1. (30 points) Evaluate the following integrals.

(a) $\displaystyle\int (x^2 - 2x)e^{x/3}\,dx$

(b) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\csc^2(x)}{\sqrt{4-\cot^2(x)}}\,dx$

(c) $\displaystyle\int \frac{\sec^4(\sqrt{x})\tan^2(\sqrt{x})}{\sqrt{x}}\,dx$

(d) $\displaystyle\int \frac{1}{(4x^2-9)^{3/2}}\,dx$

(e) $\displaystyle\int x\arctan(x)\,dx$

(f) $\displaystyle\int \frac{4x+16}{(x-2)(x^2+4)}\,dx$

2. (7 points) Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\cos(x)-1}{x\arctan(x)}$

(b) $\displaystyle\lim_{x\to 0} \left(1+\frac{1}{x^2}\right)^{x^3}$

3. Evaluate the following improper integrals or show divergence.

(a) (4 points) $\displaystyle\int_{-2}^{-1} \frac{dx}{x\sqrt{x^2-1}}$

(b) (5 points) $\displaystyle\int_{1}^{\infty} \frac{1}{x(3x+1)}\,dx$

4. (6 points) Let R be the region bounded by $x = y^2 - 6y + 8$ and $x = 4y - y^2$. Set up, but do not evaluate, an integral for the volume of the solid generated when R is rotated about each of the following lines.

(a) $x = -2$

(b) $y = 4$

5. (4 points) Find the length of the curve $y = \dfrac{1}{2}\ln(\sin(2x))$ from $x = \dfrac{\pi}{12}$ to $x = \dfrac{\pi}{4}$.

6. (5 points) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{e^y}{\csc^2(x)}$ where $y(\pi) = 0$. Give an explicit solution for y.

7. (5 points) A cup of hot coffee with a temperature of 80°C is placed in a room that has a constant temperature of 20°C. The coffee cools according to Newton's Law of Cooling, which states that the rate of change of temperature of some object (coffee) is proportional to the temperature difference between the object and its surroundings. Five minutes later, the coffee has cooled to 50°C.

(a) Write a differential equation that models the cooling of the coffee.

(b) Solve the differential equation to find an explicit expression for the temperature as a function of time.

(c) Determine the temperature of the coffee after 10 minutes. Simplify your answer.

8. (3 points) Given $\displaystyle\sum_{n=2}^{\infty} \frac{(-3)^{n-1}}{3^{2n}}$

(a) Show that the series converges.

(b) Find the sum of the series.

9. (9 points) Determine whether the following series converge of diverge. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{2n+4}{3n-1}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln(n^3+1)}{n+1}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)+5}{\sqrt{16n-1}}$

10. Determine whether the following series are absolutely convergent, conditionally convergent or divergent. Justify your answers.

(a) (3 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(n+3)!}{(n^2+3)e^{2n}}$

(b) (4 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+10}$

11. (5 points) Find the interval of convergence of the following power series.

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{4^{n+1}\sqrt{n}}$$

12. (5 points) Determine the first five non-zero terms of the Maclaurin series of $f(x) = \sqrt{2x+1}$.

13. (2 points) Given f is continuous on $[0,3]$ and $\displaystyle\int_0^3 f(x)\,dx = 4$, evaluate the following definite integral. Simplify your answer.

$$\int_1^{e^3} \frac{2+f(\ln(x))}{x}\,dx$$

14. (3 points) Fill in the following blanks with the word *must*, *might* or *cannot*, as appropriate.

(a) If $\displaystyle\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ ________________ converge.

(b) If $a_n > 0$ and $\sum a_n$ converges, then $\sum (-1)^n a_n$ ________________ converge.

(c) If the sequence $\{a_1, a_2, a_3, a_4, ...\}$ diverges, then the sequence $\{a_2, a_4, a_6, a_8, ...\}$ ________________ diverge.

Answers

1. (a) $(3x^2 - 24x + 72)e^{x/3} + C$

(b) $\pi/6$

(c) $\dfrac{2\tan^5(\sqrt{x})}{5} + \dfrac{2\tan^3(\sqrt{x})}{3} + C$

(d) $\dfrac{-x}{9\sqrt{4x^2-9}} + C$

(e) $\frac{1}{2}[x^2 \arctan(x) - x + \arctan(x)] + C$

(f) $3\ln|x-2| - \frac{3}{2}\ln|x^2+4| - \arctan(\frac{x}{2}) + C$

2. (a) $-1/2$

(b) 1

3. (a) $-\pi/3$

(b) $\ln(4/3)$

4. (a) $V = \pi \displaystyle\int_1^4 [(-y^2+4y+2)^2 - (y^2-6y+10)^2]\,dy$

(b) $V = 2\pi \displaystyle\int_1^4 (4-y)(-2y^2+10y-8)\,dy$

5. $\frac{-1}{2}\ln(2-\sqrt{3})$

6. $y = -\ln\left(\dfrac{\sin(2x) - 2x + 4 + 2\pi}{4}\right)$

7. (a) $\dfrac{dT}{dt} = k(T-20)$

(b) $T = 60\left(\dfrac{1}{2}\right)^{t/5} + 20$

(c) $T = 35°C$

8. (a) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1/3 < 1$ Therefore, the series converges by the Ratio Test

(b) $-1/36$

9. (a) Diverges be the Test for Divergence

(b) Converges by the Root Test

(c) Diverges by the Comparison Test

10. (a) Diverges by the Ratio Test

(b) Conditionally Convergent by the Limit Comparison Test and the Alternating Series Test

11. $[-5, 3)$

12. $f(x) = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{5x^4}{8} + ...$

13. 10

14. (a) might

(b) must

(c) might