201-SH2-AB - Exercises #2 - Domains and Limits by Direct Substitution

1. *Carefully read the argument below, which erroneously reaches the conclusion that $1 = 2$. For each step of the argument, either justify the step OR explain why the step is invalid. The first step has been done for you.

Assume $a = b$.

Then: $a^2 = ab$ Valid: multiplying both sides of an equation by the same value maintains the equality

$a^2 - b^2 = ab - b^2$ ____

$(a-b)(a+b) = b(a-b)$ ____

$a + b = b$ ____

$a+a+b-b = b-b+a$ ____

$a + a = a$ ____

$2a = a$ ____

$2 = 1$ ____

2. *Consider the following two functions:

$$f(x) = x \quad \text{AND} \quad g(x) = \frac{x^2}{x}$$

Are f and g the same function? Discuss and sketch graphs of both f and g.

3. Find the domain of the following functions:

a) $g(x) = 3x^2 - 7x + 4$

b) $f(x) = \sqrt{x-4}$

c) $h(x) = \dfrac{-4}{x+2}$

d) $g(x) = |x-3|$

e) $*f(x) = \dfrac{\sqrt{x}+1}{x-4}$

f) $*g(x) = \sqrt{25-x^2}$

g) $*h(x) = \sqrt{x^2-25}$

h) $f(x) = \dfrac{x^2-9}{2x^2-5x-3}$

4. Evaluate the following limits using direct substitution if possible. If not, state that direct substitution does not apply.

a) $\lim\limits_{x\to 2} 2x^3 - 4x + 7$

b) $\lim\limits_{x\to 0} \dfrac{x^2-4}{x+2}$

c) $\lim\limits_{x\to 3} \dfrac{2x^2-x-15}{x^2-5x+7}$

d) $\lim\limits_{x\to 3} \sqrt{3x^2-2}$

e) $\lim\limits_{x\to -1} \dfrac{2x-3}{x+12}$

f) $\lim\limits_{x\to 4} |x-6|$

5. Given that $f(x) = \begin{cases} x^2+2x-6 & \text{for } x < 1, \\ \sqrt{3+x^2} & \text{for } x > 1. \end{cases}$, answer the following.

(a) Consider $\lim\limits_{x\to a} f(x)$. For which value(s) of a is it necessary to evaluate the left and right handed limits separately. Briefly explain your answer.

(b) $f(1) =$

(c) $\lim\limits_{x\to 0} f(x) =$

(d) $\lim\limits_{x\to 1} f(x) =$

6. Given that $f(x) = \begin{cases} 2x - x^2 & \text{if} \quad x < 3 \\ \sqrt{x+1} & \text{if} \quad x \geqslant 3 \end{cases}$, find:

(a) $\lim\limits_{x\to 3^-} f(x)$ (b) $\lim\limits_{x\to 3^+} f(x)$ (c) $\lim\limits_{x\to 3} f(x)$ (d) $f(3)$

7. Given that $f(x) = \begin{cases} x - 3x^2 & \text{if} \quad x \leqslant 2 \\ \sqrt{x-1} & \text{if} \quad x > 2 \end{cases}$, find

(a) $\lim\limits_{x\to 2^-} f(x)$ (b) $\lim\limits_{x\to 2^+} f(x)$ (c) $\lim\limits_{x\to 2} f(x)$ (d) $f(2)$

8. Given that $f(t) = \begin{cases} t^2 & \text{for } t < -2 \\ t^2 - t & \text{for } -2 < t < 2 \\ 2t - 2 & \text{for } t \geq 2 \end{cases}$, find:

(a) $f(-3)$ (b) $f(2)$ (c) $f(3/2)$ (d) $\lim\limits_{t\to -2} f(t)$ (e) $\lim\limits_{t\to -2^+} f(t) =$ (f) $\lim\limits_{t\to 2} f(t)$

(g) $\lim\limits_{t\to 0} f(t)$

9. Given that $f(x) = \begin{cases} 3x - 5 & \text{if} \quad x < 2 \\ \sqrt{x-1} & \text{if} \quad x > 2 \end{cases}$, find:

(a) $\lim\limits_{x\to 2^-} f(x)$ (b) $\lim\limits_{x\to 2^+} f(x)$ (c) $\lim\limits_{x\to 2} f(x)$ (d) $f(2)$

10. Given that $f(x) = \begin{cases} 5x - 1 & \text{if} \quad x > 1 \\ 2 & \text{if} \quad x = 1 \\ 3x^2 + 1 & \text{if} \quad x < 1 \end{cases}$, find:

(a) $\lim\limits_{x\to 1^-} f(x)$ (b) $\lim\limits_{x\to 1^+} f(x)$ (c) $\lim\limits_{x\to 1} f(x)$ (d) $f(1)$

11. Given that $f(x) = \begin{cases} 11 - x^2 & \text{if} \quad x \leqslant -3 \\ |x+1| & \text{if} \quad x > -3 \end{cases}$, find:

(a) $\lim\limits_{x\to -3^-} f(x)$ (b) $\lim\limits_{x\to -3^+} f(x)$ (c) $\lim\limits_{x\to -3} f(x)$ (d) $f(-3)$

12. Given that $f(x) = \begin{cases} x^2 + 3 & \text{if} \quad x < -2 \\ -2x + 3 & \text{if} \quad x > -2 \end{cases}$, find:

(a) $\lim\limits_{x \to -2^-} f(x)$ (b) $\lim\limits_{x \to -2^+} f(x)$ (c) $\lim\limits_{x \to -2} f(x)$ (d) $f(-2)$

13. Given that $f(x) = \begin{cases} \sqrt{x+6} & \text{if} \quad x > 3 \\ 3 & \text{if} \quad x = 3 \\ x^2 - 6 & \text{if} \quad x < 3 \end{cases}$, find:

(a) $\lim\limits_{x \to 3^-} f(x)$ (b) $\lim\limits_{x \to 3^+} f(x)$ (c) $\lim\limits_{x \to 3} f(x)$ (d) $f(3)$

14. Given that $f(x) = \begin{cases} |x - 1| & \text{if} \quad x \leqslant -1 \\ 2x^2 & \text{if} \quad x > -1 \end{cases}$, find:

(a) $\lim\limits_{x \to -1^-} f(x)$ (b) $\lim\limits_{x \to -1^+} f(x)$ (c) $\lim\limits_{x \to -1} f(x)$ (d) $f(-1)$

15. Given that $f(x) = \begin{cases} 4x - 1 & \text{if} \quad x \leqslant 1 \\ 2 - x^2 & \text{if} \quad x > 1 \end{cases}$, find:

(a) $\lim\limits_{x \to 1^-} f(x)$ (b) $\lim\limits_{x \to 1^+} f(x)$ (c) $\lim\limits_{x \to 1} f(x)$ (d) $f(1)$

16. Given that $f(x) = \begin{cases} \sqrt{x} + 2 & \text{if} \quad x \geqslant 1 \\ 1 - x + x^2 & \text{if} \quad x < 1 \end{cases}$, find:

(a) $\lim\limits_{x \to 1^-} f(x)$ (b) $\lim\limits_{x \to 1^+} f(x)$ (c) $\lim\limits_{x \to 1} f(x)$ (d) $f(1)$

17. Given that $f(x) = \begin{cases} x^2 - 1 & \text{if} \quad x \leqslant -2 \\ 3x + 1 & \text{if} \quad x > -2 \end{cases}$, find:

(a) $\lim\limits_{x \to -2^-} f(x)$ (b) $\lim\limits_{x \to -2^+} f(x)$ (c) $\lim\limits_{x \to -2} f(x)$ (d) $f(-2)$

18. Given that $f(x) = \begin{cases} x^2 - x & \text{if} \quad x \leqslant -1 \\ x^3 & \text{if} \quad x > -1 \end{cases}$, find:

(a) $\lim\limits_{x \to -1^-} f(x)$ (b) $\lim\limits_{x \to -1^+} f(x)$ (c) $\lim\limits_{x \to -1} f(x)$ (d) $f(-1)$

Answers

1. $a^2 - b^2 = ab - b^2$: Valid: subtracted b^2 from both sides

   $(a-b)(a+b) = b(a-b)$: Valid: factored both sides

   $a + b = b$ Invalid: divided both sides by $a - b$, but since $a = b$, we know $a - b = 0$, and we are not allowed to divide by 0.

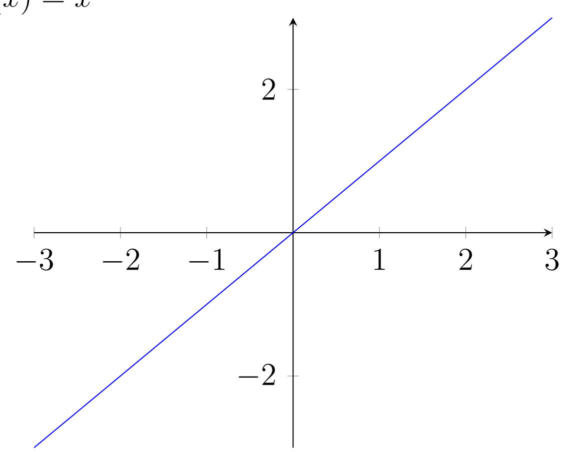
   $a + a + b - b = b - b + a$: Valid: added $-b + a = 0$ to both sides. Order of addition does not matter.

   $a + a = a$ Valid: replaced $b - b$ with 0

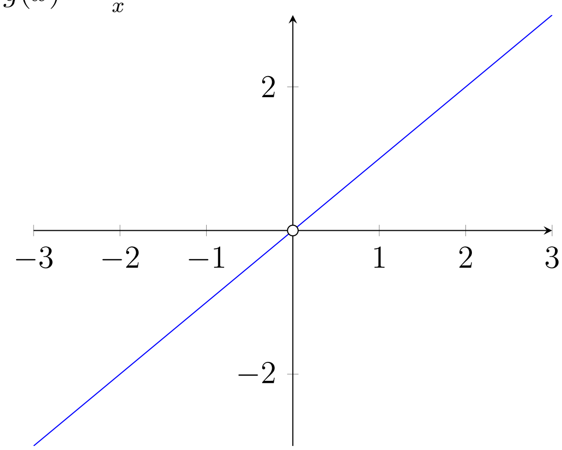
   $2a = a$ Valid: $a + a$ does, indeed, equal $2a$

   $2 = 1$ Possibly Invalid: we divided both sides by a, and we are not told that a does not equal 0, so it is possible that we divided by 0.

2. $f(x) = x$



$g(x) = \frac{x^2}{x}$



3. (a) $\mathbb{R}$
   (b) $x - 4 \geq 0$, so $x \geq 4$
   (c) $x + 2 \neq 0$, so $\mathbb{R}/\{-2\}$
   (d) $\mathbb{R}$
   (e) $x - 4 \neq 0$, so $\mathbb{R}/\{4\}$
   (f) $-5 \leq x \leq 5$ or $[-5, 5]$
   (g) $(-\infty, -5] \cup [5, \infty)$
   (h) $\mathbb{R}/\{-\frac{1}{2}, 3\}$

4. (a) 15
   (b) -2
   (c) 0
   (d) 5
   (e) $\frac{-5}{11}$
   (f) 2
5. (a) $x = 1$ only, as this is the breakpoint, where on the left the function is defined by one rule and in the right it is defined by a different rule.
   (b) dne
   (c) -6
   (d) dne
6. (a) -3 (b) 2 (c) dne (d) 2
7. (a) -10 (b) 1 (c) dne (d) -10
8. (a) 9 (b) 2 (c) $\frac{3}{4}$ (d) dne (e) 6 (f) 2 (g) 0
9. (a) 1 (b) 1 (c) 1 (d) dne
10. (a) 4 (b) 4 (c) 4 (d) 2
11. (a) 2 (b) 2 (c) 2 (d) 2
12. (a) 7 (b) 7 (c) 7 (d) dne
13. (a) 3 (b) 3 (c) 3 (d) 3
14. (a) 2 (b) 2 (c) 2 (d) 2
15. (a) 3 (b) 1 (c) dne (d) 3
16. (a) 1 (b) 3 (c) dne (d) 3
17. (a) 3 (b) -5 (c) dne (d) 3
18. (a) 2 (b) -1 (c) dne (d) 2