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Assignment - 2

PART-A (5×2=10m)

1. Given; $f(x) = |x|$ $(-\pi, \pi)$

W.K.T $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot dx$$

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} |x| \cdot dx$$

$$a_0 = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} \underbrace{|x|}_{\text{even}} \cdot \underbrace{\cos nx}_{\text{eve}} \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x \frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{(-1)^n}{n^2} - 0 - 1/n^2\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$=$ if n is odd $= -4/n^2\pi$

if, n is even $= 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx \cdot dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx \cdot dx$$

π odd/even

$$\boxed{b_n = 0}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \cdot dx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{-4}{\pi n^2}\right]\cos nx$$

If $\boxed{x=0}$ it is continuous in $(-\pi,\pi)$

$f(x) = 0$ & $\cos nx = 1$.

$$\therefore\ 0 = \pi/2 + \sum_{n=1}^{\infty}\left[\frac{-4}{\pi n^2}\right]$$

$$-\pi/2 = \sum_{n=1}^{\infty}\left[\frac{-4}{\pi n^2}\right]$$

$$\boxed{\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\right] = \frac{\pi^2}{8}}$$

2. Given; $f(x) = (l-x)^2$ in $(0,2l)$

Period is $2l$.

W.K.T $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$.

$$\therefore\ a_0 = \frac{1}{l}\int_0^{2l} f(x)\cdot dx$$

$$a_0 = \frac{1}{l}\int_0^{2l}(l-x)^2\cdot dx$$

$$= \frac{1}{l}\left[-\frac{(l-x)^3}{3}\right]_0^{2l} = \frac{1}{l}\left[\frac{l^3}{3}\right] - \frac{2l^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\cdot dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 \cos\frac{n\pi x}{l}\cdot dx$$

$u = (l-x)^2$ | $dv = \cos\frac{n\pi x}{l}$

$u' = -2(l-x)$ | $v = \sin\frac{n\pi x}{l}\cdot\frac{l}{n\pi}$

$u'' = 2$ | $v_1 = -\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^2$

| $v_2 = -\sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^3$

$$a_n = \frac{1}{l}\left[(l-x)^2\left[\sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right] - 2(l-x)\left[\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^2\right] - 2\sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^3\right]_0^{2l}$$

$$= \frac{1}{l}\left[2l\left(\frac{l}{n\pi}\right)^2 + 2l\left(\frac{l}{n\pi}\right)^2\right]$$

$$= \frac{1}{l}\left[4\frac{l^3}{n^2\pi^2}\right] = \frac{4\cdot l^2}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\cdot\sin\frac{n\pi x}{l}\cdot dx$$

$$= \frac{1}{l}\int_0^{2l}(l-x)^2\cdot\sin\frac{n\pi x}{l}\cdot dx$$

$$= \frac{1}{l}\left[(l-x)^2\left[-\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right] - 2(l-x)\left[\sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^2\right] + 2\left[\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^3\right]\right]_0^{2l}$$

$$= \frac{1}{l}\left(l^2\cdot\left[-\frac{l}{n\pi}\right] + 2\left[\frac{l}{n\pi}\right]^3 + l^2\left[\frac{l}{n\pi}\right] - 2\left[\frac{l}{n\pi}\right]^3\right)$$

$$= 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cdot\cos\frac{n\pi x}{l}$$

f(x)=

if $\boxed{x=0}$

$\therefore$ As 0 is ~~not~~ present $(0, 2l)$

it is discontinues.

$$\therefore \quad f(x) = \frac{f(0) + f(2l)}{2}$$

$$= \frac{l^2 + l^2}{2} = l^2$$

$$\therefore \quad l^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}$$

$$\frac{\cancel{2l^2}}{3} = \sum_{n=1}^{\infty} \frac{\cancel{4}^{\,2}\cancel{l^2}}{n^2\pi^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \quad \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6}$$

3. Given;

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
y	1.98	1.30	1.05	1.30	-0.88	-0.25

$$\text{let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{2} a_n \cos nx + \sum_{n=1}^{2} b_n \sin nx$$

$$\therefore \quad a_0 = \frac{2}{m} \sum y$$

$$a_n = \frac{2}{m} \sum y \cos nx$$

$$b_n = \frac{2}{m} \sum y \sin nx$$

x	y	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$	$y\cos x$	$y \sin x$	$y\cos 2x$	$y \sin 2x$
0	1.98	1	0	1	0	1.98	0	1.98	0
$\pi/3$	1.3	0.5	0.866	-0.5	0.866	0.65	1.126	-0.65	1.126
$2\pi/3$	1.05	-0.5	0.866	-0.5	-0.866	-0.525	0.909	-0.525	-0.909
π	1.3	-1	0	1	0	-1.3	0	1.3	0
$4\pi/3$	-0.88	-0.5	-0.866	0.5	0.866	0.44	0.762	0.44	-0.762
$5\pi/3$	-0.25	0.5	-0.866	0.5	-0.866	-0.125	0.217	0.125	0.217
	4.5					1.12	3.014	-2.067	-0.328

Now ; $a_0 = \frac{2}{6}(4.5) = 1.5$

$$a_1 = \frac{2}{6} \Sigma y\cos x = 1/3(1.12) = 0.3733$$

$$a_2 = \frac{2}{6} \Sigma y\cos 2x = 1/3(2.67) = \cdot 0.89$$

$$b_1 = \frac{2}{6} \Sigma y \sin x = 1/3(3.014) = 1.0048$$

$$b_2 = \frac{2}{6} \Sigma y \sin 2x = 1/3(-0.328) = -0.1093$$

Hence $f(x) = 0.75 + 0.3733\cos x + 1.0048 \sin x + 0.89\cos 2x - 0.1093 \sin 2x$

4. Given: $f(x) = k(lx - x^2)$

$\therefore$ W.K.T

$$y(x,t) = (A\cos \lambda x + B\sin \lambda x)(C\cos \lambda at + D\sin \lambda at) \quad \text{--- (1)}$$

$\therefore$ (i) $y(x,0) = 0 ; t \geq 0$

$\therefore$ $y(x,0) = (A+0)(C\cos \lambda at + D\sin \lambda at)$

$y(x,0) = 0$

$A \cdot (C\cos \lambda at + D\sin \lambda at) = 0$

$\boxed{A = 0}$

$\therefore$ Sub. A in Eq (1)

$$y(x,t) = B\sin \lambda x (C\cos \lambda at + D\sin \lambda at) \quad \text{--- (2)}$$

Applying

(ii) $y(l,t) = 0 \; ; t \geq 0$

$$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at)$$

$$y(l,t) = 0$$

$$\sin\lambda l = 0$$

$$\sin\lambda l = \sin n\pi$$

$$\boxed{\lambda = n\pi/l}$$

$\therefore$ Sub λ in Eq (2)

$$y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right] \quad \text{—— (3)}$$

(iii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \; ; \; 0 \leq x \leq l$

$$\frac{\partial}{\partial t}(y(x,t)) = B\sin\frac{n\pi x}{l}\left[-C\sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l} + D\cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right]$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = B\sin\frac{n\pi x}{l}\left[0 + D\frac{n\pi a}{l}\right]$$

$$0 = B\sin\frac{n\pi x}{l}\cdot D\,\frac{n\pi a}{l}$$

$$\boxed{D = 0}$$

Sub in Eq (3)

$$\therefore \; y(x,t) = B\sin\frac{n\pi x}{l}\cdot C\cos\frac{n\pi at}{l}$$

$$\Rightarrow y(x,t) = B_n\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l} \qquad \therefore \boxed{BC = B_n}$$

$\therefore$ general solution;

$$y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi at}{l} \quad \text{—— (4)}$$

$$\therefore \; y(x,0) = K(lx - x^2)$$

$$\therefore \; K(lx - x^2) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}.$$

This is the form of half-range sine series.

$$\therefore \quad B_n = \frac{2}{l}\int_0^l K(lx - x^2)\sin\frac{n\pi x}{l}\, dx$$

$$u = K(lx - x^2) \qquad dv = \sin\frac{n\pi x}{l}$$

$$u' = K(l - 2x) \qquad v = -\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)$$

$$u'' = K(-2) \qquad v_1 = -\sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^2$$

$$v_2 = \cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^3$$

$$B_n = \frac{2}{l}\left[K(lx - x^2)\left[-\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)\right] + K(l-2x)\left[\sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^2\right] - 2K\cos\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)^3\right]_0^l$$

$$B_n = \frac{2}{l}\left[-2K\left(\frac{l}{n\pi}\right)^3(-1)^n + 2K\left(\frac{l}{n\pi}\right)^3\right]$$

$$B_n = \frac{4Kl^2}{n^3\pi^3}\left(1 - (-1)^n\right)$$

if it is odd

$$B_n = \frac{8Kl^2}{n^3\pi^3}$$

$$\therefore \quad \boxed{y(x,t) = \sum_{n=\text{odd}} \frac{8Kl^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cdot\cos\frac{n\pi a t}{l}}$$

5. Given; $\left(\frac{\partial y}{\partial t}\right)_{t=0} = K\sin^3 \frac{\pi x}{l}$

W.K.T

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \quad \text{--- (1)}$$

Using

(i) $y(x,0) = 0 \; ; \; t \geq 0$

$\therefore \; y(x,0) = A \cdot (C\cos\lambda at + D\sin\lambda at)$

$y(x,0) = 0$

$\boxed{A = 0}$

Sub A in Eq (1)

$$\therefore \boxed{y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at)} \quad \text{--- (2)}$$

Applying

(ii) $y(l,t) = 0 \; ; \; t \geq 0$

$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at)$

$y(l,t) = 0$

$\therefore \; \sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$

$\boxed{\lambda = n\pi/l}$

$$y(x,t) = B\sin\frac{n\pi x}{l}\left[C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right] \quad \text{--- (3)}$$

iii) $y(0,t) = 0 \; ; t>0$

$$y(0,t) = B \sin \frac{n\pi x}{l}\left[C+0\right]$$

$$y(0,t) = 0$$

$$BC \sin \frac{n\pi x}{l} = 0$$

$$\boxed{C=0}$$

Sub c in Eq (3)

$$y(x,t) = B \sin \frac{n\pi x}{l} \cdot D \sin \frac{n\pi a t}{l}$$

$$\boxed{BD = B_n}$$

$$y(x,t) = B_n \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi a t}{l}$$

∴ General solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l} \quad \text{—— (4)}$$

iv) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = K \sin^3\left(\frac{\pi x}{l}\right)$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi a t}{l} \cdot \frac{n\pi a}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cdot \frac{n\pi a}{l}$$

$$K \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} B_n \cdot \sin \frac{n\pi x}{l} \cdot \frac{n\pi a}{l}$$

W.K.T $\quad \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$$

$$\frac{K}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = B_1 \cdot \frac{\pi a}{l} \cdot \sin\frac{\pi x}{l} + B_2 \cdot \frac{2\pi a}{l} \cdot \sin\frac{2\pi x}{l} + \cdots$$

B_n : By Comparing coeff on both sides

$$\therefore \frac{3K}{4} = B_1 \cdot \frac{\pi a}{l}, \quad B_2 = 0 ; \quad -\frac{K}{4} = B_3 \cdot \frac{3\pi a}{l} ; \quad B_4 = B_5 = \cdots = 0$$

$$B_1 = \frac{3Kl}{4\pi a} ; \quad B_3 = \frac{-Kl}{12\pi a}$$

$$y(x,t) = \frac{3Kl}{4\pi a} \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l} - \frac{Kl}{12\pi a} \sin\frac{3\pi x}{l} \sin\frac{3\pi a t}{l}$$

(6) Given, Initial temperature $= 10x$;

W.K.T

The proper solution is

$$u(x,t) = (A\cos\lambda x + B\sin\lambda x) e^{-\alpha^2\lambda^2 t} \quad \text{--- (1)}$$

(i) $u(0,t) = 0 ;\ \forall\ t \geq 0$

$$u(0,t) = A e^{-\alpha^2\lambda^2 t}$$

$$0 = A e^{-\alpha^2\lambda^2 t}$$

$$\boxed{A = 0}$$

$$u(x,t) = B\sin\lambda x \cdot e^{-\alpha^2\lambda^2 t} \quad \text{--- (2)}$$

Using

(ii) $u(l,t) = 0 ;\ \forall\ t \geq 0$

$$u(l,t) = B\sin\lambda l\, e^{-\alpha^2\lambda^2 t}$$

$$0 = B\sin\lambda l\, e^{-\alpha^2\lambda^2 t}$$

$$\sin\lambda l = 0$$

$$\lambda l = n\pi$$

$$\boxed{\lambda = n\pi/l}$$

$$\therefore\ u(x,t) = B\sin\frac{n\pi x}{l} \cdot e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}} \quad \text{--- (3)}$$

General solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot e^{-\frac{\alpha^2 n^2\pi^2 t}{l^2}}$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

$$10x = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

This is half range sine series.

$$B_n = \frac{2}{l}\int_0^l 10x \cdot \sin\frac{n\pi x}{l} \cdot dx$$

$$= \frac{20}{l}\int_0^l x \cdot \sin\frac{n\pi x}{l} \cdot dx$$

$$= \frac{20}{l}\left[x \cdot \left[-\cos\frac{n\pi x}{l} \cdot \left(\frac{l}{n\pi}\right)\right] + \sin\frac{n\pi x}{l} \cdot \left(\frac{l}{n\pi}\right)^2\right]_0^l$$

$$= \frac{20}{l}\left[-\frac{l^2}{n\pi}(-1)^n + \frac{l^2}{n\pi}\right]$$

$$= \frac{20l}{n\pi}\left[1-(-1)^n\right]$$

$$= \frac{40l}{n\pi}$$

$$\boxed{u(x,t) = \sum_{n=\text{odd}} \frac{40l}{n\pi} \sin\frac{n\pi x}{l} \cdot e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}}$$