Seiberg-Witten Invariants of Symplectic Manifolds

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Contents

Abstract

Inspired by the gauge theory in physics, Donaldson defined a series of diffeomorphism invariants for smooth 4-dimensional manifolds to distinguish 4-manifolds that are homeomorphic but not diffeomorphic. However, the Donaldson invariants are very difficult to compute. Seiberg and Witten defined a new kind of invariants in 1990s, called the Seiberg-Witten invariants. Most of the results proved via Donaldson theory can be reproved using Seiberg-Witten invariants and the Seiberg-Witten invariants are much easier to cumpute.

The goal of this thesis is to prove Taubes' theorem 3.1, that is, to compute the simplest Seiberg-Witten invariants of simplectic 4-manifolds. In section 1, we give a brief introduction of the differential geometry backbounds needed for understanding the Seiberg-Witten equations and Seiberg-Witten invariants. In section 2, we introduce the Seiberg-Witten equation and prove that the moduli space of its solutions is compact. In section 3, we define the Seiberg-Witten invariants and explain Taubes' Theorem 3.1, which reveals the special structure of the Seiberg-Witten invariants of symplectic 4-manifolds.

1 Preliminaries

This part serves as a crash introduction to preliminaries needed for understanding the Seiberg-Witten equations and Seiberg-Witten invariants. More details can be found in [4] and [5].

1.1 Vector Bundle, Connection and Curvature

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ in this section.

Definition 1.1. Let M be a smooth manifold. A **smooth $\mathbb{F}$-vector bundle of rank** k over M is a smooth manifold E with a surjective smooth map $\pi : E \to M$ satisfying:

1. $\forall p \in M$, $\pi^{-1}(p)$ is a k-dimensional $\mathbb{F}$-vector space.
2. $\forall p \in M$, there exists a neighborhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{F}^k$ such that the following diagram commutes and $\forall q \in U$, $\Phi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \mathbb{F}^k$ is an $\mathbb{F}$-linear isomorphism.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{F}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Remark. Given a vector bundle with a family of local trivializations, we can obtain the transition functions between different local trivializations. Conversely, given an open cover $\{U_\alpha\}$ of the smooth manifold M and a family of smooth functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{F})$ satisfying the cocyle condition $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$, one can construct a smooth vector bundle of rank k over M such that the transition functions of its local trivializations are exactly $g_{\alpha\beta}$.

Definition 1.2. Let P, M be smooth manifolds, G be a Lie group with a smooth right action on P, $\pi : P \to M$ be a smooth surjection. The triple (P, M, π) is called a **principal G-bundle** if it satisfies:

1. $\forall p \in M$, $\pi^{-1}(p) \subset P$ is an orbit of the G-action on P.

2. $\forall p \in M$, there exists a neighborhood U of p in M and a G-equivariant diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ (G has a natural right action on $U \times G$) such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Remark. 1. Similar to the case of vector bundle, given an open cover $\{U_\alpha\}$ of M, there is a correspondence between principal G-bundles over M and families of transition functions:

$$P \longleftrightarrow \{\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}\ satisfying\ \varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$$

2. Given a smooth $\mathbb{F}$-vector bundle E, we can obtain a principal $GL(k, \mathbb{F})$-bundle whose transition functions are the transition functions of E. In particular, if M is an oriented Riemannian manifold and $E = TM$, we can reduce the structure group to $SO(n)$, then TM is associated to a principal $SO(n)$-bundle, called the **frame bundle** of M.

3. Conversely, given a principal G-bundle P whose transition functions are $\{\varphi_{\alpha\beta}\}$ together with a Lie group representation $\rho : G \to GL(k, \mathbb{F})$, we can construct a smooth $\mathbb{F}$-vector bundle $E = P \times_\rho \mathbb{F}^k$ whose transition functions are $\{\rho \circ \varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{F})\}$

Definition 1.3. A **connection on a $\mathbb{F}$-vector bundle** E is an $\mathbb{F}$-linear map $d_A : \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying:

$$d_A(f\psi) = (df) \otimes \psi + f d_A\psi,\ \forall f \in C^\infty(M),\ \forall \psi \in \Gamma(E)$$

Remark. We also denote d_A by ∇_A or ∇^A. When M is a Riemannian manifold, the Levi-Civita connection in Riemannian geometry is a special connection on TM and it induces connections on $\Lambda^k T^*M$, all denoted by ∇.

To derive the local representation of connection, let's take a local trivialization (U_α, Φ_α) of the vector bundle E. On $U_\alpha \times \mathbb{F}^k$ we have constant sections $e_1 = (1, 0, \cdots, 0), \cdots, e_k = (0, \cdots, 0, 1)$. Then $d_A(e_j) = \sum_{i=1}^{k} \omega_j^i e_i$ for some $\omega_j^i \in \Omega^1(U)$. So for general local section $\sigma = \sum_{j=1}^{k} \sigma^j e_j$, $d_A(\sum_{i=1}^{k} \sigma^j e_j) = \sum_{j=1}^{k} (d\sigma^j) e_j + \sum_{i,j=1}^{k} \sigma^j \omega_j^i e_i$.

Thus, any section $\sigma \in \Gamma(E)$ has a local representative $\sigma_\alpha = \Phi_\alpha \circ \sigma|_{U_\alpha} : U_\alpha \to U_\alpha \times \mathbb{F}^k$ on U_α, and $(d_A\sigma)_\alpha = d\sigma_\alpha + \omega_\alpha\sigma_\alpha$, where (ω_j^i) is a $k \times k$ matrix of one-forms on U_α, called the **local representative of the connection** d_A **on** U_α. Then on $U_\alpha \cap U_\beta$,

$$\begin{aligned} d\sigma_\alpha + \omega_\alpha\sigma_\alpha &= g_{\alpha\beta}(d\sigma_\beta + \omega_\beta\sigma_\beta) \\ &= g_{\alpha\beta}(d(g_{\alpha\beta}^{-1}\sigma_\alpha) + \omega_\beta g_{\alpha\beta}^{-1}\sigma_\alpha) \\ &= d\sigma_\alpha + (g_{\alpha\beta} dg_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1})\sigma_\alpha \end{aligned}$$

Let $G \subset GL(k, \mathbb{F})$ be a Lie group, and $\mathfrak{g}$ be its Lie algebra. Note that if $g_{\alpha\beta} \in G, \omega_\beta \in \Gamma(T^*U_\beta \otimes \mathfrak{g})$, then $\omega_\alpha = g_{\alpha\beta} dg_{\alpha\beta}^{-1} + g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1} \in \Gamma(T^*(U_\alpha \cap U_\beta) \otimes \mathfrak{g})$. For example, if E is a complex line bundle with a Hermitian metric, then its structure group can be reduced to $G = U(1)$. A **unitary connection** on E is a connection whose local representatives take values in $Lie(U(1)) = \mathfrak{u}(1)$. In other words, these local representatives are pure imaginary 1-forms.

Inspired by the local representatives of connections on vector bundles, let's introduce connections on principal bundles. Given a principal G-bundle (P, M, π), for $x \in P$, we have the smooth map $V_x : G \to P,\ g \mapsto x \cdot g$, whose differential at identity is $v_x = dV_x\big|_e : \mathfrak{g} \to T_xP$. Then we have a short exact sequence:

$$0 \longrightarrow \mathfrak{g} \xrightarrow{v_x} T_xP \xrightarrow{\pi_*} T_{\pi(x)}M \longrightarrow 0$$

$v_x(\mathfrak{g}) \subset T_xP$ is called the **vertical subspace**. Intuitively, a connection on P is a section of the above short exact sequence, to be precise, a smooth choice of horizontal subspaces $H_x \subset T_xP$ such that $H_x \oplus v_x(\mathfrak{g}) = T_xP$ and H is a G-invariant distribution on P.

Definition 1.4. Let (P, M, π) be a principal G-bundle, $\omega \in \Omega^1(P, \mathfrak{g})$ is called a **connection on principal G-bundle P** if it satisfies:

1. $\forall x \in P$, $\omega(x) \circ v_x = id_{\mathfrak{g}}$
2. $\forall g \in G$, $R_g^*\omega = Ad(g^{-1}) \circ \omega$

It's not hard to verify that $H_x := Ker\ \omega(x)$ defines a G-invariant distribution on P and $H_x \oplus v_x(\mathfrak{g}) = T_xP$. Let $\rho : G \to GL(V)$ be a representation and $E = P \times_\rho V$. $\eta \in \Omega^k(P; V)$ is called **horizontal** if it's killed by any vertical vector $X \in v_x(\mathfrak{g})$. Define

$$\Omega^k_{basic}(P; V) := \left\{\eta \in \Omega^k(P; V) \mid \eta \textit{ is horizontal and } R_g^*\eta = \rho(g^{-1}) \circ \eta\right\}$$

We can show that there is a natural 1-1 correspondence between $\Omega^k_{basic}(P; V)$ and $\Omega^k(M; E)$.

Given a connection ω on P, we want to associate to it a connection $\nabla : \Omega^0(M; E) \to \Omega^1(M; E)$ on E. Under the natural correspondence above, it is equivalent to construct $\tilde{\nabla} : \Omega^0_{basic}(P; V) \to \Omega^1_{basic}(P; V),\ s \mapsto ds + \rho_*(\omega) \cdot s$, where $\rho_* = d\rho\big|_e : \mathfrak{g} \to \mathfrak{gl}(V)$. One can check that this gives a connection on E.

The above relation between connections on a principal bundle and connections on its associated vector bundle will come in handy when we define the $Spin^{\mathbb{C}}$ connection later.

A connection on a vector bundle can be viewed as some kind of "twisted exterior derivative" $d_A : \Omega^0(E) \to \Omega^1(E)$. As in the case of exterior derivative, we can extend d_A to be defined on all $\Omega^p(E)$ by requiring the following Leibniz rule to hold:

$$d_A(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge d_A\sigma,\ \forall \omega \in \Omega^p(M), \sigma \in \Gamma(E)$$

Definition 1.5. $\Omega = d_A \circ d_A : \Omega^0(E) \to \Omega^2(E)$ is called the **curvature of the connection** d_A.

Note that Ω is a tensor.

$$\begin{aligned}\forall f \in C^{\infty}(M), \quad \Omega(f\sigma) &= d_A \circ d_A(f\sigma) \\ &= d_A(df \otimes \sigma + f d_A \sigma) \\ &= -df \wedge d_A\sigma + df \wedge d_A \sigma + f d_A \circ d_A \sigma \\ &= f\Omega(\sigma)\end{aligned}$$

Therefore, we can derive its local representation as follows. Take local trivializations (U_α, Φ_α) and constant sections e_i as before. For any section $\sigma \in \Gamma(E)$, denote its local representative on U_α by $\sigma_\alpha = \sum\limits_{i=1}^{k} \sigma^i e_i$, then $d_A \circ d_A(\sigma_\alpha) = d_A \circ d_A(\sum\limits_{i=1}^{k} \sigma^i e_i) = \sum\limits_{i=1}^{k} \sigma^i d_A \circ d_A(e_i) = \sum\limits_{i,j=1}^{k} \sigma^i \Omega_i^j e_j = \Omega_\alpha \sigma_\alpha$, where $\Omega_\alpha = (\Omega_i^j)$ is a $k \times k$ matrix of two forms. On the other hand, $d_A \circ d_A(\sigma_\alpha) = (d + \omega_\alpha)(d\sigma_\alpha + \omega_\alpha \sigma_\alpha) = (d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha)\sigma_\alpha$. Hence $\Omega_\alpha = d\omega_\alpha + \omega_\alpha \wedge \omega_\alpha$. Moreover, $\Omega_\alpha \sigma_\alpha = g_{\alpha\beta}\Omega_\beta \sigma_\beta = g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}^{-1}\sigma_\alpha$, so $\Omega_\alpha = g_{\alpha\beta}\Omega_\beta g_{\alpha\beta}^{-1}$. In particular, when E is a complex line bundle, $\Omega_\alpha = \Omega_\beta$ defines a global two-form on M.

1.2 *Spin* and $Spin^{\mathbb{C}}$ Structure

Denote the universal cover of $SO(n)$ by $Spin(n)$. By the long exact sequence for homotopy groups of fiber bundle, we know that $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$, so $Spin(n)$ is a double cover of $SO(n)$.

Definition 1.6. Let (M, g) be an n-dimensional oriented Riemannian manifold and Fr be its frame bundle. A **Spin structure** on M is a principal $Spin(n)$-bundle $\hat{F}$ on M together with a map $\hat{F} \to Fr$ such that the following diagram commutes:

$$\begin{tikzcd} \hat{F} \times Spin(n) \arrow[r] \arrow[d] & \hat{F} \arrow[d] \arrow[dd, bend left] \\ Fr \times SO(n) \arrow[r] & Fr \arrow[d] \\ & M \end{tikzcd}$$

There is a $\mathbb{Z}_2$ action on $Spin(n) \times U(1)$, i.e. $\mathbb{Z}_2$ acts on the $Spin(n)$ factor by the deck transformation of the cover $\pi : Spin(n) \to SO(n)$ and on the $U(1)$ factor by multiplying -1. So we can define $Spin^{\mathbb{C}}(n) := (Spin(n) \times U(1))/\mathbb{Z}_2$.

Definition 1.7. Let (M, g) be an n-dimensional oriented Riemannian manifold and Fr be its frame bundle. A $Spin^{\mathbb{C}}$ **structure** on M is a principal $Spin^{\mathbb{C}}(n)$-bundle $\hat{F}$ on M together with a map $\hat{F} \to Fr$ such that the following diagram commutes:

$$\begin{tikzcd} \hat{F} \times Spin^{\mathbb{C}}(n) \arrow[r] \arrow[d] & \hat{F} \arrow[d] \arrow[dd, bend left] \\ Fr \times SO(n) \arrow[r] & Fr \arrow[d] \\ & M \end{tikzcd}$$

Spin structure does not always exist:

Theorem 1.1. *(M, g) admits a Spin structure if and only if the second Stiefel-Whitney class $w_2(TM) = 0$ in $H^2(M; \mathbb{Z}_2)$.*

But when $n = dimM = 4$, $Spin^{\mathbb{C}}$ structures always exist:

Theorem 1.2. *On any 4-dimensional oriented Riemannian manifold X, $Spin^{\mathbb{C}}$ structures exist and the set $\mathcal{S}_X$ of $Spin^{\mathbb{C}}$ structures on X is an affine space modelled on $H^2(X; \mathbb{Z})$.*

We will not prove the above two theorems. See section 3.1 of [5] for proofs.

From now on, we always assume that M is a closed oriented 4-dimensional Riemannian manifold. Denote the quanternion algebra by V. Then there is an isomorphism of $\mathbb{R}$-algebra:

$$V \cong \left\{ Q = \begin{pmatrix} t+iz & -x+iy \\ x+iy & t-iz \end{pmatrix} \;\middle|\; t, x, y, z \in \mathbb{R} \right\} = Span_{\mathbb{R}}\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{i} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Let $\langle\, , \,\rangle$ be the standard inner product on $\mathbb{R}^4$. Then through the above isomorphism, $\langle Q, Q\rangle = t^2 + x^2 + y^2 + z^2 = det\ Q$. By linear algebra we know that $SU(2) = \{Q \in V \mid \langle Q, Q\rangle = 1\}$. We have a representation:

$$\rho_0 : SU(2) \times SU(2) \to GL(V), \quad \rho_0(A_+, A_-)(Q) = A_- Q (A_+)^{-1}$$

Note that $\langle A_- Q(A_+)^{-1}, A_- Q(A_+)^{-1}\rangle = det(A_- Q(A_+)^{-1}) = det\ Q = \langle Q, Q\rangle$, and $SU(2) \times SU(2)$ is connected. So ρ_0 can be reduce to $\rho : SU(2) \times SU(2) \to SO(4) \subset GL(V)$.

Claim 1. *ρ is surjective*

Proof of Claim 1. Consider $A = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in SU(2)$,

$$\rho(A, I)(Q) = QA^{-1} = \begin{pmatrix} t+iz & -x+iy \\ x+iy & t-iz \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} e^{i\theta}(t+iz) & e^{-i\theta}(-x+iy) \\ e^{i\theta}(x+iy) & e^{-i\theta}(t-iz) \end{pmatrix}$$

$$\rho(I, A)(Q) = AQ = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} t+iz & -x+iy \\ x+iy & t-iz \end{pmatrix} = \begin{pmatrix} e^{-i\theta}(t+iz) & e^{-i\theta}(-x+iy) \\ e^{i\theta}(x+iy) & e^{i\theta}(t-iz) \end{pmatrix}$$

So (I, A) rotate the (t, z) and (x, y) planes in the same direction through the same angle θ, (A, I) rotate the (t, z) and (x, y) planes in opposite directions through the same angle θ. They together generate all the rotations in (t, z) and (x, y) planes. $\forall R \in SO(4)$, there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that R is the composition of a rotation in the $e_1 \wedge e_4$ plane with a rotation in the $e_2 \wedge e_3$ plane. To show that ρ is surjective, it suffices to show that there exists $(A_1, A_2) \in SU(2) \times SU(2)$ such that $\rho(A_1, A_2)(e_1) = \mathbf{1}, \cdots, \rho(A_1, A_2)(e_4) = \mathbf{k}$.

Since $e_1 \in SU(2)$, there exists $U_1 \in SU(2)$ such that $U_1 e_1 U_1^{-1} = diag\{e^{i\theta_1}, e^{-i\theta_1}\} =: f_1$.
Let $g_1 = diag\{e^{-i\theta_1/2}, e^{i\theta_1/2}\}$, then $g_1 f_1 g_1 = \mathbf{1}$. Next, let's deal with $e_4' = g_1 U_1 e_4 U_1^{-1} g_1$. Since $e_4' \in SU(2)$, there exists $U_4 \in SU(2)$ such that $U_4 e_4' U_4^{-1} = diag\{e^{i\theta_4}, e^{-i\theta_4}\} =: f_4$, note that orthogonality of e_1 and e_4 implies that f_4 is orthogonal to $\mathbf{1}$, so $f_4 = \pm\mathbf{k}$. If $f_4 = -\mathbf{k}$, then $\mathbf{j}(-\mathbf{k})(-\mathbf{j}) = \mathbf{k}$, $\mathbf{j}\mathbf{1}(-\mathbf{j}) = \mathbf{1}$. So we have found an element in $SU(2) \times SU(2)$ that rotates e_1 to $\mathbf{1}$ and e_4 to $\mathbf{k}$. Then $e_2 \wedge e_3$ is rotated to the plane orthogonal to (t, z) plane, that is the (x, y) plane. With an additional rotation in the (x, y) plane while fixing the (t, z) plane, we finally find $(A_1, A_2) \in SU(2) \times SU(2)$ such that $\rho(A_1, A_2)(e_1) = \mathbf{1}, \cdots, \rho(A_1, A_2)(e_4) = \mathbf{k}$. Hence we have proved the claim. $\square$

Since ρ is a surjective Lie group homomorphism and $dim\ SU(2) \times SU(2) = dim\ SO(4) = 6$, ρ is a smooth cover. $SU(2) \times SU(2)$ is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3$, thus simply-connected. So by definition $Spin(4) \cong SU(2) \times SU(2)$, $Spin^{\mathbb{C}}(4) \cong (SU(2) \times SU(2) \times U(1))/\pm 1$. So we can realize $Spin^{\mathbb{C}}(4)$ as:

$$Spin^{\mathbb{C}}(4) = \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} \middle| A_+, A_- \in SU(2), \lambda \in U(1) \right\}$$

The representation ρ extends to the representation $\rho^{\mathbb{C}} : Spin^{\mathbb{C}}(4) \to Gl(V)$ given by:

$$\rho^{\mathbb{C}} \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (Q) = (\lambda A_-)\ Q\ (\lambda A_+)^{-1}$$

We also have the following three important representations:

$$\rho_+ : Spin^{\mathbb{C}}(4) \to U(2), \quad \rho_+ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (w) = \lambda A_+ w,\ \forall w \in \mathbb{C}^2$$

$$\rho_- : Spin^{\mathbb{C}}(4) \to U(2), \quad \rho_- \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} (w) = \lambda A_- w,\ \forall w \in \mathbb{C}^2$$

$$\pi : Spin^{\mathbb{C}}(4) \to U(1), \quad \pi \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} = \lambda^2$$

Definition 1.8. Let M be an oriented Riemannian 4-manifold with a $Spin^{\mathbb{C}}$ structure. The $\mathbb{C}^2$ bundle S_+ (*resp.* S_-) associated to the representation ρ_+ (*resp.* ρ_-) is called the **positive (resp. negative) spinor bundle**. The complex line bundle L associated to the representation π is called the **determinant line bundle**. These three complex vector bundles can all be equipped with Hermitian metric.

Remark. $L \cong \Lambda^2 S_+ \cong \Lambda^2 S_-$ since they are all isomorphic to the associated line bundle of the principal $Spin^{\mathbb{C}}$-bundle with respect to the same representation π.

1.3 Clifford Multiplication and Quadratic Map

Let $V \subset End(\mathbb{C}^2)$ be the quaternion algebra. We have a complex linear map:

$$\theta : V \otimes \mathbb{C} \to End(\mathbb{C}^4), \quad \theta(Q) = \begin{pmatrix} 0 & -Q^* \\ Q & 0 \end{pmatrix}$$

Then $(\theta(Q))^2 = \begin{pmatrix} -Q^*Q & 0 \\ 0 & -QQ^* \end{pmatrix} = -detQ \cdot I = -\langle Q, Q \rangle I$. So according to the universal property of Clifford algebra, $End(\mathbb{C}^4)$ is the Clifford algebra generated by $(V \otimes \mathbb{C}, \langle\ ,\ \rangle)$ (which we will not talk about here). To write down a basis of $End(\mathbb{C}^4)$ explicitly, we take:

$$e_1 = \theta(\mathbf{1}) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad e_2 = \theta(\mathbf{k}) = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$e_3 = \theta(\mathbf{j}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \theta(\mathbf{i}) = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

Then

$$e_i e_j + e_j e_i = -2\delta_{ij} I = \begin{cases} -2I & if\ i = j \\ 0 & if\ i \neq j \end{cases}$$

Note that $I,\ e_i,\ e_i e_j\ (i < j),\ e_i e_j e_k\ (i < j < k),\ e_1 e_2 e_3 e_4$ form a basis of $End(\mathbb{C}^4)$. Therefore, we can regard the Clifford algebra as the complexified exterior algebra of V with a new multiplication. In fact, $\forall v \in V, \forall \omega \in \Lambda^k V$, we have the following useful formula:

$$v \cdot \omega = v \wedge \omega - \iota(v)\omega$$

where the interior product $\iota(v) : \Lambda^k V \to \Lambda^{k-1} V$ is defined as the adjoint of wedge product via:

$$\langle \iota(v)\omega, \theta \rangle = \langle \omega, v \wedge \theta \rangle,\ \forall \theta \in \Lambda^{k-1} V$$

We have the decomposition $\Lambda^2 V = \Lambda^2_+ V \oplus \Lambda^2_- V$, and $\Lambda^2_+ V$ has a basis:

$$e_1 e_2 + e_3 e_4 = \begin{pmatrix} -2i & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_1 e_3 + e_4 e_2 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_1 e_4 + e_2 e_3 = \begin{pmatrix} 0 & -2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence $\Lambda^2_+(V)$ corresponds to trace-free skew-Hermitian endomorphisms of $\mathbb{C}^2 \oplus 0$, which is isomorphic to $\mathfrak{su}(2)$. Similarly, $\Lambda^2_-(V)$ corresponds to trace-free skew-Hermitian endomorphisms of $0 \oplus \mathbb{C}^2$.

Since θ is a $Spin^{\mathbb{C}}(4)$-module homomorphism, it induces a bundle homomorphism (called **Clifford multiplication**) $cl : T^*M \cong TM \to End(S_+ \oplus S_-)$. This also induces bundle homomorphisms $cl : T^*M \oplus S_+ \to S_-$ and $cl : T^*M \oplus S_- \to S_+$. Moreover, we can extend Clifford multiplication to $cl : \Lambda^2 T^*M \otimes \mathbb{C} \to End(S_+ \oplus S_-)$. And it restricts to the positive Clifford multiplication $cl_+ : \Lambda^2_+ T^*M \otimes \mathbb{C} \to End(S_+)$.

Now we can define the quadratic map $q : S_+ \to i\Lambda^2_+ T^*X,\ \psi \mapsto cl_+^\dagger(\psi \otimes \psi^\dagger)$. In local coordinate, if $\psi = (a,b)^T$, then

$$\psi \otimes \psi^\dagger = (a,b)^T \otimes (\bar{a}, \bar{b}) = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}$$

whose trace free part is

$$\begin{aligned}
\psi \otimes \psi^\dagger - \frac{1}{2} tr(\psi \otimes \psi^\dagger) I = \frac{1}{2} \begin{pmatrix} |a|^2 - |b|^2 & 2a\bar{b} \\ 2b\bar{a} & |b|^2 - |a|^2 \end{pmatrix} \\
= \frac{i}{4}(|a|^2 - |b|^2)\ cl_+(e_1 \wedge e_2 + e_3 \wedge e_4) \\
+ \frac{i}{2} Im(a\bar{b})\ cl_+(e_1 \wedge e_3 + e_4 \wedge e_2) \\
+ \frac{i}{2} Re(a\bar{b})\ cl_+(e_1 \wedge e_4 + e_2 \wedge e_3)
\end{aligned}$$

Thus,

$$\begin{aligned}
q(\psi) &= cl_+^\dagger(\psi \otimes \psi^\dagger) \\
&= cl_+^\dagger(\psi \otimes \psi^\dagger - \frac{1}{2} tr(\psi \otimes \psi^\dagger) I) \\
&= i(|a|^2 - |b|^2)\ (e_1 \wedge e_2 + e_3 \wedge e_4) + 2i Im(a\bar{b})\ (e_1 \wedge e_3 + e_4 \wedge e_2) + 2i Re(a\bar{b})\ (e_1 \wedge e_4 + e_2 \wedge e_3)
\end{aligned}$$

1.4 $Spin^{\mathbb{C}}$ Connection

The representations:

$$\begin{tikzcd}
 & Spin^{\mathbb{C}}(4) \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\
SO(4) & & U(1)
\end{tikzcd}$$

induce an isomorphism of Lie algebras: $Lie(Spin^{\mathbb{C}}(4)) \cong Lie(SO(4)) \oplus Lie(U(1))$. Let $\mathcal{L}$ be the principal $U(1)$-bundle corresponding to the determinant line bundle L, then we have maps between principal bundles:

$$\begin{tikzcd}
 & \hat{F} \arrow[dl] \arrow[dr] & \\
Fr & & \mathcal{L}
\end{tikzcd}$$

The Levi-Civita connection on TM gives a canonical connection on Fr. Let A be a unitary connection on L, which gives a connection on $\mathcal{L}$. Pull these two connections back to $\hat{F}$, add them and apply the isomorphism $Lie(Spin^{\mathbb{C}}(4)) \cong Lie(SO(4)) \oplus Lie(U(1))$, then we obtain a connection on $\hat{F}$. This is called a $Spin^{\mathbb{C}}$ **connection** on $\hat{F}$, it induces $Spin^{\mathbb{C}}$ connections on spinor bundles S_+ and S_-.

Denote the $Spin^{\mathbb{C}}$ connection induced by the unitary connection A on L by d_A. In local coordinates, $(d_A\sigma)_\alpha = d\sigma_\alpha + \phi_\alpha\sigma_\alpha$, where $\phi_\alpha \in \Gamma(T^*M \otimes Lie(Spin^{\mathbb{C}}(4))) = \Gamma(T^*M \otimes (\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus i\mathbb{R}))$. So $\phi_\alpha = \frac{1}{2}aI + \frac{1}{2}\sum\limits_{i<j} \omega_{ij} e_i \cdot e_j$ where 1-forms $\omega_{ij} = -\omega_{ji} \in \Omega^1(M)$ are local representatives for the Levi-Civita connection and $a \in i\Omega^1(M)$ is the local representative for the unitary connection on L. The first $\frac{1}{2}$ comes form the square in the representation $\pi : Spin^{\mathbb{C}} \to U(1)$ and the second $\frac{1}{2}$ is explained in the following lemma:

Lemma 1.1. *([5], Lemma 3.2.4) Using the natural identification of $Lie(SO(n))$ and $Lie(Spin(n))$, the infinitesimal generator for the one parameter group $\theta \mapsto cos(\theta)1 + sin(\theta)e_ie_j \subset Spin(n)$ is $2e_i \wedge e_j$.*

So $\Omega_{ij} := d\omega_{ij} + \sum\limits_{k=1}^{4} \omega_{ik} \wedge \omega_{kj} = \sum\limits_{k,l=1}^{4} R_{ijkl} e_k^* \wedge e_l^*$.

1.5 Dirac Operator

Definition 1.9. Let A be a unitary connection on L. The **Dirac operator** D_A is the composition:

$$\Gamma(S_+ \oplus S_-) \xrightarrow{\nabla_A} \Gamma(T^*M \otimes (S_+ \oplus S_-)) \xrightarrow{cl} \Gamma(S_+ \oplus S_-)$$

Explicitly, if we pick a local orthonormal frame $\{e_i\}_{1\le i\le 4}$ of TM, then

$$\forall \psi \in \Gamma(S_+ \oplus S_-), \quad D_A(\psi) = cl(\sum_{i=1}^{4} \nabla_{e_i}^A \psi \otimes e_i^*) = \sum_{i=1}^{4} e_i \cdot \nabla_{e_i}^A \psi$$

If M is a Euclidean space and the spinor bundles are trivial, we can see that Dirac operator is the square root of the usual Laplacian, the following formula extends this intuition to the general case.

Theorem 1.3. *(Weitzenböck's formula) $D_A^2 = \nabla_A^\dagger \nabla_A + \frac{s}{4} + \frac{1}{2}cl(F_A)$ as operators on $\Gamma(S_+ \oplus S_-)$, where s is the scalar curvature of (M, g).*

Remark. If ψ is a section of S_+, then $D_A^2\psi = \nabla_A^\dagger \nabla_A \psi + \frac{s}{4}\psi + \frac{1}{2}cl_+(F_A^+)(\psi)$.

Proof. $\forall p \in M$, by choosing Gaussian normal coordinates, we have a moving frame $\{e_i\}_{1\le i\le 4}$ on a neighborhood of p such that $\nabla_{e_i} e_j(p) = 0$ and $e_i(p)$ form an orthonormal basis of T_pM. Then $\forall \psi \in \Gamma(S_+ \oplus S_-)$, at p,

$$
\begin{aligned}
D_A^2\psi &= (\sum_{i=1}^{4} e_i \cdot \nabla_{e_i}^A)(\sum_{j=1}^{4} e_j \cdot \nabla_{e_j}^A \psi) \\
&= \sum_{i,j=1}^{4} e_i e_j \nabla_{e_i}^A \nabla_{e_j}^A \psi \quad (\nabla_{e_i}^A e_j(p) = \nabla_{e_i} e_j(p) = 0) \\
&= -\sum_{i=1}^{4} \nabla_{e_i}^A \nabla_{e_i}^A \psi + \sum_{i<j}^{4} e_i e_j (\nabla_{e_i}^A \nabla_{e_j}^A - \nabla_{e_j}^A \nabla_{e_i}^A)\psi \\
&= -\sum_{i=1}^{4} \nabla_{e_i}^A \nabla_{e_i}^A \psi + \sum_{i<j} e_i e_j \Omega_A(e_i, e_j) \cdot \psi
\end{aligned}
$$

For the first term, we have the following claim:

Claim 2. $-\sum_{i=1}^{4} \nabla_{e_i}^A \nabla_{e_i}^A \psi(p) = \nabla_A^\dagger \nabla_A \psi(p)$

Proof of Claim 2. To prove the claim, firstly let's show that $\nabla_A^* \alpha = - * \nabla_A * \alpha,\ \forall \alpha \in \Gamma(T^*M \otimes (S_+ \oplus S_-))$.
By definition, $(\nabla_A \psi,\ \alpha)_{L^2} = (\psi,\ \nabla_A^* \alpha)_{L^2}$. On the other hand, $d\langle \psi(x),\ *\alpha(x) \rangle = \langle \nabla_A \psi(x),\ *\alpha(x) \rangle + \langle \psi(x),\ \nabla_A * \alpha(x) \rangle$, hence:

$$
\begin{aligned}
(\psi,\ \nabla_A^* \alpha)_{L^2} &= (\nabla_A \psi, \alpha)_{L^2} \\
&= \int_M \langle \nabla_A \psi(x),\ *\alpha(x) \rangle \\
&= -\int_M \langle \psi(x),\ \nabla_A * \alpha(x) \rangle \quad (by\ Stokes\ formula) \\
&= \langle \psi, - * \nabla_A * \alpha \rangle_{L^2}
\end{aligned}
$$

So at p, we have

$$
\begin{aligned}
\nabla_A^\dagger \nabla_A \psi &= - * \nabla_A (\sum_{i=1}^{4} \nabla_{e_i}^A \psi * (e_i^*)) \\
&= - * \sum_{i=1}^{4} \nabla_{e_i}^A \nabla_{e_i}^A \psi\ dvol \quad (de_i^*(e_j, e_k)(p) = e_j(e_i^*(e_k))(p) - e_k(e_i^*(e_j))(p) - e_i^*([e_j, e_k])(p) = 0) \\
&= -\sum_{i=1}^{4} \nabla_{e_i}^A \nabla_{e_i}^A \psi
\end{aligned}
$$

$\square$

Thus we have proved the claim. For the second term, we have:

$$
\begin{aligned}
\sum_{i<j} e_i e_j \Omega_A(e_i, e_j)\cdot\psi &= \frac{1}{2}\sum_{i<j} F_A(e_i,e_j)e_ie_j\cdot\psi + \frac{1}{8}\sum_{i,j,k,l}\Omega_{kl}(e_i,e_j)e_ie_je_ke_l\cdot\psi \\
&= \frac{1}{2}\sum_{i<j} F_A(e_i,e_j)e_ie_j\cdot\psi + \frac{1}{8}\sum_{i,j,k,l}R_{ijkl}e_ie_je_ke_l\cdot\psi \\
&= \frac{1}{2}\sum_{i<j} F_A(e_i,e_j)e_ie_j\cdot\psi + \frac{1}{8}(\sum_{i,j}R_{ijij}e_ie_je_ie_j\cdot\psi + \sum_{i,j}R_{ijji}e_ie_je_je_i\cdot\psi) \\
&= \frac{1}{2}\sum_{i<j} F_A(e_i,e_j)e_ie_j\cdot\psi + \frac{s}{4}\psi
\end{aligned}
$$

Note that in the third equality we have used the properties of R_{ijkl} to cancel terms. In particular, we have used the first Bianchi identity. Assembling all the results we get:

$$
D_A^2\psi = \nabla_A^\dagger\nabla_A\psi + \frac{1}{2}\sum_{i<j}F_A(e_i,e_j)e_ie_j\cdot\psi + \frac{s}{4}\psi = \nabla_A^\dagger\nabla_A\psi + \frac{s}{4}\psi + \frac{1}{2}cl(F_A)(\psi)
$$

$\square$

By the theory of elliptic operators, the kernels of $D_A^+ : \Gamma(S_+) \to \Gamma(S_-)$ and $D_A^- : \Gamma(S_-) \to \Gamma(S_+)$ are finite dimensional, then we may define $ind\ D_A^+ := dim_{\mathbb{C}}(Ker(D_A^+)) - dim_{\mathbb{C}}(Ker(D_A^-))$.

Theorem 1.4 (Atiyah-Singer Index Theorem for the Dirac Operator)**.**

$$
ind\ D_A^+ = \frac{1}{8}\int_M c_1(L)^2 - \frac{1}{8}\tau(M)
$$

where $\tau(M) = b_2^+(M) - b_2^-(M)$ is the signature of M.

See section 13 of [3] for a proof.

2 Seiberg-Witten Equations and Compactness of the Moduli Space

In this part, we will introduce the Seiberg-Witten equation and prove that the moduli space of its solutions is compact. The proof follows the arguments in [5].

2.1 Seiberg-Witten Equations

Let (M, g) be a closed oriented 4-dimensional Riemannian manifold with a $Spin^{\mathbb{C}}$ structure. The following equations are called the **Seiberg-Witten equations**:

$$
\begin{aligned}
&D_A^+\psi = 0 \\
&F_A^+ = q(\psi) + i\phi
\end{aligned}
\tag{2.1}
$$

where A is a unitary connection on L, $\psi \in \Gamma(S_+)$, F_A^+ is the self-dual part of F_A, $q : S_+ \to i\Lambda_+^2 T^*M$ is the quadratic map, $\phi \in \Lambda_+^2 T^*M$ is a prescribed self-dual two form, called the **perturbation term**.

2.2 Basic Properties of the Moduli Space

Define the **configuration space** $\mathcal{A} = \{(A,\psi)|A \textit{ is a unitary connection on } L, \psi \in \Gamma(S_+)\} = \{(d_{A_0}+a,\psi)|a \in i\Omega^1(M), \psi \in \Gamma(S_+)\}$, where d_{A_0} is a fixed unitary connection on L. Let $\mathcal{G} = C^\infty(M,S^1)$ be the **group of gauge transformations**, which acts on $\mathcal{A}$ by $g\cdot(d_{A_0}+a) = (d_{A_0}+a+gd(g^{-1}), g\psi)$. For any fixed $p_0 \in M$, let $\mathcal{G}_0 = \{g \in \mathcal{G}|g(p_0)=1\}$ be the **group of based gauge transformations**. Note that $\mathcal{G}_0$ acts freely on $\mathcal{A}$ but $\mathcal{G}$ doesn't.

Let $\mathcal{B} = \mathcal{A}/\mathcal{G}$ and $\tilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}_0$. Constant gauge transformations fix the points $(d_{A_0}+a,0)$ (called the **reducible elements**), so $\mathcal{B}$ has singularities at these points.

Now we can define the moduli spaces of the Seiberg-Witten equations: $\mathcal{M}_\phi = \{[A,\psi] \in \mathcal{B}|D_A^+\psi = 0, F_A^+\psi = q(\psi)+i\phi\}$ and $\tilde{\mathcal{M}}_\phi = \{[A,\psi] \in \tilde{\mathcal{B}}|D_A^+\psi = 0, F_A^+\psi = q(\psi)+i\phi\}$. These two moduli spaces are endowed with the following merits.

Theorem 2.1. *(Basic Properties of the Moduli Space)*

1. *$\mathcal{M}_\phi$ is compact.*
2. *For generic ϕ, $\tilde{\mathcal{M}}_\phi$ is a finite-dimensional orientable smooth manifold with a smooth $U(1)$-action. If $b_2^+ > 0$, then for generic ϕ, $\mathcal{M}_\phi$ is smooth and $\tilde{\mathcal{M}}_\phi \to \mathcal{M}_\phi$ is a principal S^1-bundle.*
3. *For generic ϕ, $dim(\mathcal{M}_\phi) = b_1 - b_2^+ - 1 + \frac{c_1^2-\tau}{4}$, where $\tau = b_2^+ - b_2^-$ is the signature of M, $c_1^2 = Q(c_1(L),c_1(L))$, where Q is the intersection form of M.*
4. *If $(g_0,\phi_0),(g_1,\phi_1)$ are generic (g_0,g_1 are Riemannian metrics, ϕ_0,ϕ_1 are perturbation terms in the Seiberg-Witten equations), then for a generic path (g_t,ϕ_t) connecting them, $\tilde{W} = \{t, \tilde{\mathcal{M}}_{g_t,\phi_t}\}$ is an oriented compact smooth manifold with boundary $-\tilde{\mathcal{M}}_{g_0,\phi_0} + \tilde{\mathcal{M}}_{g_1,\phi_1}$.*

Remark. If we change q to $-q$ in the Seiberg-Witten equations, then the compactness of the moduli space will fail. We will see this in the proof of Lemma 2.1.

We will prove the compactness, proof of the other three properties can be found in [5].

2.3 Proof of the Compactness

For the analysis techniques used in this part, we refer to the appendix of [1].

Lemma 2.1. *(Pointwise Bound on ψ)*
Let $s(x)$ be the scalar curvature of X at x. For any solution (A,ψ) to the Seiberg-Witten equations and any $x \in X$, we have $|\psi(x)|^2 \leq max(\max_{y\in X}(2|\phi(y)| - \frac{s(y)}{2}), 0)$.

Proof. Let's consider a point $x_0 \in X$ at which $|\psi(x)|^2$ achieves its maximum. It suffices to show that $|\psi(x_0)|^2 \leq max(2|\phi(x_0)| - \frac{s(x_0)}{2}, 0)$.

Plugging the Seiberg-Witten equations into the *Weitzenböck* formula $D_A^2(\psi) = \nabla_A^*\nabla_A(\psi) + \frac{s}{4}\psi + \frac{1}{2}cl_+(F_A^+)(\psi)$, we obtain $\nabla_A^*\nabla_A(\psi) + \frac{s}{4}\psi + \frac{|\psi|^2}{2}\psi + icl_+(\phi)(\psi) = 0$ since $cl_+(q(\psi)) = cl_+(cl_+^\dagger(\psi \otimes \psi^\dagger)) = 2(\psi \otimes \psi^\dagger - \frac{1}{2}|\psi|^2 Id)$. Taking pointwise inner product with $\psi(x)$ yields

$$\langle \nabla_A^*\nabla_A(\psi(x)), \psi(x)\rangle + \frac{s(x)}{4}|\psi(x)|^2 + \frac{|\psi(x)|^4}{2} + i\langle cl(\phi(x))(\psi(x)), \psi(x)\rangle = 0$$

In particular, since $cl(\phi)$ is skew-Hermitian, $\langle \nabla_A^*\nabla_A(\psi(x)), \psi(x)\rangle$ is real.

Since x_0 is a maximum of $|\psi(x)|^2$,

$$\begin{aligned}
0 \leq \Delta(|\psi(x_0)|^2) &= -\sum_{i=1}^{4} \frac{\partial^2}{\partial e_i^2}\langle \psi(x_0), \psi(x_0)\rangle \\
&= -\sum_i (\langle \nabla_{e_i}\nabla_{e_i}\psi(x_0), \psi(x_0)\rangle + \langle \psi(x_0), \nabla_{e_i}\nabla_{e_i}\psi(x_0)\rangle) - 2\sum_i |\nabla_{e_i}\psi(x_0)|^2 \\
&= 2Re(\langle \nabla_A^*\nabla_A\psi(x_0), \psi(x_0)\rangle) - 2\sum_i |\nabla_{e_i}\psi(x_0)|^2
\end{aligned}$$

Therefore,

$$0 \leq 2Re(\langle \nabla_A^*\nabla_A\psi(x_0), \psi(x_0)\rangle) = -\frac{s(x_0)}{4}|\psi(x_0)|^2 - \frac{|\psi(x_0)|^4}{2} - i\langle cl(\phi(x_0))(\psi(x_0)), \psi(x_0)\rangle$$

$$\Rightarrow \frac{|\psi(x_0)|^4}{2} \leq -\frac{s(x_0)}{4}|\psi(x_0)|^2 - i\langle cl(\phi(x_0))(\psi(x_0)), \psi(x_0)\rangle \leq -\frac{s(x_0)}{4}|\psi(x_0)|^2 + 2|\phi(x_0)||\psi(x_0)|^2$$

So if ψ is not identically zero, then $|\psi(x_0)|^2 \leq 2|\phi(x_0)| - \frac{s(x_0)}{2}$. $\square$

Lemma 2.2. *(L^2 Bound on $\nabla_A\psi$) There exists a constant C_0 depending only on X, ϕ such that $\|\nabla_A\psi\|_{L^2} \leq C_0$.*

Proof. Take L^2 inner product of $\nabla_A^*\nabla_A(\psi) + \frac{s}{4}\psi + \frac{|\psi|^2}{2}\psi + icl(\phi)(\psi) = 0$ with ψ. And then use the pointwise a priori bound of ψ in the Lemma 2.1 and compactness of the manifold X. $\square$

Lemma 2.3. *(Pointwise and L^2 Bounds on F_A^+)*
There exist constants k, K depending on X, ϕ such that $\forall x \in X$

$$|F_A^+(x)| \leq k, \quad \|F_A^+\|_{L^2} \leq K$$

Proof. By the second Seiberg-Witten equation,

$$|F_A^+(x)| \leq \frac{|\psi(x)|^2}{2} + |\phi(x)| \leq \frac{1}{2}\, max(\max_{y\in X}(2|\phi(y)| - \frac{s(y)}{2}), 0) + \max_{x\in X}|\phi(x)| =: k$$

So $\|F_A^+\|_{L^2} \leq k\sqrt{vol(X)} =: K$. $\square$

Lemma 2.4. *(L^2 Bound on dF_A^+)*
There is a constant C depending only on X, ϕ such that for any solution (A, ψ) to the Seiberg-Witten equations we have $\|dF_A^+\|_{L^2} \leq C$.

Proof. Let $\nabla : \Omega^2(X) = \Omega^0(X; \Lambda^2 T^*X) \to \Omega^1(X; \Lambda^2 T^*X)$ denote the Levi-Civita connection.
Let $Alt : \Omega^1(X; \Lambda^2 T^*X) \to \Omega^3(X)$ be the skew symmetrization. Since $d = Alt \circ \nabla$, it suffices to bound ∇F_A^+.
By the second Seiberg-Witten equation,

$$\nabla F_A^+ = cl^\dagger(\nabla_A(\psi) \otimes \psi^* + \psi \otimes \nabla_A \psi^* - Re\langle \nabla_A \psi, \psi \rangle Id) + i \nabla \phi$$

Since we have a priori bounds on $\|\nabla_A \psi\|_{L^2}$ and $\|\psi\|_{L^\infty}$, they together gives a bound for $\|\nabla F_A^+\|_{L^2}$. ☐

Corollary 2.1. *(L_1^2 Bound on F_A^+)*
There is a constant C_1 depending only on X, ϕ such that for any solution (A, ψ) to the Seiberg-Witten equations we have $\|F_A^+\|_{L_1^2} \leq C_1$.

Proof.

$$d \oplus d^* : L_l^2(\Lambda_+^2(T^*X)) \to L_{l-1}^2(\Lambda^3(T^*X)) \oplus L_{l-1}^2(\Lambda^1(T^*X))$$

is an elliptic operator. So by Fredholm alternative, its image is closed. $L_l^2(\Lambda_+^2(T^*X)) = \mathcal{H} \oplus \mathcal{H}^\perp$ is an orthogonal decomposition, where $\mathcal{H} = Ker(\Delta) = Ker(d \oplus d^*)$. Then $d \oplus d^* : \mathcal{H}^\perp \to Im(d \oplus d^*)$ is a continuous bijection between Banach spaces. So by the inverse map theorem, its inverse is also continuous. $\forall F \in L_l^2(\Lambda_+^2(T^*X))$, let $\Pi(F)$ be the projection of F to $\mathcal{H}^\perp$. Then there exists constants C', C'' depending on l, X such that $\|\Pi(F)\|_{L_l^2} \leq C'\|(d \oplus d^*)(F)\|_{L_{l-1}^2} \leq C''\|dF\|_{L_{l-1}^2}$. The second inequality is because $d^*F = - * d * F = - * dF$.

Let $F_A^+ = H^+ + B$ be the orthogonal decomposition mentioned above. Then $\|B\|_{L_1^2} \leq C''\|dF_A^+\|_{L^2} \leq CC''$. On the other hand, there exists a constant C''' depending only on X such that $\|H^+\|_{L_1^2} \leq C'''\|F_A^+\|_{L^2} \leq C'''K$ since the projection is continuous and $dH = 0$. Setting $C_1 = C'''K + CC''$ accomplishes the proof. ☐

Lemma 2.5. *(Gauge-Fixing Lemma)*
Fix a unitary C^∞ connection A_0 on L. Then $\forall l \geq 0$, $\exists$ constants $K', C_2 > 0$ depending on X, A_0, l such that: for any L_l^2 unitary connection A on L, there is an L_l^2 gauge transformation σ such that $\sigma^* A = A_0 + \alpha$, where $\alpha \in L_l^2(T^*X \otimes i\mathbb{R})$, $d^*\alpha = 0$ and $\|\alpha\|_{L_l^2} \leq C_2\|F_A^+\|_{L_{l-1}^2} + K'$.

Proof. Since A is a unitary connection, we can write $A = A_0 + \alpha_0$ for some $\alpha_0 \in L_l^2(T^*X \otimes i\mathbb{R})$. Denote the orthogonal complement of constant functions in $L_l^2(X; i\mathbb{R})$ by $\mathcal{I}_{L_l^2}$. For any constant function $f(x) \equiv c$, $(d^*\alpha_0, c) = (\alpha, dc) = 0$ implies $d^*\alpha_0 \in \mathcal{I}_{L_{l-1}^2}$.

Since Δ is elliptic and self-adjoint, by similar arguments as in the proof of the Corollary 2.1, we know that there is a continuous linear map $\Delta^{-1} : \mathcal{I}_{L_{l-1}^2} \to \mathcal{I}_{L_{l+1}^2}$. Let $s_0 = -\Delta^{-1}(d^*(\alpha_0)) \in L_{l+1}^2(i\mathbb{R}), \sigma_0 = exp(s_0)$. Then σ_0 is an L_{l+1}^2 guage transformation. Let $\alpha_1 = \alpha_0 + ds_0 \in L_l^2(T^*X \otimes i\mathbb{R})$. Then:

$$\sigma_0^* A = \sigma_0^{-1} d\sigma_0 + \sigma_0^{-1}(A_0 + \alpha_0)\sigma_0 = A_0 + \alpha_0 + ds_0 = A_0 + \alpha_1$$

and:

$$d^*\alpha_1 = d^*\alpha_0 - d^* d \Delta^{-1} d^* \alpha_0 = 0$$

Next, let's take up the estimates. We have an elliptic operator $d^* \oplus d^+ : L^2_l(T^*X \otimes i\mathbb{R}) \to L^2_{l-1}(i\mathbb{R} \oplus (\Lambda^2_+ T^*X \otimes i\mathbb{R}))$ whose kernel consists of the harmonic 1-forms. We can decompose α_1 into $h+\beta$, where h is the harmonic component, and β is the orthogonal complement component.

For the orthogonal component β, by similar arguments as in the proof of the Corollary 2.1, there is a constant C_2 depending on X and l such that:

$$\|\beta\|_{L^2_l} \leq C_2 \|(d^*\beta, d^+\beta)\|_{L^2_{l-1}} = C_2 \|d^+\beta\|_{L^2_{l-1}}$$

Note that $d^+\beta = d^+\alpha_1 = d^+(\sigma_0^* A - A_0) = F^+_A - F^+_{A_0}$. So

$$\|\beta\|_{L^2_l} \leq C_2 \|F^+_A - F^+_{A_0}\|_{L^2_{l-1}} \leq C_2 \|F^+_A\|_{L^2_{l-1}} + C_2 \|F^+_{A_0}\|_{L^2_{l-1}}$$

For the harmonic component h, since the quotient of the space of pure imaginary harmonic one-forms by those with periods in $2\pi i\mathbb{Z}$ is a compact torus (by Hodge theory), there is a constant K_1 depending on X, l such that for any pure imaginary harmonic one-form h, we have a decomposition $h = h_1 + h_2$ where $\|h_1\|_{L^2_l} \leq K_1$, and h_2 has periods in $2\pi i\mathbb{Z}$. Moreover, for h_2, by picking a base point $x_0 \in X$ and integrating h_2 along all smooth paths, we get a C^∞ function $\tilde{\varphi} : \tilde{X} \to i\mathbb{R}$, where $\tilde{X}$ is the universal cover of X. Since the periods of h_2 lies in $2\pi i\mathbb{Z}$, $\tilde{\varphi}$ descends to $\varphi : X \to i\mathbb{R}/2\pi i\mathbb{Z} = S^1$ and $\varphi^{-1} d\varphi = h_2$. So $h = h_1 + \varphi^{-1} d\varphi$. Set $\alpha = h_1 + \beta$. Then:

$$(\varphi^{-1})^*(A_0 + \alpha_1) = -\varphi^{-1} d\varphi + A_0 + \beta + h_1 + h_2 = A_0 + \alpha$$

So

$$(\varphi^{-1})^* \sigma_0^* A = (\varphi^{-1})^*(A_0 + \alpha_1) = A_0 + \alpha$$

$$d^*\alpha = d^* h_1 + d^*\beta = d^* h_1 + d^*\alpha_1 - d^* h = 0$$

$$\|\alpha\|_{L^2_l} \leq \|\beta\|_{L^2_l} + \|h_1\|_{L^2_l} \leq C_2 \|F^+_A\|_{L^2_{l-1}} + C_2 \|F^+_{A_0}\|_{L^2_{l-1}} + K_1$$

Set $\sigma = \sigma_0 \circ \varphi^{-1}$ and $K' = C_2 \|F^+_{A_0}\|_{L^2_{l-1}} + K_1$, then we've finished the proof. □

Corollary 2.2. *(L^2_2 Bound on α)*
Let A_0 be a fixed C^∞ unitary connection on L. There is a constant K_2 depending only on X, A_0, ϕ such that for any solution (A, ψ) to the Seiberg-Witten equations we have a connection $A' = A_0 + \alpha$ gauge equivalent to A with $d^*\alpha = 0$ and $\|\alpha\|^2_{L^2_2} \leq K_2$.

Proof. By the gauge-fixing lemma and the bound on F^+_A. □

Theorem 2.2. *(L^2_l Bound on (A, ψ))*
Suppose (A, ψ) is a solution to the Seiberg-Witten equations and that we have fixed gauge so that $A = A_0 + \alpha$ where A_0 is a fixed C^∞ unitary connection on L with $d^*\alpha = 0$ and the projection of α into the space of harmonic forms contained a given fundamental domain modulo the lattice of harmonic forms with periods in $2\pi i\mathbb{Z}$. Then for $\forall l \geq 2$, there is a constant $C(l)$ depending only on X, A_0, l, ϕ such that:

$$\|\alpha\|^2_{L^2_l} + \|\psi\|^2_{L^2_l} \leq C(l)$$

where the L^2_l-norm of ψ is taken with respect to ∇_{A_0}.

Proof. **Step 1:** Let's show that ψ is bounded in L_3^2 and α is bounded in L_4^2.

By the Dirac equation, we have $0 = D_A\psi = D_{A_0}\psi + cl(\alpha)(\psi)$. Suppose D is an elliptic operator of order 1, then $\forall 1 < p < n = 4$ and $q = \frac{np}{n-p} = \frac{4p}{4-p}$, there is a constant C depending on X, p such that

$$\|f\|_{L^q} \leq C(\|Df\|_{L^p} + \|f\|_{L^p})$$

Let $p = 2$, $D = d^+ \oplus d^*$ and $f = \alpha$. We get $\|\alpha\|_{L^4} \leq C(\|d^+\alpha\|_{L^2} + \|\alpha\|_{L^2})$. Since we know that α is bounded in L_2^2, the above inequality implies that α is bounded in L^4. Moreover, ψ is bounded pointwise, so $D_{A_0}\psi = -cl(\alpha)(\psi)$ is bounded in L^4.

Since D_{A_0} is an elliptic operator, by similar arguments as in Lemma 2.1, the component of ψ orthogonal to the kernel of D_{A_0} is bounded in L_1^4. Since ψ is bounded pointwise, the $Ker(D_{A_0})$ component of ψ is bounded in L^2, hence bounded in L_k^2 for all k by the fundamental inequality for elliptic operators. So by the Sobolev embedding theorem, the $Ker(D_{A_0})$ component of ψ is bounded in C^r for all r. Therefore, ψ is bounded in L_1^4.

Since α is bounded in L_2^2, ψ is bounded in L_1^4, by the Sobolev multiplication $L_2^2 \otimes L_1^4 \to L_1^3$ we know that $D_{A_0}\psi$ is bounded in L_1^3. Again by the fundamental inequality for elliptic operators, we see that ψ is bounded in L_2^3. Then by the Sobolev multiplication $L_2^2 \otimes L_2^3 \to L_2^2$, $D_{A_0}\psi$ is bounded in L_2^2, so ψ is bounded in L_3^2.

So by the curvature equation $F_A^+ = q(\psi) + i\phi$ and the Sobolev multiplication $L_3^2 \otimes L_3^2 \to L_3^2$, F_A^+ is bounded in L_3^2. Finally, by the gauge fixing lemma α is bounded in L_4^2.

Step 2: Bootstraping arguments.

Suppose we have obtained $L_l^2 (l \geq 3)$ bounds for α and ψ. Then F_A^+ is bounded in L_l^2 by the curvature equation and the Sobolev multiplication $L_l^2 \otimes L_l^2 \to L_l^2$. And by the gauge fixing lemma α is bounded in L_{l+1}^2.

From the Dirac equation and the Sobolev multiplication $L_l^2 \otimes L_l^2 \to L_l^2$ we deduce that $D_{A_0}\psi$ is bounded in L_l^2, hence ψ is bounded in L_{l+1}^2. So by induction on l we have proved the theorem. ☐

Corollary 2.3. *(Compactness)*
The moduli space of solutions to the Seiberg-Witten equations is compact.

Proof. Given any sequence of solutions (A_n, ψ_n), by Theorem 2.3 we may assume $A_n = A_0 + \alpha_n$ after applying proper L_3^2 gauge transformations, where A_0, α_n, ψ_n satisfy the conditions in Theorem 2.3. So by Sobolev embedding theorem they are all smooth. And by Rellich theorem, for $\forall l \geq 2$ we can find a subsequence in (α_n, ψ_n) which converges in L_l^2. To prove the compactness of the moduli space, we need to find a converge subsequence in C^∞ topology.

By Cantor's diagonal method, we can find a subsequence in (α_n, ψ_n) that converges in all L_l^2. Then by the Sobolev embedding theorem, this subsequence must converge in all C^r. So this subsequence converges in C^∞ topology to a C^∞ solution in the moduli space. ☐

3 Seiberg-Witten Invariants of Symplectic Manifolds

We're going to define the Seiberg-Witten invariants in this part. Then we will explain Taubes' Theorem 3.1, which reveals the special structure of the Seiberg-Witten invariants of symplectic 4-manifolds. The proof follows the arguments in [2].

3.1 Seiberg-Witten Invariants

Definition 3.1. Let X be a closed oriented smooth 4-manifold with $b_2^+ > 1$. We define the **Seiberg-Witten invariant** $SW_X : \mathcal{S}_X \to \mathbb{Z}$ as follows:

1. If $b_2^+ - b_1$ is even, then $SW_X := 0$.
2. If $b_2^+ - b_1$ is odd, then $dim(\mathcal{M}_{s,\phi}) = b_1 - b_2^+ - 1 + \frac{c_1^2 - \tau}{4} = b_1 - b_2^+ - 1 + 2ind(D_A^+)$ is even. Let $d(s) = \frac{1}{2}dim(\mathcal{M}_{s,\phi})$.
   - If $d(s) < 0$, then $SW_X(s) := 0$
   - If $d(s) = 0$, then $\mathcal{M}_{s,\phi}$ is a finite set of points with orientations $sgn(p) = \pm 1$. $SW_X(s) := \sum_{p \in \mathcal{M}_{s,\phi}} sgn(p)$.
   - If $d(s) > 0$, $SW_X(s) := \int_{\mathcal{M}_{s,\phi}} e^d$, where $e \in H^2(\mathcal{M};\mathbb{Z})$ is the first Chern class of the associated line bundle of $\tilde{\mathcal{M}} \to \mathcal{M}$.

Remark. We will not prove the following two facts, but they will play important roles in the proof of Taubes' theorem.

- The Seiberg-Witten invariants are well-defined, not depending on the choice of metric g and perturbation term ϕ. The proof depends on the last property in Theorem 2.1 and $b_2^+ > 1$ which allows us to perturb ϕ_t in order to avoid the emergence of reducible solutions. The freedom of choosing the perturbation term is significant in the proof of Taubes' theorem 3.1.
- There is a charge conjugation involution $s \mapsto \bar{s}$ on $\mathcal{S}_X$ which sends $c_1(L)$ to $-c_1(L)$, and $SW(\bar{s}) = \pm SW(s)$. This symmetry manifests itself in the symplectic case.

3.2 The Canonical $Spin^{\mathbb{C}}$ Structure on Symplectic 4-Manifolds

We are going to compute the Seiberg-Witten invariants for the simplest $Spin^{\mathbb{C}}$ structures of symplectic manifolds. Let X be a closed smooth 4-manifold with $b_2^+ > 1$. Moreover, suppose there is a symplectic form ω on X. Then X has a canonical orientation given by $\omega \wedge \omega$ and there is a canonical $Spin^{\mathbb{C}}$ structure $s_0 \in \mathcal{S}_X$ defined as follows.

First, ω determines a homotopy class of almost complex structures. We can pick a ω-compatible almost complex structure J, that is, a bundle map $J : TX \to TX$ such that $J^2 = -I$ and $g(u,v) := \omega(u, Jv)$ is a Riemannian metric on X. We can choose local coordinates $\phi : U \to \mathbb{R}^4,\ p \mapsto (x_1, x_2, y_1, y_2)$ such that $\omega = dx_1 \wedge dx_2 + dy_1 \wedge dy_2$, and $e_1 = \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial x_2} = J(\frac{\partial}{\partial x_1}), e_3 = \frac{\partial}{\partial y_1}, e_4 = \frac{\partial}{\partial y_2} = J(\frac{\partial}{\partial y_1})$ form a local orthonormal frame. Moreover, we have a local frame $\frac{\partial}{\partial z} = \frac{1}{2}(e_1 - ie_2), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(e_1 + ie_2), \frac{\partial}{\partial w} = \frac{1}{2}(e_3 - ie_4), \frac{\partial}{\partial \bar{w}} = \frac{1}{2}(e_3 + ie_4)$ for $TM \otimes \mathbb{C}$. Accordingly, we have a local coframe $dz = dx_1 + idx_2, d\bar{z} = dx_1 - idx_2, dw = dy_1 + idy_2, d\bar{w} = dy_1 - idy_2$ for $T^*X \otimes \mathbb{C}$. So the almost complex structure J yields a decomposition $\Lambda^* T^* X \otimes \mathbb{C} = \bigoplus_{p,q} \Lambda^p T^{1,0} \otimes \Lambda^q T^{0,1}$, where $T^{1,0}$ is the holomorphic part of $T^*X \otimes \mathbb{C}$, and $T^{0,1}$ is the antiholomorphic part of $T^*X \otimes \mathbb{C}$. Denote $K = T^{2,0} = \Lambda^2 T^{1,0}$ and $K^{-1} = T^{0,2} = \Lambda^2 T^{0,1}$.

Once we choose a $Spin^{\mathbb{C}}$ structure on X, since $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$ locally, $cl_+(\omega) : S_+ \to S_+$ splits S_+ into $\pm 2i$ eigenspaces. Let E be the subbundle corresponding to the $-2i$ eigenspace, which is a complex line bundle on X. Then we define the canonical $Spin^{\mathbb{C}}$ structure s_0 to be the one for which E is trivial. And by the classification theorem for complex line bundles $Vect_1^{\mathbb{C}}(X) \cong H^2(X;\mathbb{Z})$, so we have a 1-1 correspondence $\mathcal{S}_X \to H^2(X;\mathbb{Z}),\ s \mapsto c_1(E)$.

The Riemannian metric g on TM extends to a Hermitian metric on $TM \otimes \mathbb{C}$. So we can reduce its transition functions to $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(2)$, thus the transition functions for $T^{0,1}$ are $g_{\alpha\beta}$, and the transition functions for $K^{-1} = T^{0,2}$ are $det \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$.

There is a canonical representation

$$j : U(2) \to Spin^{\mathbb{C}}(4), \quad A \mapsto \begin{pmatrix} 1 & & \\ & det(A) & \\ & & A \end{pmatrix}$$

Then the transition functions $j \circ g_{\alpha\beta}$ yields a $Spin^{\mathbb{C}}$ structure. It coincides with the canonical $Spin^{\mathbb{C}}$ structure s_0 defined previously.

Lemma 3.1. *There are natural identifications:*

$$S_+ = (T^{0,0} \oplus T^{0,2}) \otimes E = E \oplus (K^{-1} \otimes E)$$
$$S_- = T^{0,1} \otimes E$$

and the formula for Clifford multiplication by $v \in T^*X \otimes \mathbb{C}$ *acting on* $\alpha \in T^{0,*} \otimes E$ *is:*

$$cl(v)(\alpha) = \sqrt{2}(v^{0,1} \wedge \alpha - \iota(\overline{v^{1,0}})(\alpha)) \tag{3.1}$$

Proof. For the first equation,denote by E' the $2i$ eigenspace of ω acting on S_+. Then $S_+ = E \oplus E'$. Recall that we have the extended Clifford multiplication $cl : \Lambda^2 T^* X \otimes \mathbb{C} \to End(S_+ \oplus S_-)$. So there is an isomorphism (indeed, an isometry): $K^{-1} \otimes E \to E',\ (d\bar{z} \wedge d\bar{w}) \otimes \psi \mapsto \frac{1}{2} cl(d\bar{z} \wedge d\bar{w})(\psi)$, since locally

$$\frac{1}{2} cl_+(d\bar{z} \wedge d\bar{w}) = \frac{1}{2} cl_+((e_1 \wedge e_3 - e_2 \wedge e_4) - i(e_1 \wedge e_4 + e_2 \wedge e_3)) = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$$

Similarly, for the second equation, there is an isomorphism $\Omega^{0,1} \otimes E \to S_-,\ (\lambda d\bar{z} + \mu d\bar{w}) \otimes \psi \mapsto \frac{1}{\sqrt{2}} \lambda cl(d\bar{z})(\psi) + \frac{1}{\sqrt{2}} \mu cl(d\bar{w})(\psi)$.

Once we have the above identification, we can prove the formula 3.1 by merely writing down all the local representatives and performing matrix multiplications. It's tedious and boring, so let's omit the computation. $\square$

Remark. $L \cong \Lambda^2 S_+ \cong K^{-1} \otimes E^2$. In particular, when X is equipped with the canonical $Spin^{\mathbb{C}}$ structure, we have $L \cong K^{-1}$, which can also be derived by comparing their transition functions.

By this lemma, we are able to write down the local representation of the second Seiberg-Witten equation. We can express $\psi \in \Gamma(S_+)$ as $\psi = \alpha + \beta$ where $\alpha = a \in \Gamma(E), \beta = -\frac{1}{2} b d\bar{z} \wedge d\bar{w} \in \Omega^{0,2}(X; E)$. Then

$$
\begin{aligned}
F_A^+ \psi &= q(\psi) + i\phi \\
&= i(|a|^2 - |b|^2)\ (e_1 \wedge e_2 + e_3 \wedge e_4) + 2iIm(a\bar{b})\ (e_1 \wedge e_3 + e_4 \wedge e_2) + 2iRe(a\bar{b})\ (e_1 \wedge e_4 + e_2 \wedge e_3) + i\phi \\
&= i(|\alpha|^2 - |\beta|^2)\omega + 2iIm(a\bar{b}) \frac{d\bar{z} \wedge d\bar{w} + dz \wedge dw}{2} + 2iRe(a\bar{b}) \frac{-d\bar{z} \wedge d\bar{w} + dz \wedge dw}{2i} + i\phi \\
&= i(|\alpha|^2 - |\beta|^2)\omega + a\bar{b} dz \wedge dw - \bar{a} b d\bar{z} \wedge d\bar{w} + i\phi \\
&= i(|\alpha|^2 - |\beta|^2)\omega + 2(\bar{\alpha}\beta - \alpha\bar{\beta}) + i\phi
\end{aligned}
$$

Consider the canonical $Spin^{\mathbb{C}}$ structure on X, then E is trivial and $S_+ = (X \times \mathbb{C}) \oplus K^{-1}$. Let $u_0 : X \to \mathbb{C},\ x \mapsto 1$ be a constant section of $X \times \mathbb{C}$. Then we have the following lemma:

Lemma 3.2. *There exists a unique unitary connection A_0 on K^{-1} such that $\nabla_{A_0} u_0 \in \Omega^1(X; K^{-1}) \subset \Omega^1(X; S_+)$.*

Proof. Fix a unitary connection A on K^{-1}, then any unitary connection on K^{-1} can be written as $A + a$ where $a \in i\Omega^1(X)$. Then $\nabla_{A+a} u_0 = \nabla_A u_0 + \frac{1}{2} a u_0$, where the $\frac{1}{2}$ factor comes from the pull back of connection on $L = K^{-1}$ to connection on S_+. Suppose the local representatives of the $Spin^{\mathbb{C}}$ connection ∇_A are $A_\alpha = i a_\alpha I + \frac{1}{2} \sum_{i<j} \omega_{ij} e_i \cdot e_j$ for some real-valued 1-forms $\omega_{ij} = -\omega_{ji} \in \Omega^1(M)$ and $a_\alpha \in \Omega^1(M)$. Then locally $\nabla_A u_0 = (d + A_\alpha) u_0 = A_\alpha u_0 = i a_\alpha u_0 + \frac{1}{2} \sum_{i<j} \omega_{ij} e_i \cdot e_j \cdot u_0$. We have:

$$
e_1 e_2 u_0 = e_3 e_4 u_0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} \quad e_1 e_3 u_0 = e_4 e_2 u_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
$$

$$
e_1 e_4 u_0 = e_2 e_3 u_0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -i \end{pmatrix}
$$

So the $X \times \mathbb{C}$ component of $\nabla_A u_0$ is an imaginary-valued 1-form times u_0. So $A_0 = A + a = A - 2(\nabla_A u_0)_{X \times \mathbb{C}}$ is the unique canonical connection. $\square$

Since $L \cong K^{-1} \otimes E^2$, we can write any connection A on L as $A = A_0 + 2a$, where A_0 is the canonical connection on K^{-1}, a is a connection on E.

Lemma 3.3 (Formula of the Dirac Operator)**.** *If a is a connection on E, α is a section of E, β is a section of $K^{-1} \otimes E$, then $D_{A_0+2a}(\alpha, \beta) = \sqrt{2}(\bar{\partial}_a \alpha + \bar{\partial}_a^* \beta)$*

Proof.
Step 1: First, let's assume that X is equipped with the canonical $Spin^{\mathbb{C}}$ structure and $a = 0$. Let u_0 be the constant section as in the proof of last lemma.

Claim 3. $D_{A_0} u_0 = 0$

Proof of Claim 3. To see this, note that by Leibniz rule, we have:

$$\begin{aligned}
D_{A_0}(cl(\omega)(u_0)) &= cl(\nabla_{A_0}(cl(\omega)(u_0))) \\
&= cl(cl(\nabla\omega)(u_0)) + cl(cl(\omega)(\nabla_{A_0} u_0)) \\
&= cl(d\omega + d^*\omega)(u_0) + cl(cl(\omega)(\nabla_{A_0} u_0)) \\
&= cl(cl(\omega)(\nabla_{A_0} u_0)) \quad (d\omega = 0,\ *\omega = \omega)
\end{aligned}$$

where the third equality is true for any two-form $\eta \in \Omega^2(X)$: since Clifford multiplication acts pointwise, for $\forall p \in X$, we can pick the normal coordinate at p, then $\Gamma_{ij}^k(p) = 0$ and $\nabla\eta(p) = \partial_k \eta_{ij} dx^k \otimes (dx^i \wedge dx^j)$. So:

$$\begin{aligned}
cl(cl(\nabla\eta)(\psi)) &= \partial_k \eta_{ij} cl(dx^k) cl(dx^i \wedge dx^j)(\psi) \\
&= \partial_k \eta_{ij} cl(dx^k \wedge dx^i \wedge dx^j)(\psi) - \partial_k \eta_{ij} cl(\iota(dx^k)(dx^i \wedge dx^j))(\psi) \\
&= cl(d\eta + d^*\eta)(\psi)
\end{aligned}$$

Since u_0 is in the $-2i$ eigenspace of $cl(\omega)$ and the S_+ component of $\nabla_{A_0} u_0$ is in the $2i$ eigenspace of $cl(\omega)$, we have $-2iD_{A_0} u_0 = D_{A_0}(cl(\omega)(u_0)) = cl(cl(\omega)(\nabla_{A_0} u_0)) = cl(2i\nabla_{A_0} u_0) = 2iD_{A_0} u_0$, thus $D_{A_0} u_0 = 0$. $\square$

Claim 4. $cl(d\beta) = cl(d^*\beta)$

Proof of Claim 4. Since $\beta \in \Gamma(K^{-1}) = \Omega^{0,2}(X)$,

$$\begin{aligned}
\beta &= \lambda(x) d\bar{z} d\bar{w} = \lambda(x)((e_1 \wedge e_3 - e_2 \wedge e_4) - i(e_1 \wedge e_4 + e_2 \wedge e_3)) \\
d\beta &= \frac{\partial\lambda}{\partial x_1}(-e_1 \wedge e_2 \wedge e_4 - ie_1 \wedge e_2 \wedge e_3) + \frac{\partial\lambda}{\partial x_2}(-e_1 \wedge e_2 \wedge e_3 + ie_1 \wedge e_2 \wedge e_4) \\
&\quad + \frac{\partial\lambda}{\partial x_3}(e_2 \wedge e_3 \wedge e_4 + ie_1 \wedge e_3 \wedge e_4) + \frac{\partial\lambda}{\partial x_4}(e_1 \wedge e_3 \wedge e_4 - ie_2 \wedge e_3 \wedge e_4)
\end{aligned}$$

Moreover,

$$cl_+(e_1e_2e_3 - ie_1e_2e_4) = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix} = cl_+(e_4 + ie_3)$$

$$cl_+(e_1e_3e_4 - ie_2e_3e_4) = \begin{pmatrix} 2i & 0 \\ 0 & 0 \end{pmatrix} = cl_+(e_2 + ie_1)$$

so $cl(d\beta) = cl(*d\beta) = cl(d^*\beta)$. $\square$

Claim 5. $cl(d\alpha)(u_0) = cl(\bar{\partial}\alpha)(u_0)$

Proof of Claim 5.

$$cl(dz)(\alpha) = cl_+(e_1 + ie_2)(\alpha) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = 0$$

Similarly, $cl(dw)(\alpha) = 0$. $\square$

Now we can prove the lemma in the case of canonical $Spin^{\mathbb{C}}$ structure and $a = 0$:

$$\begin{aligned} D_{A_0}(\alpha + \beta) &= D_{A_0}(cl(\alpha)(u_0)) + D_{A_0}(cl(\beta/2)(u_0)) \\ &= cl(d\alpha)(u_0) + cl(\frac{d\beta + d^*\beta}{2})(u_0) \quad \textit{since } d^*\alpha = 0 \textit{ and } D_{A_0}u_0 = 0 \\ &= cl(\bar{\partial}\alpha)(u_0) + cl(d^*\beta)(u_0) \\ &= \sqrt{2}(\bar{\partial}\alpha + \bar{\partial}^*\beta) \end{aligned}$$

Step2: For general $a \in i\Omega^1(X)$ and canonical $Spin^{\mathbb{C}}$ structure,

$$\begin{aligned} D_{A_0+2a}(\alpha, \beta) &= cl((\nabla_{A_0} + a)(\alpha, \beta)) \\ &= D_{A_0}(\alpha, \beta) + cl(a)(\alpha, \beta) \\ &= \sqrt{2}(\bar{\partial}\alpha + \bar{\partial}^*\beta) + \sqrt{2}(a^{0,1}\alpha - \iota(\overline{a^{1,0}})\beta) \\ &= \sqrt{2}(\bar{\partial}\alpha + \bar{\partial}^*\beta) + \sqrt{2}(a^{0,1}\alpha + \iota(a^{0,1})\beta) \\ &= \sqrt{2}(\bar{\partial}_a\alpha + \bar{\partial}_a^*\beta) \end{aligned}$$

For general $Spin^{\mathbb{C}}$ structures, note that the formula we want to prove is local, so by taking a local trivialization of E we can reduce the problem to the canonical case, which we have proved. $\square$

With the above preparations, we can write down the local form of the Seiberg-Witten equations. Since the Seiberg-Witten invariants does not depend on the choice of ϕ, let's take $\phi = -r\omega - iF^+_{A_0}$, where $r > 0$ is a large real number. And write $\psi = \sqrt{r}(\alpha, \beta)$. Then by the formulas for D_{A_0+2a}, F^+_A and $F^+_{A_0+2a} = F^+_{A_0} + 2F^+_a$, the Seiberg-Witten equations becomes:

$$\begin{aligned} \bar{\partial}_a\alpha &= -\bar{\partial}_a^*\beta \\ F_a^+ &= -\frac{ir}{2}(1 - |\alpha|^2 + |\beta|^2)\omega - r(\alpha\bar{\beta} - \bar{\alpha}\beta) \end{aligned}$$

Since we have the orthogonal decomposition $\Lambda^2_+(X) \otimes \mathbb{C} = \mathbb{C}\omega \oplus K \oplus K^{-1}$, the above equations implies $F_a^{0,2} = r\bar{\alpha}\beta$ and $\langle \omega, F_a \rangle = \langle \omega, F_a^+ \rangle = -\frac{ir}{2}(1 - |\alpha|^2 + |\beta|^2)\langle \omega, \omega \rangle = -ir(1 - |\alpha|^2 + |\beta|^2)$.

3.3 Taubes' Theorem

Let $c = c_1(K)$, $e = c_1(E)$. As mentioned above, we have a 1-1 correspondence $\mathcal{S}_X \to H^2(X;\mathbb{Z})$, $s \mapsto c_1(E) = c_1(K^{-\frac{1}{2}} \otimes E) + c_1(K^{\frac{1}{2}}) = \frac{1}{2}c_1(L) + \frac{1}{2}c$. So we have $SW : H^2(X;\mathbb{Z}) \to \mathbb{Z}$ and $SW(e) = \pm SW(c-e)$ by charge conjugation. We are going to prove the following theorem of Taubes:

Theorem 3.1. *(Taubes)*

1. $SW(0) = 1$
2. $SW(c) = \pm 1$
3. $SW(e) = \pm SW(c-e)$
4. *If* $SW(e) \neq 0$*, then* $0 \leq [\omega] \cdot e \leq [\omega] \cdot c$ *and equality holds if and only if* $e = 0$ *or* $e = c$*.*

Before the proof, we need the following background.

Definition 3.2. $N : \Omega^{1,0}(X) \to \Omega^{0,2}(X),\ \alpha \mapsto (d\alpha)^{0,2}$ is called the **Nijenhuis tensor**.

Remark. Nijenhuis tensor is a measure of the integrability of J. J is integrable if and only if $N = 0$.

To show that N is a tensor, it suffices to show: $\forall f \in C^\infty(X;\mathbb{C}), \forall \alpha \in \Omega^{1,0}(X), N(f\alpha) = fN(\alpha)$. This is because:

$$\begin{aligned}
N(f\alpha) &= (d(f\alpha))^{0,2} \\
&= (df \wedge \alpha)^{0,2} + (fd\alpha)^{0,2} \\
&= f(d\alpha)^{0,2} \quad \textit{since } \alpha \in \Omega^{1,0}(X) \\
&= fN(\alpha)
\end{aligned}$$

Finally, we also need another *Weitzenböck* formula:

$$2\bar{\partial}_a^*\bar{\partial}_a = \nabla_a^*\nabla_a - i\langle \omega, F_a \rangle$$

as an operator on $\Gamma(E)$ (see [1], page 212), which we will not prove. Now we can start to prove Theorem 3.1.

Proof.

$$\begin{aligned}
\bar{\partial}_a\bar{\partial}_a^*\beta &= -\bar{\partial}_a\bar{\partial}_a\alpha \\
&= -(\bar{\partial}_a(\nabla_a - \partial_a)\alpha) \\
&= -(\nabla_a\nabla_a\alpha)^{0,2} + (\nabla_a\partial_a\alpha)^{0,2} \\
&= -(F_a\alpha)^{0,2} + N(\partial_a\alpha)^{0,2} \\
&= -r|\alpha|^2\beta + N(\partial_a\alpha) \\
\Rightarrow \int_X |\bar{\partial}_a^*\beta|^2 &= \int_X \langle \beta, \bar{\partial}_a\bar{\partial}_a^*\beta \rangle = \int_X (\langle \beta, N(\partial_a\alpha)\rangle - r|\alpha|^2|\beta|^2)
\end{aligned}$$

Since N is a tensor independent of r and X is compact, we have $\|N\| = \max_{x\in X} \|N_x\| < \infty$ and is independent of r. So we have the estimate:

$$\int \langle \beta, N(\partial_a \alpha)\rangle \leq \frac{r}{2}\int |\beta|^2 + \frac{\|N\|^2}{2r}\int |\nabla_a \alpha|^2$$

Therefore, let $C = \frac{\|N\|}{2}$ be a positive constant independent of r, then:

$$\int (|\bar{\partial}_a^* \beta|^2 + r|\alpha|^2|\beta|^2 - \frac{r}{2}|\beta|^2) \leq \frac{C}{r}\int |\nabla_a \alpha|^2$$

By applying the $Weitzenböck$ formula $2\bar{\partial}_a^*\bar{\partial}_a = \nabla_a^*\nabla_a - i\langle \omega, F_a\rangle$ to α and integrate on X, we get:

$$2\int |\bar{\partial}_a \alpha|^2 = \int |\nabla_a \alpha|^2 - \int r(1-|\alpha|^2+|\beta|^2)|\alpha|^2$$

Key Step: Chern-Weil theory tells us that $-2\pi i c_1(E) = [F_a]$, which implies $-2\pi i[\omega]\cdot e = \int \omega \wedge F_a = \int *\omega \wedge F_a = \int \langle \omega, F_a\rangle = \int -ir(1-|\alpha|^2+|\beta|^2)$. So:

$$\begin{aligned}
2\pi[\omega]\cdot e &= \int r(1-|\alpha|^2+|\beta|^2) \\
&= \int |\nabla_a \alpha|^2 - \int (2|\bar{\partial}_a \alpha|^2 + r(1-|\alpha|^2+|\beta|^2)(|\alpha|^2-1)) \\
&= \int |\nabla_a \alpha|^2 - \int (2|\bar{\partial}_a^* \beta|^2 + r(1-|\alpha|^2+|\beta|^2)(|\alpha|^2-1)) \\
&\geq \int |\nabla_a \alpha|^2 + \int (2(r|\alpha|^2|\beta|^2 - \frac{r}{2}|\beta|^2 - \frac{C}{r}|\nabla_a \alpha|^2) + r(1-|\alpha|^2+|\beta|^2)(1-|\alpha|^2)) \\
&\geq \int [(1-\frac{2C}{r})|\nabla_a \alpha|^2 + r(1-|\alpha|^2)^2]
\end{aligned}$$

1. For the canonical $Spin^{\mathbb{C}}$ structure $s_0(e=0)$, the Seiberg-Witten equations have an solution $a \equiv 0, \alpha \equiv 1, \beta \equiv 0$.

2. If $[\omega]\cdot e = 0$, let r be large enough, then the above inequality tells us that $|\alpha| \equiv 1, \nabla_a \alpha \equiv 0$. Then α is a nonvanishing section of E, so E is a trivial bundle and $e = 0$. Up to a gauge transformation, we may assume $\alpha \equiv 1$. Then $0 = \nabla_a \alpha = d\alpha + a \otimes \alpha = a \otimes 1$, so $a \equiv 0$. Since $0 = 2\pi[\omega]\cdot e = \int r(1-|\alpha|^2+|\beta|^2) = \int r|\beta|^2$, $\beta \equiv 0$. Therefore, if $[\omega]\cdot e = 0$ and $SW(e) \neq 0$, we have $e = 0$, and the unique solution is $a \equiv 0, \alpha \equiv 1, \beta \equiv 0$, that is, $A = A_0, \psi = (\sqrt{r}, 0)$. Last but not least, we still need to check that this solution is cut out transversely (we leave this to the end of the proof), which tells us that $SW(0) = 1$.

3. If $[\omega]\cdot e < 0$, let r be large enough, then there is no solution to the Seiberg-Witten equations. So $SW(e) = 0$.

4. By the charge conjugation invariance, $SW(e) = \pm SW(c-e)$. So $SW(c) = \pm SW(0) = \pm 1$. If $[\omega]\cdot e > [\omega]\cdot c$, then $[\omega]\cdot(c-e) < 0$, so $SW(e) = \pm SW(c-e) = 0$. If $[\omega]\cdot e = [\omega]\cdot c$ and $SW(e) \neq 0$, then $[\omega]\cdot(c-e) = 0$ and $SW(c-e) = \pm SW(e) \neq 0$, so $c - e = 0$.

To conclude the proof, it remains to check that when $e = 0$ the unique solution is cut out transversely.

Let the **Seiberg-Witten function** be $F : \mathcal{A} \to (\Lambda^2_+ T^*X \otimes i\mathbb{R}) \oplus S_-,\ (A, \psi) \mapsto (F_A^+ - q(\psi) - \phi, D_A\psi)$. By the implicit function theorem for Banach spaces, it suffices to prove the surjectivity of the differential $DF\big|_{(A_0,(\sqrt{r},0))}$ at the unique solution.

Easy computations yield

$$DF\big|_{(A_0,(\sqrt{r},0))} = \begin{pmatrix} d^+ & -Dq\big|_{(\sqrt{r},0)} \\ \cdot\frac{1}{2}(\sqrt{r},0) & D_{A_0} \end{pmatrix}$$

where $\cdot\frac{1}{2}(\sqrt{r},0)$ sends a one-form α to $\frac{1}{2}cl(\alpha)(\sqrt{r},0) = \frac{\sqrt{r}}{\sqrt{2}}\alpha$, and:

$$\begin{aligned} Dq\big|_{(\sqrt{r},0)}(a,b) &= \begin{pmatrix} \sqrt{r} \\ 0 \end{pmatrix} \otimes (\bar{a}, \bar{b}) + \begin{pmatrix} a \\ b \end{pmatrix} \otimes (\sqrt{r},0) - \frac{\langle (a,b),(\sqrt{r},0)\rangle + \overline{\langle (a,b),(\sqrt{r},0)\rangle}}{2} Id \\ &= i\sqrt{r}\frac{Re(a)}{2}\omega + \sqrt{r}\frac{b-\bar{b}}{2} \end{aligned}$$

where the second equality follows from the identification between traceless, self-adjoint automorphisms and pure imaginary self-dual 2-forms via Clifford multiplication. Moreover, we have the following elliptic complex:

$$0 \longrightarrow \Omega^0(X; i\mathbb{R}) \xrightarrow{D_1} \Omega^1(X; i\mathbb{R}) \oplus (\Omega^0(X;\mathbb{C}) \oplus \Omega^{0,2}(X;\mathbb{C})) \xrightarrow{D_2} \Omega^2_+(X; i\mathbb{R}) \oplus \Omega^{0,1}(X;\mathbb{C}) \longrightarrow 0$$

where $D_1 = (2d, -(\cdot)\psi)$, $D_2 = DF\big|_\psi$. And the Euler characteristic of this complex equals the formal dimension of the moduli space $\mathcal{M}$, which is zero since $e = 0$ is the canonical $Spin^{\mathbb{C}}$ structure given by the almost complex structure (see the Lemma on page 102 in [4]). So in order to prove the surjectivity of $D_2 = DF\big|_{(A_0,(\sqrt{r},0))}$, we show that the first and second cohomology group of this complex are both trivial.

The formulas for D_1 and D_2 are:

$$D_1(if) = \begin{pmatrix} 2idf \\ -if \end{pmatrix}$$

$$D_2\begin{pmatrix} i\lambda \\ (a,b) \end{pmatrix} = \begin{pmatrix} d^+(i\lambda) - \frac{i\sqrt{r}}{2}Re(a)\omega - \frac{\sqrt{r}(b-\bar{b})}{2} \\ \sqrt{2}\bar{\partial}a + \sqrt{2}\bar{\partial}^*b + \frac{\sqrt{r}}{\sqrt{2}}(i\lambda)^{0,1} \end{pmatrix}$$

It's clear that $Ker(D_1) = 0$, so the first cohomology is trivial. Let's show that $Ker(D_2) = Im(D_1)$. Suppose $D_2(i\lambda,(a,b)) = 0$. Applying $\bar{\partial}$ to the equation $\sqrt{2}\bar{\partial}a + \sqrt{2}\bar{\partial}^*b + \frac{\sqrt{r}}{\sqrt{2}}(i\lambda)^{0,1} = 0$, we obtain

$$\frac{\sqrt{r}}{2}\bar{\partial}(i\lambda)^{0,1} + \bar{\partial}\bar{\partial}^*b = N(\partial a)$$

Here we haved used the formula $\bar{\partial}^2 a = -N(\partial a)$ for $a \in \Omega^0(X;\mathbb{C})$. Since $d^+(i\lambda) - \frac{i\sqrt{r}}{2}Re(a)\omega - \frac{\sqrt{r}(b-\bar{b})}{2} = 0$, we have

$$\frac{\sqrt{r}b}{2} = i(d\lambda)^{0,2} = i\bar{\partial}(\lambda^{0,1}) + iN(\lambda^{1,0})$$

By the above two equations,

$$\frac{r}{4}b + \bar{\partial}\bar{\partial}^*b = N(\partial a + i\lambda^{1,0})$$

Taking inner product with b, we get:

$$\frac{r}{4}\|b\|^2 + \|\bar{\partial}^* b\|^2 = (b, N(\partial a + i\lambda^{1,0})) \leq \frac{\mu}{2}\|b\|^2 + \frac{C}{\mu}\|\partial a + i\lambda^{1,0}\|^2$$

Take $\mu = \frac{r}{4}$ and let r go to infinity, we get $b = 0$. Write $i\lambda = \xi - \bar{\xi}$ for some $\xi \in \Omega^{1,0}(X;\mathbb{C})$. The two equations of $D_2(i\lambda, (a,b)) = 0$ now become:

$$d^+(\xi - \bar{\xi}) = \frac{i\sqrt{r}}{2} Re(a)\omega$$
$$\sqrt{2}\bar{\partial} a + \frac{\sqrt{r}}{\sqrt{2}}\bar{\xi} = 0$$

By adding elements in $Im(D_1)$ to a, we may assume $a = u$ is real-valued. So the second equation now reads $\bar{\xi} = -\frac{2}{\sqrt{r}}\bar{\partial} u$. Plug this into the first equation and note that $d^+(\partial u - \bar{\partial} u) = \frac{i}{2}\Delta u\omega$. Then we obtain $\frac{2}{r}\Delta u + u = 0$. Since Δ has a non-negative spectrum, $u = 0$. So $\xi = 0$ and $\lambda = 0$. Thus we've proved the second cohomology of the elliptic complex is trivial.

So we have proved all the statements in Taubes' theorem. $\square$

Notation

S_+	positive spinor bundle
S_-	negative spinor bundle
L	determinant line bundle
cl	Clifford multiplication
cl_+	positive Clifford multiplication
q	quadratic map
D_A	Dirac operator
D_A^+	positive Dirac operator
D_A^-	negative Dirac operator
ϕ	perturbation term in the Seiberg-Witten equations
$\mathcal{A}$	configuration space
$\mathcal{G}$	group of gauge transformations
$\mathcal{G}_0$	group of based gauge transformations
$\mathcal{M}_\phi$	moduli space of solutions to the Seiberg-Witten equations (quotient by $\mathcal{G}$)
$\tilde{\mathcal{M}}_\phi$	moduli space of solutions to the Seiberg-Witten equations (quotient by $\mathcal{G}_0$)
$\mathcal{S}_X$	set of $Spin^{\mathbb{C}}$ structures on X
SW_X	Seiberg-Witten invariant of X
ω	symplectic form
J	almost complex structure
N	Nijenhuis tensor

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