Part 1: Basic Matrix Operations

1. Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
4 X 5	4 X 5	5 X 2	4 X 2	5 X4

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) BA (b) AB^T (c) $AC+D$ (d) $E(AC)$ (e) $A-3E^T$ (f) $E(5B+A)$

2. Given $A = \begin{bmatrix} 0 & 5 & -1 \\ 3 & 2 & 1 \\ -4 & 6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -3 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 2 & 0 & 8 \\ 8 & 3 & -3 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 9 & 1 \\ 1 & -2 \\ 5 & 5 \end{bmatrix}$; find the following if defined.

(a) AD (b) B^3 (c) D^2 (d) CC^T

3. Given $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -1 \\ 4 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 6 & 1 \\ -5 & 0 \\ 2 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 2 \\ -7 & 6 \end{bmatrix}$; find the following if defined.

(a) AD (b) AC (c) A^TA-2B

4. Given $B = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 9 & -1 \\ 4 & 0 & 5 \end{bmatrix}$, $D = \begin{bmatrix} 4 & -4 \\ 0 & 0 \\ -5 & 2 \end{bmatrix}$; find the following if defined.

(a) DB (b) BD (c) $CD-B^2$

5. Given $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 & -2 \\ 1 & 2 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix}$; find the following if defined.

(a) AB (b) $2A^T-3B$ (c) $BA+3C$

6. Suppose $\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$.
Find a, b, c and d.

7. Suppose $\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 7 & 6 \\ 2 & 3 \end{bmatrix}$. Find a and b.

8. Suppose $A = \begin{bmatrix} 2 & b \\ c & 3 \end{bmatrix}$.
Find b and c so that $A^2 = \begin{bmatrix} 1 & 5 \\ -15 & 6 \end{bmatrix}$.

9. Suppose $A = \begin{bmatrix} a & 0 & -1 \\ 2 & 3 & b \end{bmatrix}$.
Find a and b so that $AA^T = \begin{bmatrix} 26 & -11 \\ -11 & 14 \end{bmatrix}$.

10. Find all the values of k, if any, that satisfy the equation: $\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$

11. Given $A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}$;

The properties of transposes below hold true in general. Verify each using the given matrices.

(a) $(A^T)^T = A$ (b) $(AB)^T = B^TA^T$ (c) $(A+B)^T = A^T + B^T$ (d) $(4C)^T = 4C^T$

Part 2: The Inverse of a Matrix

12. Given $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, D = \begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$;

    First, compute the inverse of each matrix.
    Next, noting that the properties of inverses below hold true in general, verify each using the given matrices.

    (a) $(A^T)^{-1} = (A^{-1})^T$ (b) $(A^{-1})^{-1} = A$ (c) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

13. Find the matrix A given that $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$.

14. Find the matrix A given that $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$.

15. Use row reduction to find the inverse of each matrix (if the inverse exists).

    (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ (b) $B = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$ (c) $C = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$

16. Solve the system using the inverse of the coefficient matrix.

    (a) $\begin{cases} x + y = 2 \\ 5x + 6y = 9 \end{cases}$ (b) $\begin{cases} x + 3y + z = 4 \\ 2x + 2y + z = -1 \\ 2x + 3y + z = 3 \end{cases}$ (c) $\begin{cases} x + y + z = 5 \\ x + y - 4z = 10 \\ -4x + y + z = 0 \end{cases}$

17. Suppose $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 8 & 13 & 18 \\ 14 & 19 & 24 \end{bmatrix}$. Find $a, b, c, d, e,$ and f.

18. Solve the matrix equation for X: $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$

19. Simplify the expression assuming that $A, B, C,$ and D are $n \times n$ invertible matrices.

    (a) $(A^T + B)^T (AB^T)^{-1}$
    (b) $(A^T B - B^T)^T (B^{-1} - A^{-1}) - (B - A)A^{-1}$
    (c) $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$
    (d) $(A + I)^{-1}(5B(AB)^{-1})^{-1}A^{-1}(A + I)$

20. Let A, B,C,D, and X be $n \times n$ matrices. Solve for X in the following matrix equations (all necessary matrices are assumed to be invertible). Simplify your answers as much as possible.

    (a) $B^{-1}XB = AB$
    (b) $A^{-1}X^{-1} = BA^{-1}$
    (c) $ABCXD^{-1} = 2AIB^T D^{-1}$
    (d) $B^T X^T = B + I$
    (e) $3AX + 4I = B$
    (f) $4AX - C^T = 3X$
    (g) $A^{-1}(B + X)^{-1} = A^{-1}$
    (h) $ABXA^{-1}B^{-1} = I + A$
    (i) $(A + B^{-1})^{-1}(AB)(C^T D + X) = B$

ANSWERS:

1. (a) Undefined
   (b) Defined; 4 X 4 matrix
   (c) Defined; 4 X 2 matrix
   (d) Defined; 5 X 2 matrix
   (e) Defined; 4 X 5 matrix
   (f) Defined; 5 X 5 matrix
2. (a) $AD = \begin{bmatrix} 0 & -15 \\ 34 & 4 \\ -20 & -6 \end{bmatrix}$
   (b) $B^3 = \begin{bmatrix} 95 & -66 \\ 22 & -15 \end{bmatrix}$
   (c) D^2 is undefined.
   (d) $CC^T = \begin{bmatrix} 69 & -2 \\ -2 & 82 \end{bmatrix}$
3. (a) AD is undefined.
   (b) $AC = \begin{bmatrix} -5 & 3 \\ 25 & 5 \end{bmatrix}$
   (c) $A^TA - 2B = \begin{bmatrix} 23 & 13 & -6 \\ 11 & 8 & -1 \\ -10 & -5 & 1 \end{bmatrix}$
4. (a) $DB = \begin{bmatrix} 28 & -4 \\ 0 & 0 \\ -26 & -1 \end{bmatrix}$
   (b) BD is undefined.
   (c) $CD - B^2 = \begin{bmatrix} -4 & -12 \\ 9 & -7 \end{bmatrix}$
5. (a) $AB = \begin{bmatrix} 0 & 10 & -4 \\ 1 & -3 & 4 \\ 3 & 31 & -4 \end{bmatrix}$
   (b) $2A^T - 3B = \begin{bmatrix} 4 & -17 & 16 \\ -3 & -4 & 0 \end{bmatrix}$
   (c) $BA + 3C = \begin{bmatrix} 12 & 14 \\ 25 & 17 \end{bmatrix}$
6. $a = 4,\ b = -6,\ c = -1,\ d = 1.$
7. $a = 2,\ b = 3.$
8. $b = 1, c = -3.$
9. $a = -5,\ b = 1.$
10. $k = -1.$
11. Show that the left-hand side equals the right-hand side for each part.
12. $A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}, B^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix},$
    $C^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, D^{-1} = \begin{bmatrix} -\frac{1}{2} & -2 \\ 1 & 3 \end{bmatrix},$
    Show that the left-hand side equals the right-hand side for each part.
13. $A = \begin{bmatrix} \frac{2}{7} & 1 \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix}$
14. $A = \begin{bmatrix} -\frac{9}{13} & \frac{1}{13} \\ \frac{2}{13} & -\frac{6}{13} \end{bmatrix}$
15. (a) $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$
    (b) B is not invertible
    (c) $C^{-1} \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$
    (d) $D^{-1} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{bmatrix}$
16. (a) $x = 3, y = -1$
    (b) $x = -1, y = 4, z = -7$
    (c) $x = 1, y = 5, z = -1$
17. $a = 1,\ b = 2,\ c = 3,\ d = 4,\ e = 5,\ f = 6.$
18. $X = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$
19. (a) $A(B^T)^{-1}A^{-1} + A^{-1} = (A(B^T)^{-1} + I)A^{-1}$
    (b) $B^TAB^{-1} - B^T = B^T(AB^{-1} - I)$
    (c) B^{-1}
    (d) $\frac{1}{5}I$
20. (a) $X = BA$
    (b) $X = AB^{-1}A^{-1}$
    (c) $X = 2C^{-1}B^{-1}B^T$
    (d) $X = (B + I)^TB^{-1}$
    (e) $X = \frac{1}{3}A^{-1}(B - 4I)$
    (f) $X = (4A - 3I)^{-1}C^T$
    (g) $X = I - B$
    (h) $X = (AB)^{-1}BA + A$
    (i) $X = I + (AB)^{-1} - C^TD = I + B^{-1}A^{-1} - C^TD$