1. Evaluate the following integrals.

(a) $\displaystyle\int_{-1}^{1} \frac{1}{x^2 - 2x + 5}\, dx$

(b) $\displaystyle\int \frac{1}{x(\ln x)^2}\, dx$

(c) $\displaystyle\int \cos^3(x)\sin^4(x)\ dx$

(d) $\displaystyle\int e^x \sin(2x)\ dx$

(e) $\displaystyle\int \frac{2x^3 - 9x^2 - 5x + 7}{2x^2 - 5x - 3}\, dx$

(f) $\displaystyle\int_0^1 \frac{81x^5}{\sqrt{3x^3+1}}\, dx$

(g) $\displaystyle\int_0^{\ln(2)} \frac{x}{e^x}\, dx$

(h) $\displaystyle\int \frac{\sqrt{9-4x^2}}{x}\, dx$

2. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$

(b) $\displaystyle\lim_{x\to 0} (x + e^{2x})^{1/x}$

3. Find the area of the region enclosed by the curves $y = x^2 - 4x$ and $y = -x^2 + 6x - 8$.

4. (a) Sketch the region $\mathcal{R}$ between the graphs of the functions $y = e^x$ and $y = -x$, from $x = 0$ to $x = 1$.

(b) Suppose that $\mathcal{R}$ is the base of a solid which, when sliced perpendicular to the base $\mathcal{R}$, and parallel to the y-axis, forms square cross-sections. Write down, **but do not evaluate**, an integral for the volume of this solid.

5. Solve the differential equation

$$\sqrt{1-x^2}\,\frac{dy}{dx} = \frac{1}{y}$$

given that $y = -\sqrt{\pi}$ when $x = 1/2$. Express y as a function of x.

6. The Bertalanffy equation is used by ecologists to model the growth of organisms over time. It is derived from the differential equation

$$\frac{dL}{dt} = k(L - M)$$

where L is the length of the organism at time t, M is the maximal length of the organism, and k is a rate constant.

Suppose upon first measurement a salmon is 20 cm in length. 1 year later, the salmon has grown to 50 cm in length. If the maximal length M is 80 cm, use this model to predict the length of the fish 2 years after the first measurement.

7. Determine whether the sequence

$$a_n = (-1)^n \left(\frac{e^n - n}{e^n + n}\right)$$

converges or diverges. If the sequence converges, find its limit; otherwise, explain why it diverges.

8. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2 + (-3)^n}{5^n}$

is convergent or divergent.

If it is convergent, find its sum.

9. Suppose the series $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms whose sum is 10. Determine whether the following converge or diverge, and **justify your answer**. **If possible**, find the value to which these converge.

(a) The sequence $\{a_n\}$

(b) The sequence of partial sums $\{s_N\}$, where $s_N = a_1 + a_2 + \cdots + a_N$

(c) The series $\sum_{n=1}^{\infty} e^{-a_n}$

10. Determine whether the series converges or diverges. Justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{2n+3}{4n+5}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(c) $\sum_{n=1}^{\infty} \frac{(4n^2+1)^n}{(n\pi)^{2n}}$

11. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Justify your answer.

(a) $\sum_{n=0}^{\infty} \frac{\cos n}{3^n+1}$

(b) $\sum_{n=5}^{\infty} \frac{(-1)^n}{n-2\sqrt{n}}$

12. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^3\, 5^n}$.

13. (a) Find a power series representation for

$$f(x) = \frac{1}{4-3x}$$

and write down the first four non-zero terms of the series.

(b) Use your answer from part (a) to find a power series representation for

$$g(x) = \frac{x}{(4-3x)^2}$$

Answers

1. (a) $\pi/8$

   (b) $-\dfrac{1}{\ln x}+C$

   (c) $\dfrac{1}{5}\sin^5 x-\dfrac{1}{7}\sin^7 x+C$

   (d) $\dfrac{1}{5}e^x\left(\sin(2x)-2\cos(2x)\right)+C$

   (e) $\dfrac{1}{2}x^2-2x-\ln|2x+1|-5\ln|x-3|+C$

   (f) 8

   (g) $\dfrac{1-\ln 2}{2}$

   (h) $-3\ln\left|\dfrac{3+\sqrt{9-4x^2}}{2x}\right|+\sqrt{9-4x^2}+C$

2. (a) 2

   (b) e^3

3. $\displaystyle\int_1^4 \left(-x^2+6x-8\right)-\left(x^2-4x\right)\,dx=9$

4. $\displaystyle\int_0^1 \left(e^x+x\right)^2\,dx$

5. $y=-\sqrt{2\arcsin(x)+\dfrac{2\pi}{3}}$

6. You will find $L(t)=-60\left(\dfrac{1}{2}\right)^t+80$. At $t=2$ years, $L=65$ cm.

7. $\lim\limits_{n\to\infty}|a_n|=1$, so the sequence diverges by oscillation.

8. Sum of convergent geometric series: convergent. The sum is 1/8.

9. (a) Converges to 0

   (b) Converges to the sum of the series, 10.

   (c) Diverges by the divergence test, since $e^{-a_n}\to 1\neq 0$

10. (a) Diverges by divergence test.

    (b) Converges by ratio test.

    (c) Converges by root test.

11. (a) Absolutely convergent by comparison test.

    (b) Conditionally convergent by comparison test (or limit comparison test) and alternating series test.

12. IoC: [-8,2]

13. (a) $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{3^n x^n}{4^{n+1}}=\frac{1}{4}+\frac{3x}{16}+\frac{9x^2}{64}+\frac{27x^3}{256}+\dots$

    (b) As $g(x)=\dfrac{xf'(x)}{3}$, we have that $g(x)=\displaystyle\sum_{n=1}^{\infty}\frac{n3^{n-1}x^n}{4^{n+1}}$