POLAR CODE MATHEMATICAL PROOFS

The Polar code that is introduced by “Arikan” works on a phenomenon called “Channel Polarization” and it can achieve maximum capacity for any symmetric binary-input discrete memoryless channel (BDMC).

Polar code has very low time complexity in comparison to other code for encoding and decoding of N block length code which is O(NlogN), which we will prove later. For decoding we can use SC (successive cancellation) decoding which can be further improved in performance by using SC list decoding.

- **BASIC DEFINITION FOR POLAR CODES :**

1. Binary-input discrete memoryless channel (BDMC) :

A BDM channel W can be defined as W : X → Y

Where X is a input to channel X ∈ {0,1}, Y is a output of given channel and channel has transition probabilities denoted by W(y|x), for x ∈ X and y ∈ Y.

And BDM channel will follow memoryless property when it will satisfy given equation for N different used of channel W :

$$W^N(y^N|x^N) = \prod_{\forall i} W(y_i|x_i).$$

[2]

2. Symmetric Capacity of BDMC :

For a BDM channel W, it’s Symmetric Capacity is denoted by I(W) and it can be defined as

$$I(W) \triangleq \sum_{y\in\mathcal{Y}}\sum_{x\in\mathcal{X}} \frac{1}{2} W(y|x) \log \frac{W(y|x)}{\frac{1}{2}W(y|0) + \frac{1}{2}W(y|1)}.$$

[2]

Here, I(W) is the highest rate of reliable communication over W given that input alphabet X has Uniform distribution.

I(W) will be the same as the Shannon capacity of the channel for symmetric channels which will be upper bound on the code rate for providing reliable communication efficiently.

3. <u>Bhattacharyya parameter of a BDMC :</u>

For a BDM channel W, Bhattacharyya parameter can be denoted by Z(W) which is defined by given equation:

$$Z(W) \triangleq \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}.$$

[2]

Here, Z(W) is the measure of reliability of channel W and it can be also shown as an upper bound on the probability of maximum likelihood decision error per channel use.

A BDM channel can be considered as a “good” channel when transmission rate is close to 1, i.e., Z(W) ≈ 0 when I(W) ≈ 1.

Similarly,

A BDM channel can be considered as a “bad” channel when transmission rate is close to 0, i.e., Z(W) ≈ 1 when I(W) ≈ 0.

Since W(y|x) is a probability which will take values between [0,1] so according to that Bhattacharyya parameter Z(W) will also take values in between [0,1] and in I(W) we are taking log2 of W(y|x) so because of that I(W) will also take its values in between [0,1].

From above equation of I(W) and Z(W) we have following two inequalities :

$$I(W) \geq \log \frac{2}{1 + Z(W)}$$
$$I(W) \leq \sqrt{1 - Z(W)^2}.$$

[1]

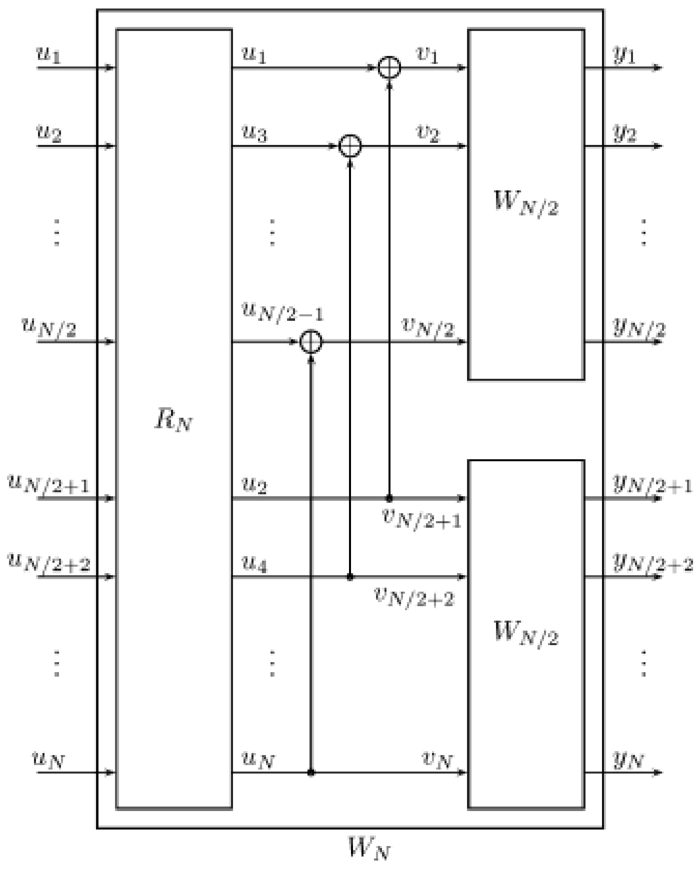
- **<u>Encoding Time Complexity :</u>**

Let X(N) be the time complexity for a channel having total length of code N, message length A, CRC bits length L, uAc as a frozen bits vector and K = A + L,

We can give encoding complexity of this channel by recurrence relation as

$$X(N) \leq 2X(N/2) + N/2 + N$$

Because we are getting the current state output vector from the previous 2 output vectors both having Length equal to divided by 2 from current output vector and we are taking modulo 2 sum of both that N/2 length vector and additional time complexity N/2 and N.



[1]

From the above figure, we can see that for simple code having length equal to 2 we can give its time complexity as X(2) = 3.

By solving this recurrence relation we can obtain

$$X(N) \leq (3/2)\, N \log N \;;\forall\; N = 2^n,\ n \geq 1$$

So, by Big O notation we can give time complexity for encoding of polar codes by **O(NlogN).**

- ## SC Decoding Time Complexity :

For SC decoding of code having parameter N as total length of code, A as length of message, L as a total CRC bits, uAc as a frozen bits vector and K = A + L is an actual message length after adding CRC bits to message having length A. We have the source vector u1N which has two parts uA random part and uAc frozen part.

This source vector is transmitted to the channel WN which gives output vector y1^N with probability WN(y1^N|u1^N) after that SC decoder will generate an estimated value of u1^N as u1^(Estimate)^N by observing y1^N & uAc. And we will consider that for all elements ui of the source vector we have decision elements (DE) which are activated in order from 1 to N.

Further if i ∈ Ac and the element ui is known then ith DE will be activated when it turn comes and will set ui^(Estimate) = ui and send this result to all succeeding DEs. or if i ∈ A then ith DE will wait until it will receive the previous decisions u1^(i-1)(Estimate) and after receiving them it will compute the likelihood ratio (LR) as :

$$L_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1}\right) \triangleq \frac{W_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1}|0\right)}{W_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1}|1\right)}$$

[1]

And decisions will be generated as follows :

$$\hat{u}_i = \begin{cases} 0, & \text{if } L_N^{(i)}(y_1^N, \hat{u}_1^{i-1}) \geq 1 \\ 1, & \text{otherwise} \end{cases}$$

[1]

After that all ui^(Estimate) sent to all succeeding DEs and its time complexity can be computed by computing time complexity of all LRs.

LR can be calculated recursively as :

$$L_N^{(2i-1)}\left(y_1^N, \hat{u}_1^{2i-2}\right) = \frac{L_{N/2}^{(i)}\left(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}\right) L_{N/2}^{(i)}\left(y_{N/2+1}^{N}, \hat{u}_{1,e}^{2i-2}\right) + 1}{L_{N/2}^{(i)}\left(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}\right) + L_{N/2}^{(i)}\left(y_{N/2+1}^{N}, \hat{u}_{1,e}^{2i-2}\right)}$$

[1]

Thus, the calculation of an LR at the length N is reduced to the calculation of two LRs at the length of N/2. This recursion can be continued down to block length 1 , at which point the LRs have the form :

$$L_1^{(1)}(y_i) = W(y_i|0)/W(y_i|1)$$

So for N = 1, L1^(1)(yi) can be computed directly by this formula.

For computation of time complexity, Let X(k) be the time complexity for decoding for $k \in \{N, N/2, N/4, \ldots, 1\}$, from the recursive LR formulas we have recursive relation as

$$X(k) \leq 2X(k/2) + \alpha$$

Where α is the worst case complexity of assembling two LRs at the length k/2 into a LR At length k. Taking X(1) as one unit, we obtain

$$X(N) \leq (1+\alpha)N = O(N)$$

We can compute overall decoding complexity from the above equation is **$O(N^2)$.**

- **<u>SCL Decoding Time Complexity:</u>**

We take εi is the event that the set where errors occurs in the decoding.

$$\mathcal{E}_i \triangleq \left\{ \left(u_1^N, y_1^N\right) \in \mathcal{X}^N \times \mathcal{Y}^N : W_N^{(i-1)}\left(y_1^N, u_1^{i-1} | u_i\right) \right.$$
$$\left. \leq W_N^{(i-1)}\left(y_1^N, u_1^{i-1} | u_i \oplus 1\right) \right\}.$$

[1]

➔ From [1] Derivation is taken that ,
P(εi) that the probability that εi would erase is

$$P(\mathcal{E}_i) \ = Z\left(W_N^{(i)}\right).$$

[1]

The parameter Z is here is the expression of the RV

$$\sqrt{\frac{W_N^{(i)}\left(Y_1^N, U_1^{i-1} | U_i \oplus 1\right)}{W_N^{(i)}\left(Y_1^N, U_1^{i-1} | U_i\right)}}$$

[1]

These RVs can be computed in complexity **O(NlogN).**

- **Conclusion:**
    - By, Using Polar codes we can easily see that Encoding and Decoding are both done in TIme complexity in **O(NlogN).**
    - This time complexity is not achieved by any channel till now.

- **References:**

[1]

Channel_Polarization_A_Method_for_Constructing_Capacity-Achieving_Codes_for_Symmetric_Binary-Input_Memoryless_Channels
Link Here

[2]
POLAR CODES FOR OPTICAL COMMUNICATIONS
By,Tufail Ahmad Link Here