Question 1: (*8 pts*)

Solve each of the following systems of equations, or show that it is inconsistent. If there are infinitely many solutions, provide the general solution.

a) $\begin{cases} x + y = 3 \\ 2x - y = 7 \\ 4x + y = 10 \end{cases}$ **b)** $\begin{cases} 2x_1 - 3x_2 + x_3 - x_4 = 1 \\ -x_1 + 2x_2 + x_4 = 1 \\ -x_2 + x_3 + x_4 = 3 \end{cases}$

Question 2: (*4 pts*)

A home baker sells three types of bread: plain, olive and raisin. One plain loaf is made using 2 ounces of sugar and is sold for 4 dollars. One olive loaf is made using 2 ounces of sugar and is sold for 5 dollars. Finally, a raisin loaf is made from 3 ounces of sugar and is sold for 8 dollars. Last week, the baker used a total of 34 ounces of sugar and earned a total of 75 dollars.

a) Define all the necessary variables and set up the system of equations to determine how many of each type of bread were sold last week. **Do not solve.**

b) Given that the augmented matrix for the system reduces as shown below, give the general solution for the system. $\left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 10 \\ 0 & 1 & 2 & 7 \end{array}\right]$

c) Find all possible realistic solutions.

Question 3: (*4 pts*)

Given the system of equations: $\begin{cases} x - y + kz = 2 \\ 3x + 7y + 2z = 7 \\ kx - ky + 25z = 3 \end{cases}$

Find all the value(s) of k (if any) such that the system has:

a) no solution. **b)** a unique solution. **c)** infinitely many solutions.

Question 4: (*3 pts*)

Given the 2×2 matrices $A = \begin{bmatrix} 1 & -3 \\ 3k-1 & k \end{bmatrix}$ and $B = \begin{bmatrix} k-1 & 5 \\ 1-2k & 0 \end{bmatrix}$, find all the value(s) of k (if any) so that $A + B$ is invertible.

Question 5: (*6 pts*)

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

a) Find adj(A). **b)** Use the adjoint found in **a)** to find A^{-1}.

Question 6: (*3 pts*)

Let A and B be invertible matrices. Solve for X in the equation $(A^T X + B)^T = B + B^T$.

Question 7: (*7 pts*)

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -5 & 3 \\ 2 & 1 & 0 \end{bmatrix}$. Find, or identify as undefined:

a) $(2I - BA)^T$ **b)** $AC + CB$ **c)** B^2

Question 8: (*7 pts*)

Suppose that $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ with $\det A = 5$, B is a 3×3 diagonal matrix with all $4'$s on the main diagonal, and C is a non-invertible 3×3 matrix. Evaluate each of the following, or state that there is not enough information to do so.

a) $\det \begin{bmatrix} g & h & i \\ 3g + 5d & 3h + 5e & 3i + 5f \\ 4a & 4b & 4c \end{bmatrix}$ **b)** $\det(2A^4B^{-1})$ **c)** $\det(A^TB + A^TC)$ **d)** $\det(AC + BC)$

Question 9: (*4 pts*)

A clothing company has three factories. Each day factory A can produce 1000 sweats and 1500 hoodies, factory B can produce 1300 sweats and 1800 hoodies, and factory C can produce 1500 sweats and 2000 hoodies. The company wants to close factory B. Is it possible to find some combination of the outputs of the other factories that will equal the output of factory B? If so, what is the combination?

Question 10: (*8 pts*)

Given the points $A(3, 1, 0)$, $B(0, 1, -2)$, and $C(1, 0, 3)$:

a) Calculate $\|\overrightarrow{AB}\|$.

b) Find a unit vector in the opposite direction of $\overrightarrow{AB}$.

c) Find parametric equations for the line passing through A and B.

d) Is the point $D(9, 1, 4)$ on the line obtained in part c)? Justify.

e) Find an equation in standard form for the plane $\mathcal{P}$ passing through the points A, B and C.

Question 11: (*6 pts*)

Given the plane $\mathcal{P} : 3x - y + 2z = 12$.

a) Find the coordinates of the x, y and z intercepts of $\mathcal{P}$, if any.

b) Find an equation in vector form for $\mathcal{P}$.

c) Is the line given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$ parallel, perpendicular, or neither to the plane $\mathcal{P}$?

Question 12: (*3 pts*)
Answer TRUE or FALSE for each part below and briefly justify:

a) If A is a 4×3 matrix, then $AX = 0$ must have only the trivial solution.

b) $\overrightarrow{0}$ can be expressed as a linear combination of the nonzero vectors $\overrightarrow{v_1}, \overrightarrow{v_2}$, and $\overrightarrow{v_3}$.

c) If A is an $n \times n$ matrix and $\det(A) = 0$, then $AX = B$ must be inconsistent.

Question 13: (*6 pts*)
In the country of Faraway, there are two internet companies: Fastlink and Slowpoke. Analysis has shown that if a person uses Fastlink for their internet, there is a 90% chance that they will use Fastlink again the following year. If a person uses Slowpoke for their internet, there is a 20% chance that they will use SlowPoke again the following year.

a) Write down a transition matrix P associated with this Markov chain.

b) If a person in Faraway uses Slowpoke for their internet this year, what is the probability that they will use Fastlink in two years?

c) Find the steady state vector $\overrightarrow{q}$ associated with the matrix P. Give your answers as fractions.

d) In the very distant future, what is the probability that someone in Faraway will use Slowpoke?

Question 14: (*5 pts*)
An open economy has two industries: Nuts and Bolts. The production of \$1 in Nuts requires \$0.40 in Nuts and \$0.20 in Bolts. The production of \$1 in Bolts requires \$0.30 in Nuts and \$0.70 in Bolts.

a) Which industries, if any, are profitable? Justify.

b) Find the production needed to meet an external demand of \$1200 for Nuts and \$1800 for Bolts.

c) What is the internal consumption when the external demand is met?

Question 15: (*3 pts*)
Consider the following matrices: $P = \begin{bmatrix} 0.2 & 0.45 \\ 0.8 & 0.65 \end{bmatrix}$ and $C = \begin{bmatrix} 0.5 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$.

a) Is P a regular transition matrix? Explain why or why not.

b) Suppose that matrix C is the consumption matrix for an open economy. Is the economy productive? Explain why or why not.

Question 16: (*5 pts*)
A closed economy produces fabric, plastic and metal. The production of \$1 of fabric requires \$0.60 of fabric, \$0.20 of plastic, and \$0.20 of metal. The production of \$1 of plastic requires \$0.50 of fabric, \$0.20 of plastic, and \$0.30 of metal. The production of \$1 of metal requires \$0.40 of fabric, \$0.10 of plastic, and of \$0.50 metal.

a) How much does each industry need to produce relative to each other in order for this economy to function?

b) If the economy produces \$12 000 worth of **plastic**, how much should be allocated to the other industries?

Question 17: (*6 pts*)
Use the Simplex Method to maximize $z = 10x_1 - 2x_2 + 8x_3$ subject to the constraints:

$$\begin{cases} 2x_1 & + & 2x_2 & + & x_3 & \leqslant & 6 \\ 2x_1 & + & 3x_2 & + & 3x_3 & \leqslant & 12 \\ -x_1 & + & x_2 & + & x_3 & \leqslant & 7 \\ x_1 & , & x_2 & , & x_3 & \geqslant & 0 \end{cases}$$

Your answer should include the maximum value of z, along with the corresponding feasible solution. If there is no maximum, identify an unbounded variable.

Question 18: (*6 pts*)
A micro-brewery produces two types of beer using wheat, malt and yeast, in different proportions. To produce one keg of the first type, they use 2kg of wheat, 16kg of malt and 100g of yeast. To produce one keg of the second type, they need 6kg of wheat, 9kg of malt and 200g of yeast. They currently have 240kg of wheat, 576kg of malt and 8200g of yeast available. The demand for their beer is high, so they will be able to sell everything. A keg of the first type sells for \$125 and a keg of the second type sells for \$90.

a) Define your variables and set up the linear program that would be used to determine the number of kegs of each type of beer the micro-brewery should produce in order to maximize their revenue. When defining your variables, please respect the order in which things were named in the problem. **DO NOT SOLVE.**

b) Give the initial simplex table, and circle the first entry that should be turned into a pivot. **DO NOT SOLVE.**

Here is the final simplex table corresponding to this problem. Use it to answer the following questions **(you should not be doing <u>any</u> calculations)**.

$$\left[\begin{array}{c|cc|c cc|c} 1 & 0 & 0 & 0 & 160/23 & 63/460 & 5130 \\ \hline 0 & 0 & 0 & 1 & 2/23 & -39/1150 & 12 \\ 0 & 1 & 0 & 0 & 2/23 & -9/2300 & 18 \\ 0 & 0 & 1 & 0 & -1/23 & 4/575 & 32 \end{array}\right]$$

c) How many kegs of each type should the micro-brewery produce in order to maximize their revenue, and what is the maximum revenue?

d) When revenue is maximized, which ingredients are not completely used, and how much is left?

Question 19: (*6 pts*)
Nadine wants to plan a surprise trip for her mom's birthday and will send the location of the trip as an encrypted message. She will encode her message with a Hill 2-cipher, using the encoding matrix $A = \begin{bmatrix} 7 & 25 \\ 3 & 6 \end{bmatrix}$.

a) Find the decryption matrix A^{-1}.

b) Multiply A and A^{-1} together to demonstrate that your decryption matrix is correct.

c) Nadine's ciphertext message to her mom is **HQCVFU**. Where will Nadine and her mom travel?

Answers:

1. a) Inconsistent b) $(-1+t,\ 0,\ 3-t,\ t),\ \ t\in\mathbb{R}$

2. a) $\begin{aligned} x_1 &= \text{\# of plain loaves} \\ x_2 &= \text{\# of olive loaves} \\ x_3 &= \text{\# of raisin loaves} \end{aligned}$ $\left\{\begin{array}{ccccccc} 2x_1 & + & 2x_2 & + & 3x_3 & = & 34 \\ 4x_1 & + & 5x_2 & + & 8x{+}_3 & = & 75 \end{array}\right.$

b) $(10+{}^1\!/\!_2\,t,\ 7-2t,\ t)$ c) $t=0\ \rightarrow(10,7,0)$ $t=2\ \rightarrow(11,3,2)$

3. a) $k=\pm5$ b) $k\neq\pm5$ c) Impossible.

4. $k\in\mathbb{R}\backslash\{0,2\}$

5. a) $\operatorname{adj}(A)=\begin{bmatrix}-1 & 3 & 7\\ 0 & 0 & 5\\ 2 & -1 & -4\end{bmatrix}$ b) $\det(A)=5$ $A^{-1}=\begin{bmatrix}{}^{-1}\!/\!_5 & {}^3\!/\!_5 & {}^7\!/\!_5\\ 0 & 0 & 1\\ {}^2\!/\!_5 & {}^{-1}\!/\!_5 & {}^{-4}\!/\!_5\end{bmatrix}$

6. $X\ =\ \left(A^T\right)^{-1}B^T$ or $\left(BA^{-1}\right)^T$

7. a) $\begin{bmatrix}-1 & 6\\ 3 & -4\end{bmatrix}$ b) Undefined, cannot multiply CB c) $\begin{bmatrix}8 & -4\\ -16 & 8\end{bmatrix}$

8. a) -100 b) $\dfrac{625}{8}$ c) Not enough information. d) 0

9. Yes, it's possible. The output of B is equivalent to $\dfrac{2}{5}$ of the output of A plus $\dfrac{3}{5}$ of the output of C.

10. a) $\|\overrightarrow{AB}\|=\sqrt{13}$ b) $\begin{bmatrix}{}^3\!/\!\sqrt{13}\\ 0\\ {}^2\!/\!\sqrt{13}\end{bmatrix}$ c) $\left\{\begin{array}{ccc} x & = & 3+3t\\ y & = & 1\\ z & = & 2t\end{array}\right.$ d) Yes. e) $2x-13y-3z=7$

11. a) x-int: $(4,0,0)$ y-int: $(0,-12,0)$ z-int: $(0,0,6)$ b) $\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 0\\ 0\end{bmatrix}+s\begin{bmatrix}{}^1\!/\!_3\\ 1\\ 0\end{bmatrix}+t\begin{bmatrix}{}^{-2}\!/\!_3\\ 0\\ 1\end{bmatrix}$

c) The line is parallel to the plane.

12. a) False b) True c) False

13. a) If State 1 = Fastlink and State 2 = Slowpoke, then $P=\begin{bmatrix}0.9 & 0.8\\ 0.1 & 0.2\end{bmatrix}$

b) 0.88 c) $\vec{q}=\begin{bmatrix}{}^8\!/\!_9\\ {}^1\!/\!_9\end{bmatrix}$ d) $\dfrac{1}{9}$

14. a) Only the Nuts industry is profitable. b) \$7500 of Nuts and \$11 000 of bolts.
c) \$6300 of Nuts and \$9200 of Bolts.

15. a) P is not even a transition matrix, the 2nd column does not add to 1.
b) Yes, $\left(I-C\right)^{-1}=\begin{bmatrix}5 & {}^5\!/\!_2\\ 5 & {}^{25}\!/\!_6\end{bmatrix}$ is non-negative.

16. a) The production of fabric must be $\frac{37}{22}$ times the production of metal, and the production of plastic must be $\frac{6}{11}$ times the production of metal. b) \$37 000 to fabric and \$22 000 to metal.

17. The maximum is $z = 39$, when $x_1 = \frac{3}{2}$, $x_2 = 0$ and $x_3 = 3$ $\left(s_1 = 0,\ s_2 = 0,\ s_3 = \frac{11}{2}\right)$

18. a) $x_1 = \#$ of kegs of Type 1 $x_2 = \#$ of kegs of Type 2.

Maximize $R = 125x_1 + 90x_2$ subject to the constraints $\begin{cases} 2x_1 + 6x_2 \leqslant 240 \\ 16x_1 + 9x_2 \leqslant 576 \\ 100x_1 + 200x_2 \leqslant 8200 \\ x_1\ ,\ x_2 \geqslant 0 \end{cases}$

b) There are 2 possible pivots: $\left[\begin{array}{c|cc|ccc|c} 1 & -125 & -90 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 6 & 1 & 0 & 0 & 240 \\ 0 & 16 & 9 & 0 & 1 & 0 & 576 \\ 0 & 100 & 200 & 0 & 0 & 1 & 8200 \end{array}\right]$

c) $x_1 = 18$ kegs of Type 1, $x_2 = 32$ kegs of Type 2. $R = \$5130$

d) There are 12kg of wheat left, everything else is used completely.

19. a) $A^{-1} = \begin{bmatrix} 14 & 11 \\ 19 & 25 \end{bmatrix}$ b) $AA^{-1} = \begin{bmatrix} 79 & 52 \\ 156 & 183 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ c) MEXICO