1. (8 points) Solve the following systems, or show that there is no solution.

(a) $\begin{cases} 3x + y + 4z = 20 \\ 2x + 3y + 2z = 5 \\ 4x + 2y + 5z = 24 \end{cases}$

(b) $\begin{cases} x_1 + 3x_2 - 4x_4 = 5 \\ -2x_1 - 6x_2 + x_3 + 10x_4 = -10 \\ 6x_1 + 18x_2 - 2x_3 - 28x_4 = 30 \end{cases}$

2. (5 points) Find the value(s) of k and h, if any, for which the system with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & h \\ 8 & 14 & k^2-4k-15 & 2h+3 \end{array}\right]$$ has:

(a) infinitely many solutions

(b) no solution

(c) a unique solution

3. (4 points) The 'Knit Happens' association received a donation of 3 colours of yarn, from which they plan to knit cardigans, sweaters and dresses. They have 84 balls of Iceberg yarn, 72 of Nutmeg yarn and 24 of Charcoal yarn. Each cardigan requires 6 Iceberg, 3 Nutmeg and 3 Charcoal. Each sweater requires 5 Iceberg, 5 Nutmeg and 1 Charcoal. Each dress requires 6 Iceberg and 8 Nutmeg. Help the guild determine how many cardigans, sweaters and dresses they can make using all the available yarn.

(a) Define your variables and set up the system needed to determine the solution. **Do not solve.**

(b) Given the general solution to the system is $\left\{\begin{array}{lcl} x_1 & = & 4+\frac{2}{3}t \\ x_2 & = & 12-2t \\ x_3 & = & t \end{array}\right\}$, find all realistic solutions.

4. (6 points) Provided $A = \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 9 & 17 \\ 1 & 1 & 8 \end{bmatrix}$, $C = \begin{bmatrix} -6 & 0 & 1 \\ 3 & 4 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$.

Find the following, or explain why it is not possible, as appropriate:

(a) C^TB

(b) a matrix X such that $AX = D$

(c) $\det(AC)$

(d) B^{-1}

5. (2 points) Given $A = \begin{bmatrix} 1 & 8 & 2 \\ 5 & 5 & 3 \\ ? & ? & ? \end{bmatrix}$ and $\operatorname{adj}(A) = \begin{bmatrix} 10 & -16 & 14 \\ -13 & 4 & 7 \\ 5 & -8 & -35 \end{bmatrix}$

(a) Perform the matrix multiplication $A\ \operatorname{adj}(A)$.

(b) What is $\det(A)$?

6. (4 points) If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$, find

(a) $\begin{vmatrix} 3a & 3b & 3c \\ g+2a & h+2b & i+2c \\ -d & -e & -f \end{vmatrix}$

(b) $\begin{vmatrix} a+3d & b+3e & c+3f \\ a+3d+g & b+3e+h & c+3f+i \\ g & h & i \end{vmatrix}$

7. (3 points) State the following as True or False, if A is 5×5 and $\det(A) = 10$. Justify your answer.

(a) Then $\det(A^T) = \dfrac{1}{10}$.

(b) The columns of A are linearly independent.

(c) Then $\det(2A) = 20$.

8. (3 points) Provided $\begin{vmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 10 & 0 \\ 5 & 2 & 4 & -1 \\ 3 & 0 & -2 & 2 \end{vmatrix} = 120$, use Cramer's Rule to solve for x_3 **only**

in the following system $\begin{cases} x_1 + 2x_3 + 3x_4 = 2 \\ -x_1 + 10x_3 = 5 \\ 5x_1 + 2x_2 + 4x_3 - x_4 = 3 \\ 3x_1 - 2x_3 + 2x_4 = 3 \end{cases}$

9. (4 points) Given the lines $L_1 : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $L_2 : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.

(a) Are the lines parallel? Justify your answer.

(b) Are the lines perpendicular? Justify your answer.

10. (7 points) Let $A = (3, -2, 4)$ and $\mathbf{v} = (-2, 5, 1)$.

(a) Find the coordinates of point B such that $\overrightarrow{BA} = \mathbf{v}$.

(b) Find a vector equation for the line L through A that is parallel to $\mathbf{v}$.

(c) Is the point $C = (-1, 8, 3)$ on the line L (from part (b))? Justify your answer.

(d) Let $\mathbf{u} = (0, 2, 1)$. Find an equation in general form ($ax + by + cz = d$) for the plane P through point A with vectors $\mathbf{u}$ and $\mathbf{v}$ that lie in the plane.

(e) Find a unit vector that is orthogonal to the plane P.

11. (3 points) Give the equation, in vector form, of the line of intersection of the planes $x + 4y + 2z = 4$ and $3x + 13y + 5z = 10$.

12. (4 points) Given $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

(a) Determine if the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is Linearly Independent or Linearly Dependent. Justify your answer.

(b) What is $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$? Is it a point, a line, a plane or $\mathbb{R}^3$?

(c) True of False? $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is identical to $\text{Span}\{\mathbf{u}, \mathbf{w}\}$.

13. (5 points) Consider the sets S_1 and S_2 described below:

$$S_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{lrcrcrcl} & x & + & y & - & 3z & = & 0 \\ \& & 2x & - & 3y & + & 14z & = & 0 \end{array} \right\}, \; S_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 = z^2 \right\}$$

Only one of the two sets above is a subspace.

(a) Which of the two sets (S_1 or S_2) is a subspace?
Justify your answer by providing a basis for that set.

(b) Which of the two sets (S_1 or S_2) is NOT a subspace?
Justify your answer by providing a counter-example to one of the properties of subspaces.

14. (5 points) Given that $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; x_1 + 2x_3 = x_2 + 2x_4 \right\}$ is a subspace of $\mathbb{R}^4$,

(a) Find a basis for S.

(b) What is the dimension of S?

15. (2 points) Provide an example of a matrix **in RREF** whose column space is 3-dimensional and whose null space is 2-dimensional, or state that no such matrix exists, as appropriate.

16. (1 point) Let $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ be a vector in the null space of the matrix $A = \begin{bmatrix} k & 5 & -1 \\ 4 & -2 & 0 \end{bmatrix}$. Find all possible values of k.

17. (1 point) Given $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, is $\mathbf{v}$ in $\operatorname{Span}\{\mathbf{u}, \mathbf{u}+\mathbf{v}\}$? Justify your answer.

18. (2 points) Find all values of k such that the vectors $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ k \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

19. (9 points) Consider the matrix A below, as well as its reduce row echelon form, R:

$$A = \begin{bmatrix} 3 & 9 & 5 & 11 & 3 \\ 2 & 6 & 3 & 7 & 1 \\ -4 & -12 & 1 & -7 & 19 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 3 & 0 & 2 & -4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Which, if any, of the following sets are valid bases for Col A?

(i) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ (ii) $\left\{ \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 19 \end{bmatrix} \right\}$

(b) What description best describes Col(A): a point, a line, a plane, or all of $\mathbb{R}^3$?

(c) Write $\begin{bmatrix} 3 \\ 1 \\ 19 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$.

(d) Find a basis for Nul(A).

(e) Give the rank of the transpose of A. In other words, find rank(A^T).

(f) Give the nullity of A^T.

20. (5 points) A simple economy is made up of two industries: power and electronics. It takes 30¢ worth of power and 10¢ worth of electronics to make 1\$ of power. Also, it takes 80¢ worth of power and 60¢ worth of electronics to make 1\$ of electronics.

(a) Find the consumption matrix associated with this economy.

(b) What production schedule will satisfy an external demand for 3000\$ of power and 250\$ of electronics?

(c) When the external demand for 3000\$ of power and 250\$ of electronics is satisfied, what is the internal consumption of each industry?

(d) TRUE or FALSE: In this economy, it would be possible to satisfy ANY external demand. Justify your answer.

21. (6 points) Elaine is trying to eat healthier and has lunch every day at the same cafeteria. If she has a hamburger on any given day, then there is a 80% probability that she will have a salad the following day. Meanwhile, if she has a salad on any given day, then there is a 30% chance that she will have a hamburger the following day.

(a) Find the transition matrix P associated with this situation.

(b) Today is Friday. If Elaine had a salad on Wednesday, what is the probability that she will have a hamburger today?

(c) Find a steady-state vector $\mathbf{q}$ for this situation. Give your final answer using simplified fractions.

(d) What is the probability that, on a day in the very distant future, Elaine will be eating a salad for lunch?

22. (6 points) Minimize the value of $z = 0x_1 + 1x_2 - 2x_3$,

subject to the constraints $\left\{ \begin{array}{rcrcrcr} 2x_1 & + & 39x_2 & + & x_3 & \leq & 11 \\ -2x_1 & + & 7x_2 & + & x_3 & \leq & 3 \\ -3x_1 & + & 5x_2 & + & x_3 & \leq & 2 \\ x_1 & \geq & 0, & x_2 & \geq 0, & x_3 & \geq\ 0 \end{array} \right\}$.

Answer this question using the Simplex Method and include a basic feasible solution, or indicate that the minimum is unbounded, if necessary.

23. (5 points) You are an international spy and you find the following message from the home office in a dropbox: **GXGNJU**. For this mission, all messages to and from the home office were encoded using a Hill 2-cipher with the encryption matrix $A = \begin{bmatrix} 3 & 1 \\ 3 & 8 \end{bmatrix}$. Decode the message to find the city to which you must fly next. You may find the following table of multiplicative inverses mod (26) helpful:

a	1	3	5	7	9	11	15	17	19	21	23	25
a^{-1}	1	9	21	15	3	19	7	23	11	5	17	25

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

ANSWERS

1. (a) $\{x = 5, y = -3, z = 2\}$ (b) $\{x_1 = 5 - 3s + 4t, x_2 = s, x_3 = -2t, x_4 = t\}$

2. (a) $k = 7,\ -3$ AND $h = 29/12$ (b) $k = 7,\ -3$ AND $h \neq 29/12$ (c) $k \neq 7,\ -3;\ h \in \mathbb{R}$

3. (a) $\left\{\begin{array}{ccccccc} 6x_1 & + & 5x_2 & + & 6x_3 & = & 84 \\ 3x_1 & + & 5x_2 & + & 8x_3 & = & 72 \\ 3x_1 & + & x_2 & + & & = & 24 \end{array}\right\}$

(x_1 : number of cardigans, x_2 : number of sweaters, x_3 : number of dresses)

(b) $\{x_1 = 4, x_2 = 12, x_3 = 0\}$ or $\{x_1 = 6, x_2 = 6, x_3 = 3\}$ or $\{x_1 = 8, x_2 = 0, x_3 = 6\}$

4. (a) $C^T B$ is undefined (b) $X = \begin{bmatrix} 19 \\ 9 \end{bmatrix}$ (c) $\det(AC)$ is undefined since AC is not square.

(d) B^{-1} does not exist.

5. (a) $\begin{bmatrix} -84 & 0 & 0 \\ 0 & -84 & 0 \\ 0 & 0 & -84 \end{bmatrix}$ (b) $\det(A) = -84$

6. (a) 18 (b) 0

7. (a) False, $\det(A^T) = 10$ (b) True since A is invertible. (c) False, $\det(2A) = 320$

8. $x_3 = 2/3$

9. (a) No, the direction vectors are not proportional.

(b) No, the dot product of the direction vectors is not zero.

10. (a) $B = \begin{bmatrix} 5 \\ -7 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + t \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ (c) No. Different values for t.

(d) $-3x - 2y + 4z = 11$ (e) $(3/\sqrt{29}, 2/\sqrt{29}, -4/\sqrt{29})$

11. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 1 \\ 1 \end{bmatrix}$

12. (a) LD (b) a plane (c) True ($\mathbf{v} = \mathbf{u} + \mathbf{w} \in \text{Span}\{\mathbf{u}, \mathbf{w}\}$)

13. (a) S_1 is a subspace with basis $\left\{ \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \right\}$

(b) S_2 is not a subspace since it is not closed under addition (many counter-examples possible).

14. (a) Many answers possible, like $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (b) 3

15. Any matrix in RREF with 3 pivots and 2 free variables will do.

16. $k = -7$

17. yes, $\mathbf{v} = (-1)\mathbf{u} + 1(\mathbf{u} + \mathbf{v})$

18. $k \neq 1/3$

19. (a) (ii) (b) a plane (c) $\begin{bmatrix} 3 \\ 1 \\ 19 \end{bmatrix} = -4 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ (d) $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

(e) 2 (f) 1

20. (a) $C = \begin{bmatrix} 0.30 & 0.80 \\ 0.10 & 0.60 \end{bmatrix}$ (b) 7000\$ of power, 2375\$ of electronics

(c) 4000\$ of power, 2125\$ of electronics (d) True because the economy is productive.

21. (a) $P = \begin{bmatrix} 0.20 & 0.30 \\ 0.80 & 0.70 \end{bmatrix}$ (b) 27% (c) $\mathbf{q} = \begin{bmatrix} 3/11 \\ 8/11 \end{bmatrix}$ (d) 8/11

22. Min $z = -14$ at $(2, 0, 7, 0, 0, 1)$

23. Dubai