Assignment #4: The Spectral Density Function and Periodogram

STA 5362: Time Series Analysis

Fall Semester

A list of files you will need to complete this assignment and where you can find them on the class Canvas web page is given below.

- `STA5362-functions.R` is the first file the "R Code" module. Before you start, be sure to source `STA5362-functions.R`.

- `assn4.R` can be found in the "Frequency Domain Time Series Analysis Methods" module under the heading R Code for Assignment #4. A link to the file can also be found in the description of Assignment #4. Documentation for the function can be found on page 4 of this homework assignment.

  Description. This problem is designed to more thoroughly investigate the periodogram. In class, the sinusoid realization was a "best case" situation. Specifically

  1. The length of the series was $n = 200$. The frequencies were $1/100 = 0.01, 1/10 = 0.1$ and $1/4 = 0.25$. For all three frequencies, complete cycles were observed. In other words, $200 \times 0.01 = 2$ complete cycles were observed for frequency 0.01; $200 \times 0.1 = 20$ complete cycles were observed for frequency 0.1; and $200 \times 0.25 = 50$ complete cycles were observed for frequency 0.25.
  2. The data was a pure signal. There was no noise.

  In the following, you will use the function `assn4` to investigate the periodogram for three cases.

  i A pure signal, but without full cycles.

  ii A signal with full cycles observed, but noise added.

  iii A signal without full cycles and with added noise.

For each model in equations (1) through answer the questions below. Include the graphs produced by the function with your answers.

1. How many cycles of the frequencies are observed in the series? (This answer will not always be an integer. You need to only answer this question once for each signal and n.)

2. What are the natural frequencies for the periodogram? Are the frequencies in the sinusoid natural frequencies? Which are and which are not? (You need to only answer this question once for each signal and n.)

3. Explain the shape of the periodogram. For what frequencies does the periodogram (try to) achieve a type of peak? What is the relation of the natural frequencies to the actual (signal) frequency?

4. If the length of the series had been so that all cycles of the frequencies are fully observed, what would the shape of the periodogram look like? Give the three smallest series lengths that result in all cycles being fully observed. For the smallest series length, run the function again. How is the shape of the periodogram changed compared to the periodogram in part (a)? (You only need to answer this question one time for each signal and each n.)

5. Knowing the amplitudes, what is the variance of the signal? If noise is added, what is the variance of the process? What is the ratio of the signal to noise variances? Explain the effect of the noise on the periodogram.

Before starting the exercises below, generate a white noise series of length 150 from a standard normal distribution. Save it in a vector `w`.

A. First, we will investigate what happens to the periodogram for a signal with a single frequency from the signal

$$x_t = 5\cos\{2\pi(t-1)(0.15)\}, \quad \text{for } t = 1, 2, \ldots, n. \tag{1}$$

   (i) Answer the questions on the first page for $n = 150$. Run the function `assn4` using

   `assn4(150, 5, 0.15).`

   (ii) Using the same model in equation (1), include noise with a variance of $\sigma_W^2 = 1$. The code will be

   `assn4(150, 5, 0.15, w)`

   (iii) Using the same model in equation (1), include noise with a variance of $\sigma_w^2 = 25$. The code will be

   `assn4(150, 5, 0.15, 5*w)`

B. Now, we will look at what happens to the periodogram for a signal with multiple frequencies. The signal is

$$\begin{aligned} x_t = 3\cos\{2\pi(t-1)(0.05)\} + 2\cos\{2\pi(t-1)(0.15)\} \\ + 4\cos\{2\pi(t-1)(0.4)\}, \quad \text{for } t = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

(i) For each frequency component, make a time plot for $n = 200$.

(ii) Now let $n = 150$. Repeat the previous questions 1(a) – 1(c) for the signal in equation (2). The code for each part will be

```
assn4(150, c(3, 2, 4), c(0.05, 0.15, 0.04))       # Part (a)
assn4(150, c(3, 2, 4), c(0.05, 0.15, 0.04), w)    # Part (b)
assn4(150, c(3, 2, 4), c(0.05, 0.15, 0.04), 5*w)  # Part (c)
```

Documentation of *R* functions for this assignment.

`assn4(n = 200, A, omega, w = rep(0, n))`

DESCRIPTION: *R* function to investigate the behavior of the periodogram.

USAGE: `assn4(n = 200, A, omega, w = rep(0, n))`

ARGUMENTS:

`n` – a positive integer scalar containing the length of the series. The default is `n = 200`.

`A` – a nonnegative real vector of length `k` containing the amplitudes of the frequency components. The maximum number of amplitudes is four. The lengths of `A` and `omega` must be equal. There is no default.

`omega` – a positive real vector with all elements between zero and 0.5, inclusive, containing the frequencies. The maximum number of frequencies is four. The lengths of `A` and `omega` must be equal. There is no default.

`w` – a real vector of length `n` containing noise to add to the sinusoid. If no noise is desired then `w = 0`, which is the default.

DETAILS: Data generated from this function will be from the model

$$x_t = \sum_{j=1}^{k} A_j \cos\{2\pi\omega_j(t-1)\} + w_t, \quad t = 1, \ldots n,$$

where k is the length of A and ω, and w_t is a noise series. If no noise is desired, then $w \equiv 0$, which is the default.

EXAMPLES:

```
assn4(A = 3, omega = 0.15)              #n = 200, 30 full cycles
assn4(n = 150, A = 3, omega = 0.15)

w <- rnorm(200)
assn4(A = c(1, 3, 10), omega = (0.01, 0.1, 0.25))      #No noise
assn4(A = c(1, 3, 10), omega = (0.01, 0.1, 0.25), w = 0.5*w)
assn4(A = c(1, 3, 10), omega = (0.01, 0.1, 0.25), w = w)

w <- w[1:150]
assn4(n = 150, A = c(1, 3, 10), omega = (0.01, 0.1, 0.25))
assn4(n = 150, A = c(1, 3, 10), omega = (0.01, 0.1, 0.25), w = w)
assn4(n = 150, A = c(1, 3, 10), omega = (0.01, 0.1, 0.25), w = 0.5*w)
```