

# Correct Mathematical Expressions and Numerical Implementation Details

## 1 Goal

Here are notes for obtaining mathematical expressions for Cubic and quadratic K-essence that are used for numerical studies for implicit scheme. I will start with action containing both, cubic and quadratic terms and then verify that ignoring cubic terms we recover equations in [1] and then follow same steps to get correct expressions for only cubic term case.

## 2 Mathematical Expressions

Horndeski's theories with nonlinear (quadratic and cubic ) kinetic terms  $\mathcal{G}(\phi, X) = -g_2 X^2 + g_3 X^3$ , with  $X = -1/2 \nabla_a \phi \nabla^a \phi$ , and all other functions, as well as the potential, set to zero for simplicity. This choice, similar to those adopted in [2], can be thought of as the two initial nonlinear terms in a Taylor expansion of the kinetic term in a  $k$ -essence theory,

$$S = \int d^4x \sqrt{-g} [R + X - g_2 X^2 + g_3 X^3] . \quad (1)$$

We will use spherically symmetric spacetime and study several cases using different initial conditions as well as the value of the quadratic and cubic couplings  $g_2, g_3$ . For simplicity we adopt Schwarzschild coordinates where the metric can be written as,

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2 . \quad (2)$$

Thus the only dynamical metric functions are the lapse function  $\alpha(t, r)$  and  $a(t, r)$ .

We follow [3] and introduce standard first order variables as used in [3],

$$\Phi \equiv \phi' , \quad \Pi \equiv \frac{a}{\alpha} \dot{\phi} , \quad (3)$$

to simplify the discussion and the numerical implementation

The action (1) will give us following equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \left[ \frac{X + \mathcal{G}(X)}{2} \right] g_{\mu\nu} + \left[ \frac{1 + \partial_X \mathcal{G}(X)}{2} \right] \nabla_\mu \phi \nabla_\nu \phi , \quad (4)$$

$$\left[ g^{\mu\nu} - \frac{\partial_{XX}^2 \mathcal{G}(X)}{1 + \partial_X \mathcal{G}(X)} \nabla^\mu \phi \nabla^\nu \phi \right] \nabla_\mu \nabla_\nu \phi = 0 \quad (5)$$

where the effective inverse metric  $\gamma^{\mu\nu}$ , is given by

$$\gamma^{\mu\nu} = g^{\mu\nu} - \frac{\partial_{XX}^2 \mathcal{G}(X)}{1 + \partial_X \mathcal{G}(X)} \nabla^\mu \phi \nabla^\nu \phi . \quad (6)$$

In the first order variables (3) we can extract from the  $rr$  and  $tt$  components of equation (4), respectively, the first order constraint equations

$$\alpha' = \frac{\alpha}{8r} [4(a^2 - 1) + r^2(\Phi^2 + \Pi^2)] - g_2 \frac{r\alpha}{16a^2} [(\Phi^2 + \Pi^2)^2 - 4\Phi^4] + g_3 \frac{r\alpha}{32a^4} [(\Pi^2 - \Phi^2)^2(\Pi^2 + 5\Phi^2)] \quad (7)$$

$$a' = \frac{a}{8r} [4(1 - a^2) + r^2(\Phi^2 + \Pi^2)] + g_2 \frac{r}{16a} [(\Phi^2 + \Pi^2)^2 - 4\Pi^4] + g_3 \frac{r}{32a^3} [(\Pi^2 - \Phi^2)^2(5\Pi^2 + \Phi^2)] . \quad (8)$$

Equation (5), in terms of the first order variables, is given by

$$\begin{aligned} \dot{\Pi} = & \frac{1}{r^2} \left( r^2 \frac{\alpha}{a} \Phi \right)' + \frac{2g_2}{a^2 + g_2(\Phi^2 - 3\Pi^2)} \frac{\alpha}{a} \left[ (\Phi^2 + \Pi^2)\Phi' - 2\Phi\Pi\Pi' \right. \\ & + \left( \frac{r}{4}\Pi^2 - \frac{a'}{a} \right) (\Phi^2 - \Pi^2)\Phi + \frac{g_2 r}{4a^2} (\Phi^2 - \Pi^2)^2 \Phi\Pi^2 + \frac{2}{r}\Phi\Pi^2 \Big] \\ & + \frac{3g_3(\Phi^2 - \Pi^2)}{4a^4 + 3g_3(5\Pi^4 - 6\Phi^2\Pi^2 + \Phi^4)} \frac{\alpha}{a} \left[ 4(\Phi^2 + \Pi^2)\Phi' - 8\Phi\Pi\Pi' \right. \\ & + \left( r\Pi^2 - \frac{4a'}{a} \right) (\Phi^2 - \Pi^2)\Phi + \frac{3g_3 r}{4a^4} (\Phi^2 - \Pi^2)^3 \Phi\Pi^2 + \frac{8}{r}\Phi\Pi^2 \Big], \end{aligned} \quad (9)$$

together with the condition that  $\partial_t \partial_r \phi = \partial_r \partial_t \phi$ , namely

$$\dot{\Phi} = \left( \frac{\alpha}{a} \Pi \right)' . \quad (10)$$

The effective inverse metric from equation (6) reads

$$\gamma^{tt} = -\frac{1}{\alpha^2} \left( 1 - \frac{\Pi^2}{a^2} \frac{8g_2 a^4 + 12g_3 a^2(\Phi^2 - \Pi^2)}{4a^4 + 4g_2 a^2(\Phi^2 - \Pi^2) + 3g_3(\Phi^2 - \Pi^2)^2} \right) \quad (11)$$

$$\gamma^{rr} = \frac{1}{a^2} \left( 1 + \frac{\Phi^2}{a^2} \frac{8g_2 a^4 + 12g_3 a^2(\Phi^2 - \Pi^2)}{4a^4 + 4g_2 a^2(\Phi^2 - \Pi^2) + 3g_3(\Phi^2 - \Pi^2)^2} \right) , \quad (12)$$

$$\gamma^{tr} = -\frac{\Pi\Phi}{a^3\alpha} \left( \frac{8g_2 a^4 + 12g_3 a^2(\Phi^2 - \Pi^2)}{4a^4 + 4g_2 a^2(\Phi^2 - \Pi^2) + 3g_3(\Phi^2 - \Pi^2)^2} \right) , \quad (13)$$

## References

- [1] Laura Bernard, Luis Lehner, and Raimon Luna. Challenges to global solutions in Horndeski's theory. *Phys. Rev. D*, 100(2):024011, 2019.
- [2] Ratindranath Akhoury, David Garfinkle, and Ryo Saotome. Gravitational collapse of k-essence. *JHEP*, 04:096, 2011.
- [3] Matthew W. Choptuik. Universality and scaling in gravitational collapse of a massless scalar field. *Phys. Rev. Lett.*, 70:9–12, 1993.