1. Evaluate the determinant of the given matrix. If the matrix is invertible, find its inverse.

(a) $A = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$ (b) $B = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$ (c) $C = \begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}$

2. Evaluate the determinant of the given matrix.

(a) $A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$ (c) $C = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$

3. Use the determinant to decide whether the given matrix is invertible.

(a) $A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $C = \begin{bmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{bmatrix}$ (d) $D = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}$

4. Find the values of k for which the matrix A is invertible.

(a) $A = \begin{bmatrix} k-3 & -4 \\ -6 & k+2 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}$

5. Find all the minors and cofactors of the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$

6. Given that $A = \begin{bmatrix} 5 & 7 & -2 \\ 0 & 3 & 1 \\ 4 & 0 & 6 \end{bmatrix}$,

(a) find $C_{2,3}$, the (2,3)-cofactor of A;
(b) find $\det(A)$;
(c) find $\operatorname{adj}(A)$;
(d) find A^{-1};
(e) use A^{-1} to solve the system:

$$\begin{cases} 5x + 7y - 2z = -3 \\ 3y + z = 0 \\ 4x + 6z = 2 \end{cases}$$

7. Given that $A = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 3 \\ 5 & 6 & 0 \end{bmatrix}$,

(a) find $C_{1,2}$, the $(1,2)$-cofactor of A;
(b) find $\det(A)$;
(c) find $\operatorname{adj}(A)$;
(d) find A^{-1};
(e) use A^{-1} to solve the system:

$$\begin{cases} 2x + 4z = 3 \\ x - y + 3z = 1 \\ 5x + 6y = 4 \end{cases}$$

8. Given $A = \begin{bmatrix} 6 & -1 & 0 \\ 0 & 3 & -1 \\ -2 & 7 & 1 \end{bmatrix}$, find the following:

(a) $M_{2,3}$, the $(2,3)$-minor of A
(b) $\det(A)$
(c) $\operatorname{adj}(A)$
(d) $A \operatorname{adj}(A)$

9. Consider the system $\begin{cases} 5x_1 + x_3 = 2 \\ 2x_1 + 3x_2 - x_3 = 0 \\ -3x_1 + x_2 = -1 \end{cases}$

(a) Write the system in the form $AX = B$.
(b) Find $\det(A)$, where A is as in the previous part.
(c) Find $\operatorname{adj}(A)$.
(d) Find $A \operatorname{adj}(A)$.
(e) Find A^{-1}.
(f) Solve the system using A^{-1}.

10. Noting that the property $\det(A) = \det(A^T)$ holds true in general, verify that it is satisfied by the given matrices.

(a) $A = \begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{bmatrix}$

11. Noting that the property $\det(kA) = k^n \det(A)$ holds true in general, verify that it is satisfied by the given matrices and constants.

(a) $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}; k = 5$

(b) $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}; k = -2$

12. Noting that the property $\det(AB) = \det(A)\det(B)$ holds true in general, verify that it is satisfied by the given matrices. Next, determine whether the equality $\det(A + B) = \det(A) + \det(B)$ holds in this case.

$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$

13. Given that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has $\det(A) = 5$, evaluate:

(a) $\det(3A)$

(b) $\det(A^4)$

(c) $\det[(2A)^{-1}]$

14. Given that $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ has $\det(A) = -3$, and suppose also that B is 3×3 and $\det(B) = 5$. evaluate:

(a) $\det(2A)$

(b) $\det(A^3)$

(c) $\det((A^{-1})^T)$

(d) $\det(5AB^{-1})$

15. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and suppose that $\det(A) = 7$.

Find the following:

(a) $\begin{vmatrix} 4d & 4e & 4f \\ 7a & 7b & 7c \\ \frac{1}{2}g & \frac{1}{2}h & \frac{1}{2}i \end{vmatrix}$

(b) $\begin{vmatrix} b & a & c \\ 2e & 2d & 2f \\ h & g & i \end{vmatrix}$

(c) $\begin{vmatrix} 2a & 3b & 5a \\ 2d & 3e & 5d \\ 2g & 3h & 5g \end{vmatrix}$

(d) $\begin{vmatrix} 2a+3d & 2b+3e & 2c+3f \\ d & e & f \\ 5g-7d & 5h-7e & 5i-7f \end{vmatrix}$

16. Suppose A, B, and C are 5×5 matrices such that $\det(A) = 5,\ \det(B) = 6$, and C is not invertible. Find the following or state that there is not enough information.

(a) $\det(A + B)$

(b) $\det(AC + BC)$

(c) $\det(AC + CB)$

(d) $\det(B^{-1} + B^{-1})$

ANSWERS:

1. (a) $det(A) = 22; A^{-1} = \begin{bmatrix} \frac{2}{11} & -\frac{5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{bmatrix}$ (b) $det(B) = 0$; B is not invertible. (c) $det(C) = 59; C^{-1} = \begin{bmatrix} -\frac{2}{59} & -\frac{7}{59} \\ \frac{7}{59} & -\frac{5}{59} \end{bmatrix}$

2. (a) $det(A) = -65$ (b) $det(B) = -123$ (c) $det(C) = -40$

3. (a) A is invertible. (b) B is invertible. (c) C is not invertible. (d) D is invertible.

4. (a) $k \neq -5, k \neq 6$ (b) $k \neq -1$

5. $M_{11} = 29 \quad M_{12} = 21 \quad M_{13} = 27$
   $C_{11} = 29 \quad C_{12} = -21 \quad C_{13} = 27$
   $M_{21} = -11 \quad M_{22} = 13 \quad M_{23} = -5$
   $C_{21} = 11 \quad C_{22} = 13 \quad C_{23} = 5$
   $M_{31} = -19 \quad M_{32} = -19 \quad M_{33} = 19$
   $C_{31} = -19 \quad C_{32} = 19 \quad C_{33} = 19$

6. (a) $C_{2,3} = 28$
   (b) $\det(A) = 142$.
   (c) $\text{adj}(A) = \begin{bmatrix} 18 & -42 & 13 \\ 4 & 38 & -5 \\ -12 & 28 & 15 \end{bmatrix}$
   (d) $A^{-1} = \dfrac{1}{142} \begin{bmatrix} 18 & -42 & 13 \\ 4 & 38 & -5 \\ -12 & 28 & 15 \end{bmatrix}$
   (e) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -14/71 \\ -11/71 \\ 33/71 \end{bmatrix}$

7. (a) $C_{1,2} = 15$.
   (b) $\det(A) = 8$.
   (c) $\text{adj}(A) = \begin{bmatrix} -18 & 24 & 4 \\ 15 & -20 & -2 \\ 11 & -12 & -2 \end{bmatrix}$
   (d) $A^{-1} = \dfrac{1}{8} \begin{bmatrix} -18 & 24 & 4 \\ 15 & -20 & -2 \\ 11 & -12 & -2 \end{bmatrix}$
   (e) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -7/4 \\ 17/8 \\ 13/8 \end{bmatrix}$

8. (a) $M_{2,3} = 40$
   (b) $\det(A) = 58$
   (c) $\text{adj}(A) = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 6 & 6 \\ 6 & -40 & 18 \end{bmatrix}$
   (d) $A\,\text{adj}(A) = \begin{bmatrix} 58 & 0 & 0 \\ 0 & 58 & 0 \\ 0 & 0 & 58 \end{bmatrix}$

9. (a) $\begin{bmatrix} 5 & 0 & 1 \\ 2 & 3 & -1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$
   (b) $\det(A) = 16$
   (c) $\text{adj}(A) = \begin{bmatrix} 1 & 1 & -3 \\ 3 & 3 & 7 \\ 11 & -5 & 15 \end{bmatrix}$
   (d) $A\,\text{adj}(A) = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$
   (e) $A^{-1} = \dfrac{1}{16} \begin{bmatrix} 1 & 1 & -3 \\ 3 & 3 & 7 \\ 11 & -5 & 15 \end{bmatrix}$
   (f) $(5/16, -1/16, 7, 16)$

10. (a) $\det(A) = \det(A^T) = -11$
    (b) $\det(A) = \det(A^T) = 101$

11. (a) $\det(5A) = 25\det(A) = -250$
    (b) $\det(-2A) = -8\det(A) = -448$

12. $\det(A)\det(B) = 10(-17) = -170 = \det(AB)$

13. (a) 45
    (b) 625
    (c) $\dfrac{1}{20}$

14. (a) -24
    (b) -27
    (c) $-1/3$
    (d) -75

15. (a) -98
    (b) -14
    (c) 0
    (d) 70

16. (a) Not enough information. For simplicity's sake, consider instead 2×2 matrices: $A = \begin{bmatrix} 0 & 5 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 6 \\ -1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$. Then $\det(AC + CB) = x^2$ (and can be any positive number).

(b) 0.

(c) Not enough information. For example, if $A = B = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\det(AC + CB) = x^2$.

(d) $16/3$