Real Analysis

$\mathbb{N} : \{1, 2, 3 \ldots\} \Rightarrow$ Set of all natural numbers

$\mathbb{Z} : \{\ldots -3, -2, -1, 0, 1, 2 \ldots\} \Rightarrow$ Set of all integers

$\mathbb{Q} : \{p/q : p$ and q are integers, $q \neq 0\}$

$\mathbb{R}$: Set of all Real numbers

* $A \subset B \Rightarrow$ Proper Subset

* $A \subseteq B \Rightarrow$ A can be a subset or $A = B$

Algebraic operations of Sets:

a) Union $\Rightarrow A \cup B$

b) Intersection $\Rightarrow A \cap B$

* Disjoint sets mean $A \cap B = \phi$
* If $A \subset B$ then $A \cap B = A$

1) Consistensy Property:

$A \subset B$, $A \cup B = B$ and $A \cap B = A$ are equivalent

a) $A \cup \phi = A$ $\quad A \cap \phi = \phi$

b) $A \cup U = U$ $\quad A \cap U = A$ {U = Universal set}

c) $A \cup A = A$ $\quad \underbrace{A \cap A = A}$

Idempotent property

d) $A \cup (A \cap B) = A$ [Absorptive property]

$A \cap (A \cap B) = A$

e) $A \cup B = B \cup A$ [Commutative property]

$A \cap B = B \cap A$

f) $A \cup (B \cup C) = (A \cup B) \cup C$ [Associative property]

Same thing for "$\cap$"

g) $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$ [Distributive property]

Complementary Set (A^c):

$$A \cup A^c = U \qquad (A^c)^c = A$$

$$A^c = U/A \text{ or } U - A$$

De-Morgan's Law:

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

Cartesian Product of Sets:

$A = \{a, b\}$ $[a, b \in A]$

$B = \{A, B\}$ $[A, B \in B]$

$A \times B = \{(a, A), (a, B), (b, A), (b, B)\}$

$A \times B \neq B \times A$

Relation on a Set:

Let $A \neq \phi$ and $B \neq \phi$

Example: $A = \{2, 3, 4, 5\}$

$B = \{4, 6, 8, 9\}$

$\rho \Rightarrow$ An elementary of A is related to element of B

Order Properties:

1) ρ is symmetric if $a \rho b \Rightarrow b \rho a$ for $a, b \in X$ [Random set]

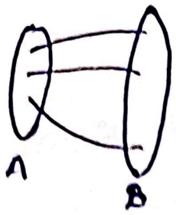
2) ρ is anti-symmetric

Definition: $A \neq \phi$ $B \neq \phi$

A relation ρ between A and B is a subset of $A \times B$. If the ordered pair $(a, b) \in \rho$ then $a \in A$ is related to $b \in B$

$(a, b) \in A \times B \Rightarrow \rho$

Functions:



Injective : One - One

Surjective : $f(A) = B$ } Both $\Rightarrow$ Bijective

Natural Numbers [IN]:

① Well - Ordering property:

Every non-empty set of natural numbers has a least element.

Proof:

Let $S \subseteq \mathbb{N}$ $S \neq \phi$. Let $K \in S$.

$T \subseteq S$, $T = \{x \in S , x \leq K\}$, $T \neq \phi$

T is finite subset of IN and therefore has a least element.

There exists $1 \leq m \leq K$ ★ m is the least element of T

If $s > K$ then $m \leq K \rightarrow m < s$ ★ K is an arbitary number from T

If $s \leq K$ then $s \in T$, $m \leq s$ ★ s is an arbitary IN

Principle of Induction:

Let $S \subseteq \mathbb{N}$, such that

(i) $1 \in S$

(ii) if $K \in S$, then $K+1 \in S$; Then $S \in \mathbb{N}$

Theorm:

Let $P(n)$ be a statement involving $n \in \mathbb{N}$

If (i) $P(1)$ is true

(ii) Consider $P(K)$ to be true then if $P(K+1)$ is true then $P(n)$ is true.

Example: ① $S_n = \frac{n(n+1)}{2}$ $\quad n \in \mathbb{N}$

$S_1 = 1$

$S_k = \frac{k(k+1)}{2}$

$S_{k+1} = (k+1) + \frac{k(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

② Show that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer

$P(1) = 6$

$P(k) = (3+\sqrt{5})^k + (3-\sqrt{5})^k$ is true

$P(k+1) = (3+\sqrt{5})^k \cdot (3+\sqrt{5}) + (3-\sqrt{5})^k (3-\sqrt{5})$

$= \frac{(3+\sqrt{5})^k}{3-\sqrt{5}} + \frac{(3-\sqrt{5})^k}{3+\sqrt{5}}$

$= (3+\sqrt{5})^{k+1} + (3-\sqrt{5})^{k+1}$

Consider $3+\sqrt{5} = a$ $\quad 3-\sqrt{5} = b$

$P(k+1) = a^{k+1} + b^{k+1}$

$= (a^k + b^k)(a+b)\ (a^{k-1} + b^{k-1})(ab)$

$= 6(a^k + b^k) - 4(a^{k-1} + b^{k-1})$

Integers:

$\mathbb{Z} = \{\ldots\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$

Rational Number:

1. Algebraic properties
2. Order properties
3. Density properties

① **Algebraic properties:**

A/ (i) $a \in \mathbb{Q}$ $b \in \mathbb{Q}$ then $a+b \in \mathbb{Q}$

(ii) $(a+b)+c = a+(b+c)$

(iii) There exists ($\exists$) a $0 \in \mathbb{Q}$ such that $a+0 = a$

(iv) $\exists$ $-a \in \mathbb{Q}$ such that $a+(-a) = 0$

(v) $a+b = b+a$ $\forall\ a, b \in \mathbb{Q}$

M/ (i) $a \in \mathbb{Q}$ $b \in \mathbb{Q}$ then $a \cdot b \in \mathbb{Q}$

(ii) $(ab)c = a(bc)$

(iii) $\exists$ $1 \in \mathbb{Q}$ such that $1 \times a = a$ $[a \in \mathbb{Q}]$

(iv) $\forall\ a \in \mathbb{Q}$ $\exists$ $\frac{1}{a} \in \mathbb{Q}$ such that $a\left(\frac{1}{a}\right) = 1$ $(a \neq 0)$

(v) $a \cdot b = b \cdot a$

D/ (i) $a(b+c) = ab + ac$

② **Order properties:**

(i) If $a, b \in \mathbb{Q}$, then one of the following is true

$a < b$ (or) $a > b$ (or) $a = b$ [Law of Trichotomy]

(ii) $a < b$ $b < c$ then $a < c$ [Law of Transitive]

(iii) $a < b \Rightarrow a+c < b+c$

(iv) $a < b \Rightarrow ac < bc$ $[c > 0]$

③ **Density property:**

If $x, y \in \mathbb{Q}$ and $x < y$

$\exists$ $r \in \mathbb{Q}$ such that $x < r < y$

$x < y \Rightarrow x+y < y+y$

$\frac{1}{2}(x+y) < \frac{1}{2}(2y) \Rightarrow \frac{x+y}{2} < y$

$x + x < y + x$

$$\boxed{x < \frac{x+y}{2} < y}$$

★ There does not exist $r \in \mathbb{Q}$ such that $r^2 = 2$

Let's say $\exists\ p, q \in \mathbb{Z}$, $q \neq 0$ such that

$\left(\frac{p}{q}\right)^2 = 2 \Rightarrow p^2 = 2q^2$

$\because p^2 =$ even, $p =$ even $= 2m$

$q^2 = 2m^2$ $\quad q^2 =$ even $\Rightarrow q =$ even

Real Numbers $[\mathbb{R}]$:

1) Algebraic property

2) Order property

3) Density property

4) Completeness property

5) Archemedian property

Theorm:

(i) $a + b = a + c$, implies that $b = c$

(ii) $a.b = a.c$ $(a \neq 0)$, implies that $b = c$

Proof:

(i) $a + b = a + c$

$-a \in \mathbb{R}$ such that $a + (-a) = 0$

$a + b + (-a) = a + c + (-a)$

$a + (-a) + b = a + (-a) + c$

$0 + b = 0 + c \Rightarrow b = c$

(ii) $a \cdot b = a \cdot c$

Let $1/a \in R$ such that $a\left(\frac{1}{a}\right) = 1$

$$a \cdot b \cdot \left(\frac{1}{a}\right) = a \cdot c \cdot \left(\frac{1}{a}\right)$$

$$a\left(\frac{1}{a}\right) \cdot b = a\left(\frac{1}{a}\right) \cdot c$$

$$1 \cdot b = c \cdot 1 \Rightarrow b = c$$

Theorm:

(i) $a \cdot 0 = 0$

(ii) $(-1)(a) = -a$

(iii) $-(-a) = a$

(iv) $1/(1/a) = a$

Proof:

(i) WKT $0 + 0 = 0$

$$a(0 + 0) = a(0)$$

$$a.0 + a.0 = a(0)$$

$$a.0 + (a.0 + (-a).0) = a(0) + (-a)(0)$$

$$a.0 = 0$$

(ii)

$$1 + (-1) = 0$$

$$1(a) + (-1)(a) = (0)a$$

$$(-a) = (-1)(a)$$

(iii)

$$a + (-a) = 0$$

$$a + (-1)a = 0$$

$$-a + [a + (-1)a] = -a + 0$$

$$-a + [a + (-1)a] + (-1)a = -a \Rightarrow 0 + (-1)a = -a$$

$-a \in \mathbb{R}$

$a + (-a) = 0$

$(-a) + -(-a) = 0$

$(-a) + (-(-a)) + a = a$

$(-a) + a + -(-a) = a$

$-(-a) = a$

(iv) $a\left(\frac{1}{a}\right) = 1 \qquad \frac{1}{a} \in \mathbb{R}$

$a \cdot \frac{1}{a} \cdot \frac{1}{1/a} = \frac{1}{1/a}$

$a = \frac{1}{1/a}$

Theorm:

Let $a, b \in \mathbb{R}$

(i) $a > 0, \; b > 0 \Rightarrow a + b > 0$ (iii) $a > 0, \; b > 0 ; \; ab > 0$

(ii) $a < 0, \; b < 0 \Rightarrow a + b < 0$ (iv) $a < 0, \; b < 0 ; \; ab > 0$

Proof:

(i) $a > 0 \qquad b \in \mathbb{R} \qquad -(-a) = a$

$a + b > b \qquad b > 0$

$a + b > 0$

Q) Prove that $1 > 0$.

$a \in \mathbb{R}$

$a^2 > 0$

$1^2 > 0$

$1 \cdot 1 > 0$

$1 > 0$

Q) Prove that $a^2 > 0$

$a > 0$ $b > 0$

$a \cdot b > 0$ $[b = a]$

$a \cdot a > 0 \Rightarrow a^2 > 0$

Absolute Value:

$a \in \mathbb{R}$

$|a| = a$ if $a > 0$

$= 0$ if $a = 0$

$= -a$ if $a < 0$

$|-a| = -(-a) = a$

Q) Prove that $|a^2| = |a|^2$

$|ab| = |a| \cdot |b|$

$b = a$

$|a \cdot a| = |a| \cdot |a|$

$|a^2| = |a|^2$

Q) Prove $|ab| = |a||b|$

$a > 0$ $b > 0 \Rightarrow ab > 0$

$|ab| = ab = |a| \cdot |b|$

$a < 0$ $b > 0 \Rightarrow ab < 0$

$|ab| = -ab = |a||b|$

Triangular ~~Identity~~ Inequality:

$\forall a, b \in \mathbb{R}$

$|a + b| \leq |a| + |b|$

$$\left.\begin{array}{l} -|a| \le a \le |a| \\ -|b| \le b \le |b| \end{array}\right\} \quad -(|a|+|b|) \le a+b \le |a|+|b|$$

$$|a+b| \le |a|+|b|$$

Archemedian Property:

If $x, y \in \mathbb{R}$ and $x, y > 0$, Then $\exists\, m \in \mathbb{N}$ such that $ny > x$

Q) If A, B, C are sets, then [You should prove both ways] [Do not use Venn diagrams]

a) $A/(B\cup C) = (A/B) \cap (A/C)$

LHS $= A - (B \cup C)$

RHS $= (A - B) \cap (A - C)$

<u>Forward proof</u>: $\forall\, x \in A \setminus (B \cup C)$

$\Rightarrow x \in A$ and $x \notin (B \cup C)$

$\Rightarrow x \in A$ and $(x \notin B \,\&\, x \notin C)$

$\Rightarrow x \in A$ and $x \notin B$; $x \in A$ and $x \notin C$

$x \in (A/B) \cap (A/C)$

<u>Backward proof</u>: $\forall\, x \in (A/B) \cap (A/C)$

$\Rightarrow x \in A$ and $x \notin B$; $x \in A$ and $x \notin C$

$\Rightarrow x \in A$ and $x \notin B$ $x \in C$

$\Rightarrow x \in A$ and $x \notin (B \cup C)$

$\Rightarrow x \in A/(B\cup C)$

b) $A/(B\cap C) = (A/B) \cup (A/C)$

<u>Forward proof</u>: $\forall\, x \in A/(B \cap C)$

$\Rightarrow x \in A$ $x \notin (B \cap C)$

$\Rightarrow x \in A$ $x \notin B$ (or) $x \notin C$

$\Rightarrow x \in A$ and $x \notin B$ (or) $x \in A$ and $x \notin C$

$\Rightarrow x \in (A/B) \cup (A/C)$

Backward proof:

$x \in A$ and $x \notin B$ (or) $x \in A$ and $x \notin C$

$\Rightarrow x \in A$ and $x \notin B$ (or) $x \notin C$

$\Rightarrow x \in A$ and $x \notin (B \cap C)$

$\Rightarrow x \in A/(B \cap C)$

$\boxed{\sim(A \wedge B) = \sim A \cup \sim B}$
$\boxed{\sim(A \cup B) = \sim A \cap \sim B}$ } DeMorgan's Laws

2) Show that $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow 2 - \epsilon < \frac{2n+1}{n+2} < 2 + \epsilon$

$$2 - \epsilon < \frac{2n+4-3}{n+2} < 2 + \epsilon$$

$$2 - \epsilon < \frac{-3}{n+2} + 2 < 2 + \epsilon$$

$$-\epsilon < \frac{-3}{n+2} < \epsilon \Rightarrow -\epsilon < \frac{-3}{n+2} \Rightarrow \epsilon > \frac{3}{n+2}$$

$$n > \frac{3}{\epsilon} - 2 \longrightarrow ①$$

$$n > \frac{-3}{\epsilon} - 2 \longrightarrow ②$$

$\because$ ② becomes subset of ①

$$n > \frac{3}{\epsilon} - 2 \Rightarrow \boxed{n \geq N > \frac{3}{\epsilon} - 2}$$

3) Prove that $\sqrt{2}$ is not a rational

Assume $\sqrt{2}$ is a rational number

$$\sqrt{2} = \frac{a}{b} \qquad GCD(a,b) = 1$$

$$a^2 = 2b^2$$

$a^2 =$ Even $\Rightarrow a =$ Even

$a = 2k = \sqrt{2}b$

$b^2 = 2K^2$

$b^2 \Rightarrow$ even $\Rightarrow$ $b =$ even

Both are even $GCD(a,b) \neq 1$

$\therefore$ Assumption is false.

4) For $n \in N$ $n^3 + 5n$ is divisible by 6.

$S(n) = n^3 + 5n$

$S(1) = 1 + 5 = 6 \Rightarrow$ Divisible by 6

$S(K) = K^3 + 5K \Rightarrow$ Assuming that $S(K)$ is divisible by 6

$S(K+1) = (K+1)^3 + 5(K+1)$

$= K^3 + 3K^2 + 3K + 5K + 1$

$= (K^3 + 5K) + [3K^2 + 3K + 5] + 1$

$= \underbrace{[K^3 + 5K]}_{\text{Divisible by 6}} + 3\underbrace{[K^2 + K + 2]}_{\text{Divisible by 6}}$

5) Let $r = a/b$ $r \in Q \cap (0,1)$ and $a \geq 1$, $b \geq 1$. (a,b) are co-primes natural numbers, then show that there exists natural number $n \geq 1$ such that $\frac{1}{n+1} \leq \frac{a}{b} \leq \frac{1}{n}$.

By Archemedian principle,

$b/a > 1$ $\exists n$

$\boxed{n < b/a < n+1} \Rightarrow \frac{1}{n+1} < \frac{a}{b} < \frac{1}{n}$

$r = \frac{a}{b} = \frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k}$

$a = 1 \Rightarrow r = 1/b$

$a = K \Rightarrow r = K/b$

$a = K+1 \Rightarrow \frac{1}{n+1} < \frac{K+1}{b} < \frac{1}{n}$

$$\frac{k+1}{b} = \frac{1}{n+1} + \left(\frac{k+1}{b} - \frac{1}{n+1}\right)$$

$$\frac{k+1}{b} = \frac{1}{n+1} + \left[\frac{(k+1)(n+1) - b}{b(n+1)}\right]$$

6) Show by induction, if x is finite set, with n numbers. Power set of X has 2^n elements.

Completeness Property of $\mathbb{R}$:

Let $S \subseteq \mathbb{R}$. An $u \in \mathbb{R}$ is said to be the upper boundary of S if $\forall x \in S \Rightarrow x \leq u$. A $v \in \mathbb{R}$ is said to the lower bound of S if $\forall y \in S \Rightarrow v \leq y$. $[S \neq \phi]$

Smallest upperbound, $\forall x \in S \Rightarrow x \leq u$ [Suprimum]

Greatest Lower bound [Infinum] $\forall x \in S \Rightarrow x \geq v$

$S = (1 \leq x \leq 2) \Rightarrow$ Closed and Bounded.

Statement: Every non-empty subset of $\mathbb{R}$ that bounded above has suprimum.

Properties of infimum and Suprimum:

Let $S \subseteq \mathbb{R}$ and $S \neq \phi$. bounded above this Sup S exists.

Let $M = \text{Sup } S$

① $x \in S \Rightarrow x \leq M$

② For each $\varepsilon > 0$, $\exists$ an element $y(\varepsilon)$ in S, such that

$M - \varepsilon < y \leq M$

③ Let $S \subseteq \mathbb{R}$ and $S \neq \phi$ which is bounded below, then

Min = Inf S exists.

① $\forall x \in S \Rightarrow x \geq m$

② For each $\epsilon > 0$ $\forall$ $y(\epsilon)$ such that $m \leq y(\epsilon) < m+\epsilon$

Archemedian Principle:

If $(x, y) \in \mathbb{R}$ and $x > y > 0$ $\exists$ $n \in \mathbb{N}$ such that $ny > x$

Proof: $\nexists n \in \mathbb{N}$ such that $ny > x$. Then $\forall k \in \mathbb{N} \Rightarrow ky \leq x$

Therefore the set $= \{ky : k \in \mathbb{N}\}$ is bounded above with x as the upper bound.

$$\sup S = b$$

Then $ky \leq b$ $\forall$ $k \in \mathbb{N}$, $b-y < b$. Then $\exists$ $p \in \mathbb{N}$ such that

$$b-y < py \leq b$$

$$(p+1)y > b$$

$(p+1) \in \mathbb{N}$ and $(p+1)y \in S$

b is not a Suprimum of S. Contradicts the initial assumption

Q) Show that $\exists$ $x \in \mathbb{R}$ and $x > 0$ such that $x^2 = 2$

Let $S = \{s \in \mathbb{R} : s \geq 0 \text{ and } s^2 < 2\}$

$S \subseteq \mathbb{R}$ and $S \neq \phi$

$\sup S$ exists let $x = \sup S$ clearly $x > 0$ and $1 \in S$ and 1 is not the upper bound $x > 1$, we shall prove $x^2 = 2$

If not, then $x^2 > 2$

Then $\frac{x^2 - 2}{2x} > 0$; $\exists$ $m \in \mathbb{N}$

$$0 < \frac{1}{m} < \frac{x^2 - 2}{2x}$$

$\frac{2x}{m} < x^2 - 2.$

$\left[x - \frac{1}{m}\right]^2 = x^2 + \frac{1}{m^2} + - \frac{2x}{m} > x^2 - \frac{2x}{m}$

$\left[x - \frac{1}{m}\right]^2 > 2$

$x - \frac{1}{m} > 0 \quad ; \quad x > 1$

$\left(x - \frac{1}{m}\right)^2 > 2$

$x - \frac{1}{m}$ is an upperband.

$x^2 < 2 \Rightarrow 2 - x^2 > 0$

$\frac{2 - x^2}{2x+1} > 0 \qquad k \in \mathbb{N}$

By Archemedian property,

$0 < \frac{1}{k} < \frac{2 - x^2}{2x+1}$

$\frac{1}{k}(2x+1) < 2 - x^2$

$\left(x + \frac{1}{k}\right)^2 = x^2 + \frac{1}{k}\left(2x + \frac{1}{k}\right)$

$\leq x^2 + \frac{1}{k}(2x+1)$

$< x^2 + 2 - x^2$

$\left(x + \frac{1}{x}\right)^2 < 2 \Rightarrow \left(x + \frac{1}{k}\right) \in S$ [Contradiction]

$\therefore x$ is an Suprimum

$x^2 \not> 2 \quad x^2 \not< 2 \quad x^2 = 2$

$y \neq x \Rightarrow y^2 = 2$

$x^2 = y^2$

$x > y$. Then $\because x > 0 \quad xy > y^2$

$x^2 > y^2$ ⨳

(i) If $x \in \mathbb{R}$ then $\exists n \in \mathbb{N}$ such that $n > x$.

Case 1: $x > 0$ take $y = 1 \Rightarrow ny > x \Rightarrow n \cdot 1 > x \Rightarrow n > x$.

Case 2: $x \leq 0$ Then $n = 1$

(ii) If $x \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$

Take $y = 1 \Rightarrow nx > y \Rightarrow 0 < \frac{1}{n} < x$

$nx > 1$

$x > \frac{1}{n}$

$x \in \mathbb{N}$, $n > 0 \Rightarrow \frac{1}{n} > 0$

(iii) If $x \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ such that $m-1 \leq x < m$

(iv) If $x \in \mathbb{R}$, $\exists m \in \mathbb{Z}$ such that $m-1 \leq x < m$.

$ny > x \quad \forall x, y \in \mathbb{R}$

Consider $y = 1$ $\quad m-1 < x < m$

$n > x$

$S = \{k,\ k > x;\ k \in \mathbb{N}\}$

$\Downarrow$

Bounded below, it has infimum

$m > x \Rightarrow m-1 \leq x$

Density Property of Real Numbers:

For all $(x, y) \in \mathbb{R}$ with $y > x$, $\exists r \in \mathbb{Q}$ such that $x < r < y$

Proof: $y > x \Rightarrow y - x > 0$

Archemedian principle $\Rightarrow 0 < \frac{1}{n} < y - x \quad n \in \mathbb{N}$

$ny - nx > 1 \Rightarrow nx + 1 < ny$

$nx \in \mathbb{R}$. Then $\exists n \in \mathbb{Z}$ $\quad m-1 \leq nx < m$

$m-1 \leq nx$ $\quad nx+1 \geq m$ $\quad mx < m < my$

$m \leq nx+1 \leq ny$ $\quad (x < \frac{m}{n} < y$

Also $\quad nx < m$

$ny > nx+1 > nx$

$\therefore y > x + \frac{1}{n} > x$

$y > x$

$\sqrt{2}y > \sqrt{2}x \Rightarrow \sqrt{2}x < r < \sqrt{2}y$

$x < \frac{r}{\sqrt{2}} < y$

$s = \frac{r}{\sqrt{2}} \notin \mathbb{Q}$

Q) Prove that $\mathbb{N}$ is not bounded above.

$\mathbb{N} \in \mathbb{R}$ and $\mathbb{N} \neq \phi$ since $1 \in \mathbb{N}$

Let $\mathbb{N}$ be bounded above $\Rightarrow \sup \mathbb{N} = u$

Then $x \in \mathbb{N} \Rightarrow x \leq u$ and $\forall \varepsilon > 0$

$\exists y \in \mathbb{N}$ such that $u - \varepsilon < y \leq u$

$\varepsilon = 1$

$k \in \mathbb{N}$

Such that $u - 1 < k \leq u$

$k + 1 > u$

$k+1 \in \mathbb{N} \Rightarrow$ Contradiction

<u>Theorm:</u>

Let $S \subseteq \mathbb{R}$ and $S \neq \phi$ bounded above. An upperbound u of S is the supremum. If $\exists s \in S$ such that $u - \varepsilon < s < u$ $\forall \varepsilon > 0$.

Proof: $u = \sup S$ $\varepsilon > 0$ Then, $u - \varepsilon$ is not an upperbound of S. Therefore $\exists$ atleast one element in S such that $s > u - \varepsilon$.

$u - \varepsilon > s \leq u$

Conversely let u be an upper bound and for each $\varepsilon > 0$ $\exists s \in S$ such that $u - \varepsilon \leq s \leq u$.

u_0 is the Supremum $u_0 < u$.

Let $\varepsilon = \frac{1}{2}(u - u_0) \Rightarrow u - \varepsilon = u_0 + \varepsilon$

$u - \varepsilon < s' \leq u \Rightarrow s' > u_0 + \varepsilon$

#TUT

Well-Ordering Property:

Every non empty subset of N has a least element

$S \subseteq N$

$k \in S$

$T \subseteq S$ such that $T : \{x : x \leq k\}$

Say m is the least element in T

$T : \{1 . . . k\}$

if $s > k$

$m < s \Rightarrow$ presence of least

$s \in N$ if $s \leq k$

$s \in T$

$m \leq s \Rightarrow$ presence of least

Q) Show that $\log_{10}^{n}$ is not a rational number if n is not a power of 10.

Proof by contradiction:

Say $\log_{10}^{n} = p/q \quad [gcd(p,q) = 1]$

$n = 10^{p/q}$

$n^q = 10^p$

It is only possible if n is 10^k

Q) Prove that $|x| + |y| + |z| \leq |x+y-z| + |y+z-x| + |z+x-y| \quad \forall\, x,y,z \in R$

$|x| = \frac{1}{2}|(x+y-z) + (x+z-y)|$

From triangle equality $\{ |a+b| \leq |a| + |b| \}$

$|x| \leq \frac{1}{2}|x+y-z| + \frac{1}{2}|x+z-y|$

Q) Let m be a non square +ve Integer. There does not exist a rational number such that $r^2 = m$ where $r \in Q$

Proof by contradiction:

Say $r = p/q \Rightarrow (p/q)^2 = m$

$p^2 = q^2 m \qquad [gcd(p,q = 1)]$

p is a multiple of m $\Rightarrow p = am$

$a^2m^2 = mq^2$

$a^2m = mq^2$

q is a multiple of m

$\therefore gcd(p.q) \neq 1$

$\therefore$ Our assumption was wrong

Q) Let $a,b,c \in R$. Then $a \times b = 0$ implies $a = 0$ (or) $b = 0$

$a \cdot \frac{1}{a} = 1 \Rightarrow (ab)\frac{1}{a} = \frac{1}{a} \cdot 0$

$\therefore b = 0$

Q) $a, b \in \mathbb{R}$. Prove $||a| - |b|| \leq |a - b|$

[Triangle Equality: $|a + b| \leq |a| + |b|$]

$|a| = |a - b + b|$ $\qquad$ $|b| = |a + b - a|$,

$|a| \leq |a - b| + |b|$ $\qquad$ $|b| \leq |a - b| + |a|$

$|a| - |b| \leq |a - b|$ $\qquad$ $|b| - |a| \leq |b - a|$

$\Rightarrow ||a| - |b|| \leq |a - b|$

Q) $a \in \mathbb{R}$ $\quad 0 \leq a \leq \frac{1}{n}$ for every Natural Number "n" then

prove $\quad$ a) $a = 0$ $\quad$ b) $a \geq 0$ $\quad$ c) $a > 0$

According to Archemedian Principle $ny > x$

$\forall n \in \mathbb{N}$ $\quad x = 1 \quad y = a$

$na > 1 \Rightarrow$ There exists atleast one n this

, contradicts the $an < 1$

$\therefore a = 0$ is the only solution

Q) Find Sup(A) and Inf(A).

a) $A = \{x \in \mathbb{R} ; x^2 < 1\}$

$x^2 - 1 < 0$

$(x+1)(x-1) < 0$

$x \in (-1, 1)$

$x > -1$ $\quad x < 1 \Rightarrow$ Sup

$\Downarrow$

Inf

(b) $A = \{n + \frac{(-1)^n}{n} ; n \in \mathbb{N}\}$

$n = 1 \Rightarrow 0 \Rightarrow$ Inf

$n = 2 \Rightarrow 2 + \frac{1}{2}$

$n = 3 \Rightarrow 3 - \frac{1}{3}$

Sup $= \infty$

Q) Prove $1^2+2^2+\dots+n^2 = \frac{2n^3+3n^2+n}{6}$ using Induction.

$S(1) = 1 = \frac{2+3+1}{6} = 1$

$S(K) = \frac{2K^3+3K^2+K}{6} = 1^2+2^2+\dots+K^2$

Assuming $S(K)$ is true.

$S(K+1) = (1^2+2^2+\dots+K^2)+(K+1)^2$

$= \frac{2K^3+3K^2+K}{6} + K^2+2K+1$

$= \frac{2K^3+3K^2+K+6K^2+12K+6}{6}$

$= \frac{2K^3+9K^2+13K+6}{6}$

$= \frac{2(K+1)^3+3(K+1)^2+(K+1)}{6}$

Q) Prove that n^3+5n is divisible by 6, $n \in N$

$S(1) = 6$

$S(K) = K^3+5n$

$S(K+1) = (K+1)^3+5(K+1)$

$= K^3+3K+3K^2+1+5K+5$

$= \underbrace{[K^3+5K]}_{\text{Divisible by } 6} + 3\underbrace{(K^2+K+2)}_{\text{Always Even}}$

Q) If $1+x>0$ then show that $(1+x)^n \geq 1+nx$

$S(1) = 2^n \geq 1+n$

$S(K) = (1+x)^K \geq 1+xK$

$S(K+1) = (1+x)^K(1+x) \geq (1+xK)+x \quad 1+x(K+1)$

$$(1+x)^k \geq 1+kx$$

$$(1+x)^{k+1} = (1+x)^k(1+x)$$

$$(1+x)^k(1+x) \geq (1+kx)(1+x)$$

$$(1+x)^{k+1} \geq 1+x+kx^2+kx$$

$$\geq 1+(k+1)x+\underbrace{kx^2}_{+ve}$$

$$1+(k+1)x+\underbrace{kx^2} \geq 1+(k+1)x$$

Young's inequality:

Prove that $p \in (1,\infty)$. We $xy \leq \frac{x^p}{p} + \frac{y^q}{q} \quad \forall x,y \in \mathbb{R}^+$ where $q = \frac{p}{p-1}$

$$\frac{1}{q} = 1-\frac{1}{p} \Rightarrow \frac{1}{p}+\frac{1}{q} = 1$$

$$\boxed{\log\left[\frac{x^p}{p}+\frac{y^q}{q}\right] \geq \frac{1}{p}\log x^p + \frac{1}{q}\log y^q}$$ [Concavity of log]

$$\log\left[\frac{x^p}{p}+\frac{y^q}{q}\right] \geq \log(xy)$$

Q) Prove that $\sqrt[n]{\prod_{j=1}^{n} x_j} \leq \frac{1}{n}\sum_{j=1}^{n} x_j \quad n \in \mathbb{N} \quad x \in \mathbb{R}^+$

$$x = \left[\frac{1}{n}\sum_{j=1}^{n} x_j\right]^{\frac{n}{n+1}} \qquad y = (x_{n+1})^{\frac{1}{n+1}}$$

$$p = 1+\frac{1}{n} = \frac{n+1}{n}$$

$$q = \frac{1}{n+1}$$

$$\log\left[\frac{1}{n}\sum_{j=1}^{n} x_j\right] \geq \frac{1}{n}\log x_1 + \frac{1}{n}\log x_2 + \dots + \frac{1}{n}\log x_n$$

$$\geq \frac{1}{n}\sum_{j=1}^{n}\log x_j$$

LHS of $S(n+1)$

$$\sqrt[n+1]{\prod_{j=1}^{n+1} x_j} = \sqrt[n+1]{\prod_{j=1}^{n} x_j} \cdot x_{n+1}^{\frac{1}{n+1}}$$

$$= \left(\sqrt[n]{\prod_{j=1}^{n} x_j}\right)^{\frac{n}{n+1}} \cdot x_{n+1}^{\frac{1}{n+1}}$$

$$\leq \underbrace{\left[\frac{1}{n}\sum_{j=1}^{n} x_j\right]^{\frac{n}{n+1}}}_{X} \cdot \underbrace{x_{n+1}^{\frac{1}{n+1}}}_{Y}$$

$$\frac{1}{n+1}\sum_{j=1}^{n} x_j + \frac{x_{n+1}}{n+1} \leq \frac{X^P}{P} + \frac{Y^q}{q}$$ [By Young's Inequality]

Q)

Hölder Inequality:

$$|x|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \quad p \in (1,\infty)$$

$$\cdot \sum_{j=1}^{n} |x_i y_j| \leq |x|_p |y|_q$$

For $p = q = 2$.

Sets in $\mathbb{R}$

Intervals:

The subset $\{x \in \mathbb{R} : a < x < b\}$ is called an open interval (a, b).

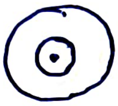
The subset $\{x \in \mathbb{R} : a \leq x \leq b\}$ is called a closed inverted $[a, b]$.

If $a = b$ then $\{a\} \Rightarrow$ Singleton set. [Always closed]

$\mathbb{R} = (-\infty, \infty) \Rightarrow$ Open

Neighbourhood:

Let $c \in \mathbb{R}$. A subset $S \subset \mathbb{R}$ is said to be a neighbourhood of c, if $\exists$ an open interval (a, b) such that $c \in (a, b) \subset S$



Theorm:

Let $c \in \mathbb{R}$, The union of two neighbour hoods of c is a neighbourhood of c itself.

Proof : Let $S_1 \subset \mathbb{R}$ and $S_2 \subset \mathbb{R}$ be two neighbourhoods of $c \in (a_1\ b_1) \subset S_1$ and $c \in (a_2\ b_2) \subset S_2$.

Then $a_1 < b_1$ $\quad a_2 < b_2$ $\quad a_1 < b_2$ $\quad a_2 < b_1$

Lets take $a_3 = \min\{a_1, a_2\}$

$b_3 = \text{Max}\{b_1 . b_2\}$

Then $(a_1\ b_1) \cup (a_2\ b_2) = (a_3\ b_3)$

Now, $(a_1\ b_1) \subset S_1 \cup S_2$ $\qquad (a_3\ b_3) = (a_1\ b_1) \cup (a_2 b_2) \subset S_1 \cup S_2$

$(a_2\ b_2) \subset S_1 \cup S_2$

$\therefore S_1 \cup S_2$ is a neighbourhood.

Theorm:

Let $c \in \mathbb{R}$. The intersection of two neighbourhoods of c is a neighbourhood of c itself.

Let's take $a_3 = \max\{a_1 \ a_2\}$

$b_3 = \min\{b_1 \ b_2\}$

Then $(a_1 \ b_1) \cap (a_2 \ b_2) = (a_3 \ b_3)$

Now, $(a_3 \ b_3) \subset (a_1 \ b_1) \subset S_1$ $(a_3 \ b_3) \subset (a_2 \ b_2) \subset S_2$

$(a_3 \ b_3) \subset S_1 \cap S_2$

Interior point:

Let $S \subset \mathbb{R}$. Some $x \in S$ is said to be an interior point of S if $\exists$ a neighbourhood $N(x)$ such that $N(x) \subset S$

$S = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

$S_0 = \phi$

Interior of $S = S_0$

$S_0 \Rightarrow$ Set of all interior points of S

Open Set:

A set $S \in \mathbb{R}$ is said to be an open set, if $x \in S$ is an interior point of S. $(a . b)$

$a < x < b$

Theorm:

Let $S \subset \mathbb{R}$. Then S is an open set if and only if $S = \text{int}(S)$

$\text{int}() \Rightarrow$ interior.

If $S = \phi$. Then $\phi = \text{int}\ \phi$

$S \neq \phi$, and let $x \in S$

x is an interior point of S.

$x \in S \Rightarrow x \in \text{int}\ S \Rightarrow S \subseteq \text{int}\ S \longrightarrow ①$

Let $y \in \text{int}(S) \Rightarrow y \in S \Rightarrow \text{int}(S) \subseteq S$

$S = \text{int}(S)$

__Theorm:__

The union of two open sets is an open set.

Proof:

Let G_1 and G_2 are two open sets in $\mathbb{R}$

Let $x \in G_1 \cup G_2$ Then $x \in G_1$ or $x \in G_2$

Let $x \in G_1$. $x \in \text{int}\ G_1$.

Therefore $\exists\ N(x)$ of x such that $N(x) \subset G_1$

$N(x) \subset G_1 \cup G_2$

Since x is arbitary, every point of $G_1 \cup G_2$ is an interior point of $G_1 \cup G_2$.

$G_1 \cup G_2 = \text{int}(G_1 \cup G_2)$

__Theorm:__

The intersection of two open sets in an Open set.

Proof [Not Correct]:

Let G_1 and G_2 are two open sets in $\mathbb{R}$

Let $x \in G_1 \cap G_2$. Then $x \in G_1$ and $x \in G_2$

Let $x \in G_1$ $x \in \text{int}(G_1)$

Therefore $\exists\ N(x)$ of x such that $N(x) \subset G_1$

$N(x) \subset G_1 \cap G_2$

Since x is arbitary, every point $G_1 \cap G_2$ is an interior point of $G_1 \cap G_2$.

$$G_1 \cap G_2 = \text{int}(G_1 \cap G_2)$$

Actual Proof:

If $G_1 \cap G_2 = \phi \Rightarrow$ Open set

$G_1 \cap G_2 \neq \phi$

Let $x \in G_1 \cap G_2 \Rightarrow x \in G_1$ and $x \in G_2$

x is an interior point of both G_1 and G_2

$\exists\ \delta_1 > 0 \quad N(x, \delta_1) \subset G_1$

Let $\delta = \text{Min}\{\delta_1, \delta_2\}$

$\text{llly} \quad \exists\ \delta_2 > 0 \quad N(x, \delta_2) \subset G_2$

$N(x, \delta) \subseteq N(x, \delta_1) \subset G_1$

$N(x, \delta) \subseteq N(x, \delta_2) \subset G_2$

$N(x, \delta) \subset G_1 \cap G_2$

$N(x, \delta) \subset G_1 \cap G_2$

Theorm:

Let S be a subset of $\mathbb{R}$. Then $\text{int}(S)$ is an open set.

Proof:

If $\text{int}(S) = \phi \Rightarrow$ is an open set

$\text{int}(S) \neq \phi$ Let $x \in \text{int}(S)$

$\exists\ N(x)$ such that $N(x) \subset S$

$y \in N(x)$

$N(x)$ is a neighbourhood of y and since $N(x) \subset S$

$\Rightarrow y \in \text{int}(S)$

Thus, $x \in \text{int}\, S$

$N(x) \subset \text{int}\, S$

i.e x is an interior point of $\text{int}(S)$ and x is an arbitrary

So $\text{int}(S)$ is an open set.

Sequence of $\mathbb{R}$:

A sequence is a function whose domain is $\mathbb{N}$. If f is such function, let $f(x) = x_n$, denote the value of f at that $n \in \mathbb{N}$

$$(x_n)_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$$

$1 \rightarrow x_1$

$2 \rightarrow x_2$

$n \rightarrow x_n$

Examples:

1) $\frac{n}{n+1}$ $\{\frac{1}{2}, \frac{2}{3}, \ldots\}$

2) $(-1)^n$ $\{-1, 1, \ldots\}$

Bounded Sequence:

A sequence x_n is said to be

(i) Bounded Above, if $\exists\, k \in \mathbb{R}$ such that $x_n \leq k \;\forall n \in \mathbb{N}$ ~~$x_n \leq k \;\forall n \in \mathbb{N}$~~

(ii) Bounded below, if $\exists\, l \in \mathbb{R}$ such that $x_n \geq l \;\forall n \in \mathbb{N}$

(iii) Bounded, if bounded below and above.

Examples:

(i) $\left(\frac{1}{n}\right) \Rightarrow$ Bounded

Convergent and Divergent Sequences:

Definitions: A sequence (x_n) is said to converge to a real number l. If for given $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $|x_n - l| < \varepsilon \;\forall n \in \mathbb{N}$

Q) Show that $\lim_{n\to\infty} \frac{1}{n} = 0$

Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$

$$0 \leq \frac{1}{N} < \varepsilon \quad \exists\, N \in \mathbb{N}$$

Thus, if $n \geq N$, we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

Q) Show that $\lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$

Let $\varepsilon \in \mathbb{R}^+$

$$\left|1 - \frac{1}{2^n} - 1\right| = \frac{1}{2^n} = \frac{1}{(1+1)^n}$$

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} \cdots + \binom{n}{n} \geq \binom{n}{0} + \binom{n}{1}$$

$$\frac{1}{2^n} \leq \frac{1}{1+n} < \frac{1}{n}$$

$$0 < \frac{1}{N} < \varepsilon \quad \forall n \in \mathbb{N}$$

$$\frac{1}{2^n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

$$\left|\left(1 - \frac{1}{2^n}\right) - 1\right| < \varepsilon \quad \forall n \in \mathbb{N}$$

Theorm:

Let s_n and t_n be sequences in $\mathbb{R}$ and let $s \in \mathbb{R}$. If for some $k \in \mathbb{R}^+$ and $N_1 \in \mathbb{N}$, we have. $|s_n - s| \leq k|t_n| \ \forall n \geq N_1$ and $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} s_n = s$.

Let $\varepsilon > 0$ is given

Since $t_n \longrightarrow 0$ as $n \longrightarrow \infty$

$\exists N_2 \in \mathbb{N}$ such that $|t_n - 0| < \frac{\varepsilon}{K} \quad \forall n \geq N_2$

$$|S_n - S| \leq K|t_n| < K \cdot \frac{\varepsilon}{K} < \varepsilon \quad \forall n \in \mathbb{N}$$

$$N = \text{Max}(N_1 . N_2)$$

<u>Uniqueness</u> <u>Theorm</u>:

Let S_n be a converging sequence. If $\lim\limits_{n \to \infty} S_n = l_1$ and $\lim\limits_{n \to \infty} S_n = l_2$, then $l_1 = l_2$.

For any given $\varepsilon \in \mathbb{R}$ $\varepsilon > 0$

$$|S_n - S| \leq \varepsilon/2 \quad \forall n \geq N_1$$

$$|S_n - l_1| \leq \varepsilon/2 \quad \forall n \geq N_2$$

$$N = \text{Max}(N_1 . N_2)$$

$$|l_1 - l_2| = |l_1 - S_n + S_n - l_2|$$

$$\leq |S_n - l_1| + |S_n - l_2| \leq \varepsilon \quad \forall n \geq N$$

$\Rightarrow$ This is true $\forall \varepsilon > 0$ $\quad N = \text{Max}(N_1, N_2)$

$l_1 = l_2$

<u>Cauchy</u> <u>Sequence</u>:

A sequence is a cauchy sequence when

$$g_{(n,m)} = |x_n - x_m| < \varepsilon \quad \forall n, m \geq n_0 \quad \exists n_0 \in \mathbb{N}$$

$\lim = 0$ $\quad \frac{1}{n} \left|\frac{1}{m} - \frac{1}{n}\right| < \left|\frac{1}{m}\right| + \left|\frac{1}{n}\right|$ $\quad \frac{1}{m} < \varepsilon/2$

$< \varepsilon$ $\quad \frac{1}{n} < \varepsilon/2$

* Cauchy Sequence's limit must be zero. and Convergent.

$x_n = (-1)^n \Rightarrow$ Not Convergent

$\hookrightarrow |x_n - x_m| < 2 \Rightarrow$ Not Cauchy $[\epsilon < 2]$

All convergent are Cauchy but not all Cauchy are convergent.

Q) $x_n = \dfrac{[\alpha] + [2\alpha] + \cdots + [n\alpha]}{n^2}$. Discuss the convergence/Divergence

[ ] $\Rightarrow$ GIF $\Rightarrow$ Greatest integer function.

Hint: $x - 1 < [x] \leq x$

$$h(x) = \frac{\alpha + 2\alpha + \cdots + n\alpha}{n^2} = \frac{\frac{n(n+1)}{2}\alpha}{n^2} = \left(\frac{n^2+n}{2n^2}\right)\alpha = \lim_{n\to\infty} = \frac{\alpha}{2}$$

$$h(x) = \frac{\alpha}{2}$$

$$f(x) = \frac{\alpha + 2\alpha \cdots + n\alpha - (n)}{n^2} = \left(\frac{\frac{n^2+n}{2} - n}{n^2}\right)\alpha = \left(\frac{n^2-n}{2n^2}\right)\alpha$$

$$\lim_{n\to\infty} \frac{n^2-n}{2n^2}\alpha = \frac{\alpha}{2}$$

$$g(x) = \frac{[\alpha] + [2\alpha] + \cdots + [n\alpha]}{n^2}$$

$$f(x) \leq g(x) \leq h(x)$$

$\because \lim_{n\to\infty} f(x) = \lim_{n\to\infty} h(x) \Rightarrow = \lim_{n\to\infty} g(x)$

$\therefore g(x) = \frac{\alpha}{2}$

Q) Show that if a subsequence $\{x_{n_k}\}$ of a Cauchy sequence $\{x_n\}$ is convergent then $\{x_n\}$ is convergent. [Hint: Triangle Inequality]

1) $|x_{n_k} - L| < \epsilon/2$

2) $|x_n - x_m| < \epsilon$
$m = n_k$

3) $|x_n - x_{n_k}| < \epsilon/2$

4) $|x_n - L| \leq \epsilon$

Q) Show that $\{x_n\}$ $x_n = \int_1^n \frac{\cos(t)}{t^2}\,dt$ is Cauchy $|x_m - x_n| < \epsilon$.

$$x_m = \int_1^m \frac{\cos(t)}{t^2}\,dt \qquad x_n = \int_1^n \frac{\cos(t)}{t^2}\,dt$$

$$|x_m - x_n| = \left| \int_n^m \frac{\cos(t)}{t^2}\,dt \right|$$

$$|x_m - x_n| \le \frac{1}{n} \left| \int_n^m \frac{\cos t}{t^2}\,dt \right| \le \int_n^m \left| \frac{\cos t}{t^2}\,dt \right| \le \int_n^m \frac{1}{t^2}\,dt$$

$$\Rightarrow \left| \frac{-1}{t} \right|_n^m \Rightarrow \frac{-1}{m} - \left(\frac{-1}{n}\right) \Rightarrow \frac{1}{n} - \frac{1}{m} > 0 \le \frac{1}{n}$$

* $\lim_{n\to\infty} \sup(x_n) \Rightarrow$ Limit Superior

$\lim_{n\to\infty} \inf(x_n) \Rightarrow$ Limit Infirieor.

Bolzano-Weistrass Theorm:

Every bounded sequence has a convergent subsequence.

Q) $|x_{n+1} - x_n| \le Ac^n$. $A>0$ $c \in (0,1)$. $\{x_n\}$ is Cauchy.

$$|x_{n+k} - x_{n+k-1} + x_{n+k-1} - x_{n+k-2} \cdots| \le Ac^{(n+k-1)} + Ac^{(n+k-2)} \cdots$$

Q) $\{x_{2n}\}$, $\{x_{2n+1}\}$. $\{x_{3n}\}$ $\Rightarrow$ Convergent.

$$\lim_{n\to\infty} x_{2n} = \alpha_1 \quad \lim_{n\to\infty} x_{2n+1} = \alpha_2 \quad \lim_{n\to\infty} x_{3n} = \alpha_3$$

Q) $$x_n = \sum_{i=1}^{n} \frac{n^2}{\sqrt{n^6+i}} = \frac{n^2}{\sqrt{n^6+n}} + \cdots + \frac{n^2}{\sqrt{n^6+1}}$$

$$\frac{n^2}{\sqrt{n^6+n}} + \frac{n^2}{\sqrt{n^6+n}} \cdots \le \frac{n^2}{\sqrt{n^6+1}} + \cdots < \frac{n^2}{\sqrt{n^6+1}} + \frac{n^2}{\sqrt{n^6+1}} \cdots$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$1 \qquad 1 \qquad 1 \Rightarrow \text{Convergent}$$

9) If $|S_n - S| \leq K|t_n|$ $K>0$ $\forall n \geq N_1$ and $\lim_{n\to\infty} t_n = 0$. Then

$\lim_{n\to\infty} S_n = S$.

Theorm:

Every convergent sequence of real numbers is bounded.

Proof:

$\lim_{n\to\infty} S_n = S \Rightarrow \forall \varepsilon > 0$ $\exists N_1 \in \mathbb{N}$ such that $|S_n - S| < \varepsilon$ $\forall n \geq N$

Let $\varepsilon = 1$

$\therefore |S_n - S| < 1$ $\forall n \geq N_1$

Triangle Inequality:

$|S_n| = |S_n - S + S| \leq |S_n - S| + |S|$

$< 1 + |S|$ $\forall n \geq N_1$

$M = \text{Max}\{|S_1|, |S_2|, \text{———} |S_N|, 1 + |S|\}$

$|S_n| \leq M$ $\forall n \in \mathbb{N}$ $\Rightarrow$ $-M \leq S_n \leq M$

The sequence is bounded.

Squeeze Theorm of Limit:

Suppose (S_n) (t_n) and (U_n) are sequences such that $S_n \leq t_n \leq U_n$ $\forall n \in \mathbb{N}$. If $\lim_{n\to\infty} S_n = \lim_{n\to\infty} U_n = l$ then $\lim_{n\to\infty} t_n = l$

Proof:

Let $\varepsilon > 0$

$\exists N_1\ N_2 \in N$ such that

$|S_n - l| < \varepsilon$ $\forall n \geq N_1$

$|U_n - l| < \varepsilon$ $\forall n \geq N_2$

$N = \text{Max}\{N_1, N_2\}$

$l - \varepsilon < S_n < l + \varepsilon$ $\forall n \geq N$

$l - \varepsilon < U_n < l + \varepsilon$

$l - \varepsilon < s_n \le t_n \le u_n < l + \varepsilon \quad \forall n \ge N$

$|t_n - l| < \varepsilon \quad \forall n \ge N$

Theorm:

Let $S \subseteq \mathbb{R}$ which is bounded above. Then $\exists$ a sequence (s_n) in S. such that $\lim_{n\to\infty} s_n = \sup(S)$

Proof:

Let $c = \sup(S) \Rightarrow c - \frac{1}{n} < s_n \le c$

$\Downarrow$ By consequence of Squeeze Theorm

$\lim_{n\to\infty} \left(c - \frac{1}{n}\right) = c \qquad \lim_{n\to\infty} s_n = c$

Consider,

$\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$

(i) $\lim_{n\to\infty} (s_n + t_n) = s + t$

(ii) $\lim_{n\to\infty} s_n t_n = st$

(iii) $\lim_{n\to\infty} s_n / t_n = s/t$

(iv) $|s_n - s| < \varepsilon/2 \quad \forall n \ge N_1$

$|t_n - t| < \varepsilon/2 \quad \forall n \ge N_2 \qquad N = \text{Max}\{N_1, N_2\}$

$|s_n + t_n - s - t| \le |s_n - s| + |t_n - t|$

$< \varepsilon \quad \forall n \ge N$

(v) $|s_n - s| < \varepsilon/2 \quad \forall n \ge N_1$

$|t_n - t| < \varepsilon/2 \quad \forall n \ge N_2 \qquad N = \text{Max}\{N_1, N_2\}$

$|s_n t_n - st| = |s_n t_n - s t_n + s t_n - st|$

$\leq |s_n t_n - s t_n| + |s t_n - s t|$

$\leq |t_n||s_n - s| + |s||t_n - t|$

$|t_n| \leq M$

$K = \text{Max}(M, |s|)$

$\leq K(|s_n - s| + |t_n - t|) < \epsilon \quad \forall n \geq N$

(iv) $|s_n - s| < \epsilon/2 \quad \forall n \geq N_1$

$|t_n - t| < \epsilon/2 \quad \forall n \geq N_2 \quad N = \text{Max}\{N_1, N_2\}$

$$\left|\frac{s_n}{t_n} - \frac{s}{t}\right| = \left|\frac{s_n}{t_n} - \frac{s}{t_n} + \frac{s}{t_n} - \frac{s}{t}\right|$$

$$\leq \left|\frac{s_n}{t_n} - \frac{s}{t_n}\right| + \left|\frac{s}{t_n} - \frac{s}{t}\right|$$

$$\leq \left|\frac{1}{t_n}\right||s_n - s| + |s|\left|\frac{1}{t_n} - \frac{1}{t}\right|$$

$$\leq \frac{1}{|t_n|}|s_n - s| + \frac{|s|}{|t_n||t|}|t - t_n|$$

Monotone Sequence:

Def: Let (s_n) be a sequence in $\mathbb{R}$

(i) (s_n) is monotonically increasing if $s_n \leq s_{n+1} \quad \forall n \in \mathbb{N}$

(ii) (s_{n+1}) is montonically decreasing if $s_n \geq s_{n+1} \quad \forall n \in \mathbb{N}$

(iii) s_n is strictly increasing if $s_n < s_{n+1} \quad \forall n \in \mathbb{N}$

(iv) s_n is strictly decreasing if $s_{n+1} < s_n \quad \forall n \in \mathbb{N}$

Theorm:

Let s_n is a bounded sequence

(i) If s_n is monotonically increasing then $\lim_{n \to \infty} s_n = \text{Sup}(s_n)$

iii) If S_n is monotonically decreasing, $\lim_{n\to\infty} S_n = \inf(S)$

Proof:

Let $S_1 = \sup(S_n)$ take $\varepsilon > 0$

$\exists\, S_{n_0}$ such that $S_1 - \varepsilon < S_{n_0}$

Since S_n is increasing.

$$S_1 - \varepsilon < S_{n_0} \leq S_n \leq S_1 < S_1 + \varepsilon$$

$$S_1 - \varepsilon < S_n < S_1 + \varepsilon \quad \forall n \geq n_0$$

$$|S_n - S_1| < \varepsilon \quad \forall n \geq n_0$$

Let $S_2 = \inf(S_n)$ take $\varepsilon > 0$

$\exists\, S_{n_0}$ such that $S_2 + \varepsilon > S_{n_0}$

Since S_n is decreasing

$$S_2 + \varepsilon > S_{n_0} \geq S_{n} \geq S_2 - \varepsilon$$

$$S_2 + \varepsilon > S_{n_0} \geq S_n \geq S_2 > S_2 - \varepsilon$$

$$S_2 + \varepsilon > S_n > S_2 - \varepsilon \quad \forall n \geq n_0$$

$$|S_n - S_2| \leq \varepsilon \quad \forall n \geq n_0$$

Theorm: A bounded monotone sequence is convergent.

Subsequences:

Def: Let (S_n) be a sequence in $\mathbb{R}$ and let $n_k \in \mathbb{N}$ such that $n_1 < n_2$. Then the sequence (S_{n_k}) is called a subsequence of S_n.

Theorm:

Let S_n be a sequence which converges to S. Then any Subsequence of S_n also converges to S.

Proof: Let $\varepsilon > 0$

$$|S_n - S| < \varepsilon \quad \forall\, n \geq N$$

Let us consider (S_{n_K}) a subsequence of (S_n)

$$|S_{n_K} - S| < \varepsilon \quad \forall\, n_K \geq N$$

Bolzano-Weistrass Theorm:

Every bounded infinite sequence in $\mathbb{R}$ has a convergent subsequence

Cauchy Sequence:

Def: A sequence S_n is cauchy if given any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $|S_n - S_m| < \varepsilon$. $\forall\, (n, m) \geq N$.

$$S_n = \frac{n+1}{n}$$

$$|S_n - S_m| = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\frac{m-n}{mn}\right| < \frac{m+n}{mn}$$

Thefore, if $m \geq n$ $\quad |S_n - S_m| < \frac{2}{n} \leq \frac{2}{N} < \varepsilon \quad \forall\, n \geq N$

$\varepsilon > 0 \longrightarrow \exists\, N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$

Theorm:

Every ~~Cauchy~~ Convergent sequence is a Cauchy sequence.

Proof: ① $|S_n - S| < \epsilon/2 \quad \forall\, n \geq N$

$\forall\, (n,m) \geq N$, we have $|S_n - S_m| = |S_n - S + S - S_m|$

$|S_n - S| + |S - S_m| < \epsilon$

② $\lim_{n\to\infty} S_n = S \Rightarrow |S_n - S| < \epsilon/2 \quad \forall\, n \geq N$

For all $(n,m) \geq N \Rightarrow |S_n - S_m| = |S_n - S + S - S_m|$

$\leq |S_n - S| + |S - S_m|$

$< \epsilon$

Theorm:

Every cauchy sequence is bounded.

Prof: Let $\epsilon = 1$ Then $\exists\, N \in \mathbb{N}$ such that

$|S_m - S_n| < 1. \quad \forall (n,m) \geq N$

Consider $K \geq N$

$|S_n| = |S_m - S_K + S_K| \leq |S_m - S_K| + |S_K|$

$< 1 + |S_K|$

Consider $M = \text{Max}\{|S_1|, |S_2|, |S_3| \ldots \ldots, |S_N|\}, |S_K| + 1\}$

$|S_n| \leq M$

$\therefore$ The sequence is bounded.

Theorm:

Every ~~convergent~~ Cauchy sequence is a convergent sequence.

Proof:

Let (S_n) be a cauchy sequence. That means (S_n) is bounded.

Now by Bolzano - Weistrass theorm, it must contain at least one subsequence (S_{n_k}) which converges to $l \in \mathbb{R}$

$$|S_n - S_m| < \varepsilon/2 \qquad \forall (n,m) \geq N_1$$

$$|S_{n_k} - l| < \varepsilon/2 \qquad \forall n_k \geq N_2$$

Consider $M = Max\{N_1 . N_2\}$

$$|S_n - l| = |S_n - S_{n_K} + S_{n_K} - l|$$

$$\leq |S_n - S_{n_K}| + |S_{n_K} - l|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$< \varepsilon$$

$\therefore$ The sequence is Convergent.

Bolzano - Weistrass Theorm:

Every bounded sequence have at least one convergent subsequen

Every bounded infinite subset of real numbers has at least one limit point.

$S \subset \mathbb{R}$ is bounded

$a_1 = \inf S$ and $b_1 = \sup S$

Then $x \in S \Rightarrow a \leq x \leq b$

$$S \subseteq [a_1, b_1] = I_1$$

Let $c_1 = \dfrac{a_1 + b_1}{2}$. Then at least one of $[a_1 . c_1]$ and $[c_1 . b_1]$ contains infinite numbers.

We take such sub interval $[a_2\ b_2]$ containing infinite elements

$I_2 = [a_2 . b_2]$

$I_2 \subset I_1$ $\quad |I_2| \leq \frac{b_1 - a_1}{2}$

$c_2 = \frac{a_2 + b_2}{2} \longrightarrow [a_2 , c_2] , [c_2 , b_2]$

$I_3 = [a_3 . b_3]$

$I_3 \subset I_2 \subset I_1$

$c_3 = \frac{a_3 + b_3}{2}$

$I_n = [a_n . b_n]$

$I_n \subset I_{n-1} - - - - \subset I_1$

Theorm on nested interval, $\exists x \in \mathbb{R}$ [precise] such that $\{x\}$

$\{x\} = \bigcap_{n=1}^{\infty} [a_n . b]$

We have to show "x" is the limit point.

Let $\varepsilon > 0$

$\inf \{ b_n - a_n , n \in \mathbb{N} \} = 0$

since $\{x\} = \bigcap_{n=1}^{\infty} [a_n . b_n]$

$0 < b_n - a_n < \varepsilon \qquad m \geq N$

$x \in I_m$ and $|b_m - a_m| < \varepsilon$

$I_m \subset N(x , \varepsilon)$

Since I_m contains infinitely many elements of $S(x,\epsilon)$ contains infinitely many elements of S for each $\epsilon > 0$

So x is a limit point of S.

Limit superior and Limit inferior:

$$\lim_{n\to\infty} \sup(S_n) \Rightarrow \inf(\{\sup(x_n) ; n \geq N\})$$

$$\lim_{n\to\infty} \inf(S_n) \Rightarrow \sup(\{\inf(x_n) : n \geq N\})$$

Q) (i) Let $\{x_n\}$ be a sequence such that $\{x_n\}$, $\{x_{2n+1}\}$, $\{x_{3n}\}$ are convergent. Prove $\{x_n\}$ is convergent.

(ii) Let sequence $\{x_n\}$ be $x_n = 1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n} - \ln(n)$. Is $\{x_n\}$ convergent or divergent.

(iii) $\dfrac{\alpha^n - \beta^n}{\alpha^n + \beta^n}$ $\forall\ \alpha, \beta \in \mathbb{R}$ and $|\alpha| \neq |\beta|$. Is it convergence or divergence

(iv) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequence. Prove that $\lim \{\sup\}(x_n - y_n) < \lim\sup(x_n) + \lim\sup(y_n)$. Does this equality hold?

Solutions:

(i) Take a subsequence $\{x_{6n}\}$ which is a subsequence of $\{x_{2n}\}$ and $\{x_{3n}\}$

$$|x_{2n+1} - l| < \epsilon \qquad n > N_1$$

Case ①: $|x_{2n} - l| < \epsilon$ $\quad n > N_0$

$n = 2k$

$$|x_n - l'| = |x_{2n} - l| < \epsilon$$

Case ②: $n = 2k+1$

(iii) Let $m \geq n$

$$|x_m - x_n| = \left|\sum_{k=1}^{m} \frac{1}{k} - \log m - \sum_{k=1}^{n} \frac{1}{k} + \log n\right|$$

$$= \left|\sum_{k=n+1}^{m} \frac{1}{k} - \log \frac{m}{n}\right|$$

$$|x_m - x_n| < \sum_{k=n+1}^{m} \frac{1}{k} < \frac{1}{n} \quad \forall (n,m) \geq N$$

x_n is Cauchy

$$\ln(x) = \int_1^t \frac{1}{t}\, dt \quad \Rightarrow \quad \frac{b-a}{b} \leq \int_1^t \frac{1}{t}\, dt \leq \frac{b-a}{a}$$

$0 < a < b$

$$\ln(n) = \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{t}\, dt = \frac{k+1-k}{k}$$

$$\ln(n) \leq 1 + \frac{1}{2} + \text{———} + \frac{1}{n}$$

$$x_n = \gamma - \ln(n)$$

x.: $x_n > 0$

$$x_{n+1} - x_n = \frac{1}{n+1} = \ln(n+1) + \ln(n)$$

$$= \frac{1}{n+1} - \int_n^{n+1} \frac{1}{t}\, dt < 0$$

x_n is a decreasing sequence bounded by 0.

(iv) $\lim\sup(x_n + y_n) < \lim\sup(x_n) + \lim\sup(y_n)$

$$x_n + y_n \leq \sup(x_n : n \geq N) + \sup(y_n : n \geq N)$$

$$\sup(x_n + y_n : n \geq N) \leq$$

$$\inf(\sup x_n + \sup y_n) \leq \inf(\sup x_n) + \inf(\sup y_n)$$

Q) $\lim\limits_{n\to\infty} \inf\left(\frac{x_{n+1}}{x_n}\right) \leq \lim\limits_{n\to\infty} \inf \sqrt[n]{x} \leq \lim\limits_{n\to\infty} \sup \sqrt[n]{x} \leq \lim\limits_{n\to\infty} \sup\left(\frac{x_{n+1}}{x_n}\right)$

$$\lim_{n\to\infty} \inf\left(\frac{x_{n+1}}{x_n}\right) = l$$

$- \varepsilon < \cancel{lotf}$

$$-\varepsilon \leq \inf\left(\frac{x_{n+1}}{x_n} : n \geq N\right) - l \leq \varepsilon$$

$$l - \varepsilon \leq \inf\left(\frac{x_{n+1}}{x_n} : n \geq N\right) \leq \frac{x_{n+1}}{x_n}$$

$$x_{n+1} \geq x_n(l-\varepsilon)$$

$$x_m \geq x_n(l-\varepsilon)$$

$$x_{n+2} \geq x_{n+1}(l-\varepsilon) \geq x_n(l-\varepsilon)^2$$

$$(x_m)^{1/n} \geq \left((x_n)^{1/n}\right)(l-\varepsilon)^{\frac{m-n}{n}}$$

Series

Example: ① $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$

$= \frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \cdots + \frac{(n+1)-n}{n\cdot n+1}$

$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}$

$= 1 - \frac{1}{n+1}$

$= \frac{n}{n+1}$

$\lim_{n\to\infty} S_n = 1$

② $S_n = 1 + 2 + 3 + \cdots + n$

$\lim_{n\to\infty} S_n = \infty$

③ $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$

$= \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}}$

$= 2$

④ $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$S_1 = 1 \qquad S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 2$

$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$

$\boxed{S_{2^n} = 1 + n\cdot\frac{1}{2}}$

$\lim_{n\to\infty} S_n = \infty$

⑤ $S_n = u_1 + u_2 + \cdots + u_n$: Convergent

$t_n = u_{m+1} + \cdots + u_{m+n}$: Convergent

$t_n = S_{m+n} - S_n$

$\sum u_n = u \quad \sum v_n = v \quad \therefore \sum u_n + \sum v_n = u+v$

<u>Cauchy's principle of Convergence:</u>

A necessary and sufficient condition for convergence of a series $\sum u_n$ is that corresponding to some $\varepsilon > 0$.

$\exists m \in \mathbb{N}$ such that

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \varepsilon \quad \forall n \geq m \text{ and } \forall p \in \mathbb{N}$$

<u>Proof:</u>

Let $S_n = u_1 + u_2 + \cdots + u_n$

Let $\sum u_n$ be convergent, then the sequence $\{S_n\}$ has to be convergent.

Therefore, $|S_{n+p} - S_n| < \varepsilon \quad \forall n \geq m$

$|u_{n+1} + u_{n+2} \cdots + u_{n+p}| < \varepsilon \quad \forall n \geq m$

$|u_{n+1}| < \varepsilon \quad \forall n \geq m$ take $p = 1$

$|u_n - 0| < \varepsilon \quad \forall n' \geq m'$

$\lim_{n \to \infty} u_n = 0$

<u>Examples:</u>

① $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots$ is Convergent

$u_n = \frac{(-1)^{n+1}}{n}$

$|S_{n+p} - S_n|$

$\Downarrow$

$$\left|\frac{1}{n+1} - \left(\frac{1}{n+2} - \frac{1}{n+3}\right)\right|$$

$$\left|\frac{1}{n+1} - \cdots \frac{(-1)^{p+n+1}}{n+p}\right| < \frac{1}{n+1}$$

* If we have a series of positive terms then it is convergent if and only if the sequence of partial sum is bounded above.

$$|S_{n+p} - S_n| < \epsilon \qquad S_{n+1} - S_n = u_{n+1} > 0$$

$\{S_n\} \Rightarrow$ Monotonically increasing sequence

$\therefore$ It has to be bounded above.

Grouping of Terms:

$$\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots - -$$

Theorm:

Let $\sum u_n$ be a series of positive terms, $\sum v_n$ is obtained from $\sum u_n$ by grouping.

(i) $\sum u_n \rightarrow S$ then $\sum v_n \rightarrow S$

Proof:

$$v_1 = u_1 + u_2 + \ldots + u_{r_1}$$

$$v_2 = u_{r_1+1} + u_{r_1+2} + \ldots + u_{r_2}$$

$$S_n = u_1 + u_2 + \ldots + u_n$$

$$t_n = v_1 + v_2 + \ldots + v_n$$

$$t_n = u_1 + u_2 + \ldots + u_{r_n} = S_{r_n}$$

Let $\sum u_n$ be convergent and the sum be S

$\{S_n\} \rightarrow S$, $\{t_n\} \rightarrow \{S_{r_n}\}$: A subsequence of $\{S_n\}$, which converges to S.

Series convergence comparision Test (Type 1):

Let Σu_n and Σv_n be two series of real numbers and $\exists m \in \mathbb{N}$ such that $u_m \le k v_n$ $\forall n \ge m$, with k being a fixed positive number.

Then (i) Σu_n is convergent if Σv_n is convergent

(ii) Σv_n is divergent if Σu_n is divergent

Proof:

(i) Let $S_n = u_1 + u_2 + u_3 + \dots + u_n$

$t_n = v_1 + v_2 + v_3 + \dots + v_n$

$$S_n - S_m = u_{m+1} + u_{m+2} + \dots + u_n \quad (n \ge m)$$

$$\le k(v_{m+1} + \dots + v_n)$$

$$\le k(t_n - t_m)$$

$$|S_n - S_m| \le k|t_n - t_m| < \epsilon \quad \forall (n,m) \ge N$$

(ii) Let us say Σu_n is divergent

Then $\{S_n\}$ is not bounded above

$$S_n - S_m \le k(t_n - t_m)$$

$$S_n \le kt_n + h \quad [h = S_m - kt_m \text{ is a finite number}]$$

If S_n is not bounded, then t_n is also not bounded

Limit for m:

Let Σu_n and Σv_n are series of +ve real numbers and $\lim_{n \to \infty} \frac{v_n}{u_n} = l$

l: Non-zero finite number. Then Σu_n and Σv_n converges and diverges together.

Proof: $l > 0$

let us choose $\epsilon > 0$ such that $l - \epsilon > 0$

$\exists m \in \mathbb{N}$ such that $l - \epsilon < \frac{v_n}{u_n} < l + \epsilon$ $\forall n \ge m$

Therefore $v_m < k u_m$ where $k = l + \epsilon$

if V_n is convergent, then U_n is also convergent $\longrightarrow$ ①

if U_n is divergent, then V_n is also divergent $\longrightarrow$ ②

$$U_n < K' V_n \quad \text{where} \quad K' = \frac{1}{1-\epsilon}$$

- If U_n is convergent then V_n is convergent — ③
- If V_n is divergent then U_n is divergent — ④

From ① and ④ we can say that U_n and V_n converge together

From ③ and ④ we can say that U_n and V_n diverge together

Theorm:

The series $\frac{1}{1^p} + \frac{1}{2^p} + \cdots$ converges for $p > 1$ and diverges for $p \leq 1$

Proof: Case - 1 : $p > 1$

Let ΣU_n be a given series with $U_n = \frac{1}{n^p}$

Let ΣV_n be obtained from ΣU_n by grouping

$$\underbrace{1}_{V_1} + \underbrace{\left(\frac{1}{2^p} + \frac{1}{3^p}\right)}_{V_2} + \underbrace{\left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)}_{V_3} + \cdots$$

$$V_2 = \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} \Rightarrow V_2 < \frac{1}{2^{p-1}}$$

$$V_3 = \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{1}{2^{2(p-1)}} \Rightarrow V_3 < \frac{1}{2^{2(p-1)}}$$

$$W_n = \left(\frac{1}{2^{p-1}}\right)^{n-1}$$

$$V_n < W_n$$

$$V_n < W_n$$

We know that W_n is convergent for $p > 1$. From comparision test, V_n is convergent if W_n is convergent.

Here W_n is convergent, so U_n is convergent

Since V_n is convergent, U_n is convergent.

Case-2: $0 < p < 1$

$$\frac{1}{2^p} > \frac{1}{2} \qquad \frac{1}{3^p} > \frac{1}{3} \qquad \frac{1}{4^p} > \frac{1}{4} \ \ldots$$

$$V_n = \frac{1}{n} \qquad W_n = \frac{1}{n^p}$$

$$V_n < W_n \quad \forall\, n \geq 2$$

WKT, V_n is divergent. From comparision test we can conclude that W_n is divergent

Since W_n is divergent, U_n is divergent

<u>Comparision</u> <u>Test</u> (Type-II):

ΣU_n and ΣV_n are two series of positive real numbers such that

$$\frac{U_{n+1}}{U_n} \leq \frac{V_{n+1}}{V_n} \quad \forall\, n, m \in \mathbb{N}$$

(i) ΣU_n is convergent, if ΣV_n is convergent

(ii) ΣV_n is divergent, if ΣU_n is divergent

<u>D'Alembert's</u> <u>Ratio</u> <u>Test</u>:

Let ΣU_n be a series of +ve real numbers and let

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$$

if $l < 1 \longrightarrow \Sigma U_n$ is ~~divergent~~ Convergent

$l > 1 \longrightarrow \Sigma U_n$ is Divergent

$l = 1 \longrightarrow$ Test fails

Cauchy's Root Test:

ΣU_n is a series of +ve real numbers and let $\lim_{n\to\infty} U_n^{1/n} = l$

(i) $l < 1 \longrightarrow \Sigma U_n$ converges

(ii) $l > 1 \longrightarrow \Sigma U_n$ diverges

D'Alembert's ratio test:

Let ΣU_n be of positive real numbers and let $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$. Then ΣU_n is convergent if $l < 1$ and divergent if $l > 1$.

Proof:

Case-I: $l < 1$

Let us choose $\varepsilon > 0$ such that

$$l + \varepsilon < 1$$

Since $\lim \frac{U_{n+1}}{U_n} = l$ $\exists\, m \in \mathbb{N}$ such that

$$l - \varepsilon < \frac{U_{n+1}}{U_n} < l + \varepsilon \quad \forall n \geq m$$

$$\left| \frac{U_{n+1}}{U_n} - l \right| < \varepsilon \quad \forall n \geq m$$

Let $l + \varepsilon = r$ Then $0 < r < 1$

We have $\frac{U_{m+1}}{U_m} < r$ $\frac{U_{m+2}}{U_{m+1}} < r$ $\frac{U_n}{U_{n-1}} < r$ $n > m$

$$\frac{U_n}{U_m} < r^{n-m} \quad \forall n > m$$

$$U_n < \left[\frac{U_m}{r^m}\right] r^n \qquad \frac{U_m}{r^m} : \text{+ve Real number}$$

$$\Sigma U_n < \frac{U_m}{U_m} \Sigma r^n \qquad \Sigma r^n : \text{Converging geometric Sequence}$$

ΣU_n is converging (Comparision Test)

Case - II : $l > 1$

$\varepsilon > 0$ such that $l - \varepsilon > 1$

$\lim \frac{u_{n+1}}{u_n} = l$ we have $\underset{= p}{l - \varepsilon} < \frac{u_{n+1}}{u_n} < l + \varepsilon \quad \forall n \geq k$

$\frac{u_{k+1}}{u_k} > p \qquad \frac{u_{k+2}}{u_{k+1}} > p \ldots \frac{u_n}{u_{n-1}} > p$

$$\frac{u_n}{u_k} > r^{n-k}$$

$$u_n > \left(\frac{u_k}{r^k}\right) r^n \quad \forall n > k$$

$\sum r^n \longrightarrow$ diverges , $\sum u_n \longrightarrow$ diverges

Cauchy's root test :

Let $\sum u_n$ be a series of positive real numbers and let

$\lim u_n^{1/n} = l \Rightarrow$ Converging if $l < 1$ Diverging if $l > 1$.

Proof :

Case- I : $l < 1$ $\quad \varepsilon > 0$ such that $\left[l + \varepsilon < 1 \right]$

$$l - \varepsilon < u_n^{1/n} < l + \varepsilon \quad \forall n \geq m$$

$(l - \varepsilon)^n < u_n < (l + \varepsilon)^n \quad l + \varepsilon = r$

$$u_n < r^n \quad \forall n \geq m$$

$\sum r^n \longrightarrow$ Converging geometric sequence $\Rightarrow \sum u_n$ is converging

Q) Given $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} - - - \quad u_n = \frac{2n-1}{n!}$

$u_{n+1} = \frac{2n+2-1}{(n+1)!} = \frac{2n+1}{(n+1)!}$

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{(n+1)(2n-1)}$

9) $x + \frac{x^2}{2!} + \frac{x^3}{3} + \cdots$ $\quad u_n = \frac{x^n}{n}$

$$u_{n+1} = \frac{x^{n+1}}{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} = \frac{nx}{(n+1)}$$

Raabe's Test:

Let $\sum u_n$ be an series of +ve real number and

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = l$$

Then convergent if $l > 1$

divergent if $l < 1$

Proof:

Case - I : $l > 1$

$$\varepsilon > 0 \longrightarrow l - \varepsilon > 1$$

$$l - \varepsilon < n\left[\frac{u_n}{u_{n+1}} - 1\right] < l + \varepsilon \quad \forall n \geq m$$

$$l - \varepsilon = r > 1$$

$$nu_n - nu_{n+1} > ru_{n+1}$$

$$nu_n - (n+1)u_{n+1} > (r-1)u_{n+1}$$

$$mu_m - (m+1)u_{m+1} > (r-1)u_{m+1}$$

$$(m+1)u_{m+1} - (m+2)u_{m+2} > (r-1)u_{m+2}$$

$$\vdots$$

$$(n-1)u_{n-1} - nu_n > (r-1)u_n \quad n > m$$

$$mU_m - nU_n > (r-1)\left[U_{m+1} + \dots + U_n\right]$$

$$U_{m+1} + U_{m+2} + \dots + U_n < \frac{1}{r-1}(mU_m - nU_n)$$

$$< \frac{1}{r-1}(mU_m)$$

$$S_n - S_m < \frac{1}{r-1}\, mU_m$$

$$S_n < \frac{1}{r-1}\, mU_m + S_m \quad \forall\, n > m$$

Partial sum is bounded above.

__Limit point:__

A limit point of a set [D] is a point $x \in D$ such that every neighbourhood of x contains atleast one point belonging to D which is not x.

Limits

Definitions:

Let $D \subset \mathbb{R}$ and $f: D \longrightarrow \mathbb{R}$ be a function. Let c be a limit point. A real number 'l' is said to be a limit of f at c. If corresponding to a preassigned $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$l - \varepsilon < f(x) < l + \varepsilon \quad \forall x \in N'(c, \delta) \cap D$$

Where $N'(c, \delta) = \{x\} \in \mathbb{R} \quad 0 < |x - c| < \delta\} = (c-\delta, c+\delta) - \{c\}$

$$\lim_{x \to c} |f(x) - l| < \varepsilon \qquad |x - c| < \delta$$

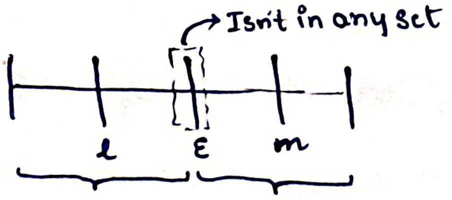
$$\lim_{x \to c} f(x) = l$$

Theorm:

Let $D \subset \mathbb{R}$ and $f: D \longrightarrow \mathbb{R}$ Let $c \in D'$. Then f can have at most one limit at c.

Proof:

Assuming $\exists$ two limits l, m $(l \neq m)$ at c for f. We assume $m > l$. $\varepsilon = \frac{m - l}{2} > 0$. Then the neighbourhoods $(l - \varepsilon, l + \varepsilon)$ and $(m - \varepsilon, m + \varepsilon)$ are disjoint.



$$(l+\varepsilon, l-\varepsilon) \cap (m - \varepsilon, m + \varepsilon) = \phi$$

Since l is a limit, of f such that $\exists \delta_1 > 0$ such that

$$l - \varepsilon < f(x) < l + \varepsilon \quad \forall x \in N(c, \delta_1) \cap D$$

$\exists \delta_2 > 0$ such that

$$m - \varepsilon < f(x) < m + \varepsilon \quad \forall x \in N(c, \delta_2) \cap D$$

take $\delta = \min(\delta_1, \delta_2)$

$l-\varepsilon < f(x) < l+\varepsilon$ and $m-\varepsilon < f(x) < m+\varepsilon$ $\forall x \in N'(c,\delta) \cap D$

$\Rightarrow$ This is a contradiction

Q) Show that $\lim_{x \to 2} f(x) = 4$ where $f(x) = \frac{x^2-4}{x-2}$, $x \neq 2$. Domain of f is $\mathbb{R}-\{2\}$. 2 is a limit point of D

where $x \in D$

$$|f(x)-4| = \left|\frac{x^2-4}{x-2} - 4\right| = |x-2|$$

Let $\varepsilon > 0$, whenever $|x-2| < \varepsilon$ and $x \in D$

satisfying

$$0 < |x-2| < \varepsilon$$

Choose $\delta = \varepsilon$

$$|f(x) - l| < \varepsilon \quad \forall \; |x-c| < \delta$$

$$\left.\begin{array}{l} |f(x) - a| < \varepsilon \\ |x - c| < \delta \end{array}\right\} \qquad \begin{array}{l} |f(x) - 4| < \varepsilon \\ |x-2| < \delta \end{array}$$

Sequential Criteria of Limits:

Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$. Let $c \in D$ and $l \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = l$. ~~iff for every sequence $\{f(x_n)\}$ converges to c~~

If for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c. the sequence $\{f(x_n)\}$ converges to l.

Proof:

$\lim_{x \to c} f(x) = l$

For an $\varepsilon > 0$ we have $\delta > 0$ such that

$l-\varepsilon < f(x) < l+\varepsilon \quad \forall x \in N'(c,\delta) \cap D$

Let $(x_n) \subseteq D - \{c\}$ converging to c

i.e $c - \delta < x_n < c + \delta \quad \forall n \geq k \in \mathbb{N}$

$l - \epsilon < f(x_n) < l + \epsilon \quad \forall n \geq k$

conversely.

Let for every sequence $\{x_n\}$ in $D - \{c\}$ converging to

we have $\lim_{x \to c} f(x) = l$.

$\exists$ a neighbourhood V of l such that for every neighbourhood W of c. $\exists x_\omega \in [W - \{c\} \cap D]$ for which $f(x_\omega) \notin V$

$W_1 = N(c.1)$

Then $\exists x_\omega \in N'(c.1) \cap D$

$f(x_1) \notin V$

Let $W_2 = N(c.1/2) \quad \exists x_2 \in N'(c.1/2) \cap D$

$f(x_2) \notin V$

$W_n = N(c.\frac{1}{n}) \quad \exists x_n \in N'(c, 1/n) \cap D$

$f(x_n) \notin V$

We therefore obtain a sequence $\{x_1\ x_2\ x_3 \ldots x_n\}$ in D such that $\lim x_n = c$ but $f(x_n)$ does not converge to l

But $|f(x_n) - l| < \epsilon \quad \exists \delta$ such that

$|x - c| < \delta$

Sequential Criteria:

$\lim_{x \to c} f(x) = l$. Then for $\epsilon > 0 \quad \exists \delta > 0$ such that $l - \epsilon < f(x) < l + \epsilon$

$\forall x \in N'(c, \delta) \cap D$

$\{x_n\} \in D - \{c\}$. Converging to c

$c - \delta < x_n < c + \delta \quad \forall\, n \geq k \in \mathbb{N}$

$l - \varepsilon < f(x) < l + \varepsilon \quad \forall\, n \geq k \longrightarrow \{f(x_n)\} \longrightarrow$ Converges to l

Conversely:

$\{x_n\} \longrightarrow c$ and $\{f(x_n)\} \longrightarrow l$

Such that $\lim_{x \to c} f(x) = l$

If not $\exists\, V$ or l such that for every neighbourhood W of c

$\exists\, x_{w} \in (W - \{c\}) \cap D$ for which $f(x_{w}) \notin V$.

$W_1 = N(c, 1) \longrightarrow x_1 \in N'(c, 1) \cap D$ such that $f(x_1) \notin V$

$W_2 = N(c, 1/2) \longrightarrow x_2 \in N'(c, 1/2) \cap D$ such that $f(x_2) \notin V$

$W_n = N(c, 1/n) \longrightarrow x_n \in N'(c, 1/n) \cap D$ such that $f(x_n) \notin V$

$\therefore\ l - \varepsilon < f(x_n) < l + \varepsilon$ is not true. $\Rightarrow$ CONTRADICTION

Theorm:

Let $D \subset \mathbb{R}$ and $f : D \longrightarrow \mathbb{R}$ let $c \in D'$, if f has a limit $l \in \mathbb{R}$ at c, then f is bounded on $N(c) \cap D$

Let $\lim_{x \to c} f(x) = l$

Let $\varepsilon = 1$

$|f(x) - l| < 1 \quad \forall\, x \in N'(c, \delta) \cap D$

But $|f(x) - l| \geq \big||f(x)| - |l|\big|$

[Reverse Triangular Inequality]

$\big|f(x) - |l|\big| \leq |f(x) - l| < 1$

$|f(x)| < 1 + |l|$

Therefore, if $c \notin D$, $|f(x)| < 1 + |l| \quad \forall\, x \in N(c, \delta) \cap D$

$\Rightarrow f(x)$ is bounded on $N(c, \delta) \cap D$

If $c \in D$

~~Let B = max{f(c)}~~

Let $B = \max\{|f(c)|+1, |l|+1\}$

Then $|f(x)| < B \quad \forall x \in N(c,\delta) \cap D$

Theorm:

Consider a $D \subset \mathbb{R}$, $f : D \longrightarrow \mathbb{R}$ and $c \in D$. $\lim_{x \to c} f(x) = l$

(i) If $l > 0$, $\exists \delta > 0$ such that $f(x) > 0 \quad \forall x \in N'(c,\delta) \cap D$

(ii) If $l < 0$

a) $\varepsilon > 0$ such that $l - \varepsilon > 0$

Since $\lim_{x \to c} f(x) = l$ $\quad \exists \delta > 0$ such that

$l - \varepsilon < f(x) < l + \varepsilon \quad \forall x \in N'(c,\delta) \cap D$

$f(x) > 0$

Theorm:

Let $D \subset \mathbb{R}$. $f : D \longrightarrow \mathbb{R}$ and $c \in D'$ such that $\lim_{x \to c} f(x) = l$. $\lim_{x \to c} g(x) = m$

(i) $\lim_{x \to c} (f+g)(m) = l + m$

(ii) $\lim_{x \to c} (f \cdot g)(n) = l \cdot m$

(i) $|f(x) - l| < \varepsilon \quad \forall x \in N'(c,\delta_1) \cap D$

$|g(x) - m| < \varepsilon \quad \forall x \in N'(c,\delta_2) \cap D$

Consider $\delta = \min(\delta_1, \delta_2)$

$|f(x) + g(x) - l - m| \leq |f(x) - l| + |g(x) - m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\forall x \in N'(c,\delta) \cap D$

(ii) $|f(x).g(x) - lm| \Rightarrow$

$$|f(x)g(x) - lg(x) + lg(x) - lm| \leq |g(x)||f(x) - l| + |l||g(x) - m|$$

(iii) $\lim_{x \to c} (f/g)(x) = l/m$

Theorm:

Let $D \subset \mathbb{R}$. $f: D \longrightarrow \mathbb{R}$ and $c \in D'$. If $f(x) \geq a \ \forall x \in D - \{c\}$ and $\lim_{x \to c} f(x) = l$. Then $l \geq a$.

Proof:

Let $\{x_n\} \in D - \{c\}$ converging to c

$$\lim_{x \to \infty} f(x_n) = l$$

$\{V_n\}$ by $V_n = a \ \forall n \in \mathbb{N}$

$$f(x_n) \geq V(x_n)$$

Theorm[Sandwhich Theorm]:

Let $D \subset \mathbb{R}$ $f: D \longrightarrow \mathbb{R}$ and $c \in D'$. $f(x) \leq g(x) \leq h(x) \ \forall x \in D - \{c\}$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$. Then $\lim_{x \to c} g(x) = l$

$$l - \varepsilon < f(x) < l + \varepsilon \quad \forall x \in N'(c, \delta_1) \cap D$$

$$l - \varepsilon < h(x) < l + \varepsilon \quad \forall x \in N'(c, \delta_2) \cap D$$

$$l - \varepsilon < f(x) \leq g(x) \leq h(x) < l + \varepsilon$$

$$l - \varepsilon < g(x) < l + \varepsilon$$

$$-\varepsilon < g(x) - l < \varepsilon \Rightarrow |g(x) - l| < \varepsilon$$

$\therefore \lim_{x \to c} g(x) = l \quad \forall x \in N'(c, \delta_1) \cap D \qquad \delta = \text{minimum}(\delta_1, \delta_2)$

$\forall x \in N'(c, \delta_2) \cap D \qquad \forall x \in N'(c, \delta) \cap D$

Cauchy's principle:

Let $D \subset \mathbb{R}$, $f: D \longrightarrow \mathbb{R}$ and $c \in D'$. A necessary and sufficient condition for the existence of $\lim_{x \to c} f(x)$ is that for a pre-assigned $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(x') - f(x'')| < \varepsilon \quad \forall x', x'' \in N'(c, \delta) \cap D$$

Let $\lim_{x \to c} f(x') = l$

$$|f(x') - l| < \varepsilon/2 \quad \forall N'(c, \delta_1) \cap D \ni x'$$

$$\lim_{x \to c} f(x'') = l$$

$$|f(x'') - l| < \varepsilon/2 \quad \forall N'(c, \delta_2) \cap D \ni x''$$

Take $\delta = \mathrm{Min}(\delta_1, \delta_2)$

$$|f(x') - f(x'')| = |f(x') - f(x'') - l + l|$$

$$\leq |f(x') - l| + |f(x'') - l| < \varepsilon \qquad \forall (x', x'') \in N'(c, \delta) \cap D$$

$\Rightarrow$ Necessary condition

Let us assume for an $\varepsilon > 0$, $\exists\, \delta > 0$, such that $|f(x') - f(x'')| < \varepsilon$ $\forall (x', x'') \in N'(c, \delta) \cap D$. Let's take $\{x_n\}$ such that $x_n \in D - \{c\}$ and $\lim_{x \to \infty} x_n = c$. $\exists\, k \in \mathbb{N}$ such that $c - \delta < x_n < c + \delta$ $\forall n \geq k$

$$x_n \in N'(c, \delta) \cap D \quad \forall n \geq k$$

By the condition ~~$f(x_n)$~~ $|f(x_n) - f(x_{n+p})| < \varepsilon$ $p \in \mathbb{N}$

This proves $\{f(x_n)\}$ is a cauchy sequence and hence convergent

We now prove that all such $\{f(x_n)\}$ converge to a common limit.

Let $\{p_n\}$ and $\{q_n\}$ in $D-\{c\}$ such that $\lt p_n = \lt q_n = c$. $\lt f(p_n) = p$

and $\lt f(q_m) = q$

Let us consider a sequence $\{x_n\}$ where $x_{2n-1} = p_n$. i.e

$\{x_n\} = \{p_1\ q_1\ p_2\ q_2 \ldots\}$ $x_{2n} = q_n$

Then $\lt_{x\to\infty} x_n = c$

Therefore $\{f(x_n)\}$ converges to l

$\{f(x_{2n-1})\}$ and $\{f(x_{2n})\}$ are converging subsequences of $\{f(x_n)\}$

$\Rightarrow p = q = l$

One-sided-Limits:

(i) Right Hand limit:

Let $D \subset \mathbb{R}$. $f: D \longmapsto \mathbb{R}$ and c be a limit point of $D_1 = D\cap(c.\infty)$ $= \{x \in D \quad x > c\}$. x is said to have a right hand limit. if for $\varepsilon > 0$ $\exists\ \delta > 0$ such that $|f(x) - l| < \varepsilon$ $\forall\ x \in N'(c.\delta) \cap D_1$

$$\lt_{x\to c^+} f(x) = l$$

(ii) Left Hand Limit:

Let $D \subset \mathbb{R}$. $f: D \longmapsto \mathbb{R}$ and c be a limit point of $D_2 = D\cap(-\infty.c)$ $= \{x \in D \quad x < c\}$. x is said to have a left hand limit. If for $\varepsilon > 0$ $\exists\ \delta > 0$ such that $|f(x) - l| < \varepsilon$ $\forall\ x \in N'(c.\delta) \cap D_2$

$$\lt_{x\to c^-} f(x) = l$$