1. (9 points) Let $A = \begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ 2 & 6 & -2 & 4 & 6 \\ 4 & 13 & -3 & 7 & 1 \end{bmatrix}$

   (a) Find a basis of $\mathrm{Col}(A)$

   (b) Find a basis of $\mathrm{Nul}(A)$.

   (c) Write 4^{th} column of A as a linear combination of 1^{st} and 3^{rd} column of A.

2. (4 points) Let A be an $n \times n$ matrix. Let I be the $n \times n$ identity matrix. Suppose $(I - A)^{-1} = I + A$. Prove $A^2 = 0$.

3. (4 points) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4. (6 points) Use linear algebra to balance the chemical equation: $NH_3 + Cl_2 \to NH_4Cl + N_2$

5. (7 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

   (a) Find $\det(A)$

   (b) Find $\mathrm{adj}(A)$

   (c) Find A^{-1} using the results from (a) and (b).

6. (4 points) Express A^{-1} of $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices and state the inverse of each of the elementary matrices you find.

7. (6 points) Let A and C be 3×3 matrices. It is given that A is symmetric, $\det(A) = -2$, and $\dim(\mathrm{Nul}(C)) = 1$.

   (a) Find $\det\left(\left(A + A^T\right)^2\right)$.

   (b) Find $\det(C + C^2)$.

8. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

   (a) Find the eigenvalues of A.

   (b) Find the eigenvectors of A.

   (c) Diagonalize A, that is, find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$.

9. (8 points) A force vector $\mathbf{F} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ N is applied to move an object along a displacement vector $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ m.

   (a) Compute the **work** W done by the force $\mathbf{F}$ in moving the object along $\mathbf{d}$.

   (b) Suppose the same force $\mathbf{F}$ is applied at a point located at position vector $\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ m from the origin (relative to a fixed pivot). Compute the **torque** $\boldsymbol{\tau}$ about the origin due to the force $\mathbf{F}$.

10. (10 points) Consider the vectors in $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

(a) Determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

(b) What is the dimension of the subspace generated by the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

(c) Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis of $\mathbb{R}^3$? Justify your answer.

11. (10 points) Define the lines $\mathcal{L}_1 : \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ and $\mathcal{L}_2 : \mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. It is known that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect.

(a) Find the intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Find an equation of the line that contains the point $(1, -1, 2)$ and is perpendicular to the plane that contains both $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Find an equation of the plane (in the form $ax + by + cz = d$) that contains both $\mathcal{L}_1$ and $\mathcal{L}_2$.

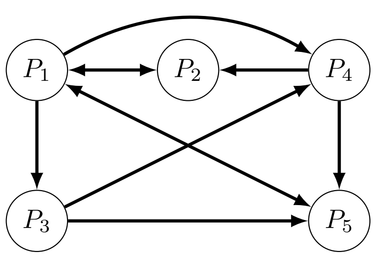
12. (4 points) Suppose two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ have the same length. Calculate $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$.

13. (7 points) Given a point $P(5, -1, 1)$ and a line $\mathcal{L} : \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$;

(a) find the point R on $\mathcal{L}$ closest to P;

(b) find the distance from P to $\mathcal{L}$.

14. (6 points) Consider the directed graph below.



(a) Find the adjacency matrix M of the directed graph.

(b) Find the total number of walks of length 2.

15. (5 points) Determine if each of the following statements is true (T) or false (F). Do not justify. An incorrect answer will eliminate a correct one, so do not guess if you are not sure.

(a) ____ If A is a square matrix, then $(A^T)^3 = (A^3)^T$.

(b) ____ If $AB = AC$ and $A \neq 0$, then $B = C$.

(c) ____ If $A \neq 0$, then $A^2 \neq 0$.

(d) ____ Suppose a matrix A has repeated eigenvalues 7, 7, 7, so $\det(xI - A) = (x - 7)^3$. Then A clearly cannot be diagonalized.

(e) ____ Two diagonalizable matrices with the same eigenvalues must be the same matrix.