Correct Mathematical Expressions and Numerical Implementation Details

1 Goal

Here are notes for obtaining mathematical expressions for Cubic and quadratic K-essence that are used for numerical studies for implicit scheme. I will start with action containing both, cubic and quadratic terms and than verify that ignoring cubic terms we recover equations in [1] and than follow same steps to get correct expressions for only cubic term case.

2 Mathematical Expressions

Horndeski's theories with nonlinear (quadratic and cubic) kinetic terms $\mathcal{G}(\phi, X) = -g_2 X^2 + g_3 X^3$, with $X = -1/2\nabla_a\phi\nabla^a\phi$, and all other functions, as well as the potential, set to zero for simplicity. This choice, similar to those adopted in [2], can be thought of as the two initial nonlinear terms in a Taylor expansion of the kinetic term in a k-essence theory,

$$S = \int d^4x\sqrt{-g}\left[R + X - g_2X^2 + g_3X^3\right] . \tag{1}$$

We will use spherically symmetric spacetime and study several cases using different initial conditions as well as the value of the quadratic and cubic couplings g_2, g_3. For simplicity we adopt Schwarzschild coordinates where the metric can be written as,

$$ds^2 = -\alpha^2dt^2 + a^2dr^2 + r^2d\Omega^2 . \tag{2}$$

Thus the only dynamical metric functions are the lapse function $\alpha(t, r)$ and $a(t, r)$.

We follow [3] and introduce standard first order variables as used in [3],

$$\Phi \equiv \phi' , \qquad \Pi \equiv \frac{a}{\alpha}\dot{\phi} , \tag{3}$$

to simplify the discussion and the numerical implementation

The action (1) will give us following equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \left[\frac{X + \mathcal{G}(X)}{2}\right] g_{\mu\nu} + \left[\frac{1 + \partial_X\mathcal{G}(X)}{2}\right]\nabla_\mu\phi\nabla_\nu\phi , \tag{4}$$

$$\left[g^{\mu\nu} - \frac{\partial^2_{XX}\mathcal{G}(X)}{1 + \partial_X\mathcal{G}(X)}\nabla^\mu\phi\nabla^\nu\phi\right]\nabla_\mu\nabla_\nu\phi = 0 \tag{5}$$

where the effective inverse metric $\gamma^{\mu\nu}$, is given by

$$\gamma^{\mu\nu} = g^{\mu\nu} - \frac{\partial^2_{XX}\mathcal{G}(X)}{1 + \partial_X\mathcal{G}(X)}\nabla^\mu\phi\nabla^\nu\phi . \tag{6}$$

In the first order variables (3) we can extract from the rr and tt components of equation (4), respectively, the first order constraint equations

$$\alpha' = \frac{\alpha}{8r}\left[4(a^2-1)+r^2(\Phi^2+\Pi^2)\right] - g_2\frac{r\alpha}{16a^2}\left[(\Phi^2+\Pi^2)^2-4\Phi^4\right]$$
$$+g_3\frac{r\,\alpha}{32a^4}\left[(\Pi^2-\Phi^2)^2(\Pi^2+5\Phi^2)\right] \tag{7}$$

$$a' = \frac{a}{8r}\left[4(1-a^2)+r^2(\Phi^2+\Pi^2)\right] + g_2\frac{r}{16a}\left[(\Phi^2+\Pi^2)^2-4\Pi^4\right]$$
$$+g_3\frac{r}{32a^3}\left[(\Pi^2-\Phi^2)^2(5\Pi^2+\Phi^2)\right]\ . \tag{8}$$

Equation (5), in terms of the first order variables, is given by

$$\dot{\Pi} = \frac{1}{r^2}\left(r^2\frac{\alpha}{a}\Phi\right)' + \frac{2g_2}{a^2+g_2\left(\Phi^2-3\Pi^2\right)}\frac{\alpha}{a}\Bigg[(\Phi^2+\Pi^2)\Phi' - 2\Phi\Pi\Pi'$$
$$+\left(\frac{r}{4}\Pi^2-\frac{a'}{a}\right)\left(\Phi^2-\Pi^2\right)\Phi + \frac{g_2 r}{4a^2}\left(\Phi^2-\Pi^2\right)^2\Phi\Pi^2 + \frac{2}{r}\Phi\Pi^2\Bigg]$$
$$+\frac{3g_3(\Phi^2-\Pi^2)}{4a^4+3g_3\left(5\Pi^4-6\Phi^2\Pi^2+\Phi^4\right)}\frac{\alpha}{a}\Bigg[4(\Phi^2+\Pi^2)\Phi' - 8\Phi\Pi\Pi'$$
$$+\left(r\Pi^2-\frac{4a'}{a}\right)(\Phi^2-\Pi^2)\Phi + \frac{3g_3 r}{4a^4}(\Phi^2-\Pi^2)^3\Phi\Pi^2 + \frac{8}{r}\Phi\Pi^2\Bigg]\ , \tag{9}$$

together with the condition that $\partial_t\partial_r\phi = \partial_r\partial_t\phi$, namely

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi\right)'\ . \tag{10}$$

The effective inverse metric from equation (6) reads

$$\gamma^{tt} = -\frac{1}{\alpha^2}\left(1-\frac{\Pi^2}{a^2}\frac{8g_2a^4+12g_3a^2(\Phi^2-\Pi^2)}{4a^4+4g_2a^2(\Phi^2-\Pi^2)+3g_3(\Phi^2-\Pi^2)^2}\right) \tag{11}$$

$$\gamma^{rr} = \frac{1}{a^2}\left(1+\frac{\Phi^2}{a^2}\frac{8g_2a^4+12g_3a^2(\Phi^2-\Pi^2)}{4a^4+4g_2a^2(\Phi^2-\Pi^2)+3g_3(\Phi^2-\Pi^2)^2}\right)\ , \tag{12}$$

$$\gamma^{tr} = -\frac{\Pi\Phi}{a^3\alpha}\left(\frac{8g_2a^4+12g_3a^2(\Phi^2-\Pi^2)}{4a^4+4g_2a^2(\Phi^2-\Pi^2)+3g_3(\Phi^2-\Pi^2)^2}\right)\ , \tag{13}$$

2.1 Wave part simplification

We will solve second order PDE so we convert Φ and Π in terms of derivatives of ϕ in evolution equation 9

$$\frac{d\Pi}{dt} = \frac{d}{dt}\left(\frac{a}{\alpha}\frac{d\phi}{dt}\right)$$

$$\frac{d\Pi}{dt} = \frac{a}{\alpha}\frac{d^2\phi}{dt^2} + \frac{d}{dt}\left(\frac{a}{\alpha}\right)\frac{d\phi}{dt}$$

and

$$\frac{1}{r^2}\left(r^2\frac{\alpha}{a}\Phi\right)' = \frac{\alpha}{a}\left(\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}\right) + \frac{d\phi}{dr}\frac{d(\alpha/a)}{dr}$$

Equation 9 with these factors with some adjustment becomes

$$\frac{\partial^2\phi(r,t)}{\partial t^2} = \frac{\alpha^2}{a^2}\left[\frac{\partial^2\phi(r,t)}{\partial r^2}+\frac{2}{r}\phi(r,t)\right] + \frac{\alpha}{a}\left[\frac{\partial\phi(r,t)}{\partial r}\frac{\partial}{\partial r}\left(\frac{\alpha}{a}\right) - \frac{\partial\phi(r,t)}{\partial t}\frac{\partial}{\partial t}\left(\frac{a}{\alpha}\right)\right] + \frac{\alpha}{a}G(r,t)$$

Where $G(r,t)$ are terms in 9 with g_2 and g_3.
We can further use $\Pi = \frac{a}{\alpha}\frac{d\phi}{dt}$

$$\frac{\partial^2\phi(r,t)}{\partial t^2} = \frac{\alpha^2}{a^2}\left[\frac{\partial^2\phi(r,t)}{\partial r^2}+\frac{2}{r}\phi(r,t)\right] - \frac{\alpha^2}{a^2}\Pi\frac{\partial}{\partial t}\left(\frac{a}{\alpha}\right) + \frac{\alpha}{a}\Phi\frac{\partial}{\partial r}\left(\frac{\alpha}{a}\right) + \frac{\alpha}{a}G(r,t)$$

References

[1] Laura Bernard, Luis Lehner, and Raimon Luna. Challenges to global solutions in Horndeski's theory. *Phys. Rev. D*, 100(2):024011, 2019.

[2] Ratindranath Akhoury, David Garfinkle, and Ryo Saotome. Gravitational collapse of k-essence. *JHEP*, 04:096, 2011.

[3] Matthew W. Choptuik. Universality and scaling in gravitational collapse of a massless scalar field. *Phys. Rev. Lett.*, 70:9–12, 1993.