TUTORIAL-4

1. Consider the ring $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ with ordinary addition and multiplication. Prove that R is a field and identify its multiplicative identity and zero-divisor status.

2. Consider the ring homomorphism $\phi : \mathbb{Z}[x] \to \mathbb{R}$ defined by $\phi(f(x)) = f(\sqrt{2})$. Find the kernel of ϕ and describe its structure.

3. Let $I = \{\text{multiples of } 6\}$ and $J = \{\text{multiples of } 10\}$ in $\mathbb{Z}$. Compute $I \cap J$ and $I + J$. What do these represent in terms of divisibility?

4. In polynomial ring $\mathbb{R}[x, y]$, explain why:

   - $\langle x, y \rangle$ is a prime ideal
   - $\langle x^2 - y \rangle$ is a prime ideal
   - $\langle x^2, y^2 \rangle$ is NOT a prime ideal

5. Let $R = \mathbb{Z}[x]$ and $I = \langle 2, x \rangle$. Describe an element of R/I and explain how two cosets $a + I$ and $b + I$ are equal.

6. In $\mathbb{Z}[x]$, consider the ideal $I = \langle x^2 - 2 \rangle$. Show that $\mathbb{Z}[x]/\langle x^2 - 2 \rangle \cong \mathbb{Z}[\sqrt{2}]$ and explain the isomorphism explicitly.

7. Show that the quotient ring $(\mathbb{Z}/\langle n \rangle)/(\langle a \rangle + \langle n \rangle)$ has a specific structure related to divisors of n. Give examples for $n = 12$ and $n = 20$.

BONUS

1. Let I be an ideal in ring R. Prove that there exists an ideal J in R such that $I \cap J = \{0\}$ and $I + J = R$ if and only if I is one of several possible types (investigate this condition).

2. Prove the correspondence theorem: If I is an ideal of ring R, there is a bijection between ideals of R containing I and ideals of R/I. Apply this to factor rings of $\mathbb{Z}$.