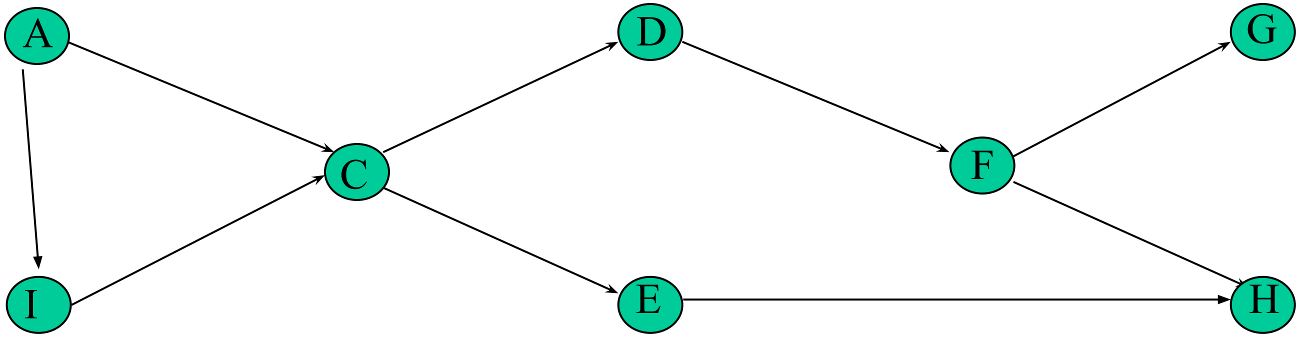

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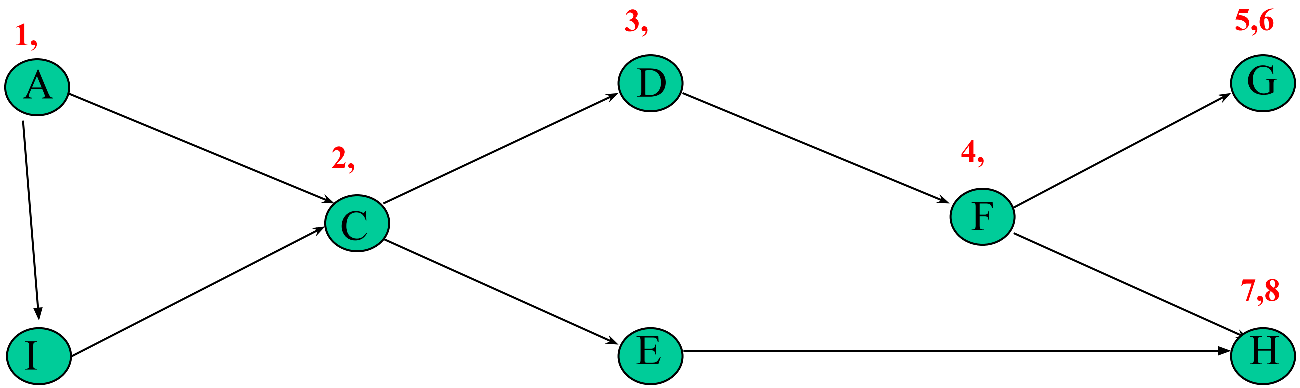

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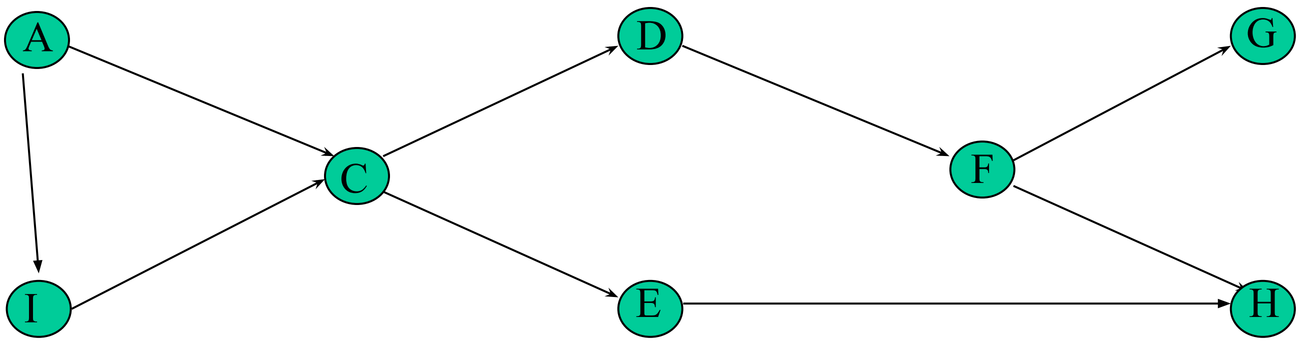

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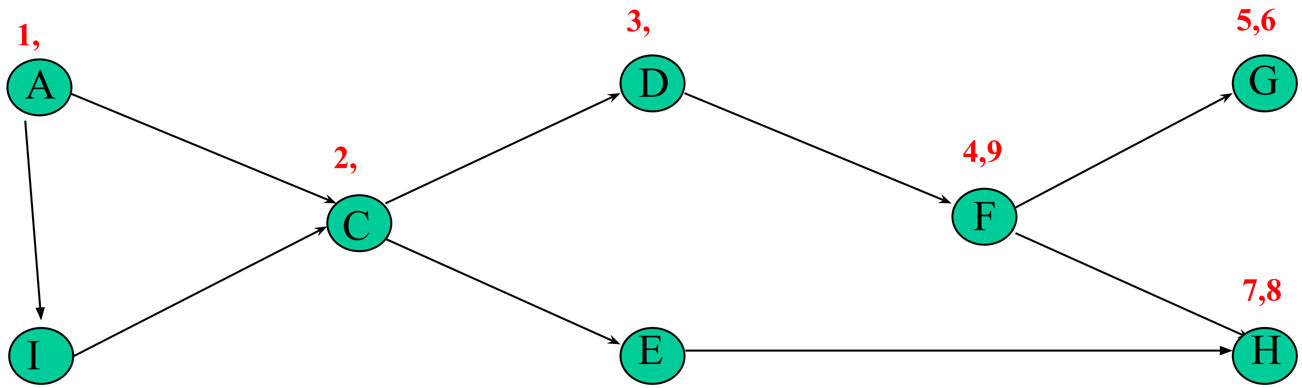

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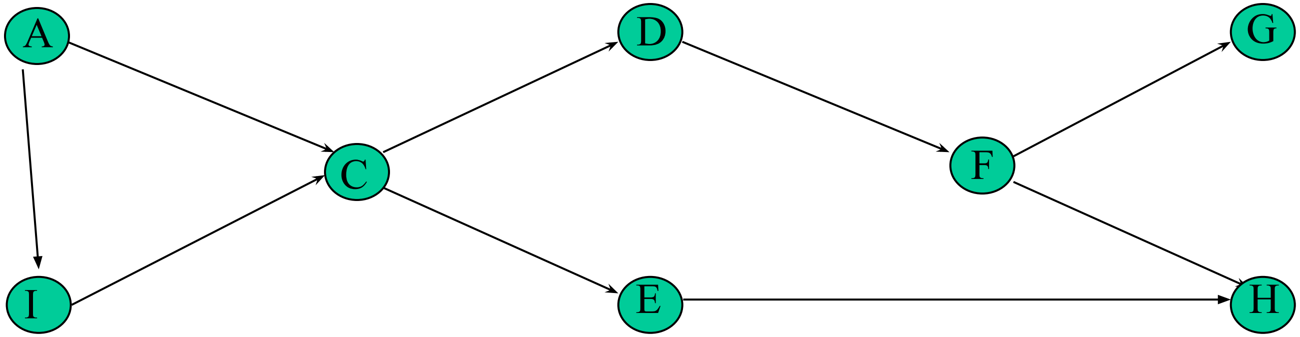

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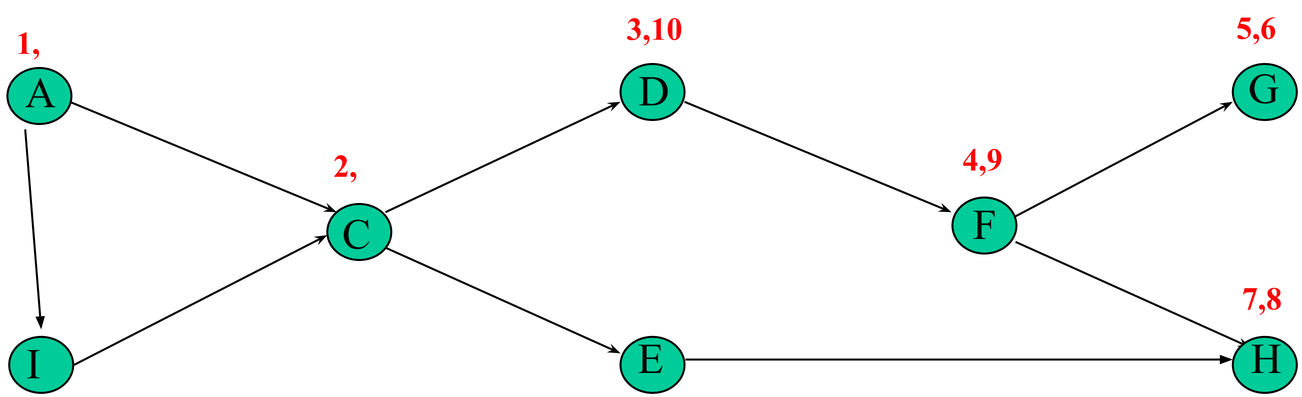

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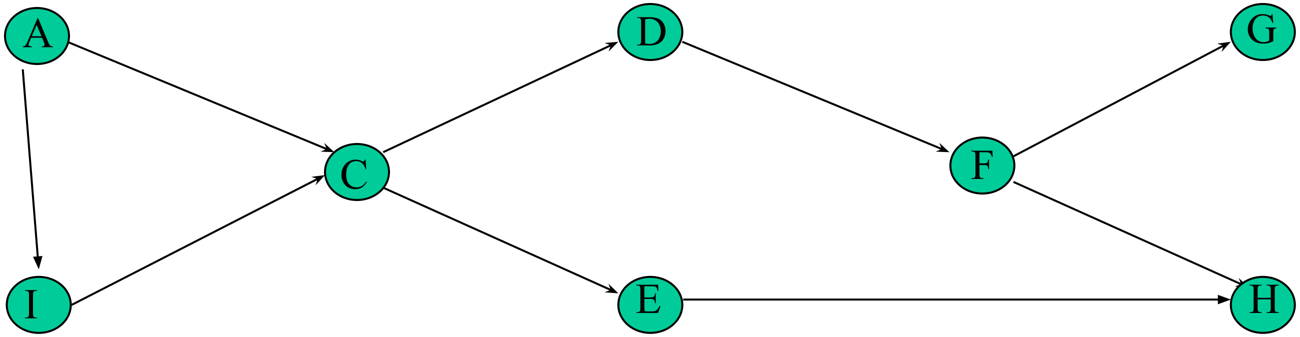

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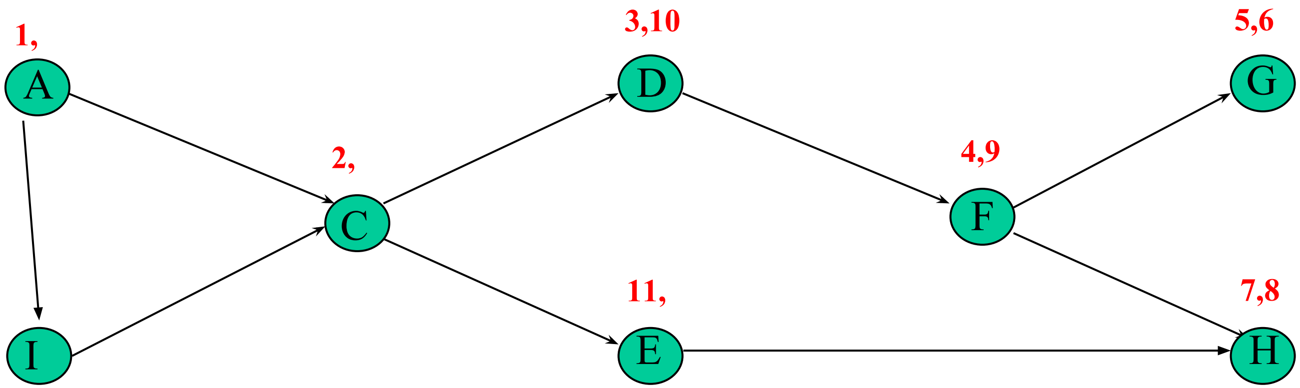

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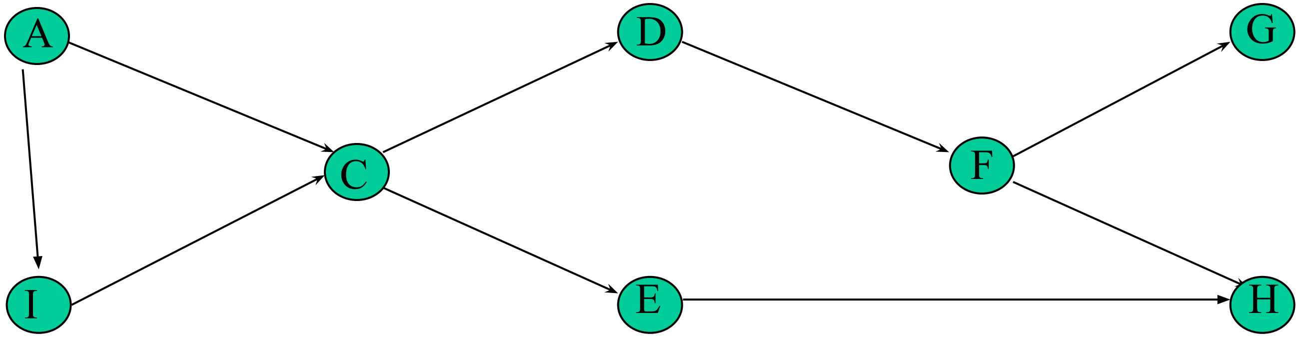
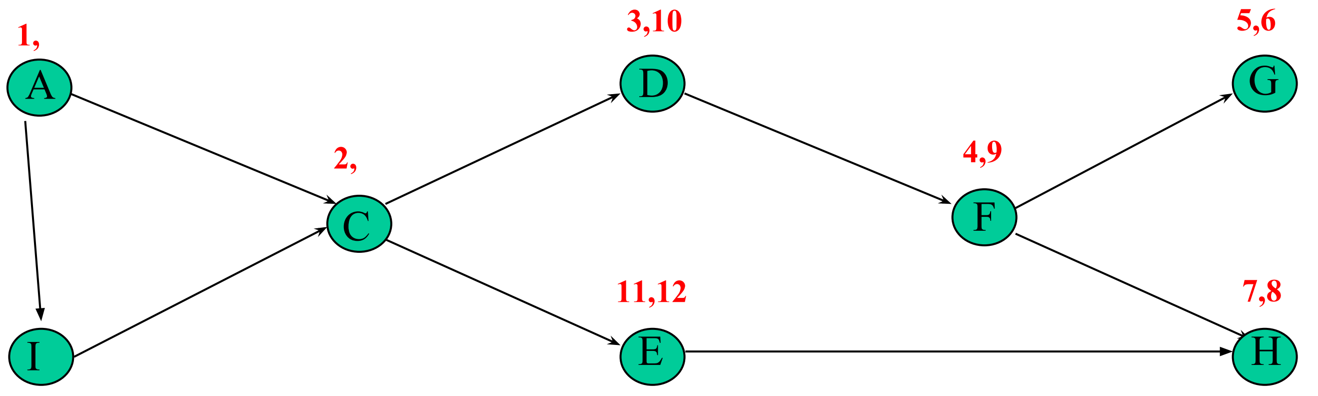


- a) Indicate the pre and post numbers of the nodes.
- Solution



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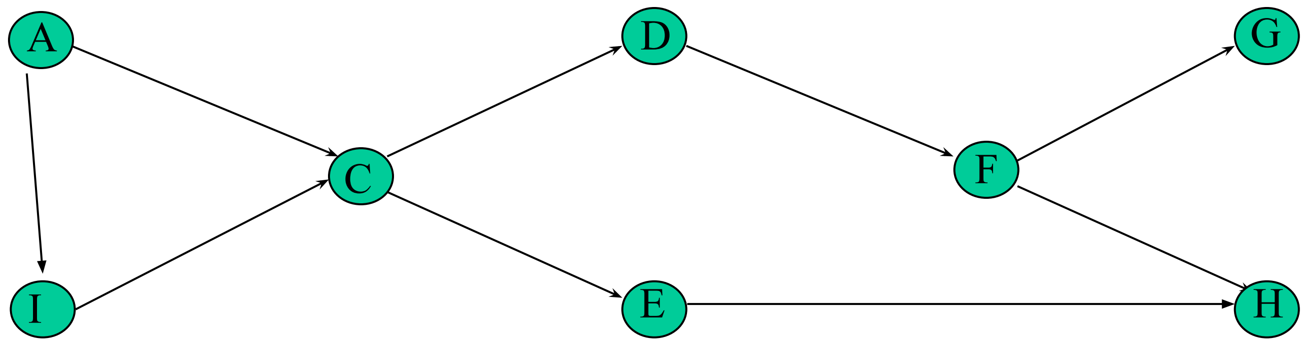

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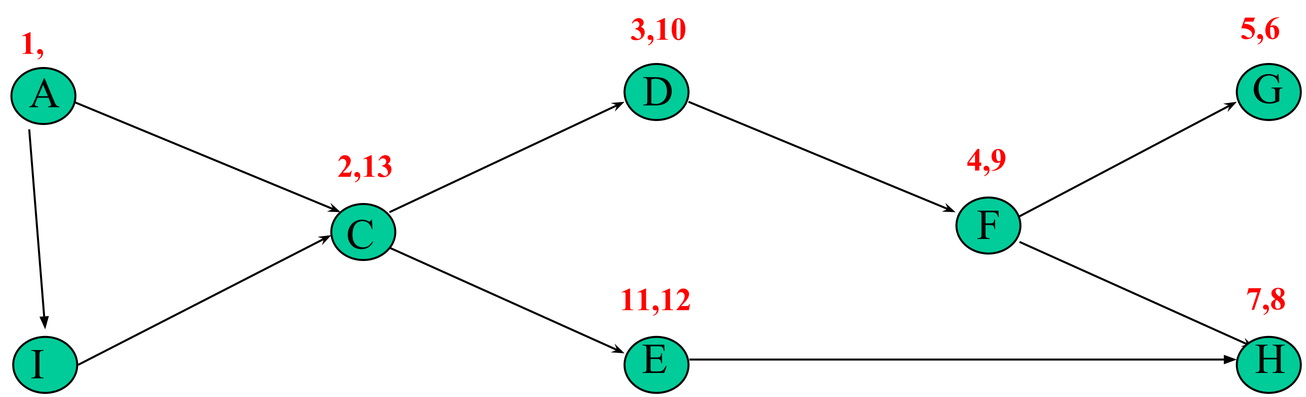

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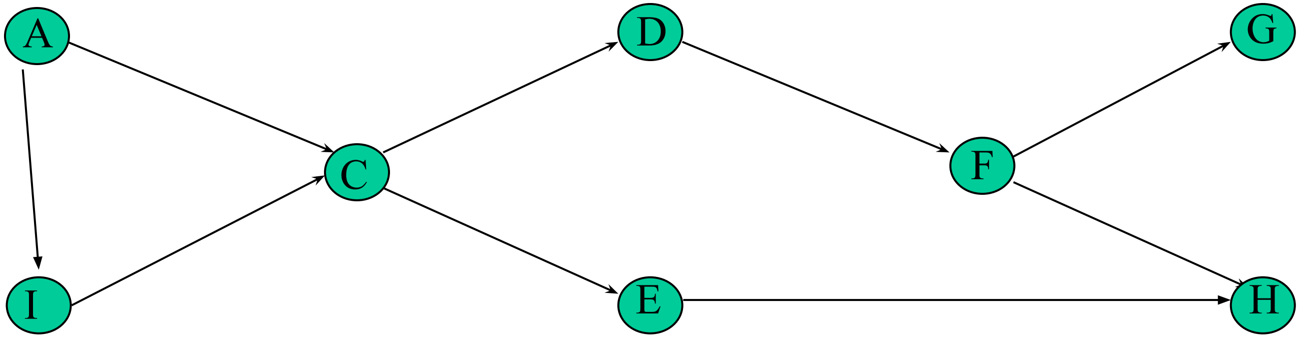

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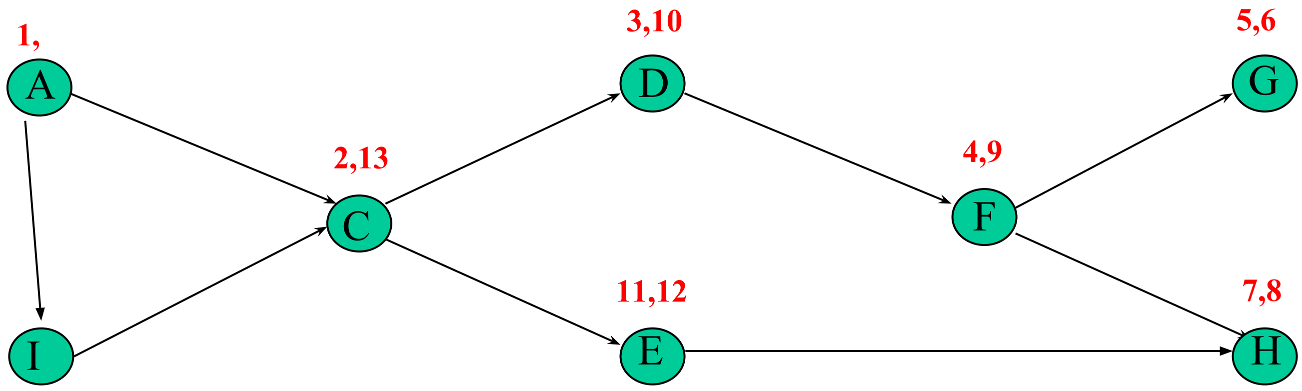

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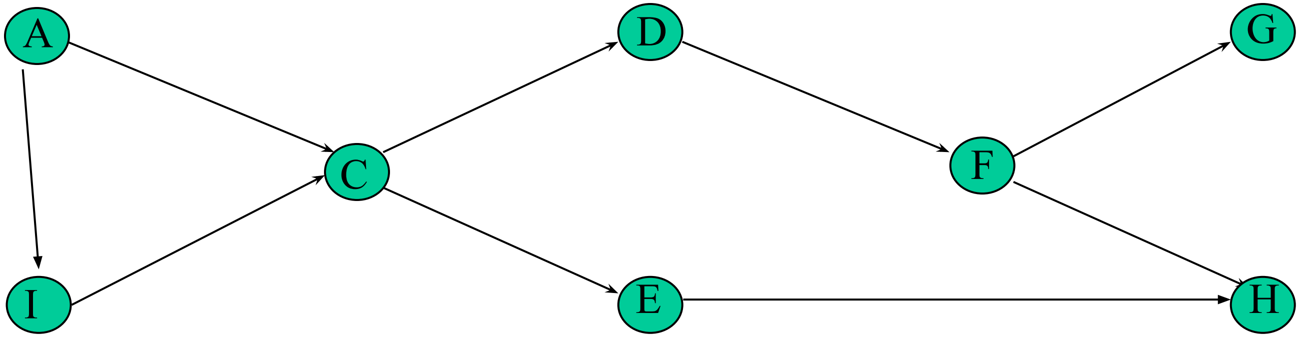

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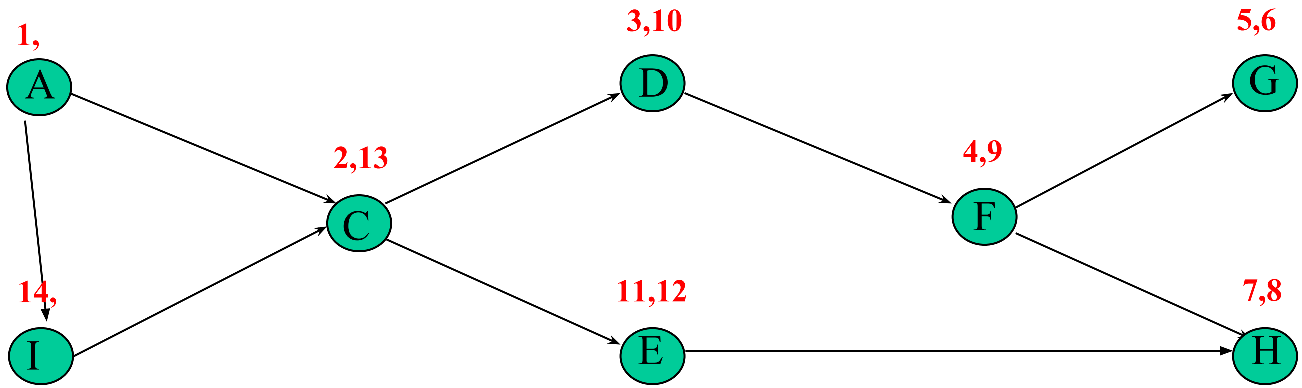

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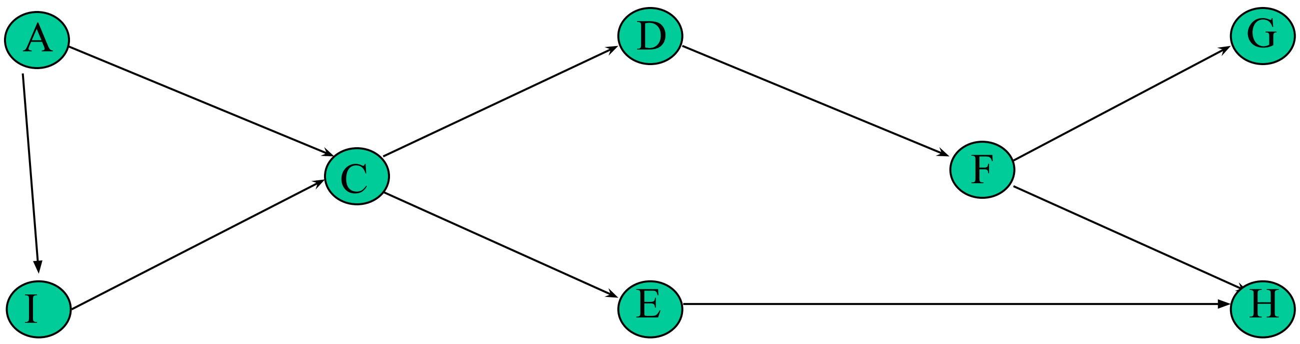

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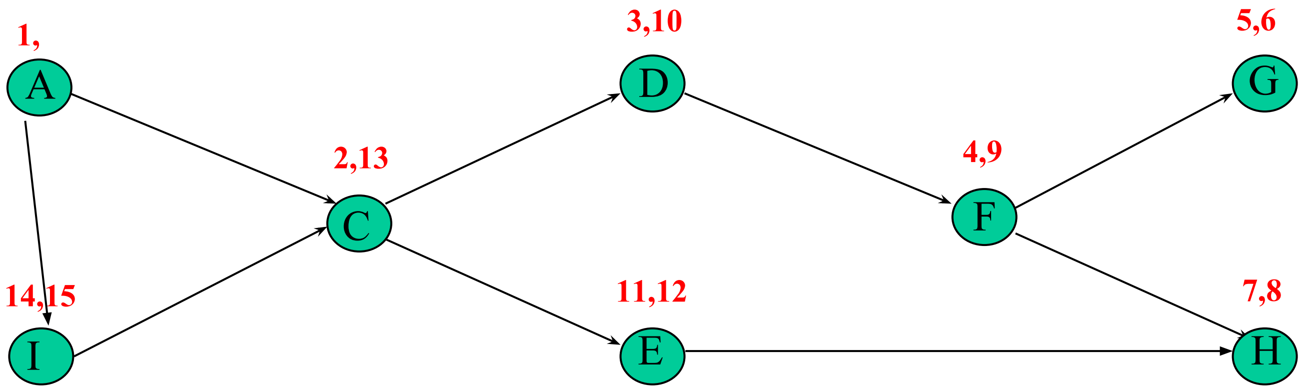

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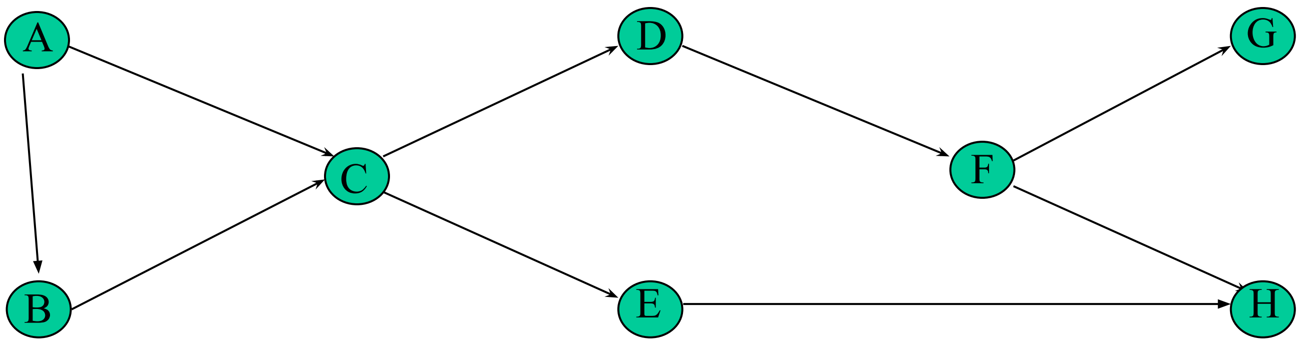

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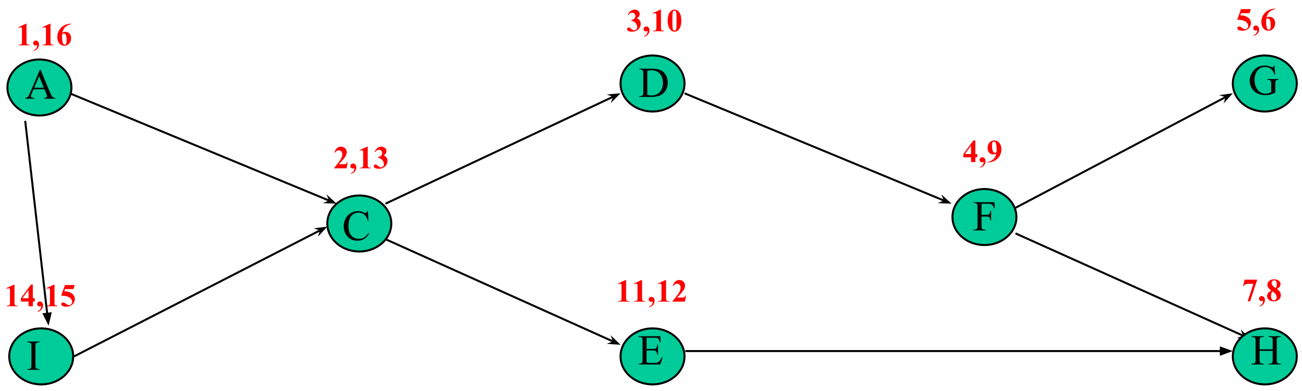

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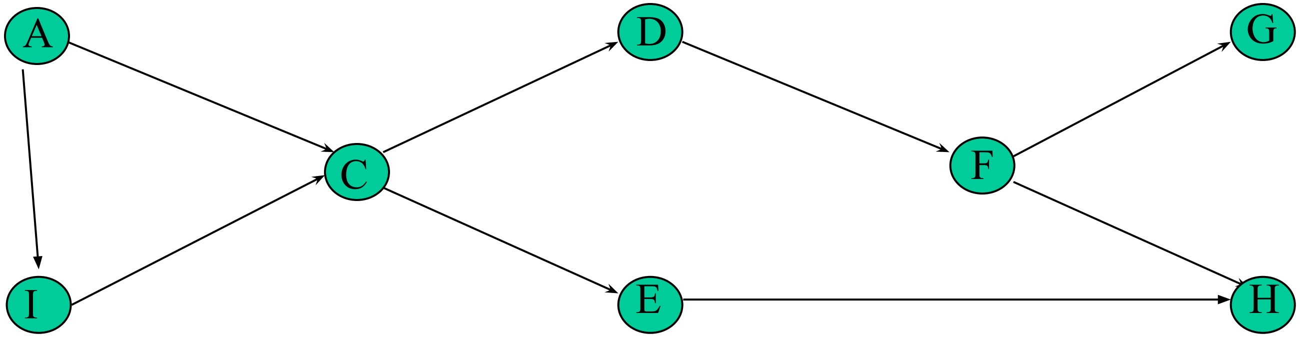

- b) What are the sources and sinks of the graph?

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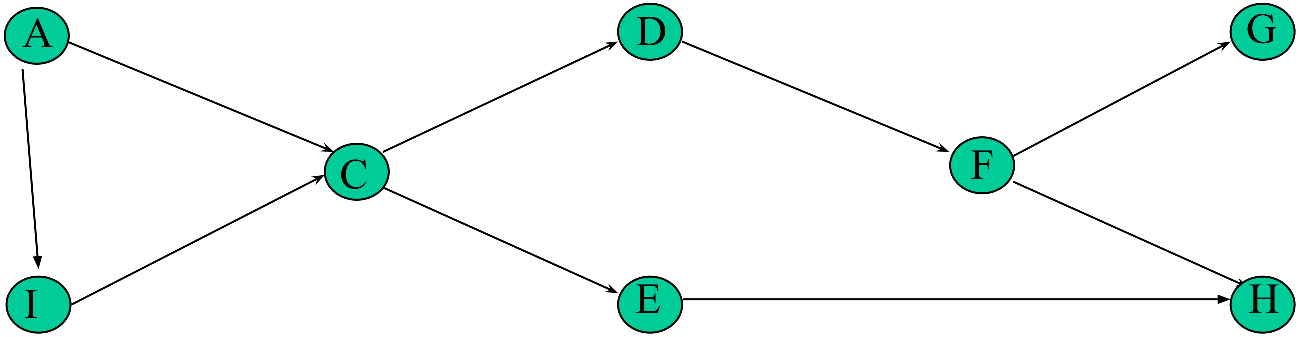

- b) What are the sources and sinks of the graph?
- Solution
  - From the DFS performed in the previous step we are aware that vertex A is a source node since it has the highest post number.

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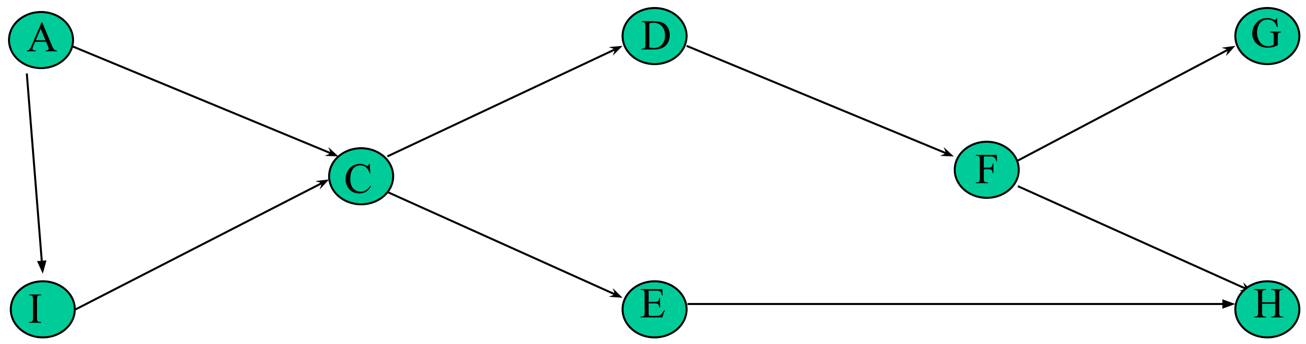

- b) What are the sources and sinks of the graph?
- Solution
  - From the DFS performed in the previous step, we are aware that vertex G has the lowest post number. Thus, vertex G is a sink.

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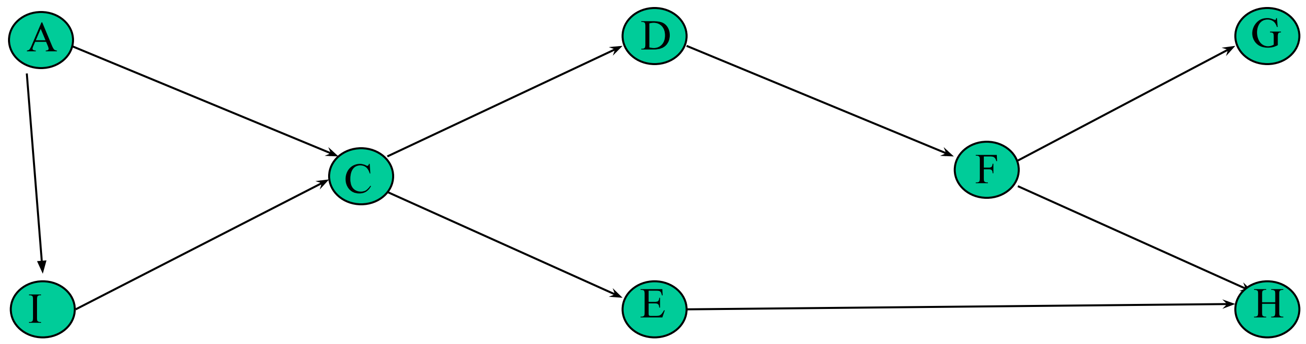

- b) What are the sources and sinks of the graph?
- Solution
  - From the DFS performed in the previous step, we are aware that vertex G has the lowest post number. Thus, vertex G is a sink.
  - Also, in the above DFS while visiting the vertices in the adjacency list of vertex F we could have visited vertex H before vertex G. This would have lead to vertex H having the least post number. Thus, vertex H can be a sink as well.

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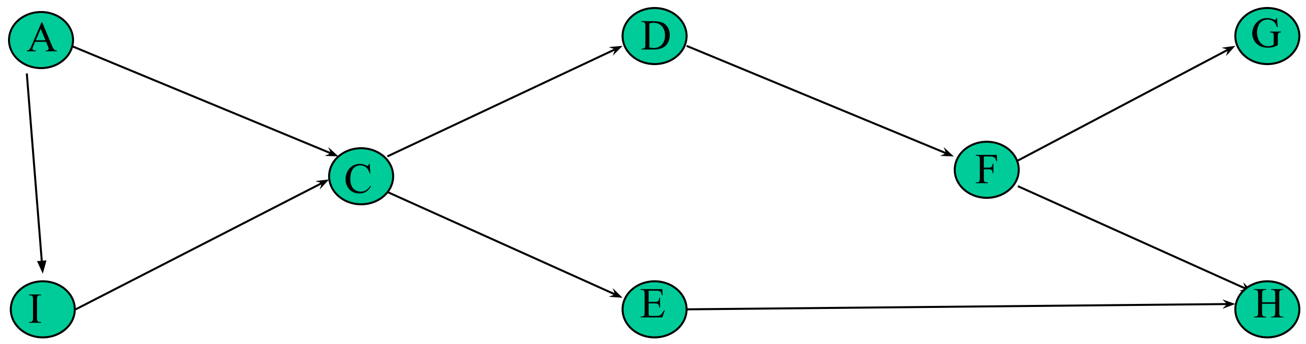

- c) What topological ordering is found by the algorithm?
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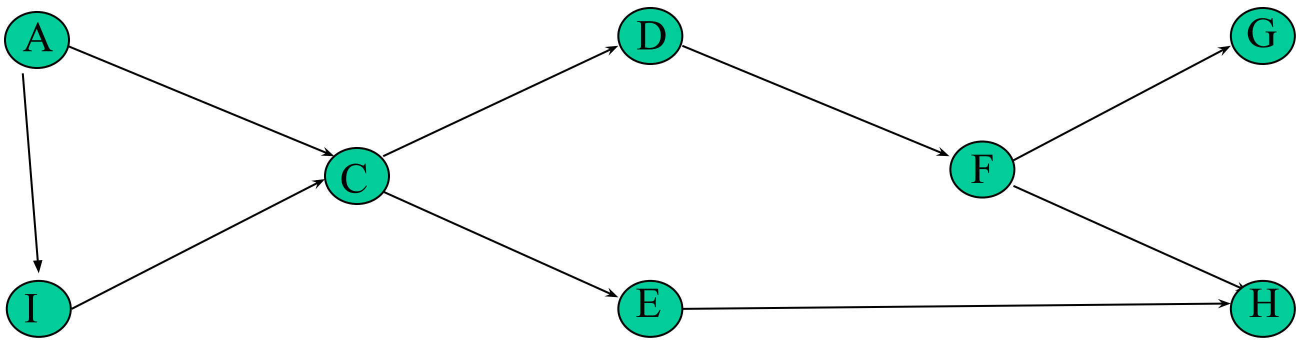

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