**1.** (8 points) Solve each of the following systems or show that it is inconsistent. If there are infinitely many solutions, provide the general solution.

a) $\begin{cases} -3x + 4y - 2z = 5 \\ 2x + 3y + 3z = 16 \\ x + y = 9 \end{cases}$ b) $\begin{cases} x_1 + 2x_3 = 7 \\ 3x_1 - x_2 + 3x_3 + x_4 = 9 \\ -2x_1 + x_2 - x_3 = -19 \end{cases}$

**2.** (4 points) The North Pole Factory makes toy cars and toy planes and toy trains. Each product requires wheels and screws as summarized in the following table:

| | Wheels | Screws |
|---|---|---|
| Toy car | 4 | 4 |
| Toy plane | 4 | 8 |
| Toy train | 8 | 11 |

If there are 136 wheels and 232 screws available, how many of each toy can be made so that all the parts are used?

(a) Define all the necessary variables and set up the system of equations for the problem.

(b) Given that the augmented matrix for the system reduces as shown below, give the general solution for the system: $\left[\begin{array}{ccc|c} 1 & 0 & 5/4 & 10 \\ 0 & 1 & 3/4 & 24 \end{array}\right]$ and give all possible realistic solutions.

**3.** (3 points) Find all (if any) values of $k$ such that the system: $\begin{cases} x + 9y - 4z = -2 \\ (k-5)y = 2k - 10 \\ (k-7)z = k^2 - 1 \end{cases}$

(a) has no solutions

(b) has infinitely many solutions

(c) has a unique solution

**4.** (6 points) Consider the system $\begin{cases} x_1 - x_2 + 2x_3 = -17 \\ 3x_1 + 4x_3 = 0 \\ -2x_1 - 3x_2 + 5x_3 = 34 \end{cases}$

(a) Write the system in the form $AX = B$

(b) Find $\operatorname{adj}(A)$.

(c) Compute $\det(A)$.

(d) Using your previous results, find $A^{-1}$.

(e) Solve the system using $A^{-1}$.

**5.** (3 points) Let $A$, $B$ and $C$ be invertible matrices. Solve for $X$ in the equation $C(2X + B)^{-1}A = B$.

**6.** (6 points) Given $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 3 \\ 8 & 2 & -5 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$,

compute the following, if defined. If it is not defined, briefly explain why.

a) $C^{-1}$ b) $6A^{-1} - C^T$ c) $AB$ d) $BA$

**7.** (7 points) If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} a & b & c \\ 3g+4a & 3h+4b & 3i+4c \\ 2d & 2e & 2f \end{bmatrix}$,

Additionally, $\det(A) = 4$, and $C$ is a $3 \times 3$ non-invertible matrix;

Find the following or state that there is not enough information:

(a) $\det(B)$

(b) $\det(A - B)$

(c) $\det(AC)$

(d) $\det(3A^{-1})$

(e) How many solutions are there to the system $CX = \mathbf{0}$?

**8.** (4 points) The Pasta Bros is a company with three locations owned by brothers Pino, Pasquale, and Pietro. Each brother manages a different location. The company specializes in producing fresh linguini, spaghetti, and fettucini. The daily output at Pino's location is 8 tons of linguini, 16 tons of spaghetti, and 24 tons of fettucini. The daily output at Pasquale's location is 15 tons of linguini, 35 tons of spaghetti, and 25 tons of fettucini. The daily output at Pietro's location is 10 tons of linguini, 22 tons of spaghetti, and 22 tons of fettucini. Pietro has decided to retire and close his location. Is there some combination of the outputs from the other two locations that will equal the output of Pietro's location? If yes, what is the combination?

**9.** (8 points) Let $\vec{u} = \langle 4, 2, 0 \rangle, \vec{v} = \langle -2, 4, 3 \rangle$, and $\vec{w} = \langle 3, -1, -2 \rangle$.

a) Evaluate $2||3\vec{u}|| - ||\vec{v} - 2\vec{w}||$.

b) Find a vector of length 2 that is oppositely directed to $\vec{w}$.

c) Find the value(s) of $k$ such that $\vec{r} = \langle 5, k, 2 \rangle$ is orthogonal to $\vec{w}$.

d) Do $\vec{u}, \vec{v}, \vec{w}$ form an orthogonal set? Justify your answer.

**10.** (5 points) Given the points $P(3, -2, -3), Q(4, -1, 0)$, and $R(1, -2, 4)$:
a) Find a vector equation for the line parallel to $\vec{PQ}$ that is passing through $R$. b) Is the point $(2, -1, 5)$ on the line in part a)? Justify your answer. c) Find a standard equation $(ax + by + cz = d)$ for the plane passing through the points $P, Q$, and $R$.

**11.** (2 points) Find an equation of the plane $3x - 7y - 5z = 6$ in parametric form.

**12.** (3 points) For each of the following, answer TRUE or FALSE and briefly justify your answer. If false, providing a counter example will suffice.

(a) A homogeneous system with more unknowns than equations will have infinitely many solutions.

(b) If a system $AX = 0$ has infinitely many solutions, then $AX = B$ must also have infinitely many solutions for any $B$.

(c) If the determinant of a matrix $A$ is 0, then one of the rows (or columns) must be a multiple of another row (or column).

**13.** (3 points) Consider the plane $\mathcal{P} : -5x + 3z = 30$.

a) Find all intercepts of the plane (if any).

b) Give an equation of a plane (in point-normal form) that's orthogonal to $\mathcal{P}$ and contains $(1, 5, -9)$.

**14.** (3 points) Consider the system: $\begin{cases} 3x + hy + 5z = 1 \\ -6x + 4y - 10z = k \end{cases}$

Find the value(s) of $h$ and $k$ such that the planes:

a) are parallel.

b) intersect in a line.

**15.** (2 points) Let $L$ be a line with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} + t \begin{bmatrix} -7 \\ 3 \\ -1 \end{bmatrix}$

and $P$ be a plane with equation $-4x - 6y + 10z = 12$.

Is the line $L$ parallel to $P$, perpendicular to $P$, or neither? Justify your answer.

**16.** (6 points) In a study on cigarette use in the town of Asheville with a population of 3600 people, researchers determined that 65% of people who self declared as smokers on a given year, will continue smoking the next year, whereas 90% of people who said they were non smokers will still be non smokers the following year. Assuming the population and trends remain constant year after year, answer the following questions.

(a) Draw a transition diagram and give a transition matrix associated with this situation.

(b) If 30 percent of the population this year are smokers, give the initial state vector and determine how many smokers we expect to have in Asheville 2 years from now?

(c) Find the steady state vector for this Markov chain. Give your answer using fractions. Over the long run, how many people in Asheville do we expect to be non-smokers?

**17.** (5 points) A closed economy produces aluminium, copper, and iron. The production of \$1 worth of aluminium requires 80 cents worth of aluminium, 10 cents worth of copper, and 10 cents of iron. To produce \$1 worth of copper, it takes 20 cents worth of aluminium, 60 cents worth of copper, and 20 cents of iron. Finally, to produce \$1 worth of iron, 30 cents worth of aluminium, 20 cents worth of copper, and 50 cents of iron are required.

a) How much does each industry need to produce relative to each other in order to function?

b) If the economy produces 12 000 worth of iron, how much aluminium and copper should it produce?

**18.** (5 points) An open economy produces food and housing. The production of \$1 worth of food requires 30 cents worth of food and 20 cents worth of housing. The production of \$1 worth of housing requires 40 cents worth of food and 60 cents worth of housing.

(a) Which industries, if any, are profitable? Justify.

(b) Construct the consumption matrix, $C$

(c) Is this economy productive?

(d) What production schedule is required to satisfy an external demand of \$45 000 dollars worth of food and \$72 000 dollars worth of housing?

(e) In this scenario, what is the internal consumption?

**19.** (2 points) Determine if $A = \begin{bmatrix} 5 & 2 \\ 5 & 15 \end{bmatrix}$ is a valid encoding matrix for a Hill-2 cipher. Briefly justify your answer.

**20.** (5 points) Your friends have been using the encoding matrix $A = \begin{bmatrix} 1 & 1 \\ 6 & 3 \end{bmatrix}$ to encrypt secret messages for each other. While you are studying in the library, one of them drops a note to tell you where to meet them after the final exam. The message simply reads **OVWGXI**

a) Find the decryption matrix $A^{-1}$, and check that it is correct by evaluating the product $A^{-1}A$.

b) Where are you supposed to meet up?

**21.** (5 points) Use the Simplex Method to minimize $z = -4x_1 + 7x_2 - 3x_3 - 2x_4$ subject to the constraints

$$\begin{cases} x_1 - 3x_2 + x_3 + 5x_4 \leqslant 8 \\ 2x_1 + 2x_2 - 2x_3 + 6x_4 \leqslant 12 \\ 3x_1 - 10x_2 + x_3 - 5x_4 \leqslant 30 \\ x_1, \ x_2, \ x_3, \ x_4 \geqslant 0 \end{cases}$$

Your answer should include the minimum value of $z$, along with the corresponding feasible solution. If there is no minimum, identify an unbounded variable.

**22.** (5 points) A real estate developer offers three types of houses.

The Type 1 house needs 500m$^2$ of land, an initial investment of \$45 000 and 720 hours of work.

The Type 2 house needs 500m$^2$ of land, an initial investment of \$45 000 and 540 hours of work.

The Type 3 house needs 1000m$^2$ of land, an initial investment of \$60 000 and 720 hours of work.

The developer has 30 000m$^2$ of land, \$2 400 000 of capital and 32 400 work-hours available, and his profit on the houses is \$16 000 for Type 1, \$12 400 for Type 2 and \$19 200 for Type 3.

**a)** Define your variables and set up the linear program that would be used to determine the number of houses of each type the developer should build in order to maximize his profit. When defining $x_1$ etc., please respect the order in which things were named in the question. **Do not solve.**

**b)** Give the initial simplex table. **Do not solve.**

Below is the final simplex table corresponding to this linear program. Use it to answer the questions that follow.

$$\left[\begin{array}{c|ccc|ccc|c} 1 & 0 & 400 & 0 & 32/5 & 0 & 160/9 & 768\,000 \\ \hline 0 & 0 & 1/4 & 1 & 1/500 & 0 & -1/720 & 15 \\ 0 & 0 & 7500 & 0 & -30 & 1 & -125/3 & 150\,000 \\ 0 & 1 & 1/2 & 0 & -1/500 & 0 & 1/360 & 30 \end{array}\right]$$

**c)** How many houses of each type should the developer build in order to maximize his profit?

**d)** What is the maximum profit?

**e)** When profit is maximized, how much land, capital and work-hours are unused?

---

**Answers:**

1. a) $(5, 4, -2)$ b) $(7-2t, -5-3t, t, -17)$, $t \in \mathbb{R}$

2. a) $\begin{array}{rcl} x & = & \#\text{ of toy cars} \\ y & = & \#\text{ of toy planes} \\ z & = & \#\text{ of toy trains} \end{array}$ $\left\{\begin{array}{rcrcrcr} 4x & + & 4y & + & 8z & = & 136 \\ 4x & + & 8y & + & 11z & = & 232 \end{array}\right.$

   b) $\begin{array}{rcl} x & = & 10 - {}^{5}\!/\!_{4}t \\ y & = & 24 - {}^{3}\!/\!_{4}t \\ z & = & t \end{array}$ $\begin{array}{rcl} t = 0 & \to & (10, 24, 0) \\ t = 4 & \to & (5, 21, 4) \\ t = 8 & \to & (0, 18, 8) \end{array}$

3. a) $k = 7$ b) $k = 5$ c) $k \in \mathbb{R}\backslash\{5, 7\}$

4. a) $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ -2 & -3 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -17 \\ 0 \\ 34 \end{bmatrix}$ b) $\begin{bmatrix} 12 & -1 & -4 \\ -23 & 9 & 2 \\ -9 & 5 & 3 \end{bmatrix}$ c) 17 d) $\begin{bmatrix} 12/17 & -1/17 & -4/17 \\ -23/17 & 9/17 & 2/17 \\ -9/17 & 5/17 & 3/17 \end{bmatrix}$ e) $\begin{bmatrix} -20 \\ 27 \\ 15 \end{bmatrix}$

5. $X = \dfrac{1}{2}\left(AB^{-1}C - B\right)$

6. a) Undefined, $\det(C) = 0$ b) $\begin{bmatrix} 2 & -4 \\ -2 & -2 \end{bmatrix}$ c) $\begin{bmatrix} -2 & 0 & 6 \\ 23 & 6 & -12 \end{bmatrix}$ d) Undefined, #Col B $\neq$ #Row A

7. a) $-24$ b) 0 c) 0 d) $\dfrac{27}{4}$ e) Infinitely many.

8. Yes, Pietro's output is equivalent to $\dfrac{1}{2}$ a day of Pino's plus $\dfrac{2}{5}$ of a day of Pasquale's.

9. a) $12\sqrt{5} - \sqrt{149}$ b) $-\dfrac{2}{\sqrt{14}}\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$ c) $k = 11$ d) No, both $\vec{u} \cdot \vec{w}$ and $\vec{v} \cdot \vec{w}$ are not 0.

10. a) $\begin{bmatrix}1\\-2\\4\end{bmatrix}+t\begin{bmatrix}1\\1\\3\end{bmatrix}$ b) No. c) $7x-13y+2z=41$

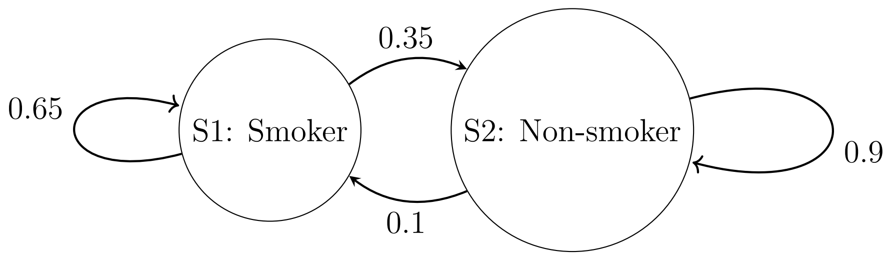
11. $\begin{cases} x &=& 2+\frac{7}{3}s+\frac{5}{3}t \\ y &=& s \\ z &=& t \end{cases}$

12. a) True b) False c) False

13. a) $x$-int: $(-6,0,0)$ $y$-int: none $z$-int: $(0,0,10)$ b) $3(x-1)+a(y-5)+5(z+9)=0$ ($a$ can be any number)

14. a) $h=-2$, $k\neq-2$ (if $k=-2$, the planes are the same) b) $h\neq-2$, $k\in\mathbb{R}$

15. The line is parallel to the plane.

16. a) 

$P=\begin{bmatrix}0.65 & 0.1\\0.35 & 0.9\end{bmatrix}$

b) $\vec{x_0}=\begin{bmatrix}0.3\\0.7\end{bmatrix}$ 885 smokers c) $\vec{q}=\begin{bmatrix}2/9\\7/9\end{bmatrix}$ 800 smokers, 2800 non-smokers.

17. a) The production of aluminium should be $\frac{8}{3}$ times the production of iron, and the production of copper should be $\frac{7}{6}$ times the production of iron. b) \$32 000 of aluminium and \$14 000 or copper.

18. a) Only food is profitable. b) $C=\begin{bmatrix}0.3 & 0.4\\0.2 & 0.6\end{bmatrix}$ c) Yes, the row-sums are all less than 1.

d) \$234 000 of food and 297 000 of housing. e) \$189 000 of food and \$225 000 of housing.

19. $\det(A)=13$, which does not have a multiplicative inverse, so NO.

20. a) $A^{-1}=\begin{bmatrix}25 & 9\\2 & 17\end{bmatrix}$ b) ANNIE'S

21. Final table: $\left[\begin{array}{c|cccc|ccc|c} 1 & 0 & 3 & 0 & -17 & -7/2 & -1/4 & 0 & -31 \\ \hline 0 & 0 & -2 & 1 & 1 & 1/2 & -1/4 & 0 & 1 \\ 0 & 1 & -1 & 0 & 4 & 1/2 & 1/4 & 0 & 7 \\ 0 & 0 & -5 & 0 & -18 & -2 & -1/2 & 1 & 8 \end{array}\right]$ No minimum, $x_2$ is unbounded.

22. a) $x_1 = \#$ of Type 1 houses, $x_2 = \#$ of Type 2 houses, $x_3 = \#$ of Type 3 houses.

Maximize $P = 16\,000x_1 + 12\,400x_2 + 19\,200x_3$

subject to the constraints $\begin{cases} 500x_1 + 500x_2 + 1000x_3 \leqslant 30\,000 \\ 45\,000x_1 + 45\,000x_2 + 60\,000x_3 \leqslant 2\,400\,000 \\ 720x_1 + 540x_2 + 720x_3 \leqslant 32\,400 \\ x_1, x_2, x_3 \geqslant 0 \end{cases}$

b) $\left[\begin{array}{c|ccc|ccc|c} 1 & -16\,000 & -12\,400 & -19\,200 & 0 & 0 & 0 & 0 \\ \hline 0 & 500 & 500 & 1000 & 1 & 0 & 0 & 30\,000 \\ 0 & 45\,000 & 45\,000 & 60\,000 & 0 & 1 & 0 & 2\,400\,000 \\ 0 & 720 & 540 & 720 & 0 & 0 & 1 & 32\,400 \end{array}\right]$

c) $x_1 = 30$ $x_2 = 0$ $x_3 = 15$ d) $P = 768\,000$

e) $s_1 = 0$ land, $s_2 = \$150\,000$ capital and $s_3 = 0$ work-hour left unused.