Reduced Order Modeling for SH Wave Propagation

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Outline

Reduced Order Methods: Overview

- Parametrized PDE:
  $\mathcal{A}(u(\boldsymbol{\mu});\boldsymbol{\mu}) = f \rightsquigarrow \boldsymbol{A}_h(\boldsymbol{\mu})u_h(\boldsymbol{\mu}) = f \rightsquigarrow \boldsymbol{A}_N(\boldsymbol{\mu})u_N(\boldsymbol{\mu}) = f$

- Input parameters:

$$\mu \text{ (physical, geometrical,etc.)}$$

- $()_h$: "truth" high order method (FEM, FV, FD) - to be accelerated

- $()_N$: reduced order method - the accelerator

- The reduced order method does not replace the full order method, but rather builds upon it

Reduced Order Methods: Overview

- parameter domain: $\mathbb{P}^Q = [\mu^q_{min}, \mu^q_{max}]^Q$, where $1 \leq q \leq Q$
- parameter sample: $s_N = \left\{\mu^{(1)}, ..., \mu^{(N)}\right\}$ using greedy algorithm or other ways
- snapshots: solve full order $\left\{u_h(\mu^1), ..., u_h(\mu^N)\right\}$
- reduced order space: $V_N \subset V_h : \dim V_N = N$, $\dim V_h = \mathcal{N}$, with $N \leq \mathcal{N}$.
  $V_N = \mathrm{span}\left\{u_h(\mu^{(n)}), 1 \leq n \leq N\right\}$
- Offline: very expensive preprocessing (high order): basis calculation (done once) after suitable parameters sampling (greedy, POD,...)
- Online: extremely fast (reduced order): real time input-output evaluation

$$\mu \longrightarrow u_N(\mu)$$

Reduced Order Methods: Overview

- Construct a **Snapshot Matrix** from high-fidelity solutions.
- Apply **Singular Value Decomposition (SVD)**.
- Retain only a few dominant modes for an efficient representation.

$$\boldsymbol{A}_N = \mathcal{Z}^T \boldsymbol{A}_h \mathcal{Z}$$

Why Reduced Order Methods?

- High-fidelity simulations (FEM, FDM) are computationally expensive.
- Reduced Order Modeling (ROM) approximates solutions in a lower-dimensional space.
- Significant reduction in computational cost while preserving accuracy.

Why Reduced Order Methods in seismology?

- Seismic wave simulations (FEM, FDM) require **high computational resources**, especially for large-scale domains and high-resolution models.
- **Real-time ground motion prediction** is critical for earthquake early warning systems and seismic hazard assessment.
- Traditional high-fidelity methods become infeasible for **parametric studies**, such as varying source locations, soil properties, or wave velocities.
- Reduced Order Modeling (ROM) provides an efficient alternative by approximating solutions in a **lower-dimensional space**.
- ROM enables **fast and reliable seismic simulations**, significantly reducing computational costs while preserving accuracy.

Mathematical Model for SH Waves

- Seismic wave equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \mu \Delta u = f \quad \text{in } \Omega \tag{1}$$

- Initial conditions:

$$u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}|_{t=0} = 0 \quad \text{in } \Omega \tag{2}$$

- Delta-source model:

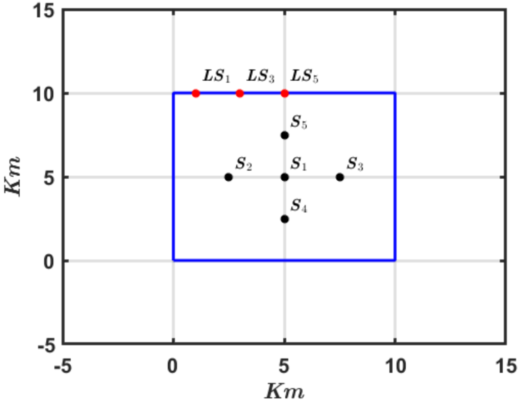
$$f = M(t)\delta(x - x_0) \tag{3}$$

- Function $M(t)$ approximates $\delta(t)$:

$$M(t) = M_0 \frac{t}{\tau^2} e^{-t/\tau}, \quad \lim_{\tau \to 0} M(t) = \delta(t), \tag{4}$$

with $\tau = 0.05s$.

Mathematical Model for SH Waves: Problem Domain



Mathematical Model for SH Waves

- Boundary conditions
  - **Free surface condition:**

$$\frac{\partial u}{\partial n}\bigg|_{\Gamma} = 0 \quad \text{(on the ground)} \tag{5}$$

  - **Absorbing boundary condition (non-reflecting):**

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial n} \quad \text{on } \Gamma \tag{6}$$

  - Absorbing boundaries allow truncation of an infinite domain into a finite computational one.
  - $c = \sqrt{\frac{\mu}{\rho}}$ is the propagation speed.
  - ρ is the density, μ is Lame's parameter, u is displacement.

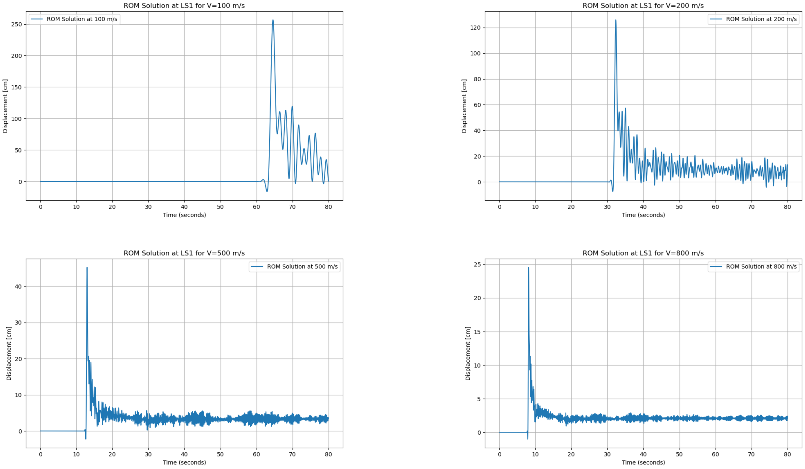
Numerical Method (FOM) and Problem Domain

- **Space discretization:** Finite Element Method (FEM) with second-order approximation.
- **Time discretization:** Leapfrog scheme (explicit method).
- **Mesh details:**
  - Approximately 30,000 triangular elements.
  - Second-order approximation functions.
- **Time steps:** 40,000 timesteps

Case 1: Varying SH wave Velocity

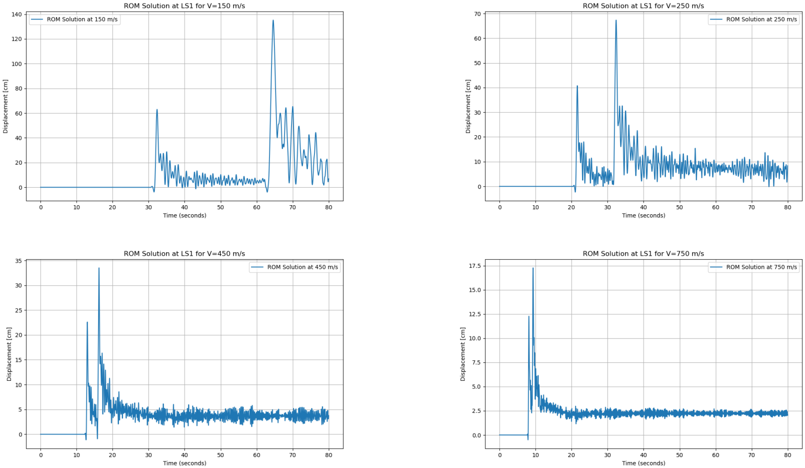
- ROM is applied with **velocity** as the only varying parameter.
- **Focus:** ROM analysis on shear wave velocity.
- **Velocity Range:** 100m/s to 1000m/s (used for snapshot data generation).
- Analyzed for velocities used in snapshot generation.
- Tested on additional velocities beyond Full Order Model (FOM) computations.

Case 1: Error Analysis



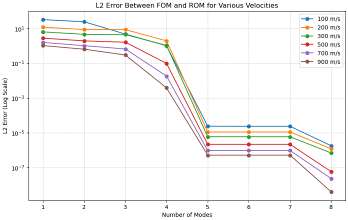
ROM Solutions at LS1 for V = 100, 200, 500, 800m/s (included in FOM computation)

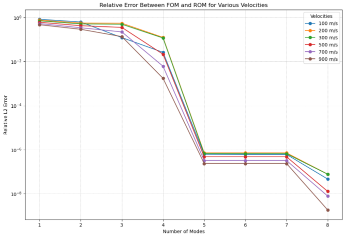
Case 1: Error Analysis



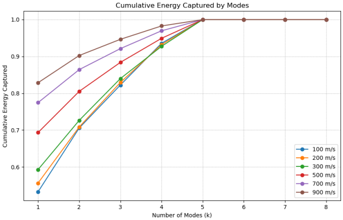
ROM Solutions at LS1 for V = 150, 250, 450, 750m/s (NOT included in FOM computation)

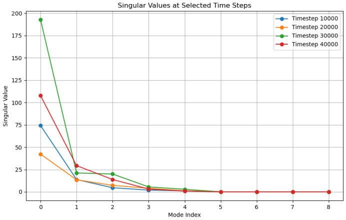
Case 1: Energy distribution





Error comparison between FOM and ROM



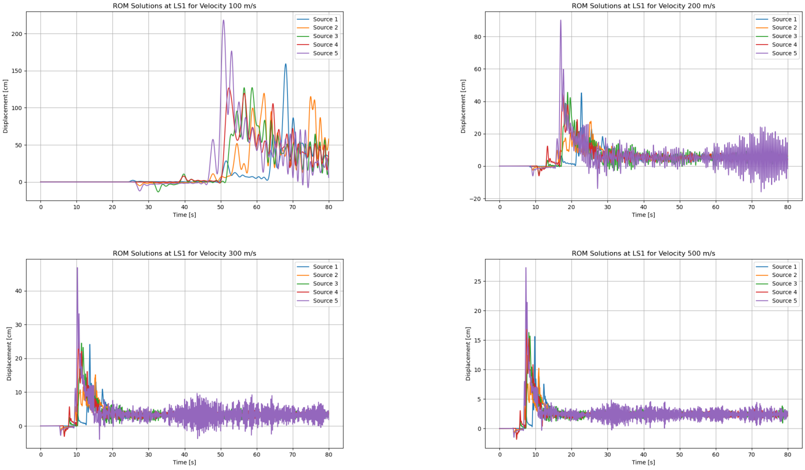


Energy distribution(left) and singular value decomposition across modes(right)

Case 2: Velocity and Source Location

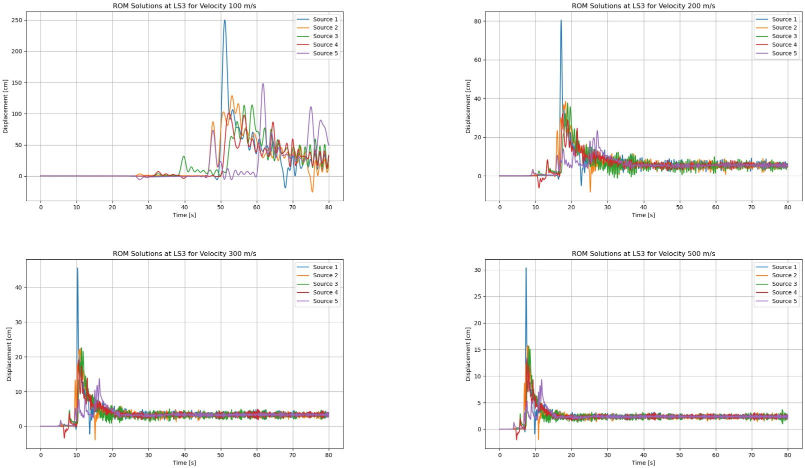
- two parameters: source location and velocity
- source position (center, left, right, bottom, top)
- velocities $[100, 1000]$
- ROM variation across sources and velocities
- source1 (center), source2 (left), source3 (right), source4 (bottom), source5 (top)

Case 2: Error Analysis



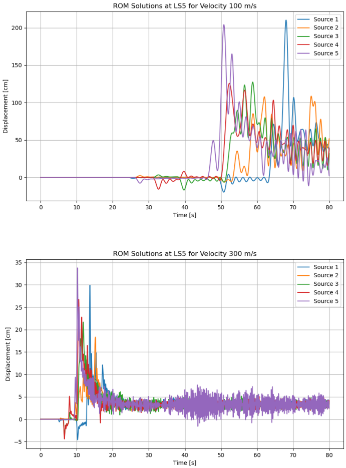
ROM Solutions at LS1 for V = 100, 200, 300, 500m/s (included in FOM computation) at different source location

Case 2: Error Analysis



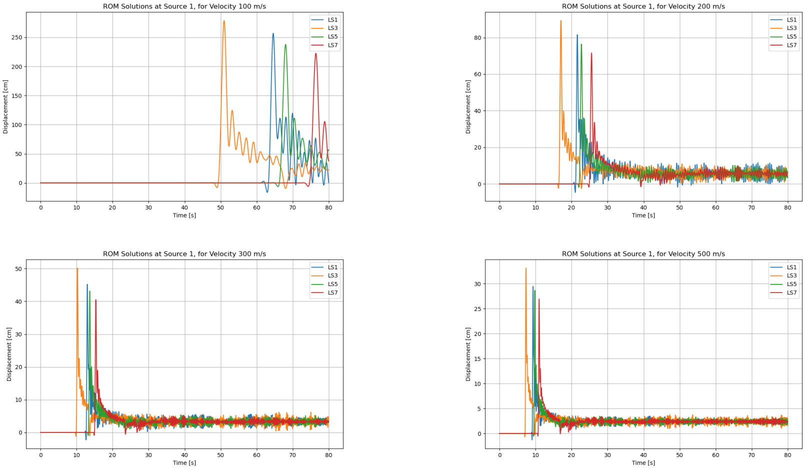
ROM Solutions at LS3 for V = 100, 200, 300, 500m/s (included in FOM computation) at different source location

Case 2: Error Analysis



ROM Solutions at LS5 for V = 100, 200, 300, 500m/s (included in FOM computation) at different source location

Case 2: Error Analysis



ROM Solutions at different LS points with source at center and V = 100, 200, 300, 500m/s

Conclusion

- ROM provides a computationally efficient alternative for seismic simulations.
- Significant reduction in computation time while maintaining accuracy.
- Validation through comparison with full-order FEM simulations.
- ROM effectively reduces computational cost while maintaining accuracy.
- Comparison of FOM and ROM shows minimal error, validating the reduced model.
- Cumulative energy plots demonstrate ROM captures essential wave dynamics.

Future Work: Machine Learning Integration

- Explore **neural networks** to learn ROM basis functions adaptively.
- build a relationship between parameter values and ROM coefficients using **neural networks** to improve results for unseen parameters
- Develop **data-driven ROM** for improved generalization.
- Hybrid approach combining **ROM and Deep Learning** for real-time predictions.
- ROM for non-homogeneous domain (different material properties)

THANK YOU