1. (9 points) Let $A = \begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ 2 & 6 & -2 & 4 & 6 \\ 4 & 13 & -3 & 7 & 1 \end{bmatrix}$

   (a) Find a basis of $\mathrm{Col}(A)$

   (b) Find a basis of $\mathrm{Nul}(A)$.

   (c) Write 4^{th} column of A as a linear combination of 1^{st} and 3^{rd} column of A.

Solution:

$$\begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ 2 & 6 & -2 & 4 & 6 \\ 4 & 13 & -3 & 7 & 1 \end{bmatrix} \xrightarrow[R_3=R_3-4R_1]{R_2=R_2-2R_1} \begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -11 \end{bmatrix} \xrightarrow{R_1=R_1-3R_3} \begin{bmatrix} 1 & 0 & -4 & 5 & 36 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -11 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -4 & 5 & 36 \\ 0 & 1 & 1 & -1 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So a basis of $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 13 \end{bmatrix} \right\}$.

The solutions of $A\mathbf{x} = \mathbf{0}$ are

$$\begin{cases} x_1 = 4s - 5t - 36u \\ x_2 = -s + t + 11u \\ x_3 = s \\ x_4 = t \\ x_5 = u \end{cases}$$

In parametric vector form:

$$\mathbf{x} = s \begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -36 \\ 11 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So a basis of $\mathrm{Nul}(A)$ is

$$\left\{ \begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -36 \\ 11 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

As for part (c),

$$\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$$

2. (4 points) Let A be an $n \times n$ matrix. Let I be the $n \times n$ identity matrix. Suppose $(I - A)^{-1} = I + A$. Prove $A^2 = 0$.

Solution: One has $(I-A)(I+A)=I$, hence $I-A^2=I$, so $A^2=0$.

3. (4 points) Find the inverse of $A=\begin{bmatrix}1&0&3&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$

Solution:

$$\begin{bmatrix}1&0&3&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&0&1&0&0&1&0\\0&0&1&0&0&0&0&1\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_3}\begin{bmatrix}1&0&3&0&1&0&0&0\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&1\\0&0&0&1&0&0&1&0\end{bmatrix}\xrightarrow{R_1=R_1-3R_3}$$
$$\begin{bmatrix}1&0&0&0&1&0&0&-3\\0&1&0&0&0&1&0&0\\0&0&1&0&0&0&0&1\\0&0&0&1&0&0&1&0\end{bmatrix}$$

Hence

$$A^{-1}=\begin{bmatrix}1&0&0&-3\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$$

4. (6 points) Use linear algebra to balance the chemical equation: $NH_3+Cl_2\rightarrow NH_4Cl+N_2$

Solution:

$$8\,NH_3+3\,Cl_2\rightarrow 6\,NH_4Cl+N_2$$

5. (7 points) Let $A=\begin{bmatrix}1&1&1\\1&2&2\\1&1&3\end{bmatrix}$

(a) Find $\det(A)$

Solution:

$$\det(A)=2$$

(b) Find $\operatorname{adj}(A)$

Solution:

$$\operatorname{adj}(A)=\begin{bmatrix}4&-1&-1\\-2&2&0\\0&-1&1\end{bmatrix}^T=\begin{bmatrix}4&-2&0\\-1&2&-1\\-1&0&1\end{bmatrix}$$

(c) Find A^{-1} using the results from (a) and (b).

Solution:

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -2 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

6. (4 points) Express A^{-1} of $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices and state the inverse of each of the elementary matrices you find.

Solution:

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_1 = R_1 \div 2} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So define

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

and

$$E_3 E_2 E_1 A = I$$

so

$$A^{-1} = E_3 E_2 E_1$$

We also have

$$E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

7. (6 points) Let A and C be 3×3 matrices. It is given that A is symmetric, $\det(A) = -2$, and $\dim(\mathrm{Nul}(C)) = 1$.

   (a) Find $\det\left(\left(A + A^T\right)^2\right)$.

   Solution: Since A is symmetric, $A^T = A$, hence

   $$\det\left(\left(A + A^T\right)^2\right) = \det\left((2A)^2\right) = \det\left(4A^2\right) = 4^3\left(\det(A)\right)^2 = 4^3 \cdot (-2)^2 = 256$$

   (b) Find $\det(C + C^2)$.

   Solution: Since $\dim(\mathrm{Nul}(C)) \neq 0$, $\det(C) = 0$, and hence

   $$\det(C + C^2) = \det\left(C(I + C)\right) = \det(C) \cdot \det(I + C) = 0$$

8. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

   (a) Find the eigenvalues of A.

Solution: Characteristic polynomial of A is

$$\det(xI - A) = \begin{vmatrix} x-1 & -2 \\ -3 & x-2 \end{vmatrix} = x^2 - 3x - 4$$

The eigenvalues are the roots of the characteristic polynomial, which are $\lambda_1 = 4$ and $\lambda_2 = -1$.

(b) Find the eigenvectors of A.

Solution: Solve the equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$ for each eigenvalue λ.

- For $\lambda_1 = 4$:

$$(4I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This leads to the system of equations:

$$\begin{cases} 3x_1 - 2x_2 = 0 \\ -3x_1 + 2x_2 = 0 \end{cases}$$

whose solution is:

$$x_1 = \frac{2}{3}x_2$$

where $x_2 = t$ is a free variable. Therefore, $x_1 = \frac{2}{3}t$ and the eigenvectors corresponding to $\lambda_1 = 4$ are:

$$\mathbf{v}_1 = t \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

- For $\lambda_1 = -1$:

$$(-1I - A)\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This leads to the system of equations:

$$\begin{cases} -2x_1 - 2x_2 = 0 \\ -3x_1 - 3x_2 = 0 \end{cases}$$

whose solution is:

$$x_1 = -x_2$$

where $x_2 = t$ is a free variable. Therefore, $x_1 = -t$ and the eigenvectors corresponding to $\lambda_1 = 4$ are:

$$\mathbf{v}_1 = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(c) Diagonalize A, that is, find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.

Solution: D is the diagonal matrix of eigenvalues and P is the matrix of eigenvectors:

$$D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix}$$

We can verify the equation $P^{-1}AP = D$ by solving $AP = PD$:

$$AP = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} & 1 \\ 4 & -1 \end{bmatrix}$$

$$PD = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} & 1 \\ 4 & -1 \end{bmatrix}$$

9. (8 points) A force vector $\mathbf{F} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ N is applied to move an object along a displacement vector $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ m.

   (a) Compute the **work** W done by the force $\mathbf{F}$ in moving the object along $\mathbf{d}$.

   (b) Suppose the same force $\mathbf{F}$ is applied at a point located at position vector $\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ m from the origin (relative to a fixed pivot). Compute the **torque** $\boldsymbol{\tau}$ about the origin due to the force $\mathbf{F}$.

Solution:

(a) Use the **dot product** to compute work:

$$W = \mathbf{F} \cdot \mathbf{d} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 4 \cdot 2 + 3 \cdot 1 + 0 \cdot 0 = 11\,\mathrm{J}$$

(b) Use the **cross product** to compute torque:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix} \mathrm{N} \cdot \mathrm{m}$$

10. (10 points) Consider the vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

   (a) Determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

   (b) What is the dimension of the subspace generated by the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   (c) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of $\mathbb{R}^3$? Justify your answer.

Solution: Row reduce the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Since the reduced row echelon form has 2 pivots and 1 free variable, we conclude the following:

(a) The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly **dependent**.

(b) The dimension of the subspace generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is 2.

(c) The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis of $\mathbb{R}^3$ because it is not linearly independent.

11. (10 points) Define the lines $\mathcal{L}_1 : \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ and $\mathcal{L}_2 : \mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. It is known that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect.

(a) Find the intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Solution: To find the intersection, we need to solve the following linear system:

$$\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow s \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} - t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

Use Gauss-Jordan elimination:

$$\begin{bmatrix} -3 & -2 & -5 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix} \xrightarrow[R_3 = R_3 + 3R_1]{R_2 = R_2 - R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & -5 & -5 \end{bmatrix} \xrightarrow[R_3 = R_3 + 5R_2]{R_1 = R_1 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So one has solution

$$\begin{cases} s = 1 \\ t = 1 \end{cases}$$

Hence the intersection is the point $(0, 1, 3)$.

(b) Find an equation of the line that contains the point $(1, -1, 2)$ and is perpendicular to the plane that contains both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Solution: Any direction vector $\mathbf{v}$ of the line we are looking for is orthogonal to both direction vectors of $\mathcal{L}_1$ and $\mathcal{L}_2$. So we can choose

$$\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are direction vectors of lines $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Hence we find

$$\mathbf{v} = \langle -3, 1, 1 \rangle \times \langle 2, 0, 1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k} = \langle 1, 5, -2 \rangle$$

Hence a parametric vector equation of the line is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

(c) Find an equation of the plane (in the form $ax + by + cz = d$) that contains both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Solution: The vector $\mathbf{v}$ we find in part (b) is also a normal vector of the plane we are looking for. A point on the line $\mathcal{L}_1$ is $(3, 0, 2)$, which is also on the plane.
Therefore, an equation of the plane is

$$\begin{aligned} \langle x-3, y, z-2\rangle \cdot \langle 1, 5, -2\rangle &= 0 \\ x - 3 + 5y - 2(z-2) &= 0 \\ x + 5y - 2z &= -1 \end{aligned}$$

12. (4 points) Suppose two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ have the same length. Calculate $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$.

Solution: Since $\mathbf{u}$ and $\mathbf{v}$ have the same length, $\|\mathbf{u}\| = \|\mathbf{v}\|$, hence

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0$$

13. (7 points) Given a point $P(5, -1, 1)$ and a line $\mathcal{L} : \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$;

(a) find the point R on $\mathcal{L}$ closest to P;

Solution: A direction vector of the line $\mathcal{L}$ is

$$\mathbf{v} = \langle -1, -1, 0\rangle$$

Choose a point $Q(1, -1, 0)$ on the line $\mathcal{L}$. Then

$$\overrightarrow{QR} = \mathrm{Proj}_{\mathbf{v}}^{\overrightarrow{QP}} = \frac{\overrightarrow{QP}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\langle 4,0,1\rangle\cdot\langle -1,-1,0\rangle}{\langle -1,-1,0\rangle\cdot\langle -1,-1,0\rangle}\langle -1,-1,0\rangle = -2\langle -1,-1,0\rangle$$

Therefore

$$R = Q - 2\langle -1,-1,0\rangle = \langle 1,-1,0\rangle - 2\langle -1,-1,0\rangle = \langle 3,1,0\rangle$$

(b) find the distance from P to $\mathcal{L}$.

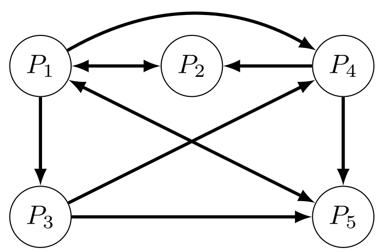
Solution: First, we need to find

$$\mathrm{Perp}_{\mathbf{v}}^{\overrightarrow{QP}} = \overrightarrow{QP} - \mathrm{Proj}_{\mathbf{v}}^{\overrightarrow{QP}} = \langle 4,0,1\rangle + 2\langle -1,-1,0\rangle = \langle 2,-2,1\rangle$$

The distance d is

$$d = \|\,\mathrm{Perp}_{\mathbf{v}}^{\overrightarrow{QP}}\,\| = 3$$

14. (6 points) Consider the directed graph below.



(a) Find the adjacency matrix M of the directed graph.

Solution: $M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) Find the total number of walks of length 2.

Solution:

$$M^2 = \begin{bmatrix} 2 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Therefore total the number of closed walks of length 2 is the sum of all entries in M^2, which is 19

15. (5 points) Determine if each of the following statements is true (T) or false (F). Do not justify. An incorrect answer will eliminate a correct one, so do not guess if you are not sure.

(a) __**T**__ If A is a square matrix, then $(A^T)^3 = (A^3)^T$.

(b) __**F**__ If $AB = AC$ and $A \neq 0$, then $B = C$.

(c) __**F**__ If $A \neq 0$, then $A^2 \neq 0$.

(d) __**F**__ Suppose a matrix A has repeated eigenvalues 7, 7, 7, so $\det(xI - A) = (x-7)^3$. Then A clearly cannot be diagonalized.

(e) __**F**__ Two diagonalizable matrices with the same eigenvalues must be the same matrix.