1. (5 points) Solve the system or show that it has no solution:

$$\begin{cases} 2x + 6y - 2z = -10 \\ 2x + 5y + 3z = 16 \\ 3x - 7y - 4z = -4 \end{cases}$$

2. (5 points) A bakery is well known for three types of cakes: Balanced Cakes, Fruity Cakes and Nutty Cakes. The main ingredients aside from flour, eggs, sugar and water are fruit and nuts. Each Balanced Cake uses 200 grams of fruit and 200 grams of nuts. Each Fruity Cake uses 400 grams of fruit and no nuts. Each Nutty Cake uses 100 grams of fruit and 500 grams of nuts. The baker has 3000 grams of fruit and 1000 grams of nuts on hand and wants to use up all of the ingredients. How many Balanced, Fruity and Nutty cakes should the baker make? List all realistic possibilities, considering complete cakes only.

3. (4 points) Given the system: $\begin{cases} x + 2y = 1 \\ 2x + (k^2 - 5)y = k - 1 \end{cases}$

Find the value(s) of k such that the system has:

(a) a unique solution

(b) infinitely many solutions

(c) no solution

4. (3 points) Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -9 \\ -1 & 2 \end{bmatrix}$. Find, or identify as undefined:

(a) A^{-1}

(b) B^{-1}

(c) $(I - 2B)^T$

5. Let A, B and C be 3×3 matrices. You know the following things: $\det(A) = 5$. B is an upper triangular matrix, and the entries on its main diagonal are all 2's. C is not invertible. Evaluate each of the following, or state if there is not enough information to do so.

(a) (1 point) $\det(A^{-1})$

(b) (1 point) $\det(B)$

(c) (1 point) $\det(AC)$

(d) (1 point) $\det(A + C)$

(e) (2 points) $\det((4A^T)^{-1})$

(f) (1 point) $\operatorname{rank}(A)$

6. (3 points) If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 13$, find $\det \begin{bmatrix} 3d & 3e & 3f \\ a & b & c \\ g + 5a & h + 5b & i + 5c \end{bmatrix}$.

7. (2 points) A, B, C and X are square matrices, with $AX + B = CX$. Solve for X in terms of A, B and C (assume that any needed matrices are invertible).

8. Given the linear system: $\begin{cases} -x_1 + 2x_2 + 3x_3 - 2x_4 = -1 \\ x_1 - 3x_2 - 3x_3 + 4x_4 = 1 \\ x_1 - 2x_2 - 2x_3 + 5x_4 = 1 \\ 2x_1 - 4x_2 - 6x_3 + 7x_4 = 2 \end{cases}$

(a) (3 points) Find the determinant of the coefficient matrix, and use it to show that Cramer's Rule applies.

(b) (2 points) Use Cramer's Rule to solve for x_2 **only**. (Do **not** solve for x_1, x_3 or x_4!)

9. (5 points) Let $A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 9 & 3 \\ 1 & 2 & -2 \end{bmatrix}$.

(a) Find $\text{adj}\,(A)$.

(b) Use $\text{adj}\,(A)$ to find A^{-1}.

10. (2 points) On the graph below, draw the line $\mathcal{L} : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
(Hint: you could start by plotting two points on the line.)

11. Given the points $A(-1, 4, 5)$ and $B(5, 1, 7)$, find:

(a) (1 point) the vector $\overrightarrow{AB}$

(b) (2 points) a unit vector parallel to $\overrightarrow{AB}$

(c) (1 point) an equation (in any appropriate form) for the line that passes through A and B

12. (4 points) Find an equation in standard form ($ax + by + cz = d$) for the plane

$$\mathcal{P} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$$

13. Given the plane $\mathcal{P} : x + 7y - 3z = 4$,

(a) (2 points) Find an equation in vector form for $\mathcal{P}$

(b) (2 points) Find an equation (in any appropriate form) for the line $\mathcal{L}$ which is perpendicular to $\mathcal{P}$ and passes through the point $(9, 2, -5)$

14. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$.

(a) (2 points) Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent or linearly dependent? Justify your answer.

(b) (2 points) Circle the best description of $\text{Span}\,\{\vec{u}, \vec{v}, \vec{w}\}$ and justify your answer:
the empty set **a point** **a line** $\mathbb{R}^2$ $\mathbb{R}^3$

(c) (3 points) Let $\vec{a} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$. Express $\vec{a}$ as a linear combination of $\vec{u}$, $\vec{v}$, and $\vec{w}$ in **two different ways**, or show that it is not possible to do so.

15. Given the planes $\mathcal{P}_1 : x - 4y + 6z = 3$ and $\mathcal{P}_2 : -2x + 10y + 7z = -5$,

(a) (3 points) Determine if $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel, perpendicular, or neither.

 Question 15 continues on the next page.

(b) (3 points) Find the line of intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$, or show that they don't intersect. (If there is a line of intersection, express it in any appropriate form.)

16. Given that M is a 4×7 matrix, fill in the blanks:

(a) (1 point) _____ $\leq \text{rank}\,(M) \leq$ _____

(b) (1 point) _____ $\leq \text{nullity}\,(M) \leq$ _____

17. The matrix $A = \begin{bmatrix} 2 & -4 & a & -3 & -2 \\ -1 & 2 & b & 1 & -2 \\ 3 & 1 & c & 2 & 15 \\ 5 & 3 & d & 4 & 25 \end{bmatrix}$ reduces to $R = \begin{bmatrix} 1 & 0 & -4 & 0 & 2 \\ 0 & 1 & 5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) (1 point) Find a basis for $\text{Col}\,(A)$.

(b) (2 points) Find a basis for $\text{Nul}\,(A)$.

(c) (1 point) Find a basis for $\text{Col}\,(R)$.

(d) (2 points) Find a, b, c, and d.

(e) (1 point) Find $\text{rank}\,(A)$.

(f) (1 point) Find $\text{nullity}\,(A^T)$.

18. (8 points) Action Figures 3-D Printing Incorporated sells 3-D printed, hand-painted animal superhero models. A model of Awesome Cat takes 1 hour to print, 5 hours to hand-paint, and uses 7 units of material. A model of Fantastic Dog takes 1 hour to print, 3 hours to hand-paint, and uses 3 units of material. A model of Incredible Hamster takes 2 hours to print, 1 hour to hand-paint, and uses 1 unit of material. The company makes \$10 of profit on each Awesome Cat, \$60 of profit on each Fantastic Dog, and \$40 of profit on each Incredible Hamster.

The company has 100 hours available on the printer, 90 hours available for hand-painting, and 120 units of material in stock.

(a) How many of each model should they make to maximize profit, and what is the maximum profit?

(b) When profit is maximized, how many units of material are left over, if any?

19. (5 points) An economy has two interdependent sectors: technology and agriculture. To produce \$1 of technology requires \$0.40 of technology and \$0.10 of agriculture. To produce \$1 of agriculture requires \$0.10 of technology and \$0.15 of agriculture. There is also an external demand for \$14 000 of technology and \$3000 of agriculture.

(a) Which, if any, of the industries are profitable? Justify.

(b) How much must each industry produce in order to meet the external demand?

20. (6 points) In a certain city, there are three cell phone service providers: Shark, Octopus and Dolphin. (Aquatic names are trendy right now in that city!) At the end of each year, all the contracts expire and each customer chooses whether to stay with the same provider or to switch. Market research shows that if a customer subscribes to Shark in a given year, there's a 90% chance they'll stay with Shark the next year and a 10% chance they'll switch to Octopus, but they definitely won't switch to Dolphin. On the other hand, an Octopus subscriber has a 50% chance of staying with Octopus, a 40% chance of switching to Shark, and a 10% chance of switching to Dolphin. Finally, a Dolphin subscriber has a 10% chance of switching to Shark and a 30% chance of switching to Octopus.

If the current trends continue, what percent of customers will be subscribing to Shark in the long term?

21. (5 points) Alice is going to send her friends a secret message to tell them what she's planning to do after the Linear Algebra exam. She will encode her message with a Hill 2-cipher, using the encoding matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$.

(a) Find the decryption matrix A^{-1}.

(b) Multiply A and A^{-1} together to demonstrate that your decryption matrix is correct.

(c) Alice's ciphertext message to her friends is ZNHKRA. What is she planning to do after the exam?

The following tables may prove useful in answering the Cryptography question:

The alphabet:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

Reciprocals mod 26:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

Answers:

1. $(3, -1, 5)$

2. Two possibilities: 5 balanced, 5 fruity, 0 nutty; or 0 balanced, 7 fruity, 2 nutty

3. a) $k \neq -3, 3$ b) $k = 3$ c) $k = -3$

4. a) undefined b) $\begin{bmatrix} -2/3 & -3 \\ -1/3 & -1 \end{bmatrix}$ c) $\begin{bmatrix} -5 & 2 \\ 18 & -3 \end{bmatrix}$

5. a) $\frac{1}{5}$ b) 8 c) 0 d) not enough info e) $\frac{1}{320}$ f) 3

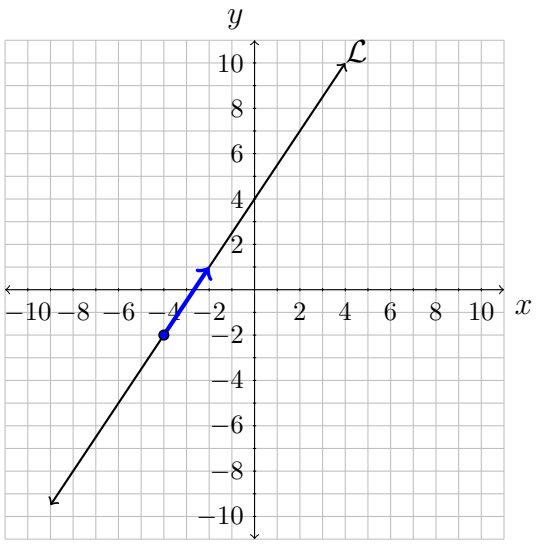
6. -39

7. $-(A - C)^{-1}B$

8. a) The determinant is 3; Cramer's rule applies since the determinant $\neq 0$. b) 0

9. a) $\text{adj}\,(A) = \begin{bmatrix} -24 & 18 & -33 \\ 9 & -7 & 12 \\ -3 & 2 & -3 \end{bmatrix}$ b) $A^{-1} = \begin{bmatrix} 8 & -6 & 11 \\ -3 & 7/3 & -4 \\ 1 & -2/3 & 1 \end{bmatrix}$

 Question 21 continues on the next page.

10. 

11. a) $(6, -3, 2)$ b) $(6/7, -3/7, 2/7)$ c) $\vec{x} = (-1, 4, 5) + t(6, -3, 2)$

12. $9x - 30y + 8z = 91$

13. a) $\vec{x} = (4, 0, 0) + s(-7, 1, 0) + t(3, 0, 1)$ b) $\vec{x} = (9, 2, -5) + t(1, 7, -3)$

14. a) Linearly dependent b) $\mathbb{R}^2$ c) There are infinitely many possibilities. For example: $\vec{a} = -10\vec{u} + 0\vec{v} + 3\vec{w}$, or $\vec{a} = -12\vec{u} + \vec{v} + 3\vec{w}$.

15. a) perpendicular b) $\vec{x} = (5, 1/2, 0) + t(-44, -19/2, 1)$

16. a) $0 \leq \text{rank}\,(M) \leq 4$ b) $3 \leq \text{nullity}\,(M) \leq 7$

17. a) $\{(2, -1, 3, 5), (-4, 2, 1, 3), (-3, 1, 2, 4)\}$ b) $\{(4, -5, 1, 0, 0), (-2, 3, 0, -6, 1)\}$
c) $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$ d) $a = -28$, $b = 14$, $c = -7$, $d = -5$ e) $\text{rank}\,(A) = 3$ f) $\text{nullity}\,(A^T) = 1$

18. a) The maximum profit is \$2640. Make 0 Awesome Cats, 16 Fantastic Dogs, and 42 Incredible Hamsters. b) 30 units of material

19. a) Both industries are profitable. b) \$24 400 of technology and \$6400 of agriculture

20. About 77%

21. TRAVEL