Measures of Dependence

Time Series Analysis

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Measures of Dependence

- The most general representation of a time series is that is it a collection of n random variables at arbitrary time points $t_1, t_2, \ldots, t_n$, for any positive integer n.
- The time series $\{X_{t_j} : j = 1, \ldots, n\}$ is completely described (in the stochastic sense) via the joint cumulative probability distribution function

$$F_{t_1,\ldots,t_n}(c_1, \ldots, c_n) = P\left(X_{t_1} \leq c_1, \cdots, X_{t_n} \leq c_n\right).$$

- This mapping of dimension n-space to $[0,1]$ completely describes the process, but is unweildly. Because of the complexity, the marginal $F_{t_j}(x)$ for each X_{t_j} may be useful. Additionally,

$$f_{t_j}(x) = \frac{\partial F_{t_j}(x)}{\partial x}, \quad \text{and} \quad F_{t_j}(x) = P(X_{t_j} \leq x).$$

Mean Function of a Time Series $\{X_t\}$

A descriptive measure of the process is the *mean function*

$$\mu_{X,t} = \mathrm{E}\left[X_t\right] = \int_{-\infty}^{\infty} x f_t(x)\, dx.$$

The mean is the *ensemble mean*.

- **Moving Average Series.** For all t,

$$\mu_{V,t} = \mathrm{E}\left[V_t\right] = \frac{1}{3}\mathrm{E}\left[W_{t+1} + W_t + W_{t-1}\right] = 0.$$

- **Random Walk with Drift.** For all t

$$\mu_{X,t} = \mathrm{E}\left[\delta t + \sum_{j=1}^{t} W_j\right] = \delta t.$$

- **Signal Plus Noise.** For all t,

$$\mu_{X,t} = \mathrm{E}\left[2\cos\left\{2\pi\left(\frac{t+15}{50}\right)\right\} + W_t\right] = 2\cos\left(\frac{t+15}{50}\right).$$

Autocovariance Function of a Time Series $\{X_t\}$

Autocovariance Function

Let μ_t represent the mean function of the time series X_t. Then the *autocovariance function* $\gamma_X(s,t)$ is defined as the second-moment product

$$\gamma_X(s,t) = \mathrm{Cov}\left(X_s,\, X_t\right) = \mathrm{E}\left[(X_s - \mu_s)(X_t - \mu_t)\right],$$

for all s and t. Note that $\gamma_X(s,t) = \gamma_X(t,s)$ for all s and t.

Autocovariance Function of a Time Series $\{X_t\}$.

- The autocovariance measures the *linear* dependence between two points of the same process at different times.
- If $s = t$, then the autocovariance reduces to the variance (function) of the process.
- The autocovariance function can be difficult to interpret, especially since there is no guaranty that two points that are h-units spaced in time have the same autocovariance as any other two points that are also h-units apart in time.

Autocovariance Function of White Noise

- Recall for white noise W_t, $\mathrm{E}\left[W_t\right] = 0$.
- By definition, white noise is a series of *uncorrelated* random variables. Therefore

$$\gamma_W(s,t) = \mathrm{Cov}\left(W_s, W_t\right) = \begin{cases} \sigma_W^2 & s = t, \\ 0 & s \neq t. \end{cases}$$

- Notice there are two cases, $s = t$ and $s \neq t$.

Autocovariance of Moving Average Example

$$\gamma(s,t) = \text{Cov}\left(\frac{1}{3}(W_{s-1} + W_s + W_{s+1}),\ \frac{1}{3}(W_{t-1} + W_t + W_{t+1})\right)$$

There are four cases to consider

i $s = t$

ii $s = t + 1$ or $s = t - 1$ ($|t - s| = 1$)

iii $s = t + 2$ or $s = t - 2$ ($|t - s| = 2$)

iv $|t - s| > 2$.

$$\gamma(s,t) = \begin{cases} \frac{3}{9}\sigma_W^2 & s = t \\ \frac{2}{9}\sigma_W^2 & |t-s| = 1 \\ \frac{1}{9}\sigma_W^2 & |t-s| = 2 \\ 0 & |t-s| = 3, 4, \ldots \end{cases}$$

Autocovariance of a Random Walk

For the random walk, we can write

$$X_t = \delta t + \sum_{j=1}^{t} W_t.$$

- The δt shift doesn't affect the autocovariance, so

$$\gamma(s,t) = \mathrm{Cov}\left(\sum_{j=1}^{s} W_j, \sum_{j=1}^{t} W_j\right) = \min\{s,t\}\sigma_W^2.$$

- The variance is

$$\gamma(t,t) = t\sigma_W^2.$$

The Autocorrelation Function of a Time Series $\{X_t\}$

Autocorrelation Function

Let $\gamma(s,t)$ represent the autocovariance function of a time series $\{X_t\}$. Then the *autocorrelation function* of X_t is

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}.$$

The autocorrelation function is $|\rho(s,t)| \leq 1$ for all s and t.

Cross-Covariance and Cross-Correlation Functions

Cross-Covariance and Cross-Correlation Functions

Let X_t and Y_t be two time series with mean functions $\mu_{X,t}$ and $\mu_{Y,t}$, respectively.

1. The *cross-covariance function* between X_s and Y_t is

$$\gamma_{X,Y}(s,t) = \mathrm{Cov}\,(X_s,\, Y_t) = \mathrm{E}\left[(X_s - \mu_{X,s})(Y_t - \mu_{Y,t})\right].$$

2. The *cross-correlation function* between X_s and Y_t is

$$\rho_{X,Y}(s,t) = \frac{\gamma_{X,Y}(s,t)}{\sqrt{\gamma_X(s,s)\gamma_Y(t,t)}}.$$

Cross-Covariance and Cross-Correlation

- The cross-covariance and cross-correlation change both as a function of X and Y and of s and t.
- The two functions are *not* symmetric around time or around X and Y.
- These ideas are extendible to vector series that are of higher dimensions than two.