1. Solve each of the following systems or show that it is inconsistent.

(a) $\begin{cases} 2x - 3y - z = 3 \\ 10x - 3y + 8z = 27 \\ -x - 4y - 4z = -7 \end{cases}$

(b) $\begin{cases} 5x_1 - x_2 - 16x_3 + 3x_4 = 41 \\ -6x_1 + x_2 + 18x_3 - 3x_4 = -46 \\ 4x_1 - 8x_3 + 2x_4 = 28 \end{cases}$

2. How many solutions do each of the systems of equations associated with the following augmented matrices have? You do not need to state what the solutions are, if any.

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 2 & 3 & 7 & -4 & 5 \\ 0 & 1 & 5 & 3 & 0 \\ 0 & 0 & 10 & 8 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

3. Given the system $\begin{cases} x + 2z = 5 \\ x + y + hz = 0 \\ -x + y + 3z = 3k \end{cases}$

Find the value(s) of h and k for which the system has

(a) infinitely many solutions.

(b) no solution.

(c) a unique solution.

4. The rocky horror pharmacy performs some basic medical procedures for a small fee. Individuals can book an appointment with a 'nurse' to get blood drawn, receive a vaccination, or get their blood pressure checked. When drawing blood, the nurse will spend 2 minutes on the procedure and use 3 bandages. When administering a vaccine, the 'nurse' will spend 4 minutes on the procedure and use 8 bandages (yikes!). And because of some sub-par equipment, when taking a patient's blood pressure, the nurse will spend 3 minutes on the procedure and use 3 bandages. Yesterday, the 'nurse' spent a total of 108 minutes performing these medical procedures and used 186 bandages. Find all possible combinations of the three procedures (blood draws, vaccinations, and blood pressure checks) that might account for this use of time and bandages.

(a) Define your variables and set up the system. **Do not solve.**

(b) Given the reduced row echelon matrix to the system is $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 30 \\ 0 & 1 & \frac{-3}{4} & 12 \end{array}\right]$, find all realistic solutions.

5. Calculate the following or state that they do not exist. Given $A = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 5 \end{bmatrix}$

(a) A^{-1}

(b) A^2

(c) BA

(d) $3AC + C$

(e) the size of a matrix D such that $A^T C + BD$ exists.

6. Given that all matrices are 5×5 and are invertible. Solve for X.

$$AXB + B = A$$

7. Given that $\det(C) = 0$, $\det(D) = -2$, $\det(E) = 2$, and that all are 3×3 matrices. Evaluate or state that there is not enough information for the following.

(a) $\det(2E^{-1}D)$

(b) $\det(DC + CE)$

8. Given $A = \begin{bmatrix} 1 & 0 & 3 \\ 5 & 1 & -2 \\ -3 & 1 & 0 \end{bmatrix}$ find A^{-1}, by using the adjoint of A.

9. Given $\begin{vmatrix} 1 & 3 & 1 & 5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 4 & -3 \\ 0 & 4 & 0 & 5 \end{vmatrix} = -32.$

Use Cramer's Rule to solve for x_1 only in the system with the following augmented matrix. $\left[\begin{array}{cccc|c} 1 & 3 & 1 & 5 & 2 \\ 0 & 1 & -1 & 4 & 4 \\ 0 & 0 & 4 & -3 & 0 \\ 0 & 4 & 0 & 5 & -3 \end{array}\right]$

10. Given the points $P_1(1,3,2), P_2(3,-1,4), P_3(3,5,-2)$ and $P_4(1,3,-2)$

(a) Evaluate $||\overrightarrow{P_1P_2}||$

(b) Find the **general (standard)** equation of the plane with points P_1, P_2 and P_3

(c) Is P_4 on the plane?

11. Find the equation of a plane that contains the point $(-8,3,4)$ and is **parallel** to both of the lines

$\mathcal{L}_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} t$ and $\mathcal{L}_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} t$.

12. Provided that $\left[\begin{array}{ccc|c} -5 & -2 & 7 & -29 \\ 3 & 1 & -5 & 17 \end{array}\right]$ reduces to $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 4 & 2 \end{array}\right]$

(a) Write the vector equation of the intersection of the planes $-5x - 2y + 7z = -29$ and $3x + y - 5z = 17$.

(b) Is the intersection found in part (a) a point, line, a plane or all of $\mathbb{R}^3$?

13. Given $\mathcal{P}_1 : 8x - 12y + 10z = 2$, $\mathcal{P}_2 : 6x - 9y + kz = 7$

(a) Find the value of k that makes $\mathcal{P}_1$ and $\mathcal{P}_2$ parallel, if possible.

(b) Find a **unit vector** that is perpendicular to $\mathcal{P}_1$.

14. Given $\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}, \overrightarrow{v_3} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}, \overrightarrow{v_4} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}, \overrightarrow{v_5} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

(a) Which of the sets are Linearly Independent / Linearly Dependent. **Justify.**

i. $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$

ii. $\{\overrightarrow{v_1}, \overrightarrow{v_4}\}$

iii. $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_5}\}$

iv. $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_5}\}$

(b) Is span$(\{\overrightarrow{v_1}, \overrightarrow{v_2}\})$ a point, a line, a plane or all of $\mathbb{R}^3$?

(c) Is span$(\{\overrightarrow{v_1}, \overrightarrow{v_4}\})$ a point, a line, a plane or all of $\mathbb{R}^3$?

(d) Does span$(\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}) =$ span$(\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \overrightarrow{v_4}\})$? **Justify.**

15. Let $A = \begin{bmatrix} 1 & 1 & 1 & -2 & 3 & 9 \\ -4 & 0 & a & 8 & 3 & 9 \\ 3 & 5 & 7 & -6 & 1 & 3 \\ 1 & 3 & 5 & -2 & b & -6 \end{bmatrix}$ where the columns of A are $\overrightarrow{a_1}, \overrightarrow{a_2}$....

Furthermore A reduces to $\begin{bmatrix} 1 & 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Give a basis for Col(A).

(b) Give a basis for the **null space** of A. What is the **nullity** of A?

(c) Write $\overrightarrow{a_3}$ as a linear combination of $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$, if possible.

(d) Find the values of a and b.

16. If A is a 6×7 matrix, and the general solution to $A\overrightarrow{x} = \overrightarrow{b}$ has 2 free variables.

(a) What is the rank of A?

(b) How many solutions does $A\overrightarrow{x} = \overrightarrow{0}$ have?

(c) What is the rank of A^T and nullity of A^T?

17. Let $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ be a vector in the nullspace of $\begin{bmatrix} k & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$. Find all possible values of k.

18. A simple economy is made up of two industries: water and power. It takes \$ 0.5 of water and \$ 0.3 of power to make \$ 1 of water. Also, it takes \$ 0.7 of water and \$ 0.4 of power to make \$ 1 of power.

(a) Find the consumption matrix associated with this economy.

(b) Which (if any) of the two industries are profitable? **Justify your answer**.

(c) What production schedule will satisfy an external demand for \$ 90 of water and \$ 270 of power?

(d) In the above scenario what is the internal consumption of each industry?

19. In the simple world of Sera, it is either sunny or cloudy. If it is currently sunny in Sera then there is a 40 % chance it will be sunny tomorrow. If it is currently cloudy in Sera then there is only a 30 % chance of being cloudy tomorrow.

(a) Find the transition matrix **and** the initial state vector, if it is currently sunny in Sera.

(b) If there is a 5 % chance that Sera will be sunny on Dec 20th, what is the probability it will be cloudy on Dec 22nd?

(c) Find a steady-state vector $\overrightarrow{q}$ for this situation. Give your final answer using simplified fractions.

(d) What is the probability that, on a day in the very distant future, Sera will have a sunny day?

A B C	D E F	G H I	J K L	M N O	P Q R	S T U	V W X	Y Z
1 2 3	4 5 6	7 8 9	10 11 12	13 14 15	16 17 18	19 20 21	22 23 24	25 26

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

20. Given the encryption matrix $A = \begin{bmatrix} 3 & 14 \\ 2 & 11 \end{bmatrix}$.

(a) Find the decryption matrix.

(b) Check your answer to part a by multiplying it with A.

(c) Decode the word: 'JQ YX QM'

21. State whether the following are 'True' or 'False'.

(a) If $A + B$ exists, then AB must exist.

(b) If A is a 4×4 matrix and the nullity of A is 2, then $\det(A)$=0.

(c) If $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ are vectors in $\mathbb{R}^3$ and $\overrightarrow{w} \in \text{span}\{\overrightarrow{u}, \overrightarrow{v}\}$ then the span $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ must be a plane.

(d) If A is a 3×3 matrix and $A\overrightarrow{x} = \overrightarrow{0}$ has infinitely many solutions, then the set of columns of A must be linearly independent.

ANSWERS:

1. (a) $(3, 1, 0)$

(b) $(5 + 2p, -4 - 6p, p, 4)$

2. (a) No solns

(b) One solution

3. (a) $h = 7$ and $k = -10/3$

(b) $h = 7$ and $k \neq -10/3$

(c) $h \neq 7$

4. (a) $\left[\begin{array}{ccc|c} 2 & 4 & 3 & 108 \\ 3 & 8 & 3 & 186 \end{array}\right]$

(b) $(30, 12, 0), (18, 15, 4)$ and $(6, 18, 8)$

5. (a) $\dfrac{1}{14}\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 16 & 4 \\ 3 & 13 \end{bmatrix}$

(c) not possible

(d) $\begin{bmatrix} 67 & 14 & 81 \\ -1 & 18 & 17 \end{bmatrix}$

(e) 1×3

6. $X = B^{-1} - A^{-1}$

7. (a) -8

(b) not enough information

8. $A^{-1} = \dfrac{1}{26}\begin{bmatrix} 2 & 3 & -3 \\ 6 & 9 & 17 \\ 8 & -1 & 1 \end{bmatrix}$

9. $\dfrac{-1}{2}$

10. (a) $2\sqrt{6}$

(b) $x + y + z = 6$

(c) No

11. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} s + \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} t + \begin{bmatrix} -8 \\ 3 \\ 4 \end{bmatrix}$

12. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} s + \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$

(b) A line

13. (a) $k = \frac{15}{2}$

(b) $\dfrac{1}{\sqrt{308}}\begin{bmatrix} 8 \\ -12 \\ 10 \end{bmatrix}$

14. (a)
i. LI
ii. LD
iii. LI
iv. LD

(b) Plane

(c) Line

(d) Yes, both span $\mathbb{R}^3$

15. (a) $\left\{ \begin{bmatrix} -1 \\ -4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ b \end{bmatrix} \right\}$ or equivalent.

(b) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \right\}$ or equivalent.
nullity is 3.

(c) $\overrightarrow{a_3} = -\overrightarrow{a_1} + 2\overrightarrow{a_2}$

(d) $a = 4, b = -2$

16. (a) 5

(b) Infinite

(c) rank of A^T is 5 and nullity of A^T is 1?

17. $k = 2$.

18. (a) $C = \begin{bmatrix} 0.5 & 0.7 \\ 0.3 & 0.4 \end{bmatrix}$

(b) Water

(c) Water \$ 2700, Power \$ 1800

(d) Water \$ 2610, Power \$ 1530

19. (a) $P = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) 50.55 %

(c) $q = \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix}$

(d) $7/13 \approx 53.85$ %

20. (a) $A^{-1} = \begin{bmatrix} 23 & 18 \\ 10 & 11 \end{bmatrix}$

(b) $A \times A^{-1} = I$

(c) PASTA

21. (a) False

(b) True

(c) False

(d) False