1. For each of the following matrices, determine if it is a Markov transition matrix. If so, determine if it is regular.

   (a) $\begin{bmatrix} 0.2 & 0.66 \\ 0.8 & 0.33 \end{bmatrix}$

   (b) $\begin{bmatrix} 0.7 & 0 \\ 0.3 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 0.8 & 1 \\ 0.2 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   (e) $\begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.4 & 0.7 & 0.1 \\ 0.4 & 0 & 0.5 \end{bmatrix}$

2. In each of the following cases, find the state vector $\mathbf{x_3}$, given the transition matrix and state vector provided.

   (a) $P = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$ and $\mathbf{x_0} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

   (b) $P = \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$ and $\mathbf{x_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   (c) $P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.6 & 0.8 & 0.3 \\ 0.1 & 0 & 0.2 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$

3. For each of the following regular transition matrices, find the associated steady state vector.

   (a) $\begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix}$

   (b) $\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

   (c) $\begin{bmatrix} 0.3 & 0.1 & 0.5 \\ 0.4 & 0.8 & 0.1 \\ 0.3 & 0.1 & 0.4 \end{bmatrix}$

4. (a) Given $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$, find P^3.

   (b) Use your result from part (a) to answer the following question:

   If my cat is awake at some moment, there is a $1/3$ chance that she will be awake an hour from then. If my cat is asleep at some moment, there is a $1/2$ chance that she will be awake an hour from then.

   My cat is awake right now. What is the chance that she will be asleep three hours from now?

5. At a certain college, on any given day some students buy lunch in the cafeteria and the rest bring a lunch from home. It is known that if a student eats in the cafeteria today, there is a 34% chance that they will eat in the cafeteria tomorrow, and if they bring a lunch from home today, there is a 44% chance that they will bring a lunch from home tomorrow.

   On a day in the distant future, what % of students do we expect to be buying their lunch in the cafeteria, and what % of students will be bringing a lunch from home?

6. At a certain company, on any given day some employees are at work and the rest are absent. It is known that if an employee is at work today, there is a 80% chance that they will be at work tomorrow (and a 20% chance that they will be absent tomorrow), and if the employee is absent today, there is a 75% chance that they will be at work tomorrow (and a 25% chance that they will be absent tomorrow).

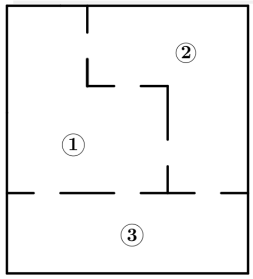
   Draw the associated transition diagram. Assuming the number of exmployees stays consistent at 1800, in the long term, how many employees do we expect to be at work each day? How many do we expect to stay home?

7. In a certain urban area the overall population remains stable at $14\,000$ inhabitants. However, every year 10% of the people who live in the urban core move out to the suburbs, and 25% of the people who live in the suburbs move in to the urban core.

   In the long term, how many people do we expect to live in the urban core, and how many do we expect to live in the suburbs?

8. There is a 45% probability that any given child will grow up to become a doctor if their father is a doctor. Meanwhile, a child whose father is *not* a doctor is only 5% likely to become a doctor when they grow up. Dr. Peter Cummings has many sons, but no daughters. What is the probability that his first grandchild will grow up to become a doctor?

9. A rat is placed in a maze set up as in the illustration below:



When changing chambers (numbered as chambers 1, 2, and 3 in the image), the rat will choose a doorway at random from among those it can see. For example, if the rat is in Chamber 1, it can see five doorways, three of which will lead into Chamber 2, and two of which leads into Chamber 3. That means that there is a $3/5$ probability that the next Chamber that the rat is to occupy will be Chamber 2, and a $2/5$ probability that it will occupy Chamber 3 next.

   (a) Draw the transition diagram and find a transition matrix P associated with a rat's movement from one chamber to another.

   (b) If a rat is currently in Chamber 2, what is the probability that it will be in Chamber 1, Chamber 2, and Chamber 3, respectively, after changing chamber twice?

   (c) What is the long-term probability that the rat will be in Chamber 1?

10. History has shown that a student who asks for an extension on a homework assignment in Linear Algebra is 40% likely to also need an extension on the next homework assignment. Meanwhile, a student who does not ask for an extension on their current homework assignment is only 10% likely to ask for an extension on their next homework assignment. If there is a 75% chance that Sophie will be able to complete Assignment 4 on time, what is the probability that she will need an extension on Assignment 6?

11. Although it isn't true that all cats have nine lives, something strange has been happening to felines across Canada. A virus swept through the feline population in January 2015 and, since then, the rules of life and death haven't applied as reliably as in previous years. On any given month after January 2015, a living cat will have a 90% chance of also being alive the next month. However, since the mysterious virus hit, it has been noted that a cat that is dead on one month will have a 98% chance of remaining dead next month, but also a 2% chance of spontaneously coming back to life (assuming that the cat died after the virus had struck).

   (a) Find a transition matrix P associated with this situation.

   (b) If Mr. Whiskers (your neighbour's cat) is alive today, what is the probability that Mr. Whiskers will be dead two months from now?

   (c) Assuming that no new cats are born, what percentage of the cat population from January 2015 is expected to be alive, and what percentage of the cat population is expected to be dead in the distant future?

ANSWERS:

1. (a) NOT a Markov matrix.
   (b) Markov Matrix and NOT regular.
   (c) Markov Matrix and regular
   (d) Markov Matrix and NOT regular
   (e) Markov Matrix and regular

2. (a) $\mathbf{x_3} = \begin{bmatrix} 72/125 \\ 53/125 \end{bmatrix}$

   (b) $\mathbf{x_3} = \begin{bmatrix} 9/25 \\ 16/25 \end{bmatrix}$

   (c) $\mathbf{x_3} = \begin{bmatrix} 7/20 \\ 27/50 \\ 11/100 \end{bmatrix}$

3. Steady state vectors:

   (a) $\begin{bmatrix} 9/16 \\ 7/16 \end{bmatrix}$

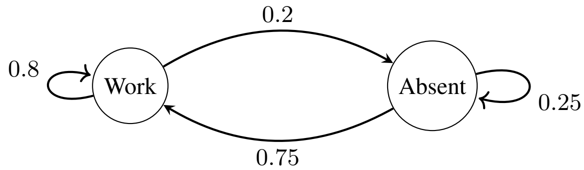
   (b) $\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$

   (c) $\begin{bmatrix} 11/48 \\ 9/16 \\ 5/24 \end{bmatrix}$

4. (a) $\begin{bmatrix} 23/54 & 31/72 \\ 31/54 & 41/72 \end{bmatrix}$

   (b) There's a $31/54$ chance (or about 57.4%) that my cat will be asleep three hours from now.

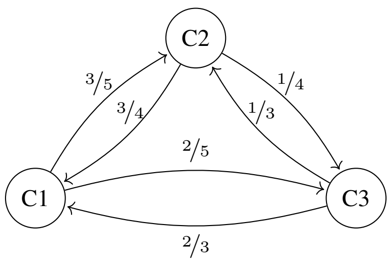
5. Approximately 45.9% will be buying lunch in the cafeteria, and 54.1% will be bringing a lunch from home.

6. 

Approximately 1421 will be at work on any given day ($^{15}/_{19}$ of all employees), so 379 will stay home.

7. 10000 will live in the urban core ($^{5}/_{7}$ of all inhabitants) with the remaining 4000 in the suburbs.

8. There is a 23% chance that Dr. Cummings' first grandchild will grow up to become a doctor.

9. (a) 

$$P = \begin{bmatrix} 0 & ^{3}/_{4} & ^{2}/_{3} \\ ^{3}/_{5} & 0 & ^{1}/_{3} \\ ^{2}/_{5} & ^{1}/_{4} & 0 \end{bmatrix}$$

(b) There is a $^{1}/_{6}$ probability that the rat will be in Chamber 1, a $^{8}/_{15}$ probability that it will be in Chamber 2, and a $^{3}/_{10}$ probability that it will be in Chamber 3.

(c) There is a $^{5}/_{12}$ probability that the rat will be in Chamber 1 long-term. (Note that this means that, over the course of an extended period of time, the rat will end up in Chamber 1 roughly $^{5}/_{12}$ of the time.)

10. There is a $^{61}/_{400}$ (15.25%) chance that Sophie will need an extension on Assignment 6.

11. (a) $P = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$

(b) There is a 18.8% chance that Mr. Whiskers will be dead two months from now.

(c) At any given point in the distant future, $^{1}/_{6}$ of the cat population from January 2015 would be alive, and $^{5}/_{6}$ would be dead.