(8) 1. Given below is the the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$.

$$[A|\mathbf{b}] = \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 1 & 1 \\ 2 & 5 & 3 & 0 & 1 \\ -1 & -3 & -2 & 1 & 0 \\ 0 & -1 & -1 & 2 & 1 \end{array}\right]$$

(a) Determine whether $\mathbf{x} = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ is a solution to the system.

(b) Find the general solution of this system in parametric-vector form.

(c) Find a basis for $\operatorname{Col} A$.

(d) What is the general solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$?

(e) Find a basis for $\operatorname{Nul} A$.

(f) Write the fourth column of A as a linear combination of the first three columns of A.

(6) 2. Let

$$A = \begin{bmatrix} 1 & 1 & a \\ 1 & a & a \\ a & a & a \\ a & a & a^2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a^2 - 2a \end{bmatrix}$$

(a) For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have no solution ?

(b) For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have a unique solution ?

(c) For what value(s) of a does the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions ?

(6) 3. Use linear algebra to balance the chemical equation:

$$\mathrm{B_2S_3} + \mathrm{H_2O} \longrightarrow \mathrm{H_3BO_3} + \mathrm{H_2S}$$

(6) 4. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 15 & 0 & 1 \end{bmatrix}$.

(a) Find the inverse of A.

(b) Write A as a product of elementary matrices.

(7) 5. Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$.

(a) Evaluate $A^T A$ and find $\left(A^T A\right)^{-1}$.

(b) Evaluate AA^T and show that AA^T is not invertible.

(c) Prove that if A is any $m \times n$ matrix, then $A^T A$ and AA^T are both symmetric.

(3) 6. Find the **rank** and **nullity** (dimension of null space) of each matrix A described below.

(a) A is a 5×5 elementary matrix.

(b) A is a 5×7 matrix such that A has linearly independent rows.

(c) A is a **non-zero** 2×2 matrix such that A^2 is the **zero** matrix.

(2) 7. Suppose A is an $m \times n$ matrix and that there is a matrix C such that $AC = I$. Show that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^m$. What can you conclude about the rank of A?

(5) 8. Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix}$

(a) Find $\det A$.

(b) What is $\det(-2A^{-1}A^TA)$?

(4) 9. Suppose A, B and C are $n \times n$ matrices such that $ABCA = I$.

(a) Use determinants to explain why A, B and C are invertible.

(b) Find C^{-1} in terms of A and B (in simplest form).

(7) 10. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Given that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$

(a) Find the eigenvectors of A.

(b) Diagonalize A. Specifically, find matrices D and P such that $AP = PD$

(c) Find A^{1000}.

(8) 11. Consider the lines $\mathcal{L}_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ -4 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathcal{L}_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 9 \end{bmatrix} + t \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$

(a) Find the coordinates of the point of intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Let $\mathcal{P}$ be the plane that contains the point $Q(2, 1, 1)$ and is orthogonal to the line $\mathcal{L}_1$. Give the equation (in $ax + by + cz = d$ form) of this plane.

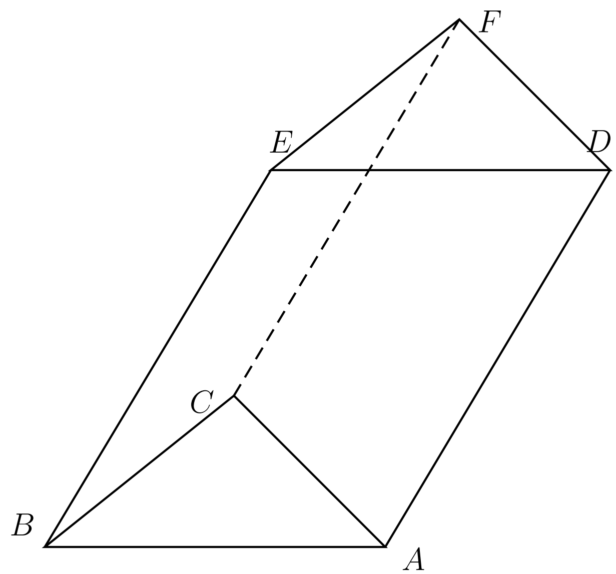
(c) Find the cosine of the angle between $\mathcal{L}_1$ and $\mathcal{L}_2$.

(7) 12. Consider the prism in $\mathbb{R}^3$ (Note that a prism can be seen as half a parallelepiped.) whose triangular base has vertices at the points $A(0,1,3)$, $B(2,-1,3)$, and $C(1,1,5)$. Furthermore assume that another vertex of this prism is at $D(4,7,10)$. (See the image below).

(a) Find a parametric vector equation for the line through A and B.

(b) Find the area of triangle $\triangle ABC$.

(c) Find the volume of the prism. (Note that $\overrightarrow{AD}$ is not necessarily orthogonal to $\triangle ABC$.)



(3) 13. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be unit vectors in $\mathbb{R}^n$. Furthermore, let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be orthogonal to each other. Simplify the following.

$$\text{Proj}_{\mathbf{u}+\mathbf{w}}(\mathbf{u}-2\mathbf{v})$$

(7) 14. Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq y \right\}$

(a) Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in V?

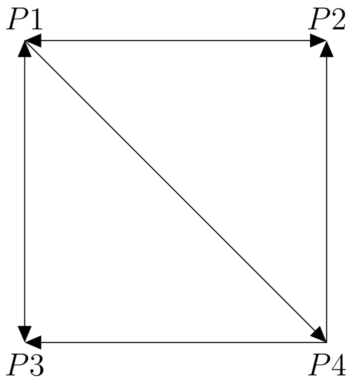
(b) Is V closed under vector addition? Justify.

(c) Is V closed under scalar multiplication? Justify.

(d) Is there a non-zero vector $\mathbf{u}$ such that Span$\{\mathbf{u}\}$ is contained in V? If yes give such a $\mathbf{u}$, if no, explain why

(e) Is V a subspace of $\mathbb{R}^2$?

(6) 15. Given the following directed graph



(a) Determine the adjacency matrix of the given graph, M.

(b) Given that $M^2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$, find M^4.

(c) How many walks of length 4 are there from $P4$ to $P1$?

(d) How many total walks of length 4 are there?

(e) How many **closed** walks of length 4 are there?

(3) 16. A pump handle has a pivot at $P(1, 1, -1)$ and extends to $Q(6, 1, -6)$ (m). A force $\mathbf{F} = (1, 0, -10)$ (N) is applied at Q. Find the torque about the pivot that is produced.

(2) 17. A constant force $\mathbf{F} = (40, 30)$ (N) is used to move a sled horizontally 10 m. Calculate the work done.

(10) 18. Fill in the blanks. The missing word is **might**, **must** or **cannot**. Justify your answers.

(a) If $A^2 + 3A = 2I$, then A __________ be invertible.

(b) If $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$ for three given points A, B, and C in $\mathbb{R}^n$, then Span$\{\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}\}$ __________ be three-dimensional.

(c) Two lines in $\mathbb{R}^3$ that are orthogonal to a third line __________ be parallel.

(d) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent vectors in $\mathbb{R}^n$, then $\mathbf{a}, \mathbf{2a} + \mathbf{3b}, \mathbf{a} - \mathbf{3c}$ __________ be linearly independent.

(e) If A is a square matrix, then A and A^T __________ have the same eigenvalues.

Answers

1. (a) Yes since it satisfies $A\mathbf{x} = \mathbf{b}$.

(b) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

(c) $\{\mathbf{a}_1, \mathbf{a}_2\}$

(d) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

(e) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(f) $\mathbf{a}_4 = 5\mathbf{a}_1 - 2\mathbf{a}_2 + 0\mathbf{a}_3$

2. (a) $a \neq 0$ and $a \neq 3$

 (b) $a = 3$

 (c) $a = 0$

3.

$$\mathrm{B_2S_3 + 6H_2O \longrightarrow 2H_3BO_3 + 3H_2S}$$

4. (a) $A^{-1} = \begin{bmatrix} 0 & 1/3 & 0 \\ 1 & 0 & 0 \\ 0 & -5 & 1 \end{bmatrix}$.

 (b) Many answers possible, e.g., $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

5. (a) $A^TA = \begin{bmatrix} 2 & 1 \\ 1 & 14 \end{bmatrix}$

 $(A^TA)^{-1} = \frac{1}{27} \begin{bmatrix} 14 & -1 \\ -1 & 2 \end{bmatrix}$

 (b) $AA^T = \begin{bmatrix} 10 & 3 & 5 \\ 3 & 1 & 2 \\ 5 & 2 & 5 \end{bmatrix}$

 $\text{rank}(AA^T) = \text{rank}(A^TA) = 2$; or, show that $\det(AA^T) = 0$.

 (c) $(A^TA)^T = A^T(A^T)^T = A^TA$ and $(AA^T)^T = (A^T)^TA^T = AA^T$

6. (a) $\text{rank}A = 5$ and $\text{nullity}A = 0$

 (b) $\text{rank}A = 5$ and $\text{nullity}A = 2$

 (c) $\text{rank}A = 1$ and $\text{nullity}A = 1$

7. $(AC)\mathbf{b} = \mathbf{b}$ or $A(C\mathbf{b}) = \mathbf{b}$ which implies for every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{x} = C\mathbf{b}$ is a solution. Therefore, there is a pivot position in every row of A so $\text{rank}A = m$.

8. (a) $\det A = -14$

 (b) $\det(-2A^{-1}A^TA) = -224$

9. (a) $|A||B||C||A| = |I|$ which implies $|A|^2|B||C| = 1$
   So $|A| \neq 0$, $|B| \neq 0$, $|C| \neq 0$
   (b) $C^{-1} = A^2B$

10. (a) The eigenvectors are $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-4\\0\\1\end{bmatrix}$
   (b) $P = \begin{bmatrix}1 & 1 & -4\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $D = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}$
   (c) $A^{1000} = I$

11. (a) $(-1, -4, -1)$
   (b) $-x + 2y + 3z = 3$
   (c) $\cos\theta = \dfrac{13}{14\sqrt{3}}$

12. (a) $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\1\\3\end{bmatrix} + t\begin{bmatrix}2\\-2\\0\end{bmatrix}$
   (b) Area=3
   (c) Volume=13

13.
$$\text{Proj}_{\mathbf{u}+\mathbf{w}}(\mathbf{u} - 2\mathbf{v}) = \frac{1}{2}(\mathbf{u} + \mathbf{w})$$

14. (a) Yes since $0 \geq 0$
   (b) Yes since $x_1 \geq y_1$ and $x_2 \geq y_2$ implies that $x_1 + x_2 \geq y_1 + y_2$. So, if $(x_1, y_1) \in V$ and $(x_2, y_2) \in V$ then $(x_1 + x_2, y_1 + y_2) \in V$.
   (c) No, for example $\mathbf{v} = \begin{bmatrix}3\\2\end{bmatrix} \in V$ yet $-\mathbf{v}$ is not in V.
   (d) Yes, $\mathbf{u} = \begin{bmatrix}1\\1\end{bmatrix}$ or any nonzero multiple of $\mathbf{u}$.
   (e) No, by Part (c).

15. (a) $M = \begin{bmatrix}0 & 1 & 1 & 1\\1 & 0 & 0 & 0\\1 & 0 & 0 & 0\\0 & 1 & 1 & 0\end{bmatrix}$
   (b) $M^4 = M^2M^2 = \begin{bmatrix}4 & 4 & 4 & 2\\2 & 2 & 2 & 2\\2 & 2 & 2 & 2\\4 & 2 & 2 & 0\end{bmatrix}$
   (c) 4

(d) 38

(e) 8

16. Let $\mathbf{r} = \overrightarrow{PQ}$, then $\mathbf{T} = \mathbf{F} \times \mathbf{r} = (1, 0, -10) \times (5, 0, -5) = (0, -45, 0)$. Its magnitude is 45 Nm.

17. $W = \mathbf{F} \cdot \mathbf{d} = (40, 30) \cdot (10, 0) = 400$ J

18. (a) **must**, since A is a square matrix and $A(A + 3I)/2 = I$, by IMT A is invertible.

    (b) **cannot**, since the given 3 vectors are linearly dependent.

    (c) **might**, since they can be parallel or skew lines.

    (d) **must**, since $x_1\mathbf{a} + x_2(\mathbf{2a} + \mathbf{3b}) + x_3(\mathbf{a} - \mathbf{3c}) = \mathbf{0}$ has only the trivial solution.

    (e) **must**, since $(A - \lambda I)^T = A^T - \lambda I$ so $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$ which implies they have the same characterisric polynomial and therefore the same eigenvalues.