1. [8 points] Solve the following systems or show that the system has no solution. Give parametric solutions where applicable.

   a. $\begin{cases} 2x - 3y + 4z = 2 \\ 4x + y + 2z = 2 \\ x - y + 3z = 3 \end{cases}$

   b. $\begin{cases} x_1 - 2x_2 + x_3 + 8\,x_4 = 3 \\ -2x_1 + 4x_2 + 3x_3 - \,x_4 = 29 \\ 3x_1 - 6x_2 + 2x_3 + 21\,x_4 = 2 \end{cases}$

2. [4 points] David makes plush animals from rags. A bat requires 20 grams of stuffing and 2 rags. A giant snake requires 240 grams of stuffing and 26 rags. A kangaroo requires 80 grams of stuffing and 9 rags. David has 1560 grams of stuffing and 174 rags. He wants to determine how many of each plush animal he can make if he uses all of his stuffing and rags.

   a. Define your variables and set up the system. **Do not solve**.

   b. Given the general solution to the system as $(-30 + 2t, 9 - \frac{1}{2}t, t\ )$, find all realistic solutions.

3. [4 points] Consider the linear system given by the following augmented matrix

   $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 2 & -1 & h+1 \\ 3 & 4 & k^2+1 & 3 \end{array}\right]$ For what value(s) of h and k will the system have:

   a. a unique solution

   b. no solution

   c. infinitely many solutions

4. [6 points] Given $A = \begin{bmatrix} 2 & 7 \\ -1 & 3 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -3 & 2 \\ -1 & 5 & 8 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -5 \\ -6 & 15 \end{bmatrix}$. Calculate the following, or state that they are undefined.

   a. AB

   b. $AC+B^T$

   c. BC

   d. $C^{-1}B$

5. [4 points] Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$, find:

   a. $Adj(A)$

   b. $Det(A)$

   c. A^{-1} by using your answers to a. and b.

6. [3 points] Given the linear system $\begin{cases} 4x_1 - x_2 + 3x_3 = 1 \\ 6x_1 + 2\,x_2 - x_3 = 0 \\ 3x_1 + 3x_2 + 2x_3 = -1 \end{cases}$, and given that $\begin{vmatrix} 4 & -1 & 3 \\ 6 & 2 & -1 \\ 3 & 3 & 2 \end{vmatrix} = 79$,

   use Cramer's Rule to solve for $\boldsymbol{x_2}$ **only**.

7. [6 points] Suppose that A, B and C are 5 × 5 matrices such that det(B) = 4, det(C) = −3, and that the linear system $A\vec{X} = \vec{D}$ has no solutions for some vector $\vec{D}$.

   Evaluate the following determinants, or state that there isn't enough information:

   a. det (B^3C^T)
   b. det $(2BC^{-1})$
   c. det (A)
   d. det (A + B)

8. [4 points] Let A= $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 15$, evaluate $\begin{vmatrix} a & 7 & b & c \\ 2g & 10 & 2h & 2i \\ 0 & 3 & 0 & 0 \\ 4a+d & 9 & 4b+e & 4c+f \end{vmatrix}$

9. [2 points] Let L be the vector equation of a line in R^2: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \end{bmatrix}$. For each equation below, determine if it is also a vector equation of the line L.

   a. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} + t\begin{bmatrix} 8 \\ 10 \end{bmatrix}$
   b. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

10. [5 points] Let P be the plane in R^3 whose standard form is $4x - y + kz = 11$ and let L be the line defined by the vector equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 3 \\ -19 \end{bmatrix} + t\begin{bmatrix} 1 \\ k \\ 10 \end{bmatrix}$.

    a. Give the value(s) of k, if any, for which the line L is parallel to the plane P.
    b. Give the value(s) of k, if any, for which the line L is perpendicular to the plane P.
    c. Give the value(s) of k, if any, for which the plane P contains the point (1, 1, 1).

11. [3 points] Consider a plane that contains the points P (7, 3, −1), Q (3, 2, 2), and R (6, −1, −2), give a standard equation (ax + by + cz = d) for the plane.

12. [3 points] Give a vector equation for the line of intersection of the planes $-x + 3y + 8z = 6$ and $4x + y + 7z = -11$.

13. [6 points] Let $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}-2\\-4\\-2\end{bmatrix}$ $and\ \vec{w} = \begin{bmatrix}1\\1\\7\end{bmatrix}$.

a. Is the set $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent or linearly dependent? Justify.

b. Express $\vec{w}$ as a linear combination of $\vec{u}\ and\ \vec{v}$, or state that it is not possible.

c. Circle the word below that describes Span $\{\vec{u}, \vec{v}, \vec{w}\}$:

POINT LINE PLANE R^2 R^3

d. Provide a basis for Span $\{\vec{u}, \vec{v}, \vec{w}\}$.

e. What is the dimension of Span $\{\vec{u}, \vec{v}, \vec{w}\}$?

14. [9 points] The matrix $A = \begin{bmatrix} 2 & -1 & a & 1 & -2 & 3 \\ 1 & -2 & b & 0 & 0 & 1 \\ 3 & 1 & c & 4 & -8 & 7 \\ -1 & 4 & d & 6 & -12 & 5 \end{bmatrix}$ reduces to $R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Let $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}, \overrightarrow{a_4}, \overrightarrow{a_5}, \overrightarrow{a_6}$ represents the columns of A.

a. Choose a basis for Col(A).

b. Find the values of a, b, c, and d.

c. Determine whether each set is linearly independent or linearly dependent:

i. $\{\overrightarrow{a_1}, \overrightarrow{a_4}, \overrightarrow{a_5}\}$

ii. $\{\overrightarrow{a_2}, \overrightarrow{a_4}, \overrightarrow{a_6}\}$

iii. $\{\overrightarrow{a_5}\}$

d. Choose a basis for Nul(A).

e. Let $\vec{v} = \begin{bmatrix}1\\2\\3\\1\\2\\3\end{bmatrix}$ and suppose that A$\vec{v} = \vec{w}$ for some vector $\vec{w}$. Give a general solution (in vector form) to A$\vec{x} = \vec{w}$.

15. [4 points] Suppose A is a 3 × 5 matrix, and that Col(A) is a plane in R^k.

a. What is the value of k?

b. What is the dim $\text{col}(A)$?

c. What is the rank of A?

d. What is nullity of A^T ?

16. [3 points] In the parts that follow, assume that all vectors are in R^3. Fill in each blank below with the appropriate word from the following list: MUST, MIGHT, or CANNOT.
 (each word may appear once, more than once, or not at all).
    a. If $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, then $\{\vec{u}, \vec{v}, \vec{w}\}$_____________ be a basis for R^3.
    b. If $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent, then $\{\vec{u}, \vec{v}\}$ _______________be linearly independent.
    c. If $col(A)$is linearly independent, and A = $[\vec{u}\ \vec{v}\ \vec{w}]$, then Nul(A) ___________contain only the zero vector.

17. [5 points] An economy has two industries: Razzle and Dazzle. To produce \$1 of Razzle requires \$0.60 of Razzle and \$0.40 of Dazzle. To produce \$1 of Dazzle requires \$0.30 of Razzle and \$0.50 of Dazzle.

    a. Find a consumption matrix C associated with this economy.
    b. Which industries, if any, are profitable? Justify your answer.
    c. Given an external demand for \$120 of Razzle and \$400 of Dazzle, how much of each industry should be produced to meet it?
    d. Find the internal consumption when the demand is met.

The following tables may prove useful in answering the Cryptography question:

The alphabet:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	0

Reciprocals mod 26:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

18. [6 points] Pat has written down the name of their crush but first encrypted it with a Hill 2-Cipher.
The encoding matrix used was $A = \begin{bmatrix} 3 & 2 \\ 1 & 11 \end{bmatrix}$ and the encoded name is MGBGGA.
    a. Find the decryption matrix.
    b. Decode the name of the crush.

19. [6 points] Like clockwork, Gary goes out to lunch only on the first day of the month. And he goes to either the Krusty Krab or the Chum Bucket. If he goes to the Krusty Krab, then there's a 60% probability that he'll return there the following month. If he goes to the Chum Bucket, then there's a 30% chance that he'll return there the following month.

    a. Draw the transition diagram
    b. Find the transition matrix associated with this situation.

c. If Gary went to the Chum Bucket in February, what is the probability that he'll eat there in April? be sure to provide the initial state vector as well as the state vector associated with the month of April.

d. Find a steady-state vector. In the long run, how often will Gary eat at the Krusty Krab?

20. [2 points] A furniture company makes three different types of furniture: sideboards, tables and chairs. Two machines are used in the production - a jigsaw and a lathe. The manufacture of a sideboard requires 1 hour on the jigsaw and 2 hours on the lathe. A table requires 4 hours on the jigsaw and none on the lathe. A chair requires 2 hours on the jigsaw and 8 hours on the lathe.
The jigsaw can only operate 100 hours per week and the lathe for 40 hours per week. The profit made on a sideboard is $30, $60 on a table and $40 on a chair. They want to determine how many of each type of furniture to produce in order to maximize profit. Give the objective function and list all constraints. Identify the variables. DO NOT SOLVE.

21. [5 points] Minimize and provide a feasible solution including all slack variables for
$p = -5x + y - 2z$ subject to

$$x + y - z \leq 60,$$
$$2y + 6z \leq 48,$$
$$2x - 3z \leq 140$$
$$x \geq 0, y \geq 0, z \geq 0$$

Answers

1)a. (x, y, z) = $(-1/2, 1, 3/2)$ b. $\begin{cases} x_1 = -4 + 2t - 5s \\ x_2 = t \\ x_3 = 7 - 3s \\ x_4 = s \end{cases}$

2) a. x= # bats, y= # snakes, z= # kangaroos. $\begin{cases} 20x + 240y + 80z = 1560 \\ 2x + 26y + 9z = 174 \end{cases}$ b. 2 bats, 1 snake, 16 kangaroos

OR 6 bats, 0 snakes, 18 kangaroos

3) a. unique solution if $k \neq \pm 3$ (for any h ∈ R) b. no solution if k = ±3 and $h \neq 8$ c. infinitely many solutions if k = ±3 and h = 8

4) a. $\begin{bmatrix} -7 & 29 & 60 \\ -3 & 18 & 22 \\ -4 & 20 & 32 \end{bmatrix}$ b. $\begin{bmatrix} -38 & 94 \\ -23 & 55 \\ -24 & 68 \end{bmatrix}$ c. $\begin{bmatrix} 6 & -15 \\ -36 & 90 \\ -44 & 110 \end{bmatrix}$ c. Not possible d. Undefined as det(C) = 0

5) a. $adj(\text{A}) = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -5 \\ 3 & -2 & 1 \end{bmatrix}$ b. $det(\text{A}) = -4$ c. $\text{A}^{-1} = -\frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -5 \\ 3 & -2 & 1 \end{bmatrix}$

6) −37/79

7)a. -192 b. $\frac{-128}{3}$ c.0 d. $not\ enough\ info$

8) 90

9) a) It is an equation of the line L. b) It is not an equation of the line L.

10) a. $k = \frac{-4}{9}$ b. no such k-value exists c. $k = 8$

11) 13x − 7y + 15z = 55

12)a. $\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}\right\}$

13) a. Linearly dependent since $\vec{u}$ and $\vec{v}$ are multiples. b. No, since $\vec{w}$ is not a multiple of $\vec{u}$ and $\vec{v}$.

c. PLANE d. $\{\vec{u}, \vec{w}\}$ or $\{\vec{v}, \vec{w}\}$ are two possibilities. e. 2

14)a. $\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_4}\}$ b. a = 0, b = −3, c = 5, d = 7 c. i. Linearly dependent ii. Linearly independent iii) Linearly independent

d. $\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ e. $X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} + r\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

15) a. k = 3 b. rank of A = 2 c. Rank(A)= 2 d. nullity of A^T= 1

16)a. Cannot b. Must c. Must

17)a. $C = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.5 \end{bmatrix}$ b. Only Dazzle is profitable. c. The industry should produce \$2250 of Razzle and \$2600 of Dazzle d) The internal consumption would be \$2130 of Razzle and \$2200 of Dazzle.

18) a. $A^{-1} = \begin{bmatrix} 23 & 10 \\ 5 & 11 \end{bmatrix}$ b. ELLIOT

19)a. $P = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$ b. 37% c. $q = \begin{bmatrix} 7/11 \\ 4/11 \end{bmatrix}$ d. 7/11

20) $x = number\ of\ sideboards, y = number\ of\ tables\ and\ z = number\ of\ chairs$ P= profit

Maximize $P = 30x + 60y + 40z$ subject to

$$\begin{cases} x + 4y + 2z \le 100 \\ 2x \qquad + 8z \le 40 \end{cases}$$

21) Optimal Solution: min p = -356; x = 68, y = 0, z = 8