[Marks]

(10) 1. Let $A = \begin{bmatrix} 1 & 3 & -1 & 2 & 3 \\ 2 & 6 & -2 & 4 & 6 \\ 4 & 13 & -3 & 7 & 1 \end{bmatrix}$

(a) Find a basis of Col(A)

(b) Find a basis of Nul(A).

(c) Find a basis of Row(A).

(d) Write 4$^{\text{th}}$ column of A as a linear combination of 1$^{\text{st}}$ and 3$^{\text{rd}}$ column of A.

(4) 2. Let A be an $n \times n$ matrix. Let I be the $n \times n$ identity matrix. Suppose $(I - A)^{-1} = I + A$. Prove $A^2 = 0$.

(4) 3. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(6) 4. Use linear algebra to balance the chemical equation: $NH_3 + Cl_2 \to NH_4Cl + N_2$

(7) 5. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

(a) Find det(A)

(b) Find adj(A)

(c) Find A^{-1} using the results from (a) and (b).

(4) 6. Express A^{-1} of $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices and state the inverse of each of the elementary matrices you find.

(6) 7. Let A and C be 3×3 matrices. It is given that A is symmetric, $\det(A) = -2$, and $\dim(\text{Nul}(C)) = 1$.

(a) Find $\det\left(\left(A + A^T\right)^2\right)$.

(b) Find $\det(C + C^2)$.

(8) 8. Define a linear transformation $T : \mathbb{P}_3 \to \mathbb{R}^3$ by $T(p(x)) = \begin{bmatrix} p(1) \\ p(2) \\ 0 \end{bmatrix}$ for $p(x)$ in $\mathbb{P}_3$. Find a basis of the kernel of T.

(6) 9. Let R be the rotation in $\mathbb{R}^2$ by the angle $-\frac{5\pi}{4}$. Let S be the horizontal shear such that $S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(a) Find the standard matrix of R

(b) Find the standard matrix of S

(c) Find the standard matrix of $R \circ S$

(8) 10. Let $T : \mathbb{R}^n \to \mathbb{R}^4$ be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A such that $A\mathbf{x} = \mathbf{0}$ has only solution $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

[Marks]

(a) Find n.

(b) Find the size of A.

(c) Is T one-to-one?

(d) Is T onto?

(10) 11. Define the lines $\mathcal{L}_1 : \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ and $\mathcal{L}_2 : \mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. It is known that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect.

(a) Find the intersection of $\mathcal{L}_1$ and $\mathcal{L}_2$.

(b) Find an equation of the line that contains the point $(1, -1, 2)$ and is perpendicular to the plane that contains both $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Find an equation of the plane (in the form $ax + by + cz = d$) that contains both $\mathcal{L}_1$ and $\mathcal{L}_2$.

(4) 12. Suppose two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ have the same norm. Calculate $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$.

(6) 13. Define $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a^2 = d^2 \right\}$.

(a) Show that H is not closed under vector addition.

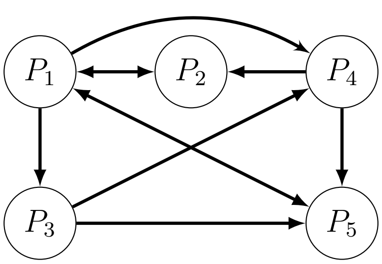
(b) Show that H is closed under scalar multiplication.

(7) 14. Given a point $P(5, -1, 1)$ and a line $\mathcal{L} : \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$;

(a) find the point R on $\mathcal{L}$ closest to P;

(b) find the distance from P to $\mathcal{L}$.

(6) 15. Consider the directed graph below.



(a) Find the adjacency matrix M of the directed graph.

(b) Find the total number of walks of length 2.

(4) 16. Determine if each of the following statements is true (T) or false (F). Do not justify.

(a) If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$. __________

(b) If A is a square matrix, then $(A^T)^3 = (A^3)^T$. __________

(c) If $AB = AC$ and $A \neq 0$, then $B = C$. __________

(d) If $A \neq 0$, then $A^2 \neq 0$. __________

[Marks]

Answers:

1. (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 13 \end{bmatrix} \right\}$ (not unique)

   (b) $\left\{ \begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -36 \\ 11 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ (not unique)

   (c) $\{\langle 1, 0, -4, 5, 36\rangle, \langle 0, 1, 1, -1, -11\rangle\}$ (not unique)

   (d) $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$

2. One has $(I - A)(I + A) = I$, hence $I - A^2 = I$, so $A^2 = 0$.

3. $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4. $8\,\mathrm{NH_3} + 3\,\mathrm{Cl_2} \to 6\,\mathrm{NH_4Cl} + \mathrm{N_2}$

5. (a) 2

   (b) $\begin{bmatrix} 4 & -2 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

   (c) $\frac{1}{2}\begin{bmatrix} 4 & -2 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

6. $A^{-1} = E_3 E_2 E_1$, where

$$E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix},\ E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

   We also have

$$E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ E_3^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

   The answer is not unique depending on the elementary row operations one chooses.

7. (a) 256

   (b) 0

8. $\{2 - 3x + x^2, 6 - 7x + x^3\}$ (not unique)

9. (a) $\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

10. (a) 3

    (b) 4×3

    (c) Yes because the kernel of T is trivial or the rank of A is 3 which is equal to the dimension of the domain of T.

    (d) No because the rank of A is not equal to the dimension of the codomain of T.

11. (a) $(0, 1, 3)$

    (b) $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$ (not unique)

    (c) $x + 5y - 2z = -1$

12. 0

13. (a) It is enough to provide a counter-example. Let $A_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \in H$ and $A_2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \in H$. One has $A_1 + A_2 = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \notin H$.

    (b) Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H$. Let λ be any real number. Then

$$\lambda A = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$$

    To prove $\lambda A \in H$, it is enough to show that $(\lambda a)^2 = (\lambda d)^2$, which is equivalent to $\lambda^2(a^2 - d^2) = 0$. Since $A \in H$, one must have $a^2 = d^2$, therefore $a^2 - d^2 = 0$, which implies $\lambda^2(a^2 - d^2) = 0$, and hence $\lambda A \in H$. So H is closed under scalar multiplication.

[Marks]

14. (a) $(3, 1, 0)$
    (b) 3

15. (a) $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$
    (b) 19

16. (a) T (b) T (c) F (d) F