Networks, Signals & Systems

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1. Signals

- Natural / Artificial variations that carry some info.

OR

$$\boxed{f(\text{independent variables}) = \text{Some value of dependent variable}}$$

Eg: · Seismo signals
Optical signals
And so on.

- Temp. at 2PM $= f(x, y, z)$ ← pt. in space

- Photos − i.e, brightness of each pixel $= f(r, g, b)$

- In a system,

$$x(t) \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y(t)$$

NOTE that $\left.\begin{array}{l} y(t)\ \forall\ t \in \mathbb{R} \\ x(t)\ \forall\ t \in \mathbb{R} \end{array}\right\}$ are entire signals.

BUT in "$y(t)$ is a fn of $x(t)$",
$y(t)$ is an output at t
$x(t)$ is an entire signal.

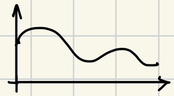
a) Ways to classify signals

① Real-valued & complex signals

$$x \quad \underset{\text{time}}{\mathbb{R}} \longrightarrow \underset{\text{complex}}{\mathbb{C}}$$

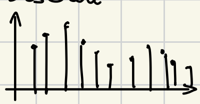
② Continuous Time & Discrete Time

- Continuous Time: The signal is defined at every spot.



Notation: $x(t)$

- Discrete Time: The signal is defined at specific time intervals.



Notation: $x[n]$ n^{th} step

③ Even and Odd signals

- Even : $x(t) = x(-t)$ $\forall\ t \in \mathbb{R}$.
- Odd : $x(t) = -x(-t)$ $\forall\ t \in \mathbb{R}$

④ Periodic and Aperiodic Signals

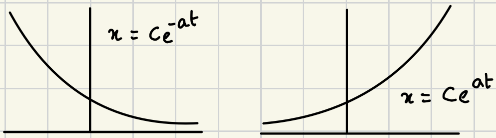
$x(T+t) = x(T)$ Periodic

$\underbrace{x(T+t) \neq x(T)}_{\text{for ALL } t \in \mathbb{R}}$ Aperiodic

b) Different functions to represent signals

① Real Exponential signals

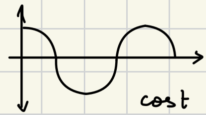
$$\boxed{x(t) = Ce^{at}} \quad , \ a, C \in \mathbb{R}$$



② Real Sinusoids

$$\boxed{x(t) = A\cos(\omega_0 t + \varphi)}$$

$$\omega_0 = \frac{2\pi}{T_0}$$



③ Complex Exponential Signals

$$\boxed{z(t) = Ce^{at}} \ , \ a, C \in \mathbb{C}$$

Suppose $C = |c| e^{j\theta}$

Then $\boxed{z(t) = Ce^{j\theta + at}}$, $a \in \mathbb{C}$, $C \in \mathbb{R}$

④ Complex Sinusoids

$$\boxed{z(t) = Ce^{j(\omega t + \varphi)}}$$

Split into $Re(z)$ & $Im(z)$.

$Re(z) = \cos(\omega t + \varphi)$

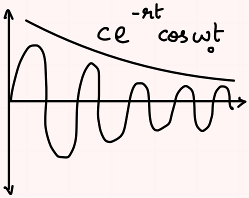
$Im(z) = \sin(\omega t + \varphi)$

Graphs of complex exponentials

Take the real part.

$Re(z) = |c|\, e^{rt} \cos(\omega_0 t + \varphi)$

Basically, its a cos fn. with amplitude changing with time.



c) Some imp. quantities associated with signals

① Total Energy

$$\boxed{E_\infty \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \, dt}$$

OR

$$\boxed{E_\infty \triangleq \int_{-\infty}^{\infty} x(t) \cdot \bar{x}(t) \, dt}$$

The magnitude of the signal is taken to avoid getting a complex value for energy.

② Average Power

$$\boxed{P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt}$$

- Note that if $t_\infty < \infty$, then it is called a finite energy power signal.

HW: How do nonzero periodic signals have ∞ energy?

NOTE: If $E_\infty = $ finite,

$P_\infty = 0.$

Why?

We know

$$0 \le \int_{-T}^{T} |x(t)|^2 dt \le E_\infty$$

dividing by $2T$,

$$0 < P_\infty \le \frac{E_\infty}{2T}$$

And in limits terms,

$$P_\infty = \lim_{T \to \infty} \frac{E_\infty}{2T}$$

$$= 0$$

③ Power of A Periodic Signal

$$P_\infty \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Now, if $x(t)$ is periodic, so is $x(t)^2$.

$$|x(t + kT_0)|^2 = |x(t)|^2 \ \forall \ k \in \mathbb{Z}$$

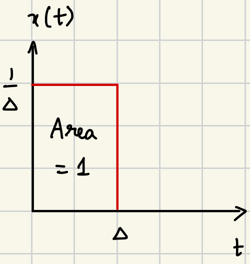
So for periodic signal $x(t)$,

$$P_\infty = \lim_{k \to \infty} \frac{1}{2kT_0} \int_{-kT_0}^{kT_0} |x(t)|^2 dt$$

d) Special signals

① Unit Impulse / Impulse signal

$$x_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & \forall \ t \in [0, \Delta] \\ 0, & \forall \text{ other } t. \end{cases}$$



Defined:

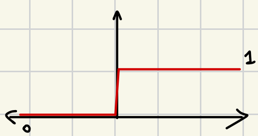
$$\boxed{\delta(t) \triangleq \lim_{\Delta \to 0} x_\Delta(t)}$$

So it's like an extremely small jerk lasting for ~0 sec.

② Unit Step Function

$$u(t) \triangleq \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$= \begin{cases} 0 & \forall \ t < 0 \\ 1 & \forall \ t > 0 \end{cases}$$

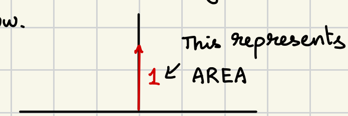


A more formal definition:

$$\boxed{\delta(t) = 0 \quad \forall \ t \neq 0} \quad \& \quad \boxed{\int_{-\infty}^{\infty} \delta(t) dt = 1}$$

Dirac - Delta Function

... and we represent it using an arrow.



NOTE

Do NOT write

$$u(t) \triangleq \int_{-\infty}^{t} \delta(t) dt.$$

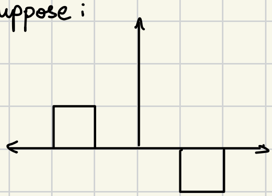
It is as nonsense as

$$u(2) \triangleq \int_{-\infty}^{2} \delta(2) d2$$

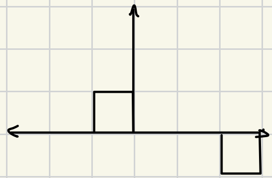
e) Transforming Signals

① Time Shifts

Suppose :

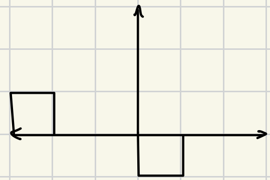


$x(t)$



$x(t - t_0)$

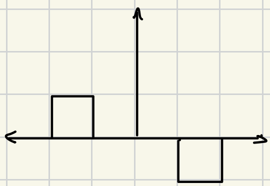
Forward Shift



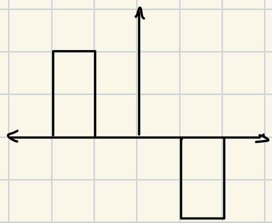
$x(t + t_0)$

Backward Shift

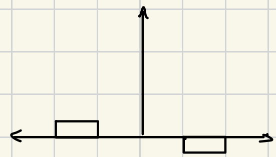
② Amplitude Modification



$x(t)$

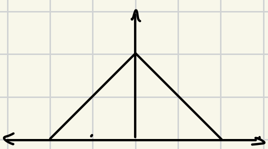


$\alpha x(t)\ ,\ \alpha > 1$

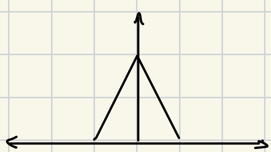


$\alpha x(t)\ ,\ \alpha < 1$

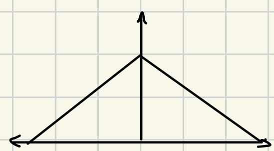
③ Time Scaling



$x(t)$



$x(2t)$



$x(t/2)$

f) Some Examples of Systems

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

Eg: Whatever transformations we discussed above, i.e :

- → Time Shift
- → Amp modulation
- → Time scaling

NOTE

$$u(t) \triangleq \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$\downarrow$

Value at t

$$u(t) \triangleq \int_{-\infty}^{t} \delta(\tau)\, d\tau \quad \forall\, t \in \mathbb{R}$$

$\downarrow$

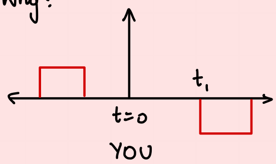
Entire signal

NOTE:

$x(t - t_0)$ = Time delay

$x(t + t_0)$ = Time advance

Why?



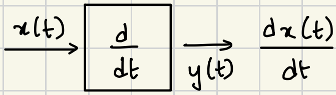
You move forward with time. So you see the signal at $t = t_1$.

But for $x(t - t_0)$, the signal shifts ahead t_0. So you observe the signal at $t = t_0 + t_1$.

Hence, it is a time delay.

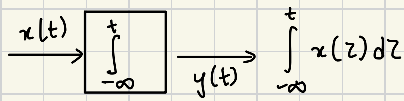
Other Systems:

① Differentiator



Captures the rate of change of a signal.

② Integrator / Accumulator



Accumulates the output values of the system until time t.

NOTE: Smoothening a signal / Moving average

Let

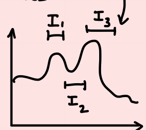
$$y(t) = \frac{\int_{t-\delta}^{t+\delta} x(\tau)\,d\tau}{2\delta}$$

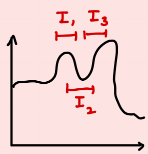
(Where $\tau \in [t-\delta, t+\delta]$

The graph of $y(t)$ will now look like a smoothened version of $x(t)$, since we are basically replacing small intervals of values with their average.

Then how do we still get a continuous graph?

Instead of taking intervals like this...





Take them like this.

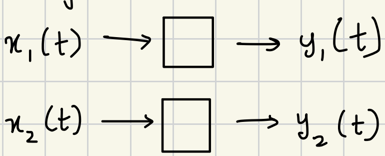
g) Some important classes of systems

① LTIs (Linear Time - Invariant Systems)

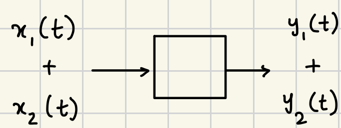
They satisfy the property of

ⓐ additivity.

Eg:



So if it's an LTI,



ⓑ Scaling

Eg: $x(t) \longrightarrow \boxed{} \longrightarrow y(t)$ then $a \cdot x(t) \longrightarrow \boxed{} \longrightarrow a \cdot y(t)$

ⓒ Time Invariance

If

$x(t) \longrightarrow \boxed{} \longrightarrow y(t)$ then $x(t-t_0) \longrightarrow \boxed{}$ $y(t-t_0)$

(It is called linear because it follows additivity and scaling.)

② Causal Systems

If $y(\tau)$ depends on $x(\tau)$ $\forall$ $\tau \leq t$

BUT NOT ON $\tau > t$

③ Stable Systems

If $|y(t)| <$ const. 1 $\forall t$ whenever $|x(t)| <$ const. 2 $\forall t$ ...

Then system is stable.

$y(t) = e^t x^2(t)$

Prove that this is causal.

Sol: $y(t)$ depends on $x(t)$, for $\tau \leq t$

HENCE the system is causal.

NOTE that causality does NOT depend on $x(t)$. We always measure causality only with reference to wherever the input starts.

2. Harmonics of A Sinusoid

- $e^{j\omega_0 t} = e^{j\left(\frac{2\pi}{T}\right)t} = \cos\omega_0 t + j\sin\omega_0 t$

  (Both are periodic.)

  $f_0 = \frac{1}{T}$ (frequency of the above sinusoid)

  So any fn. of the form $e^{jk\omega_0 t}$ $[k \in \mathbb{Z} - \{0\}]$ is called the $|k|^{th}$ harmonic of $e^{j\omega_0 t}$.

- Now, let

$$\underbrace{x(t)}_{\text{"Our biryani"}} \triangleq \sum_{k \in \mathbb{Z}} \underbrace{a_k}_{\text{Depends on } k} e^{jk\omega_0 t} \qquad (\text{where } a_k \in \mathbb{C}\ \forall k)$$

Obviously, $x(t)$ is periodic.

But can we say the reverse is true? i.e, can any periodic signal be written as a sum of sinusoids?

Yes!

[But with caveats:
- $x(t)$ must have finite energy in any time - period
- $x(t)$ must be continuous]

a) Fourier Series

Let $x(t)$ have fundamental angular frequency $\omega = \frac{2\pi}{T}$.

Then $x(t)$ can be expressed as

$$\boxed{x(t) = \sum a_k e^{jk\omega t}} \quad \text{for some } a_k \in \mathbb{C}\ \forall\, k \in \mathbb{Z}.$$

For some ω, let

$$x(t) = \sum_{k \in \mathbb{Z}} a_k e^{jk\omega t} dt \quad , \text{ where } T \text{ is some time interval of length } T.$$

$$x(t)e^{-jk'\omega t} = \sum_{k \in \mathbb{Z}} a_k e^{j(k-k')\omega t} dt$$

Integrating on both sides,

$$\int_0^T x(t)e^{-jk'\omega t} = \int_0^T \sum a_k e^{j(k-k')\omega t} dt$$

$$= \sum \left(\int_0^T a_k e^{j(k-k')\omega t} dt \right)$$

Think about this integral. Since time period $= \frac{2\pi}{(k-k')\omega_0}$,

① if $k \neq k'$, we will integrate over $(k'-k)$ time periods.

$\Rightarrow$ Integral is 0.

② if $k' = k$, $e^{j(k-k')\omega t} = 0$.

$\Rightarrow$ Integral becomes T.

Since $\int e^{j(k'-k)\omega t} dt = \begin{cases} 0, & k \neq k' \\ T & k = k' \end{cases}$

$= a_{k'} \cdot T$ (since for every other k, integral is 0.)

Hence,

$$\boxed{a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega t} dt}$$

Using this, the coeff. of any specific term can be found.

Eg: $x(t) = \sin^2 500t$

Write it using FS coeffs.

Sol: $\sin^2 500t = \frac{1 - \cos 1000t}{2}$

$$= \frac{1}{2} - \left(\frac{e^{j1000t} + e^{-j1000t}}{2} \right) \frac{1}{2}$$

$$= \frac{-1}{4} e^{-j1000t} + \frac{1}{2} - \frac{1}{4} e^{j1000t}$$

- Assume $x(t)$ is real.

$$x(t) = \sum_{k \in \mathbb{Z}} a_k e^{jk\omega t} \quad (\text{FS of } x(t))$$

$$\overline{x(t)} = \sum_{k \in \mathbb{Z}} a_k^* e^{-jk\omega t} \quad (\text{conjugate of } x(t))$$

$$\overline{x(t)} = \sum_{k' \in \mathbb{Z}} a_{-k'}^* e^{jk'\omega t} \quad (\text{FS of } x^*(t))$$

(Replacing k with $-k'$)

- Notice how a_k and a_{-k}^* are conjugates.
- The k^{th} coeff. of $x^*(t)$ = conjugate of $(-k)^{th}$ FS coeff. on $x(t)$.
- If $x^*(t) = x(t)$ [i.e $x(t) \in \mathbb{R}$]

$$\boxed{a_k = a_{-k}^*} \quad \forall k$$

(Conjugate Symmetry of FS coeffs.)

- Note that Fourier series can also be used to reconstruct signals.

In

$$x(t) = \sum_{k \in \mathbb{Z}} a_k e^{jk\omega t}$$

a_k depends on $k, \omega, x(t)$

$e^{jk\omega t}$ depends on k, ω.

If $x(t)$ is known, we can find its FS representation $\Leftrightarrow$ FS coefficients

- Now someone need not transmit the entire signal $x(t)$ – transmitting all a_k would allow the reciever to reconstruct $x(t)$ themselves.

Note: If $x(t)$ is real, why is it written as a sum of complex numbers?

$$x(t) = \sum_{k\in\mathbb{Z}} a_k e^{jk\omega t}$$

$$= a_0 + \sum_{k>0} \left(a_k e^{jk\omega t} + a_{-k} e^{-jk\omega t}\right)$$

$$= a_0 + \sum_{k>0} \left(\underbrace{a_k e^{jk\omega t}}_{z} + \underbrace{a_k^* e^{-jk\omega t}}_{\bar{z}}\right)$$

Since $z + \bar{z} = 2\,\mathrm{Re}(z)$,

$$= a_0 + \sum 2\,\mathrm{Re}\left(a_k e^{jk\omega t}\right)$$

Let $a_k = |a_k| e^{j\phi_k}$ $\left(\phi_k = \tan^{-1}\left(\frac{\mathrm{Im}(a_k)}{\mathrm{Re}(a_k)}\right) \right)$

Then

$$x(t) = a_0 + \sum 2\,\mathrm{Re}\left(|a_k|\, e^{j(\phi_k + k\omega t)}\right)$$

$$= a_0 + 2\sum |a_k| \cos\left(\phi_k + k\omega t\right)$$

Which is real.

So from what we've done so far, the diff. forms of an FS are:

- $x(t) = \sum_{k\in\mathbb{Z}} a_k e^{jk\omega t}$
- $x(t) = \sum_{k\in\mathbb{Z}} a_k^* e^{-jk\omega t}$
- $x(t) = a_0 + 2\sum_{k=1}^{\infty} |a_k| \cos(k\omega_0 t + \phi_k)$
- $x(t) = a_0 + 2\sum_{k=1}^{\infty} B_k \cos k\omega_0 t - C_k \sin k\omega_0 t$ $\quad$ (if $a_k = B_k + jC_k$)

b) Properties of FS expansions

① Linearity

If $A(t)$ & $B(t)$ both have same angular frequency ω.

$$A(t) = \sum a_k e^{jk\omega t}$$

$$B(t) = \sum b_k e^{jk\omega t}$$

And let a $C(t) = A(t) + B(t)$.

Then if $c(t) = \sum c_k e^{jk\omega t}$,

Then $c_k = a_k + b_k \;\; \forall\, k \in \mathbb{Z}$

② Multiplication

$$A(t) = \sum a_k e^{jk\omega t}$$

$$B(t) = \sum b_k e^{jk\omega t}$$

Then if $c(t) = \sum c_k e^{jk\omega t}$,

And let $C(t) = A(t) \cdot B(t)$

$$= \left(\sum a_n e^{jn\omega t}\right)\left(\sum a_m e^{jm\omega t}\right)$$

So $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Which means $C(t)$ becomes

$$C(t) = \sum \left(\sum_{k=0}^{n} a_k b_{n-k} \right) e^{jn\omega t}$$

③ Convolution

$$c_k = \sum_{l=0}^{k} a_l b_{k-l}$$

This kind of a series is called a CONVOLUTION.

Let $x(t), y(t)$ have a period T.

Then if

$$z(t) = \int_T x(\tau) y(t-\tau)\, d\tau$$

$z(t)$ is called a periodic convolution of $x(t)$ & $y(t)$.

$z(t)$ ALSO has period T.

$$z(t) = \sum C_k e^{j\omega t}$$

Then $\boxed{C_k = T a_k b_k}$

Consider an example:

We want to find out how much smoke was emitted from fireworks over time T.

$S(t)$ = smoke from a firework at time t after ignition

$F(t)$ = no. of fireworks ignited at time t.

Now to find total smoke at a time:

① $t=0$

$$Y(0) = F(0) \cdot S(0)$$

② $t=1$

$$Y(1) = F(1) S(0) + F(0) S(1)$$

... and so on. So:

$$Y(t) = \sum_{\tau=0}^{t} F(\tau) S(t-\tau)$$

Or if fn. is continuous,

$$z(t) = \int_0^t x(\tau) y(t-\tau)\, d\tau$$

c) Laplace Transform

Recall the FS of a periodic signal.

The k^{th} FS coeff (which multiplies w/ $e^{jk\omega t}$)

$$a_k = \frac{1}{T} \int x(t) e^{-jk\omega t}\, dt$$

- For any signal (not necessarily periodic), we define:

$$\boxed{X(s) \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$$

The value of $X(s)$ is the Laplace Transform of $x(t)$ at complex frequency s, where

$$\boxed{s = \sigma + j\omega}$$

- The Laplace Transform basically converts $x(t)$ (from t - domain) to $X(s)$ (to s - domain)
- We define a 'Region of Convergence' for s, which make the integral a finite value.

Note: The sifting property of convolutions

Let's assume we take

$$y(t) = \underbrace{x(t)}_{\text{Scaling Const.}}\ \underbrace{\delta(t-t_0)}_{\text{Impulse}}$$

Then,

$$y(t) = \begin{cases} 0, & t \neq t_0 \\ x(t_0)\,\delta(0), & t = t_0 \end{cases}$$

Now take $z(t) = x(t_0)\,\delta(t-t_0)$

Then

$$z(t) = 0 \begin{cases} 0, & t \neq t_0 \\ x(t_0)\,\delta(0), & t = t_0 \end{cases}$$

Now if you integrate both,

$$\int y(t) = \int_T x(\tau)\,\delta(t-\tau)\,d\tau$$

$$\int z(t) = \int_T x(t)\,\delta(t-\tau)\,d\tau$$

$$= x(t) \int_T \delta(t-\tau)\,d\tau$$

Notice how $y(t)$ and $z(t)$ are basically the same thing.

Which means:

$$\boxed{\int_T x(\tau)\, y(t-\tau)\,d\tau = x(t) \int_T y(t-\tau)\,d\tau}$$

You should see now that Laplace Transform is basically a convolution too.

Let:

$$X(s) = \int_T \delta(t-0)\, e^{-st}\, dt$$

$$= e^{0} \int_T \delta(t-0) \quad \left(\text{Applying the sifting property}\right)$$

$$= 1$$

Eg: $e^{-at}u(t)$ $(a \in \mathbb{R})$

Apply LT on this.

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty}$$

BUT $s = \sigma + j\omega$.

So

$$X(s) = \left[\frac{e^{-(\sigma+a)t} \cdot e^{-j\omega t}}{-(s+a)} \right]$$

Now we can compare σ & a.

If $\sigma + a > 0$, $e^{-(\sigma+a)}$ converges.

$$\boxed{X(s) = \frac{1}{s+a}.}$$

If $\sigma + a < 0$, $-(\sigma + a)t > 0$.

Which means for

$t \to \infty$,

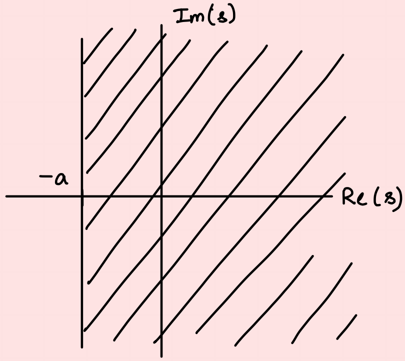
$e^{-(\sigma+a)t} \to \infty$

So $X(s)$ becomes unbounded.

Finding the ROC:

The ROC would be $\{s : Re(s) < a\}$

ROC diagram



d) Properties of Laplace Transform

① Convolution in time domain

$$\boxed{LT\left(x(t) * y(t)\right) = X(s)\, Y(s)}$$

if $x(t) \to X(s)$ has ROC $= R_1$ $(\in \mathbb{C})$

$y(t) \to Y(s)$ has ROC $= R_2$

Note that $*$ is the convolution symbol.

The ROC of the combination contains $R_1 \cap R_2$.

② Linearity

$x(t) \rightarrow X(s)$

$y(t) \rightarrow Y(s)$

Then $\alpha x(t) + \beta y(t) \longrightarrow \alpha X(s) + \beta Y(s)$

③ Derivative

If $x(t) \xrightarrow{LT} X(s)$, then $\frac{d}{dt} x(t) \xrightarrow{LT} s \cdot X(s)$

e) Inverse Laplace Transform

We use the following formula:

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{s=\sigma+j(-\infty)}^{\sigma+j(+\infty)} X(s) e^{st} \, ds}$$

where σ is any $Re(s)$ $\forall s \in$ ROC of $X(s)$

f) Fourier Transform

Let $x(t)$ be any signal.

The Fourier Transform of $x(t)$ is a signal in the ω-domain (or f-domain)

$$\tilde{X}(\omega) = FT[x(t)]$$

$$= X(j\omega)$$

$$= LT(X(s)) \; [\text{at } s = j\omega]$$

$$= \int_{-\infty}^{\infty} x(t) e^{-st} \, dt \quad \text{at } s = j\omega.$$

$$= \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} \, dt$$

Note : LTI and convolution

Let $h(t)$ be the output to an input signal being $\delta(t)$.

Now if the input signal were instead $x(t)$,

$$\boxed{y(t) = x(t) * h(t)}$$

But why ?

<u>Proof</u> :

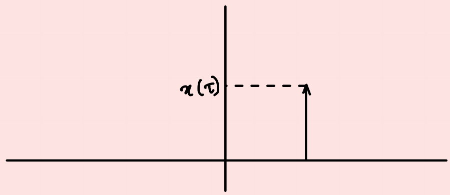
Since it is already established that :

$$\delta(t) \longrightarrow h(t)$$

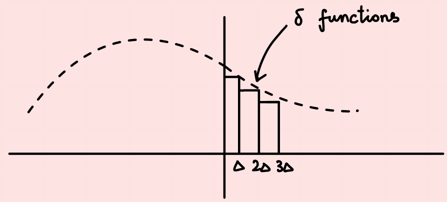
Then $\quad c \cdot \delta(t) \longrightarrow c \cdot h(t) \quad$ [LTI.]

Now let us represent $x(t)$ as "slices" of a scaled $\delta(t)$ at every t. for example, a slice at $t = \tau$ could be :

$$x(\tau) \cdot \delta(t - \tau)$$



$x(\tau)\,\delta(t-\tau)$



$x(t)$

Now, output is:

$$x(\tau)\,\delta(t-\tau) \longrightarrow x(\tau)\,h(t-\tau)$$

Integrating over all τ,

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= x(t) * h(t).$$

Note that this ONLY applies to LTI systems cause otherwise we can't say for sure that a sum of inputs guarantees a sum of their respective outputs.

① Response of LTI systems to exponential sinusoids (e^{st})

Output of the system is

$$y(t) = h(t) * e^{st}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st}\left(\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau\right)$$

$$= H(s)\, e^{st}$$

Where $H(s)$ is the LT of $h(t)$ (at specific s.)

If you notice, the whole operation does nothing but scale e^{st};

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(s)\, e^{st}$$

This correlates back to a property of vectors called eigenfunctions.

Eigenvectors

If $A\vec{v} = \lambda \vec{v}$ is true, where:

A = Transformation

λ = Some scalar

An s-domain view of LTI systems

$$\begin{tikzcd} x(t) \arrow[r, "\text{Output}"] \arrow[d, "\text{LT}"] & x(t) * h(t) \arrow[d, "\text{LT}"] \\ X(s) \arrow[r, "\text{Output}"] & X(s)\, H(s) \end{tikzcd}$$

Poles and Zeroes of LT

Poles ($\times$) : Where the fn. goes to ∞.

Zeroes (0) : Where the fn. goes to 0.

Eg: $$X(s) = \frac{s}{s+a}$$

$s = 0$ is a zero.

$s = -a$ is a pole.

② **Important LT pairs**

- $\delta(t) \xrightarrow{\text{LT}} 1$, ROC $= \mathbb{C}$
- $a \in \mathbb{R}$,

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}, \quad \text{ROC} = \{ s \in \mathbb{C},\ \text{Re}(s) > -a \}$$

$$-e^{-at}u(-t) \longleftrightarrow \frac{1}{s+a}, \quad \text{ROC} = \{ s \in \mathbb{C},\ \text{Re}(s) < -a \}$$

Eg: $$X(s) = \frac{1}{(s+1)(s+2)}, \quad \text{Re}(s) > -1$$

$$= \underbrace{\frac{1}{s+1}}_{\text{ROC is } s > -1} - \underbrace{\frac{1}{s+2}}_{\text{ROC is } s > -2}$$

Note that the ROC of $X(s)$ contains the ROC of both transforms.

③ LTI systems described by linear differential eq.s.

A) Explicit

- For an arbitary input $x(t)$,

$$y(t) = (\text{Bla Bla})\, x(t)$$

- Via Impulse Response:

By putting $x(t) = \delta(t)$, we get $h(t)$.

Since $y(t) = x(t) * h(t)$

- $H(s)$ with ROC.

Then we can use 2 methods to find response to $x(t)$.

→ Find $h(t)$, then find $x(t) * h(t)$

→ Find $X(s)$, then find $Y(s) = X(s)\,H(s)$

Then find $\text{ILT}(Y(s)) = y(t)$.

B) Implicit

We can use differential equations.

Eg: $\frac{d}{dt} y(t) + 2y(t) = x(t)$

Sol: Apply LT on both sides.

$\Rightarrow sY(s) + 2Y(s) = X(s)$

$\Rightarrow Y(s)(s+2) = X(s)$

so $Y(s) = \frac{1}{s+2} X(s)$

so $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+2}$

From given ROC and that $H(s) = \frac{1}{s+2}$,

$$\text{ILT} \longrightarrow h(t) = e^{-2t} u(t)$$