```
In[ ]:= Quit[]

In[ ]:= (*Ansatz*)
dim = 4;
x = {t, r, θ, φ};
BISERIES[Z_] := Z /. Union[{S2'[θ]^2 → 4 S2[θ] (1 - S2[θ]), S2''[θ] → 2 (1 - 2 S2[θ])}] // Simplify
gdd = {{-α[t, r]^2, 0, 0, 0}, {0, A[t, r]^2, 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 S2[θ]}};
(*gdowndown or metric*)
guu = Simplify[Inverse[gdd]];
g = Simplify[Det[gdd]];
SQRTg = Sqrt[-g] // Simplify ;
ϕ = ϕ0[t, r];

In[ ]:= (*Metric tensors*)
christ = BISERIES[Table[1/2 Sum[
      guu[[l4, l1]] (D[gdd[[l2, l1]], x[[l3]]] + D[gdd[[l1, l3]], x[[l2]]] - D[gdd[[l2, l3]], x[[l1]]]),
      {l1, 1, dim}], {l2, 1, dim}, {l3, 1, dim}, {l4, 1, dim}]];
 (*which index is up index?* Its the final index*)

In[ ]:= Ruddd = BISERIES[Table[D[christ[[l2, m2, l1]], x[[m1]]] - D[christ[[l2, m1, l1]], x[[m2]]] +
     Sum[-christ[[k, m2, l1]] × christ[[l2, m1, k]] + christ[[k, m1, l1]] × christ[[l2, m2, k]],
      {k, 1, dim}], {l1, 1, dim}, {l2, 1, dim}, {m1, 1, dim}, {m2, 1, dim}]];
(* R^l1_l2 m1 m2 = Γ^l1_l2 m2,m1 - Γ^l1_l2 m1,m2 - Γ^l1_k m2 Γ^k_l2 m1 + Γ^l1_k m1 Γ^k_l2 m2 *)

In[ ]:= Rdddd = BISERIES[Table[Sum[gdd[[a1, b1]] × Ruddd[[b1, a2, a3, a4]], {b1, dim}],
    {a1, dim}, {a2, dim}, {a3, dim}, {a4, dim}]];
Rdd = BISERIES[Table[Sum[Ruddd[[k, l1, k, l2]], {k, 1, dim}], {l1, 1, dim}, {l2, 1, dim}]];
R = BISERIES[Sum[Rdd[[l1, l2]] × guu[[l1, l2]], {l1, 1, dim}, {l2, 1, dim}]];

In[ ]:= X  = -1 / 2 Sum[guu[[b1, b2]] × D[ϕ, x[[b1]]] × D[ϕ, x[[b2]]], {b1, dim}, {b2, dim}];

In[ ]:= Clear[f]
(*Scalar field Stress-Energy Tensor*)
TϕTTdd = BISERIES[
   Table[1/2 ( gdd[[a1, a2]] * (X - β X ^ 2 + γ X ^ 3)  + (1 - 2 β X + 3 γ X ^ 2) D[ϕ, x[[a1]]] × D[ϕ, x[[a2]]]) ,
    {a1, dim}, {a2, dim}]];(*added cubic term in K(x)*)
```

In[ ]:=
```
(*Equations of motion*)
BOXphi =
  BISERIES[1/SQRTg Sum[D[SQRTg guu[[i, j]] × D[ϕ, x[[j]]], x[[i]]], {i, 1, dim}, {j, 1, dim}]];
RHS = 0;
(*myRHS = -2β/(1 - 2β X)
  Sum[Sum[guu[[i, k]] guu[j, l]D[ϕ, x[[k]]]  D[ϕ, x[[l]]] ,{k, 1, dim},{l, 1, dim}]
     D[D[ϕ, x[[i]]], x[[j]]] +Sum[christ[[ i, j, h]] D[ϕ, x[[h]]],
     {h, 1, dim}] , {i, 1, dim}, {j, 1,dim}]; ()*)
Edd = BISERIES[Rdd - 1/2 gdd R - TϕTTdd];
Eϕ = BISERIES[BOXphi - RHS];

(*d2xxG2(X)/(1+ dxG2(X)) d^u phi d^v phi Del_u[Del_v[phi]] =
   -2b/(1-2bX) d/du phi d/dv phi (nabla [d_u phi - Tabc ]])*)
```

In[ ]:=
```
myRHS = (-2 β + 6 γ X) / (1 - 2 β X + 3 γ X ^ 2)
   Sum[ Sum[guu[[a, i]] × D[ϕ, x[[a]]], {a, 1, dim} ] × Sum[guu[[b, j]] × D[ϕ, x[[b]]], {b, 1, dim}]
     (D[ϕ, {x[[j]]}, {x[[i]]} ]  -  Sum[christ[[ i, j, k]] × D[ϕ, x[[k]]], {k, 1, dim} ]) , {i,
     1, dim}, {j, 1, dim}];
```

In[ ]:=
```
myEϕ = BISERIES[BOXphi - myRHS] (*dtPI*)
```

Out[ ]=

$$-\frac{A^{(0,1)}[t,r]\,\phi 0^{(0,1)}[t,r]}{A[t,r]^3}+\frac{\left(\frac{2}{r}+\frac{\alpha^{(0,1)}[t,r]}{\alpha[t,r]}\right)\phi 0^{(0,1)}[t,r]+\phi 0^{(0,2)}[t,r]}{A[t,r]^2}-\frac{A^{(1,0)}[t,r]\,\phi 0^{(1,0)}[t,r]}{A[t,r]\,\alpha[t,r]^2}+\frac{\alpha^{(1,0)}[t,r]\,\phi 0^{(1,0)}[t,r]-\alpha[t,r]\,\phi 0^{(2,0)}[t,r]}{\alpha[t,r]^3}+$$

$$\Big(4\left(3\gamma\,\alpha[t,r]^2\,\phi 0^{(0,1)}[t,r]^2+A[t,r]^2\left(2\beta\,\alpha[t,r]^2-3\gamma\,\phi 0^{(1,0)}[t,r]^2\right)\right)$$
$$\Big(\alpha[t,r]^5\,\phi 0^{(0,1)}[t,r]^2\left(-A^{(0,1)}[t,r]\,\phi 0^{(0,1)}[t,r]+A[t,r]\,\phi 0^{(0,2)}[t,r]\right)+$$
$$A[t,r]^3\,\alpha[t,r]^2\,\alpha^{(0,1)}[t,r]\,\phi 0^{(0,1)}[t,r]\,\phi 0^{(1,0)}[t,r]^2-A[t,r]^5\,\alpha^{(1,0)}[t,r]\,\phi 0^{(1,0)}[t,r]^3+$$
$$A[t,r]^2\,\alpha[t,r]^3\,\phi 0^{(0,1)}[t,r]\,\phi 0^{(1,0)}[t,r]\left(\phi 0^{(0,1)}[t,r]\,A^{(1,0)}[t,r]-2A[t,r]\,\phi 0^{(1,1)}[t,r]\right)+$$
$$A[t,r]^5\,\alpha[t,r]\,\phi 0^{(1,0)}[t,r]^2\,\phi 0^{(2,0)}[t,r]\Big)\Big)\Big/$$
$$\Big(3\gamma\,A[t,r]^3\,\alpha[t,r]^7\,\phi 0^{(0,1)}[t,r]^4+2A[t,r]^5\,\alpha[t,r]^5\,\phi 0^{(0,1)}[t,r]^2\left(2\beta\,\alpha[t,r]^2-3\gamma\,\phi 0^{(1,0)}[t,r]^2\right)+$$
$$A[t,r]^7\left(4\alpha[t,r]^7-4\beta\,\alpha[t,r]^5\,\phi 0^{(1,0)}[t,r]^2+3\gamma\,\alpha[t,r]^3\,\phi 0^{(1,0)}[t,r]^4\right)\Big)$$

```
(*dphi/dr = PHI Q here   and dphi/dt = alpha/a PI*)
```

In[ ]:=
```
sostϕt = {D[ϕ0[t, r], t] → α[t, r] / A[t, r] × P[t, r]}; (*dphi/dt = alpha/a P *)
sostϕr = {D[ϕ0[t, r], r] → Q[t, r]}; (*dphi/dr = Q*)
(*Solve for dPHI/dt =  f (A, alpha, PI, PI', alpha', A')*)
```

In[ ]:=
```
EQ1 = (D[ϕ0[t, r], {t}, {r}] /. D[sostϕr , t]) - (D[ϕ0[t, r], {t}, {r}] /. D[sostϕt , r]) // Simplify ;
(*D[f, {x},{y}] = d^2f/dxdy*)
sQt = Solve[EQ1 == 0, D[Q[t, r], t]][[1]] //
  Simplify (*here we get expression of dPHI/dt pf PHI dot*)
```

Out[ ]=
$$\left\{Q^{(1,0)}[t,\, r] \to \frac{A[t,\, r] \times \alpha[t,\, r]\, P^{(0,1)}[t,\, r] + P[t,\, r]\left(-\alpha[t,\, r]\, A^{(0,1)}[t,\, r] + A[t,\, r]\, \alpha^{(0,1)}[t,\, r]\right)}{A[t,\, r]^2}\right\}$$

```
(*here we compute dPI/dt  = f(PHI, A, alp and there derivs )*
   Ephi is equation of motion for the scalar field)
```

In[ ]:=
```
EP1 = Collect[myEϕ //. Union[sostϕt , sostϕr , D[sostϕt , t], D[sostϕr , r]], {D[P[t, r], t],
    D[α[t, r], {t, 2}], D[α[t, r], {t, 1}], D[A[t, r], r], D[Q[t, r], r], r, η}, Simplify];
sPt = Solve[EP1 == 0, D[P[t, r] , t]][[1]] // Simplify
```

Out[ ]=
$$\begin{aligned}
\{P^{(1,0)}[t,\, r] \to \big(&-3\, r\, \gamma\, Q[t,\, r]\left(P[t,\, r]^4 - 6\, P[t,\, r]^2\, Q[t,\, r]^2 + 5\, Q[t,\, r]^4\right) \alpha[t,\, r]\, A^{(0,1)}[t,\, r] + \\
&4\, A[t,\, r]^5\left(r\, \alpha[t,\, r]\, Q^{(0,1)}[t,\, r] + Q[t,\, r]\left(2\, \alpha[t,\, r] + r\, \alpha^{(0,1)}[t,\, r]\right)\right) + \\
&3\, \gamma\, A[t,\, r]\left(P[t,\, r]^2 - Q[t,\, r]^2\right)\big(P[t,\, r]^2\left(r\, \alpha[t,\, r]\, Q^{(0,1)}[t,\, r] + Q[t,\, r]\left(2\, \alpha[t,\, r] - 3\, r\, \alpha^{(0,1)}[t,\, r]\right)\right) - \\
&\quad Q[t,\, r]^2\left(5\, r\, \alpha[t,\, r]\, Q^{(0,1)}[t,\, r] + Q[t,\, r]\left(2\, \alpha[t,\, r] + r\, \alpha^{(0,1)}[t,\, r]\right)\right) + \\
&\quad 4\, r\, P[t,\, r]^3\, A^{(1,0)}[t,\, r] - 4\, r\, P[t,\, r]\, Q[t,\, r]^2\, A^{(1,0)}[t,\, r]\big) + \\
&4\, \beta\, A[t,\, r]^3\big(P[t,\, r]^2\left(-r\, \alpha[t,\, r]\, Q^{(0,1)}[t,\, r] + Q[t,\, r]\left(-2\, \alpha[t,\, r] + r\, \alpha^{(0,1)}[t,\, r]\right)\right) + \\
&\quad Q[t,\, r]^2\left(3\, r\, \alpha[t,\, r]\, Q^{(0,1)}[t,\, r] + Q[t,\, r]\left(2\, \alpha[t,\, r] + r\, \alpha^{(0,1)}[t,\, r]\right)\right) - \\
&\quad 2\, r\, P[t,\, r]^3\, A^{(1,0)}[t,\, r] + 2\, r\, P[t,\, r]\, Q[t,\, r]^2\, A^{(1,0)}[t,\, r]\big) - \\
&4\, r\, A[t,\, r]^4\, Q[t,\, r]\left(\alpha[t,\, r]\, A^{(0,1)}[t,\, r] + 4\, \beta\, P[t,\, r]\, \phi 0^{(1,1)}[t,\, r]\right) + \\
&4\, r\, A[t,\, r]^2\, Q[t,\, r]\big(\beta\, P[t,\, r]^2\, \alpha[t,\, r]\, A^{(0,1)}[t,\, r] - 3\, \beta\, Q[t,\, r]^2\, \alpha[t,\, r]\, A^{(0,1)}[t,\, r] + \\
&\quad 6\, \gamma\, P[t,\, r]^3\, \phi 0^{(1,1)}[t,\, r] - 6\, \gamma\, P[t,\, r]\, Q[t,\, r]^2\, \phi 0^{(1,1)}[t,\, r]\big)\big) / \big(r\, A[t,\, r]^2 \\
&\left(4\, A[t,\, r]^4 + 4\, \beta\, A[t,\, r]^2\left(-3\, P[t,\, r]^2 + Q[t,\, r]^2\right) + 3\, \gamma\left(5\, P[t,\, r]^4 - 6\, P[t,\, r]^2\, Q[t,\, r]^2 + Q[t,\, r]^4\right)\right)\big)\}
\end{aligned}$$

In[ ]:=
```
RHSterm = (-3 r γ Q[t, r] (P[t, r]^4 - 6 P[t, r]^2 Q[t, r]^2 + 5 Q[t, r]^4) α[t, r] A^(0,1)[t, r] +
    4 A[t, r]^5 (r α[t, r] Q^(0,1)[t, r] + Q[t, r] (2 α[t, r] + r α^(0,1)[t, r])) + 3 γ A[t, r]
     (P[t, r]^2 - Q[t, r]^2) (P[t, r]^2 (r α[t, r] Q^(0,1)[t, r] + Q[t, r] (2 α[t, r] - 3 r α^(0,1)[t, r])) -
      Q[t, r]^2 (5 r α[t, r] Q^(0,1)[t, r] + Q[t, r] (2 α[t, r] + r α^(0,1)[t, r])) +
      4 r P[t, r]^3 A^(1,0)[t, r] - 4 r P[t, r] Q[t, r]^2 A^(1,0)[t, r]) +
    4 β A[t, r]^3 (P[t, r]^2 (-r α[t, r] Q^(0,1)[t, r] + Q[t, r] (-2 α[t, r] + r α^(0,1)[t, r])) +
      Q[t, r]^2 (3 r α[t, r] Q^(0,1)[t, r] + Q[t, r] (2 α[t, r] + r α^(0,1)[t, r])) -
      2 r P[t, r]^3 A^(1,0)[t, r] + 2 r P[t, r] Q[t, r]^2 A^(1,0)[t, r]) -
    4 r A[t, r]^4 Q[t, r] (α[t, r] A^(0,1)[t, r] + 4 β P[t, r] ϕ0^(1,1)[t, r]) +
    4 r A[t, r]^2 Q[t, r] (β P[t, r]^2 α[t, r] A^(0,1)[t, r] - 3 β Q[t, r]^2 α[t, r] A^(0,1)[t, r] +
      6 γ P[t, r]^3 ϕ0^(1,1)[t, r] - 6 γ P[t, r] Q[t, r]^2 ϕ0^(1,1)[t, r])) / (r A[t, r]^2
    (4 A[t, r]^4 + 4 β A[t, r]^2 (-3 P[t, r]^2 + Q[t, r]^2) + 3 γ (5 P[t, r]^4 - 6 P[t, r]^2 Q[t, r]^2 + Q[t, r]^4)));

(*We want to separate wave equation part *)
```

In[ ]:=
```
FirstTerm = α[t, r] / A[t, r] Q^(0,1)[t, r] + 2 / r α[t, r] / A[t, r] × Q[t, r] +
   Q[t, r] ((α^(0,1)[t, r] A[t, r] - α[t, r] A^(0,1)[t, r]) / A[t, r]^2);
```

In[ ]:=
```
KessenceTerms = RHSterm - FirstTerm // FullSimplify
```

Out[ ]=
```
-((4 (2 β A[t, r]^2 + 3 γ (-P[t, r]^2 + Q[t, r]^2)) (r Q[t, r] (P[t, r]^2 + Q[t, r]^2) α[t, r] A^(0,1)[t, r] +
      A[t, r] (α[t, r] (-2 P[t, r]^2 Q[t, r] - r (P[t, r]^2 + Q[t, r]^2) Q^(0,1)[t, r]) -
         2 r P[t, r]^2 Q[t, r] α^(0,1)[t, r] + r P[t, r] (P[t, r]^2 - Q[t, r]^2) A^(1,0)[t, r]) +
      2 r A[t, r]^2 P[t, r] × Q[t, r] ϕ0^(1,1)[t, r])) / (r A[t, r]^2
    (4 A[t, r]^4 + 4 β A[t, r]^2 (-3 P[t, r]^2 + Q[t, r]^2) + 3 γ (5 P[t, r]^4 - 6 P[t, r]^2 Q[t, r]^2 + Q[t, r]^4))))
```

```
(*dA/dt  is obtained below in the notebooks usig Gtr = Ttr from EFEqs* )
```

In[ ]:=
```
alltransformKessence =
  KessenceTerms /. {A^(1,0)[t, r] → 1/(16 A[t, r]^4) r P[t, r] × Q[t, r] (4 A[t, r]^4 +
        3 γ (P[t, r]^2 - Q[t, r]^2)^2 + 4 β A[t, r]^2 (-P[t, r]^2 + Q[t, r]^2)) α[t, r],  ϕ0^(1,1)[t, r] ->
      (A[t, r] × α[t, r] P^(0,1)[t, r] + P[t, r] (-α[t, r] A^(0,1)[t, r] + A[t, r] α^(0,1)[t, r])) / A[t, r]^2} //
   FullSimplify ;
```

In[ ]:=
```
Collect[alltransformKessence , {β, γ}]
```

Out[ ]=
```
((2 β A[t, r]^2 + 3 γ (-P[t, r]^2 + Q[t, r]^2)) α[t, r]
   (Q[t, r] (4 r^2 β A[t, r]^2 P[t, r]^2 (P[t, r]^2 - Q[t, r]^2)^2 - 3 r^2 γ P[t, r]^2 (P[t, r]^2 - Q[t, r]^2)^3 +
        16 r A[t, r]^3 (P[t, r]^2 - Q[t, r]^2) A^(0,1)[t, r] +
        4 A[t, r]^4 P[t, r] (-r^2 P[t, r]^3 + P[t, r] (8 + r^2 Q[t, r]^2) - 8 r P^(0,1)[t, r])) +
     16 r A[t, r]^4 (P[t, r]^2 + Q[t, r]^2) Q^(0,1)[t, r])) / (4 r A[t, r]^5
   (4 A[t, r]^4 + 4 β A[t, r]^2 (-3 P[t, r]^2 + Q[t, r]^2) + 3 γ (5 P[t, r]^4 - 6 P[t, r]^2 Q[t, r]^2 + Q[t, r]^4)))
```

In[ ]:=
```
KessenceTerms /. {β -> 0, γ -> 0} // FullSimplify
 (*to check wave part is separated correctly*)
```

Out[ ]=
```
0
```

In[ ]:=
```
QuadKEssenceTerms = alltransformKessence /. γ -> 0 // FullSimplify
```

Out[ ]=
```
(β α[t, r] (Q[t, r] (r^2 β P[t, r]^2 (P[t, r]^2 - Q[t, r]^2)^2 + 4 r A[t, r] (P[t, r]^2 - Q[t, r]^2) A^(0,1)[t, r] +
        A[t, r]^2 P[t, r] (-r^2 P[t, r]^3 + P[t, r] (8 + r^2 Q[t, r]^2) - 8 r P^(0,1)[t, r])) +
     4 r A[t, r]^2 (P[t, r]^2 + Q[t, r]^2) Q^(0,1)[t, r])) / (2 r A[t, r]^3 (A[t, r]^2 + β (-3 P[t, r]^2 + Q[t, r]^2)))
```

In[ ]:=
```
(*Taken from Bernard et al paper Eq 40 kessence terms part*)
```

In[ ]:=
```
LauraQuadKessence  = 2 β / (A[t, r]^2 + β (-3 P[t, r]^2 + Q[t, r]^2))
   α[t, r] / A[t, r] ((P[t, r]^2 + Q[t, r]^2) Q^(0,1)[t, r] - 2 Q[t, r] × P[t, r] P^(0,1)[t, r] +
     (r / 4 P[t, r]^2 - A^(0,1)[t, r] / A[t, r]) (Q[t, r]^2 - P[t, r]^2) Q[t, r] + β r / (4 A[t, r]^2)
     (Q[t, r]^4 - 2 P[t, r]^2 Q[t, r]^2 + P[t, r]^4) Q[t, r] P[t, r]^2 + 2 / r Q[t, r] P[t, r]^2)
```

Out[ ]=

$$\left(2\,\beta\,\alpha[t,r]\left(\frac{2\,P[t,r]^2\,Q[t,r]}{r}+\frac{r\,\beta\,P[t,r]^2\,Q[t,r]\left(P[t,r]^4-2\,P[t,r]^2\,Q[t,r]^2+Q[t,r]^4\right)}{4\,A[t,r]^2}+Q[t,r]\left(-P[t,r]^2+Q[t,r]^2\right)\left(\frac{1}{4}\,r\,P[t,r]^2-\frac{A^{(0,1)}[t,r]}{A[t,r]}\right)-2\,P[t,r]\times Q[t,r]\,P^{(0,1)}[t,r]+\left(P[t,r]^2+Q[t,r]^2\right)Q^{(0,1)}[t,r]\right)\right)\Big/\left(A[t,r]\left(A[t,r]^2+\beta\left(-3\,P[t,r]^2+Q[t,r]^2\right)\right)\right)$$

In[ ]:=
```
QuadKEssenceTerms - LauraQuadKessence // FullSimplify
```

Out[ ]= 0

In[ ]:=
```
CubicKEssenceTerms = alltransformKessence /. {β -> 0} // Simplify
```

Out[ ]=

$$\left(3\,\gamma\left(-P[t,r]^2+Q[t,r]^2\right)\alpha[t,r]\left(Q[t,r]\left(-3\,r^2\,\gamma\,P[t,r]^2\left(P[t,r]^2-Q[t,r]^2\right)^3+16\,r\,A[t,r]^3\left(P[t,r]^2-Q[t,r]^2\right)A^{(0,1)}[t,r]-4\,A[t,r]^4\,P[t,r]\left(r^2\,P[t,r]^3-P[t,r]\left(8+r^2\,Q[t,r]^2\right)+8\,r\,P^{(0,1)}[t,r]\right)\right)+16\,r\,A[t,r]^4\left(P[t,r]^2+Q[t,r]^2\right)Q^{(0,1)}[t,r]\right)\right)\Big/\left(4\,r\,A[t,r]^5\left(4\,A[t,r]^4+3\,\gamma\left(5\,P[t,r]^4-6\,P[t,r]^2\,Q[t,r]^2+Q[t,r]^4\right)\right)\right)$$

```
myCubicKessence  = (3 γ (Q[t, r]^2 - P[t, r]^2) /
      (4 A[t, r]^4 + 3 γ (5 P[t, r]^4 - 6 P[t, r]^2 × Q[t, r]^2 + Q[t, r]^4)) (α[t, r] / A[t, r]))
   (4 (P[t, r]^2 + Q[t, r]^2) Q^(0,1)[t, r]  -  8 Q[t, r] × P[t, r] P^(0,1)[t, r]  +
     (r P[t, r]^2 - 4 A^(0,1)[t, r] / A[t, r]) (Q[t, r]^2 - P[t, r]^2) Q[t, r] +
     (3 r γ P[t, r]^2 Q[t, r] (-P[t, r]^2 + Q[t, r]^2)^3 / (4 A[t, r]^4)) + (8 P[t, r]^2 × Q[t, r]) / r  );
```

In[ ]:=
```
CubicKEssenceTerms - myCubicKessence // FullSimplify
```

Out[ ]= 0

Case Minkowski

General case

```
(*Constraint need for dA/dt in dPI/dt  or evolution*)
```

```
In[ ]:= Ett = Collect[Edd[[1, 1]] //. Union[sostϕt , sostϕr , D[sostϕt , t], D[sostϕr , r], D[sostϕr , t]],
          {D[B[t, r], r], η, r}, Simplify];
        Etr = Collect[Edd[[1, 2]] //. Union[sostϕt , sostϕr , D[sostϕt , t],
             D[sostϕr , r], D[sostϕr , t], sQt], {D[B[t, r], t], η, r}, Simplify];
        Err = Collect[Edd[[2, 2]] //. Union[sostϕt , sostϕr , D[sostϕt , t], D[sostϕr , r], D[sostϕr , t]],
          {D[A[t, r], r], η, r}, Simplify];
        Eθθ = Collect[Edd[[3, 3]] //. Union[sostϕt , sostϕr , D[sostϕt , t], D[sostϕr , r],
             D[sostϕr , t], sQt], {D[B[t, r], {t, 2}], D[Q[t, r], r], D[P[t, r], t],
           D[A[t, r], {r, 2}], D[B[t, r], t], D[A[t, r], t], D[A[t, r], r], η, r}, Simplify];

In[ ]:= Ealp = Collect[Err, {D[α[t, r], r], r}, Simplify];
        EA = Collect[Ett, {D[A[t, r], r], r}, Simplify];

In[ ]:= Solve[Ealp == 0, α^(0,1)[t, r]] // FullSimplify
```

Out[ ]=

$$\left\{\left\{\alpha^{(0,1)}[t,r] \to \frac{1}{32\, r\, A[t,r]^4}\Big(16\, A[t,r]^6 + r^2\, \gamma\, \left(P[t,r]^2 - Q[t,r]^2\right)^2 \left(P[t,r]^2 + 5\, Q[t,r]^2\right) - 2\, r^2\, \beta\, A[t,r]^2 \left(P[t,r]^4 + 2\, P[t,r]^2\, Q[t,r]^2 - 3\, Q[t,r]^4\right) + 4\, A[t,r]^4 \left(-4 + r^2 \left(P[t,r]^2 + Q[t,r]^2\right)\right)\Big)\, \alpha[t,r]\right\}\right\}$$

In[ ]:= AlpEquation =

$$\frac{1}{32\, r\, A[t,r]^4}\Big(16\, A[t,r]^6 + r^2\, \gamma \left(P[t,r]^2 - Q[t,r]^2\right)^2 \left(P[t,r]^2 + 5\, Q[t,r]^2\right) - 2\, r^2\, \beta\, A[t,r]^2 \left(P[t,r]^4 + 2\, P[t,r]^2\, Q[t,r]^2 - 3\, Q[t,r]^4\right) + 4\, A[t,r]^4 \left(-4 + r^2 \left(P[t,r]^2 + Q[t,r]^2\right)\right)\Big)\, \alpha[t,r];$$

```
In[ ]:= Collect[AlpEquation // FullSimplify , {β, γ}]
```

Out[ ]=

$$\frac{r\, \gamma \left(P[t,r]^2 - Q[t,r]^2\right)^2 \left(P[t,r]^2 + 5\, Q[t,r]^2\right) \alpha[t,r]}{32\, A[t,r]^4} - \frac{r\, \beta \left(P[t,r]^4 + 2\, P[t,r]^2\, Q[t,r]^2 - 3\, Q[t,r]^4\right) \alpha[t,r]}{16\, A[t,r]^2} + \frac{\left(16\, A[t,r]^6 + 4\, A[t,r]^4 \left(-4 + r^2 \left(P[t,r]^2 + Q[t,r]^2\right)\right)\right) \alpha[t,r]}{32\, r\, A[t,r]^4}$$

In[ ]:= myAlpEq = α[t, r] / (8 r) (4 (A[t, r]^2 - 1) + r ^ 2 (P[t, r]^2 + Q[t, r]^2)) -

$$\beta\, \frac{r\ \alpha[t,r]}{16\, A[t,r]^2} \left(\left(P[t,r]^2 + Q[t,r]^2\right) {}^\wedge 2 - 4\, Q[t,r]^4\right) + \gamma\, \frac{r\ \alpha[t,r]}{32\, A[t,r]^4} \left(\left(P[t,r]^2 - Q[t,r]^2\right) {}^\wedge 2 \left(P[t,r]^2 + 5\, Q[t,r]^2\right)\right);$$

```
In[ ]:= AlpEquation - myAlpEq // FullSimplify

Out[ ]= 0
```

In[ ]:=
```
QuadAlpConstraint = Collect[AlpEquation /. {γ -> 0} // FullSimplify , β]
```

Out[ ]=
$$-\frac{r\,\beta\left(P[t,r]^4+2\,P[t,r]^2\,Q[t,r]^2-3\,Q[t,r]^4\right)\alpha[t,r]}{16\,A[t,r]^2}+\frac{\left(8\,A[t,r]^4+2\,A[t,r]^2\left(-4+r^2\left(P[t,r]^2+Q[t,r]^2\right)\right)\right)\alpha[t,r]}{16\,r\,A[t,r]^2}$$

In[ ]:=
```
CubicAlpConstraint = Collect[AlpEquation /. {β -> 0} // FullSimplify , γ]
```

Out[ ]=
$$\frac{r\,\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2\left(P[t,r]^2+5\,Q[t,r]^2\right)\alpha[t,r]}{32\,A[t,r]^4}+\frac{\left(16\,A[t,r]^6+4\,A[t,r]^4\left(-4+r^2\left(P[t,r]^2+Q[t,r]^2\right)\right)\right)\alpha[t,r]}{32\,r\,A[t,r]^4}$$

In[ ]:=
```
Solve[EA == 0, A^(0,1)[t, r]] // FullSimplify
```

Out[ ]=
$$\Big\{\Big\{A^{(0,1)}[t,r]\to\frac{1}{32\,r\,A[t,r]^3}\Big(-16\,A[t,r]^6+r^2\,\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2\left(5\,P[t,r]^2+Q[t,r]^2\right)+2\,r^2\,\beta\,A[t,r]^2\left(-3\,P[t,r]^4+2\,P[t,r]^2\,Q[t,r]^2+Q[t,r]^4\right)+4\,A[t,r]^4\left(4+r^2\left(P[t,r]^2+Q[t,r]^2\right)\right)\Big)\Big\}\Big\}$$

In[ ]:=
```
AEquation =
  1/(32 r A[t, r]^3) (-16 A[t, r]^6 + r^2 γ (P[t, r]^2 - Q[t, r]^2)^2 (5 P[t, r]^2 + Q[t, r]^2) + 2 r^2 β A[t, r]^2
    (-3 P[t, r]^4 + 2 P[t, r]^2 Q[t, r]^2 + Q[t, r]^4) + 4 A[t, r]^4 (4 + r^2 (P[t, r]^2 + Q[t, r]^2)));
```

In[ ]:=
```
Collect[AEquation // FullSimplify , {β, γ}]
```

Out[ ]=
$$\frac{r\,\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2\left(5\,P[t,r]^2+Q[t,r]^2\right)}{32\,A[t,r]^3}+\frac{r\,\beta\left(-3\,P[t,r]^4+2\,P[t,r]^2\,Q[t,r]^2+Q[t,r]^4\right)}{16\,A[t,r]}+\frac{-16\,A[t,r]^6+4\,A[t,r]^4\left(4+r^2\left(P[t,r]^2+Q[t,r]^2\right)\right)}{32\,r\,A[t,r]^3}$$

In[ ]:=
```
myAEq = A[t, r] / (8 r) (4 (1 - A[t, r]^2 ) + r ^ 2 (P[t, r]^2 + Q[t, r]^2)) +
   β r/(16 A[t, r]) ((P[t, r]^2 + Q[t, r]^2) ^ 2 - 4 P[t, r]^4)  +
   γ r/(32 A[t, r]^3) ((P[t, r]^2 - Q[t, r]^2) ^ 2 (5 P[t, r]^2 +  Q[t, r]^2 ));
```

In[ ]:=
```
AEquation - myAEq // FullSimplify
```

Out[ ]= 0

In[ ]:= `QuadAConstraint = Collect[AEquation /. {γ -> 0} // FullSimplify , β]`

Out[ ]=
$$\frac{r\,\beta\left(-3\,P[t,r]^4+2\,P[t,r]^2\,Q[t,r]^2+Q[t,r]^4\right)}{16\,A[t,r]}+\frac{-8\,A[t,r]^4+2\,A[t,r]^2\left(4+r^2\left(P[t,r]^2+Q[t,r]^2\right)\right)}{16\,r\,A[t,r]}$$

In[ ]:= `CubicAConstraint = Collect[AEquation /. {β -> 0} // FullSimplify , γ]`

Out[ ]=
$$\frac{r\,\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2\left(5\,P[t,r]^2+Q[t,r]^2\right)}{32\,A[t,r]^3}+\frac{-16\,A[t,r]^6+4\,A[t,r]^4\left(4+r^2\left(P[t,r]^2+Q[t,r]^2\right)\right)}{32\,r\,A[t,r]^3}$$

In[ ]:= `EqdtA = Collect[Etr, {D[A[t, r], t], r}, Simplify]`

Out[ ]=
$$-\frac{P[t,r]\times Q[t,r]\left(4\,A[t,r]^4+3\,\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2+4\,\beta\,A[t,r]^2\left(-P[t,r]^2+Q[t,r]^2\right)\right)\alpha[t,r]}{8\,A[t,r]^5}+\frac{2\,A^{(1,0)}[t,r]}{r\,A[t,r]}$$

In[ ]:= `Solve[EqdtA == 0, A^(1,0)[t, r]] // FullSimplify`

Out[ ]=
$$\left\{\left\{A^{(1,0)}[t,r]\to\frac{r\,P[t,r]\times Q[t,r]\left(4\,A[t,r]^4+3\,\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2+4\,\beta\,A[t,r]^2\left(-P[t,r]^2+Q[t,r]^2\right)\right)\alpha[t,r]}{16\,A[t,r]^4}\right\}\right\}$$

In[ ]:= `dAdt = Collect[`
$$\frac{r\,P[t,r]\times Q[t,r]\left(4\,A[t,r]^4+3\,\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2+4\,\beta\,A[t,r]^2\left(-P[t,r]^2+Q[t,r]^2\right)\right)\alpha[t,r]}{16\,A[t,r]^4},$$
`{β, γ}]`

Out[ ]=
$$\frac{1}{4}\,r\,P[t,r]\times Q[t,r]\times\alpha[t,r]+\frac{3\,r\,\gamma\,P[t,r]\times Q[t,r]\left(P[t,r]^2-Q[t,r]^2\right)^2\alpha[t,r]}{16\,A[t,r]^4}+\frac{r\,\beta\,P[t,r]\times Q[t,r]\left(-P[t,r]^2+Q[t,r]^2\right)\alpha[t,r]}{4\,A[t,r]^2}$$

In[ ]:= `CubicdAdt = dAdt /. {β -> 0}`

Out[ ]=
$$\frac{1}{4}\,r\,P[t,r]\times Q[t,r]\times\alpha[t,r]+\frac{3\,r\,\gamma\,P[t,r]\times Q[t,r]\left(P[t,r]^2-Q[t,r]^2\right)^2\alpha[t,r]}{16\,A[t,r]^4}$$

In[ ]:= `QuaddAdt = dAdt /. {γ -> 0}`

Out[ ]=
$$\frac{1}{4}\,r\,P[t,r]\times Q[t,r]\times\alpha[t,r]+\frac{r\,\beta\,P[t,r]\times Q[t,r]\left(-P[t,r]^2+Q[t,r]^2\right)\alpha[t,r]}{4\,A[t,r]^2}$$

In[ ]:=
```
sostϕr
sostϕt
```

Out[ ]= $\{\phi 0^{(0,1)}[t, r] \to Q[t, r]\}$

Out[ ]= $\left\{\phi 0^{(1,0)}[t, r] \to \frac{P[t, r] \times \alpha[t, r]}{A[t, r]}\right\}$

In[ ]:=
```
Clear[ϕ0]
```

In[ ]:=
```
Amp = 1 / 100;
rc = 20;
var = 2;
alp0 = 1;
A0 = 1;
```

$$\phi in = Amp\ Exp\left[-\left(\frac{r - rc}{var}\right)^2\right]$$

```
P0 = A[t, r] / α[t, r] × D[ϕin, t]
Q0 = D[ϕin, r]
```

Out[ ]= $\frac{1}{100} e^{-\frac{1}{4}(-20+r)^2}$

Out[ ]= 0

Out[ ]= $\frac{1}{200} e^{-\frac{1}{4}(-20+r)^2} (20 - r)$

In[ ]:=

$$\phi in = Amp \left(\frac{r}{var}\right)^2 Exp\left[-\left(\frac{r - rc}{var}\right)^2\right]$$

```
P0 = A[t, r] / α[t, r] × D[ϕin, t]
```

$$Q0 = \frac{D[\phi in, r]}{\phi in} \ // \ Simplify$$

Out[ ]= $\frac{1}{400} e^{-\frac{1}{4}(-20+r)^2} r^2$

Out[ ]= 0

Out[ ]= $10 + \frac{2}{r} - \frac{r}{2}$

```
(*Initially  Alp, a*)
```

In[ ]:=
```
Ealp0 = Ealp //. {A[t, r] → A[r], α[t, r] → α[r],
    D[A[t, r], r] → A '[r], D[α[t, r], r] → α '[r], P[t, r] → P0, Q[t, r] → Q0}
EA0 = EA //. {A[t, r] → A[r], α[t, r] → α[r], D[A[t, r], r] → A '[r],
    D[α[t, r], r] → α '[r], P[t, r] → P0, Q[t, r] → Q0}
```

Out[ ]=
$$\frac{1-A[r]^2}{r^2}-\frac{\frac{e^{-\frac{3}{2}(-20+r)^2}(20-r)^6\gamma}{12\,800\,000\,000\,000}+\frac{3e^{-(-20+r)^2}(20-r)^4\beta A[r]^2}{800\,000\,000}+\frac{e^{-\frac{1}{2}(-20+r)^2}(20-r)^2A[r]^4}{10\,000}}{16A[r]^4}+\frac{2\alpha'[r]}{r\,\alpha[r]}$$

Out[ ]=
$$\frac{\left(1-\frac{1}{A[r]^2}\right)\alpha[r]^2}{r^2}-\frac{\left(\frac{e^{-\frac{3}{2}(-20+r)^2}(20-r)^6\gamma}{64\,000\,000\,000\,000}+\frac{e^{-(-20+r)^2}(20-r)^4\beta A[r]^2}{800\,000\,000}+\frac{e^{-\frac{1}{2}(-20+r)^2}(20-r)^2A[r]^4}{10\,000}\right)\alpha[r]^2}{16A[r]^6}+\frac{2\alpha[r]^2A'[r]}{r\,A[r]^3}$$

```
(*Effective inverse metric *)
```

In[ ]:=
```
gutt = -1 / α[t, r]^2
```

Out[ ]=
$$-\frac{1}{\alpha[t,r]^2}$$

In[ ]:=
```
gurr = 1 / A[t, r] ^2
```

Out[ ]=
$$\frac{1}{A[t,r]^2}$$

In[ ]:=
```
sostϕr
```

Out[ ]=
$$\left\{\phi 0^{(0,1)}[t,r]\to Q[t,r]\right\}$$

In[ ]:=
```
sostϕt
```

Out[ ]=
$$\left\{\phi 0^{(1,0)}[t,r]\to \frac{P[t,r]\times\alpha[t,r]}{A[t,r]}\right\}$$

In[ ]:=
```
Xv = 1 / (2 A[t, r]^2) (P[t, r]^2 - Q[t, r]^2)
```

Out[ ]=
$$\frac{P[t,r]^2-Q[t,r]^2}{2A[t,r]^2}$$

In[ ]:=
```
Yutt = gutt - (-2 β + 6 γ Xv) / (1 - 2 β Xv + 3 γ Xv^2) (gutt)^2 (ϕ0^(1,0)[t, r])^2 /.
    {ϕ0^(1,0)[t, r] -> α[t, r] / A[t, r] × P[t, r]} // FullSimplify
```

Out[ ]=
$$\frac{-1+\frac{4P[t,r]^2\left(2\beta A[t,r]^2+3\gamma\left(-P[t,r]^2+Q[t,r]^2\right)\right)}{4A[t,r]^4+3\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2+4\beta A[t,r]^2\left(-P[t,r]^2+Q[t,r]^2\right)}}{\alpha[t,r]^2}$$

In[ ]:=
```
Yurr =  gurr - (-2 β + 6 γ Xv) / (1 - 2 β Xv + 3 γ Xv^2) (gurr)^2 (ϕ0^(0,1)[t, r])^2 /.
    {ϕ0^(0,1)[t, r] -> Q[t, r]} // FullSimplify
```

Out[ ]=
$$\frac{1}{A[t,r]^2}+\frac{4Q[t,r]^2\left(2\beta A[t,r]^2+3\gamma\left(-P[t,r]^2+Q[t,r]^2\right)\right)}{4A[t,r]^6+3\gamma A[t,r]^2\left(P[t,r]^2-Q[t,r]^2\right)^2+4\beta A[t,r]^4\left(-P[t,r]^2+Q[t,r]^2\right)}$$

In[ ]:=
```
Yutr = 0 - (-2 β + 6 γ Xv) / (1 - 2 β Xv + 3 γ Xv ^ 2) gurr ϕ0^(0,1)[t, r] gutt ϕ0^(1,0)[t, r] /.
  {ϕ0^(0,1)[t, r] -> Q[t, r], ϕ0^(1,0)[t, r] -> α[t, r] / A[t, r] × P[t, r]}
```

Out[ ]=
$$\frac{P[t,r]\times Q[t,r]\left(-2\beta+\frac{3\gamma\left(P[t,r]^2-Q[t,r]^2\right)}{A[t,r]^2}\right)}{A[t,r]^3\left(1-\frac{\beta\left(P[t,r]^2-Q[t,r]^2\right)}{A[t,r]^2}+\frac{3\gamma\left(P[t,r]^2-Q[t,r]^2\right)^2}{4A[t,r]^4}\right)\alpha[t,r]}$$

In[ ]:=
```
Yutt /. γ -> 0
```

Out[ ]=
$$\frac{-1+\frac{8\beta A[t,r]^2 P[t,r]^2}{4A[t,r]^4+4\beta A[t,r]^2\left(-P[t,r]^2+Q[t,r]^2\right)}}{\alpha[t,r]^2}$$