1. Evaluate the following limits. Use $-\infty$, ∞, or "does not exist" wherever appropriate.

[3] (a) $\displaystyle\lim_{x\to -1} \frac{x^2-4x-5}{3x^2-2x-5}$

[3] (b) $\displaystyle\lim_{x\to 0^-} \frac{\sin(x)\cos(x)}{x^2}$

[3] (c) $\displaystyle\lim_{x\to 5^-} \frac{|2x-10|}{x^2-25}$

[3] (d) $\displaystyle\lim_{x\to \infty} \frac{6x+4\cos(x)}{3x}$

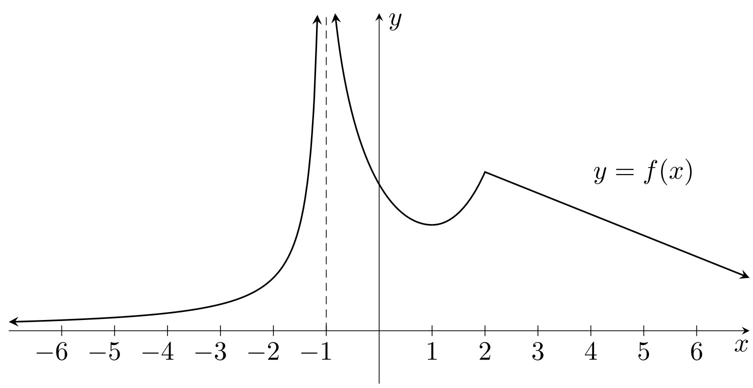
[4] **2.** Let $f(x) = \begin{cases} \dfrac{\dfrac{1}{x^2+1} - \dfrac{1}{5}}{x-2} & \text{if } x \neq 2 \\ a & \text{if } x = 2 \end{cases}$

(a) Find the value(s) of a which will make the function $f(x)$ continuous everywhere, or show that no such value exists.

(b) If $a = 2024$, what type of discontinuity does $f(x)$ have at $x = 2$ (infinite, removable or jump)?

[4] **3.** Find all horizontal asymptotes of the function $f(x) = \dfrac{5x^3-14x}{2x^3-7} + \arctan(3x)$.

4. Let $f(x)$ be the the function whose graph is shown below:



[1] (a) List the value(s) where f is nondifferentiable.

[2] (b) Put $f'(-4)$, $f'(-2)$, $f'(1)$ and $f'(4)$ in increasing order by filling in the blanks:

_______ < _______ < _______ < _______

[4] **5.** Let $f(x) = \sqrt{3x^2-9}$. Use the limit definition of the derivative to find $f'(x)$.

6. Find the derivative $\dfrac{dy}{dx}$ of each of the following. Do not simplify your answers.

[3] (a) $y = x^2 + 3^x + 4x + \dfrac{5}{x^2} + \dfrac{x}{6} + 7$

[3] (b) $y = \ln(x^3 + e^{\pi x})$

[3] (c) $y = \sqrt[5]{(2x^2+3)\sin(2x)}$

 Question 6 continues on the next page.

[3] (d) $y = \dfrac{\operatorname{arcsec}(x)}{\sec(x)}$

[3] (e) $y = x^{x+\tan(x)}$

[4] **7.** Let $f(x) = \dfrac{x^2 - 21}{x - 5}$. Find all values of x at which $f(x)$ has a horizontal tangent line.

[4] **8.** Find an equation of the tangent line to $f(x) = \dfrac{3}{x} - \dfrac{\pi}{4} + \arctan(2x)$ when $x = \dfrac{1}{2}$.

[4] **9.** Find the slope of the tangent line to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(3, -1)$.

[6] **10.** A hot air balloon rising vertically is tracked by an observer on the ground located 90 metres from the lift-off point. At a certain moment, the angle between the observer's line-of-sight and the horizontal is $\frac{\pi}{6}$ radians, and it is changing at a rate of $\frac{1}{40}$ radians per second. How fast is the balloon rising at this moment, in metres per second?

11. You are given the following function f and its first two derivatives:

$$f(x) = \frac{-x^2 - x + 2}{x^2 - 2}, \qquad f'(x) = \frac{x^2 + 2}{(x^2 - 2)^2}, \qquad f''(x) = \frac{-2x(x^2 + 6)}{(x^2 - 2)^3}$$

(a) Investigate each of the following:

[1] i. domain and intercepts

[2] ii. horizontal and vertical asymptotes

[2] iii. intervals of increase and decrease, and local extrema

[3] iv. intervals of upwards and downwards concavity, and points of inflection

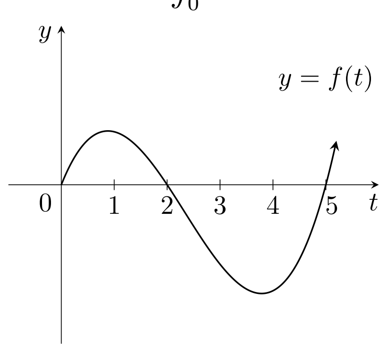
[2] (b) Sketch the graph of f, labelling all notable features.

[6] **12.** An aquaculture business wants to construct a cylindrical water tank with an open top. The volume of the tank needs to be 1000 m^3. The material for the base costs \$1 per square metre, and the material for the sides costs \$8 per square metre. What radius should the tank have in order to minimize the cost of the materials?

Note: the volume of a cylinder is $V = \pi r^2 h$, where r is the base radius and h is the height.

[4] **13.** Find $f(x)$ if $f(1) = \frac{2}{3}$ and $f'(x) = \sqrt{x} + \dfrac{1}{\sqrt{1 - x^2}}$.

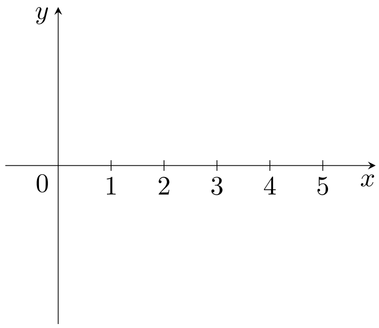
14. Let $g(x) = \displaystyle\int_0^x f(t)\,dt$, where $f(t)$ is the function pictured below:



[2] (a) Is $g(5)$ greater than $g(2)$, less than $g(2)$, or equal to $g(2)$? Explain.

 Question 14 continues on the next page.

[1] (b) Draw a rough sketch of $g'(x)$ on the interval $x \in [0, 5]$.



[4] **15.** Evaluate $\displaystyle\int_0^2 (2x + x^3)\, dx$ by using a limit of Riemann sums.

The following summation formulas are provided for reference:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

16. Evaluate the integrals.

[3] (a) $\displaystyle\int \frac{(2x+5)^2}{x}\, dx$

[3] (b) $\displaystyle\int \frac{\sin(x) + \cos^3(x)}{\cos^2(x)}\, dx$

[3] (c) $\displaystyle\int_{-8}^{1} \left|\sqrt[3]{x}\right|\, dx$

[4] **17.** Find the area of the region between the curve $y = 12 - 3x^2$ and the x-axis.

Answers:

1. (a) $\dfrac{3}{4}$ (b) $-\infty$ (c) $-\dfrac{1}{5}$ (d) 2

2. (a) $-\dfrac{4}{25}$ (b) removable

3. $y = \dfrac{5-\pi}{2}$ and $y = \dfrac{5+\pi}{2}$

4. (a) $x = -1, 2$ (b) $f'(4) < f'(1) < f'(-4) < f'(-2)$

5. $f'(x) = \lim\limits_{h\to 0} \dfrac{\sqrt{3(x+h)^2-9} - \sqrt{3x^2-9}}{h} = \dfrac{3x}{\sqrt{3x^2-9}}$

6. (a) $y' = 2x + 3^x \ln 3 + 4 - 10x^{-3} + \frac{1}{6}$ (b) $y' = \dfrac{3x^2 + \pi e^{\pi x}}{x^3 + e^{\pi x}}$

(c) $y' = \frac{1}{5}((2x^2+3)\sin(2x))^{-4/5}\,(4x\sin(2x) + 2(2x^2+3)\cos(2x))$ or, if you chose to use logarithmic differentiation, your unsimplified answer could be $y' = \sqrt[5]{(2x^2+3)\sin(2x)}\left(\dfrac{4x}{5(2x^2+3)} + \dfrac{2\cos(2x)}{5\sin(2x)}\right)$

(d) $y' = \dfrac{\dfrac{\sec x}{x\sqrt{x^2-1}} - \operatorname{arcsec} x\, \sec x\, \tan x}{\sec^2 x}$ or, if you chose to rewrite $\dfrac{1}{\sec x}$ as $\cos x$ before differentiating, your unsimplified answer could be $y' = \dfrac{\cos x}{x\sqrt{x^2-1}} - \operatorname{arcsec} x\, \sin x$

(e) $y' = x^{(x+\tan x)}\left[(1+\sec^2 x)\,\ln x + \dfrac{x+\tan x}{x}\right]$

7. $x = 3, 7$

8. $y = -11x + \frac{23}{2}$

9. $y' = \dfrac{9}{13}$

10. 3 m/s

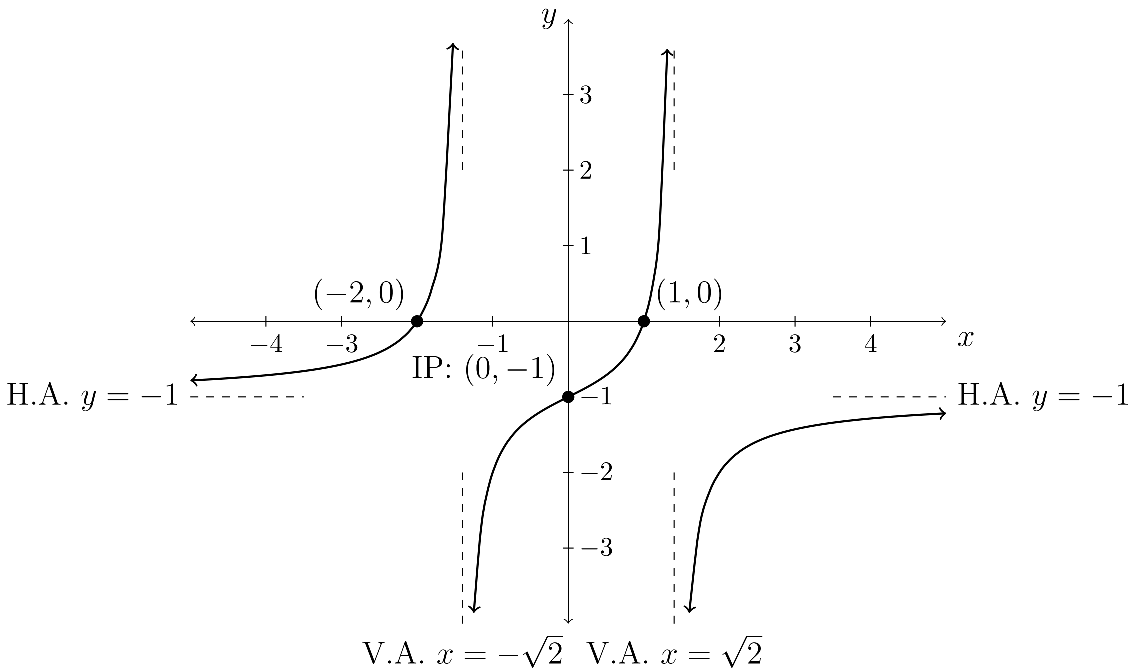
11. (a) i. Domain: $x \in (-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)$ Intercepts: $(-2,0), (1,0), (0,-1)$

ii. HA $y = -1$, VA $x = -\sqrt{2}$, $x = \sqrt{2}$

iii. f is increasing on its whole domain and has no local extrema

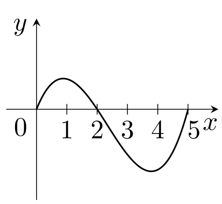
iv. C upwards on $x \in (-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$; C downwards on $x \in (-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$
IP at $(0, -1)$

(b)



12. $r = \dfrac{20}{\sqrt[3]{\pi}}$ m

13. $f(x) = \frac{2}{3}x^{3/2} + \arcsin x - \frac{\pi}{2}$

14. (a) $g(5) < g(2)$ (b)  since $g'(x) = f(x)$.

15. $\displaystyle\int_0^2 (2x + x^3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{4i}{n} + \frac{8i^3}{n^3}\right)\frac{2}{n}\right] = 8$

16. (a) $2x^2 + 20x + 25\ln|x| + C$ (b) $\sec x + \sin x + C$ (c) $\dfrac{51}{4}$

17. 32 units^2