Dhirubhai Ambani Institute of Information and Communication

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POLAR CODES

(CT-216)

Group 4, Sub-group 3

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HONOUR CODE

We declare that

- The work that we are presenting is our own work.
- We have not copied the work (the code, the results, etc.) that someone else has done.
- Concepts, understanding and insights we will be describing are our own.
- We make this pledge truthfully. We know that violation of this solemn pledge can carry grave consequences.

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I. Introduction

In 1948, Shannon first defined the notion of ‘Channel Capacity’. Maximum possible data rate R_b that can be achieved given the transmitted power P_s and bandwidth B is bounded by the following equation. The noise added by the channel is the Additive White Gaussian Noise (AWGN) whose power is denoted by P_n. This equation is known as the Channel Capacity C bound.

$$R_B \leq C = B \times log_2(1 + P_S / P_n)$$

Optimized codes that perform close to this Channel Capacity are needed to be performed. A number of codes were thus developed that could empirically achieve rates very close to capacity, namely Turbo Codes and LDPC Codes. However, the search for a more efficient code that could achieve encoding and decoding at a lower time complexity and simpler implementation continued.

II. Arikan’s Polar Codes

Erdal Arikan first introduced the concept of Polar codes in 2008. They are the first family of error-correcting codes to have achieved Shannon’s Channel Capacity on any symmetric binary-input discrete memoryless channel (B-DMC) asymptotically. They utilize low coding complexity of O(N log N), where N is the block length. Larger block length implies better error performance. This, however, brings in more latency and complexity. They are based on the phenomenon of polarization and implemented recursively, where a channel W is transformed into two virtual channels W_1 and W_2.

III. The principle of Polarization

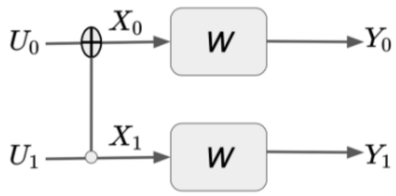
The main idea behind polar codes is to send the message bits in such a way that a fraction of these bits are efficiently transmitted through a noiseless channel and the remaining through a noisy channel. This is the concept of Polarization. Essentially, it transforms the channels in such a way that a channel of non-zero capacity is converted into two extreme channels, thus aggregating and redistributing its capacity. It is to be noted that this process conserves the total capacity. One of the channels is stripped of some part of its capacity while the other is awarded more part of the total capacity. The good channels are used to transmit bits while the bad channels are discarded for transmission. As this number of transformation channels increases, the fraction of reliable bits approaches channel capacity. This technique is information lossless. Channel capacity for the given channel is given as I(W).

This operation is done in two steps :

1) Channel Combining - In this step, given N copies of channels are combined using an XOR gate, where $N = 2^n$. The total channel capacity is combined.
2) Channel Splitting - In the second step, the combined multi-input channels are split back into a set of N binary-input channels.

   At infinite length, the bit channels will polarize to become either infinitely upgraded (totally reliable - $I(W^+)=1$) or infinitely degraded (totally unreliable - $I(W^-)=0$).

   We model this to a BEC channel with an error probability of p. For a BEC channel, the maximum achievable capacity is 1-p.



Combination of two input bits
[1]

New Channels
$W_0 : U_0 \rightarrow (Y_0, Y_1)$
$W_1 : U_1 \rightarrow (Y_0, Y_1, U_0)$

Let the total capacity of the BEC channel be represented as I(W) and the capacity of the compound channel $C(W^2)$.

$$C(W^2) = 2 \times I(W) = 2 \times (1 - p)$$

The Receiver estimates u_0 as follows:

$$\hat{u}_0 = y_0 \oplus y_1$$

u_0 can only be decoded when neither y_1 or y_2 is absent. Therefore, the erasure probability of the induced bit-channel W^- becomes :

$$p_1 = 1 - (1 - p)^2 = 2p - p^2$$

Channel capacity of W^- is given as :

$$I(W^-) = 1 - p_1 = 1 - 2p + p^2$$

This capacity is worse than that of the original channel W.

u_1 can only be decoded as long as neither y1 or y2 is erased. Therefore, the erasure probability and capacity of W^+ are given as follows :

$$p_2 = p^2$$

$$I(W^+) = 1 - p_2 = 1 - p^2$$

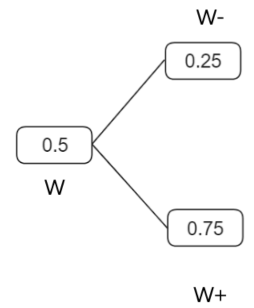
As is evident, the following inequality takes place :

$$I(W^-) \leq I(W) \leq I(W^+)$$

The above equation can be rewritten as :

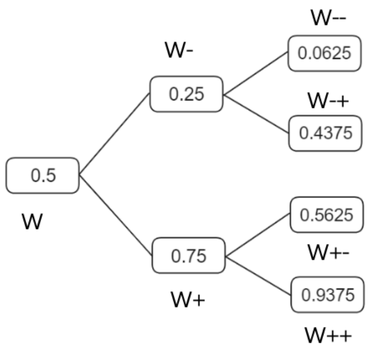
$$I(W^-) + I(W^+) = 2 \times I(W)$$

For example, a BEC(0.5) is taken for length N=2.



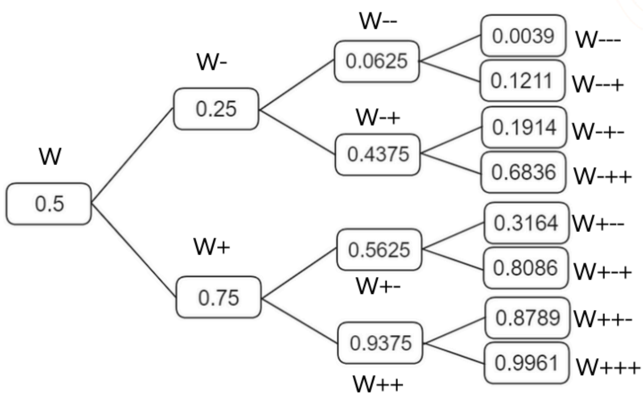
BEC(0.5) Modeled W with length N=2

When the encoder length is increased to 4, the following changes are observed.



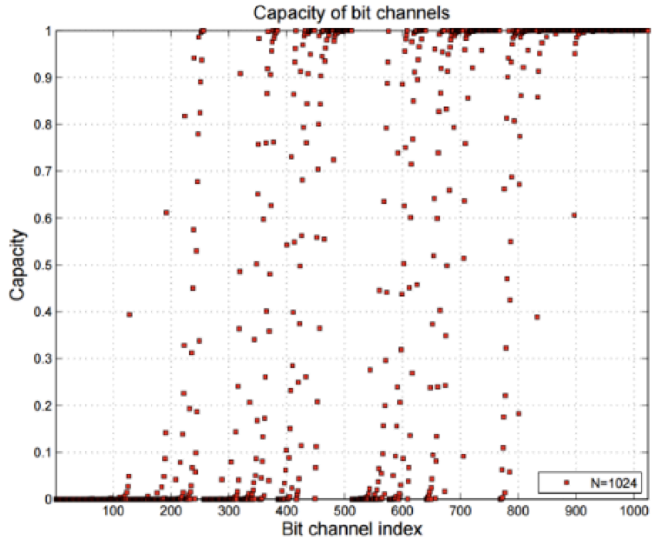
BEC(0.5) Modeled W with length N=4

When N becomes 8, the bad channels become worse and the good channels become better.

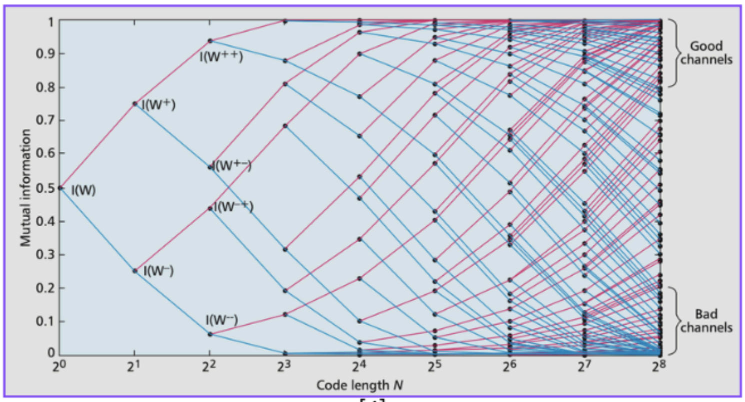


BEC(0.5) Modelled W with length N=8

At higher values of N, the extent of polarization increases with a higher number of channels concentrating over Capacity = 0 and Capacity =1.
This is depicted through the help of the diagrams below.



[5]



[4]

<u>Polar codes achieve Shannon Capacity asymptotically.</u>

It can be proven that the resultant coding rate R is equal to the Channel Capacity I(W) when N is infinitely large. The number of perfect channels as a result of polarization will be :

$$k = N \times I(W)$$

The remaining (N-k) channels will have 0 capacity. The resulting coding rate will be given as:

$$\mathrm{R} = \frac{k}{N} = \frac{NI(W)}{N} = I(W)$$

Hence,we have proved that the coding rate is equivalent to the channel capacity asymptotically.

IV. ENCODING

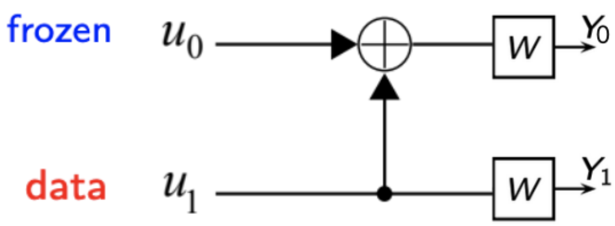
Let the block length of the encoder be N and the message be of k bits. Polar code encoding works on the concept of reliable bits and frozen bits. A reliability sequence is the sequence of 1024 bit-channel positions that are arranged in the increasing order of their reliability. Out of these positions, the most reliable k positions are chosen. Information bits are transmitted through these good channels. These bits are the reliable bits. Input to the remaining (N-k) positions is frozen (set as 0). The information bits are transmitted with the help of these frozen bits. The polar encoding process may be generalized as the following :

$$X_N = u_N F_N$$

Here, X_N is the generated codeword. u_N is a tall vector of the input bits. F_N is the nth Kronecker product of the (2 × 2) kernel matrix F.

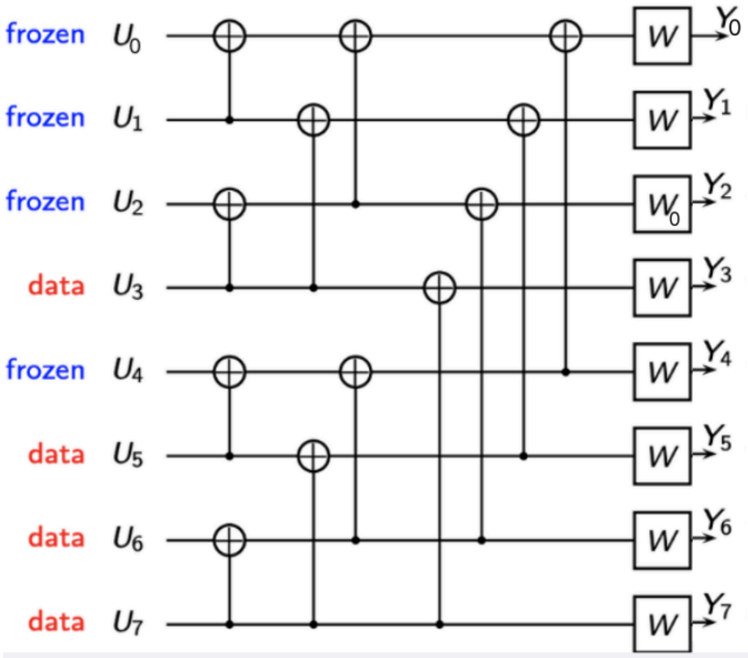
$$\mathbf{F} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This forms a recursive structure with depth $n = \log_2 N$. It utilizes a total of N/2 XOR gates. Coding operation has a complexity of $O(N \log_2 N)$.
Encoding operation of N=2 with input bits as u_0 and u_1 is shown in the figure below.



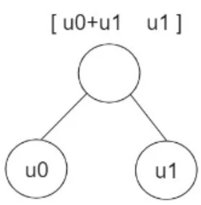
[6]

Similarly, an encoder of length N = 8 is shown below.



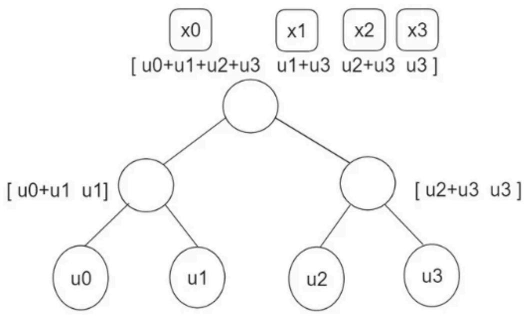
[6]

Another way to implement polar code encoding is through a recursion tree. This method allows for a systematic way to construct the encoding process for the codes.Encoding starts at the leaf nodes. The leaf nodes represent the original information bits. At each level of the tree, adjacent nodes are combined to form new nodes. The first element of this new node is the XOR of the left and right child nodes. The second element is the right child node passed as it is, unchanged. This process is repeated until the root node is reached. For a 2-bit length encoding, this scheme is shown in the following figure.



Polar encoder of length 2

Similarly, the polar encoder of length 4 is depicted in the hfigure below.

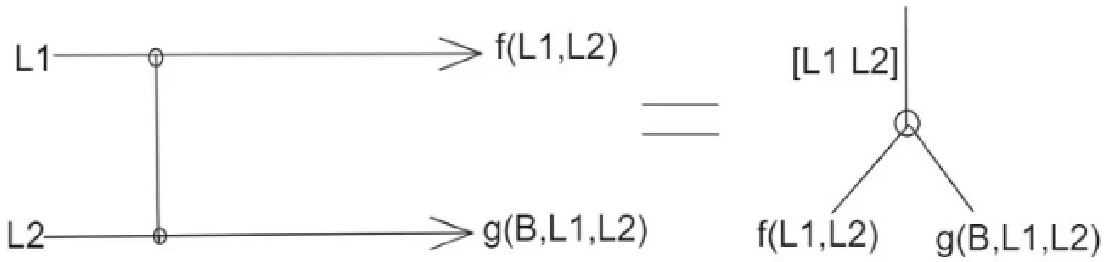


Polar encoder of length 4.

V. DECODING

Decoding is basically a process that decodes the received message at the receiver side. Polar decoding is based on beliefs. The receiver receives the beliefs of the message. Here, the Successive Cancellation List Decoding is used at the receiver. A List stores the messages of length that is given in list size.Each and every message bit received at the receiver is to be decoded .

The main parts in SCL Decoding are the MinSum function which works on the principle of Single Parity Check (SPC) , G function which works on the principle on repetition code AND Path metric [PM] that stores all the paths that are lesser than or equal to List size .



$$f(L_1, L_2) = sign(L_1)sign(L_2)min(|L_1|, |L_2|)$$

$$g(\beta, L_1, L_2) = (1 - 2\beta)L_1 + L_2$$

As is shown above, the function L and G are utilized for getting the original message bits.

For depth n = $\log_2(N)$-1, we have a common process for decoding that is mentioned below.

2-bit Decoder :

- r_0 and r_1 are the received bits.

-> SISO decode u_0 first(SPC)

- $L(u_0) = f(r_0,r_1)$
- $\hat{u}_0=0$ if $L(u_0)\geqslant 0$ else $\hat{u}_0=1$

->Given $\hat{u}_0$ decode u1 (REP)

- if $\hat{u}_0 = 0$, $L(u_1) = r_0+r_1$
- if $\hat{u}_0 = 1$, $L(u_1) = r_1-r_0$

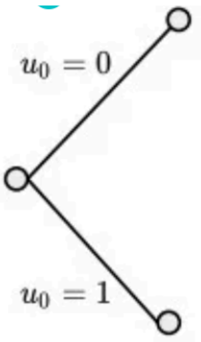
In SCL Decoding, a Decision Metric(DM) is also included.

This DM stores all the paths as given below.

- If $L(u_i)\geqslant 0$; $\hat{u}_i=0$ has $DM_i=0$, $\hat{u}_i=1$ has $DM_i = |L(u_i)|$
- If $L(u_i)< 0$; $\hat{u}_i=1$ has $DM_i=0$, $\hat{u}_i=0$ has $DM_i = |L(u_i)|$

DM stores paths even if the bit is frozen and assigns the only way to proceed further.

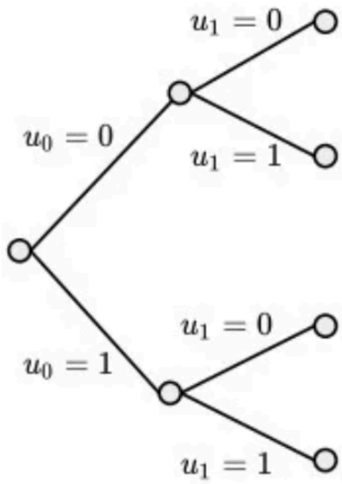
The whole process continues as demonstrated below,
(The size of the list is taken as 4)



$$u_0 = \begin{cases} 0 \\ 1 \end{cases}$$

[1]

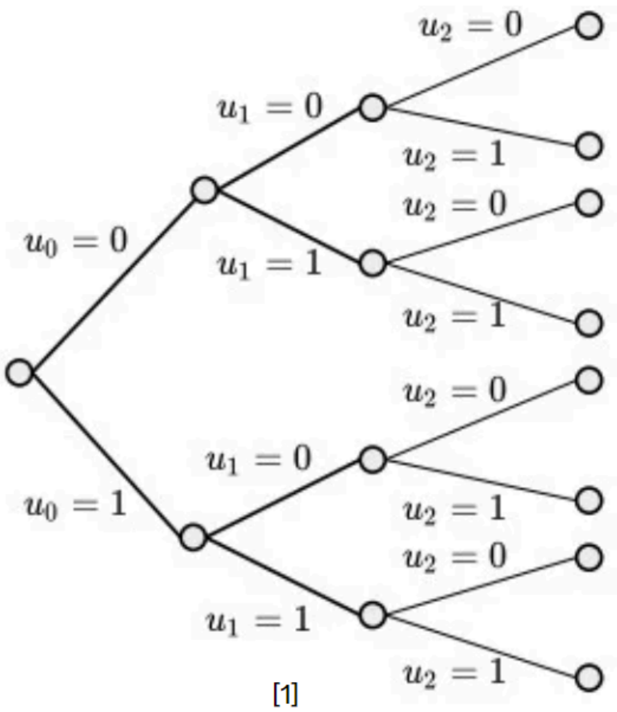
- At leaf node, if the bit is not frozen, it splits into two different paths. It is assigned as shown above.



$$u_0 u_1 = \begin{cases} 00 \\ 01 \\ 10 \\ 11 \end{cases}$$

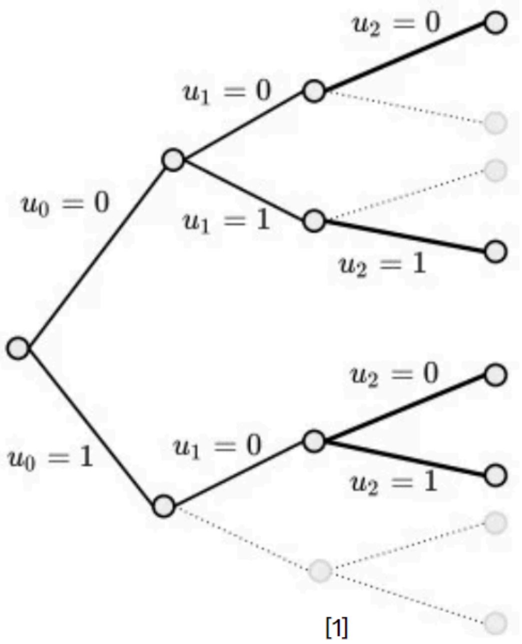
[1]

- Further, we arrive at the u1 node and assume it is not frozen. Now, it has four ways to split up. Since the size of the list that can store the paths is 4 , and there are four paths available, all paths in PM are taken.



[1]

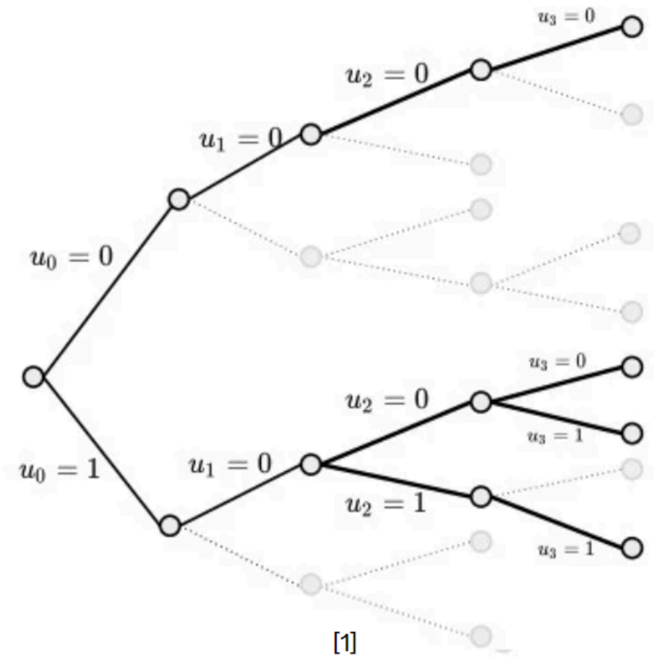
- Now, at node u2, if the bit is not frozen, then there will be 8 paths to be taken, but the most minimal path is taken. Since the size is 4, 4 paths are selected.



[1]

$$u_0 u_1 u_2 = \begin{cases} 000 \\ 011 \\ 100 \\ 101 \end{cases}$$

- Assume that these paths are the most optimum paths and we continue with these four paths and delete all other paths from the list.

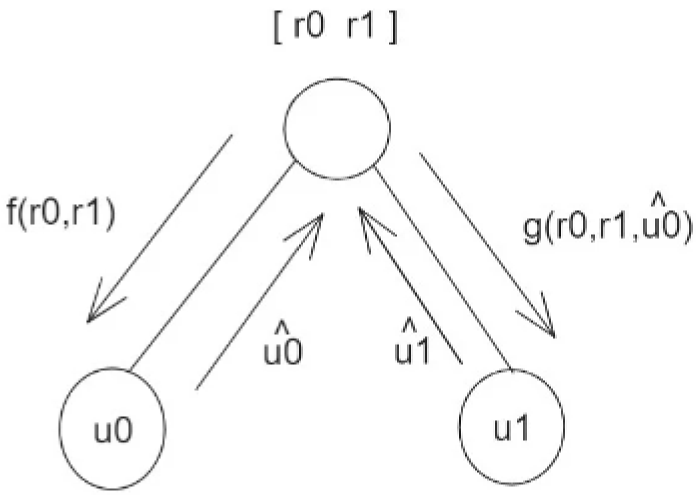


$$u_0 u_1 u_2 u_3 = \begin{cases} 0000 \\ 1000 \\ 1001 \\ 1011 \end{cases}$$

[1]

- Assume this is the last stage. Minimum paths are chosen according to the decoding algorithm.Most optimum paths are chosen which are almost the same as the original message. The path that has the minimum PM is chosen and that is the answer.

  At every depth = Log2(N) - 1 stage, the following process occurs.



- First, the beliefs travel to the node. After applying the Function F, the estimated value of that leaf node is received. Further it can be used for calculating the leaf node that is next to it. The G function is used with the help of the beliefs and estimated values of the previous bit.

  The time complexity of both SCL encoding and decoding is already proved above as O(NlogN) .

VI. REFERENCES

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