1. (6 points) Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ k^2 \end{pmatrix}$ and $\mathbf{v}_4 = \begin{pmatrix} 2 \\ 4 \\ k \end{pmatrix}$.

(a) For what values of k is $\mathbf{v}_4$ in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(b) For what values of k is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent?

2. Let A be a 3×5 and set $\mathbf{a}_j = \text{col}_j(A)$ for $j = 1, \ldots, 5$. Suppose that

$$A \sim \begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 6 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

(a) (3 points) Give a basis for $\text{Nul}(A)$.

(b) (3 points) Given that $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{a}_3 = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$, find the entries of $\mathbf{a}_1$.

(c) (8 points) Indicate each statement as either true or false. (No explanations needed.)

i. $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5\}$ is a basis for $\text{Col}(A)$. .... (TRUE or FALSE)

ii. $\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ is linearly independent. .... (TRUE or FALSE)

iii. For each $\mathbf{b} \in \mathbb{R}^3$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution. .... (TRUE or FALSE)

iv. $\text{Col}(A) = \mathbb{R}^5$ .... (TRUE or FALSE)

v. $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ .... (TRUE or FALSE)

vi. $\text{Span}\{\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\} = \text{Span}\{\mathbf{a}_3, \mathbf{a}_5\}$ .... (TRUE or FALSE)

vii. $\text{rank}(A^T) = 2$ .... (TRUE or FALSE)

viii. $\text{row}_1(A^T A) \in \text{Col}(A^T)$ .... (TRUE or FALSE)

3. (4 points) Find the point on the line given by $(x, y, z) = (1, -1, -1) + t(5, 2, 2)$, $t \in \mathbb{R}$, that is closest to the point $(10, 20, 5)$.

4. (4 points) Find the distance between the plane $3x + y - 5z = 30$ and the point $(10, -20, 10)$.

5. Consider the two lines given by the parametric vector equations

$$\mathbf{x} = s \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}, \ s \in \mathbb{R} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 7 \\ 8 \\ 31 \end{pmatrix} + t \begin{pmatrix} 4 \\ -4 \\ 7 \end{pmatrix}, \ t \in \mathbb{R}$$

(a) (4 points) Find the point of intersection between the lines.

(b) (2 points) Find the $\cos(\alpha)$, where α is the acute angle between the lines.

6. (6 points) Find numbers a, b, and c such that the function $f(x) = a(x-1)^2 + bx^2 + c(x+1)^2$ satisfies $f(-1) = 24$, $f(0) = 4$, and $f(2) = 12$.

7. (6 points) Let A, B, and X be 5×5 invertible matrices such that $A^T X^{-1} B = 7I$. Given that

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 25 & 70 \end{pmatrix},$$

find the $(3,4)$-entry of X.

8. (6 points) Find the inverse of the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}$.

9. (6 points) Use Cramer's rule to solve for x in the linear system $\begin{cases} x + y + z = 1 \\ x + y - z = 0 \\ 2x - 3y - 3z = 0 \end{cases}$.

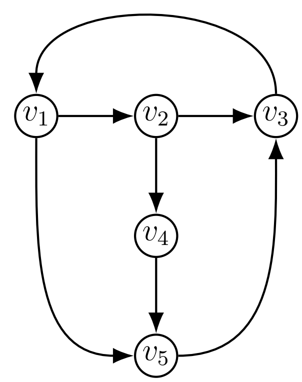
10. Consider the subset W of $\mathbb{R}^3$ given by $W = \{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 0\}$.

(a) (3 points) List three vectors from W which form a linearly independent set.

(b) (3 points) Is W closed under vector addition? Justify your answer.

(c) (3 points) Is W closed under scalar multiplication? Justify your answer.

11. Consider the given directed graph.



(a) (2 points) Find the adjacency matrix A for this directed graph.

(b) Given that $A^6 = \begin{pmatrix} 4 & 1 & 0 & 0 & 3 \\ 0 & 2 & 3 & 1 & 2 \\ 1 & 0 & 4 & 2 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 & 2 \end{pmatrix}$ find the following:

i. (1 point) The number of closed walks of length 6.

ii. (1 point) The number of walks of length 7 that end at v_4.

iii. (2 points) The number of walks of length 12 that start at v_4 and end at v_5.

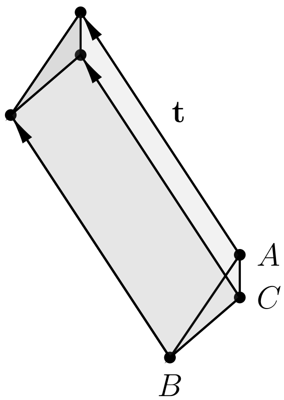
12. Consider the points $A(2,2,3)$, $B(1,0,0)$ and $C(2,2,2)$.

(a) (4 points) Find an equation, in normal form, of the plane that contains the points A, B, and C.

(b) (3 points) Find area of the triangle whose vertices are A, B, and C.

(c) (5 points) Find the point between B and C that is 1 unit away from B.

(d) (5 points) Let $\mathbf{t} = (2, -3, 6)$. A polyhedron with two parallel triangular faces is shown below. Find its volume.



13. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $y = x$, and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the horizontal shear such that $S(2, 3) = (4, 3)$.

(a) (4 points) Find the standard matrices for R, S, and $R \circ S$.

(b) (3 points) Give an equation, in normal form, for the image of $y = 2x + 3$ under S

(c) (3 points) Give an equation, in normal form, for the image of $y = 5$ under $R \circ S$.

Answers

1. (a) $k \in \mathbb{R} \setminus \{-2\}$

(b) $k \in \mathbb{R} \setminus \{-2, 2\}$

2. (a) $\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -6 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

(b) $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

(c) i. True

ii. False

iii. True

iv. False

v. True

vi. True

 Question 2 continues on the next page.

vii. False

viii. True

3. $(16, 5, 5)$.

4. $2\sqrt{35}$.

5. (a) $(3, 12, 24)$

(b) $\dfrac{44}{81}$

6. $a = 5$, $b = 4$, $c = -1$

7. 5

8. $\dfrac{1}{2}\begin{pmatrix} 0 & -1 & 2 \\ -4 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$.

9. $\dfrac{3}{5}$

10. (a) $(1, -1, 0)$, $(0, 1, -1)$, $(1, 0, -1)$

(b) No. *Justification omitted.*

(c) Yes. *Justification omitted.*

11. (a) $\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

(b) i. 12

ii. 5

iii. 8

12. (a) $2x - y = 2$

(b) $\dfrac{\sqrt{5}}{2}$

(c) $(4/3, 2/3, 2/3)$

(d) $7/2$

13. (a) $(R) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $(S) = \begin{pmatrix} 1 & 2/3 \\ 0 & 1 \end{pmatrix}$, $(R \circ S) = \begin{pmatrix} 0 & 1 \\ 1 & 2/3 \end{pmatrix}$

(b) $6x - 7y = -9$

(c) $x = 5$