1. (8 points) Give the solution set for each of the following systems, or indicate that no solution exists, as appropriate.

(a) $\begin{cases} 6x_1 + 4x_2 - 8x_3 + 6x_4 = 24 \\ 3x_1 + 4x_2 - 2x_3 + 6x_4 = 21 \\ 2x_1 + 3x_2 - x_3 + 2x_4 = 3 \end{cases}$

(b) $\begin{cases} 3x_1 - x_2 + 7x_3 = -11 \\ -2x_1 + x_2 - 5x_3 = 8 \\ 3x_1 + 2x_2 + 4x_3 = -10 \\ -2x_1 + 2x_2 - 6x_3 = 10 \end{cases}$

2. (6 points) For the system $\begin{cases} x_1 + 5x_3 = -2 \\ -x_1 + 3x_2 + x_3 = 8 \\ x_1 + kx_2 + 12x_3 = h \end{cases}$, find the value(s) of h and k for which the system has

(a) Infinitely many solutions.

(b) No solution.

(c) A unique solution.

3. (3 points) The Funky Fruit Smoothie Company is producing smoothies out of mango, banana and orange. To produce one Bahama smoothie, it takes 6 mangos, 7 bananas, and 5 oranges. To produce one Miami smoothie, it takes 3 mangos, 2 bananas, and 1 orange. Finally, to make a Venezuela smoothie, it takes 2 bananas and 2 oranges. The company has 24 mangos, 46 bananas and 38 oranges on hand.

(a) Set up a linear system to determine the numbers of Bahama, Miami, and Venezuela smoothies that can be produced in order to use up all the ingredients. **Do not solve** this system.

(b) Assuming that the solution to this system is $\begin{cases} x_1 = 10 - \frac{2}{3}t \\ x_2 = -12 + \frac{4}{3}t \\ x_3 = t \end{cases}$, $t \in \mathbb{R}$, and knowing that only complete smoothies can be produced, determine all the realistic solutions to this system.

4. (5 points) Consider $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 0 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & -4 \end{bmatrix}$. Find the following, or state that the calculation is undefined, as appropriate.

(a) $(CB)^{-1}$.

(b) $A^T C^T$.

(c) The matrix X for which $A^{-1}X = C$.

5. (3 points) Given $A = \begin{bmatrix} 1 & 2y \\ 4 & 0 \\ 5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x^2 & 5 \\ 2 & 0 & 1 \end{bmatrix}$, find all value(s), if any, of x and y so that AB is symmetric.

6. (5 points) An economy has two industries: Tics and Tacs.
To produce \$1 of Tics requires 20¢ of Tics and \$1 of Tacs. To produce \$1 of Tacs requires 10¢ of Tics and 70¢ of Tacs.

(a) Find the consumption matrix C associated with this economy.

(b) Which of the two industries are profitable? Justify your answer.

(c) Given an external demand for \$1400 of Tics and \$2800 of Tacs, how much of each industry should be produced to meet it?

(d) Find the internal consumption when demand is met.

7. (7 points) Let A, B, and C be 3×3 matrices. Assume A is non-invertible, $\det(B) = 5$, and $\det(C) = -\frac{4}{3}$. Find the following, or state that there is not enough information. Justify all of your answers by showing your work.

(a) $\det(3B^{-1}C^2)$

(b) $\det(AB + AC)$

(c) $\operatorname{rank}(B)$

(d) $\det(A + B)$

8. (6 points) The matrix $\begin{bmatrix} -3 & -5 & 4 & 8 \\ 1 & -1 & 2 & 9 \\ 6 & 2 & -2 & 9 \\ 9 & 13 & 0 & 8 \end{bmatrix}$ has a determinant of 16.
Use Cramer's Rule to solve for x_2 **only** in the system of linear equations

$$\left\{ \begin{array}{rcrcrcrcr} -3x_1 & - & 5x_2 & + & 4x_3 & + & 8x_4 & = & 2 \\ x_1 & - & x_2 & + & 2x_3 & + & 9x_4 & = & -4 \\ 6x_1 & + & 2x_2 & - & 2x_3 & + & 9x_4 & = & 4 \\ 9x_1 & + & 13x_2 & & & + & 8x_4 & = & 3 \end{array} \right\}$$

9. (2 points) Let A and B be an $n \times n$ matrices. Answer True or False. If False, explain your answer.

a) If $\det(A) = 0$, then the system of linear equations $AX = B$ must have no solution.

b) If $\det(AB) \neq 0$, then both A and B are necessarily invertible matrices.

10. (5 points) Consider the planes $\mathcal{P}_1 : -2x + y + 3z = 2$ and $\mathcal{P}_2 : 3x + hy + kz = 4$.

(a) Give the vector equation of a line through the origin that is orthogonal to the plane $\mathcal{P}_1$.

(b) Find possible values of h and k for which the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel, or state that no such values exist, as appropriate.

(c) Find one possible set of values of h and k for which the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are perpendicular, or state that no such values exist, as appropriate. (Note: Many correct answers exist.)

11. (6 points) Consider the points $P_1(2, 3, 5)$, $P_2(4, -2, 3)$ and $P_3(3, -4, 7)$.

(a) Find $\|\overrightarrow{P_1P_2}\|$

Question 11 continues on the next page.

(b) Find a vector equation of the plane containing the points P_1, P_2, and P_3.

(c) Find an equation of the plane containing the points P_1, P_2, and P_3 in general form $(ax + by + cz = d)$.

12. (3 points) Suppose A is $m \times n$ and that $\dim(\operatorname{Col}(A)) = 4$.

(a) Suppose that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. What is the value of n?

(b) Give the rank of A^T.

(c) Now suppose that the null space of A^T is a line through the origin. What is the value of m?

13. (1 point) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$.

Is $\mathbf{u}$ in $\operatorname{Nul}(A)$? Justify your answer.

14. (6 points) Given the vectors $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 8 \end{bmatrix}$,

(a) For which value(s) of k is vector $\begin{bmatrix} 4 \\ -2 \\ k \end{bmatrix}$ in $S = \operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?

(b) Find a basis for S.

(c) Describe $S = \operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. If the span is a line, give its equation in vector form. If the span is a plane, give its equation in general form $(ax + by + cz = d)$.

15. (3 points) Determine if the following set S is a subspace of $\mathbb{R}^3$. Justify your answer.

$$S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; z = x^3 \right\}.$$

16. (3 points) Given that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5 \;\middle|\; \begin{cases} x_1 = c \\ x_2 = e \\ x_3 = g, \text{with } c, e, g, p \in \mathbb{R} \\ x_4 = e \\ x_5 = p \end{cases} \right\}$ is a subspace of $\mathbb{R}^5$,

(a) Find a basis for S.

(b) What is the dimension of S?

17. (2 points) Given $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$, and $\mathbf{u_4}$ vectors from $\mathbb{R}^n$, fill in the blanks with the appropriate word from the following list: **MUST, MIGHT** or **CANNOT**.
If $S = \operatorname{Span}\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\} = \operatorname{Span}\{\mathbf{u_1}, \mathbf{u_4}\}$, then

(a) $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ ____________ be linearly independent.

(b) $\mathbf{u_3}$ ____________ be a linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$.

18. (8 points) Given the vectors $\mathbf{u_1} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{u_2} = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{u_3} = \begin{bmatrix} -12 \\ 0 \\ -9 \end{bmatrix}$, $\mathbf{u_4} = \begin{bmatrix} 24 \\ 10 \\ 13 \end{bmatrix}$ and $\mathbf{u_5} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and the fact that the matrix R below is the reduced row echeclon form of the matrix A, answer the following questions.

$$A = \begin{bmatrix} 4 & 8 & -12 & 24 & 0 \\ 0 & 2 & 0 & 10 & 1 \\ 3 & 5 & -9 & 13 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find a unit vector parallel to $\mathbf{u_1}$.

(b) Express the vector $\mathbf{u_4}$ as a linear combination of the vectors $\mathbf{u_1}$ and $\mathbf{u_2}$.

(c) Express the vector $\mathbf{u_2}$ as a linear combination of the vectors $\mathbf{u_1}$ and $\mathbf{u_4}$.

(d) Determine whether each of the following set is linearly independent or linearly dependent.

i. $\{\mathbf{u_1}, \mathbf{u_2}\}$

ii. $\{\mathbf{u_2}, \mathbf{u_4}, \mathbf{u_5}\}$

(e) Give a basis for $\text{Nul}(A)$.

19. (6 points) John got a message from his super paranoid mom about where to meet.
Given a Hill 2-cipher with encryption matrix $A = \begin{bmatrix} 11 & 1 \\ 5 & 2 \end{bmatrix}$, decrypt the following message to figure out what he should do:

NMYRRW

You may find the following table of multiplicative inverses mod (26) helpful:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

20. (6 points) A small vegetarian sandwich shop serves only two kinds of sandwiches: falafel and tofu. The shop observes that if a customer orders a falafel sandwich, there is a 70% chance that she will order a falafel sandwich on their next visit. If the customer orders a tofu sandwich, there is a 40% chance that they will order a falafel sandwich on their next visit.

(a) Give a transition matrix P associated with this situation.

(b) Sally goes to the sandwich shop once a week. If she ordered a falafel sandwich 2 weeks ago, what is the probability that she will order a tofu sandwich this week?

(c) Find a steady state vector associated with the matrix P from part (a). Your answer should be given using fractions.

21. (6 points) Use the Simplex Method to find a basic feasible solution that maximizes $z = 2x+7y$ subject to the following constraints:

$$\begin{cases} -4x + 2y \leq 8 \\ -2x + 4y \leq 40 \\ 2x + y \leq 5 \\ x, y \geq 0 \end{cases}$$

 Question 21 continues on the next page.

ANSWERS

1. (a) $\{x_1 = 1+2t, x_2 = -3-t, x_3 = t, x_4 = 5\}$ (b) No solution
2. (a) $k=\frac{7}{2}$ and $h=5$ (b) $k=\frac{7}{2}$ and $h \neq 5$ (c) $k \neq \frac{7}{2}$ and h can have any value
3. (a) $\begin{cases} 6x + 3y + 0z = 24 \\ 7x + 2y + 2z = 46 \\ 5x + y + 2z = 38 \end{cases}$ (b) $\{x=4$ Bahamas, $y=0$ Miamis, $z=9$ Venezuelas$\}$, $\{x=2$ Bahamas, $y=4$ Miamis, $z=12$ Venezuelas$\}$, and $\{x=0$ Bahamas, $y=8$ Miamis, $z=15$ Venezuelas$\}$
4. (a) $\begin{bmatrix} 10/39 & 1/13 \\ 1/26 & -1/26 \end{bmatrix}$ (b) undefined (c) $\begin{bmatrix} -1 & 0 & 4 \\ 7 & 6 & -12 \end{bmatrix}$
5. $x = \pm 2, y = 1$
6. (a) $C = \begin{bmatrix} 0.20 & 0.10 \\ 1.00 & 0.70 \end{bmatrix}$ (b) Tics are not profitable, as they spend \$1.20 to produce \$1 of output. Tacs are profitable, as they spend only 80¢ to produce \$1 of output. (c) \$5000 of Tics and \$26000 of Tacs should be produced. (d) The economy consumes \$3600 of Tics and \$23200 of Tacs.

7. (a) $\frac{48}{5}$ (b) 0 (c) 3 (d) not enough information
8. -140
9. (a) False. (b) True.
10. (a) $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ (b) $h = \frac{-3}{2}, k = \frac{-9}{2}$ (c) $h=3, k=1$ (multiple answers possible)
11. (a) $\sqrt{33}$ (b) $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + s \begin{bmatrix} 2 \\ -5 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}$ (c) $8x+2y+3z=37$
12. (a) $n=4$ (b) 4 (c) $m=5$
13. Yes.
14. (a) $k=\frac{22}{3}$ (b) $\{\mathbf{u}, \mathbf{v}\}$ (other answers possible) (c) S is a plane with equation $5x-y-3z=0$.
15. S is not a subspace since it is not closed under addition, nor is it closed under scalar multiplication. (There exist many possible counter-examples that can be provided as justification in each case, but only one counter-example to one of these two properties is necessary in order to obtain full marks.)
16. (a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (b) 4
17. (a) CANNOT (b) MIGHT
18. (a) $\begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix}$ (b) $\mathbf{u_4} = -4\mathbf{u_1} + 5\mathbf{u_2}$ (c) $\mathbf{u_2} = \frac{4}{5}\mathbf{u_1} + \frac{1}{5}\mathbf{u_4}$ (d) (i) linearly independent, (ii) linearly independent (e) $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$
19. GO HOME

20. (a) $P = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$ (b) 39% (c) $\begin{bmatrix} 4/7 \\ 3/7 \end{bmatrix}$

21. Max $z = 32$ when $\{x = \frac{1}{4}, y = \frac{9}{2}, s_1 = 0, s_2 = \frac{45}{2}, s_3 = 0\}$