1. (4 points) Find the equation of the form $p(x) = a + bx + cx^2$ of the quadratic polynomial that passes through the points $(-1, 6)$, $(1, -8)$, and $(2, -9)$.

2. (3 points) Consider the following augmented matrix of a linear system.

$$\left[\begin{array}{cc|c} 1 & 0 & h \\ 0 & h^2-4 & 8-4h \end{array}\right]$$

Find all values of h such that the system has:

(a) A unique solution
(b) Infinitely many solutions
(c) No solution

3. (3 points) It is given that $[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5\,]$ is row equivalent to $\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$.

Determine whether the following sets are linearly independent or linearly dependent. Justify.

(a) $\{\mathbf{v}_1,\ \mathbf{v}_3\}$

(b) $\{\mathbf{v}_3,\ \mathbf{v}_4\}$

(c) $\{\mathbf{v}_2,\ \mathbf{v}_4,\ \mathbf{v}_5\}$

4. (3 points) Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 3 & x & 1 & 0 & 14 \\ 2 & 6 & y & 1 & 11 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

If R is the reduced row echelon form (RREF) of A, determine the missing values x and y.

5. (a) (2 points) Find a basis for the line $y = 4x$ in $\mathbb{R}^2$.

(b) (2 points) Write a specific matrix A whose column space is the line $y = 4x$ in $\mathbb{R}^2$ and whose null space is in $\mathbb{R}^4$.

6. Suppose that $R = \begin{bmatrix} 1 & 0 & -8 & 0 & 0 & 9 \\ 0 & 1 & 10 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is the reduced row echelon form (RREF) of

$A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4 \quad \mathbf{a}_5 \quad \mathbf{a}_6\,]$. (Note: Matrix A is not given. You only have the reduced form R)

(a) (4 points) Fill in the blank with the word *must, might*, or *cannot*, as appropriate.

(i) The set $\{\,\mathbf{a}_1,\ \mathbf{a}_3,\ \mathbf{a}_6\}$ ________________ form a basis for $\mathrm{Col}(A)$.

(ii) The top three rows of A ________________ form a basis for $\mathrm{Row}(A)$.

(iii) $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ________________ be in $\mathrm{Col}(A)$.

(iv) $\begin{bmatrix} 2 \\ 0 \\ -16 \\ 0 \\ 0 \\ 18 \end{bmatrix}$ ________________ be in $\mathrm{Row}(A)$.

(b) (2 points) Find a basis for $\mathrm{Nul}(A)$.

7. (4 points) Given $A = \begin{bmatrix} 2 & 0 & -5 \\ -3 & 3 & 2 \\ -3 & k & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$, use Cramer's Rule to find the value(s) of k such the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution where $x_3 = 2$.

8. (2 points) Let M be a 3×3 matrix such that $M^2 = 4I$. Find all possible values for $\det(M)$.

9. (3 points) Let $A = \begin{bmatrix} 0 & -2 \\ 4 & 1 \end{bmatrix}$. Find a matrix X such that $AX = A^T$.

10. (4 points) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects points through the y-axis. Let $S: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix}$. Find the standard matrix for $T \circ S$.

11. Let $\mathcal{W}$ be the subspace of $\mathbb{M}_{2\times 2}$ where the sum of the entries in the first row is equal to twice the sum of the entries in the second column.

(a) (4 points) Find a basis for $\mathcal{W}$.

(b) (1 point) What is the dimension of $\mathcal{W}$?

12. (4 points) Let $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 4 & -6 & 0 \\ 0 & 2 & 1 \\ 0 & 8 & k \end{bmatrix}$.

For each statement below, determine <u>all values of k</u> (if any) that would make the statement true.

(i) T is an *onto* transformation.

(ii) The column space of A is a plane.

(iii) The kernel of T is plane.

(iv) T is a linear transformation.

13. Let $\mathcal{H} = \{1 + x,\ 3 + 2x - 2x^2,\ -2 + 4x^2\}$ and let $p(x) = 3 + x - 4x^2$.

(a) (1 point) Is $p(x)$ in $\mathcal{H}$? Justify.

(b) (3 points) Is $p(x)$ in $\text{Span}(\mathcal{H})$? Justify.

(c) (1 point) Find a basis for $\text{Span}(\mathcal{H})$.

14. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ k \\ k^2 \end{bmatrix}$.

(a) (2 points) Find all values for k such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) (1 point) Find a unit vector that is parallel to $\mathbf{u}$.

15. Consider the plane $\mathcal{P} : x - 2y + z = 3$ and the line $\mathcal{L} : \mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

(a) (2 points) Determine the intersection of $\mathcal{L}$ and $\mathcal{P}$.

(b) (3 points) Find an equation of the form $ax + by + cz = d$ for the plane that contains $\mathcal{L}$ and is perpendicular to $\mathcal{P}$.

16. (2 points) Find the cosine of the acute angle between the vector $\begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}$, positioned at the origin, and the y-axis.

17. (2 points) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^4$ such that $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$ and $\mathbf{u} \cdot \mathbf{v} = 1$, find $\|\mathbf{u} - \mathbf{v}\|$.

18. (4 points) Let $\mathcal{L}$ be the line $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and let P be the point $(1, -4, -2)$. Find the point on $\mathcal{L}$ closest to P.

19. (4 points) Fill in the blank with the word *must, might*, or *cannot*, as appropriate.

(a) If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, then $\mathbf{w}$ ______________ be a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) If A can be written as a product of elementary matrices, then $A\mathbf{x} = \mathbf{b}$ ______________ have infinitely many solutions.

(c) Given $A = [\,\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\,]$, the matrix equation $A\mathbf{x} = \mathbf{a}_3$ ______________ be consistent.

(d) Two lines in $\mathbb{R}^3$ that are perpendicular to a third line ______________ be parallel.

ANSWERS

1. $p(x) = -3 - 7x + 2x^2$

2. (a) $h \in \mathbb{R}\backslash\{-2, 2\}$ (b) $h = 2$ (c) $h = -2$

3. (a) Linearly dependent (b) Linearly independent (c) Linearly dependent

4. $x = 9,\ y = 3$

5. (a) $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right\}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 4 & 4 & 0 & 8 \end{bmatrix}$ (other answers exist)

6. (a) (i) must (ii) might (iii) might (iv) must

(b) $\left\{\begin{bmatrix} 8 \\ -10 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right\}$ (other answers exist)

7. $k = \dfrac{42}{11}$

8. $\det(M) \in \{-8, 8\}$

9. $\begin{bmatrix} -1/2 & 3/4 \\ 0 & -2 \end{bmatrix}$

10. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ -1 & 1 \end{bmatrix}$

11. (a) $\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ (other answers exist) (b) 3

12. (i) $k \in \mathbb{R} \setminus \{4\}$ (ii) $k = 4$ (iii) Not possible (iv) k is any real number

13. (a) No. It is not one of the three vectors in $\mathcal{H}$.

(b) Yes. $3 + x - 4x^2 = -3(1 + x) + 2(3 + 2x - 2x^2) + 0(-2 + 4x^2)$

(c) $\{\,1 + x,\ 3 + 2x - 2x^2\,\}$ (other answers exist)

14. (a) $k = -2$ or $k = 3$ (b) $\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

15. (a) $t = -4$: $(-13, -5, 6)$ (b) $y + 2z = 7$

16. $\cos(\theta) = \dfrac{1}{\sqrt{6}}$

17. $\sqrt{11}$

18. $(0, -5, 1)$

19. (a) might (b) cannot (c) must (d) might