Invitation to Dynamical Systems

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Solutions to Problems

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Chapter 1

Answers for §1.1

(1.1.1) This system can be described by three numbers: x, the distance the ball has traveled horizontally, y the height of the ball above the ground, v_y the velocity (upwards) of the ball. We also need to know v_x, the horizontal velocity of the ball, but this doesn't change, so we don't need to include it as a state variable (but it wouldn't be wrong, per se, to do so).

The state vector is: $\begin{bmatrix} x \\ y \\ v_y \end{bmatrix}$. The dynamics of the situation are described by the equations

$$\begin{aligned} x' &= v_x \\ y' &= v_y \\ v_y' &= -g \end{aligned}$$

where v_x and g are constants.

(1.1.2)

(a) $f^2(x) = 4x$; $f^3(x) = 8x$.

(b) $f^2(x) = 9x - 8$; $f^3(x) = 27x - 26$.

(c) $f^2(x) = x^4 - 6x^2 - 6$; $f^3(x) = x^8 - 12x^6 + 48x^4 - 72x^2 + 33$.

(d) $f^2(x) = \sqrt{1 + \sqrt{1 + x}}$; $f^3(x) = \sqrt{1 + \sqrt{1 + \sqrt{1 + x}}}$.

(e) $f^2(x) = 2^{2^x}$; $f^3(x) = 2^{2^{2^x}}$.

(1.1.3)

(a) 0

(b) -2186

(c) 1934791555410494424614913

(d) approximately 1.61744

(e) The computer couldn't handle this one!

(1.1.4) $x(1) = 6,\ x(2) = 18,\ x(3) = 54,\ x(4) = 162,\ x(k) = 2 \cdot 3^k$

(1.1.5) $x(k) = b \cdot a^k$

(1.1.6) For $x(t) = 2e^{3t}$, $x'(t) = 6e^{3t} = 3(2e^{3t}) = 3x(t)$, and $x(0) = 2 \cdot e^0 = 2$, so the given solution checks out.

(1.1.7) $x(t) = be^{at}$. $x'(t) = a(be^{at}) = ax(t)$, and $x(0) = b \cdot e^0 = b$, so the answer checks out.
If $a = 0$, $x(t) = b$. $x'(t) = 0 = 0 \cdot b = ax(t)$, and $x(0) = b$, so our answer still works.
If $b = 0$, $x(t) = 0$. $x'(t) = 0 = a \cdot 0 = ax(t)$, and $x(0) = 0 = b$, so our answer still works.

(1.1.8) Let s represent time, t. We now have three state variables x_1, x_2, s and the system is:

$$\begin{aligned} x_1' &= 3x_1 + (2-s)x_2 \\ x_2' &= x_1 x_2 - s \\ s' &= 1 \end{aligned}$$

(1.1.9) Introduce a new state variable j (representing time, k). The system is:

$$\begin{aligned} x_1(k+1) &= 2x_1(k) + j(k)x_2(k) \\ x_2(k+1) &= x_1(k) - j(k) - 3x_2(k) \\ j(k+1) &= j(k) + 1 \end{aligned}$$

(1.1.10) Run `discrete` (in `sys` directory) using `collatz`. See the file `collatz.m`.

Answers for §1.2

(1.2.2) Iterating $f(x) = \sin(\cos x)$ starting from any value will lead to 0.69482. As the sine and cosine buttons are pressed, one alternately sees the values 0.69482 and 0.768169.

(1.2.3)

$$v' = -\frac{k}{m}x^3, \quad x' = v$$

(1.2.4) See figure.

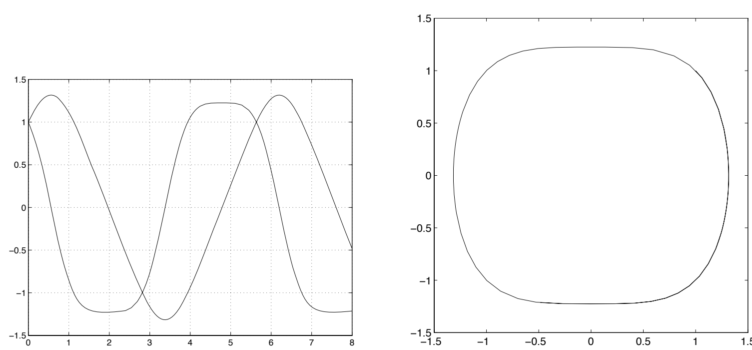

Figure 1: Figures for problem 1.2.4

(1.2.5) $c(k)$ = checking, $s(k)$ = savings, $r(k)$ = retirement, $l(k)$ = car loan
p_c = amount of paycheck
p_e = amount of rent, utilities, and other expenses including monthly fee
p_s = amount deposited into savings
p_r = amount deposited into retirement
p_l = amount of loan payment
c_s = rate on savings, c_r = rate on retirement, c_l = rate on loan

$$\begin{aligned} c(k+1) &= c(k) + p_c - p_e - p_s - p_r - p_l \\ s(k+1) &= (1+c_s)s(k) + p_s \\ r(k+1) &= (1+c_r)r(k) + p_r \\ l(k+1) &= (1+c_l)l(k) - p_l \end{aligned}$$

(1.2.6) If s, the savings rate, increases, so does the per-capita capitalization. If the population rate or the depreciation rate goes up, then the per-capita capitalization level drops.

(1.2.7) Yes, if labor and capital are doubled in this new model, so is output.
The new system is $k' = sAk^a - (d+\rho)k$.

(1.2.8) MODEL I: birth rate= $\frac{b}{1+x}$, $\frac{dx}{dt} = \frac{bx}{1+x} - px^2$.
Fixed points: $x = 0$, $x = -1/2 + \sqrt{1 + 4b/p}$
MODEL II: birth rate= be^{-x} , $\frac{dx}{dt} = bxe^{-x} - px^2$.
Fixed points: $x = 0$, $x =$(sol'n to $x = \frac{b}{p}e^{-x}$)

(1.2.9) $w \equiv$ scavengers, $x \equiv$ herbivores, $y \equiv$ carnivores, $z \equiv$ top level carnivores

$$\begin{aligned} w' &= -aw + ewy - fwz \\ x' &= bx - gxy - hxz \\ y' &= -cy + ixy - jyz \\ z' &= -dz + kxy + lyz, \end{aligned}$$

where $a, b, \ldots, l$ are all constants

(1.2.11)

(a) For $x(0) = 1$, $x(k) \to 0.78540$

(b) For $x(0) = 1$, $x(k) \to 1.30633$

Note: This equation has an infinite number of solutions. -3.09641 and 4.70332 are two others.

(c) For $x(0) = 2$, $x(k) \to 1.70998$

(d) For $x(0) = 1$, $x(k) \to -1.40119$

(1.2.12)

(a) $n = 10 \Rightarrow y(1) = 4.48436$; $n = 100 \Rightarrow y(1) = 4.76203$.

(b) $n = 10 \Rightarrow y(1) = 0.18528$; $n = 100 \Rightarrow y(1) = 0.02019$.

(c) $n = 10 \Rightarrow y(1) = -28.76543$; $n = 100 \Rightarrow y(1) \ll -10^{100000}$.

(1.2.13) The starting Fibonacci vector should be $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) $x(k+1) = y(k)$, $y(k+1) = x(k)y(k)$.

(b) $w(k+1) = x(k)$, $z(k+1) = y(k)$
$x(k+1) = z(k)x(k)$, $y(k+1) = w(k) + z(k) + y(k)$.

(c) $x' = y$, $y' = 3y - 2x + 2$.

Chapter 2

Answers for §2.1

(2.1.1)

(a) $x(k) = k + 1$.

(b) $x(k) = 2$.

(c) $x(k) = 2(1 - (3/2)^k)$.

(d) $x(k) = \begin{cases} 4 & \text{if } k \text{ even} \\ -3 & \text{if } k \text{ odd} \end{cases}$.

(e) $x(k) = \frac{5}{6}(-\frac{1}{2})^k + \frac{2}{3}$.

(2.1.2)

(a) $x(k) \to \infty$, no fixed point.

(b) System 'stuck' at 2 (or any other x_0).

(c) $x(k) \to -\infty$; does not approach fixed point at 2.

(d) System oscillates between -3 and 4; fixed point at $1/2$.

(e) $x(k) \to 2/3$, which is the fixed point.

(2.1.3)

(a) $x(t) = e^t$.

(b) $x(t) = t$.

(c) $x(t) = 1$.

(d) $x(t) = e^{-t} + 1$.

(e) $x(t) = \frac{3}{2}(e^{2t} - 1)$.

(2.1.4)

(a) $x(t) \to \infty$; the fixed point is 0.

(b) $x(t) \to \infty$, no fixed point.

(c) System 'stuck' at 1 (or any other x_0).

(d) $x(t) \to 1$, which is the fixed point.

(e) $x(t) \to \infty$; fixed point at $-3/2$.

(2.1.5) If $x(t) = e^{at}\left(x_0 + \frac{b}{a}\right) - \frac{b}{a}$, then $\frac{dx(t)}{dt} = e^{at}(ax_0 + b)$. On the other hand, if we take this expression for $x(t)$ and substitute into $x' = ax + b$, we get...

$$\begin{aligned} x'(t) &= a\left[e^{at}\left(x_0 + \frac{b}{a}\right) - \frac{b}{a}\right] + b \\ &= \left[e^{at}(ax_0 + b) - b\right] + b \\ &= e^{at}(ax_0 + b)\ , \end{aligned}$$

which is what we got for $\frac{dx(t)}{dt}$, as desired. Using this expression for $x(t)$, we see that $x(0) = e^0(x_0 + \frac{b}{a}) - \frac{b}{a} = x_0$, also as desired, so the given expression for $x(t)$ is the desired solution.

(2.1.6) $f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b = rx + s$, where $r = ac$ and $s = ad + b$. Apply Table 2.2 to see that $x(t)$ is bounded if (1) $ac < 0$, (2) $ac > 0$ and $x_0 = -\frac{ad+b}{ac}$, or (3) $ac = 0$ and $b = -ad$. Otherwise, $x(t) \to \pm\infty$.
If $ac = 0$, then x is 'stuck' at x_0 if $b = -ad$, and blows up otherwise.
If $ac \neq 0$, then there is a fixed point at $-\frac{ad+b}{ac}$.

(2.1.7) $x(t) = e^{ct}\left(x_0 - \frac{r}{c}\right) + \frac{r}{c}$.
If $x_0 > r/c$, the fish population explodes. If $x_0 = r/c$, the population stays constant. If $x_0 < r/c$, the fish population eventually dies out.

Answers for §2.2

(2.2.1)

(a) $1, 3\ ;\ \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$

(b) $1, -1\ ;\ \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$

(c) $0, -2\ ;\ \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$

(d) $\frac{1\pm\sqrt{5}}{2};\quad \begin{bmatrix} \frac{1\pm\sqrt{5}}{2} \\ 1 \end{bmatrix}.$

(2.2.2)

(a) $A^k = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix}\ ;\ e^{At} = \begin{bmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{bmatrix}.$

(b) $A^k = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & , k \text{ odd} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & , k \text{ even} \end{cases}\ ;\ e^{At} = \frac{1}{2}\begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}.$

(c) $A^0 = I\ ,\ A^k = \begin{bmatrix} -(-2)^{k-1} & (-2)^{k-1} \\ (-2)^{k-1} & -(-2)^{k-1} \end{bmatrix}$ for $k > 0\ ;\ e^{At} = \frac{1}{2}\begin{bmatrix} 1 + e^{-2t} & 1 - e^{-2t} \\ 1 - e^{-2t} & 1 + e^{-2t} \end{bmatrix}.$

(d) Let $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$. Then

$$A^k = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha^k & 0 \\ 0 & \beta^k \end{bmatrix} \cdot \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix}$$

and

$$e^{At} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix}.$$

(2.2.3) Let a and b be the eigenvalues of A. So $\det(xI-A) = (x-a)(x-b) = x^2-(a+b)x+ab$ is A's characteristic polynomial. Notice this is $x^2 - tx + d$ where t and d are the trace and determinant of A. So the roots are

$$x = \frac{t \pm \sqrt{t^2-4d}}{2}.$$

Since $d > 0$ we know that $t^2 - 4d < t^2$ so the two eigenvalues are either complex (with $t/2 < 0$ as real part) or real and both negative. Therefore $\mathbf{0}$ is a stable fixed point of $\mathbf{x}' = A\mathbf{x}$.

(2.2.4) Let $\mathbf{u}(k) = \mathbf{x}(k) + (A-I)^{-1}\mathbf{b}$. Modest matrix algebra now shows that $\mathbf{u}(k+1) = A\mathbf{u}(k)$. This transformation requires that $A - I$ be invertible, or equivalently, that 1 is not an eignevalue of A.

(2.2.5) Many possible answers including $A = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(2.2.6) Complex roots of a real polynomial (and thus the complex eigenvalues of a real matrix) always occur in pairs of the form $(a+bi, a-bi)$. These two points are symmetric about the x (real)-axis in the complex plain.

(2.2.7) If A diagonalizes, it has n linearly independent eigenvectors, so we may write $\mathbf{v} = c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$. From this we get

$$\begin{aligned} A\mathbf{v} &= A(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n) \\ &= c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_1\mathbf{x}_n, \\ &\vdots \\ A^k\mathbf{v} &= c_1\lambda_1^k\mathbf{x}_1 + \cdots + c_n\lambda_1^k\mathbf{x}_n. \end{aligned}$$

Assuming that λ_1 is $\lambda_{\max}$, the $c_1\lambda_1^k\mathbf{x}_1$ term will dominate this expression as k gets large, so each multiplication by A is more nearly $A(c_1\lambda_1^k\mathbf{x}_1) = c_1\lambda_1^{k+1}\mathbf{x}_1$, i.e. a multiplication by λ_1. Note that, after a large number of iterations (say k), the 'current' value for $\mathbf{v}$ will approximate $c_1\lambda_1^k\mathbf{x}_1$, so in this process, $\mathbf{v}$ approaches (a scalar multiple of) $\mathbf{x}_1$, the eigenvector of λ_1.
Note: Taking $\mathbf{v}$ to be a 'randomly chosen' vector implies that $c_1 \neq 0$ is very likely.

(2.2.8) $A(\mathbf{v}') = (A\mathbf{v})' = \begin{bmatrix} x' + 2y' + 3z' \\ 4x' + 5y' + 6z' \\ 7x' + 8y' + 9z' \end{bmatrix}$.

(2.2.9) $\mathbf{x}(k) = \begin{bmatrix} 4^k - 2^{k+1} \\ 4^k + 2^{k+1} - 1 \end{bmatrix}$.

(2.2.10) $\mathbf{x}(k) = \begin{bmatrix} \frac{5}{4}3^k - \frac{k}{2} - \frac{9}{4} \\ \frac{5}{4}3^k + \frac{k}{2} + \frac{3}{4} \end{bmatrix}.$

(2.2.11) $\mathbf{x}(t) = \begin{bmatrix} e^{3t} - 2e^t \\ e^{3t} + 2e^t - 1 \end{bmatrix}.$

(2.2.12) $\mathbf{x}(t) = \begin{bmatrix} 3e^{2t}\sin t - e^{2t}\cos t \\ e^{2t}\sin t + 3e^{2t}\cos t - 1 \end{bmatrix}.$

(2.2.13) For $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, and $e^{At} \to \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, so $\mathbf{x}(t) \to \frac{1}{2}\begin{bmatrix} x_1(0) + x_2(0) \\ x_1(0) + x_2(0) \end{bmatrix}$.
For $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, $\mathbf{x}(t) \to \infty$.

(2.2.14) For the system $\mathbf{x}' = A\mathbf{x}$, we have $x_1(t) \to 0$ and $x_2(t)$ and $x_3(t)$ oscillating.
For the system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ we have $x_1(t) \to 2$ and x_2 and x_3 oscillating.

(2.2.15) For the system $\mathbf{x}' = A\mathbf{x}$, we have all $x_i(t)$ oscillating.
For the system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ we have $|\mathbf{x}(t)| \to \infty$. In particular $x_1(t)$ explodes but $x_2(t)$ and $x_3(t)$ oscillate.

(2.2.16) For $\mathbf{x}(k+1) = A\mathbf{x}(k)$, if $\mathbf{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathbf{x}(k)$ cycles through the sequence $\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} b \\ -a \end{bmatrix} \to \begin{bmatrix} -a \\ -b \end{bmatrix} \to \begin{bmatrix} -b \\ a \end{bmatrix} \to \begin{bmatrix} a \\ b \end{bmatrix} \to \cdots$.
For $\mathbf{x}(k+1) = A\mathbf{x}(k) + \mathbf{b}$, $\mathbf{x}(k)$ cycles through the sequence $\begin{bmatrix} a \\ b \end{bmatrix} \to \begin{bmatrix} b+2 \\ -a+1 \end{bmatrix} \to \begin{bmatrix} -a+3 \\ -b-1 \end{bmatrix} \to \begin{bmatrix} -b+1 \\ a-2 \end{bmatrix} \to \begin{bmatrix} a \\ b \end{bmatrix} \to \cdots$.

(2.2.17) The first system has $x_1(k)$ constant and $x_2(k)$ oscillating between two values.
The second system has $|\mathbf{x}(k)| \to \infty$ ($x_1(k)$ explodes).

(2.2.18) In the first system, we have $x_1(k)$ constant, but $x_2(k)$ and $x_3(k)$ tend to 0.
In the second system, we have $|\mathbf{x}(k)| \to \infty$ since $x_1(k) \to \infty$.

(2.2.19) In the first system, we have $x_1(k)$ constant, but $x_2(k)$ and $x_3(k)$ tend to 0.
In the second system, we have $x_1(k)$ constant, but $x_2(k)$ and $x_3(k)$ tending to $\frac{20}{3}$ and $\frac{5}{3}$ respectively.

(2.2.20) See the figure.
The eigenvalues of A are $\frac{7}{20} \pm \frac{\sqrt{39}}{20}$, which have an absolute value of $\frac{\sqrt{22}}{10} \approx 0.469 < 1$, so Table 2.3 tells us that $\mathbf{x}(k)$ will converge to $\tilde{\mathbf{x}} = (I - A)^{-1}\mathbf{b} = \begin{bmatrix} 3/13 \\ -5/15 \end{bmatrix}$. This is the fixed point of the system.

Answers for §2.3

(2.3.1) The eigenvalues of A are the roots of the real polynomial $\det(A - \lambda I)$, and

$$\begin{aligned} \det(A^T - \lambda I) &= \det(A^T - \lambda I^T) \\ &= \det[A^T - (\lambda I)^T] \\ &= \det[(A - \lambda I)^T] \\ &= \det(A - \lambda I). \end{aligned}$$

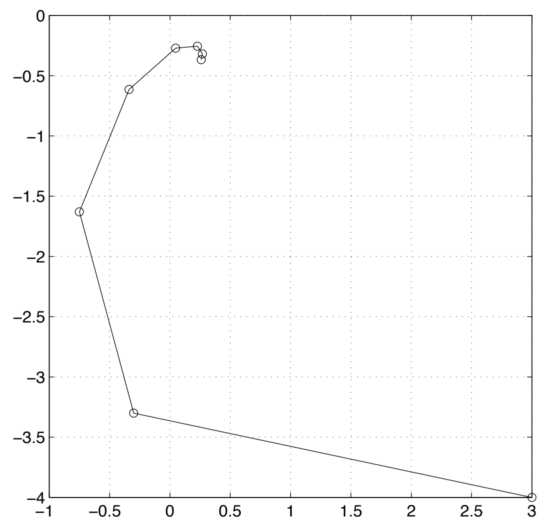

Figure 2: Problem 2.2.20

Since A and A^T have the same polynomial, they have the same eigenvalues.
For $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, A has eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, while A^T has eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(2.3.3)

$$P_1^{512} \approx P_1^{513} = \begin{bmatrix} .15067 & .08610 & .19372 & .24250 & .05438 & .27264 \\ .15067 & .08610 & .19372 & .24250 & .05438 & .27264 \\ .15067 & .08610 & .19372 & .24250 & .05438 & .27264 \\ .15067 & .08610 & .19372 & .24250 & .05438 & .27264 \\ .15067 & .08610 & .19372 & .24250 & .05438 & .27264 \\ .15067 & .08610 & .19372 & .24250 & .05438 & .27264 \end{bmatrix}$$

$$P_2^{512} = \begin{bmatrix} .59259 & 0 & .40741 & 0 & 0 \\ 0 & .20370 & 0 & .29630 & .50000 \\ .59259 & 0 & .40741 & 0 & 0 \\ 0 & .20370 & 0 & .29630 & .50000 \\ 0 & .20370 & 0 & .29630 & .50000 \end{bmatrix},$$

$$P_2^{513} = \begin{bmatrix} 0 & .20370 & 0 & .29630 & .50000 \\ .59259 & 0 & .40741 & 0 & 0 \\ 0 & .20370 & 0 & .29630 & .50000 \\ .59259 & 0 & .40741 & 0 & 0 \\ .59259 & 0 & .40741 & 0 & 0 \end{bmatrix}$$

$$P_3^{512} \approx P_3^{513} = \begin{bmatrix} 0 & .13228 & 0 & .08232 & .33069 & .27440 & 0 & .18032 \\ 0 & .28571 & 0 & 0 & .71429 & 0 & 0 & 0 \\ 0 & .10582 & 0 & .09651 & .26455 & .32171 & 0 & .21141 \\ 0 & 0 & 0 & .15328 & 0 & .51095 & 0 & .33577 \\ 0 & .28571 & 0 & 0 & .71429 & 0 & 0 & 0 \\ 0 & 0 & 0 & .15328 & 0 & .51095 & 0 & .33577 \\ 0 & .18519 & 0 & .05393 & .46296 & .17978 & 0 & .11814 \\ 0 & 0 & 0 & .15328 & 0 & .51095 & 0 & .33577 \end{bmatrix}$$

(2.3.4) Results should (roughly) approximate

(a) [.151 .086 .194 .243 .054 .273]

(b) [.296 .102 .204 .148 .250]

(c) [0 .286 0 0 .714 0 0 0] or [0 0 0 .153 0 .511 0 .336]

(2.3.5)

(a) [.15067 .08610 .19372 .24250 .05438 .27264]. This is the result seen in (2.3.3) and (2.3.4). It is the steady state distribution of the Markov chain and the fixed point of the corresponding deterministic dynamical system.

(b) [.29630 .10185 .20370 .14815 .25000]. This is the result from (2.3.4) and half the values seen in (2.3.3). It is the stable vector of the Markov chain (though not the steady state ... the chain is periodic, so on any given iteration, an occupation probability may be 0 or twice that seen here) and the fixed point of the dynamical system.

(c) [0 .13192 0 .08251 .32979 .27504 0 .18074] and

[0 $-$.11223 0 .09307 $-$.28057 .31025 0 .20388] (for example).

Note that, while this could be *any* basis for the eigenspace of the eigenvalue 1, [0 .28571 0 0 .71429 0 0 0] and [0 0 0 .15328 0 .51095 0 .33577] also constitute a valid basis. Any vector in this eigenspace is a stable vector and a fixed point, and the two vectors from (2.3.4) are the steady states of the recurrent classes. Note that every row in (2.3.3) is in the span of this eigenspace.

(2.3.6)

(a) Contains a single aperiodic recurrent class (the chain moves more or less freely among all states).

(b) Contains a single period-two recurrent class (the chain alternates between states $\{1, 3\}$ (on even iterations) and $\{2, 4, 5\}$ (on odd iterations)).

(c) Contains one transient and two recurrent classes (the chain spends a finite amount of time in $\{1, 3, 7\}$ and then lives the rest of its life in either $\{2, 5\}$ or $\{4, 6, 8\}$).

(2.3.7) If P is a Markov transition matrix, then every row sum is 1. If e is the n-vector of all 1's, then $Ae = e = 1 \times e$, so 1 is an eigenvalue of P.

(2.3.8) $\mathbf{p}(m) = \mathbf{p}(0)P^m$, so $\mathbf{p}(m)^T = (P^T)^m \mathbf{p}(0)^T$. Note that 1 is the unique eigenvalue of maximum absolute value for P^T, and that $\mathbf{p}(m)$ is found by repeatedly multiplying by P^T. Now see the solution to (2.2.D) (particularly the first 'Note') to see why $\mathbf{p}(m)$ approaches (settles down to) the eigenvector of P^T corresponding to 1 (scaled to be a probability vector, since P^T times any probability vector (i.e. $\mathbf{p}(0)$) is always another probability vector).

Chapter 3

Answers for §3.1

(3.1.1) and .2)

(a) $x = 2$ and $x = -1$: both unstable.

(b) $x = 0$: stable.

(c) $x = 1$ and $x = -1$: both marginally stable.

(d) $x = 0$:unstable; $x = 1$: stable.

(e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: unstable.

(3.1.3)

(a) $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

(b) $x = n\pi,\ \forall n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

(c) $x = 0$

(d) $x = -1$ and $x = 1$

(e) $x = 0$

(f) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

(3.1.4) Set $k' = 0$ to give $sA\sqrt{k} - (d + \rho)k = 0$ and solve for k.

(3.1.5) In the discrete case: $\mathbf{x}, \mathbf{y}$ distinct fixed points $\Rightarrow A\mathbf{x} + \mathbf{b} = \mathbf{x},\ A\mathbf{y} + \mathbf{b} = \mathbf{y} \Rightarrow A\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) + \mathbf{b} = \frac{1}{2}(A\mathbf{x} + \mathbf{b}) + \frac{1}{2}(A\mathbf{y} + \mathbf{b}) = \frac{\mathbf{x}+\mathbf{y}}{2} \Rightarrow \frac{\mathbf{x}+\mathbf{y}}{2}$ is also a fixed point (distict from $\mathbf{x}$ and $\mathbf{y}$). Thus the system cannot have exactly two fixed points.
In the continuous case: $\mathbf{x}, \mathbf{y}$ distinct fixed points $\Rightarrow A\mathbf{x} + \mathbf{b} = \mathbf{0},\ A\mathbf{y} + \mathbf{b} = \mathbf{0} \Rightarrow A\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) + \mathbf{b} = \frac{1}{2}(A\mathbf{x} + \mathbf{b}) + \frac{1}{2}(A\mathbf{y} + \mathbf{b}) = \mathbf{0} \Rightarrow \frac{\mathbf{x}+\mathbf{y}}{2}$ is also a fixed point (distict from $\mathbf{x}$ and $\mathbf{y}$). Thus the system cannot have exactly two fixed points.
(In fact, in both cases, *any* affine combination of fixed points is also a fixed point.)

(3.1.6) See the figure.

(3.1.7) If $t = 1$, then all points in $[0, 2\pi)$ are fixed points. For t an integer greater than one, the fixed points are $\left\{0, \frac{2\pi}{t-1}, \frac{4\pi}{t-1}, \ldots, \frac{2\pi(t-2)}{t-1}\right\}$. In particular, for $t = 3$, they are $\{0, \pi\}$ (for $t = 2$, 0 is the single fixed point).

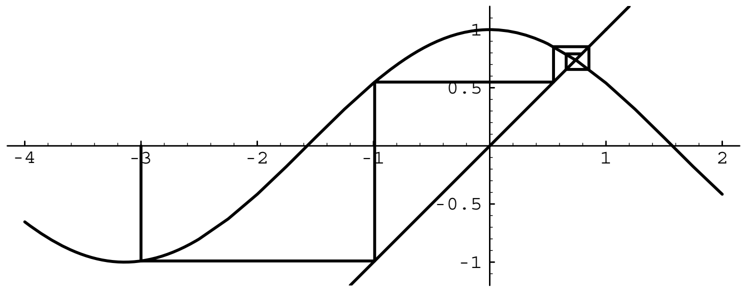

Figure 3: Problem 3.1.6

(3.1.8)

(a) $x(t) = 2e^{-3t}$

(b) $x(t) = \sqrt{4t+1}$

(c) $x(t) = 2e^t - 1$

(d) $x(t) = -1$. (Note: Change $\ln(x+1) = t + C$ to $x + 1 = e^C e^t = ke^t$ prior to solving for the constant $k =$"e^C".)

(e) $x(t) = \tan t$

(f) $x(t) = \left(\frac{t}{2} + 1\right)^2$

Answers for §3.2

(3.2.1)

(a) $x = 0.73909$: stable.

(b) $x = -1$: unstable.

(c) $x = 1 - \frac{\sqrt{3}}{2}$: stable; $x = 1 + \frac{\sqrt{3}}{2}$: unstable.

(d) $x = 0$: stable; $x = 2.51286$: unstable.

(e) $x = -1.49578$: stable; $x = 0$: unstable; $x = 1.49578$: stable.

(f) $x = 0.91486$: unstable.

(g) $x = 1.30296$: unstable.

(h) $x = 1.31594$: stable.

(i) No fixed point.

(3.2.2)

(a) $x = 0$: stable.

(b) $x = 0$: unstable.

(c) $x = \frac{-3-\sqrt{17}}{2}$: stable; $x = \frac{-3+\sqrt{17}}{2}$: unstable.

(d) $x = 0$: unstable; $x = 1$: stable; $x = 2$: unstable.

(e) No fixed points.

(f) $x = 0$: stable.

(g) $x = 0$: stable; $x = 2$: unstable.

(h) $x = 2 - \sqrt{3}$: stable; $x = 2 + \sqrt{3}$: unstable.

(i) $x = -\frac{\pi}{2} + 2\pi n$, $\forall n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$: all unstable.

(3.2.3)

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: unstable.

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: unstable; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$: unstable.

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$: unstable; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}$: stable.

(d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 - 2\sqrt{19} \\ 4 - \sqrt{19} \end{bmatrix}$: stable; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 + 2\sqrt{19} \\ 4 + \sqrt{19} \end{bmatrix}$: unstable.

(e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.29586 \\ 0.25917 \end{bmatrix}$: stable; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12.71321 \\ 2.54264 \end{bmatrix}$: unstable.

(3.2.4) Let $f(k) = sA\sqrt{k} - (d+\rho)k$, so $f'(k) = -d - \rho + \frac{As}{2\sqrt{k}}$ which, at $k = [sA/(d+\rho)]^2$ equals $-(d+\rho)/2 < 0$, so the fixed point is stable.

(3.2.5)

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$: stable.

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: unstable.

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: stable; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: unstable.

(d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} n\pi \\ -n\pi \end{bmatrix}$, $\forall n \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$: all unstable.

(e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.20448 \\ 0.40896 \end{bmatrix}$: unstable; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.83315 \\ 5.66630 \end{bmatrix}$: stable.

(3.2.6) (a) stable, (b) unstable, (c) semistable, (d) semistable, (e) stable, (f) marginally stable (neutral), (g) unstable.

(3.2.7) (a) semistable, (b) semistable, (c) unstable, (d) stable.

(3.2.8) Discrete:

$f'(\tilde{x}) = 1$	$f''(\tilde{x}) = 0, f'''(\tilde{x}) < 0$	Stable
	$f''(\tilde{x}) \neq 0$ or $f'''(\tilde{x}) > 0$	Unstable
	$f''(\tilde{x}) = f'''(\tilde{x}) = 0$	Test fails
$f'(\tilde{x}) = -1$	$f''(\tilde{x}) = 0, f'''(\tilde{x}) > 0$	Stable
	$f''(\tilde{x}) \neq 0$ or $f'''(\tilde{x}) < 0$	Unstable
	$f''(\tilde{x}) = f'''(\tilde{x}) = 0$	Test fails

Continuous:

$f'(\tilde{x}) = 0$	$f''(\tilde{x}) = 0, f'''(\tilde{x}) < 0$	Stable
	$f''(\tilde{x}) \neq 0$ or $f'''(\tilde{x}) > 0$	Unstable
	$f''(\tilde{x}) = f'''(\tilde{x}) = 0$	Test fails

Note that this test does work on $x' = x^3$ and $x' = -x^3$.

Answers for §3.3

(3.3.1)

(a) $V = x^2 + y^2$, $\frac{dV}{dt} = -2x^6$.

(b) $V = (x+1)^2 + y^2$, $\frac{dV}{dt} = -2(x+1)^2$.

(c) $V = x^2$, $\frac{dV}{dt} = -2x^3 \sin^{-1} x$.

(3.3.2) $x_i \equiv$ extension of *spring* i ($x_i > 0 \Rightarrow$ extended, etc.)
$v_i \equiv$ velocity of *block* i ($v_i > 0 \Rightarrow$ downward movement, etc.)
$h_i \equiv$ height of *block* i (value unimportant; $\frac{dh_i}{dt} = -\frac{dv_i}{dt}$ used to find $\frac{dE}{dt}$).

(a) No friction:

$$\begin{aligned} x_1' &= v_1 \\ v_1' &= g - \frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}x_2 \\ x_2' &= v_2 - v_1 \\ v_2' &= g - \frac{k_2}{m_2}x_2 \end{aligned}$$

Fixed point: $[x_1, x_2, v_1, v_2] = \left[\frac{m_1 g}{k_1}\left(1 + \frac{m_2}{m_1}\right), \frac{m_2 g}{k_2}, 0, 0\right]$
$E = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 x_2^2 + \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + m_1 g h_1 + m_2 g h_2$, $\frac{dE}{dt} = 0$. The fixed point is marginally stable.

(b) Normal (proportional to velocity) friction:

$$\begin{aligned} x_1' &= v_1 \\ v_1' &= g - \frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}x_2 - \mu_1 v_1 \\ x_2' &= v_2 - v_1 \\ v_2' &= g - \frac{k_2}{m_2}x_2 - \mu_2 v_2 \end{aligned}$$

Same fixed point and same expression for E. $\frac{dE}{dt} = -m_1\mu_1 v_1^2 - m_2\mu_2 v_2^2$. This is less than 0 "almost everywhere" (meets requirements), so the fixed point is stable.

(c) Weird (proportional to velocity cubed) friction:

$$\begin{aligned} x_1' &= v_1 \\ v_1' &= g - \frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}x_2 - \mu_1 v_1^3 \\ x_2' &= v_2 - v_1 \\ v_2' &= g - \frac{k_2}{m_2}x_2 - \mu_2 v_2^3 \end{aligned}$$

Same fixed point and same expression for E. $\frac{dE}{dt} = -m_1\mu_1 v_1^4 - m_2\mu_2 v_2^4$. This is less than 0 "almost everywhere" (meets requirements), so the fixed point is stable.

(3.3.3) THIS ISN'T RIGHT
$\tilde{\mathbf{x}}$ is a stable fixed point if there exists a $V(\mathbf{x})$ such that:
(1) $V(\mathbf{x}) > 0\ \forall \mathbf{x} \neq \tilde{\mathbf{x}}$, and $V(\tilde{\mathbf{x}}) \leq 0$
(2) $\exists k \in \{1, 2, \ldots\}$ such that $V(f^k(\mathbf{x})) < V(\mathbf{x})\ \forall \mathbf{x}$ within some fixed distance of $\tilde{\mathbf{x}}$ but $\neq \tilde{\mathbf{x}}$.

(a) $V(x) = x^2$:

(1) $V(x) > 0 \forall\ x \neq 0 \checkmark$;

(2) $V(f(x)) = (\tan^{-1} x)^2 < x^2 = V(x)\ \forall x \neq 0 \checkmark$.

(b) $V(x, y) = x^2 + y^2$:

(1) $V(x, y) > 0\ \forall (x, y) \neq (0, 0) \checkmark$;

(2) $(k = 2)\ V(f^2(x, y)) = \frac{1}{4}(x^2 + y^2) < x^2 + y^2 = V(x, y)\ \forall (x, y) \neq (0, 0)$.

(3.3.4)

(a) Gradient system. Take $h = \frac{5}{2}x_1^2 - 7x_1x_2 + 5x_2^2$. $\mathbf{0}$ is a stable fixed point.

(b) Gradient system. Take $h(\mathbf{x}) = \frac{1}{2}\left(e^{x_1^2}x_2^2 + (x_2 - x_1)^2\right)$. $\mathbf{0}$ is a stable fixed point.

(c) Gradient system. Take $h(\mathbf{x}) = \frac{1}{3}x_1^3 - x_1^2x_2 + x_1x_2^2 - \frac{1}{3}x_2^3$. $\mathbf{0}$ is an unstable fixed point.

(d) Not a gradient system.

(e) Not a gradient system.

(3.3.5) Let $\mathbf{x}' = A\mathbf{x}$ and suppose this is a gradient system, i.e., $A\mathbf{x} = f(\mathbf{x}) = -\nabla h(\mathbf{x})$. Then we must have that

$$a_{ij} = \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} = a_{ji}$$

so $A = A^T$, i.e., A is symmetric.

(3.3.6) Any one-dimensional system $x' = f(x)$ is a gradient system, where $h = -\int f(x)dx$.

Answers for §3.4

(3.4.1)

(a) For $x(0) = -1$, $x(k) \to -.61803$ (at $k = 3$)
For $x(0) = 1$, $x(k) \to 1.61803$ (at $k = 4$)

(b) For $x(0) = -2$, $x(k) \to -2.35619$ (at $k = 2$)
For $x(0) = 1$, $x(k) \to .78540$ (at $k = 2$)

(c) 1.465571.

(d) -1.40962 and 0.63673265.

(e) No solutions.

(3.4.2)

(a) For $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{x}(k) \to \begin{bmatrix} 2.46063 \\ 1.71619 \end{bmatrix}$ (at $k = 5$)

(b) One answer: $(x, y) = (1.8146621, 0.54129370)$.

(c) $(x, y) = (-0.46898994, 0.8832035)$ or $(0.8832035, -0.46898994)$.

(d) $(x, y, z) = (5.2282, 0.42778, -2.8906)$, or $(3.3055, 0.73628, -1.9473)$, or $(-1.8473, -0.89254, 0.7269)$, or $(0.43503, 0.52865, 1.4566)$.

(3.4.3) Yes: recall that we wanted to find a such that $-1 < g'(\tilde{\mathbf{x}}) = 1 + af'(\tilde{\mathbf{x}}) < 1$, and that $a = -\frac{1}{f'(\mathbf{x}(k))}$ is used to approximate an a where $g'(\tilde{\mathbf{x}}) = 0$. So long as our initial guess is reasonably close, we will still have $-1 < 1 + \frac{f(\tilde{\mathbf{x}})}{f'(\mathbf{x})} < 1$ even if $f'(\mathbf{x})$ is only updated on every other iteration, so the method still works.

(3.4.4)

(a) Since $a \neq 0$, $f(x) = 0$ iff $g(x) = xe^{af(x)} = xe^0 = x$ $\surd$.

(b) $g'(x) = e^{af(x)}[1 + axf'(x)]$.

(c) $g'(\tilde{x}) = 0 \;\Leftrightarrow\; 1 + a\tilde{x}f'(\tilde{x}) = 0 \;\Leftrightarrow\; a = -\frac{1}{\tilde{x}f'(\tilde{x})}$.

(d) $g(x) = x \exp\left(-\frac{f(x)}{xf'(x)}\right)$.

(e)

$$g'(x) = \exp\left(-\frac{f(x)}{xf'(x)}\right)\left[1 - x\frac{xf'(x)^2 - f(x)[f'(x) + xf''(x)]}{x^2 f'(x)^2}\right],$$

and since $f(\tilde{x}) = 0$,

$$g'(\tilde{x}) = \exp(0)\left[1 - \tilde{x}\frac{\tilde{x}f'(\tilde{x})^2}{\tilde{x}^2 f'(\tilde{x})^2}\right] = [1 - 1] = 0.$$

(f) Applying this method to equations from (3.4.1),we get ...

(1) For $x(0) = -1$, $x(k) \to -.61803$ (at $k = 4$)

For $x(0) = 1$, $x(k) \to 1.61803$ (at $k = 6$)

(2) For $x(0) = -2$, $x(k) \to -2.35619$ (at $k = 3$)

For $x(0) = 1$, $x(k) \to .78540$ (at $k = 3$)

(g) The new method requires one less addition, but two more multiplications and one exponentiation. It also seems to be a bit slower.

(h) Under this method, $x(k)$ can never change sign, so a possitive $x(0)$ would never converge to a negative root. If the only root is negative, the method could converge to 0, blow up, or simply wander around, depending on f and $x(0)$.

Chapter 4

Answers for §4.1

(4.1.1) For $k_1 = k_2 = m_1 = m_2 = 1$,
$x_1' = v_1$, $v_1' = g - x_1^3 + x_2^3$, $x_2' = v_2 - v_1$, $v_2' = g - x_2^3$.
See the figure for plots of x_1 and x_2.

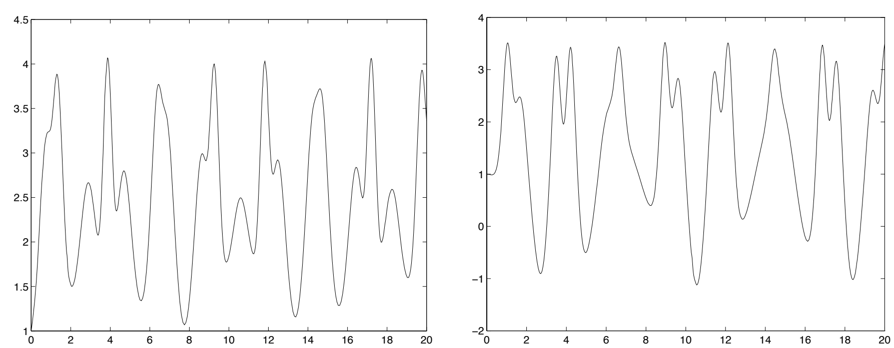

Figure 4: Problem 4.1.1

(4.1.2) $x' = v$, $v' = -x^3$. See the figure.
As noted in the chapter, two dimensional systems (such as this one) cannot exhibit chaotic behavior.

(4.1.3) For (a) the system is

$$\begin{aligned} x' &= y \\ y' &= -\mu(x^2 - 1)y - x \end{aligned}$$

For (b) see the figure.

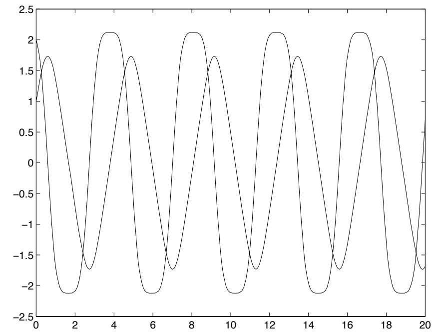

Figure 5: Problem 4.1.2

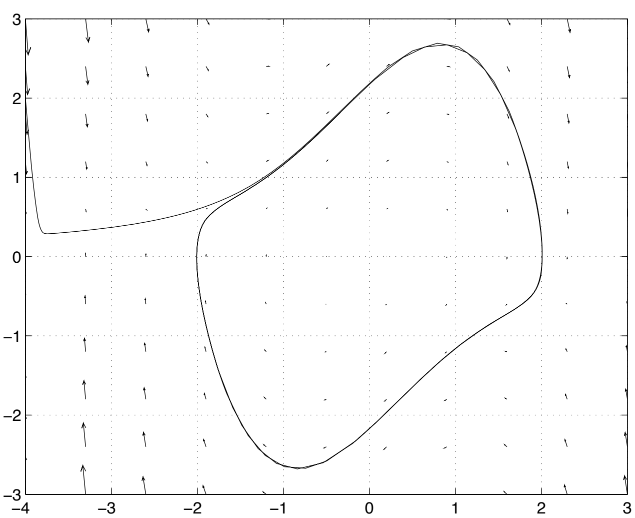

Figure 6: Problem 4.1.3.

(4.1.4) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, eigenvalues: $\{-22.828, 11.828, -2.667\}$

$\begin{bmatrix} -\sqrt{72} \\ -\sqrt{72} \\ 27 \end{bmatrix}$, eigenvalues: $\{-13.855, .094 + 10.195i, .094 - 10.195i\}$

$\begin{bmatrix} \sqrt{72} \\ \sqrt{72} \\ 27 \end{bmatrix}$, eigenvalues: $\{-13.855, .094 + 10.195i, .094 - 10.195i\}$

Note that all sets of eigenvalues constain elements with positive real parts, so all three fixed points are unstable.

(4.1.5)

(a) $Df_a(\mathbf{0}) = \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix}$, bifurcation occurs at $a = 0$. See the figure.

(b) $Df_a(\mathbf{0}) = \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix}$, bifurcation occurs at $a = 0$. See the figure.

(4.1.6)

(a) $\mathbf{x}(t) = \begin{bmatrix} \sin t \\ \cos t \\ \sin \sqrt{2}t \\ \cos \sqrt{2}t \end{bmatrix}$

(b) For (a) see the figure. The solid curve is $x_1(t)$ and the dotted curve is $x_3(t)$.

Note: If $\mathbf{x}_0 = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, then $\mathbf{x}(t) = \begin{bmatrix} -r\cos(t+\theta) \\ r\sin(t+\theta) \\ -s\cos(\sqrt{2}t+\phi) \\ s\sin(\sqrt{2}t+\phi) \end{bmatrix}$, where $r = \sqrt{a^2+b^2}$, $s = \sqrt{c^2+d^2}$, θ is the angle of the point (a,b), and ϕ is the angle of the point (c,d). Observe that, regardless of the particular values of a, b, c, and d, x_1 and x_2 stay in a permanent circular orbit of radius r, and x_3 and x_4 stay in a permanent circular orbit of radius s. For the given $\mathbf{x}_0, r = s = 1$, $\theta = \phi = \pi/2$.

(c) As seen above, $\mathbf{x}(t)$ neither a) approaches a fixed point nor b) tends towards infinity. Now suppose $\mathbf{x}(t)$ tends towards a periodic orbit. Then there is stable periodic orbit which it tended towards, and being periodic, there would exist t_1 and t_2 such that $\mathbf{x}(t_1) = \mathbf{x}(t_2)$. Using $\mathbf{x}(t)$ from (1), $\sin t_1 = \sin t_2$ and $\cos t_1 = \cos t_2$, so we must have $t_1 = t_2 + 2n\pi$ for some integer n. Similarly, $\sin\sqrt{2}t_1 = \sin\sqrt{2}t_2$ and $\cos\sqrt{2}t_1 = \cos\sqrt{2}t_2$, so we must have $\sqrt{2}t_1 = \sqrt{2}t_2 + 2m\pi$ for some integer m. This gives $t_1 = t_2 + \sqrt{2}m\pi$, and from above, we must have $\sqrt{2}m\pi = 2n\pi$, i.e. that $\sqrt{2} = m/n$, which says that $\sqrt{2}$ is a rational number. This is false, so our original supposition that $\mathbf{x}(t)$ tends towards a periodic orbit must be false.

(d) As "Note"ed above, x_1 and x_2 fall into a circular (periodic) orbit together, as do x_3 and x_4. These two orbits are independent of each other, and thus constitute periodic subsystems.

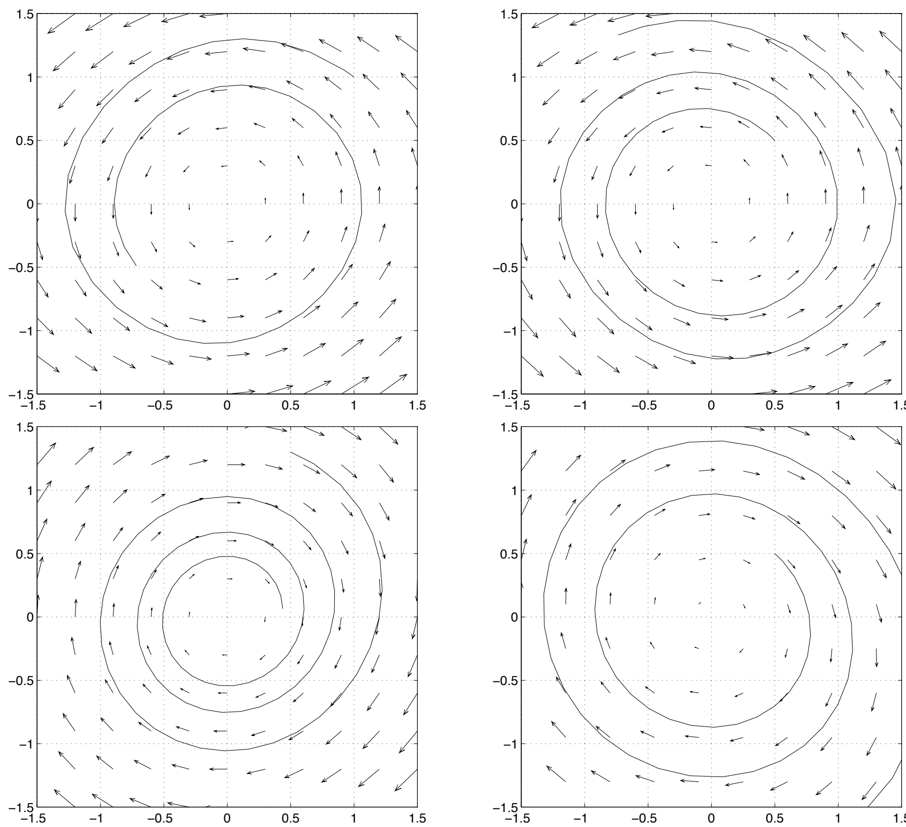

Figure 7: Problem 4.1.5; (a) is above and (b) is below

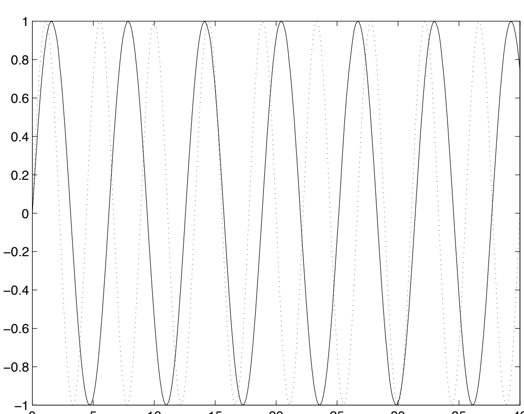

Figure 8: Problem 4.1.6

(e) Since x_1 and x_2 are independent of x_3 and x_4, they behave as a two-dimensional system, and thus (like all two-dimensional systems) cannot exibit chaotic behavior. Similarly for x_3 and x_4. Thus $\mathbf{x}(t)$ as a whole is not chaotic. In particular, using the formula for $\mathbf{x}(t)$ from the "Note", we can see that small changes in initial conditions only create small changes in the orbit radii and initial angles.

Answers for §4.2

(4.2.1) Read *periodic point*$^{stability}_{prime period}$, i.e $.55801^s_2$ is a stable point of prime period 2. Of course, all points of prime period 1 (fixed points) also have periods 2 and 3.

(a) 0^u_1 , $.55801^s_2$, $.67742^u_1$, $.76457^s_2$

(b) $.54257^u_1$, $.90098^u_3$, 1^u_3 , 1.47948^u_2 , 1.73773^u_3 , 2^u_3 , 2.45743^u_1 , 2.85385^u_2 , 3^u_3 , 3.02796^u_3

(c) -1.44619^u_1 , -1.00116^s_3 , $-.96708^u_3$, $-.37948^u_2$, $-.00307^s_3$, $.08580^u_3$, $.69147^u_1$, 1.13420^u_2 , 1.31525^u_3 , 1.32449^s_3

(d) $.73909^s_1$

(e) -2.93810^s_1 , -2.66318^u_1 , -1.81888^u_3 , -1.28895^u_3 , $-.89605^u_2$, $-.73664^u_3$, $.83438^u_3$, 1.17012^u_1 , 1.87411^u_2 , 2.01490^u_3 , 2.22219^u_3

(f) $-.97937^s_1$, $-.88773^u_1$, $-.60629^u_3$, $-.42965^u_3$, $-.29868^u_2$, $-.24555^u_3$, $.27813^u_3$, $.39004^u_1$, $.62470^u_2$, $.67163^u_3$, $.74073^u_3$

(g) 0^u_1

(h) -1.84141^s_1 , 1.14619^u_1

(i) $.61906^s_1$, 1.51213^u_1

(j) -2.33112^s_2 , 0^u_1 , 2.33112^s_2

(k) -2.61803^s_2 , -1.21341^u_1 , $-.38197^s_2$

(l) -2^u_1 , $-10/9^u_3$, $-6/7^u_3$, $-2/5^u_2$, $-2/9^u_3$, $2/7^u_3$, $2/3^u_1$, $6/5^u_2$, $10/7^u_3$, $14/9^u_3$

(4.2.2)

(a) $a = -e$: period doubling, $a = 1/e$: tangent

(b)
$a = -2.61778$: period doubling, $a = -2.26183$: tangent,
$a = -1$: period doubling, $a = 1$: tangent,
$a = 2.26183$: period doubling, $a = 2.61778$: period doubling

(c)
$a = -2.61778$: period doubling, $a = -2.26183$: tangent,
$a = -1$: period doubling, $a = 1$: tangent,
$a = 2.26183$: period doubling, $a = 2.61778$: period doubling

(d) Note: all of the following bifurcation points repeat themselves with a period of 2π.

$a = -2.25565$: tangent,	$a = -1.20845$: period doubling,
$a = -.47596$: period doubling,	$a = 1.16049$: period doubling,
$a = 1.98110$: period doubling,	$a = 3.61755$: period doubling
$a = 4.35004$: period doubling,	$a = 5.39724$: tangent

(e) $a = .15343$: period doubling

(4.2.3) A computer generated bifurcation diagram for the family of functions $f_a(x) = ax(1-x)$ is shown in the figure. Horizontal axis runs from $a = 2$ to $a = 4$ and vertical from $x = 0$ to $x = 1$.
There is a tangent bifurcation at $a = 1$. There are period doubling bifurcations at $a = 3$ and $a = 1 + \sqrt{6} \approx 3.45$.

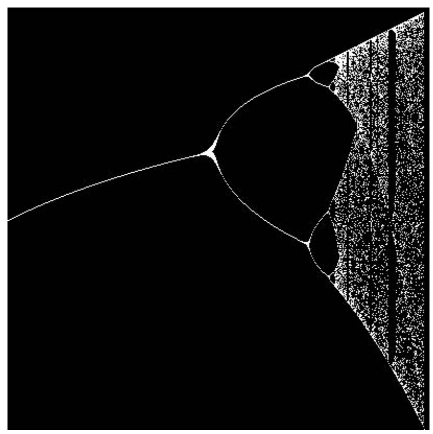

Figure 9: Problem 4.2.3

(4.2.4) We are given that $2 \triangleright 1$. We want to prove $4 \triangleright 1$. Let $f: \mathbf{R} \to \mathbf{R}$ be continuous with a point p of prime period 4. We must prove that f has a fixed point.
Therefore p is a point of prime period 2 of f^2. Since $2 \triangleright 1$, f^2 has a fixed point q. Now either q is a fixed point of f as well (in which case we're done) or q is a point of prime period 2 of f. But since $2 \triangleright 1$, we again conclude that f has a fixed point.

(4.2.5) Suppose $a \triangleright b$ and $b \triangleright c$. Let f be a continuous function $f: \mathbf{R} \to \mathbf{R}$ with points of prime period a. Since $a \triangleright b$, we know that f has points of prime period b, and since $b \triangleright c$ we know that f has points of prime period c. Thus $a \triangleright c$.

(4.2.6) In all of the following, (x, y, z) may be any permutation of (A, B, C).
Fixed Points: $xxx\ldots$
Prime Period 2: $xyzxyz\ldots$
Prime Period 3: $xyxy\ldots$, $xxyzzyxxy\ldots$, $xyyxzzxyy\ldots$, $xyzzyxxyz\ldots$

(4.2.7) $f(x) = x^2 - 1.3$. On successive evaluations, we get $-1.29962 \to 0.38902 \to -1.14866 \to 0.019430 \to -1.29962$, each of which has prime period 4.

(4.2.8) If x has period p and prime period q, then q must divide p. If p is prime, only 1 and p divide p, so x must have prime period 1 or p.

(4.2.9) $g(a) = f(a) - a = b - a > 0$, $g(b) = f(b) - b = a - b < 0$, so by the Intermediate Value Theorem, there exists $c \in (a, b)$ such that $g(c) = 0$. Thus $g(c) = f(c) - c = 0$, i.e. $f(c) = c$ is a fixed point of f.

Answers for §4.3

(4.3.1) If $x = 0.d_1d_2\cdots$, then $\sigma(x) = 0.d_2d_3\cdots$, $\sigma^2(x) = 0.d_3d_4\cdots$, $\ldots$,$\sigma^k(x) = 0.d_{k+1}d_{k+2}\cdots$. On the other hand, $2x = d_1.d_2d_3\cdots$, $2^2x = d_1d_2.d_3d_4\cdots$, $\ldots$,$2^kx = d_1d_2\cdots d_k.d_{k+1}d_{k+2}\cdots$, and $2^kx \bmod 1 = 0.d_{k+1}d_{k+2}\cdots = \sigma^k(x)$.

(4.3.2) $\{x : \sigma^k(x) = x\} = \{x : x = 0.\overline{d_1d_2\cdots d_k}\ , d_1, \ldots, d_k \text{ not all } 1\}$

(4.3.3) See the figure showing the first 60 iterations of σ starting with $x_0 = \pi$.

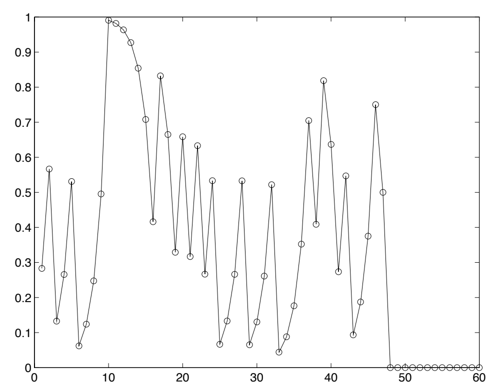

Figure 10: Problem 4.3.3

(4.3.4) As seen in (4.3.2), any repeated length n sequence of 1's and 0's (except the sequence of all 1's) constitutes a point of period n, so there are $2^n - 1$ such points. x has period n iff x has prime period d and $d|n$. Thus

$$\begin{aligned}
\text{(number of points with period } n) &= \sum_{d|n} \text{(number of points with } \textit{prime} \text{ period } d) \\
2^n - 1 &= \sum_{d|n} f(d).
\end{aligned}$$

The formula now follows by Mobius inversion (see standard number theory texts or combinatorics texts).

(4.3.5) If we label the cards $0, 1, \ldots, 51$, a perfect riffle shuffle is

$$\begin{array}{ll}
0 \to 0 & 26 \to 1 \\
1 \to 2 & 27 \to 3 \\
\vdots & \vdots \\
25 \to 50 & 51 \to 51.
\end{array}$$

Call this permutation σ. Notice that $\sigma(51) = 51$, so $\sigma^8(51) = 51$, and for $x \neq 51, \sigma(x) = 2x$ mod 51. Similarly to (4.3.1), $\sigma^8(x) = 2^8 x \bmod 51 = 256x \bmod 51 = [51 \times (5x) + x] \bmod 51 = x \bmod 51 = x$, so $\sigma^8(x) = x\ \forall x$, as desired.

(4.3.6) For $x \in [0,1)$, write x in ternary notation: $x = 0.d_1 d_2 \cdots$, where $d_i \in \{0,1,2\}\ \forall i$. Then $f(x) = 0.d_2 d_3 \cdots$.
Points with prime period k are, for the given $k \ldots$
$k = 1 : x = 0\ ,\ 1/2$
$k = 2 : x = 1/8\ ,\ 1/4\ ,\ 3/8\ ,\ 5/8\ ,\ 3/4\ ,\ 7/8$
$k = 3 : x = 1/26\ ,\ 2/26\ ,\ \ldots\ ,\ 12/26\ ,\ 14/26\ ,\ 15/26\ ,\ \ldots\ ,\ 25/26$
$k = 4 : \{x = n/80 : n \in \{1, 2, \ldots, 79\}\ ,\ 10 \not| n\}$

(4.3.7) To divide a size N deck into three piles, randomly generate $x_1, x_2, \ldots, x_N$ each in [0,1). Let pile A have $|\{x_i : x_i \in [0, 1/3)\}|$ cards, pile B have $|\{x_i : x_i \in [1/3, 2/3)\}|$ cards, and pile C have $|\{x_i : x_i \in [2/3, 1)\}|$ cards. For riffling these piles together, if piles A, B and C have a, b, and c cards left in them, respectively, then a card falls from pile A with probability $\frac{a}{a+b+c}$, etc.
Equivalently, order $x_1, \ldots, x_N$ (generated as above) and then assign them to the cards in that order. Apply $f(x) = 3x \bmod 1$ to each number to find the new order of the cards after a shuffle.
Define $P_N(k) \equiv \text{Prob}\big\{\text{each card is in its own length } \frac{1}{3^k} \text{ interval}\big\} = \big(1 - \frac{1}{3^k}\big) \times \big(1 - \frac{2}{3^k}\big) \times \cdots \times \big(1 - \frac{N-1}{3^k}\big)$. For $N = 216$, $P(11) = 0.877$, $P(12) = 0.957$, $P(13) = 0.986$, $P(14) = 0.995$. Based on these simple calculations, 12 to 14 shuffles are sufficient to guarantee a thorough mixing.

Chapter 5

Answers for §5.1

(5.1.1) (a) yes, (b) yes, (c) yes, (d) yes, (e) no, (f) no, (g) no, (h) no.

(5.1.2) All real numbers in [0,1] whose pentary (base 5) expansions may be written using only 0's and 4's (allowing $0.\cdots 1_5$ to be written as $0.\cdots 0\bar{4}_5$).

(5.1.3) In base 3, $f(x) = \begin{cases} 10x_3 & , \quad x \leq .\bar{1}_3 \\ 10(1-x)_3 & , \quad x > .\bar{1}_3 \end{cases}$ (where $.\bar{1}_3 = 1/2$). B = Cantor's set. For $x \in B$, if $x = .0\ldots, f(x)$ moves the decimal point one place to the right; if $x = .2\ldots, f(x)$ switches all 0's and 2's in x, and then moves the decimal point one place to the right. Notice that in both cases, $f(x) \in B$.

(5.1.4)

(a) bounded, closed, compact.

(b) not bounded, closed, not compact.

(c) not bounded, closed, not compact.

(d) not bounded, not closed, not compact.

(e) not bounded, closed, not compact.

(f) not bounded, closed, not compact.

(g) bounded, closed, compact.

(h) bounded, not closed, not compact.

(i) bounded, closed, compact.

(j) bounded, closed, compact.

(5.1.5) Several examples: $[0,1]$, $(0,1]$, $(0,1)$, $[0,1)$, $[0,\infty)$, $(0,\infty)$, $\emptyset$, $\mathbf{R}$.

(5.1.6) Cantor's set, $\mathbf{Q}$ (the rationals), $\mathbf{Z}$ (the integers), any finite subset. A singleton set $\{x\}$ is connected and (vacuously) totally disconnected.

Answers for §5.2

(5.2.1) The result looks like Sierpiński's triangle.

(5.2.2) The set contains all $(x,y) \in [0,1]\times[0,1]$ such that x and y don't share a common ternary digit containing a 1 (i.e. if x's 5th ternary digit is a 1, then y cannot have a 1 as its 5th digit. However, $x = .0101\overline{01}_3, y = .12\overline{1012}_3$ is a valid member of this set).

Answers for §5.3

(5.3.1)

(a) Not a contraction map.

(b) $s = 1/2$.

(c) Not a contraction map.

(d) Not a contraction map.

(e) $s = 1/2$.

(f) $s = \sqrt{\frac{2}{e}}$.

(5.3.2) Suppose $x_1, x_2, \cdots$ is a sequence converging to x. This means that $|x_n - x| \to 0$ as $n \to \infty$. Now $|f(x_n) - f(x)| \le s|x_n - x|$, so, as $n \to \infty$ $f(x_n) \to f(x)$. Therefore f is continuous.

(5.3.3) $f(x) = \frac{1}{2}|x|$ is a contraction map (with $s = 1/2$), but is not differentiable at $x = 0$.

(5.3.4) Let $f\colon \mathbf{R} \to \mathbf{R}$ be differentiable and suppose $|f'(x)| < s$ for all x where $s < 1$. Let $a, b \in \mathbf{R}$. Then

$$\left|\frac{f(a) - f(b)}{a - b}\right| = |f'(c)|$$

for some c between a and b. Since $|f'(c)| \le s$ we have $|f(a) - f(b)| \le s|a - b|$, and therefore f is contractive.

(5.3.5) A sketch of the graph of the function $f(x) = x - \frac{1}{2}\log{(e^x + 1)}$ is shown in the figure. The derivative is $f'(x) = 1 - e^x/(2 + 2e^x)$, so $0 < f'(x) < 1$. However, the graph $y = f(x)$ is asymptotic to $y = x$ as $x \to -\infty$. Thus for a, b very negative $f(a) - f(b)$ is nearly $a - b$ (or as close as we want, so there is no $s < 1$ for which $|f(a) - f(b)| \le s|a - b|$

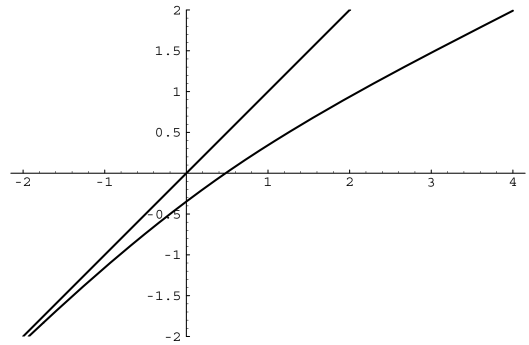

Figure 11: Problem 5.3.5

(5.3.6) $\vec{d}(J,K) = \sqrt{2} - 1$, $\vec{d}(K,J) = 0$, $d(J,K) = \sqrt{2} - 1$.

(5.3.7) $\vec{d}(J,K) = 0$, $\vec{d}(K,J) = 1$, $d(J,K) = 1$

(5.3.8) $f([1,2]) = [\sqrt{2}, 1.5]$, so $f : [1,2] \to [1,2]$, and $f(p/q) = \frac{p^2+2q^2}{2pq} \in \mathbf{Q}$, so $f : \mathbf{Q} \to \mathbf{Q}$. Thus $x \in \mathcal{X} \Rightarrow f(x) \in \mathcal{X}$, as desired.
$f'(x) = \frac{1}{2} - \frac{1}{x^2}$ ranges from $-\frac{1}{2}$ to $\frac{1}{4}$ for $x \in [1,2]$, so $s = \frac{1}{2}$, and thus f is a contraction mapping. However, since any fixed point of $f : \mathcal{X} \to \mathcal{X}$ must also be a fixed point of $f : [1,2] \to [1,2]$, and $f : [1,2] \to [1,2]$ has a unique fixed point $\tilde{x} = \sqrt{2} \notin \mathcal{X}$, it follows that $f : \mathcal{X} \to \mathcal{X}$ has no fixed point.

Answers for §5.4

(5.4.1)

(a) $[0,3]$.

(b) $[\frac{1}{2}, 1] \cup [2,4]$.

(c) $[-1,1]$.

(d) Cantor's set.

(e) See the figure.

(f) See the figure.

(5.4.2) This is embedded in `fractory`.

(5.4.3) $f_1(x) = \frac{1-t}{2}x$, $f_2(x) = \frac{1-t}{2}x + \frac{1+t}{2}$

(5.4.4)

$$A = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}, \ \mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}, \ \mathbf{b}_3 = \begin{bmatrix} 2/3 \\ 0 \end{bmatrix},$$

$$\mathbf{b}_4 = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}, \ \mathbf{b}_5 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}, \ \mathbf{b}_6 = \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}, \ \mathbf{b}_7 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}, \ \mathbf{b}_8 = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}.$$

The IFS is $f_i(\mathbf{x}) = A\mathbf{x} + \mathbf{b}_i, i = 1, \ldots, 8$.

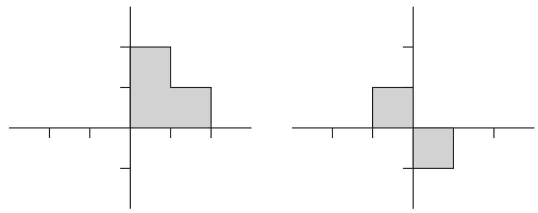

Figure 12: Problem 5.4.1 (e) [left] and (f) [right]

(5.4.5) $[0,1]$.

Answers for §5.5

(5.5.1) The described procedure is equivalent to the IFS

$$f_1(x) = \frac{x}{3}\ ,\quad f_2(x) = \frac{x}{3} + \frac{2}{3}.$$

Making the replacements $h \to 0, t \to 2$, the coin flips *ththhtt*... would, for instance, generate the points $.2, .02, .202, .0202, .00202, .200202, .2200202$, etc. This is the randomized algorithm for generating the Cantor Set.

(5.5.2) Since $\{\mathbf{x}_1\} \in \mathcal{H}^2$, the Contraction Mapping Theorem for Complete Metric Spaces tells us that if K is the attractor of F, then $F^k(\{\mathbf{x}_1\}) \to K$ as $k \to \infty$. Define $a_k \equiv F^k(\{\mathbf{x}_1\})$, and $a_0 \equiv \{\mathbf{x}_1\}$, so that $a_k \to K$ as $k \to \infty$. Since we also see that $\mathbf{x}_1 \in a_k \Rightarrow \mathbf{x}_1 \in f_1(a_k) \Rightarrow \mathbf{x}_1 \in F(a_k) = a_{k+1}$, it follows inductively that $\mathbf{x}_1 \in a_0\ ,\ \ \mathbf{x}_1 \in a_1\ ,\ \ \mathbf{x}_1 \in a_2\ , \ldots$. Since $\mathbf{x}_1 \in a_k$ for all k, $\mathbf{x}_1$ must also be contained in the limit of the a_k's, i.e. $\mathbf{x}_1 \in K$. Similarly, $\mathbf{x}_2, \mathbf{x}_3 \in K$.

(5.5.3) F has contractivity $s_F \equiv \max_{1 \le i \le k} s_i < 1$. If we call the fractal K,

$$\begin{aligned} d(\mathbf{0}, K) &= 1 \\ d(F(\mathbf{0}), F(K)) = d(F(\mathbf{0}), K) &\le s_F \\ &\vdots \\ d(F^{50}(\mathbf{0}), K) &\le s_F^{50}. \end{aligned}$$

Since the 50th point generated (call it $\mathbf{x}_{50}$) is $\in F^{50}(0)$, $d(\mathbf{x}_{50}, K) \le s_f^{50}$.
If $s_F \le 0.88$, 50 is reasonable ($0.88^{50} \approx \frac{1}{600}$). However, if $s_F = 0.95$, $s_F^{50} = .07695 \approx \frac{1}{13}$, which is a noticeable distance for *any* level of computer screen resolution. $0.95^{500} = 7.2745 \times 10^{-12}$, on the other hand, is *not* a noticeable distance under *any* resolution.

Answers for §5.6

(5.6.1) $d = \frac{\log 8}{\log 3} \approx 1.89279$.

(5.6.2) $N_\Box\left(C_t, \left(\frac{1-t}{2}\right)^k\right) = 2^k$, so $d_{C_t} = \frac{\log 2}{\log \frac{2}{1-t}}$

(5.6.3) $C_{1/2}$ (from (5.6.B)) has dimension exactly $1/2$.

(5.6.4) Clearly a triangle of side length 1 can be covered by a square of side length 1. From the figure we see that a square of side length 1 can be covered by 5 triangles. Thus, $N_{\square}(K,r) \le N_{\triangle}(K,r) \le 5N_{\square}(K,r)$. Given this, the proof proceeds identically to the one in the text showing the equivalency of unit disks and boxes.

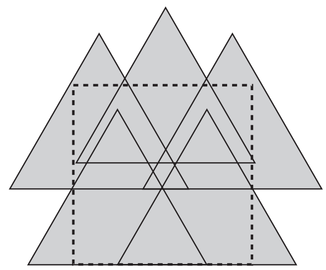

Figure 13: Problem 5.6.4

(5.6.5) Doubling the size does not change the fractal dimension; it is still d.

(5.6.6) $d = \log 4/\log 3 \approx 1.2619$.

(5.6.7) $\dim(K_1 \cup K_2) = \max\{d_1, d_2\}$; $\dim(K_1 \cap K_2) \le \min\{d_1, d_2\}$.

(5.6.8) No. Example: If K_i is the ith iteration in generating the Koch Snowflake K, $d(K_i, K) \to 0$ as $i \to \infty$, but $d_i = 1\ \forall i$ (K_i is just a curve in the plane) while $d = \log 4/\log 3 > 1$.

(5.6.9) Let S_n and B_n be the unit hypercube and ball, respectively, in $\mathbf{R}^n$. $\mathbf{x} \in S_n$ iff $|x_i| \le 1/2$, $i = 1, 2, \ldots, n$. $\mathbf{x} \in B_n$ iff $\sum_{i=1}^n x_i^2 \le 1$.
(1) For $\mathbf{x} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \in S_4$, $\sum_{i=1}^4 x_i^2 = 1$, so $\mathbf{x} \in B_4$.
(2) For $\mathbf{x} = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}) \in S_5$, $\sum_{i=1}^5 x_i^2 = \frac{\sqrt{5}}{2} > 1$, so $\mathbf{x} \notin B_5$.

(5.6.10) Since

$$\frac{\log_a x}{\log_a y} = \log_x y$$

we have

$$\frac{\log_a N(K,r)}{\log_a(1/r)} = \log_{1/r} N(K,r) = \frac{\log_b N(K,r)}{\log_b(1/r)}$$

so it doesn't matter what base log we use in the definition of dimension.

Chapter 6

Answers for §6.1

(6.1.1) The unit circle, i.e., $\{z : |z| \le 1\}$.

(6.1.2) $f(z) = z^2 - 6$, so $f(-2) = 4 - 6 = -2$. Thus $f^k(-2) \not\to \infty$, so $-2 \in B_{-6}$. Also, $f(2) = -2$, so $2 \in B_{-6}$. Also $f(\pm 2\sqrt{2}) = 8 - 6 = 2$, so $\pm 2\sqrt{2} \in B_{-6}$. Another fixed point of f is 3, so $3 \in B_{-6}$. Also $\pm\sqrt{3}$.

(6.1.3) Let $f(z) = z^2 - 1 + 3i$. The fixed points of f are $-1 + i$ and $2 - i$, so they are in B_{-1+3i}. Also $-2 + i$ and $1 - i$ are in B_{-1+3i}.

Answers for §6.2

(6.2.1) $\mathcal{M}$ is symmetric about the x-axis because $c \in \mathcal{M}$ if and only if $\overline{c} \in \mathcal{M}$ (the conjugate of c). This is true because $\overline{z^2 + c} = \overline{z}^2 + \overline{c}$. Thus

$$\overline{f_c^k(0)} = f_{\overline{c}}^k(0)$$

and one explodes iff the other does.

Answers for §6.3

(6.3.1) Newton's method to solve $z^3 - 1 = 0$ is to iterate the function

$$g(z) = z - \frac{z^3 - 1}{3z^2}$$

but notice that $g(0)$ is undefined (0 denominator).

(6.3.2) To solve $x^2 - 3 = 0$ we iterate $f(x) = x - (x^2 - 3)/(2x)$. Notice that $f(i) = -i$ and $f(-i) = i$, so i is a point of prime period 2, and Newton's method doesn't convege to a root. Notice that $f(iy)/i = (-3+y^2)/(2y)$. If we iterate f starting at $x = 1.01i$, we get a sequence of purely imaginary values plotted in the figure.

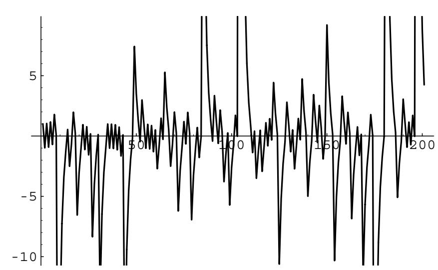

Figure 14: Problem 6.3.2

The value i is an unstable periodic point.
If we iterate starting at $x = i\sqrt{3}$ we see that $f(i\sqrt{3}) = 0$, and then $f^2(i\sqrt{3})$ is undefined.

Answers for §6.4

(6.4.1) For any x, y, $|g_1(x) - g_1(y)| = |x/b - y/b| = |x - y|/|b|$, so g_1 has contractivity $|1/b|$. Similarly, $|g_2(x) - g_2(y)| = |(x+1)/b - (y+1)/b| = |(x+1) - (y+1)|/|b| = |x - y|/|b|$, so g_2 also has contractivity $|1/b|$.

Proof of similitudes: Let $c = 1/b$, and write z and c in polar notation, i.e. $z = re^{i\theta}$ and $c = se^{i\phi}$, so that (r, θ) and (s, ϕ) are the polar coordinates for the complex vectors z and c. Then $zc = rse^{i(\theta+\phi)}$ has polar coordinates $(rs, \theta + \phi)$, and thus multiplication by c (i.e. the function $g_1(z)$) constitutes a dilation (by a factor of s) and a rotation (through an angle ϕ). g_1 is therefore a similitude.
Similarly, $g_2(z) = zc + c$ consists of the above dilation and rotation, plus the translation by c, so it is also a similitude.

(6.4.2) $g_1(z)$ and $g_2(z)$ are both contraction maps on the complex plane (alternately, on $\mathbf{R}^2$), and thus are also contraction maps on H^2, the set of compact sets of $\mathbf{R}^2$. Then we have that $G = g_1 \cup g_2$ is also a contraction map on H^2 (see Section 5.4). As noted in the text, then, $C(b)$ is the unique atractive fixed point of G by the Contraction Mapping Theorem, and since G is a function in H^2, we must have that $C(b)$ is also in H^2, i.e. $C(b)$ is a compact set.

(6.4.3) $\log 2/\log b = \log_b 2$. This follows from seeing $C(b)$ as the attractor of just touching (indeed, disjoint) similitudes with contractivity $1/b$.

(6.4.4) If $1 < b \leq 2$, then $C(b) = \left[0, \frac{1}{1-b}\right]$.

(6.4.5) If $1 < b \leq 2$, then members of $C(b)$ may have more than one representation, but for $b > 2$, members of $C(b)$ have a unique representation.

(6.4.6) $\log 2/\log 3$.