using the leap frog method (see equation (4))

$$U_m^{n+1} = U_m^{n-1} - pU_m^n\,(U_{m+1}^n - U_{m-1}^n). \tag{52}$$

We assume that initial data is available for $n = 0$ and $n = 1$ and that $u(0, t) = 0$ leading to $U_0^n = 0$ $(n \geqslant 0)$. In a local, linearized stability analysis, the non-linear coefficient U_m^n is 'frozen' at some value $\bar{U}$ and, since the leap frog method is stable for linear problems provided the C.F.L. condition is satisfied, we obtain $0 < p|\bar{U}| \leqslant 1$, that is

$$0 < pU_m^n \leqslant 1. \tag{53}$$

This is then used as a guide to the stability of the scheme (52). To demonstrate the inadequacy of this result it is sufficient to show that the method is unstable for one particular choice of initial data. Following Fornberg (1973), choose

$$U_m^0 = (-1)^{m+1}\,\epsilon \sin(\tfrac{1}{3}m\pi) \qquad (m \geqslant 0),$$

and $U_m^1 = \alpha U_m^0$, where $\alpha > 1$ and ϵ (> 0) is sufficiently small so as not to violate the assumptions under which (52) was linearized. It can now be shown that there is a solution of (52) in the form

$$U_m^n = c_n\,(-1)^{m+1}\,\epsilon \sin \tfrac{1}{3}n\pi \tag{54}$$

provided c_n satisfies the recurrence relation

$$c_{n+1} - c_{n-1} = \gamma c_n^{\,2} \qquad (n \geqslant 1),$$

where $c_0 = 1$, $c_1 = \alpha$ and $\gamma = 2\epsilon p/\sqrt{3}$. Using an inductive argument, it can now be shown that

$$c_{2n} \geqslant (1+\gamma)^n \text{ and } c_{2n+1} \geqslant (1+\gamma)^n \qquad (n \geqslant 0),$$

and so $c_n \to \infty$ as $n \to \infty$. We conclude from (54) that the method is therefore unstable for every choice of p and ϵ (no matter how small).

Since $u\partial u/\partial x = \tfrac{1}{2}\partial(u^2)/\partial x$, equation (51) may be written in the alternative form

$$\frac{\partial u}{\partial t} + \tfrac{1}{3}u\frac{\partial u}{\partial x} + \tfrac{1}{3}\frac{\partial u^2}{\partial x} = 0 \tag{55}$$

for which the leap frog method becomes

$$U_m^{n+1} = U_m^{n-1} + \tfrac{1}{3}p\{U_m^n(U_{m+1}^n - U_{m-1}^n) + (U_{m+1}^n)^2 - (U_{m-1}^n)^2\}. \tag{56}$$

$$= U_m^{n-1} + \tfrac{1}{3}p(U_{m+1}^n + U_m^n + U_{m-1}^n)(U_{m+1}^n - U_{m-1}^n).$$

Equation (55) is known as the conservation form of (51) and its discrete form (56) is much less susceptible to the non-linear instability experienced by (52). This example demonstrates the care which has to be exercised in the approximation of non-linear terms in hyperbolic equations.

4.11 Second-order equations in one space dimension

A general second-order linear partial differential equation in two independent variables x, t can be written in the form

$$a\frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial^2 u}{\partial t \partial x} + c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial u}{\partial t} + e\frac{\partial u}{\partial x} + fu = g, \tag{57}$$

where the coefficients $a, b, c, \ldots, g$ are functions of the independent variables x and t. Equation (57) is said to be *hyperbolic* if

$$b^2 - ac > 0.$$

The simplest hyperbolic equation of this type is the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{58}$$

and this will now be studied in some detail to illustrate the type of initial and boundary conditions required to determine a unique solution of an hyperbolic equation. A solution of equation (58) can be written in the form

$$u = F(x + t) + G(x - t), \tag{59}$$

where F, G are arbitrary differential functions.

Now *Cauchy's initial value problem* consists of equation (58) together with the initial conditions $u(x, 0) = f(x)$ and $\partial u/\partial t(x, 0) = g(x)$, for $-\infty < x < +\infty$. It is easy to show that the solution of this problem is given by

$$u(x, t) = \tfrac{1}{2}[f(x + t) + f(x - t) + \int_{x-t}^{x+t} g(\xi)\mathrm{d}\xi]. \tag{60}$$

This is one of the few non-trivial initial or boundary value problems which can be solved in a comparatively simple manner.

It is worth interpreting equation (60) in the light of the diagram in figure 3. Here the value of the solution at the point (x_0, t_0) depends on the initial data on the part of the x-axis between A and B, the points where the lines $x \pm t$ = constant through P cut the x-axis. The lines $x \pm t$ = constant are called the *characteristic curves* of the wave equation and the triangle PAB is called the *domain of dependence* of the point (x_0, t_0). These concepts generalize to all hyperbolic problems involving two independent variables, and the method of characteristics (see Garabedian (1964),

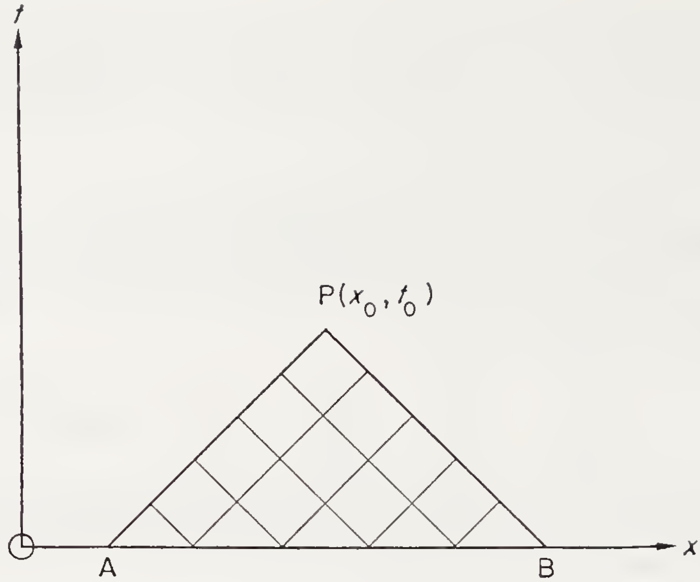

Figure 3

Jeffrey and Tanuiti (1964), Courant and Hilbert (1962), etc.) is undoubtedly the most effective method for solving hyperbolic problems in one space dimension. It might appear, therefore, that there is no role for finite difference methods to play in solving hyperbolic equations involving two independent variables. Nevertheless, it is necessary to consider finite difference methods for hyperbolic equations in one space dimension in order to instigate finite difference methods for hyperbolic equations in higher dimensions, where characteristic methods are less satisfactory.

Returning to the Cauchy problem involving the wave equation (58), the standard difference replacement of equation (58) on a rectangular grid is

$$\frac{1}{k^2}(U_m^{n+1} - 2U_m^n + U_m^{n-1}) - \frac{1}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) = 0, \tag{61}$$

where k and h are the grid sizes in the time and distance co-ordinates respectively, and the difference formula (61) refers to the grid point $x = mh$, $t = nk$, where m and n are integers. The above formula can be rewritten in the explicit form

$$U_m^{n+1} = 2(1 - p^2)U_m^n + p^2(U_{m+1}^n + U_{m-1}^n) - U_m^{n-1},$$

where $p = k/h$ is the mesh ratio. If this formula is expanded by Taylor's theorem about the grid point (mh, nk) the result

$$k^2\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right) + \tfrac{1}{12}k^2h^2(p^2 - 1)\frac{\partial^4 u}{\partial x^4}$$
$$+ \tfrac{1}{360}k^2h^4(p^4 - 1)\frac{\partial^6 u}{\partial x^6} + \ldots = 0$$

is obtained, and so the truncation error is

$$k^2 h^2\left[\tfrac{1}{12}(p^2-1)\frac{\partial^4 u}{\partial x^4}+\tfrac{1}{360}h^2(p^4-1)\frac{\partial^6 u}{\partial x^6}+\ldots\right].$$

The truncation error vanishes completely when $p = 1$, and so the difference formula

$$U_m^{n+1} = U_{m+1}^n + U_{m-1}^n - U_m^{n-1}$$

is an exact difference representation of the wave equation.

Assume that the functions f and g are prescribed on an interval $0 \leqslant x \leqslant 2Mh$ of the x-axis, where M is an integer. Then the initial conditions define U_m^0 and $U_m^1 (m = 0, 1, 2, \ldots, 2M)$ in the following manner:

$$\left.\begin{aligned} &U_m^0 = f(mh) \\ \text{and}\quad &U_m^1 = (1-p^2)f(mh) + \tfrac{1}{2}p^2[f(m+1)h + f(m-1)h] + kg(mh) \\ &\qquad\qquad (m = 0, 1, 2, \ldots, 2M). \end{aligned}\right\} \quad (62)$$

The latter formula is obtained from the Taylor expansion

$$\begin{aligned} u_m^1 &= u_m^0 + k\left(\frac{\partial u}{\partial t}\right)_m^0 + \tfrac{1}{2}k^2\left(\frac{\partial^2 u}{\partial t^2}\right)_m^0 + \ldots \\ &= f(mh) + kg(mh) + \tfrac{1}{2}k^2\left(\frac{\partial^2 u}{\partial x^2}\right)_m^0 + \ldots \\ &\approx f(mh) + kg(mh) + \tfrac{1}{2}p^2[f((m+1)h) + f((m-1)h) - 2f(mh)]. \end{aligned}$$

We now use formula (61) with $n = 1, 2, 3 \ldots$ to extend the initial values (62) to larger values of n. As illustrated in figure 4, the explicit calculation using formula (61) with $n = 1, 2, 3 \ldots$, together with values (62), comes to a halt at $n = N$ with the grid points, at which values of U have been calculated, lying inside a triangle. Hence the grid point Q, which is the apex of the triangle, has a *domain of dependence* on the x-axis which lies in the interval $0 \leqslant x \leqslant 2Mh$. This time the domain of dependence of the difference equation depends on the mesh ratio p and not on the characteristic curves through Q, as was the case with the differential equation. In this simple problem, it follows that three distinct cases arise for the difference problem:

(i) $0 < p < 1$. Here the domain of dependence for the difference equation includes that for the differential equation.

(ii) $p = 1$. The domains coincide, and the difference system takes the particularly simple form.

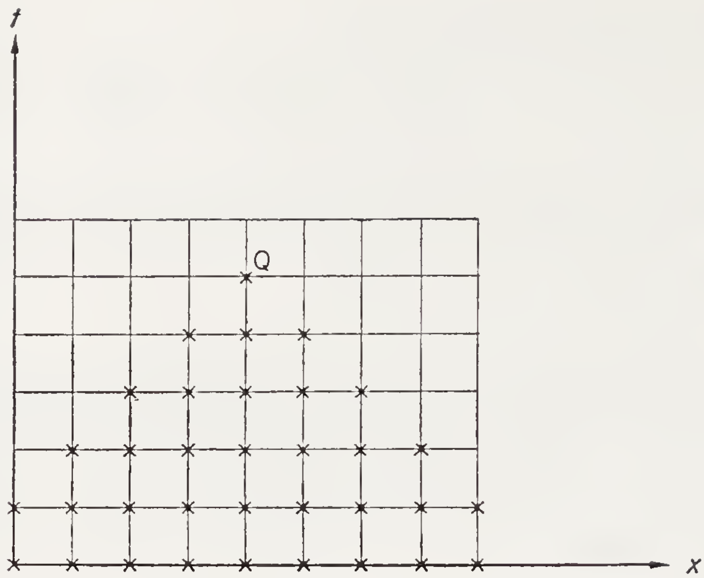

Figure 4

$$\left.\begin{aligned} U_m^0 &= f(mh) \\ U_m^1 &= \tfrac{1}{2}[f((m+1)h) + f((m-1)h] + kg(mh) \end{aligned}\right\} (m = 0,1,\ldots,2M)$$

$$U_m^{n+1} = U_{m+1}^n + U_{m-1}^n - U_m^{n-1} \quad (n = 1,2,\ldots,N).$$

(iii) $p > 1$. The domain of dependence for the difference equation lies inside that for the differential equation.

In each of the three cases, *provided p is kept constant,* the domain of dependence of the difference equation remains constant as $h, k \to 0$. The significance of the correspondence or otherwise of the domains of dependence of second-order hyperbolic equations and their associated explicit difference replacements is the substance of the Courant–Friedrichs–Lewy (C.F.L.) condition, which has already been discussed for first-order hyperbolic systems.

It is relatively easy to show that the domain of dependence for the difference equation must not lie inside that for the differential equation, i.e. $p \not> 1$. For if it does, it follows that the solution of the difference equation at a grid point $X = mh$, $T = nk$ is independent of the initial data of the problem which lies outside the domain of dependence of the difference equation but inside the domain of dependence of the differential equation. Alteration of this initial data will modify the solution of the differential equation, but leave the solution of the difference equation unaltered. This, of course, continues to apply when $h, k \to 0$ with p remaining constant and (X, T) remaining a *fixed* point. When $0 < p < 1$, thedomain of dependence of the difference equation exceeds that of the differential equation, and it can be shown (Courant, Friedrichs and Lewy, (1928)), that the contribution to the solution of the difference equation at the grid point $X = mh$, $T = nk$ due to the extra initial data outside the domain of dependence of the differential equation tends to zero as h, $k \to 0$, p remaining constant and (X, T) remaining a *fixed* point. Thus the C.F.L. condition states that *the domain of dependence of the difference equation must include the domain of dependence of the differential equation.*

Example 12,. Solve the difference equation

$$U_m^{n+1} = U_{m+1}^n + U_{m-1}^n - U_m^{n-1}$$

on a square grid $(k = h)$ *in the* (x, t) *plane.*

Figure 5

Consider the grid to be split up into two subgrids indicated by dots and crosses respectively as in figure 5(a). Since the difference equation connects the function values over each subgrid separately, we consider only the subgrid formed by the dots illustrated in figure 5(b). This is an arbitrary choice, and any result obtained can come equally from the subgrid formed by the crosses. Through a grid point P, the lines $x \pm t =$ constant meet the second row of dots at the points A and B respectively. The difference equation can be rewritten in the form

$$U_m^{n+1} - U_{m-1}^n = U_{m+1}^n - U_m^{n-1},$$

and if in figure 5 (b), we take $A \equiv (0, k)$, $B \equiv (8h, k)$ and $P \equiv (4h, 5k)$ where of course $k = h$, a simple calculation using the difference equation in this form gives

$$\begin{aligned} U_P - U_A &= (U_P - U_3^4) + (U_3^4 - U_2^3) + (U_2^3 - U_1^2) + (U_1^2 - U_A) \\ &= (U_B - U_7^0) + (U_6^1 - U_5^0) + (U_4^1 - U_3^0) + (U_2^1 - U_1^0), \end{aligned}$$

where the right-hand side of this expression contains nodal values on the first and second rows only. Now let $h \to 0$, and the prescribed values on the first and second rows converge to a function $f(x)$, while the difference quotients of the type $(1/\sqrt{2}h)(U_B - U_7^0)$, $(1/\sqrt{2}h)(U_6^1 - U_5^0)$, etc., converge to the function $\gamma_+(x)$. If P remains fixed at the point (x, t) as $h \to 0$, then the result

$$u_P = f(x - t) + \frac{1}{\sqrt{2}} \int_{x-t}^{x+t} \gamma_+(\xi) d\xi$$

is obtained. Alternatively, by writing the difference equation in the form

$$U_m^{n+1} - U_{m+1}^n = U_{m-1}^n - U_m^{n-1},$$

we can obtain the result

$$u_P = f(x+t) + \frac{1}{\sqrt{2}} \int_{x-t}^{x+t} \gamma_-(\xi)\mathrm{d}\xi\cdot$$

where

$$\frac{1}{\sqrt{2}} [\gamma_+(\xi) + \gamma_-(\xi)] = g(\xi).$$

Hence by averaging the results obtained for u_P, we get

$$u_P = \tfrac{1}{2}[f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(\xi)\mathrm{d}\xi],$$

which corresponds exactly with equation (60) the solution of the differential equation.

It is instructive now to look at the stability of the difference equation (61). Following von Neumann we look at solutions of (61) which have the form

$$U_m^n = \mathrm{e}^{\alpha nk}\,\mathrm{e}^{i\beta mh} \tag{63}$$

where β is real and α is complex. Substituting (63) into (61) leads to

$$\mathrm{e}^{2\alpha k} + 2(2p^2 \sin^2 \tfrac{1}{2}\beta h - 1)\mathrm{e}^{\alpha k} + 1 = 0$$

which gives

$$\mathrm{e}^{\alpha k} = 1 - 2p^2 \sin^2 \tfrac{1}{2}\beta h \pm 2p \sin \tfrac{1}{2}\beta h\, (p^2 \sin^2 \tfrac{1}{2}\beta h - 1)^{1/2}. \tag{64}$$

The von Neumann necessary condition for stability is

$$|\mathrm{e}^{\alpha k}| \leqslant 1,$$

and so we examine the two cases

(i) $p^2 \sin^2 \tfrac{1}{2}\beta h \leqslant 1$. This gives

$$\mathrm{e}^{\alpha k} = (1 - 2p^2 \sin^2 \tfrac{1}{2}\beta h) \pm i\, 2p \sin \tfrac{1}{2}\beta h\, (1 - p^2 \sin^2 \tfrac{1}{2}\beta h)^{1/2},$$

and so

$$|\mathrm{e}^{\alpha k}| = 1.$$

(ii) $p^2 \sin^2 \frac{1}{2}\beta h > 1$. This leads to

$$|e^{\alpha k}| > 1,$$

from (64), if we consider the root with the negative sign.

Thus from (i), the difference equation (61) is stable for all β if $p \leqslant 1$.

Exercise

10. By writing (61) as the two-level system

$$U_m^{n+1} = 2(1-p^2)U_m^n + p^2(U_{m+1}^2 + U_{m-1}^n) - V_m^n,$$

$$V_m^{n+1} = U_m^n,$$

obtain the amplification matrix, and hence show that (61) is stable if $p \leqslant 1$.

A more general hyperbolic equation in one space variable is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(a(x, t)\frac{\partial u}{\partial x}\right) + b(x, t)\frac{\partial u}{\partial x} + c(x, t)u, \tag{65}$$

where $a(x, t) > 0$. As in the section on parabolic equations, the self-adjoint second-order term $\partial/\partial x\,[a(x, t)\partial u/\partial x]$ is replaced by

$$A_{m+1}^n U_{m+1}^n - (A_{m+1}^n + A_m^n)U_m^n + A_m^n U_{m-1}^n,$$

where

$$A_m^n = \frac{1}{h}\left[\int_{x_{m-1}}^{x_m} \frac{dx}{a(x, nk)}\right]^{-1}.$$

This leads to the explicit difference replacement of equation (65) given by

$$U_m^{n+1} = k^2\left[\left(A_{m+1} + \frac{b_m^n}{2h}\right)U_{m+1}^n + \left(A_m - \frac{b_m^n}{2h}\right)U_{m-1}^n\right] +$$

$$[2 - k^2(A_{m+1} + A_m - c_m^n)]\,U_m^n - U_m^{n-1} + 0(k^4 + k^2h^2). \tag{66}$$

An *implicit* difference approximation to (58) is

$$[U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}] + 2[U_{m+1}^n - 2U_m^n + U_{m-1}^n] +$$

$$[U_{m+1}^{n-1} - 2U_m^{n-1} + U_{m-1}^{n-1}] = \frac{4}{p^2}[U_m^{n+1} - 2U_m^n + U_m^{n-1}],$$

which is stable for all $p > 0$. This is most conveniently written in the form

$$-(U_{m+1}^{n+1}+U_{m-1}^{n+1})+2\left(1+\frac{2}{p^2}\right)U_m^{n+1}=+2\left[(U_{m+1}^{n}+U_{m-1}^{n})-\right.$$

$$\left.2\left(1-\frac{2}{p^2}\right)U_m^n\right]+\left[(U_{m+1}^{n-1}+U_{m-1}^{n-1})-2\left(1+\frac{2}{p^2}\right)U_m^{n-1}\right], \tag{67}$$

and a method based on formula (67) requires the solution of a tridiagonal system at each time step. This is easily accomplished using the technique outlined in section 2.5 of chapter 2.

The consistency of a finite difference approximation to a hyperbolic equation is now mentioned briefly. A difference approximation to a hyperbolic equation is *consistent* if

$$\frac{\text{Truncation error}}{k^2} \to 0 \text{ as } h, k \to 0.$$

Exercise

11. Show that formula (67) is a consistent approximation to the wave equation.

4.12 Second-order equations in two space dimensions

An arbitrary homogeneous *linear* hyperbolic equation of second order in *two space variables* can be transformed into the differential equation

$$\frac{\partial^2 u}{\partial t^2}-\left(a\frac{\partial^2 u}{\partial x_1^2}+2b\frac{\partial^2 u}{\partial x_1\partial x_2}+c\frac{\partial^2 u}{\partial x_2^2}\right)$$

$$+d\frac{\partial u}{\partial t}+e\frac{\partial u}{\partial x_1}+f\frac{\partial u}{\partial x_2}+gu=0, \tag{68}$$

where the coefficients are functions of the independent variables x_1, x_2 and t and satisfy the conditions

$$a > 0, c > 0, \quad ac - b^2 > 0.$$

The most important example of equation (68) is the wave equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}=0, \tag{69}$$

and this will now be used to illustrate the connection between the domains of dependence of equation (69) and an explicit finite difference approximation to equation (69), with respect to the Cauchy initial value problem. It is conveneient to consider the latter to consist of equation (69) together with the initial conditions

$$\left.\begin{aligned} u(x_1, x_2, 0) &= f(x_1, x_2) \\ \text{and} \qquad & \\ \frac{\partial u}{\partial t}(x_1, x_2, 0) &= g(x_1, x_2) \end{aligned}\right\} \quad (-\infty < x_1, x_2 < +\infty).$$

The domain of dependence in the (x_1, x_2) plane of the differential equation (69) for a point $P(X_1, X_2, T)$ is the circle

$$(x_1 - X_1)^2 + (x_2 - X_2)^2 \leqslant T^2,$$

which is cut from the (x_1, x_2) plane by the circular cone with apex angle $\frac{1}{4}\pi$, vertex at P, and axis parallel to the t-axis. This cone is called the *characteristic cone* and plays a similar role to the characteristic lines $x \pm t =$ constant of the corresponding problem in one space dimension. The solution (60) for the latter problem also has a counterpart in two space dimensions, but the formula is now much more difficult (see Courant and Hilbert (1962), chapter 3, section 6.2).

The most natural difference approximation to equation (69) is

$$\frac{1}{k^2}(U_{l,m}^{n+1} - 2U_{l,m}^n + U_{l,m}^{n-1}) - \frac{1}{h^2}(U_{l+1,m}^n - 2U_{l,m}^n + U_{l-1,m}^n) -$$

$$\frac{1}{h^2}(U_{l,m+1}^n - 2U_{l,m}^n + U_{l,m-1}^n) = 0, \quad (70)$$

where k and h are the grid sizes in the time and distance co-ordinates respectively, and the difference formula (70) refers to the grid point

$$x_1 = lh, \quad x_2 = mh, \quad t = nk,$$

where l, m and n are integers. The above formula can be written in the explicit form

$$U_{l,m}^{n+1} = 2(1 - 2p^2)U_{l,m}^n +$$

$$p^2(U_{l+1,m}^n + U_{l-1,m}^n + U_{l,m+1}^n + U_{l,m-1}^n) - U_{l,m}^{n-1}, \quad (71)$$

where the mesh ratio $p = k/h$. If this formula is expanded by Taylor's theorem about the grid point (lh, mh, nk), it is easily shown that the principal part of the truncation error is given by

$$\tfrac{1}{12}k^2h^2\left[(p^2 - 1)\left(\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^4 u}{\partial x_2^4}\right) + 2p^2\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2}\right]. \quad (72)$$

Unlike the one space-dimensional case, there is no value of p which causes the truncation error to vanish identically and so there is no difference replacement of the form (71) which is an exact representation of the wave equation in two space dimensions.

The explicit difference formula (71) allows the value of the function u at a grid point $P(X_1, X_2, T)$ to be expressed uniquely in terms of the values of the function at certain points of two initial planes $t = 0, k$. These initial values are given by

$$U^0_{l,m} = f(lh, mh)$$

and

$$U^1_{l,m} = (1 - 2p^2)\, f(lh, mh) + \tfrac{1}{2}p^2 \,[f((l+1)h, mh) + f((l-1)h, mh) + f(lh, (m+1)h) + f(lh, (m-1)h)] + kg(lh, mh),$$

respectively. The grid points which influence the value of U at the point P lie inside a pyramid which cuts out from the two initial planes $t = 0, k$ two rhombuses as domains of dependence. If $h, k \to 0$, with p remaining constant, the grid points which influence the value of U at $P(X_1, X_2, T)$, *which remains fixed,* continue to lie inside the above pyramid, and the domain of dependence of P is a rhombus cut out on the (x_1, x_2) plane by this pyramid. The Courant–Friedrichs–Lewy condition for the difference scheme (71) to be convergent for all smooth initial data is that the rhombus of dependence of the difference scheme must contain the circle of dependence of the differential equation in its interior. There is no loss of generality if we consider the grid point P to lie on the t-axis and so $X_1 = X_2 = 0$. If $T = nk$, then the domain of dependence for the differential equation is given by $x_1^2 + x_2^2 \leqslant n^2k^2$, and so in figure 6, OQ$=nk$. Now the rhombus of dependence for the difference equation, shown in figure 6, has OR $=$ OS $= nh$, and so the C.F.L. condition is satisfied provided the rhombus includes the circle, and this is the case provided

$$nk \leqslant \frac{1}{\sqrt{2}}\, nh$$

or

$$p \leqslant \frac{1}{\sqrt{2}}. \tag{73}$$

This compares unfavourably with the C.F.L. condition of $p \leqslant 1$ for the one space-dimensional wave equation. In fact, it is easy to show that for the wave equation in s space dimensions, the C.F.L. condition for the explicit difference equation of the type (71) is

$$p \leqslant \frac{1}{\sqrt{s}},$$

a condition which grows more troublesome as the number of space dimensions increases.

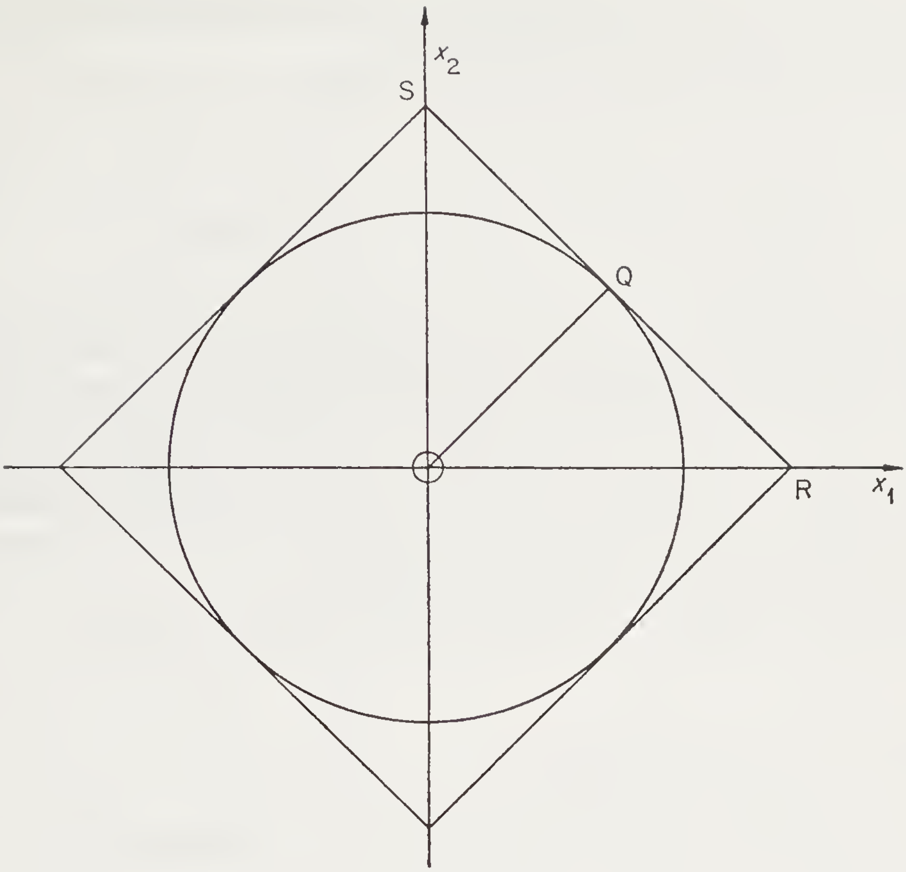

Figure 6

Condition (73) also guarantees the stability of formula (71). The proof of this is so similar to that given for the one space-dimensional case, that no details will be given.

Exercise

12. Using the standard nine point high accuracy difference replacement of

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2},$$

derive an explicit difference formula for the wave equation in two space dimensions. Determine the stability and C.F.L. conditions for this formula.

Once again it is possible to improve stability by considering implicit schemes. These implicit schemes will only be of value, however, if they can be used to calculate the solution in a relatively simple manner. In general, in two or more space dimensions this will not be the case, unless the difference operator at the most advanced time level can be factorized, and the difference equation rewritten as two or more simpler difference equations. This technique which was used with great success in parabolic equations can be employed with almost equal success in hyperbolic equations involving two or more space dimensions.

Consider an implicit difference formula in the form

$$[1 + a(\delta_{x1}^2 + \delta_{x2}^2) + d\delta_{x1}^2\delta_{x2}^2]\, U_{l,m}^{n+1} = [2 - b(\delta_{x_1}^2 + \delta_{x2}^2) -$$

$$e\delta_{x1}^2\delta_{x2}^2]\, U_{l,m}^{n} - [1 + c(\delta_{x1}^2 + \delta_{x2}^2) + f\delta_{x1}^2\delta_{x2}^2]\, U_{l,m}^{n-1}, \qquad (74)$$

where the coefficients a, b, c, d, e, f are to be chosen so that formula (74) is an adequate replacement of equation (69), and can be used in a comparatively simple manner to calculate the solution. An implicit formula like (74), whether factorized or not, cannot be used to solve a pure initial value problem. It is, however, a natural formula to use for the solution of an initial boundary value problem where u and $\partial u/\partial t$ are given at $t=0$ for $0 \leqslant x_1,\ x_2 \leqslant 1$, and u is given on the four side boundaries $x_1 = 0, 1$ $(0 \leqslant x_2 \leqslant 1)$ and $x_2 = 0, 1$ $(0 \leqslant x_1 \leqslant 1)$ for $t > 0$. This is the problem of a vibrating square membrane fixed round its perimeter.

The difference operator at the advanced time level can be factorized if

$$d = a^2,$$

and formula (74) can be rewritten as

$$(1 + a\delta_{x_1}^2)(1 + a\delta_{x_2}^2)U^{n+1} = [2 - b(\delta_{x_1}^2 + \delta_{x_2}^2) - e\delta_{x_1}^2\delta_{x_2}^2]\, U^n -$$

$$[1 + c(\delta_{x_1}^2 + \delta_{x_2}^2) + f\delta_{x_1}^2\delta_{x_1}^2]\, U^{n-1}, \qquad (75)$$

where U^s is written for $U_{l,m}^s$ $(s = n + 1, n, n - 1)$. The terms in formula (75) can be expanded in the Taylor series in terms of u and its derivatives at the grid point (lh, mh, nk), the even derivatives with respect to t being replaced by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \qquad \frac{\partial^4 u}{\partial t^4} = \frac{\partial^4 u}{\partial x_1^4} + 2\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u}{\partial x_2^4}, \text{ etc.,}$$

from equation (69). The expansions up to and including terms involving h^6 are

$$u_{l,m}^{n+1} - 2u_{l,m}^{n} + u_{l,m}^{n-1} = p^2 C_2 + \tfrac{1}{12}p^4 C_4 + \tfrac{1}{6}p^4 D_4 + \tfrac{1}{720}p^6 C_6 + \tfrac{1}{240}p^6 D_6,$$

$$(\delta_{x_1}^2 + \delta_{x_2}^2)u_{l,m}^{n\pm1} = C_2 \pm pC_3 + \tfrac{1}{2}(p^2 + \tfrac{1}{6})C_4 + p^2 D_4 \pm$$

$$\tfrac{1}{6}p(p^2 + \tfrac{1}{2})C_5 \pm \tfrac{1}{3}p^3 D^5 +$$

$$(\tfrac{1}{24}p^4 + \tfrac{1}{24}p^3 + \tfrac{1}{360})C_6 + \tfrac{1}{8}p^2(p^2 + \tfrac{1}{3})D_6,$$

$$\delta_{x_1}^2\delta_{x_2}^2 u_{l,m}^{n\pm1} = D_4 \pm pD_5 + \tfrac{1}{2}(p^2 + \tfrac{1}{180})D_6,$$

$$(\delta_{x_1}^2 + \delta_{x_2}^2)u_{l,m}^n = C_2 + \tfrac{1}{12}C_4 + \tfrac{1}{360}C_6,$$

$$\delta_{x_1}^2\delta_{x_2}^2 u_{l,m}^n = D_4 + \tfrac{1}{360}D_6,$$

where

$$C_2 = h^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right),$$

$$C_3 = h^3\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right),$$

$$C_4 = h^4\left(\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^4 u}{\partial x_2^4}\right),$$

$$C_5 = h^5\frac{\partial}{\partial t}\left(\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^4 u}{\partial x_2^4}\right),$$

$$C_6 = h^6\left(\frac{\partial^6 u}{\partial x_1^6} + \frac{\partial^6 u}{\partial x_2^6}\right),$$

$$D_4 = h^4\frac{\partial^4 u}{\partial x_1^2\partial x_2^2},$$

$$D_5 = h^5\frac{\partial^5 u}{\partial t\partial x_1^2\partial x_2^2},$$

$$D_6 = h^6\frac{\partial^4}{\partial x_1^2\partial x_2^2}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right).$$

Lees (1962) suggested formula (75) with

$$a = c = -\eta p^2,\quad b = -(1-2\eta)p^2,\quad e = -2\eta^2 p^4,\quad f = \eta^2 p^4,$$

where η is a parameter. Substitution of the above Taylor expansions leads to a principal truncation error of

$$-[\{(\eta - \tfrac{1}{12})p^4 + \tfrac{1}{12}p^2\}\,C_4 + 2(\eta - \tfrac{1}{12})p^4 D_4].$$

Fairweather and Mitchell (1965) suggested formula (75) with

$$a = c = \tfrac{1}{12}(1-p^2),\quad b = -\tfrac{1}{6}(1+5p^2),\quad e = -\tfrac{1}{72}(1 + 10p^2 + p^4),$$

$$f = \tfrac{1}{144}(1-p^2)^2.$$

This leads to a principal truncation error of

$$-\tfrac{1}{180}p^2[(p^4 - \tfrac{3}{4})C_6 + \tfrac{7}{4}(p^4 - \tfrac{29}{21})D_6].$$

and so is an order h^2 more accurate than the method of Lees. A formula of intermediate accuracy was obtained by Samarskii (1964a), where the coefficients were

$$a = c = \tfrac{1}{12}(1 - 6p^2), \quad b = -\tfrac{1}{6}, \quad e = -\tfrac{1}{72}(1 + 36p^4),$$

$$f = \tfrac{1}{144}(1 - 6p^2)^2.$$

Exercise

13. Show that, in the methods of Lees, Samarskii, and Fairweather and Mitchell, the principal truncation errors are

$$0(h^4 + k^4), \quad 0(h^6 + k^4), \text{ and } 0(h^6 + k^6)$$

respectively.

The methods mentioned above have

$$b = -(2a + p^2), \; c = a.$$

This guarantees an accuracy of at least $0(h^3 + k^3)$. Using these values, together with $f = a^2$, formula (75) can be rewritten in the form

$$(1 + a\delta_{x_1}^2)(1 + a\delta_{x_2}^2)[U^{n+1} + U^{n-1} - 2U^n] =$$

$$[p^2(\delta_{x_1}^2 + \delta_{x_2}^2) - (e + 2a^2)\delta_{x_1}^2\delta_{x_2}^2]U^n, \tag{76}$$

and so the methods of Lees, Samarskii and Fairweather and Mitchell become

$$(1 - \eta p^2\delta_{x_1}^2)(1 - \eta p^2\delta_{x_2}^2)(U^{n+1} + U^{n-1} - 2U^n) = p^2(\delta_{x_1}^2 + \delta_{x_2}^2)U^n, \tag{76a}$$

$$[1 + \tfrac{1}{12}(1 - 6p^2)\delta_{x_1}^2][1 + \tfrac{1}{12}(1 - 6p^2)\delta_{x_2}^2](U^{n+1} + U^{n-1} - 2U^n) =$$

$$p^2[(\delta_{x_1}^2 + \delta_{x_2}^2) + \tfrac{1}{6}\delta_{x_1}^2\delta_{x_2}^2]U^n \tag{76b}$$

and

$$[1 + \tfrac{1}{12}(1 - p^2)\delta_{x_1}^2][1 + \tfrac{1}{12}(1 - p^2)\delta_{x_2}^2](U^{n+1} + U^{n-1} - 2U^n) =$$

$$p^2[(\delta_{x_1}^2 + \delta_{x_2}^2) + \tfrac{1}{6}\delta_{x_1}^2\delta_{x_2}^2]U^n \tag{76c}$$

respectively. The stability of these formulae is discussed in the respective articles.

We now rewrite (76) in the form

$$(1 + a\delta_{x_1}^2)(1 + a\delta_{x_2}^2)(U^{n+1} + U^{n-1} - 2U^n) = p^2[\sum_i \delta_{x_i}^2 + \beta\delta_{x_1}^2\delta_{x_2}^2]U^n, \tag{77}$$

where

$$\beta = -\frac{1}{p^2}(e + 2a^2).$$

Various splittings of formula (77) will be given which ease the computational procedure. The most obvious, but least economical, splitting from the point of view of computation is that of D'Yakonov and is given by

$$\left.\begin{aligned} &(1 + a\delta_{x_1}^2)U^{n+1\,*} = p^2[\sum_i \delta_{x_i}^2 + \beta\delta_{x_1}^2\delta_{x_2}^2]U^n + \\ &\qquad\qquad (1 + a\delta_{x_1}^2)(1 + a\delta_{x_2}^2)(2U^n - U^{n-1}) \\ &\text{and} \\ &(1 + a\delta_{x_2}^2)U^{n+1} = U^{n+1\,*} \end{aligned}\right\} \tag{78}$$

where $U^{n+1\,*}$ is an intermediate value. This requires the computation of $\delta_{x_1}^2 U$, $\delta_{x_2}^2 U$, and $\delta_{x_1}^2\delta_{x_2}^2 U$ at each grid point of the net. A more economical splitting due to Lees is

$$\left.\begin{aligned} &(1 + a\delta_{x_1}^2)(U^{n+1\,*} - 2U^n + U^{n-1}) = p^2\left(\sum_i \delta_{x_i}^2 - \frac{\beta}{a}\delta_{x_2}^2\right)U^n \\ &\text{and} \\ &(1 + a\delta_{x_2}^2)U^{n+1} = U^{n+1\,*} + \left(2a + p^2\frac{\beta}{a}\right)\delta_{x_2}^2 U^n - a\delta_{x_2}^2 U^{n-1}, \end{aligned}\right\} \tag{79a}$$

which may be written in the simplified form

$$\left.\begin{aligned} &(1 + a\delta_{x_1}^2)U^{n+1\,*} = p^2\left(\sum_i \delta_{x_i}^2 - \frac{\beta}{a}\delta_{x_2}^2\right)U^n \\ &\text{and} \\ &(1 + a\delta_{x_2}^2)(U^{n+1} - 2U^n + U^{n-1}) = U^{n+1\,*} + p^2\frac{\beta}{a}\delta_{x_2}^2 U^n, \end{aligned}\right\} \tag{79b}$$

where $U^{n+1\,*}$ is a different intermediate value. Two other possible splittings are

$$\left.\begin{aligned} &(1 + a\delta_{x_1}^2)U^{n+1\,*} = p^2[\sum_i \delta_{x_i}^2 + \beta\delta_{x_1}^2\delta_{x_2}^2]U^n \\ &\text{and} \\ &(1 + a\delta_{x_2}^2)(U^{n+1} - 2U^n + U^{n-1}) = U^{n+1\,*}, \end{aligned}\right\} \tag{80}$$

and

$$(1+a\delta_{x_1}^2)U^{n+1\,*}=\frac{p^2}{a}[-1+(a-\beta)\delta_{x_2}^2]\,U^n$$

and (81)

$$(1+a\delta_{x_2}^2)(U^{n+1}-2U^n+U^{n-1})=U^{n+1\,*}+\frac{p^2}{a}(1+\beta\delta_{x_2}^2)U^n,$$

respectively. Further factorizations are given in Gourlay (1977) and Mitchell (1971).

Exercise

14. Show that the elimination of $U^{n+1\,*}$ in equations (79b), (80) and (81) respectively recovers formula (77).

Example 13. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2},$$

together with the initial conditions

$$u=\sin \pi x_1 \sin \pi x_2,$$

$$\frac{\partial u}{\partial t}=0,$$

for $0\leqslant x_1, x_2\leqslant 1, t=0$, *and the boundary condition*

$$u=0,$$

on the boundary of the unit square for $t>0$. *The theoretical solution of this problem is*

$$u=\sin \pi x_1 \sin \pi x_2 \cos 2\pi t,$$

within the region $0\leqslant x_1, x_2\leqslant 1, t>0$. *Use the split method of Lees given by the equations of (79a) (or 79b) with* $a=-\eta p^2$ *and* $\beta=0$ *to obtain a numerical solution of the above problem.*

Table 4

$t \backslash \eta$	¼	½	1
0.3	− 0.009353	− 0.018065	− 0.034928
0.6	− 0.010148	− 0.020010	− 0.040224
0.9	+ 0.025128	+ 0.047913	+ 0.090206
1.2	+ 0.037914	+ 0.074418	+ 0.148135
1.5	− 0.019930	− 0.035643	− 0.057948
1.8	− 0.069506	− 0.134848	− 0.261908
2.1	− 0.011155	− 0.027693	− 0.076422
2.4	+ 0.087278	+ 0.165541	+ 0.306223
2.7	+ 0.062022	+ 0.128138	+ 0.276733
3.0	− 0.076253	− 0.137287	− 0.224734