Growth Notes

Jason Hall

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Contents

As with anything and everything I produce, the usual caveats apply. These notes are mostly directed at studying for prelims (and in particular Erzo's section of the prelim). Where I felt it helpful I have provided some intuition, where I felt it was obvious I have not. If you find any errors, please do let me know, and I will fix them.

1 Continuous Time

1.1 Introduction

Consider the problem of a consumer maximizing his utility subject to some lifetime budget constraint.

$$\max \int_0^\infty e^{-\rho t} u(c_t) dt$$

Subject to:

$$\int_0^\infty e^{-rt} c_t \leq \mathcal{W}$$

Where $\mathcal{W}$ is some stock of $t = 0$ wealth, and $u(\cdot)$ is assumed to have all the properties we normally associate with instantaneous utility. This is the continuous time formulation of the cake-eating problem with saving.

This has a familiar solution. After formulating the Lagrangian and differentiating with respect to c_t, one recovers:

$$e^{-\rho t} u'(c_t) = \lambda e^{-rt}$$

$$-\rho t + ln(u'(c_t)) = ln(\lambda) - rt$$

Differentiate this function with respect to time[1], and one recovers

$$-\rho + \frac{u''(c_t)\dot{c}_t}{u'(c_t)} = -r$$

Or equivalently:

$$r = \rho + \left[\underbrace{-\frac{u''(c_t)}{u'(c_t)}}_{\text{Arrow-Pratt measure of risk aversion}} \times \dot{c}_t \right]$$

For reference, take $u(c_t) = \ln(c_t)$, so that $u'(c_t) = \frac{1}{c_t}$ and $u''(c_t) = -\frac{1}{c_t^2}$.

Then we have that:

$$\frac{u''(c_t)}{u'(c_t)} = -\frac{c_t}{c_t^2} = -\frac{1}{c_t}$$

This gives us the Euler equation that appears most often in Erzo's slides:

$$r = \rho + \frac{\dot{c}_t}{c_t}$$

With some small difference because I have imposed that $\forall t \; r_t = r$, as opposed to more generally allowing it to vary and writing the budget constraint as:

[1] Note that I always follow the convention of denoting time derivatives with dot notation, so the derivative of consumption c_t with respect to time is written as $\dot{c}_t$.

$$\int_0^\infty e^{-\int_0^t r_s ds} c_t dt \leq \mathcal{W}$$

To derive this Euler equation, we formulated a Lagrangian. But in general, suppose that the state also evolves dynamically according some choices by the agent. Then we do not have a single set of constraints or even a countable collection of constraints. We have a set of constraints with the same cardinality as the positive real numbers. As a result, we generally turn to a different set of dynamic programming techniques. These are the continuous time counterparts of familiar discrete time techniques.

As with discrete time, we begin by defining a value function. Consider the following maximization problem.

$$V(t,x) = \max_{(\alpha_s)_{\{s\geq t\}}} \int_t^\infty e^{-\rho(s-t)} u(\alpha_s, x_s) ds$$

Subject to a law of motion:

$$\dot{x}_t = \mu(\alpha_t, x_t)$$

And an initial constraint:

$$x_0 = x$$

In this problem, α_s is called the control and x_s is called the state.

α_s can be thought of as akin to consumption in the standard income fluctuation problem. By picking your choice of current consumption, subject to your wealth today (i.e. the state), you uniquely pin down your wealth tomorrow, at least in the deterministic case. It is in this sense that c_t is a control: it *controls* your wealth tomorrow. Consume a lot today and you will be poorer tomorrow.

We would like something like a recursive formulation of this problem. To do so, we will proceed by essentially converting the problem into something that looks like a discrete problem.

Specifically, suppose that dt is a small increment of time, so small that the state is constant over the increment. Then the above problem is equivalent to:

$$V(t, x_t) = \max \int_t^{t+\Delta t} e^{-\rho(s-t)} u(\alpha_s, x) ds + e^{\rho(\Delta t)} V(t+\Delta t, x_{t+\Delta t})$$

Subtract $e^{-\rho \Delta t} V(t, x_t)$ from both sides,

$$(1 - e^{-\rho\Delta t}) V(t, x_t) = \max \int_t^{t+\Delta t} e^{-\rho(s-t)} u(\alpha_s, x) ds + e^{-\rho(\Delta t)} \left[V(t+\Delta t, x_{t+\Delta t}) - V(t,x)\right]$$

Multiply the right hand side by $\frac{x_{t+\Delta t} - x_t}{x_{t+\Delta t} - x_t}$ and divide through by Δt

$$\frac{(1 - e^{-\rho\Delta t})}{\Delta t} V(t, x_t) = \max \int_t^{t+\Delta t} \frac{e^{-\rho(s-t)}}{\Delta t} u(\alpha_s, x) ds + e^{-\rho(\Delta t)} \left[\frac{V(t+\Delta t, x_{t+\Delta t}) - V(t,x)}{x_{t+\Delta t} - x_t} \frac{x_{t+\Delta t} - x_t}{\Delta t}\right]$$

Now take limits in Δt to recover

$$\rho V(t, x_t) = \max u(\alpha_t, x_t) + \frac{\partial V}{\partial x}\dot{x}_t$$

This is very sloppy I will rewrite when I have time.

An alternative way to derive the HJB is via the use of Leibnitz. By the principle of optimality, we can write:

$$V(t_0, x) = \max \int_{t_0}^{t} e^{-\rho(s-t_o)} u(\alpha_s, x) ds + e^{-\rho(t-t_0)} V(t, x_t)$$

Differentiating this with respect to t, we recover that:

$$0 = \max e^{-\rho(t-t_0)} u(\alpha_t, x) - \rho e^{-\rho(t-t_0)} V(t, x_t) + e^{-\rho(t-t_0)} \left[\frac{\partial V}{\partial t} + \sum_{n=1}^{N} \frac{\partial V}{\partial x_n} \frac{dx_n}{dt} \right]$$

Moving some stuff around, taking limits as $t \to t_0$, and using the fact that the state evolves according to μ, we can rewrite this as:

$$\rho V(t, x_t) = \dot{V} + \max_{\alpha_t} \left[u(\alpha_t, x_t) + \sum_{n=1}^{N} \frac{\partial V}{\partial x_n} \mu_{n,t}(\alpha_t, x_t) \right]$$

This is the standard continuous time Hamilton-Jacobi-Bellman Equation.

Consider the continuous time formulation of the Neoclassical growth model.

$$\max \int_0^\infty e^{-\rho t} u(c_t) dt$$

Subject to:

$$\dot{k}_t = F(k) - \delta k - c$$

The Hamilton-Jacobi-Bellman equation for this model is:

$$\rho V(k) = \max_c u(c) + V'(k)[F(k) - \delta k - c]$$

Similarly, consider a basic consumption savings problem without uncertainty.

$$\max \int_0^\infty e^{-\rho t} u(c_t) dt$$

Subject to:

$$\dot{a}_t = r a_t + y - c_t$$

and an initial condition.

The HJB for this is:

$$\rho V(a) = \max_c u(c) + V'(a)[ra + y - c]$$

I think you get the idea.

Generalizing this to stochastic dynamic programs is difficult but only in the sense that stochastic calculus is in general difficult. As a practical matter, given a knowledge of stochastic integrals and Ito's lemma, the result is almost immediate. To see this, recall the multidimensional version of Ito's Lemma.

Given a diffusion process $dX_t = \mu dt + \sigma dW_t$, and a C^2 scalar valued function $f(\dot{)}$, we have that:

$$df(X_t) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{N} \frac{\partial f}{\partial X_i} dX_{t,i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial X_i \partial X_j} dX_{t,i} dX_{t,j}$$

Rewriting this using the definition of the diffusion process, we have:

$$\begin{aligned}
df(X_t) &= \frac{\partial f}{\partial t}dt + \sum_{i=1}^{N} \frac{\partial f}{\partial X_i}(\mu_i dt + \sigma_i dW_{t,i}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial X_i \partial X_j}(\mu_i dt + \sigma_i dW_{t,i})(\mu_j dt + \sigma_i dW_{t,j}) \\
&= \frac{\partial f}{\partial t}dt + \sum_{i=1}^{N} \frac{\partial f}{\partial X_i}(\mu_i dt + \sigma_i dW_{t,i}) + \dots \\
&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial X_i \partial X_j}(\mu_i \mu_j \underbrace{dt^2}_{=0} + \mu_j \sigma_i \underbrace{dt dW_{t,i}}_{=0} + \mu_i \sigma_j \underbrace{dt dW_{t,j}}_{=0} + \sigma_i \sigma_j \underbrace{dW_{t,i} dW_{t,j})}_{=dt} \\
&= \left(\frac{\partial f}{\partial t} + \sum_{i=1}^{N} \mu_i \frac{\partial f}{\partial X_i} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_i \sigma_j \frac{\partial^2 f}{\partial X_i \partial X_j} \right) dt + \sum_{i=1}^{N} \frac{\partial f}{\partial X_i} \sigma_i dW_{t,i}
\end{aligned}$$

1.2 Hamiltonians

Recall that the HJB equation takes the form:

$$\rho V(t, x_t) = \dot{V}_t + \max \left[u(\alpha, x) + \sum_{i=1}^{N} \mu_{t,n}(x, \alpha) \frac{\partial V}{\partial x_n} \right]$$

The objects encoded in $\{\frac{\partial V}{\partial x_n}\}_{i=1}^{N}$ are very special. They are called **costates** and they are shadow prices, as in the Lagrangian formulation. Similarly, they are typically denoted as $\{\lambda_i, t\}_{i=1}^{n}$. With this notation, we can write the Hamiltonian:

$$\mathcal{H}(\underbrace{\alpha}_{\text{Control}}, \underbrace{x}_{\text{State}}, \underbrace{\lambda}_{\text{Costate}}) = \max_{\alpha} \left[u(\alpha, x) + \sum_{i=1}^{N} \mu_{t,n}(x, \alpha) \lambda_{n,t} \right]$$

2 Growing Like China

2.1 The Environment

Suppose that we have an economy with two kinds of individuals, workers and entrepreneurs. We are interested in an overlapping generations model. Adopting the notation that $c_{t,1}$ that refers to the consumption of an individual of generation t in the first period of their live, and x_{t+1} denotes the bequest of generation t to generation $t+1$, workers have preferences of the form:

$$\mathcal{U}_t = \ln(c_{t,1}) + \beta \ln(c_{t,2})$$

While entrepreneurs have preferences of the form

$$\mathcal{U}_t = \ln(c_{t,1}) + \beta \ln(c_{t,2}) + \omega \ln(x_{t+1})$$

The budget constraint of a worker is:

$$c_{t,1} + s_t \leq w_t$$
$$c_{t,2} \leq s_t(1+R)$$

Where r is the interest rate. The economy is small, so that the this r is determined by the world interest rate.

The budget constraint of the entrepreneur is

$$c_{t,1} + s_t \leq x_t$$

$$c_{t,2} + x_{t+1} \leq \rho_t s_t$$

Where ρ_t denotes the return on capital invested by a generation t entrepreneur when they are young.

Capital depreciates fully in each period. There are two kinds of firms in this world. E-firms are operated by entrepreneurs, and thus have a fixed capital stock dependent on the savings of the entrepreneur when they are young, and have access to a Cobb Douglas technology of the form:[2]

$$y_{E,t} = \chi^{1-\alpha} k_{E,t}^{\alpha} n_{E,t}^{1-\alpha}$$

On the other hand, there are F-firms that are neoclassical in their entirety:

$$y_{F,t} = k_{F,t}^{\alpha} n_{F,t}^{1-\alpha}$$

F-firms pick both capital and labor.

[2] Note that in SSZ, to access the more productive technology, they face a moral hazard problem with managers, so that there is a wedge on their output. This is important for later parts of the question.

2.2 Savings Rates

The problem of a worker is:

$$\max c_{t,1} + \beta c_{t,2}$$

Subject to:

$$c_{t,1} + \frac{c_{t,2}}{1+r} \leq w_t$$

This is a standard two period model, and thus has the typical result that in the first period one consumes $\frac{1}{1+\beta}$ of the wage, and saves a fraction $\frac{\beta}{1+\beta}$.

The problem of an entrepreneur is:

$$\max c_{t,1} + \beta c_{t,2} + \omega x_{t+1}$$

Subject to:

$$c_{t,1} + \frac{c_{t,2} + x_{t+1}}{\rho} \leq x_t$$

This has the also familiar result in the first period one consumes a constant fraction $\frac{1}{1+\beta+\omega}$ and saves $\frac{\beta+\omega}{1+\beta+\omega}$. In the second period, an individual will consume a fraction $\frac{\beta}{\beta+\omega}$ of their assets, while they will pass as a bequest $\frac{\omega}{\beta+\omega}$. One easy way to see this is to treat this as a sequence of two-step budgeting problems, but the algebra is fairly straightforward anyways.

2.3 F-firms

Suppose as we have that the world interest rate is some exogeneous constant r. A no-arbitrage argument implies that the return to capital must exactly equal the world interest rate.[3]

The problem of a F-firm is thus:

$$\max K_{F,t}^{\alpha} N_{F,t}^{1-\alpha} - w_t N_{F,t} - R K_{F,t}$$

This implies the standard first order conditions:

$$R = \alpha \left(\frac{N_{F,t}}{K_{F,t}} \right)^{1-\alpha}$$

$$w_t = (1-\alpha) \left(\frac{K_{F,t}}{N_{F,t}} \right)^{\alpha}$$

Manipulating the first condition, we can see that:

$$\frac{K_{F,t}}{N_{F,t}} = \left(\frac{R}{\alpha} \right)^{\frac{1}{\alpha-1}}$$

The wage rate is thus:

[3] Note the standard no-arbitrage condition for investments is that $r_t^k - \delta = r_t$. When $\delta = 1$, this implies that $r_t^K = 1 + r_t = R_t$.

$$w_t = (1-\alpha)\left(\frac{R}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$

Note that this is constant. The core mechanism of the model is thus, when the F-firms are hiring someone, they uniquely pin down the wage rate. The E-firms will pick their employment based on the capital that they have available, and the fact that this wage rate is pinned down by the large F-firms essentially provides a reserve army of the *employed* - a large labor pool that they can just comfortably draw from. As a consequence, since they have a more productive technology, they will accumulate more capital and thus labor over time, and the F-firms will gradually shrink.

2.4 E-firms

The problem of an E-firm with capital stock $K_{E,t}$ is:

$$\max \chi^{1-\alpha} K_{E,t}^{\alpha} N_{E,t}^{1-\alpha} - w_t N_{E,t}$$

The first order condition in this problem is:

$$N_{E,t}^{\alpha} = \frac{(1-\alpha)\chi^{1-\alpha}}{w_t} K_{E,t}^{\alpha}$$

$$N_{E,t}^{\alpha} = \left(\frac{\alpha}{R}\right)^{\frac{\alpha}{\alpha-1}} \chi^{1-\alpha} K_{E,t}^{\alpha}$$

$$N_{E,t} = \left(\frac{\alpha}{R}\right)^{\frac{1}{\alpha-1}} \chi^{\frac{1-\alpha}{\alpha}} K_{E,t}$$

So given a constant R, employment is a constant function of $K_{E,t}$.
Output of E firms given $K_{E,t}$ is then:

$$Y_{E,t} = \chi^{1-\alpha} \frac{R}{\alpha} \chi^{\frac{(1-\alpha)^2}{\alpha}} K_{E,t}$$

Now note that

$$\frac{(1-\alpha)^2}{\alpha} = \frac{1}{\alpha} - 2 + \alpha$$

So the above simplifies to:

$$Y_{E,t} = \chi^{\frac{\alpha-1}{\alpha}} \frac{R}{\alpha} K_{E,t}$$

The E-firm wage bill is then:

$$w_t N_{E,t} = (1-\alpha)\chi^{\frac{1-\alpha}{\alpha}} \left(\frac{R}{\alpha}\right) K_{E,t}$$

The value of the firm is then:

$$\begin{aligned} V_E(K_{E,t}) &= \alpha \chi^{\frac{(1-\alpha)}{\alpha}} \frac{R}{\alpha} K_{E,t} \\ &= \chi^{\frac{(1-\alpha)}{\alpha}} R K_{E,t} \\ &> R K_{E,t} \end{aligned}$$

So the entrepreneurs have a savings advantage by virtue of their ability to operate these firms. Hence their savings will grow faster over time than a (similar infinitely lived) worker agent.

Their return is ρ is thus:

$$\rho \equiv \chi^{\frac{1-\alpha}{\alpha}} R$$

2.5 The Growth of the Private Sector

We know that x_{t+1} is shared across generations. Fix a capital stock $K_{E,t}$.

From this, we know that:

$$x_{t+1} = \frac{\omega}{\beta+\omega}\rho K_{E,t}$$

$$K_{E,t+1} = \frac{\beta+\omega}{1+\beta+\omega}x_{t+1}$$

Hence

$$K_{E,t+1} = \frac{\beta+\omega}{1+\beta+\omega}\frac{\omega}{\beta+\omega}\rho K_{E,t}$$

And thus that:

$$\frac{K_{E,t+1}}{K_{E,t}} = \frac{\omega}{1+\beta+\omega}\rho$$

From this we see that if ρ (and by extension χ) is too small, then the E firms will not expand. Rather they will die out. Hence in order to explain the dynamics of growth, it must be that the productivity advantage of the private sector is large in some sense. Note that again, growth in the private sector is based on the ability of these firms to draw workers from the public sector at some constant wage rate independent of their marginal product of labor in the E sector. When this ability goes away because essentially the F firms have disappeared, then you simply have a neoclassical economy.

2.6 The Transition

What is aggregate GDP during the transition? It is

$$Y_t = Y_{E,t} + Y_{K,t}$$

We know that

$$Y_{E,t} = \chi^{\frac{\alpha-1}{\alpha}}\frac{R}{\alpha}K_{E,t}$$

From our expression for $N_{E,t}$ in terms of $K_{E,t}$ we know that this is equivalent to:

$$Y_{E,t} = \chi^{\frac{\alpha-1}{\alpha}}\frac{R}{\alpha}\left(\frac{\alpha}{R}\right)^{\frac{1}{\alpha-1}}\chi^{\frac{1-\alpha}{\alpha}}N_{E,t} = \left(\frac{R}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}N_{E,t}$$

From the market clearing condition, we know that

$$N_{F,t} = N - N_{E,t}$$

Since

$$K_{F,t} = N_{F,t} \left(\frac{R}{\alpha} \right)^{\frac{1}{\alpha-1}}$$

We know that:

$$K_{F,t} = (N - N_{E,t}) \left(\frac{R}{\alpha} \right)^{\frac{1}{\alpha-1}}$$

Hence:

$$Y_{F,t} = \left[(N - N_{E,t}) \left(\frac{R}{\alpha} \right)^{\frac{1}{\alpha-1}} \right]^{\alpha} (N - N_{E,t})^{1-\alpha} = \left(\frac{R}{\alpha} \right)^{\frac{\alpha}{\alpha-1}} (N - N_{E,t})$$

Combining these two expressions, we see that during the transition *there is no growth*. Output is constant.

$$Y_t = \left(\frac{R}{\alpha} \right)^{\frac{\alpha}{\alpha-1}} N$$

This is very different from SSZ, where there is growth. The reason is that in SSZ, there is managerial compensation that introduces a wedge on the marginal product of labor. This means that the marginal product of labor in the entrepreneurial sector is the marginal product of labor in the neoclassical sector divided by a factor $1 - \psi$, where $\psi \in (0, 1)$. As a consequence, the growth of the E-sector moves resources from a less distorted to a more distorted one. We know from Basu and Fernald that this results in growth, but intuitively, we are taking labor from a place where it will produce less and moving it to one where it will produce more. The E-sector is more valuable in that sense. In this version of the model, the marginal products are equated across sectors, so there is no incremental improvement from taking labor in one sector and moving it to the other at the margin.

3 Cost Minimization

Let $F(x_1, x_2, \ldots, x_n)$ be the production function of a firm.

Consider the cost minimization problem of the firm subject to a vector of factor prices $(w_1, w_2, \ldots, w_n)$

$$\min \sum_{i=1}^{n} w_i x_i$$

Subject to:

$$F(x_1, x_2, \ldots, x_n) \geq \bar{Y}$$

Call this problem $C^*(w_1, \ldots, w_n, \bar{Y})$.

This problem admits a Lagrangian formulation:

$$\Lambda(x_1, \ldots, x_n; w_1, \ldots, w_n, \bar{Y}) = \sum_{i=1}^{n} w_i x_i - \lambda \left(F(x_1, \ldots, x_n) - \bar{Y} \right)$$

The first order condition with respect to an arbitrary input is given by:

$$(x_i) : w_i - \lambda F_i(\mathbf{x}) = 0 \implies w_i = \lambda F_i(\mathbf{x})$$

The envelope theorem tells us that a given $\bar{Y}$, the choice of factor mix is optimal and hence for small perturbations, the agent will continue to use factors in the same proportion. Mathematically, this means that

$$C'(\bar{Y}) = \lambda$$

One can see this by rewriting the Lagrangian in a form that is parameterized by $\bar{Y}$:

$$\Lambda(\bar{Y}) = \sum_{i=1}^{n} w_i x_i^*(\bar{Y}) - \lambda \left(F(\mathbf{x}^*(\bar{Y}) - \bar{Y} \right)$$

Differentiating with respect to $\bar{Y}$, we recover:

$$\Lambda'(\bar{Y}) = \sum_{i=1}^{n} w_i \frac{dx_i}{d\bar{Y}} - \lambda \left(\sum_{i=1}^{n} F_i(\mathbf{x}(\bar{Y}) \frac{dx_i}{d\bar{Y}} \right) + \lambda$$

This can be rewritten as:

$$\Lambda'(\bar{Y}) = \sum_{i=1}^{n} \underbrace{\left[w_i - \lambda F_i(\mathbf{x}(\bar{Y}) \right]}_{0 \text{ by FOC}} \frac{dx_i}{d\bar{Y}} + \lambda$$

By the equivalence of the Lagrangian and the original cost minimization problem, we have the result.

Now let us suppose that the firm charges a markup $\mu > 1$ over marginal cost, so that $P = \mu C'(\bar{Y})$. It follows that $P = \mu\lambda$, and thus that $\lambda = \frac{P}{\mu}$.

From the first order condition of the firm with respect to input i, we have that:

$$w_i = \frac{P}{\mu} \frac{\partial F}{\partial x_i}$$

And hence

$$\underbrace{\frac{w_i x_i}{PY}}_{\text{Cost share of factor } i} = \frac{1}{\mu} \underbrace{\frac{\partial F}{\partial x_i}\frac{x_i}{Y}}_{\text{Output elasticity w.r.t factor } i}$$

Now let us make an additional assumption on FF, namely that it is homogeneous of degree η with $\eta > 0$. Recall that a function $f(\mathbf{x})$ is homogeneous of degree η when, for all $\lambda > 0$, we have that$f(\lambda\mathbf{x}) = \lambda^{\eta} f(\mathbf{x})$

By Euler's Homogeneity Theorem, we have that:

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} x_i = \eta F(\mathbf{x})$$

Summing both the left hand side and the right hand side of the expression above, we have that:

$$\frac{1}{PY}\sum_{i=1}^{n} w_i x_i = \frac{1}{\mu Y}\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}$$

This is equivalent to:

$$\frac{C(Y)}{PY} = \frac{\eta}{\mu}$$

4 Modern and Traditional Production

4.1 The setup

Imagine a world with two types of technologies, one used by modern workers that is Cobb-Douglas:

$$Y_M = \left(\frac{l_{M,u}}{1-\alpha}\right)^{1-\alpha} \left(\frac{l_{M,s}}{\alpha}\right)^{\alpha}$$

And one used by traditional workers that is linear:

$$Y_T = l_{T,u}$$

Aggregate utility is Weighted CES, so that the problem of the representative household is:

$$\max \left((1-\gamma) Y_T^{\frac{\eta-1}{\eta}} + \gamma Y_m^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}$$

Subject to:

$$P_T Y_T + (P_M e^{\tau}) Y_M \leq w_u(1-l_s) + w_s l_s + T$$

The government budget constraint dictates that:

$$P_M(e^{\tau}-1)Y_M = T$$

Labor supply is determined by:

$$\ln\left(\frac{l_s}{1-l_s}\right) = \ln(\Sigma) + \gamma \ln\left(\frac{w_s}{w_u}\right)$$

4.2 Unit Cost

The first task is to determine the unit cost functions, i.e to solve the cost minimization problem for $\bar{Y} = 1$ given that these functions are constant returns to scale.

The unit cost function for the traditional sector is obvious:

$$\min w_u l_{T,u}$$

Subject to:

$$l_u \geq 1$$

Hence, the unit cost function

$$C_T^*(w_u, w_s; 1) = w_u = P_T \text{ with the last equality following from zero profits}$$

The unit cost function for the modern sector is:

$$\min w_u l_{M,u} + w_s l_{M,s}$$

Subject to:

$$\underbrace{\frac{(1-\alpha)^{\alpha-1}}{\alpha^\alpha}}_{\Gamma} l_{M,u}^{1-\alpha} l_{M,s}^{\alpha} \geq 1$$

First order conditions are:

$$(l_{M,u}) : w_u = \lambda(1-\alpha)\left(\frac{l_{M,s}}{l_{M,u}}\right)^\alpha \Gamma$$

$$(l_{M,s}) : w_s = \lambda\alpha\left(\frac{l_{M,u}}{l_{M,s}}\right)^{1-\alpha} \Gamma$$

$$(\lambda) : \left(\frac{l_{M,u}}{1-\alpha}\right)^{1-\alpha}\left(\frac{l_{M,s}}{\alpha}\right)^{\alpha} = 1$$

Dividing the first by the second recovers that:

$$\frac{w_u}{w_s} = \frac{l_{M,s}}{\alpha} \times \frac{1-\alpha}{l_{M,u}}$$

More familiarly

$$\frac{w_u}{w_s} = \frac{1-\alpha}{\alpha} \times \frac{l_{M,s}}{l_{M,u}}$$

$$\frac{l_{M,u}}{1-\alpha} = \frac{w_s}{w_u}\frac{l_{M,s}}{\alpha}$$

Plugging this into (λ), we have that:

$$\left(\frac{w_s}{w_u}\right)^{1-\alpha}\frac{l_{M,s}}{\alpha} = 1$$

$$l_{M,s} = \alpha\left(\frac{w_u}{w_s}\right)^{1-\alpha}$$

From here we then have that

$$l_{M,u} = (1-\alpha)\left(\frac{w_s}{w_u}\right)^{\alpha}$$

Plugging this into the cost function, we have that

$$C_M^*(w_s, w_u; 1) = \alpha w_u^{1-\alpha} w_s^\alpha + (1-\alpha) w_u^{1-\alpha} w_s^\alpha = w_u^{1-\alpha} w_s^\alpha = P_m$$

Which was what was wanted.

4.3 The Consumer Problem

The consumer problem is as above. Taking first order conditions, we have that:

$$(Y_T) : (1-\gamma)U^{\frac{1}{\eta}} Y_T^{-\frac{1}{\eta}} = \lambda P_T$$

$$(Y_M) : \gamma U^{\frac{1}{\eta}} Y_M^{-\frac{1}{\eta}} = \lambda P_M e^{\tau}$$

Divide the latter by the former to recover:

$$\frac{\gamma}{1-\gamma} \frac{Y_M^{-\frac{1}{\eta}}}{Y_T^{-\frac{1}{\eta}}} = \frac{P_M e^{\tau}}{P_T}$$

$$\frac{\gamma}{1-\gamma} \frac{Y_T^{\frac{1}{\eta}}}{Y_M^{\frac{1}{\eta}}} = \frac{P_M e^{\tau}}{P_T}$$

$$\frac{Y_T^{\frac{1}{\eta}}}{Y_M^{\frac{1}{\eta}}} = \left[\frac{1-\gamma}{\gamma} \frac{P_M e^{\tau}}{P_T}\right]$$

$$\frac{Y_M}{Y_T} = \left[\frac{1-\gamma}{\gamma} \frac{P_M e^{\tau}}{P_T}\right]^{-\eta}$$

$$\frac{Y_M}{Y_T} = \left[\frac{1-\gamma}{\gamma} \frac{w_u^{1-\alpha} w_s^{\alpha} e^{\tau}}{w_u}\right]^{-\eta}$$

$$\frac{Y_M}{Y_T} = \left[\frac{1-\gamma}{\gamma} \left(\frac{w_s}{w_u}\right)^{\alpha} e^{\tau}\right]^{-\eta}$$

Do more algebra to find the price index is simply:

$$P = \left(\gamma^{\eta}(P_M e^{\tau})^{1-\eta} + (1-\gamma)^{\eta}(P_T)^{1-\eta}\right)^{\frac{1}{1-\eta}}$$

4.4 Real GDP

Let Real GDP Y be defined by

$$Y \equiv \frac{P_T Y_T + P_M e^{\tau} Y_M}{P}$$

The total derivative of Y is:

$$dY = \frac{P_T}{P} dY_T + \frac{P_M e^{\tau}}{P} dY_M$$

$$\frac{Y}{Y} dY = \frac{P_T}{P} dY_T + \frac{P_M e^{\tau}}{P} dY_M$$

$$d\ln(Y) = \frac{P_T}{PY} dY_T + \frac{P_M e^{\tau}}{PY} dY_M$$

$$d\ln(Y) = \frac{P_T Y_T}{PY} d\ln(Y_T) + \frac{P_M e^\tau Y_M}{PY} d\ln(Y_M)$$

Since $PY = P_T Y_T + P_M e^\tau Y_M$, we can define:

$$s_M = \frac{P_M e^\tau Y_M}{P_T Y_T + P_M e^\tau Y_M}$$

Hence we have:

$$d\ln(Y) = (1 - s_M) d\ln(Y_T) + s_M d\ln(Y_M)$$

Which was what was wanted.

4.5 Labor

By the population identity, we have that:

$$l_{T,u} + l_{M,u} + l_{M,s} = L$$

Treating this as a function of L and totally differentiating, we have:

$$dL = dl_{T,u} + dl_{M,u} + dl_{M,s}$$

$$\frac{L}{L} dL = \frac{l_{T,u}}{l_{T,u}} dl_{T,u} + \frac{l_{M,u}}{l_{M,u}} dl_{M,u} + \frac{l_{M,s}}{l_{M,s}} dl_{M,s}$$

$$d\ln(L) = e_{T,u} d\ln(l_{T,u}) + e_{M,u} d\ln(l_{M,u}) + e_{M,s} d\ln(l_{M,s})$$

Since LL is constant, it must be that

$$d\ln(L) = 0$$

, so we have:

$$d\ln(l_{T,u}) + e_{M,u} d\ln(l_{M,u}) + e_{M,s} d\ln(l_{M,s}) = 0$$

4.6 The Ratio

The claim is that:

$$d\ln(Y) = (s_M - e_M) d\ln\left(\frac{Y_M}{Y_T}\right)$$

We have from

$$d\ln(Y) = (1 - s_M) d\ln(Y_T) + s_M d\ln(Y_M)$$

That immediately

$$d\ln(Y) = d\ln(Y_T) + s_M d\ln\left(\frac{Y_M}{Y_T}\right)$$

So what we need to show is that

$$d\ln(Y_T) = -e_M d\ln\left(\frac{Y_M}{Y_T}\right)$$
$$d\ln(Y_T) = -e_M d\ln(Y_M) + e_M d\ln(Y_T)$$
$$d\ln(Y_T) = -(e_{M,u} + e_{M,s}) d\ln(Y_M)$$
$$d\ln(Y_T) = -e_{M,u} d\ln(Y_M) - e_{M,s} d\ln(Y_M)$$

Noting that

$$d\ln(Y_T) = d\ln(l_{T,u})$$

And that:

$$d\ln(Y_M) = (1-\alpha)d\ln(l_{M,u}) + \alpha d\ln(l_{M,s})$$

We have that:

$$d\ln(Y_T) = -e_{M,u}\left[(1-\alpha)d\ln(l_{M,u}) + \alpha d\ln(l_{M,s})\right] - e_{M,s}\left[(1-\alpha)d\ln(l_{M,u}) + \alpha d\ln(l_{M,s})\right]$$

$$d\ln(Y_T) = -e_{M,u}(1-\alpha)d\ln(l_{M,u}) - e_{M,u}\alpha d\ln(l_{M,s}) - e_{M,s}(1-\alpha)d\ln(l_{M,u}) - e_{M,s}\alpha d\ln(l_{M,s})$$

What remains is to claim that $e_{M,u} d\ln(l_{M,s}) = e_{M,s} d\ln(l_{M,u})$.

I think that this follows heuristically from the fact that the factor shares will remain constant so long as prices are invariant, necessitating a constant growth rate of factors along the expansion path.

In any event, this is algebra, and less interesting than the economic content of the expression. Let us suppose that $\tau > 0$ or $\frac{w_s}{w_u} > 1$.

Recall that

$s_m = \frac{P_m e^\tau Y_M}{P_m e^\tau Y_M + P_T Y_T}$

$s_m = \frac{e^\tau (w_s l_{M,s} + w_u l_{M,u})}{e^\tau (w_u l_{M,u} + w_s l_{M,s}) + w_u l_{T,u}}$

Now recall that

$e_m = \frac{l_{M,s} + l_{M,u}}{l_{M,s} + l_{M,u} + l_{T,u}}$

Let us first suppose that $e^\tau = 1$, so that $\tau = 0$. If $\frac{w_s}{w_u} > 1$, then, the following establishes the desired result:

Lemma: *Let $a, b, c \in \mathbb{R}_{++}$. Then $\forall \alpha \in (1, \infty)$, we have that $\frac{a+b}{a+b+c} < \frac{\alpha a+b}{\alpha a+b+c}$*

Proof. Let a, b, c, α be given as above. Observe that $(\alpha a + b)(a + b + c) = \alpha a^2 + \alpha ab + \alpha ac + ab + b^2 + bc$

Further, observe that:

$(a + b)(\alpha a + b + c) = \alpha a^2 + \alpha ab + ab + b^2 + ac + bc$

Hence:

$(a + b)(\alpha a + b + c) < (\alpha a + b)(a + b + c)$

And thus that:

$\frac{a+b}{a+b+c} < \frac{\alpha a+b}{\alpha a+b+c}$ ■

By the Lemma, the result follows. The result for $e^{\tau} > 1$ follows from a trivial corollary.[4]

As to why there is a reallocation gain, the intuition there is as follows. To the extent that we observe a markup in a given sector, this markup is due to a restriction of output more than would be optimal *ceteris paribus*. As a result, a flow of resources into a sector that is too small induces a production response that is comparatively larger, as the marginal product of those resources is higher.

4.7 Skilled Labor

Recall that:

$$\frac{Y_M}{Y_T} = \left[\frac{1-\gamma}{\gamma}\left(\frac{w_s}{w_u}\right)^{\alpha} e^{\tau}\right]^{-\eta}$$

Applying the natural logarithm, we have:

$$\ln\left(\frac{Y_M}{Y_T}\right) = \ln\left(\left[\frac{1-\gamma}{\gamma}\left(\frac{w_s}{w_u}\right)^{\alpha} e^{\tau}\right]^{-\eta}\right)$$

$$\ln\left(\frac{Y_M}{Y_T}\right) = -\eta\ln\left(\frac{1-\gamma}{\gamma}\right) - \eta\alpha\ln(\left(\frac{w_s}{w_u}\right) - \eta\ln(e^{\tau})$$

$$\ln\left(\frac{Y_M}{Y_T}\right) = -\eta\ln\left(\frac{1-\gamma}{\gamma}\right) - \eta\alpha\ln\left(\frac{w_s}{w_u}\right) - \eta\tau$$

Now suppose that τ is fixed. The total derivative of this function is then the extraordinarily simple:

$$d\ln\left(\frac{Y_M}{Y_T}\right) = -\eta\alpha d\ln\left(\frac{w_s}{w_u}\right)$$

From this result, we have that:

$$d\ln(Y) = (s_M - e_M)\left[-\eta\alpha \times d\ln\left(\frac{w_s}{w_u}\right)\right]$$

[4]The proof of the corollary is merely to take $b = \frac{\hat{b}}{\alpha}$.

5 Combinatorial Romer

These notes solve Erzo's version of Romer 1990.

The economic environment is as follows.

There is a unit measure of agents, with preferences

$$U(C) = \int_0^\infty e^{-\rho t} ln(C_t)dt$$

With

$$C_t = \left(\int_0^\Omega \sum_{j=1}^{J_{\omega,t}} c_{j,\omega,t}^{\frac{\epsilon-1}{\epsilon}} d\omega\right)^{\frac{\epsilon}{\epsilon-1}}$$

where $J_{\omega,t} \in \mathbb{N}$,$J_{\omega,0} \in 2^{\mathbb{N}\cup\{0\}}$ $\rho > 0$, and $\epsilon > 1$

Call $J_t \equiv \frac{1}{\Omega}\int_0^\Omega J_{\omega,t} d\omega$ the average number of goods across industries (where an industry is named by $\omega \in \Omega$).

5.1 Firm Problem

Note that there is monopolistic competition in this model, so we must consider the problem of a final goods producer and an individual intermediate goods producer.

The final goods producer problem is given by:

$$\max P_t \left(\int_0^\Omega \sum_{j=1}^{J_{\omega,t}} c_{j,\omega,t}^{\frac{\epsilon-1}{\epsilon}} d\omega\right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^\Omega \sum_{j=1}^{J_{\omega,t}} p_{j,\omega,t} c_{j,\omega,t} d\omega$$

Differentiating with respect to $c_{j,\omega,t}$, we recover the following condition:

$$P_t C_t^{\frac{1}{\epsilon}} c_{j,\omega,t}^{-\frac{1}{\epsilon}} = p_{j,\omega,t}$$

Solving for $c_{j,\omega,t}$, we recover

$$c_{j,\omega,t} = \left(\frac{P_t}{p_{j,\omega,t}}\right)^\epsilon C_t$$

This characterizes the demand for intermediate goods, which is internalized by each intermediates producer. Using this expression and the aggregation constraint for C_t, we can derive the ideal price index P_t:

$$C_t = \left(\int_0^\Omega \sum_{j=1}^{J_{\omega,t}} \left[\left(\frac{P_t}{p_{j,\omega,t}}\right)^\epsilon C_t\right]^{\frac{\epsilon-1}{\epsilon}} d\omega\right)^{\frac{\epsilon}{\epsilon-1}}$$

$$C_t = C_t \left(\int_0^\Omega \sum_{j=1}^{J_{\omega,t}} \left(\frac{p_{j,\omega,t}}{P_t}\right)^{1-\epsilon} d\omega\right)^{\frac{\epsilon}{\epsilon-1}}$$

$$P_t^{-\epsilon} = \left(\int_0^\Omega \sum_{j=1}^{J_{\omega,t}} (p_{j,\omega,t})^{1-\epsilon} d\omega\right)^{\frac{\epsilon}{\epsilon-1}}$$

$$P_t = \left(\int_0^{\Omega} \sum_{j=1}^{J_{\omega,t}} {p_{j,\omega,t}}^{1-\epsilon} d\omega \right)^{\frac{1}{1-\epsilon}}$$

The problem of an intermediate good producer with technology $y_{j,\omega,t} = l_{j,\omega,t}$ is given by:

$$\max P_t C_t^{\frac{1}{\epsilon}} c_{j,\omega,t}^{1-\frac{1}{\epsilon}} - P_t w_t c_{j,\omega,t}$$

Differentiating with respect to $c_{j,\omega,t}$, we recover:

$$\frac{\epsilon - 1}{\epsilon} P_t C_t^{\frac{1}{\epsilon}} c_{j,\omega,t}^{-\frac{1}{\epsilon}} = P_t w_t$$

Substituting back in using the conditional factor demand, we now have:

$$p_{j,\omega,t} = \frac{\epsilon}{\epsilon - 1} P_t w_t$$

We can also see that

$$l_{j,\omega,t} = c_{j,\omega,t} = \left[\frac{\epsilon - 1}{\epsilon} \right]^{\epsilon} C_t w_t^{-\epsilon}$$

Now we substitute into the price index to recover:

$$P_t = \left(\int_0^{\Omega} \sum_{j=1}^{J_{\omega,t}} \left(\frac{\epsilon}{\epsilon - 1} P_t w_t \right)^{1-\epsilon} d\omega \right)^{\frac{1}{1-\epsilon}}$$

$$P_t = \frac{\epsilon}{\epsilon - 1} P_t w_t \left(\int_0^{\Omega} \sum_{j=1}^{J_{\omega,t}} 1 d\omega \right)^{\frac{1}{1-\epsilon}}$$

$$1 = \frac{\epsilon}{\epsilon - 1} w_t \left(\int_0^{\Omega} J_{\omega,t} d\omega \right)^{\frac{1}{1-\epsilon}}$$

$$\frac{\epsilon - 1}{\epsilon} = w_t \left(J_t \Omega \right)^{\frac{1}{1-\epsilon}}$$

$$w_t = \frac{\epsilon - 1}{\epsilon} \left(J_t \Omega \right)^{\frac{1}{\epsilon-1}}$$

To get real labor returns we exploit the symmetry of the problem and recover:

$$w_t l_t = \left[\frac{\epsilon - 1}{\epsilon} \right] \frac{C_t}{(J_t \Omega)^{\frac{1}{1-\epsilon}}} (J_t \Omega)^{\frac{\epsilon}{1-\epsilon}} = \left[\frac{\epsilon - 1}{\epsilon} \right] \frac{C_t}{J_t \Omega}$$

Variable profits are of course the residual:

$$v_t = \left[\frac{1}{\epsilon} \right] \frac{C_t}{\Omega J_t}$$

5.2 Blueprints

In this economy, an innovation arrives in an industry, and then doubles the number of products in that industry. Imagine this as like cupholders doubling the number of unique cars, since every car can be produced with a cupholder or without.

In equilibrium it must be that

$$r_t q_t = v_t + \dot{q}_t$$

Where note that I am using dot notation to denote time derivatives.

5.3 Labor Supply

Individuals in this economy supply both labor and entrepreneurial services. This is done according to a Roy model. Broadly, suppose that individuals are characterized by an (x, y) drawn from a distribution Ψ with density ψ, where x can be thought of as their labor productivity and y can be thought of as their entrepreneurial skill.

Note that once an entrepreneur discovers an idea, they receive $q_t J_{\omega,t}$. Since there is no directed innovation in this model an entrepreneur's expected return is $q_t J_t$. Let the skill premium on entrepreneurial ability relative to wages be given by

$$s_t \equiv \frac{q_t J_t}{w_t}$$

A worker at time t picks whether to supply entrepreneurial ability or wages by comparing their own return on entrepreneurial services given their entrepreneurial skill y and entrepreneurial return $q_t J_t$ and comparing it to the wage they could command given the piece rate wage w_t and their labor productivity x. Mathematically, this means they consider:

$$w_t \max\{x, s_t y\}$$

If $x > s_t y$, they choose to supply labor, and if $x \leq s_t y$, they supply entrepreneurial services. This means that the aggregate supply of effective labor is given by:

$$\mathcal{L}(s_t) = \int_0^\infty \int_{s_t y}^\infty x\psi(x, y)dxdy$$

The aggregate supply of entrepreneurial skill is given by:

$$\mathcal{E}(s_t) = \int_0^\infty \int_0^{s_t y} y\psi(x, y)dxdy$$

5.4 Growth

Recall that everytime an entrepreneur has an idea, the number of products in that market doubles. This means that $J_{\omega,t} = 2J_{\omega,t-}$ and further that

$$\underbrace{J_{\omega,t}}_{\text{Ideas in industry } \omega \text{ at } t} = \underbrace{J_{\omega_0}}_{\text{Initial products}} \times \underbrace{2^{N_{\omega,t}}}_{2^{\text{Number of shocks to industry}}}$$

$N_{\omega,t}$ is Poisson with mean $A_t \equiv \frac{1}{\Omega} \int_0^t \mathcal{E}(s_u)du$

By definition of $J_{\omega,t}$ we have:

$$\mathbb{E}[J_{\omega,t}] = J_{\omega,0}\mathbb{E}[2^{N_{\omega,t}}]$$

Which is of course equivalent to:

$$J_{\omega,0}\sum_{n=0}^{\infty} 2^n \frac{A_t^n e^{-A_t}}{n!} = J_{\omega,0}e^{A_t} \underbrace{\sum_{n=0}^{\infty} \frac{(2A_t)^n e^{-2A_t}}{n!}}_{\text{Poisson with mean } 2A_t} = J_{\omega,0}e^{A_t}$$

Among other things this means that

$$\dot{J}_t = \underbrace{\frac{\mathcal{E}(s_t)}{\Omega}}_{\text{Average Entrepreneurial Effort}} \underbrace{\times J_t}_{\text{Average number of products in an industry}}$$

5.5 Equilibrium

We have the following static equilibrium conditions:

$$C_t = (\Omega J_t)^{\frac{1}{\epsilon-1}}\mathcal{L}(s_t)$$

Which is derived from our expression for $w_t l_{j,\omega,t}$ above and the fact that $\int_0^\Omega \sum_{j=1}^{J_{\omega,t}} l_{j,\omega,t} d\omega = \mathcal{L}(s_t)$

$$w_t = \left(\frac{\epsilon - 1}{\epsilon}\right)(\Omega J_t)^{\frac{1}{\epsilon-1}}$$

$$s_t = \frac{\mathcal{L}(s_t)}{1-\frac{1}{\epsilon}}\frac{q_t J_t}{C_t}$$

Which follows from the definition of s_t and our expression for the wage.

We also have a dynamic expression for the number of products in this economy:

$$(\dot{\Omega J_t}) = \frac{\mathcal{E}(s_t)}{\Omega}\Omega J_t$$

The only expressions we need to derive now are an expression for the real price of blueprints over time, given in terms of marginal utility.

That is to say that we need to find an expression for $\dot{\left(\frac{q_t}{c_t}\right)}$.

To do this, recall that first that the continuous time deterministic Euler is given by:

$$r_t = \rho + \frac{\dot{C}_t}{C_t}$$

From the no-arbitrage condition for the price of blueprints above we then have that:

$$(\rho + \frac{\dot{C}_t}{C_t})q_t = \frac{1}{\epsilon}\frac{C_t}{J_t} + \dot{q}_t$$

$$\rho q_t - \frac{1}{\epsilon}\frac{C_t}{\Omega J_t} = \dot{q}_t - \frac{q_t \dot{C}_t}{C_t}$$

$$\rho\frac{q_t}{C_t}-\frac{1}{\epsilon}\frac{1}{\Omega J_t}=\underbrace{\frac{\dot{q}_t}{C_t}-\frac{q_t\dot{C}_t}{C_t^2}}_{\left(\dot{\frac{q_t}{C_t}}\right)}$$

5.6 Solving the Model

To solve this, we will first combine the two dynamic problems:

$$\left(\dot{\frac{q_t\Omega J_t}{C_t}}\right)=\left(\dot{\frac{q_t}{C_t}}\right)\Omega J_t+\frac{q_t}{C_t}(\dot{\Omega J_t})$$

$$\left(\dot{\frac{q_t}{C_t}}\right)\Omega J_t+\frac{q_t}{C_t}(\dot{\Omega J_t})=(\Omega J_t)\left[\rho\frac{q_t}{C_t}-\frac{1}{\epsilon}\frac{1}{\Omega J_t}\right]+\frac{q_t}{C_t}\frac{\mathcal{E}(s_t)}{\Omega}\Omega J_t$$

$$(\Omega J_t)\left[\rho\frac{q_t}{C_t}-\frac{1}{\epsilon}\frac{1}{\Omega J_t}\right]+\frac{q_t}{C_t}\frac{\mathcal{E}(s_t)}{\Omega}\Omega J_t=\left(\rho+\frac{\mathcal{E}(s_t)}{\Omega}\right)\frac{q_t\Omega J_t}{C_t}-\frac{1}{\epsilon}$$

Setting this quantity to zero as in a balanced growth path implies that:

$$\left(\rho+\frac{\mathcal{E}(s_t)}{\Omega}\right)\frac{q_t\Omega J_t}{C_t}=\frac{1}{\epsilon}$$

Since the relative return of entrepreneurial ability to labor productivity must be constant along the balanced growth path, we also know that:

$$s=\frac{\epsilon}{\epsilon-1}\mathcal{L}(s)\frac{q_tJ_t}{C_t}$$

Now note that

$$\left(\rho+\frac{\mathcal{E}(s)}{\Omega}\right)=\frac{1}{\epsilon}\frac{C_t}{q_t\Omega s}$$

$$\left(\rho+\frac{\mathcal{E}(s)}{\Omega}\right)=\frac{1}{\epsilon}\frac{\epsilon}{\epsilon-1}\frac{\mathcal{L}(s)}{\Omega J_t}=\frac{1}{\epsilon-1}\frac{\mathcal{L}(s)}{\Omega s}$$

5.7 Planning Problem

Recall that $C_t=(\Omega J_t)^{\frac{1}{\epsilon-1}}\mathcal{L}(s_t)$, and that J_t grows according to $\dot{J}_t=\frac{\mathcal{E}(s_t)}{\Omega}J_t$

The Hamiltonian (after normalizing the measure of industries Ω to 1) is then given by:

$$\mathcal{H}(J,\lambda)=\max_s\underbrace{\frac{ln(J)}{\epsilon-1}+ln(\mathcal{L}(s))}_{\text{Maximize } C_t}+\underbrace{\lambda J\mathcal{E}(s)}_{\text{Subject to law of motion for } J_t}$$

Differentiating with respect to s we have:

$$\frac{\mathcal{L}'(s_t)}{\mathcal{L}(s_t)}+\lambda_tJ_t\mathcal{E}'(s_t)=0$$

Recall that

$$\mathcal{E}(s_t) = \int_0^\infty \int_0^{s_t y} y\psi(x,y)dxdy$$

Differentiating with respect to to s_t, we then have that:

$$\mathcal{E}'(s_t) = \int_0^\infty \frac{d}{ds_t} \int_0^{s_t y} y\psi(x,y)dxdy$$

By Leibnitz we have:

$$\mathcal{E}'(s_t) = \int_0^\infty y^2\psi(s_t y, y)dy$$

Similarly for $\mathcal{L}(s_t)$, we have:

$$\mathcal{L}(s_t) = \int_0^\infty \int_{s_t y}^\infty x\psi(x,y)dxdy$$

Reordering the integrals and correcting the bounds, we have:

$$\mathcal{L}(s_t) = \int_0^\infty \int_0^{\frac{x}{s_t}} x\psi(x,y)dydx$$

Again differentiating, we have:

$$\mathcal{L}'(s_t) = \int_0^\infty x\frac{d}{ds_t} \int_0^{\frac{x}{s_t}} \psi(x,y)dydx$$

$$\mathcal{L}'(s_t) = \int_0^\infty -\frac{x^2}{s_t^2}\psi(x,\frac{x}{s_t})dx$$

Now apply a change of variables $y = \frac{x}{s_t} \iff s_t y = x$, and hence $dx = s_t dy$

$$\mathcal{L}'(s_t) = \int_0^\infty -\frac{x^2}{s_t^2}\psi(x,\frac{x}{s_t})dx$$

$$\mathcal{L}'(s_t) = \int_0^\infty -\frac{(s_t y)^2}{s_t^2}\psi(s_t y, y)s_t dy$$

$$\mathcal{L}'(s_t) = -s_t \int_0^\infty y^2\psi(s_t y, y)dy$$

Thus we have:

$$s_t\mathcal{E}'(s_t) + \mathcal{L}'(s_t) = 0$$

Using this fact, we can reexpress the Hamiltonian first order condition as:

$$\frac{1}{\mathcal{L}(s_t)} = \frac{\lambda_t J_t}{s_t}$$

Now, under the dynamics of this single state and co-state Hamiltonian, we know that dynamics are given by:

$$\dot{J}_t = J_t\mathcal{E}(s_t)$$

$$\dot{\lambda_t} = \rho\lambda_t - \frac{1}{\epsilon - 1}\frac{1}{J_t} + \lambda_t \mathcal{E}(s_t)$$

With the additional condition that:

$$\underbrace{\frac{s_t}{\mathcal{L}(s_t)}}_{\text{Marginal benefit of more consumption today}} = \underbrace{\lambda_t J_t}_{\text{Marginal benefit from having more blueprints in the future}}$$

Now note that by the product rule, we have that:

$$(\dot{\lambda_t J_t}) = \rho\lambda_t J_t - \frac{1}{\epsilon - 1}$$

Along an optimal path, this quantity must be equal to zero, so we must have that:

$$\lambda_t J_t = \frac{1}{\rho}\frac{1}{\epsilon - 1}$$

Immediately we then see that:

$$\frac{s^*}{\mathcal{L}(s^*)} = \frac{1}{\rho}\frac{1}{\epsilon - 1} > \frac{1}{\rho + \mathcal{E}(s)}\frac{1}{\epsilon - 1} = \frac{s}{\mathcal{L}(s)}$$

In other words, the planner wants more research done. The intuition is that every time someone introduces a new set of goods, that doubles the amount of goods in an industry. When someone after that has a new idea, that means that there are 4 times as many goods in that industry now. As a result, the first person only gets profits from essentially $J_{\omega,t}$ products, but the second person gets profits from $2 \times J_{\omega,t}$ products. Nothing changed other than the order of who recieved the Poisson shock, so in some sense it is optimal for inventors to "wait" for product markets to get large. This collective waiting is inefficient.

5.8 Adding Population Growth

In this section, Erzo adds population growth to Romer's model. Population growth is modeled as $H_t = H_0 e^{\eta t}$. Utility is changed to be:

$$U(c) = \int_0^\infty e^{-\rho t} H_t ln(\frac{C_t}{H_t})dt$$

The entrepreneurial skill and labor productivity distribution is untouched, so that $\mathcal{L}(s_t)$ and $\mathcal{E}(s_t)$. We will however let entrepreneurial ability be devoted to either starting new industries E_t or increasing the number of products in pre-existing industries M_t. This means that the provision of effective labor and effective entrepreneurial ability is given by:

$$L_t = \mathcal{L}(s_t)H_t$$

$$E_t + M_t = \mathcal{E}(s_t)H_t$$

In addition, with exogeneous probability δ a whole industry dies. Hence we have the following law of motion for the size of industries:

$$\dot{\Omega}_t = E_t - \delta\Omega_t$$

The average number of products in a given industry evolves according to:

$$\dot{J}_t = \frac{M_t}{\Omega_t} J_t$$

Note that the firm problems are unchanged, so the labor share and profit share are unchanged. From the fact that we consider symmetric allocations of labor and thus that

$$\int_0^\Omega \sum_{j=1}^{J_{\omega,t}} l_t d\omega = L_t = \mathcal{L}(s_t) H_t$$

We have that

$$l_t \Omega_t J_t = \mathcal{L}(s_t) H_t$$

Using factor shares to labor, we have:

$$w_t \mathcal{L}(s_t) = \left(\frac{\epsilon - 1}{\epsilon}\right) \frac{C_t}{H_t}$$

Variable profits are given by:

$$v_t = \frac{1}{\epsilon} \frac{C_t}{H_t} \frac{H_t}{\Omega_t J_t}$$

And the relative return is given by:

$$s_t = \frac{q_t J_t}{w_t} = q_t J_t \frac{H_t}{C_t} \frac{\epsilon \mathcal{L}(s_t)}{\epsilon - 1}$$

The price of blueprints is given by the following no arbitrage:

$$r_t q_t = \underbrace{v_t}_{\text{Flow profits}} + \underbrace{\dot{q}_t - \delta q_t}_{\text{Expected capital gains}}$$

The Euler here is given by:

$$r_t = \rho + \frac{\dot{\left(\frac{C_t}{H_t}\right)}}{\frac{C_t}{H_t}}$$

We now find the differential equation governing the marginal utility value of blueprints:

$$\dot{\left(\frac{q_t}{\frac{C_t}{H_t}}\right)} = \frac{\dot{q}_t}{\frac{C_t}{H_t}} - q_t \frac{\dot{\frac{C_t}{H_t}}}{\frac{C_t}{H_t}^2} = \frac{(r_t + \delta) q_t - v_t}{\frac{C_t}{H_t}} - \frac{q_t (r_t - \rho)}{\frac{C_t}{H_t}}$$

Now that the r_t cancels so we have:

$$\dot{\left(\frac{q_t}{\frac{C_t}{H_t}}\right)} = (\rho + \delta) \frac{q_t}{C_t H_t} - \frac{1}{\epsilon} \frac{H_t}{\Omega J_t}$$

We do as we did before and use the product rule to find:

$$\dot{\left(\frac{q_t J_t}{\frac{C_t}{H_t}}\right)} = (\rho + \delta + \frac{M_t}{\Omega_t})\frac{q_t J_t}{\frac{C_t}{H_t}} - \frac{1}{\epsilon}\frac{H_t}{\Omega_t}$$

Now we can collect all of the equilibrium conditions:

1. Law of motion for number of industries per capita:

$$\dot{\left(\frac{\Omega_t}{H_t}\right)} = \frac{E_t}{H_t} - (\eta + \delta)\frac{\Omega_t}{H_t}$$

2. Law of motion for average number of products per industry:

$$\dot{J}_t = \frac{M_t}{\Omega_t} J_t$$

3. Evolution of value of entrepreneurial activity:

$$\dot{\left(\frac{q_t J_t}{\frac{C_t}{H_t}}\right)} = (\rho + \delta + \frac{M_t}{\Omega_t})\frac{q_t J_t}{\frac{C_t}{H_t}} - \frac{1}{\epsilon}\frac{H_t}{\Omega_t}$$

4. Market clearing in entrepreneurial market:

$$\frac{M_t}{H_t} + \frac{E_t}{H_t} = \mathcal{E}(s_t)$$

5. Market clearing in labor:

$$s_t = \frac{\epsilon}{\epsilon - 1}\mathcal{L}(s_t)\frac{q_t J_t}{\frac{C_t}{H_t}}$$

All that remains is to solve for the balanced growth path. To do so, suppose that such a path exists with stable relative price $s_t = s$ and average number of industries per capita $\frac{\Omega_t}{H_t} = \frac{\Omega}{H}$. This gives us restriction on conditions (1) and (4):

$$\frac{E_t}{H_t} = (\eta + \delta)\frac{\Omega}{H}$$

$$\frac{M_t}{H_t} + \frac{E_t}{H_t} = \mathcal{E}(s)$$

Together, we then have

$$\frac{M_t}{H_t} + (\eta + \delta)\frac{\Omega_t}{H_t} = \mathcal{E}(s)$$

$$\frac{M_t}{H_t} = \mathcal{E}(s) - (\eta + \delta)\frac{\Omega}{H}$$

From setting (3) equal to 0, we have that:

$$(\rho + \delta + \frac{M_t}{\Omega_t})\frac{q_t J_t}{\frac{C_t}{H_t}} = \frac{1}{\epsilon}\frac{H_t}{\Omega_t}$$

Solving (5) for the price delivers us that:

$$\frac{q_t J_t}{\frac{C_t}{H_t}} = \frac{\epsilon - 1}{\epsilon}\frac{s}{\mathcal{L}(s)}$$

And hence:

$$(\rho + \delta + \frac{M_t}{\Omega_t})\frac{\epsilon - 1}{\epsilon}\frac{s}{\mathcal{L}(s)} = \frac{1}{\epsilon}\frac{H}{\Omega}$$

Which can be rewritten as:

$$\rho + \delta + \mathcal{E}(s)\frac{H}{\Omega} = \eta + \delta + \frac{1}{\epsilon - 1}\frac{\mathcal{L}(s)}{s}\frac{H}{\Omega}$$

We also require that $\frac{M_t}{H_t} \geq 0$, so we have:

$$\mathcal{E}(s) - (\eta + \delta)\frac{\Omega}{H} \geq 0$$

The growth rate of average industry size is given by:

$$ln(\dot{J_t}) = \mathcal{E}(s)\frac{H}{\Omega} - (\eta + \delta) \geq 0$$

Which implies the total number of products grows at the same rate.

$$ln(\dot{\Omega_t J_t}) = \mathcal{E}(s)\frac{H}{\Omega} - (\eta + \delta) \geq 0$$

As a special case, suppose that average industry size is constant, then the above reduces to

$$\mathcal{E}(s)\frac{H}{\Omega} = (\eta + \delta)$$

And we can substitute this into the other balanced growth expression to recover:

$$\rho + \delta = \frac{1}{\epsilon - 1}\frac{\mathcal{L}(s)}{s}\frac{H}{\Omega}$$

Substituting for $\frac{H}{\Omega}$

$$\frac{\rho + \delta}{\eta + \delta} = \frac{1}{\epsilon - 1}\frac{\mathcal{L}(s)}{s}\frac{1}{\mathcal{E}(s)}$$

6 Multi-Sector AK

6.1 Setup

Utility in this world is

$$U(C) = \int_0^\infty e^{-\rho t} \ln(C_t) dt$$

Where

$$C_t = \left(\sum_{n=1}^{N} C_{n,t}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

That is, aggregate consumption is a CES combination of goods in N separate industries with elasticity of substitution across them of ϵ.

Each sector n has a Cobb-Douglas production technology given by:

$$C_{n,t} = X_{n,t}^{1-\alpha} L_{n,t}^{\alpha}$$

And the labor market clearing condition is:

$$\sum_{i=1}^{N} L_{n,t} \leq L$$

Let $K_t \equiv [K_{1,t}, \ldots, K_{N,t}]'$ denote the collection of sector specific capital and $X_t \equiv [X_{1,t}, \ldots, X_{N,t}]'$ denote the collection of time t flow investments into each sector's capital stock. To fully specify the law of motion for aggregate capital K_t, we need to specify the degree to which one sector's capital can be switched to be used in another sector instaneously (the degree to which we can turn swords into plowshares if you will). One natural choice would be to specify $A \in \mathbb{R}^{N \times N} = \mathbb{I}_N$. This assumption is akin to assuming that each sector's capital is unique and cannot be repurposed. This is **not** the assumption that Erzo makes. Erzo specifies that:

$$A \in \mathbb{R}^{N \times N} : \forall i, j \in \{1, 2, \ldots, N\} \; A_{i,j} > 0$$

That is that A is a $N \times N$ matrix with **strictly positive** real entries. The strictly positive assumption here plays a crucial role, as will be discussed later. Importantly, we can interpret A in a similar manner to how we would interpret a stochastic matrix. Namely, the rows correspond to the transition probabilities from one type of capital to another type of capital, so that, for instance $A_{1,1}$ is akin to the probability that the capital stays in type 1 today, while $A_{1,2}$ is akin to the probability that capital changes from type 2 to type 1.[5]

This implies that the law of motion for capital is given by:

$$\dot{K}_t = AK_t - X_t$$

This is a vector-valued differential equation, or alternatively a N-coupled system of ordinary differential equations.

[5] To see this, note that A is $n \times n$, our vector of capital is $n \times 1$, which means that the first entry in our system of differential equations is going to $\dot{K}_{1,t} = \sum_{n=1}^{N} A_{1,n} K_{n,t} - X_{1,t}$.

6.2 An Arbitrary Sector

As with any model with differentiated goods, we begin by characterizing the individual demands that an individual sector faces, given a final goods producer's problem.

This problem is:

$$\max P_t \left(\sum_{i=1}^{N} C_{n,t}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}} - \sum_{n=1}^{N} p_{n,t} C_{n,t}$$

The first order condition with respect to the choice of an arbitrary sector specific consumption good is:

$$P_t C_t^{\frac{1}{\epsilon}} C_{n,t}^{-\frac{1}{\epsilon}} = p_{n,t}$$

This implies that

$$C_{n,t} = C_t \left(\frac{P_t}{p_{n,t}} \right)^{\epsilon}$$

Next we derive an aggregate price index, given individual aggregate demands.

Recall that

$$C_t = \left(\sum_{n=1}^{N} C_{n,t}^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

Hence

$$C_t = \left(\sum_{n=1}^{N} (C_t \left(\frac{P_t}{p_{n,t}} \right)^{\epsilon})^{\frac{\epsilon-1}{\epsilon}} \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$1 = \left(\sum_{n=1}^{N} \left(\frac{P_t}{p_{n,t}} \right)^{\epsilon-1} \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$1 = P_t^{\epsilon} \left(\sum_{n=1}^{N} (p_{n,t})^{1-\epsilon} \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$P_t^{-\epsilon} = \left(\sum_{n=1}^{N} (p_{n,t})^{1-\epsilon} \right)^{\frac{\epsilon}{\epsilon-1}}$$

$$P_t = \left(\sum_{n=1}^{N} p_{n,t}^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}$$

The next step is to consider the problem of an intermediate goods producer. In this case, Erzo has in mind that each sector is competitive, so intra-sector profits are driven to zero.[6] To put it differently, the representative firm in each sector does not consider the pricing power that they

[6] This is why the factor demands are not premultiplied by $\frac{\epsilon-1}{\epsilon}$.

have, given the elasticity across sectors. In this world, then, the profit maximization problem of a sector is given by:

$$\max p_{n,t}C_{n,t} - v_{n,t}X_{n,t} - w_t L_{n,t}$$

Subject to:

$$C_{n,t} = X_{n,t}^{1-\alpha} L_{n,t}^{\alpha}$$

This is a standard problem, with the standard factor demands.

The first order condition with respect to $X_{n,t}$ is:

$$p_{n,t} \times \underbrace{(1-\alpha)\left(\frac{L_{n,t}}{X_{n,t}}\right)^{\alpha}}_{\text{MPK}} = v_{n,t}$$

This gives a factor demand

$$v_{n,t}X_{n,t} = (1-\alpha)p_{n,t}C_{n,t}$$

The factor demand for labor is of course also:

$$w_t L_{n,t} = \alpha p_{n,t} C_{n,t}$$

Using the isoelastic demand of the final goods producer, we have that:

$$w_t L_{n,t} = \alpha P_t C_t^{\frac{1}{\epsilon}} C_{n,t}^{1-\frac{1}{\epsilon}}$$

Summing across sectors, we have:

$$w_t L_t = \alpha P_t C_t^{\frac{1}{\epsilon}} \underbrace{\sum_{n=1}^{N} C_{n,t}^{1-\frac{1}{\epsilon}}}_{C_t^{1-\frac{1}{\epsilon}}}$$

Hence

$$w_t L_t = \alpha P_t C_t$$

Now we shall take a brief aside. Suppose that we have a generic Cobb-Douglas firm. It's cost minimization problem is:

$$\min v_{n,t}X_{n,t} + w_t L_{n,t}$$

Subject to:

$$X_{n,t}^{1-\alpha} L_{n,t}^{\alpha} \geq \bar{Y}$$

FOCs are:

$$(X_{n,t}) : v_{n,t} + \lambda \left[(1-\alpha)\left(\frac{L_{n,t}}{X_{n,t}}\right)^{\alpha}\right] = 0$$

$$(L_{n,t}) : w_t + \lambda \left[\alpha \left(\frac{X_{n,t}}{L_{n,t}} \right)^{1-\alpha} \right] = 0$$

Divide the two FOCs by each other, so that:

$$\frac{v_{n,t}}{w_t} = \frac{1-\alpha}{\alpha} \frac{L_{n,t}}{X_{n,t}}$$

Or more familiarly,

$$\frac{v_{n,t} X_{n,t}}{w_t L_{n,t}} = \frac{1-\alpha}{\alpha}$$

$$\frac{v_{n,t} X_{n,t}}{1-\alpha} = \frac{w_t L_{n,t}}{\alpha}$$

Suppose that $\bar{Y} = 1$, which is without loss, given that the technology is constant returns. One would recover (from this, and from the zero profit condition):

$$p_{n,t} = \left(\frac{v_{n,t}}{1-\alpha} \right)^{1-\alpha} \left(\frac{w_t}{\alpha} \right)^{\alpha}$$

In real terms, this is:

$$\frac{p_{n,t}}{P_t} = \left(\frac{v_{n,t}/P_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{w_t/P_t}{\alpha} \right)^{\alpha}$$

Now, from this, we can reexpress the aggregate price level in terms of factor prices:

$$P_t = \left(\sum_{n=1}^{N} (P_t \frac{p_{n,t}}{P_t})^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}$$

$$1 = \left(\sum_{n=1}^{N} \left(\frac{p_{n,t}}{P_t} \right)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}$$

$$1 = \left(\sum_{n=1}^{N} \left(\left(\frac{v_{n,t}/P_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{w_t/P_t}{\alpha} \right)^{\alpha} \right)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}$$

$$1 = \left(\sum_{n=1}^{N} \left(\left(\frac{v_{n,t}/P_t}{1-\alpha} \right)^{1-\alpha} \right)^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}} \left(\frac{w_t/P_t}{\alpha} \right)^{\alpha}$$

6.3 Pricing Things

If we want things to make sense (in the sense that we have some kind of production of every good), we need to have $X_{n,t} > 0\ \forall n, t$. Hence this is the case that is worth considering.

As with all pricing problems, we start from a no-arbitrage condition. For this economy that is:

$$\underbrace{r_t \frac{q_{n,t}}{P_t}}_{\text{Required return}} = \underbrace{\frac{q_t' a_n}{P_t}}_{\text{Change in price due to capital transitioning}} + \underbrace{\dot{\left(\frac{q_{n,t}}{P_t}\right)}}_{\text{Capital Gains}}$$

Note that a_n is the n-th column vector of A. The row vectors describe something akin to transition probabilities into that type of capital, so the column vectors describe something akin to transition probabilities out of that sector.

We also have the familiar consumer Euler in the form of

$$r_t = \rho + \frac{\dot{C}_t}{C_t}$$

The game here is of course to eliminate the interest rate, and find the differential equation for marginal utility weighted prices, that is $\dot{\left(\frac{q_{n,t}}{P_t C_t}\right)}$

Begin by noting that

$$\dot{\left(\frac{q_{n,t}}{P_t C_t}\right)} = \frac{\dot{\left(\frac{q_{n,t}}{P_t}\right)}}{C_t} - \frac{\frac{q_{n,t}}{P_t}\dot{C}_t}{C_t^2}$$

$$\dot{\left(\frac{q_{n,t}}{P_t C_t}\right)} = r_t \frac{q_{n,t}}{P_t C_t} - \frac{q_t' a_n}{P_t C_t} - \frac{(r_t - \rho) q_{n,t}}{P_t C_t}$$

$$\dot{\left(\frac{q_{n,t}}{P_t C_t}\right)} = -\frac{q_t' a_n}{P_t C_t} + \rho \frac{q_{n,t}}{P_t C_t}$$

$$\dot{\left(\frac{q_{n,t}}{P_t C_t}\right)} = \rho \frac{q_{n,t}}{P_t C_t} - \frac{q_t' a_n}{P_t C_t}$$

Stacking these conditions, we have the following differential equation:

$$\dot{\left(\frac{q_t}{P_t C_t}\right)} = \rho \frac{q_t}{P_t C_t} - \frac{A' q_t}{P_t C_t}$$

6.4 Balanced Growth

Let us suppose that the relative price of capital across sectors is constant, and that

$$X_t = X e^{\kappa t}$$

This immediately implies that

$$C_t = C e^{(1-\alpha)\kappa t}$$

I.e. that C_t grows according to the factor intensity of capital. This should not be surprising given that L is taken to be fixed in this model, so the only source of growth here is capital accumulation.

The next thing to observe is that:

$$\frac{v_t}{P_t} = \frac{v}{P} \times e^{-\alpha\kappa t}$$

This follows from the fact that the ratio of prices of the various kinds of capital are constant, and hence all must grow at some common rate. The fact that this rate is $e^{-\alpha\kappa t}$ follows from the fact that:

$$\frac{v_{n,t}}{P_t} = \underbrace{(1-\alpha)(\frac{p_{n,t}}{P_t})^{1-\epsilon}}_{\text{Pinned down ex ante}} \qquad \underbrace{\frac{C_t}{X_{n,t}}}_{\text{Ratio of growth rates according to } (1-\alpha)\kappa \text{ and } \kappa}$$

Note that since $v_{n,t} = q_{n,t}$, we also have the same statement about the price of capital, not just the factor price, and hence

$$\frac{q_t}{P_t} = \frac{q}{P} e^{-\alpha\kappa t}$$

$$\frac{q_t}{P_t C_t} = \frac{q}{PC} e^{(-\alpha-(1-\alpha))\kappa t}$$

$$\frac{q_t}{P_t C_t} = \frac{q}{PC} e^{-\kappa t}$$

Hence we have that:

$$\left(\frac{\dot{q_t}}{P_t C_t}\right) = -\kappa \frac{q}{PC} e^{-\kappa t} = -\kappa \frac{q_t}{P_t C_t}$$

This is important, because from the pricing equation, we have:

$$\left(\frac{\dot{q_t}}{P_t C_t}\right) = \rho \frac{q_t}{P_t C_t} - \frac{A' q_t}{P_t C_t}$$

Hence, in order for us to have balanced growth, we must have:

$$-\kappa \frac{q_t}{P_t C_t} = \rho \frac{q_t}{P_t C_t} - \frac{A' q_t}{P_t C_t}$$

$$\frac{A' q_t}{P_t C_t} = \rho \frac{q_t}{P_t C_t} + \kappa \frac{q_t}{P_t C_t}$$

$$A' \frac{q_t}{P_t C_t} = (\rho + \kappa) \frac{q_t}{P_t C_t}$$

More familiarly again, we have:

$$\left(\frac{q_t}{P_t C_t}\right)' A = (\rho + \kappa) \left(\frac{q_t}{P_t C_t}\right)'$$

What does this say? It says that $\rho+\kappa$ is an eigenvalue of A and that $\frac{q_t}{P_t C_t}$ is its associated left eigenvector (or rather the transpose of).[7]

With this in hand, let us further suppose that

$$X_t = xK_t$$

, so that we have:

$$\dot{K}_t = (A - x\mathbb{I}_n)\, K_t$$

From the prior conjecture that $X_t = Xe^{\kappa t}$, we also then know that $K_t = Ke^{\kappa t}$, which further simplifies the above to:

$$0 = (A - (\kappa + x)\mathbb{I}_n)\, K$$

Now we reach the point at which the assumption that the matrix A has all positive entries comes into play.

Perron-Frobenius Theorem: *Suppose that* $\mathbf{A}$ *is a strictly positive (and hence real-valued) square matrix. Then the following are true:*[a]

1. *There exists* $r \in \mathbb{R}_{++}$ *such that* r *is an eigenvalue of* $\mathbf{A}$ *and any other eigenvalue* λ *of* $\mathbf{A}$ *has the property that* $|\lambda| < r$. *We call* r *the Perron root of* $\mathbf{A}$ *and note that it trivially coincides with the spectral radius of* $\mathbf{A}$.

2. *Such* r *is simple (i.e. has geometric multiplicity* 1*).*

3. *There exists an eigenvector for* r *that is strictly positive (in the vector sense). We call this vector the Perron eigenvector.*

4. *There exists no other eigenvectors of* $\mathbf{A}$ *that are non-negative (in the vector sense).*

5. $\lim_{k\to\infty} \frac{\mathbf{A}^k}{r^k} = \mathbf{v}\mathbf{w}'$, *where* $\mathbf{v}$ *and* $\mathbf{w}$ *are normalized so that* $\mathbf{w}'\mathbf{v} = 1$. *This matrix is the projection onto the eigenspace associated with* r*, and hence we call this projection the Perron projection.*

[a] Note that this is not the true theorem. This is the baby version, the real version is about irreducible non-negative matrices.

Since we want that the vector of capitals to be strictly positive, it is necessarily the case that $\kappa + x = r_A$, and the vector of capitals the associated Perron eigenvector.

Taking stock, we know that the following statements are true:

[7] Recall that a left eigenvector is a vector u such that, for some eigenvalue λ, the following equality holds: $u'A = \lambda u'$. Contrast this with a right eigenvector (sometimes called a principal eigenvector, or more commonly simply referred to as an eigenvector), which solves the problem of $Au = \lambda u$. Note that while left and right eigenvalues are the same, left and right eigenvectors need not be, and in general won't be. Also note that a left eigenvector of a matrix A' associated with eigenvalue λ, will be a right eigenvector of A associated with the same eigenvalue λ. This is the easiest way to remember the distinction: a right eigenvector is the normal eigenvector of a matrix, while a left eigenvector is a (right) eigenvector of the transpose of that same matrix. A stationary distribution of a Markov Chain is the left eigenvector associated with the eigenvalue 1, which is also the spectral radius of the transition matrix.

1. Steady state capital is the Perron eigenvector of A, so that:
$$AK = r_A K$$
2. Capital prices are a left eigenvector associated with $\rho + \kappa$, so that:
$$q'A = q'(\rho + \kappa)$$

Hence we have that
$$q'AK = q' r_A K$$
And
$$q'AK = q'(\rho + \kappa)K$$
Finally, we have that:
$$r_A q'K = (\rho + \kappa) q'K$$
And thus that:
$$\rho + \kappa = r_A$$
Since r_A is the Perron root of A, we know from the equilibrium conditions that $r_A = \kappa + x$, and thus we can rewrite the above as:
$$\rho + \kappa = x + \kappa$$
$$\rho = x$$
$$\kappa = r_A - \rho$$
To find the other things, we simply need to note that:
$$\frac{L_n}{L_m} = \left(\frac{p_{n,t}}{p_{m,t}}\right)^{1-\epsilon}$$
$$\frac{p_{n,t}}{p_{m,t}} = \left(\frac{v_n}{v_m}\right)^{1-\alpha}$$
$$\frac{v_n}{v_m} = \frac{q_n}{q_m}$$
From the fact that:
$$\frac{q'X_t}{P_t C_t} = (1-\alpha)\left(\frac{p_{n,t}}{P_t}\right)^{1-\epsilon}$$
Now from standard Cobb-Douglas logic, we know that the total capital share of output must be $1-\alpha$, the output elasticity with respect to capital.
$$\frac{q'X}{PC} = 1 - \alpha$$
And then consumption follows from the X_n and L_n expressions in the obvious way.

7 Grossman and Helpman

The environment in GH is as follows:

1. Preferences:
$$U(C) = \int_0^\infty e^{-\rho t} H ln(\frac{C_t}{H}) dt$$
2. Consumption aggregator:
$$C_t = \exp\left(\int_0^1 ln(C_{\omega,t}) d\omega\right)$$
3. Price Index:
$$P_t = \exp\left(\int_0^1 ln(p_{\omega,t}) d\omega\right)$$
4. Intermediate Goods Technology:
$$y_{\omega,t} = z_{\omega,t} l_{\omega,t}$$

7.1 Technological Growth

Note that both Grossman-Helpman and Klette-Kortum make use of a "competitive fringe" to constrain the pricing behavior of individual monopolists. The story here is as follows. Suppose that entrepreneurs invest in order to improve a time tt blueprint. When they improve the blueprint, they get a monopoly in that industry, as they are suddenly more productive than all the competition. The old, less productive, technology becomes common knowledge and anyone can produce using it.

More specifically, $z_{\omega,t} = z_{\omega,0}\lambda^{N_{\omega,t}}$, where $N_{\omega,t}$ is a Poisson random variable, with Poisson rate $\gamma m_{\omega,t}$, where $m_{\omega,t}$ is labor devoted to replication. When there is a shock, the new productivity is $z_{\omega,t} = \lambda z_{\omega,t-}$.

This means that the competitive fringe has marginal cost given by $\frac{p_{\omega,t}}{P_t} = \lambda \frac{w_t}{z_{\omega,t}}$.

7.2 The Firm Problem

The Problem of a Final Goods producer here is given by:

$$\max P_t \exp\left(\int_0^1 ln(C_{\omega,t}) d\omega\right) - \int_0^1 p_{\omega,t} C_{\omega,t}$$

The first order condition with respect to $C_{\omega,t}$ is:

$$\frac{P_t}{C_{\omega,t}} C_t = p_{\omega,t}$$

In a symmetric equilibrium, we know that $C_{\omega,t} = C_{\omega',t}$

This implies that $C_t = C_{\omega,t}$, through the consumption aggregator.

Now consider the problem of the intermediate's good producer:

$$\max \left(\frac{p}{P_t} - \frac{w_t}{z_{\omega,t}}\right) C_{\omega,t}$$

Substituting $C_{\omega,t}$ for $\frac{P_t}{p_{\omega,t}}C_t$, this problem is equivalent to:

$$\max\left(1-\frac{w_t}{z_{\omega,t}}\frac{P_t}{p_{\omega,t}}\right)C_t$$

Noting that the objective is increasing in $p_{\omega,t}$, we know that the limit price will bind, so thus we have that:

$$p_{\omega,t}=\lambda\frac{P_t w_t}{z_{\omega,t}}$$

This implies that the monopolist's variable profits are:

$$v_t=\left(1-\frac{1}{\lambda}\right)C_t$$

To find the real wage, we substitute into the price index:

$$P_t=\exp\left(\int_0^1 ln(\lambda\frac{P_t w_t}{z_{\omega,t}})d\omega\right)$$

$$P_t=P_t\lambda w_t\exp\left(\int_0^1 -ln(z_{\omega,t})d\omega\right)$$

$$\frac{1}{w_t}=\lambda\exp\left(\int_0^1 -ln(z_{\omega,t})d\omega\right)$$

$$\frac{1}{w_t}=\lambda\frac{1}{\exp\left(\int_0^1 ln(z_{\omega,t})d\omega\right)}$$

$$w_t=\frac{1}{\lambda}\exp\left(\int_0^1 ln(z_{\omega,t})d\omega\right)$$

Since we consider symmetric allocations, we know that $l_{\omega,t}=l_t$, and thus from the production technology and the isoelastic demand curve, we know that

$$l_t=\frac{C_{\omega,t}}{z_{\omega,t}}=\frac{P_tC_t}{p_{\omega,t}z_{\omega,t}}=\frac{C_t}{\lambda w_t}$$

From market clearing we then have that:

$$C_t=\lambda w_t L_t$$

7.3 Blueprints

Innovation in this world is zero-sum in the sense that if you gain an industry, someone else must lose an industry. Under a symmetric allocation so that $m_{\omega,t}=E_t$, the rate of loss will be γE_t. The price of blueprints must reflect the fact that your monopoly is almost surely time limited. As a consequence, the no-arbitrage condition for the price of blueprints q_t is:

$$r_tq_t=\underbrace{v_t}_{\text{Flow profits}}+\underbrace{\dot{q}_t}_{\text{Capital Gains}}-\underbrace{\gamma E_tq_t}_{\text{Expected loss of monopoly}}$$

Now we derive the differential equation that governs changes in the marginal utility weighted price of blueprints (which again given log utility is simply $\dot{\left(\frac{q_t}{c_t}\right)}$), using the no-arbitrage condition above and the Euler equation, which in this environment is simply $r_t = \rho + \frac{\dot{C}_t}{C_t}$

$$\dot{\left(\frac{q_t}{C_t}\right)} = \frac{\dot{q}_t}{C_t} - \frac{q_t \dot{C}_t}{C_t^2}$$

$$\frac{\dot{q}_t}{C_t} - \frac{q_t \dot{C}_t}{C_t^2} = \frac{r_t q_t - v_t + \gamma E_t q_t}{C_t} - \frac{q_t}{C_t}(r_t - \rho)$$

$$\frac{r_t q_t - v_t + \gamma E_t q_t}{C_t} - \frac{q_t}{C_t}(r_t - \rho) = (\rho + \gamma E_t)\left(\frac{q_t}{c_t}\right) - \frac{\lambda - 1}{\lambda}$$

Hence:

$$\dot{\left(\frac{q_t}{c_t}\right)} = (\rho + \gamma E_t)\left(\frac{q_t}{c_t}\right) - \frac{\lambda - 1}{\lambda}$$

7.4 Equilibrium

We have the following set of equilibrium conditions:

1. Labor share:
$$C_t = \lambda w_t L_t$$
2. Labor Market Clearing:
$$L_t + E_t = H$$
3. Free entry:
$$\gamma q_t \leq w_t \text{ with equality if } E_t = m_{\omega,t} > 0$$
4. Price of blueprints:
$$\dot{\left(\frac{q_t}{c_t}\right)} = (\rho + \gamma E_t)\left(\frac{q_t}{c_t}\right) - \frac{\lambda - 1}{\lambda}$$

To solve the model, suppose that $E_t \in (0, \infty)$, so that the third condition holds with equality and we have that

$$\frac{1}{\lambda} = \gamma \frac{q_t L_t}{C_t} = \gamma \frac{q_t(H - E_t)}{C_t} \iff \gamma \frac{q_t H}{C_t} - \frac{1}{\lambda} = \gamma E_t \frac{q_t}{C_t}$$

Substituting this expression into condition 4, we have that:

$$\dot{\left(\frac{q_t}{c_t}\right)} = (\rho + \gamma H)\left(\frac{q_t}{c_t}\right) - 1$$

Setting the differential equation to zero, we have that:

$$\frac{q_t}{C_t} = \frac{1}{\rho + \gamma H}$$

Now we need to check that this equilibrium is feasible in the sense that L and E are both positive numbers.

$$L = \frac{C_t}{\lambda w_t} = \frac{1}{\lambda\gamma}\frac{C_t}{q_t} = \frac{\rho + \gamma H}{\lambda\gamma}$$

$$E = H - L = \frac{\lambda\gamma H - \rho - \gamma H}{\lambda\gamma} = \frac{(\lambda - 1)\gamma H - \rho}{\lambda\gamma}$$

Consistency with our conjecture requires that:

$$\rho < (\lambda - 1)\gamma H$$

Suppose that this does not hold, so that $E_t = 0$. Then the price of a blueprint q_t along the balanced growth path is simply

$$\frac{q_t}{C_t} = \frac{1}{\rho}\left(1 - \frac{1}{\lambda}\right)$$

Noting that $\gamma q_t \leq w_t$, we thus have that:

$$\frac{1}{\lambda H} = \frac{w_t}{C_t} \geq \gamma\frac{q_t}{C_t} = \frac{\gamma}{\rho}\left(1 - \frac{1}{\lambda}\right)$$

This reduces to:

$$\rho \geq \frac{\gamma\lambda H(\lambda - 1)}{\lambda} = (\lambda - 1)\gamma H$$

7.5 Determining Consumption

First we shall derive the result in the economy without the Roy model.

Recall that we derived the following closed form expression for w_t:

$$w_t = \frac{1}{\lambda}\exp\left(\int_0^1 ln(z_{\omega,t})d\omega\right)$$

Since $z_{\omega,t} = z_{\omega_0}\lambda^{N_{\omega,t}}$, we have that:

$$ln(z_{\omega,t}) = ln(z_{\omega,0}) + N_{\omega,t}ln(\lambda)$$

Integrating across the continuum of industries and assuming a heuristic law of large numbers for the continuum, we have:

$$\int_0^1 ln(z_{\omega,t})d\omega = \int_0^1 ln(z_{\omega,0})d\omega + \underbrace{\gamma Eln(\lambda)t}_{\text{Expectation of Poisson with mean } \gamma Et}$$

Hence we have that:

$$w_t = \frac{1}{\lambda}w_0\exp(\gamma Eln(\lambda)t)$$

And using market clearing, we have that:

$$C_t = Lw_0\exp(\gamma Eln(\lambda)t)$$

7.6 Planning Problem

First note that in each period, the planner solves:

$$\max_{l_{\omega,t}(\cdot)} \int_0^1 ln(z_{\omega,t} l_{\omega,t}) d\omega$$

Subject to:

$$\int_0^1 l_{\omega,t} d\omega \leq L_t$$

We know that in a symmetric allocation, this reduces:

$$ln(L_t) + \int_0^1 ln(z_{\omega,t}) d\omega$$

Further we know that

$$\mathbb{E}_0 \left[\int_0^1 ln(z_{\omega,t}) - ln(z_{\omega,0}) d\omega \right] = \gamma ln(\lambda) \int_0^t E_s ds$$

From our calculation in the prior section.
Note that

$$\mathbb{E}_0 \left[\int_0^1 ln(z_{\omega,t}) - ln(z_{\omega,0}) d\omega \right] = \mathbb{E}_0 \left[\int_0^1 ln(z_{\omega,t}) \right] - \alpha \text{ where } \alpha \text{ is a constant}$$

This together implies that the Planner Problem can be given by:

$$\max \mathbb{E}_0 \left[e^{-\rho t} \left(ln(L_t) + \gamma ln(\lambda) \int_0^t E_s ds \right) \right]$$

Subject to:

$$\forall t \; L_T + E_t \leq H$$

Applying a change of variables with $K_t = \int_0^t E_s ds$, we can rewrite this problem as:

$$\max \mathbb{E}_0 \left[e^{-\rho t} \left(ln(H - E_t) + \gamma ln(\lambda) K_t \right) \right]$$

Subject to:

$$\dot{K}_t = E_t$$

The Hamiltonian for this is:

$$\mathcal{H}(K, \mu) = \max_E \underbrace{ln(H - E) + \gamma ln(\lambda) K}_{\text{Flow utility}} + \underbrace{\mu E}_{\text{Constraint}}$$

The dynamics are given by:

$$\dot{K}_t = E_t$$

$$\dot{\mu}_t = \rho\mu_t - \gamma\ln(\lambda)$$

If $E_t > 0$, then we have that $\frac{1}{H-E_t} \geq \mu_t$ which holds with equality if $E_t > 0$.
Setting $\dot{\mu}_t$ to zero, we have that

$$\mu_t = \frac{1}{\rho}\left(\gamma\ln(\lambda)\right)$$

Hence we have that
$\frac{1}{H-E_t} = \frac{1}{\rho}\gamma\ln(\lambda)$, which implies that:

$$\rho = \gamma\ln(\lambda)(H - E_t)$$

And thus

$$E_t = \frac{\gamma\ln(\lambda)H - \rho}{\gamma\ln(\lambda)} = H - \frac{\rho}{\gamma\ln(\lambda)}$$

Recall that the competitive allocation has

$$E_t = \frac{(\lambda-1)\gamma H - \rho}{\lambda\gamma} = \frac{\lambda-1}{\lambda}H - \frac{\rho}{(\lambda H)}$$

The growth rate (assuming that E_t is interior) in both cases is given by: $\gamma\ln(\lambda)E_t$, so we have a socially optimal growth rate of:

$$\gamma\ln(\lambda)H - \rho = \rho\left(\ln(\lambda)\frac{\gamma H}{\rho} - 1\right)$$

And a competitive growth rate of:

$$\gamma\ln(\lambda)\frac{\lambda-1}{\lambda}H - \frac{\ln(\lambda)\rho}{\lambda} = \rho\left(\frac{\ln(\lambda)}{\lambda}\right)\left(\frac{(\lambda-1)\gamma H}{\rho} - 1\right)$$

We can ask when competitive growth is slower than efficient growth, so that:

$$\rho\left(\ln(\lambda)\frac{\gamma H}{\rho} - 1\right) > \rho\left(\frac{\ln(\lambda)}{\lambda}\right)\left(\frac{(\lambda-1)\gamma H}{\rho} - 1\right)$$

Then

$$\left(\ln(\lambda)\frac{\gamma H}{\rho} - 1\right) > \left(\frac{\ln(\lambda)}{\lambda}\right)\left(\frac{(\lambda-1)\gamma H}{\rho} - 1\right)$$

$$\left(\ln(\lambda)\frac{\gamma H}{\rho}\right) > \frac{\ln(\lambda)(\lambda-1)\gamma H}{\lambda\rho} - \frac{\ln(\lambda)}{\lambda} + 1$$

$$\frac{\gamma H}{\rho} > \frac{(\lambda-1)\gamma H}{\lambda\rho} - \frac{1}{\lambda} + \frac{1}{\ln(\lambda)}$$

$$\frac{\gamma H}{\lambda\rho} > \frac{1}{\ln(\lambda)} - \frac{1}{\lambda}$$

$$1 + \frac{\gamma H}{\rho} > \frac{\lambda}{\ln(\lambda)}$$

7.7 Adding A Roy Model

Suppose that there is a smooth two dimensional distribution Ψ on $\mathbb{R}^2_+$, with density ψ. Let the relative price of blueprints to the wage be given by $s_t \equiv \frac{q_t}{w_t}$. Then

$$L_t = H\mathcal{L}(s_t) \text{ and } E_t = H\mathcal{E}(s_t)$$

Where $\mathcal{L}(s_t) = \int_0^\infty \int_{s_t y}^\infty x\psi(x,y)dxdy$ and $\mathcal{E}(s_t) = \int_0^\infty \int_0^{s_t y} y\psi(x,y)dxdy$

The resulting equilibrium conditions in this new environment are thus:

1. Labor share:
$$C_t = \lambda w_t H\mathcal{L}(s_t) \iff 1 = \lambda H \frac{q_t}{C_t}\frac{\mathcal{L}(s_t)}{s_t}$$
2. Labor Market Clearing:
$$\mathcal{L}(s_t) + \mathcal{E}(s_t) = 1$$
3. Free entry:
$$\gamma q_t \leq w_t \text{ with equality if } E_t > 0$$
4. Price of Blueprints:
$$\dot{\left(\frac{q_t}{c_t}\right)} = (\rho + H\mathcal{E}(s_t))\left(\frac{q_t}{c_t}\right) - \frac{\lambda - 1}{\lambda}$$

Note that where before the rate of monopoly decay was γE_t, now it is determined by $\mathcal{E}(s_t)$.

7.8 Klette-Kortum

Klette-Kortum is basically Grossman with a small twist: now monopolists can leverage their existing blueprints in order to grow into new industries. The way this is modeled is that the incumbent in industry ω hires units of labor $x_{\omega,t}$ to generate productivity improvements for some other randomly selected good ω' randomly at some rate $f(x_{\omega,t})$, which we shall assume to be symmetric across industries. In a symmetric allocation $x_{\omega,t} = M_t$, and monopolists lose their monopolies at some rate $d_t \equiv f(M_t) + E_t$.

This implies a no-arbitrage condition for the price of blueprints given by:

$$r_t q_t = \underbrace{v_t}_{\text{Flow profits}} + \underbrace{\max_x\{q_t f(x) - w_t x\}}_{\text{Expected value of blueprint net of research costs}} + \underbrace{\dot{q}_t}_{\text{Capital gains}} - \underbrace{d_t q_t}_{\text{Expected loss of monopoly}}$$

Note that we must have that $q_t f'(x) = w_t$

Recall also that we derived that $C_t = \lambda w_t L_t$ and $v_t = \left(1 - \frac{1}{\lambda}\right) C_t$

Now we shall make use of the Euler equation: $r_t = \rho + \frac{\dot{C}_t}{C_t}$ and derive again a differential equation governing the price of blueprints.

$$\left(\rho + \frac{\dot{C}_t}{C_t}\right) q_t = \left(1 - \frac{1}{\lambda}\right) C_t + q_t f(M_t) - q_t f'(M_t) M_t + \dot{q}_t - d_t q_t$$

$$\rho\left(\frac{q_t}{C_t}\right) + \frac{\dot{C}_t q_t}{C_t^2} = \left(1 - \frac{1}{\lambda}\right) + \frac{q_t}{C_t} f(M_t) - \frac{q_t}{C_t} f'(M_t) M_t + \frac{\dot{q}_t}{C_t} - d_t \frac{q_t}{C_t}$$

$$\underbrace{\frac{\dot{q}_t}{C_t} - \frac{\dot{C}_t q_t}{C_t^2}}_{\left(\dot{\frac{q_t}{C_t}}\right)} = (\rho + d_t - f(M_t) + f'(M_t)M_t)\frac{q_t}{c_t} - \left(1 - \frac{1}{\lambda}\right)$$

Equilibrium conditions are then:

1. Labor share:
$$C_t = \lambda w_t L_t \iff s_t = \frac{q_t}{C_t}\lambda L_t$$

2. Labor market clearing:
$$L_t + M_t = H\mathcal{L}(s_t)$$

3. Research optimality:
$$w_t = q_t f'(M_t) \iff 1 = s_t f'(M_t)$$

4. Price of blueprints:
$$\left(\dot{\frac{q_t}{C_t}}\right) = (\rho + d_t - f(M_t) + f'(M_t)M_t)\frac{q_t}{c_t} - \left(1 - \frac{1}{\lambda}\right)$$

To find the balanced growth path, suppose that $\left(\dot{\frac{q_t}{C_t}}\right) = 0$, which implies that:

$$\frac{q}{C} = \frac{1 - \frac{1}{\lambda}}{(\rho + d - f(M) + f'(M)M)} = \frac{1 - \frac{1}{\lambda}}{(\rho + H\mathcal{E}(s) + f'(M)M)}$$

$$\frac{q}{C} = \frac{1}{\lambda}\frac{\lambda - 1}{(\rho + H\mathcal{E}(s) + f'(M)M)}$$

$$q = \frac{C}{\lambda}\frac{\lambda - 1}{(\rho + H\mathcal{E}(s) + f'(M)M)}$$

Since $\frac{C_t}{\lambda} = \frac{q_t}{s_t}L_t$, this is equivalent to:

$$q = \frac{qL_t}{s}\frac{\lambda - 1}{(\rho + H\mathcal{E}(s) + f'(M)M)}$$

$$s = \frac{(\lambda - 1)L_t}{(\rho + H\mathcal{E}(s) + f'(M)M)}$$

$$s = \frac{(\lambda - 1)(H_t\mathcal{L}(s) - M_t)}{(\rho + H\mathcal{E}(s) + f'(M)M)}$$

$$s = \frac{(\lambda - 1)(H\mathcal{L}(s) - M)}{\rho + H\mathcal{E}(S) + f'(M)M}$$

Note that Klette-Kortum will generate a "realistic" size distribution of firms, but the problem is that it takes an absurdly long amount of time to get to the size of large firms we see after say 40 years, conditional on those firms growing large in the first place.

8 Dixit Stiglitz Efficiency

This section of notes introduces Erzo's simple model of firm growth and talks about static vs. dynamic Dixit-Stiglitz efficiency.

The baseline environment has preferences of the form:

$$U(C) = \int_0^\infty e^{-\rho t} H_t ln(c_t) dt \text{ where } c_t = \frac{C_t}{H_t}$$

Consumption C_t is an aggregate of the form:

$$C_t = \left[\int_0^{N_t} C_{\omega,t}^{\frac{\epsilon-1}{\epsilon}} d\omega \right]^{\frac{\epsilon}{\epsilon-1}}$$

Note that because this is a CRS aggregator, we can equivalently express this in terms of per capita consumption:

$$c_t = \left[\int_0^{N_t} c_{\omega,t}^{\frac{\epsilon-1}{\epsilon}} d\omega \right]^{\frac{\epsilon}{\epsilon-1}}$$

Note that the number of varieties in the CES aggregator is an endogenous object. This is one of the mechanisms by which this model will generate growth. The other mechanism is population growth, which takes the following form:

$$H_t = H_0 e^{\eta t}$$

8.1 Firms

There is implicitly a competitive final goods producer.

At each point in time t, their problem is:

$$\max_{c_{\omega,t}} P_t \left(\left[\int_0^{N_t} C_{\omega,t}^{\frac{\epsilon-1}{\epsilon}} d\omega \right]^{\frac{\epsilon}{\epsilon-1}} \right) - \int_0^{N_t} p_{\omega,t} C_{\omega,t} d\omega$$

Taking first order conditions with respect to $C_{\omega,t}$, we end up with:

$$P_t C_t^{\frac{1}{\epsilon}} C_{\omega,t}^{-\frac{1}{\epsilon}} = p_{\omega,t}$$

To calculate the price, solve for $C_{\omega,t} = \left(\frac{P_t}{p_{\omega,t}}\right)^\epsilon C_t$. Plugging this into the price index, we recover:

$$C_t = \left[\int_0^{N_t} (\left(\frac{P_t}{p_{\omega,t}}\right)^\epsilon C_t)^{\frac{\epsilon-1}{\epsilon}} d\omega \right]^{\frac{\epsilon}{\epsilon-1}}$$

$$1 = \left[\int_0^{N_t} (\left(\frac{P_t}{p_{\omega,t}}\right))^{\epsilon-1} d\omega \right]^{\frac{\epsilon}{\epsilon-1}}$$

$$P_t = \left[\int_0^{N_t} p_{\omega,t}^{1-\epsilon} d\omega \right]^{\frac{1}{1-\epsilon}}$$

Intermediate goods producers are monopolists in their products, have technology $c_{\omega,t} = zl_{\omega,t}$ and face the isoelastic demand curve from the final goods producer, so their problem is given by:

$$\max_{C_{\omega,t}} P_t C_t^{\frac{1}{\epsilon}} C_{\omega,t}^{1-\frac{1}{\epsilon}} - C_{\omega,t}\frac{w_t}{z}$$

The first order condition is:

$$\left(1-\frac{1}{\epsilon}\right) P_t \left(\frac{C_t}{C_{\omega,t}}\right)^{\frac{1}{\epsilon}} = \frac{w_t}{z}$$

This implies that the monopolist sets the price:

$$p_{\omega,t} = \frac{\epsilon}{\epsilon-1}\frac{w_t}{z}$$

The "real" price of the good is then:

$$\frac{p_{\omega,t}}{P_t} = \frac{\epsilon}{\epsilon-1}\frac{w_t}{P_t z}$$

Assuming a symmetric equilibrium and plugging the intermediate price into the price index, we then have that:

$$P_t = \left[\int_0^{N_t}\left(\frac{\epsilon}{\epsilon-1}\frac{w_t}{z}\right)^{1-\epsilon} d\omega\right]^{\frac{1}{1-\epsilon}}$$

$$P_t = \frac{\epsilon}{\epsilon-1}\frac{w_t}{z}N_t^{\frac{1}{1-\epsilon}}$$

$$\frac{w_t}{P_t} = \frac{\epsilon-1}{\epsilon}zN_t^{\frac{1}{\epsilon-1}}$$

Now note that from the firm's problem, we know that

$$C_{\omega,t} = zl_t = \left(\frac{p_{\omega,t}}{P_t}\right)^{-\epsilon} C_t$$

And hence:

$$l_t = \left(\frac{p_{\omega,t}}{P_t}\right)^{-\epsilon}\frac{C_t}{z}$$

$$l_t = \left(\frac{\epsilon}{\epsilon-1}\frac{w_t}{P_t z}\right)^{-\epsilon}\frac{C_t}{z}$$

lt=N1tCtzl$_t = N_t^{\frac{\epsilon}{1-\epsilon}}\frac{C_t}{z}$

Finally, we have that the labor share

$$\frac{w_t l_t}{P_t} = \frac{\epsilon-1}{\epsilon}\frac{C_t}{N_t}$$

[8]

[8] Note that $L_t = N_t l_t$ in general for symmetric equilibria.

Since

$$N_t^{\frac{\epsilon}{1-\epsilon}} N_t^{\frac{1}{\epsilon-1}} = N_t^{\frac{\epsilon}{1-\epsilon}-\frac{1}{1-\epsilon}} = N_t^{\frac{\epsilon-1}{1-\epsilon}} = N_t^{-1}$$

This of course implies that variable profits are:

$$\frac{v_t}{P_t} = \frac{1}{\epsilon}\frac{C_t}{N_t}$$

8.2 Blueprints

To produce requires a technology. How are these technologies generated? They are generated in two ways.

The first way is that current owners of a blueprint can hire labor to generate new blueprints from an existing one. This reproduction technology has a fixed factor in the form of the existing blueprint: $G(1, m_t) = g(m_t)$, where m_t is the labor hired to replicate the blueprint at time t. Another way to put this is given a supply of replication labor m_t, $g(m_t)$ is the instantaneous probability that a new blueprint will be generated.

The second way is that entrepreneurs will generate according to their entrepreneurial skill.

The first one is what is important to price the blueprints today, as it implies a no-arbitrage condition given by:

$$\underbrace{r_t q_t}_{\text{Required return}} = \max_{m_t} \underbrace{q_t g(m_t) - \frac{w_t}{P_t} m_t}_{\text{Value of replication activity}} + \underbrace{\frac{v_t}{P_t}}_{\text{Flow profits}} + \underbrace{\dot{q}_t}_{\text{Capital gains}}$$

$g(\cdot)$ is assumed to be differentiable, so m_t^* is characterized by $\frac{w_t}{P_t} = q_t g'(m_t^*)$

It is also convenient re-express flow profits in terms of the wage by noting that:

$$\frac{v_t}{P_t} = \frac{1}{\epsilon - 1}\frac{w_t l_t}{P_t}$$

Hence we can rewrite the no-arbitrage condition as:

$$r_t q_t = \max_{m_t} q_t g(m_t) - \frac{w_t}{P_t}(m_t - \frac{l_t}{\epsilon - 1}) + \dot{q}_t$$

I will now derive the differential equation governing the value of blueprints. To do this, again we recall that:

$$\dot{\left(\frac{q_t}{c_t}\right)} = \frac{\dot{q}_t}{c_t} - \frac{q_t \dot{c}_t}{c_t^2}$$

$$r_t = \rho + \frac{\dot{c}_t}{c_t}$$

Abusing notation and implicitly constraining that m_t is chosen optimally so we can disregard the maximum operator, we have:

$$(\rho + \frac{\dot{c}_t}{c_t})q_t = q_t g(m_t) - \frac{w_t}{P_t}(m_t - \frac{l_t}{\epsilon - 1}) + \dot{q}_t$$

$$(\rho + \frac{\dot{c}_t}{c_t})q_t = q_t g(m_t) - q_t g'(m_t)m_t + \frac{w_t l_t}{P_t(\epsilon - 1)} + \dot{q}_t$$

$$(\rho + \frac{\dot{c}_t}{c_t})q_t = q_t g(m_t) - q_t g'(m_t)m_t + \frac{1}{\epsilon}\frac{c_t H_t}{N_t} + \dot{q}_t$$

$$\underbrace{\frac{\dot{q}_t}{c_t} - \frac{q_t \dot{c}_t}{c_t^2}}_{\dot{\left(\frac{q_t}{c_t}\right)}} = (\rho - g(m_t) + g'(m_t)m_t)\,\frac{q_t}{c_t} - \frac{1}{\epsilon}\frac{H_t}{N_t}$$

Recall that the second source of blueprint generation was the supply of entrepreneurial labor. Both this and traditional labor are supplied to a traditional Roy model. Specifically, suppose that each individual is named by a draw from a two dimensional smooth talent distribution Ψ with density ψ, where x is an individual's entrepreneurial skill and y is an individual's labor productivity. An individual will choose to supply entrepreneurial labor if, given their skill and productivity $qx > wy$, or their return from inventing is higher than their return from working for others.

Define $s = \frac{q}{w}$ to be the "inventor's premium". It follows then that the supply of entrepreneurial skill and effective labor are (respectively):

$$\mathcal{E}(s) = \int_0^\infty \int_0^{sx} x\psi(x,y)dydx$$

and

$$\mathcal{L}(s) = \int_0^\infty \int_{sx}^\infty y\psi(x,y)dydx$$

Doing a bit of work (see the Romer notes), one can recover that

$$\mathcal{E}'(s) = \int_0^\infty x^2\psi(x,sx)dx$$

$$\mathcal{L}'(s) = -s\int_0^\infty x^2\psi(x,sx)dx$$

Note that in order to show this you rewrite the bounds and later apply a change of variables. Why is this important? Because this is crucial for characterizing the growth of the blueprints. Specifically, we now know that:

$$\dot{N}_t = \underbrace{g(m_t)N_t}_{\text{Growth from existing blueprint}} + \underbrace{H_t\mathcal{E}(s)}_{\text{Growth from new blueprints}}$$

Noting that $\dot{H}_t = \eta H_t$, We can rewrite this in per-capita terms as:

$$\dot{\left(\frac{N_t}{H_t}\right)} = g(m_t)\frac{N_t}{H_t} + \mathcal{E}(s) - \eta\frac{N_t}{H_t} = (g(m_t) - \eta)\frac{N_t}{H_t} + \mathcal{E}(s)$$

Since $\mathcal{E}(s) > 0$, we can immediately see that in order to have a balanced growth path, it must be that the other term is smaller than zero. Since $\frac{N_t}{H_t} > 0$, this means that $\eta > g(m_t)$.

8.3 Equilibrium

We have 5 equilibrium conditions:

1. Differential equation for blueprints per capita:
$$\left(\dot{\frac{N_t}{H_t}}\right) = g(m_t)\frac{N_t}{H_t} + \mathcal{E}(s) - \eta\frac{N_t}{H_t} = (g(m_t) - \eta)\frac{N_t}{H_t} + \mathcal{E}(s)$$

2. Differential equation for value of blueprints:
$$\left(\dot{\frac{q_t}{c_t}}\right) = (\rho - g(m_t) + g'(m_t)m_t)\,\frac{q_t}{c_t} - \frac{1}{\epsilon}\frac{H_t}{N_t}$$

3. Labor Share:
$$\frac{w_t l_t}{P_t} = \frac{\epsilon - 1}{\epsilon}\frac{c_t H_t}{N_t}$$

4. Optimality for Research Labor:
$$\frac{w_t}{P_t} = q_t g'(m_t)$$

5. Labor Market Clearing:
$$H_t \mathcal{L}(s_t) = (l_t + m_t)N_t$$

A balanced growth path will have:

$$-\mathcal{E}(s) = (g(m_t) - \eta)\frac{N_t}{H_t}$$

$$\frac{1}{\epsilon}H_t N_t = (\rho - g(m_t) + g'(m_t)m_t)\frac{q_t}{c_t}$$

Let $s \equiv \frac{1}{g'(m)}$ be defined such that $m(s)$ and $l(s)$ solve:

$$\frac{1}{\rho - g(m)}\left(\frac{l}{\epsilon - 1} - m\right)$$

Then we have demand for blueprints:

$$\frac{N}{H} = \frac{\mathcal{L}(s)}{l(s) + m(s)}$$

and supply:

$$\frac{N}{H} = \frac{\mathcal{E}(s)}{\eta - g(m(s))}$$

These two curves intersect, and that intersection point is our equilibrium.

8.4 Static Efficiency

Typically in these models, fixed costs for blueprint production are introduced in one of two ways. Either they take the fixed cost in terms of units of the final good, or they say that there is a fixed cost in terms of labor. Despite these seeming similar, there is a very significant difference between the two implications. When the fixed cost is in terms of the final good, the outcomes are inefficient, and there is an underprovision of variety. When the fixed cost is in terms of units of labor, the outcome is efficient.

To see this, let us first solve a simple problem with a fixed cost in terms of units of the final good, an intermediate production technology given by $Y_\omega = l_\omega$, and inelastic labor supply L. The final good is produced by a CES aggregator, so with a symmetric allocation given by: $\frac{L}{N}$, we have that

$$Y = \frac{L}{N} N^{\frac{\epsilon}{\epsilon-1}} = LN^{\frac{\epsilon}{\epsilon-1}-1} = LN^{\frac{l}{\epsilon-1}}$$

Now consider the planner problem given by:

$$\max Y - NF$$

subject to:

$$Y = LN^{\frac{1}{\epsilon-1}}$$

Hence we can rewrite the problem as:

$$\max LN^{\frac{1}{\epsilon-1}} - NF$$

Differentiating with respect to N, we have that:

$$\frac{1}{\epsilon-1} LN^{\frac{1}{\epsilon-1}} = NF$$

Doing some algebra, we have that:

$$N^{\frac{\epsilon-2}{\epsilon-1}} = \frac{1}{\epsilon-1}\frac{L}{F}$$

$$N = \left(\frac{1}{\epsilon-1}\frac{L}{F}\right)^{\frac{\epsilon-1}{\epsilon-2}}$$

Now let's consider the competitive allocation.

Given a standard CES aggregator and a linear production technology, we know that

$$p = \frac{\epsilon}{\epsilon-1} w$$

I.E. that wages are marked down (or prices are marked up) as a fixed function of marginal cost.

We know that the labor share in this economy is:

$$wL = \frac{\epsilon-1}{\epsilon} PY$$

And that profits are:

$$vN = \frac{1}{\epsilon} PY$$

Now note that the problem of the final goods producer in this economy is:

$$\max P \left(\int_0^N Y_\omega^{\frac{\epsilon-1}{\epsilon}} d\omega \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^N pY_\omega d\omega$$

The free entry condition says that:

$$\underbrace{v}_{\text{Nominal variable profits}} = \underbrace{Pf}_{\text{Nominal fixed cost per firm}}$$

Or equivalently

$$\frac{v}{P} = f$$

Which implies that

Note that $F = f$ and hence from the profit condition, we have that:

$$PFN = \frac{1}{\epsilon} PY$$

And hence:

$$Y = \epsilon NF$$

$$N = \left(\frac{L}{\epsilon F} \right)^{\frac{\epsilon-1}{\epsilon-2}}$$

Hence, when the fixed cost is in term of the final good, there is an underprovision of variety.

Next let's suppose that instead of paying a fixed cost in terms of the final good, so that $Y = C + NF$, there is a fixed cost in labor, so that$L + FN = H$

Repeating the same logic for the aggregator, we have that $Y = LN^{\frac{1}{\epsilon-1}}$, so that the planners problem becomes:

$$\max N^{\frac{1}{\epsilon-1}} (H - FN)$$

Suppose that $\epsilon > 1$, so that the objective is concave and thus the solution is characterized by the FOC. We can rewrite the objective as:

$$HN^{\frac{1}{\epsilon-1}} - FN^{\frac{1}{\epsilon-1}+1}$$

Differentiating:

$$\frac{1}{\epsilon-1} HN^{\frac{1}{\epsilon-1}-1} - \frac{\epsilon}{\epsilon-1} FN^{\frac{1}{\epsilon-1}} = 0$$

$$HN^{\frac{1}{\epsilon-1}-1} = \epsilon FN^{\frac{1}{\epsilon-1}}$$

$$\frac{H}{N} = F\epsilon$$

$$N = \frac{H}{F\epsilon}$$

Now consider the competitive allocation.
Price indices and quantities are as before, so $P = N^{\frac{1}{1-\epsilon}}p$, and $C = N^{\frac{\epsilon}{\epsilon-1}}\frac{L}{N}$
Factor shares are again $vN = \frac{1}{\epsilon}PC$ and $wL = \frac{\epsilon-1}{\epsilon}PC$
It must be that

$$\frac{vN}{wL} = \frac{1}{\epsilon - 1}, \; \frac{v}{w} = F$$

Using the labor market clearing $L + FN = H$, one can show that:

$$N = \frac{H}{F\epsilon}$$

Hence the fixed cost in terms of labor will generate efficient outcomes. One can show that this generalizes nicely to the Roy model version, where entrepreneurial labor N is endogenously determined by the skill distribution.

8.5 Dynamic Efficiency

More interesting than the static efficiency results are efficiency results in dynamic economies. So, consider an economy along the lines above (itself a deterministic version of the Luttmer 2011 economy), but with a unit productivity (i.e. that $z = 1$). As we showed above, consumption is given by:

$$l_t = N_t^{\frac{\epsilon}{1-\epsilon}} C_t$$

Since $L_t = N_t l_t$, we have that:

$$L_t = N_t^{\frac{\epsilon}{1-\epsilon}} N_t C_t$$

$$L_t = N_t^{\frac{\epsilon}{1-\epsilon}+1} C_t$$

$$L_t = N_t^{\frac{1}{1-\epsilon}+1} C_t$$

$$C_t = L_t N_t^{\frac{1}{\epsilon-1}}$$

$$c_t = \frac{L_t}{H_t} N_t^{\frac{1}{\epsilon-1}}$$

The number of blueprints grows according to:

$$\dot{N}_t = g(m_t)N_t + E_t$$

And market clearing is:

$$L_t + m_t N_t = \mathcal{L}(s_t) H_t$$

$$E_t = \mathcal{E}(S_t)H_t$$

From this, we can write the Hamiltonian as:

$$\mathcal{H}_t(N, \lambda) = \max_{m,s} H_t \ln(c) + \lambda(g(m)N + E)$$

$$\mathcal{H}_t(N, \lambda) = \max_{m,s} H_t \left(\frac{ln(N)}{\epsilon - 1} + \ln\left(\frac{L_t}{H_t}\right)\right) + \lambda(g(m)N + E)$$

$$\mathcal{H}_t(N, \lambda) = \max_{m,s} H_t \left(\frac{ln(N)}{\epsilon - 1} + \ln\left(\mathcal{L}(s) - \frac{mN}{H_t}\right)\right) + \lambda(g(m)N + \mathcal{E}(s)H_t)$$

This maximization has first order conditions given by:

$$(m) : -N_t \frac{H_t}{H_t\mathcal{L}(s_t) - m_t N_t} + \lambda_t g'(m_t)N_t = 0$$

$$(s) : \frac{H_t^2 \mathcal{L}'(s_t)}{H_t\mathcal{L}(s_t) - m_t N_t} + \lambda_t \mathcal{E}'(s_t)H_t = 0$$

These can be rewritten as:

$$\frac{H_t \mathcal{L}'(s_t)}{H_t\mathcal{L}(s_t) - m_t N_t} = -\lambda_t \mathcal{E}'(s_t)$$

$$\frac{H_t}{H_t\mathcal{L}(s_t) - m_t N_t} = -\lambda_t \frac{\mathcal{E}'(s_t)}{\mathcal{L}'(s_t)}$$

Recall that $s_t\mathcal{E}'(s_t) + \mathcal{L}'(s_t) = 0$, and hence we have:

$$s_t = -\frac{\mathcal{L}'(s_t)}{\mathcal{E}'(s_t)}$$

This means we can rewrite the above condition as:

$$\frac{H_t s_t}{H_t\mathcal{L}(s_t) - m_t N_t} = \lambda_t$$

We can substitute this into the first order condition for m_t to recover:

$$1 = s_t g'(m_t)$$

These two conditions characterize the instantaneous choice of the planner.
The Hamiltonian dynamics are always of the form:

$$\dot{N}_t = \frac{\partial \mathcal{H}_t}{\partial \lambda}$$

$$\dot{\lambda}_t = \rho\lambda_t - \frac{\partial \mathcal{H}_t}{\partial N}$$

Taking these first order conditions we again have that:

$$\dot{N}_t = g(m)N + \mathcal{E}(s)H_t$$

$$\dot{\lambda}_t = \rho\lambda_t - \frac{H_t}{N_t(\epsilon - 1)} + m_t \frac{H_t}{H_t \mathcal{L}(s_t) - m_t N_t} - \lambda_t g(m_t)$$

$$\dot{\lambda}_t = \rho\lambda_t - \frac{H_t}{N_t(\epsilon - 1)} + \frac{\lambda_t m_t}{s_t} - \lambda_t g(m_t)$$

$$\dot{\lambda}_t = \rho\lambda_t - \frac{H_t}{N_t(\epsilon - 1)} + \lambda_t g'(m_t) m_t - \lambda_t g(m_t)$$

$$\dot{\lambda}_t = (\rho - g(m_t) + g'(m_t) m_t)\lambda_t - \frac{H_t}{N_t(\epsilon - 1)}$$

This should look similar to:

$$\dot{\left(\frac{q_t}{c_t}\right)} = (\rho - g(m_t) + g'(m_t) m_t) \frac{q_t}{c_t} - \frac{1}{\epsilon} \frac{H_t}{N_t}$$

Specifically, let $\hat{\lambda}_t \equiv \frac{\epsilon - 1}{\epsilon} \lambda_t$
Then:

$$\dot{\hat{\lambda}}_t = (\rho - g(m_t) + g'(m_t) m_t)\hat{\lambda}_t - \frac{1}{\epsilon} \frac{H_t}{N_t}$$

So take

$$\frac{q_t}{c_t} = \frac{\epsilon - 1}{\epsilon} \lambda_t$$

and:

$$\frac{w_t}{c_t} = \frac{1}{s_t} \frac{\epsilon - 1}{\epsilon} \lambda_t$$

Hence, the competitive equilibrium is efficient, despite the fact that there are markups in this economy.

Why? Because the product market distortions are homogeneous across goods and “labor” is supplied inelastically.

9 Ordinary Differential Equations

A first order ordinary differential equation is an equation of the form:

$$\dot{x}(t) = f(t, x(t)) \tag{1}$$

Where as always I notate time derivatives with dots.

Sometimes you will see this written in *implicit form* as:

$$F(t, x(t), \dot{x}(t)) = 0 \tag{2}$$

The simplest ODEs take the form:

$$f(t, x) = g(x)h(t) \tag{3}$$

In this case, we have that:

$$\int_{x_0}^{x} \frac{d\chi}{h(\chi)} = \int_{t_0}^{t} g(s)ds \tag{4}$$

The simplest and most famous example is the example of exponential growth, which is a differential equation of the form:

$$\dot{x} = \alpha x$$

In this case, we have that

$$\frac{\dot{x}}{x} = \alpha$$

$$x = x_0 e^{\alpha(t-t_0)}$$

The next important result that we have is that of the Cauchy formula. Recall that a linear first-order ODE is has the form:

$$\dot{x} + g(t)x = h(t) \tag{5}$$

In this case, we proceed by noting that if $h(t) = 0$, this is a separable ODE, and hence we could write:

$$x_h(t; C) = Cexp\left[-\int_{t_0}^{t} g(s)ds\right] \tag{6}$$

This x_h is a called a homogenous solution. To recover the particular solution, we set $x_p(t) = x_h(t; C(t))$. Note then that we can plug this into the general form, to recover that:

In this case, it follows that:

$$\dot{x} = \dot{C}(t)exp\left[-\int_{t_0}^{t} g(s)ds\right] - C_t g(t)exp\left[-\int_{t=0}^{t} g(s)ds\right]$$

Noting that

$$g(t)x = C(t)g(t)exp\left[-\int_{t_0}^{t} g(s)ds\right]$$

We see that the above equation simplifies to:

$$\dot{C}(t) = h(t) \exp\left[\int_{t_0}^{t} g(s)ds\right]$$

This is again a separable equation, so we can recover the solution directly as:

$$C(t) = C(0) + \int_{t_0}^{t} h(s) \exp\left[\int_{t_0}^{s} g(\theta)d\theta\right] ds \tag{7}$$

Typically, we set $C(0) = 0$

From here, we recover the Cauchy Formula, which says that the solution to (5) has the following form:

$$x(t) = \left(x_0 + \int_{t_0}^{t} h(s) \exp\left[\int_{t_0}^{s} g(\theta)d\theta\right] ds\right) \exp\left[-\int_{t_0}^{t} g(s)ds\right] \tag{8}$$