[Marks]

(8) 1. Let $A = \begin{bmatrix} 0 & 1 & 2 & 2 & -2 \\ 1 & 0 & 3 & 0 & 4 \\ -1 & 3 & 3 & 0 & -10 \end{bmatrix}$. Its row reduced echelon form $R = \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

- (a) Solve the linear system $A\mathbf{x} = \mathbf{0}$. Write the solution in parametric vector form.
- (b) Find a basis of $\mathrm{Row}(A)$.
- (c) Find a basis of $\mathrm{Col}(A)$.
- (d) Write the last column of A as a linear combination of the other columns of A.

(6) 2. Use linear algebra to balance the chemical equation: $C_3H_8 + O_2 \rightarrow CO_2 + H_2O$.

(7) 3. Let the matrix $A = \begin{bmatrix} 1 & -2 & -1 & -2 \\ 2 & -1 & 4 & 5 \\ 4 & 1 & 2 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$.

- (a) Use row operations to find A^{-1} or to show that A^{-1} does not exist.
- (b) Find two elementary matrices E_1 and E_2 such that $E_2E_1A = \begin{bmatrix} 1 & -2 & -1 & -2 \\ 2 & -1 & 4 & 5 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 4 & 1 & 2 & 1 \end{bmatrix}$

(6) 4. Let $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}$.

- (a) Find $\mathrm{adj}(A)$.
- (b) Compute $\det(A)$.
- (c) Find A^{-1} using $\mathrm{adj}(A)$.

(5) 5. Solve for the matrix X where

$$\left(\begin{bmatrix} 2 & 5 \\ 2 & 6 \end{bmatrix} X^T \right)^{-1} = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$$

(9) 6. Let A be a symmetric 4×4 matrix with $\det(A) = -3$. Let B be a 4×4 matrix with $\det(B) = 2$.

- (a) Find $\det((A^{-1})^3)$.
- (b) Find $\det(\mathrm{adj}(A^{-1}B))$.
- (c) Find $\det(A + A^T)$.

(8) 7. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $S\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.
Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation of $-\frac{\pi}{4}$ about the origin.

- (a) Find the standard matrix of S.
- (b) Find the standard matrix of $R \circ S$.

(6) 8. We are given the lines $\mathcal{L}_1 : \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\mathcal{L}_2 : \mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Find an equation of the plane (in the form $ax + by + cz = d$) that contains $\mathcal{L}_2$ and is parallel to $\mathcal{L}_1$.

(6) 9. Find an equation of the plane (in the form $ax + by + cz = d$) that contains the point $M(1, -1, 2)$ and is perpendicular to both planes $x + 2y - z = 2$ and $x - y - z = 4$.

[Marks]

(8) 10. Define $H = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in M_{2\times 2} : xy = zw \right\}$.

(a) Show that H is not closed under vector addition.

(b) Show that H is closed under scalar multiplication.

(5) 11. Find a basis for the subspace $V = \{p(x) \in \mathbb{P}_2 : p(2) = p'(1)\}$.

(5) 12. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$. Suppose that $\mathbf{u}$ is a unit vector, $\|\mathbf{v}\| = 5$ and $\mathbf{u} \cdot \mathbf{v} = -3$. Find all values of k, if any, for which the vectors $\mathbf{u} + k\mathbf{v}$ and $\mathbf{v} + 2\mathbf{u}$ are orthogonal.

(9) 13. Given a point $P(8, 5, 0)$ and a line $\mathcal{L} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$,

(a) find the point Q on $\mathcal{L}$ closest to P;

(b) find the distance from P to $\mathcal{L}$.

(8) 14. Given points $A(0, 0, 1), B(1, 1, 2)$, $C(4, 6, 5)$, and $D(-1, 4, k)$,

(a) find the area of triangle ABC;

(b) find an equation of the plane (in the form $ax + by + cz = d$) that contains A, B and C;

(c) find k such that $\overrightarrow{AD}$ is orthogonal to $\overrightarrow{AC}$.

(4) 15. Fill in the blank with the word *must*, *might*, or *cannot*, as appropriate.

(a) If two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then $\text{Span}\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ ____________ be equal to $\text{Span}\{\mathbf{u}, \mathbf{v}\}$.

(b) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then T ____________ be a rotation.

(c) If A is a 3×3 matrix, then $\text{Col}(A)$ and $\text{Nul}(A)$ ____________ both be planes in $\mathbb{R}^3$.

(d) Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. If $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent and $\{\mathbf{u}, \mathbf{w}\}$ is linearly dependent, then $\{\mathbf{v}, \mathbf{w}\}$ ____________ be linearly dependent.

ANSWERS

1. (a) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (b) $\left\{ \begin{pmatrix} 1 & 0 & 3 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ (d) $\begin{bmatrix} -2 \\ 4 \\ -10 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

2. $C_3H_8 + 5\,O_2 \rightarrow 3\,CO_2 + 4\,H_2O$

3. (a) A^{-1} does not exist

Question 15 continues on the next page.

[Marks]

(b) $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ OR $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4. (a) $\operatorname{adj}(A) = \begin{bmatrix} 7 & 1 & -3 \\ 4 & 8 & 2 \\ -2 & -4 & 12 \end{bmatrix}$ (b) 26 (c) $A^{-1} = \frac{1}{26}\begin{bmatrix} 7 & 1 & -3 \\ 4 & 8 & 2 \\ -2 & -4 & 12 \end{bmatrix}$

5. $X = \begin{bmatrix} -25 & 9 \\ \frac{39}{2} & -7 \end{bmatrix}$

6. (a) $\frac{-1}{27}$ (b) $\frac{-8}{27}$ (c) -48

7. (a) $\begin{bmatrix} -2 & 2 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -\frac{3\sqrt{2}}{2} & \sqrt{2} \\ \frac{\sqrt{2}}{2} & -\sqrt{2} \end{bmatrix}$

8. $3x - 5y + z = 7$

9. $x + z = 3$

10. (a) $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in H$ and $A_2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \in H$, but $A_1 + A_2 \notin H$ (for example)

(b) For any $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in H$ (so $xy = zw$) and any scalar λ, $\lambda A = \begin{bmatrix} \lambda x & \lambda y \\ \lambda z & \lambda w \end{bmatrix}$, $\lambda A \in H$ since $(\lambda x)(\lambda y) = \lambda^2(xy) = \lambda^2(zw) = (\lambda z)(\lambda w)$.

11. $\left\{-\frac{1}{2}x^2 + x, -\frac{1}{2}x^2 + 1\right\}$ (other answers possible)

12. $k = \frac{1}{19}$

13. (a) $Q(0, 5, 4)$ (b) $4\sqrt{5}$

14. (a) $\sqrt{2}$ (b) $-x + z = 1$ (c) $k = -4$

15. (a) MUST (b) MIGHT (c) CANNOT (d) MIGHT

This is the end of the examination.