1. (6 points) Let

$$A = \begin{bmatrix} 1 & 2 & 6 & 4 \\ 0 & 2 & 5 & 3 \\ 2 & 6 & 17 & 11 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix}.$$

(a) Solve $A\mathbf{x} = \mathbf{b}$.

(b) Find a basis for $\mathrm{Col}(A)$.

(c) Find a basis for $\mathrm{Nul}(A)$.

2. (5 points) Given that the eigenvalues of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

are 2, 1, and -1, find a matrix P such that $P^{-1}AP$ is diagonal.

3. (8 points) Consider the matrix

$$A^{-1} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1/2 & 0 \\ 1 & 0 & 12 \end{bmatrix}.$$

(a) Find A.

(b) Express A as a product of elementary matrices.

(c) Find $\mathrm{adj}(A)$.

4. (5 points) Suppose the $n \times n$ matrix A_n has 3's along its main diagonal, 2's along the diagonal below, a 2 in the $(1, n)$ position, and everything else zero. For example:

$$A_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

(a) Find by cofactors of row 1 or otherwise the determinant of A_4.

(b) Find the determinant of A_n for $n > 4$.

5. (4 points) Use Cramer's Rule to solve the following linear system for x_3 only:

$$\begin{cases} -3x_1 + 6x_2 + 2x_3 + x_4 = 0 \\ \quad\quad - 2x_2 + 2x_3 - x_4 = 0 \\ -2x_1 - x_2 + 2x_3 \quad\quad = 1 \\ -x_1 - 2x_2 + x_3 \quad\quad = 0 \end{cases}$$

6. (10 points) Let P be the plane $2x - y + 2z = -7$, and let A be the point $(-4, 6, -1)$.

(a) Find an equation for the line through A that is perpendicular to P.

(b) Find the point on P that is closest to A.

(c) What is the distance from A to P?

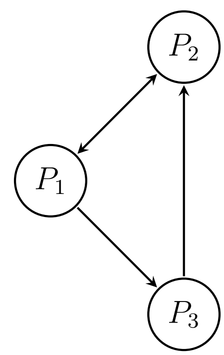
(d) Find an equation of the form $ax + by + cz = d$ of the plane through A that is parallel to P.

7. (5 points) Solve for the matrix X in $(X^{-1} + A)^T = AB$ where $A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$.

8. (5 points) Consider the graph given below.

(a) Find M, the adjacency matrix.

(b) Find M^3.



(c) Use your answer in (b) to determine the total number of walks of length three.

9. (3 points) Compute the determinant of

$$A = \begin{bmatrix} 3a + 2b & 3c + 2d \\ abd - b^2c & ad^2 - bcd \end{bmatrix},$$

given that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 5$. (You may find it helpful to factor the entries wherever possible.)

10. (8 points) Given the lines $L_1 : \mathbf{x} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + r \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ and $L_2 : \mathbf{x} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

(a) Find the distance between the parallel lines L_1 and L_2.

(b) Let L_3 be the line $\mathbf{x} = \begin{bmatrix} 3 \\ -4 \\ -6 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.
Find the point of intersection of L_2 and L_3.

(c) Find the cosine of the angle between L_2 and L_3. Note: the angle in question is the *smaller* of the two angles between the lines' respective direction vectors.

11. (6 points) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 5 \\ c \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(a) For which value(s) of c are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

(b) Find $\mathbf{v} \times \mathbf{w}$.

(c) For which value(s) of c does the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ equal one?

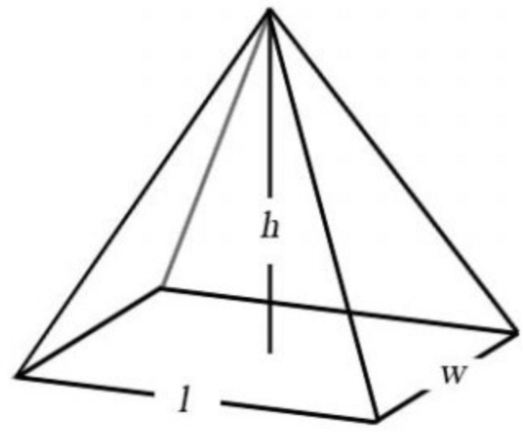
12. (4 points) Given that $\|\mathbf{x}\| = 1$, $\|\mathbf{y}\| = 3$, and $\mathbf{x} \cdot \mathbf{y} = 2$, find

(a) $4\mathbf{x} \cdot (\mathbf{y} - 2\mathbf{x})$

(b) $\|\mathbf{x} + \mathbf{y}\|$

13. (3 points) A rigid bar extends from $A(1, -2, 0)$ to $B(-3, 2, 1)$. Find the torque at A when a force of $5N$ is applied at B in the direction of $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

14. (2 points) The pyramid shown has $l = 5$ m, $w = 2$ m, and $h = 4$ m. An object is moved along one of the slanted edges from a bottom corner to the peak by applying a force of 15 N applied vertically (up). How much work is done?



15. (6 points) Let

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a + b = 2c - d \right\}.$$

(a) Find one vector that is in U and one vector that is in $\mathbb{R}^4$ but not in U.

(b) Find a basis of U.

(c) What is the dimension of U?

16. (4 points) If A and B are $m \times n$ matrices, is $U = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = B\mathbf{x}\}$ a subspace of $\mathbb{R}^n$? Justify your answer.

17. (4 points) Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be an independent set in $\mathbb{R}^n$. Determine whether the following sets are independent or dependent. Justify your answers.

1. $\{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$
2. $\{\mathbf{u} + \mathbf{w}, \mathbf{v} + \mathbf{w}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$

18. (12 points) Mark each of the following as true or false. Justify each answer. Zero marks will be awarded for unjustified answers.

(a) If A is $m \times n$ then $\text{rank}(A) + \dim(\text{Nul}(A^T)) = n$.

(b) Two diagonalizable matrices A and B with the same eigenvalues and eigenvectors must be the same matrix.

(c) If A, B, and $A+B$ are invertible matrices then $(A+B)^{-1} = A^{-1} + B^{-1}$.

(d) If the augmented matrix $[A\ \mathbf{b}]$ has a row of zeros then the equation $A\mathbf{x} = \mathbf{b}$ has infinite solutions.

(e) If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are vectors in $\mathbb{R}^n$ and the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$ has only the trivial solution then the matrix $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ is invertible.

(f) For nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have $\|\text{proj}_{\mathbf{v}}\,\mathbf{u}\| = |\mathbf{u}\cdot\mathbf{w}|$ where $\mathbf{w}$ is a unit vector parallel to $\mathbf{v}$.

ANSWERS

1.

(a) $$\mathbf{x} = \begin{bmatrix} 10 \\ -3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}, \quad t, s \in \mathbb{R}.$$

(b) $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \right\}.$$

(c) $$\left\{ \begin{bmatrix} -2 \\ -5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

2.

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \qquad P^{-1}AP = \text{diag}(2, 1, -1).$$

3. (a)

$$A = \begin{bmatrix} -6 & 0 & 1 \\ 0 & -2 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

(b) An infinity of correct answers, here.

(c)

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 12 \end{bmatrix}$$

4. (a) 65

(b) $3^n + (-1)^{n-1}2^n$

5. $-10/3$

6. (a) $(x, y, z) = (-4, 6, -1) + t(2, -1, 2)$

(b) $(-2,\ 5,\ 1)$

(c) 3

(d) $2x - y + 2z = -16$

7. $X = \begin{bmatrix} \dfrac{3}{50} & -\dfrac{2}{75} \\ -\dfrac{4}{25} & -\dfrac{1}{25} \end{bmatrix}$

8. (a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(c) 7

9. 75

10.

(a) Distance between L_1 and L_2 : $\sqrt{10}$

(b) Intersection point of L_2 and L_3 : (0, 2, 0

(c) Cosine of the angle between L_2 and L_3 : $\dfrac{4}{9}$

11.

(a) Orthogonality condition: $c = -6$

(b) $\mathbf{v} \times \mathbf{w} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

(c) Volume condition: $c = 4$ or $c = 6$

12.

(a) $$\begin{aligned} 4\mathbf{x}\cdot(\mathbf{y}-2\mathbf{x}) &= 4(\mathbf{x}\cdot\mathbf{y}-2\mathbf{x}\cdot\mathbf{x}) \\ &= 4(2-2(1)) \\ &= 0 \end{aligned}$$

(b) $$\begin{aligned} \|\mathbf{x}+\mathbf{y}\| &= \sqrt{\|\mathbf{x}\|^2+\|\mathbf{y}\|^2+2(\mathbf{x}\cdot\mathbf{y})} \\ &= \sqrt{1^2+3^2+2(2)} \\ &= \sqrt{14} \end{aligned}$$

(f)

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = (\mathbf{u}\cdot\tfrac{\mathbf{v}}{\|\mathbf{v}\|})\tfrac{\mathbf{v}}{\|\mathbf{v}\|}.$$

Let $\mathbf{w} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$, which is a unit vector parallel to $\mathbf{v}$. So,

$$\|\operatorname{proj}_{\mathbf{v}}\mathbf{u}\| = \|(\mathbf{u}\cdot\mathbf{w})\mathbf{w}\| = |\mathbf{u}\cdot\mathbf{w}|\,\|\mathbf{w}\| = |\mathbf{u}\cdot\mathbf{w}|.$$

13. $\begin{bmatrix}15\\10\\20\end{bmatrix}\ N\cdot m$

14.

15. (a) $\begin{pmatrix}2\\0\\1\\0\end{pmatrix}\in U,\quad \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\notin U.$

(b) $\{\,(-1,1,0,0),\ (2,0,1,0),\ (-1,0,0,1)\,\}$

(c) $\dim U = 3$

16. Yes.

17. (a) Dependent.

(b) Independent.

18. (a) False. Consider $A = \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}$. $1+1\neq 3$.

(b) False. Consider $A = \begin{bmatrix}1&0\\0&2\end{bmatrix}$ and $B = \begin{bmatrix}2&0\\0&1\end{bmatrix}$.

(c) False. Let $A = B = \begin{bmatrix}1&0\\0&1\end{bmatrix}$. $(A+B)^{-1} = \frac{1}{2}I_2$ while $A^{-1}+B^{-1} = 2I_2$.

(d) False. Consider $[A\ \mathbf{b}] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

(e) True. This equation having only trivial solution $\implies$ A has a pivot for every column $\implies$ A is invertible.