Problem 3.5. What is the negation of these statements?
(a) Jan is rich and happy.
(b) If Kilam was born yesterday, then pigs fly.
(c) Niaz was born yesterday and pigs can't fly.
(d) Kilam's phone has at least 8GB of RAM.
(e) If Kilam is in pajamas, then all lights are off.
(f) Every student is a friend of another student.
(g) Some student is a friend of another student.
(h) All Kilam's friends are big and strong.

Solution :
(a) "Jan is rich and happy." can be represented by the expression $A \wedge B$, where A represents being rich and B represents being happy. $A \wedge B$ is only true when both A and B are true, and false otherwise. This means that the negation of that would only be false when both A and B are true, and true otherwise. This negation can be represented with the expression $\neg A \vee \neg B$, which is true unless A is true and B is true. In words, this expression can be written as "Jan is not rich or is not happy."
(b) "If Kilam was born yesterday, then pigs fly." is the implication $A \to B$, where A represents Kilam being born yesterday and B represents pigs flying. An implication is always true unless A is true and B is false. So, the negation of an implication would always be false unless A is true and B is false. This can be represented mathematically as $A \wedge \neg B$, which is false unless A is true and B is false. In words, this expression can be written as "Kilam was born yesterday and pigs don't fly."
(c) "Niaz was born yesterday and pigs can't fly." can be represented by the expression $A \wedge \neg B$, where A represents Niaz being born yesterday and B representing pigs flying. The expression is only true when A is true and B is false, and false otherwise. The negation of that expression would then be only false when A is true and B is false, and true otherwise. The negation can be represented as an implication, as implications are only false when A is true and B is false. Thus, the implication for this scenario would be $A \to B$. In words, this implication can be written as "If Niaz was born yesterday, then pigs fly."
(d) "Kilam's phone has at least 8GB of RAM." can be represented by the expression $A \geq B$, where A represents the amount of RAM that Kilam's phone has, and B representing 8GB. The negation of this would be $A < B$. In words, this expression can be represented as "Kilam's phone has less than 8GB of RAM."
(e) "If Kilam is in pajamas, then all lights are off." can be represented by the implication $A \to \neg B$, where A represents Kilam being in pajamas B represents all lights being on. This implication is always true unless A is true and B is true. So, the negation of this implication would always be false unless A is true and B is true. This can be represented with the expression $A \wedge B$, which is only true when A is true and B is true. In words, this expression can be written as "Kilam is in pajamas and all lights are on."
(f) "Every student is a friend of another student." can be represented by the expression $\forall a \in A : P(a)$, where a is a student, the set A represents all students, and the predicate $P(a) =$ "Student a is a friend of another student". The negation of this expression would have to flip the quantifier ($\forall \to \exists$) and negate the predicate ($\neg P(a)$). The resulting expression would then be $\exists a \in A : \neg P(a)$. In words, this expression can be written as "Some student is not a friend of another student."
(g) "Some student is a friend of another student." can be represented by the expression $\exists a \in A : P(a)$, where a is a student, the set A represents all students, and the predicate $P(a) =$ "Student a is a friend of another student". The negation of this expression would have to flip the quantifier ($\exists \to \forall$) and negate the predicate ($\neg P(a)$). The resulting expression would then be $\forall a \in A : \neg P(a)$. In words, this expression can be written as "Every student is not a friend of another student."
(h) "All Kilam's friends are big and strong" can be represented by the expression $\forall a \in A : B \wedge C$, where a is one of Kilam's friends, the set A represents all of Kilam's friends, B represents being big, and C represents being strong. The negation of this expression would have to flip the quantifier ($\forall \to \exists$) and negate the predicate ($\neg(B \wedge C) \to \neg B \vee \neg C$). The resulting expression would then be $\exists a \in A : \neg B \vee \neg C$. In words, this expression can be written as "One of Kilam's friends is not big or is not strong."

Problem 3.56. In which (if any) of the domains $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are these claims T. (x and y can have different domains.)
(a) $\exists x : x^2 = 4$
(b) $\exists x : x^2 = 2$
(c) $\forall x : (\exists y : x^2 = y)$
(d) $\forall y : (\exists x : x^2 = y)$

Solution :
(a) Solving the equation $x^2 = 4$ gives the solution $x = 2$. 2 is a natural number, an integer, a rational number, and is a real number, which means that it is within the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Thus, the domain of x is the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.
(b) Solving the equation $x^2 = 2$ gives the solution $x = \sqrt{2}$. $\sqrt{2}$ is an irrational number, which means it is only within the set $\mathbb{R}$. Thus, the domain of x is the set $\mathbb{R}$.
(c) In this equation, y can have a different domain depending on the domain of x. The domain of x includes the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, as any number can be squared. If the domain of x is the set $\mathbb{N}$, then the domain of y will be the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. This is because the square of a natural number will always give a natural number. Natural numbers are also integers, rational numbers, and real numbers, so the domain of y will be the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. If the domain of x is the set $\mathbb{Z}$, then the domain of y will be the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. This is because the square of an integer will always give an integer. Integers are also rational numbers and real numbers, so the domain of y will be the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. If the domain of x is the set $\mathbb{Q}$, then the domain of y will be the sets $\mathbb{Q}$, $\mathbb{R}$. This is because the square of a rational number will always give a rational number. Rational numbers are also real numbers, so the domain of y will be the sets $\mathbb{Q}$, $\mathbb{R}$. If the domain of x is the set $\mathbb{R}$, then the domain of y will be the set $\mathbb{R}$. This is because the square of a real number will always give a real number. Real numbers cannot always be represented as a natural number, integer, or a rational number, so the domain of y will be the set $\mathbb{R}$.
(d) In this equation, x can have a different domain depending on the domain of y. Solving the equation for x will give the solution $x = \sqrt{y}$. Negative numbers do not have real square roots, which means the domain of y cannot include the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, as they all include negative numbers. The only set that does not contain negative numbers is the set $\mathbb{N}$, so the domain of y is the set $\mathbb{N}$. For all elements in this domain of y, x can be a natural number, integer, rational number, or real number, but it will only always ever be a real number, as the square root of a natural number can sometimes result in an irrational number. Thus, the domain of x is the set $\mathbb{R}$.

Problem 3.58 (Hempel's Paradox). Do you believe in induction? Consider the claim "ALL ravens are black."
(a) You observe a black raven. Does that strengthen your belief that "ALL ravens are black?" Is it a proof?
(b) You observe a white sock. Does that strengthen your belief that "ALL non-black things are not ravens?"
(c) Show that "ALL ravens are black." is logically equivalent to "ALL non-black things are not ravens." So, does inductive logic suggest that observing a white sock strengthens your belief that "ALL ravens are black?" Hmm...

Solution :
(a) Observing a black raven would strengthen my belief that "ALL ravens are black" as the raven is not a counter-example that disproves my belief. It is not a proof, however, as I have only seen one raven (which is black) and not all the ravens in the world.
(b) Observing a white sock would strengthen my belief that "ALL non-black things are not ravens" as the sock is a non-black object and it is not a raven, so it isn't a counter-example that disproves my belief.
(c) "ALL ravens are black" is always true unless a raven is not black. "ALL non-black things are not ravens" is always true unless a non-black thing is a raven. A raven that is not black would be a non-black thing, which means "ALL non-black things are not ravens" would only be false when a raven is not black. Since "ALL ravens are black" and "ALL non-black things are not ravens" are both true unless a raven is not black, then that means they are logically equivalent statements. Since the statements are logically equivalent and a white sock would strengthen my belief that "ALL non-black things are not ravens", a white sock would also strengthen my belief that "ALL ravens are black", even though I didn't observe a raven (inductive logic!).

Problem 4.13. Prove by contradiction:
(i) If you cover an 8 x 8 chessboard with 32 dominos, some pair of adjacent dominos must form a 2 x 2 square.

Solution : Assuming some pair of adjacent dominos on an 8 x 8 chessboard cannot form a 2 x 2 square, let's try to cover the chessboard with 32 dominos. However, when attempting to cover the board with dominos, in some situations involving a 2 x 2 square on the chessboard, you are faced with only 2 possible options. The first option is that you have to leave a square behind on the chessboard and not cover it with a domino, so that you don't end up creating a 2 x 2 square of adjacent dominos. However, this option results in you not being able to completely cover the chessboard and you cannot use up all the 32 dominos to cover the chessboard. The second option is to place dominos directly adjacent to each other anyways, creating a 2 x 2 square, but you are able to cover the chessboard completely and you can use all the 32 dominos. Both options form a contradiction with the assumption, as you have to cover the 8 x 8 chessboard with 32 dominoes and some pair of adjacent dominos cannot form a 2 x 2 square. Thus, the assumption is false, and the original statement is true, which means some pair of adjacent dominos must form a 2 x 2 square if you cover an 8 x 8 chessboard with 32 dominos.

Problem 4.42. In each case, prove (or disprove) a relationship between the sets.
(d) $A = \{4k + 1, k \in \mathbb{Z}\}$ and $B = \{4k + 5, k \in \mathbb{Z}\}$. Prove or disprove $A = B$.
(e) $A = \{12m + 21n, m, n \in \mathbb{Z}\}$. Prove or disprove $A = \mathbb{Z}$.
(f) $A = \{12m + 25n, m, n \in \mathbb{Z}\}$. Prove or disprove $A = \mathbb{Z}$.

Solution :
(d) The set $A = \{4k + 1\}$ can be re-written as $A = \{2(2k) + 1\}$. Since $k \in \mathbb{Z}$, set A consists of only odd numbers. The set $B = \{4k + 5\}$ can also be re-written, as $B = \{2(2k + 2) + 1\}$. Set B also consists of odd numbers. Since integers can only ever be odd or even, sets A and B have to be equivalent as they both only consist of odd integers.
(e) The set $A = \{12m + 21n\}$ can be re-written as $A = \{3(4m + 7n)\}$. In the re-written definition of set A, we can see that it only contains multiples of 3, since $m, n \in \mathbb{Z}$. This means that set A cannot be equivalent to set $\mathbb{Z}$, as set A only consists of multiples of 3, while set $\mathbb{Z}$ consists of all integers.
(f) The set $A = \{12m + 25n\}$ can be re-written as $A = \{2(6m + 12n) + n\}$. In the re-written definition of set A, we can see that, since $m, n \in \mathbb{Z}$, set A contains even numbers plus an integer. If the integer is odd, then the element in set A would be odd as even numbers plus an odd number results in an odd number. If the integer is even, then the element in set A would be even as even numbers plus an even number results in an even number. Since integers can only be odd or even, set A can represent all integers, which would mean $A = \mathbb{Z}$.