RA Assignment - 2

i) Cauchy but not monotone.

Let the sequence be $\{x_n\}$ such that $x_n = \frac{1}{n}\cos \pi n \ \forall n \in \mathbb{N}$.

$$x_{n+1} - x_n = \begin{cases} 0 - \frac{1}{n}, & n \text{ is odd \& of the form } (4k+1),\ k \in \\ 0 + \frac{1}{n}, & n \text{ is odd \& of the form } (4k-1),\ k \in \\ \vdots & \text{(similarly for } n \to \text{even).} \end{cases}$$

$-\frac{1}{n} < 0$ and $\frac{1}{n} > 0$

$\Rightarrow \{x_n\}$ is not monotone.

But, $\lim_{n \to \infty} \frac{1}{n}\cos \pi n = 0$, since $\frac{1}{n} \to 0$ and $\cos \pi n$ is bounded.

$\therefore \{x_n\}$ is convergent.

Since $\{x_n\}$ is convergent, it is Cauchy.

$\therefore \{x_n\}$ is Cauchy but not monotone.

ii) Monotone but not Cauchy.

Let the sequence $\{x_n\}$ be the sequence of natural numbers, ie, $\{x_n : x_n = n \ \forall n \in \mathbb{N}\}$

$\Rightarrow |x_m - x_n| \geq 1 \ \forall m, n \in \mathbb{N}$

$\Rightarrow |x_m - x_n| \not< \varepsilon \ \forall m, n \geq N_0 \in \mathbb{N}, \ \forall \varepsilon > 0$.

$\therefore \{x_n\}$ is not Cauchy

$x_{n+1} - x_n = 1 > 0$

$\therefore \{x_n\}$ is monotonically increasing.

$\therefore \{x_n\}$ is Monotonic but not Cauchy.

iii) Bounded but not Cauchy :

Let $\{x_n\}$ be a sequence such that,

$$x_n = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$$

Clearly $|x_n| = 1 \ \forall n \in \mathbb{N} \Rightarrow \{x_n\}$ is bounded.

But,

$$|x_m - x_n| = \begin{cases} 0, & m, n \text{ are of same parity} \\ 2, & m, n \text{ are of different parity} \end{cases}$$

$\Rightarrow |x_m - x_n| \nless \varepsilon \ \forall m, n \geq N_0 \in \mathbb{N} \ \forall \varepsilon > 0$.

$\therefore \{x_n\}$ is not Cauchy.

$\therefore \{x_n\}$ is Bounded but not Cauchy.

2. $f: \mathbb{R} \mapsto \mathbb{R}$, $f(x) = \dfrac{x^3}{1+x^2}$.

Take $c \in \mathbb{R}$ such that $|x-c| < \delta$, then;

$$|f(x) - f(c)| = \left| \frac{x^3}{1+x^2} - \frac{c^3}{1+c^2} \right|$$

$$= \left| \frac{x(1+x^2) - x}{1+x^2} - \frac{c(1+c^2) - c}{1+c^2} \right|$$

$$= \left| x - c - \left(\frac{x}{1+x^2} - \frac{c}{1+c^2} \right) \right|$$

$$\leq \underbrace{|x-c|}_{<\delta} + \left| \frac{x}{1+x^2} - \frac{c}{1+x^2} \right|$$

$$\left| \frac{x}{1+x^2} - \frac{c}{1+c^2} \right| = \left| \frac{x + xc^2 - c - cx^2}{(1+x^2)(1+c^2)} \right|$$

$$= \left| \frac{xc(c-x) - 1(c-x)}{(1+x^2)(1+c^2)} \right|$$

$$= \left| \frac{(xc-1)(c-x)}{(1+x^2)(1+c^2)} \right|$$

$$= \left| \frac{(xc-1)(x-c)}{(1+x^2)(1+c^2)} \right|$$

$$\leq \delta \left| \frac{xc-1}{(1+x^2)(1+c^2)} \right|$$

Now, $\left| \dfrac{xc-1}{(1+x^2)(1+c^2)} \right|$ is always going to be a finite quantity for finite x. (Numerator clearly finite for finite x and c, and denominator is always greater than or equal to 1).

Therefore, we can take $\delta \left| \frac{xc - 1}{(1+x^2)(1+c^2)} \right|$ as $\varepsilon' > 0$.

$$\Rightarrow \left| \frac{x}{1+x^2} - \frac{c}{1+c^2} \right| \leq \varepsilon'$$

$$\Rightarrow |f(x) - f(c)| \leq |x - c| + \varepsilon'$$

$$= |f(x) - f(c)| < \delta + \varepsilon'$$

Taking $\varepsilon = \delta + \varepsilon'$, $\Rightarrow |f(x) - f(c)| < \varepsilon$

$\therefore$ $f(x)$ is continuous since $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$ $(\delta, \varepsilon > 0)$.
and c is any point in $\mathbb{R}$.

For uniform continuity,

~~A function is said to be uniformly continuous if $\forall x, y$, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$.~~

A function is said to be uniformly continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall x, y \in$ Domain of f, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

To prove ~~contin~~ uniform continuity,

$$f'(x) = \frac{(3x^2)(1+x^2) - (x^3)(2x)}{(1+x^2)^2}$$

$$= \frac{3x^2 + 3x^4 - 2x^4}{(1+x^2)^2} = \frac{3x^2 + x^4}{(1+x^2)^2}$$

$$\Rightarrow f'(x) = \frac{x^2(3+x^2)}{(1+x^2)^2}$$

$$= \frac{2x^2}{(1+x^2)^2} + \frac{x^2\cancel{(1+x^2)}}{(1+x^2)^{\cancel{2}}}$$

$$f'(x) = \frac{2x^2}{(1+x^2)^2} + \frac{x^2}{1+x^2}$$

Clearly ; $2x^2 > (1+x^2)^2 \ \forall\ x \geq 1$ and $x^2 > 1+x^2 \ \forall\ x$.

$\therefore$ ~~$f'(x)$ is bounded finite also~~ f

$\therefore$ $f'(x)$ is finite $\forall\ x \in \mathbb{R}$. Since $f'(x)$ is finite, let the bound of $f'(x)$ be M, ie, $|f'(x)| \leq M \ \forall\ x \in \mathbb{R}$.

W.k.t by Mean Value theorem,

$$f'(c) = \frac{|f(a) - f(b)|}{|a-b|}$$

where $c \in [a, b]$.

$$\Rightarrow \frac{|f(a) - f(b)|}{|a-b|} \leq M$$

$$\Rightarrow |f(a) - f(b)| \leq M|a-b|$$

$$\Rightarrow |f(x) - f(y)| \leq M|x-y|$$

Take $\delta = \frac{\varepsilon}{M}$, then

$$\Rightarrow |f(x) - f(y)| < \varepsilon$$

$\therefore$ $f(x)$ is uniformly continuous.

4. <u>Given:</u> $f: \mathbb{R} \mapsto \mathbb{R}$ is continuous, $K \in \mathbb{R}$.
<u>To Prove:</u> $f^{-1}(K)$ is a closed set.

Since $f(x)$ is continuous, by definition we can say that, if a set $U \subseteq \mathbb{R}$ is open, then $f^{-1}(U)$ is also open.

IIIly if $U \subseteq \mathbb{R}$ is closed, then $f^{-1}(U)$ is also closed.

The singleton set $\{k\}$ is closed in $\mathbb{R}$.

Therefore, $f^{-1}(\{k\})$ will also be closed in $\mathbb{R}$.

5. ~~To~~ 1) To Prove: ~~Wt~~ Any subset of a nowhere dense set is nowhere dense.

Let A be a nowhere dense set in X, then,

$\forall$ open $G \subseteq X$, $(X \setminus \bar{A}) \cap G \neq \phi$. ($\bar{A}$ = closure of A)

Let $B \subseteq A$, then, $\bar{B} \subseteq \bar{A}$ (~~Also~~ By definition of closed sets)

$\Rightarrow \quad X \setminus \bar{A} \subseteq X \setminus \bar{B}$

$\Rightarrow \quad (X \setminus \bar{A}) \cap G \subseteq (X \setminus \bar{B}) \cap G$

Since $(X \setminus \bar{A}) \cap G \neq \phi$, Clearly $(X \setminus \bar{B}) \cap G \neq \phi$

$\therefore$ B is nowhere dense.

Thus, any subset of a nowhere dense set is also nowhere dense.

2) To Prove: The union of finitely many nowhere dense sets is nowhere dense.

Let A_i be a collection of nowhere dense ~~d~~ sets, with $i \in \mathbb{N}$ and $i \leq n \in \mathbb{N}$.

$\Rightarrow$ ~~[illegible]~~ ~~[illegible]~~ $(X \setminus \bar{A}_1) \cap G \neq \phi$

$(X \setminus \bar{A}_2) \cap G \neq \phi$

$\vdots$

$(X \setminus \bar{A}_n) \cap G \neq \phi$

Let $B_2 = A_1 \cup A_2$ ∴ then $\overline{B_2} = \bar{A_1} \cup \bar{A_2}$

Then,
$$(X \setminus \overline{B_2}) \cap G = (X \setminus (\bar{A_1} \cup \bar{A_2})) \cap G$$
$$= ((X \setminus \bar{A_1}) \cap G) \cup ((X \setminus \bar{A_2}) \cap G)$$
$$\neq \phi.$$

Ill^{ly} define $B_n = \bigcup_{k=1}^{n} A_k$, then $\overline{B_n} = \bigcup_{k=1}^{n} \bar{A_k}$

$$\Rightarrow (X \setminus \overline{B_n}) \cap G = X \setminus (\bar{A_1} \cup \bar{A_2} \cup A_3 \cdots \cup \bar{A_n}) \cap G$$
$$= ((X \setminus \bar{A_1}) \cap G) \cup ((X \setminus \bar{A_2}) \cap G) \cdots$$
$$\cdots \cup ((X \setminus \bar{A_n}) \cap G).$$
$$\neq \phi.$$

$\Rightarrow B_n$ is nowhere dense.

∴ The union of finitely many nowhere dense sets is nowhere dense.

3) <u>To Prove:</u> The closure of a nowhere dense set is nowhere dense.

Let $B = \bar{A}$, where A is a nowhere dense set

We know that $\overline{(\bar{A})} = \bar{A}$, since $\bar{A}$ contains all the limit points of itself.

$$\Rightarrow (X \setminus \bar{B}) \cap G = (X \setminus \bar{\bar{A}}) \cap G$$
$$= (X \setminus \bar{A}) \cap G$$
$$\neq \phi$$

$\Rightarrow B$ is nowhere dense

∴ The closure of a nowhere dense set is nowhere dense.

4) Given: X has no isolated points.
To Prove: Every finite set in X is nowhere dense.

Let $\{x\}$ be a singleton set in X $\Rightarrow$ $\{x\}$ is closed.

~~$\Rightarrow$ $(X \setminus \overline{\{x\}}) \cap G) \neq \phi$ as for the equality to ho~~

$\Rightarrow$ $\forall$ open $G \subseteq X$, $(X \setminus \overline{\{x\}}) \cap G \neq \phi$; as for the equality to hold, $G = \{x\}$, which is not possible since G should be open.

$\Rightarrow$ $\{x\}$ is nowhere dense in X.

Since any finite set can be represented as a union of finitely many singleton sets, by (2), any finite set in X is nowhere dense.

6. Given: $\{a_n\}$ is a sequence.
$\{b_n\}$ is nondecreasing convergent sequence of positive no's.
$|a_{n+1} - a_n| \leq b_{n+1} - b_n$
To Prove: $\{a_n\}$ is Cauchy.

Since $\{b_n\}$ is convergent, it is Cauchy.

$\therefore$ $|b_m - b_n| < \varepsilon$ $\forall$ $m, n \geq N_0 \in \mathbb{N}$
$=$ $b_m - b_n < \varepsilon$ $\forall$ $m, n \geq N_0 \in \mathbb{N}$ (Since $b_n > 0$ $\forall$ $n \in \mathbb{N}$ and it is non decreasing)

Take $m > n+1$, then

$$|a_{n+1} - a_n| \leq b_{n+1} - b_n \leq b_m - b_n < \varepsilon \quad \forall\, m,n \geq N_0$$

($b_m \geq b_{n+1}$ since the seq. is non decreasing)

$$\Rightarrow |a_{n+1} - a_n| < \varepsilon \quad \forall\, n \geq N_0 \quad \text{—(1)}$$

From (1) we can say that, (also let $m = n+k$)

$$|a_{n+2} - a_{n+1}| < \varepsilon$$
$$|a_{n+3} - a_{n+2}| < \varepsilon$$
$$\vdots$$
$$\Rightarrow |a_{n+k} - a_{n+k-1}| < \varepsilon$$

By Δle inequality we get,

$$|a_{n+k} - a_{n+k-1} + a_{n+k-1} - a_{n+k-2} \cdots + a_{n+1} - a_n|$$
$$\leq |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \cdots$$
$$< k\varepsilon = \varepsilon'$$

$$\Rightarrow |a_{n+k} - a_n| < \varepsilon'$$

$$\Rightarrow |a_m - a_n| < \varepsilon' \quad \forall\, m,n \geq N_0 \in \mathbb{N}.$$

$\therefore$ $\{a_n\}$ is Cauchy.

7. Given: $f: X \mapsto Y$ is continuous.
To Prove: $f(\overline{E}) \subseteq \overline{f(E)} \ \forall \ E \subset X$ ($\overline{E}$ - closure of E)

Since $f(x)$ is continuous, the limit points of E are going to be mapped to the limit points of $f(E)$.

To prove that, let $\{x_n\}$ be a sequence in E such that it converges to $x^* \in \overline{E}$.

$\therefore$ x^* is a limit point of E.

As $f(x)$ is continuous, $\{f(x_n)\}$ will converge to $f(x^*)$, (Sequential definition of continuity).

Since $\{x_n\} \subseteq E$, $\{f(x_n)\} \subseteq f(E)$
$\Rightarrow$ $f(x^*)$ is a limit point of $f(E)$

$\therefore$ The limit points of E are mapped to the limit points of $f(E)$.

Since $\overline{f(E)}$ contains $f(E)$ and the limit points of $f(E)$.

$\forall \ x \in \overline{E} \ \exists \ f(x) \in \overline{f(E)}$.

$\Rightarrow$ $f(\overline{E}) \subseteq \overline{f(E)}$

3. <u>Given</u> :- $\{a_n\}, \{b_n\}$ are bounded sequences such that $\lim\sup a_n < 0$ and $\lim\sup b_n < 0$.

$\{c_n\}$ is a sequence such that $c_n = a_n \cdot b_n \ \forall n \in \mathbb{N}$

<u>To Prove</u> :- $\lim\sup c_n = \lim\inf a_n \cdot \lim\inf b_n$.

Since $\lim\sup a_n, \lim\sup b_n < 0$,

$$a_n, b_n < 0 \ \forall n \geq N_0 \in \mathbb{N}$$

$$\Rightarrow c_n > 0 \ \forall n \geq N_0 \in \mathbb{N}$$

$$\Rightarrow \lim\inf a_n, \lim\inf b_n < 0$$

$$\Rightarrow \lim\sup a_n \cdot \lim\sup b_n \leq \lim\inf a_n \cdot \lim\inf b_n$$

~~$\lim\sup c_n = \lim\sup(a_n \cdot b_n)$~~

~~Since $\lim\inf a_n$ and $\lim\inf b_n$~~

By definition of lim inf and lim sup,

$$\lim\sup a_n \geq a_n \geq \lim\inf a_n \ \forall n \geq N_1 \in \mathbb{N}$$
$$\lim\sup b_n \geq b_n \geq \lim\inf b_n \ \forall n \geq N_2 \in \mathbb{N}$$

$$\Rightarrow \lim\sup a_n \cdot \lim\sup b_n \leq a_n b_n \leq \lim\inf a_n \cdot \lim\inf b_n \quad \forall n \geq \max(N_1, N_2)$$

$$\Rightarrow c_n \leq \lim\inf a_n \cdot \lim\inf b_n \ \forall n \geq \max(N_1, N_2).$$

Since c_n is bounded (a_n, b_n are bounded $\Rightarrow a_n \cdot b_n$ is bounded) and $\lim\inf a_n \cdot \lim\inf b_n$ is its upper bound after a certain no. of terms, we can say that,

$$\Rightarrow \boxed{\lim\sup c_n \leq \lim\inf a_n \cdot \lim\inf b_n}$$