1. (8 points) You are given the following system of linear equations:

$$\begin{aligned} x_1 - x_2 - 7x_3 - x_4 &= -2 \\ 2x_1 - 6x_3 + 4x_4 &= 0 \\ x_1 - 3x_3 + 3x_4 &= 1 \end{aligned}$$

(a) Find the general solution in parametric vector form.

(b) Let A be the coefficient matrix of this sytem. Find a specific solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ such that the first entry of $\mathbf{x}$ is 1.

(c) Does there exist a vector $\mathbf{c} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{c}$ has no solution?

(d) Write the fourth column of A as a linear combination of the first three columns of A, or explain why it cannot be done.

2. (5 points) Consider the matrix equation given below. Find the value(s) of k, if any, such that the corresponding system of linear equations has

(a) no solution

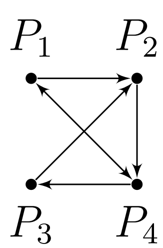
(b) infinite solutions

(c) one solution

$$\begin{bmatrix} -1 & k+3 \\ k & -18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ -36 \end{bmatrix}$$

3. (1 point) Find a 3×3 matrix A that is symmetric, nonzero, and not diagonal.

4. (5 points) You are given the following directed graph:



(a) Write the adjacency matrix M for this graph.

(b) Use matrix multiplication to determine how many walks of length four there are from P_4 to P_1.

5. (3 points) Set up, **but do not solve**, a linear system that can be used to balance the chemical equation given below. In particular, find the coefficient matrix of this linear system.

$$H_3PO_4 + Ca(OH)_2 \rightarrow Ca_3(PO_4)_2 + H_2O$$

6. (5 points) (a) Let $A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 7 & -6 \\ 2 & -3 & 0 \end{bmatrix}$. Find A^{-1} using row reduction.

(b) What is $(A^T)^{-1}$? (Note that no row reduction is needed here.)

7. (3 points) Let $A = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix}$. Write A as a product of elementary matrices.

8. (4 points) Find matrix A such that

$$(2A - 4\,[-2 \;\; 1 \;\; 0])^T = 3A^T + [3 \;\; 4 \;\; -1]^T$$

9. (7 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

(a) What are the eigenvalues of A?

(b) Find two linearly independent eigenvectors of A.

(c) Diagonalize A. That is, find an invertible matrix P such that $P^{-1}AP$ is diagonal.

10. (6 points) Use Cramer's Rule to solve the following linear system for x_3 only:

$$\begin{aligned} -x_1 - 3x_2 + 2x_3 - x_4 &= 0 \\ -x_1 + 2x_2 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 0 \\ -2x_1 + 7x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

11. (9 points) Consider the points $A(1, 0, 2)$, $B(3, -1, 1)$, and $C(0, 4, 3)$.

(a) Find the area of the triangle ABC.

(b) Find the cosine of the angle at A in the triangle ABC.

(c) Find the projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.

(d) Is $\overrightarrow{AB}$ orthogonal to $\overrightarrow{AC}$? Justify.

12. (4 points) Find the point on $\mathcal{L} : \mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ that is closest to $A(1, 1, 1)$.

13. (5 points) Find an equation of the form $ax + by + cz = d$ for each of the following planes:

(a) The plane that contains the origin and is parallel to the plane with equation $3x - y + z = 4$.

(b) The plane that contains the origin and also contains the line $\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

14. (5 points) Consider the nonparallel lines $L_1 : \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 11 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ and $L_2 : \mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Find the points A (on L_1) and B (on L_2) that are closest together.

15. (4 points) Given $\mathbf{x} \cdot \mathbf{y} = 4$, $(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = -5$ and $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = 21$, find $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$.

16. (5 points) Let $A = \begin{pmatrix} 3 & 2 & 0 & 1 & 5 \\ 1 & 0 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & 3 \\ -4 & -3 & 1 & -1 & 2 \end{pmatrix}$. It is given that A row reduces to

$R = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Find bases of $\text{col}(A)$ and $\text{null}(A)$ and give their dimensions.

17. (6 points) For each of the following sets of vectors (i) determine whether the set is linearly independent and (ii) state whether the span of the set is a line, a plane, or all of $\mathbb{R}^3$.

(a) $\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

18. (7 points) Determine whether U is a subspace of $\mathbb{R}^4$. Explain your conclusions. If U is a subspace, give a basis of U.

(a) $U = \left\{ \begin{pmatrix} a - b + 2 \\ b + c \\ 0 \\ a + 2c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$

(b) $U = \left\{ (a, b, c, d) \middle| \begin{array}{r} a - b + c + 2d = 0 \\ 2a + b - 4c + d = 0 \end{array} \right\}$

19. (8 points) Mark each of the following as true or false. Justify each answer. Zero marks will be awarded for unjustified answers.

(a) If A satisfies the equation $X^2 = X$ then $I - A$ also satisfies this equation.

(b) If $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ is a basis of a subspace W of $\mathbb{R}^n$ and $\mathbf{x}_1, \mathbf{x}_2 \in W$ are independent then $\mathbf{x}_1, \mathbf{x}_2, \mathbf{b}_3$ is a basis of W.

(c) If the $n \times n$ matrix A satisfies $A^3 + 2A^2 + A - 5I = 0$ then A is invertible.

(d) If $A^T = -A$ then A^2 is symmetric.

ANSWERS

1. (a) $\mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ 1 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -4/3 \\ 1/3 \\ 0 \end{bmatrix}$

(c) No, since $\text{rank}(A) = m$.

(d) Impossible. System is inconsistent.

2. (a) $k = 3$
   (b) $k = -6$
   (c) $k \notin \{-6, 3\}$

3. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. (a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$
   (b) 2

5. $\begin{bmatrix} 3 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 4 & 2 & -8 & -1 \\ 0 & 1 & -3 & 0 \end{bmatrix}$ (Row order: H,P,O,Ca)

6. (a) $\begin{bmatrix} -18 & -3 & 5 \\ -12 & -2 & 3 \\ -5 & -1 & 1 \end{bmatrix}$
   (b) $\begin{bmatrix} -18 & -12 & -5 \\ -3 & -2 & -1 \\ 5 & 3 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 5 & -8 & 1 \end{bmatrix}$

9. (a) $\lambda = 0, 3$
   (b) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
   (c) $\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$

10. $x_3 = \frac{-10}{3}$

11. (a) $\frac{\sqrt{59}}{2}$
    (b) $\frac{-7}{6\sqrt{3}}$
    (c) $\frac{-7}{18} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$
    (d) No. Their dot product is not zero.

12. $(\frac{3}{2}, 2, \frac{5}{2})$

13. (a) $3x - y + z = 0$
    (b) $7x - y - 10z = 0$

14. $A(2, 5, 9)$, $B(0, 2, 3)$

15. $\|\mathbf{x}\| = 2$, $\|\mathbf{y}\| = 3$

16. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \end{bmatrix} \right\}$, $\dim(\mathrm{col}(A)) = 3$

    $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, $\dim(\mathrm{null}(A)) = 2$

17. (a) linearly dependent, span is a plane
    (b) linearly independent, span is $\mathbb{R}^3$

18. (a) U is a subspace of $\mathbb{R}^4$. A basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$.
    (b) U is a subspace of $\mathbb{R}^4$. A basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

19. (a) True. Expand $(I - A)^2$ to get $I - A$.
    (b) False. Counterexample:
    $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$,
    $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{b}_3\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.
    (c) True. Rearrange the equation to get
    $A\left(\frac{1}{5}(A^2 + 2A + I)\right) = I$
    and then rearrange to get
    $\frac{1}{5}(A^2 + 2A + I)A = I$.
    Conclude that $A^{-1} = \frac{1}{5}(A^2 + 2A + I)$.
    (d) True. $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$.