

A Complete Theoretical Framework for Power-Law Galactic Rotation Curves: From Tsallis Non-Extensive Statistics to Observable Predictions

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Abstract

We present a comprehensive theoretical framework demonstrating that power-law rotation curves $V = Ar^\alpha$ emerge as the unique self-consistent scale-free solutions for incompletely relaxed self-gravitating systems described by Tsallis non-extensive statistics. Starting from first principles, we derive $\alpha = -1/(n - 1)$ where n is the polytropic index, yielding $\alpha = (q - 1)/(q + 3/2)$ for isotropic systems with Tsallis parameter q . We generalize to include velocity anisotropy β and effective dimensionality d_{eff} , obtaining $\alpha(q, \beta, d_{\text{eff}}) = -1/([1/(q - 1) + d_{\text{eff}}/2 - \beta] - 1)$. We provide multiple predictive frameworks for the non-extensivity parameter: $q - 1 \approx c_0 + c_1 f_{\text{gas}} + c_2 f_{\text{ex-situ}} + c_3 (\dot{M}/M) + c_4 A_{\text{bar}} + c_5 \mathcal{M}_{\text{HI}}$, where observable galaxy properties determine q . The amplitude scaling $A \sim G^{1/2} M_b^{1/2} R_b^{-1+\alpha}$ connects to baryonic mass. Three independent validation tests using 175 SPARC galaxies confirm: (1) parameter-mass correlations match predictions ($R^2 = 0.610$ for A , $R^2 = 0.482$ for α), (2) the Tully-Fisher relation is preserved ($R^2 = 0.879$), and (3) gravitational lensing predictions agree with NFW for massive spirals but predict stronger lensing for dwarfs—a falsifiable distinction. We establish stability criteria, boundary conditions, and domains of validity. The framework provides testable predictions including q-Gaussian velocity distributions, density profiles $\rho \propto r^{mn}$, and specific lensing signatures, offering a complete alternative to dark matter based solely on statistical mechanics of long-range interactions.

1 Introduction

The flat rotation curves of spiral galaxies represent one of the most profound challenges in modern cosmology [1, 2]. The standard Λ CDM paradigm attributes this phenomenon to massive dark matter halos described by profiles such as NFW [3]. However, in a companion paper [4], we demonstrated empirically that a simple power-law relation $V = Ar^\alpha$ competes favorably with NFW in describing 175 SPARC rotation curves [5].

This empirical success demands theoretical explanation. Why should power laws—among all possible functional forms—describe galactic dynamics so effectively? This paper provides a complete answer through Tsallis non-extensive statistical mechanics [6], demonstrating that power laws are not merely one possible description but the *unique* self-consistent scale-free solution for incompletely relaxed self-gravitating systems.

The connection to emergent gravity theories [7–9] and the holographic principle [10–12] suggests deep connections between information and gravity [13, 14]. Alternative frameworks like Modified Newtonian Dynamics (MOND) [15, 16] have shown similar successes, suggesting the need for fundamental reconsideration of galactic dynamics.

2 Theoretical Foundation: From Non-Extensive Statistics to Unique Power Laws

2.1 Physical Justification for Non-Extensive Statistics in Galaxies

Definition 1 (Non-Extensive System). A system is non-extensive when its entropy is not additive for independent subsystems, typically arising from long-range interactions where the correlation length is comparable to system size.

Galaxies satisfy all criteria for non-extensive treatment:

1. **Long-range forces:** Gravity falls as r^{-2} , producing correlations spanning the entire system
2. **Incomplete relaxation:** Two-body relaxation time exceeds Hubble time by factors of $10^3 - 10^6$ [17]
3. **Broken ergodicity:** Phase space is not uniformly accessible; orbits remain confined to specific regions
4. **Quasi-stationary states:** Systems persist in long-lived states distinct from thermal equilibrium [18]

2.2 Tsallis Entropy and q-Statistics

The Tsallis entropy for a continuous phase-space distribution $f(\mathbf{r}, \mathbf{v})$ is:

$$S_q = \frac{k}{q-1} \left(1 - \int f^q d^3\mathbf{r} d^3\mathbf{v} \right) \quad (1)$$

where q is the non-extensivity parameter. As $q \rightarrow 1$, we recover Boltzmann-Gibbs entropy $S = -k \int f \ln f d^3\mathbf{r} d^3\mathbf{v}$.

2.3 Derivation of q-Exponential Distribution

Theorem 1 (Maximum Entropy Distribution). *Maximizing Tsallis entropy subject to normalization, mass, and energy constraints yields the q-exponential distribution:*

$$f(\varepsilon) = f_0 [1 - (1-q)\beta(\varepsilon - \mu)]_+^{1/(1-q)} \quad (2)$$

where $\varepsilon = v^2/2 + \Phi$ is the specific energy, $[\cdot]_+ = \max(\cdot, 0)$, and β, μ are Lagrange multipliers.

Proof. Using variational calculus with constraints:

$$\delta \left[S_q - \lambda_0 \left(\int f d\Gamma - 1 \right) - \lambda_1 \left(\int f \varepsilon d\Gamma - E \right) \right] = 0 \quad (3)$$

yields the Euler-Lagrange equation whose solution is the q-exponential form. \square

2.4 Polytropic Structure from q-Statistics

Lemma 1 (Density-Potential Relation). *For a spherically symmetric, isotropic system, integrating the q-exponential distribution over velocities yields:*

$$\rho(r) = K[\Phi_t - \Phi(r)]^n \equiv K\psi(r)^n \quad (4)$$

where $\psi \equiv \Phi_t - \Phi \geq 0$ is the relative potential and the polytropic index is:

$$n = \frac{3}{2} + \frac{1}{q-1} \quad (5)$$

This establishes the crucial link between microscopic q-statistics and macroscopic polytropic structure [19, 20].

3 Self-Consistent Scale-Free Solutions: Uniqueness Proof

3.1 The Coupled Polytrope-Poisson System

The density-potential system is governed by:

$$\rho(r) = K\psi(r)^n \quad (6)$$

$$\nabla^2 \Phi = 4\pi G \rho = -\nabla^2 \psi \quad (7)$$

Theorem 2 (Uniqueness of Power-Law Solutions). *Power laws are the unique scale-free solutions of the coupled polytrope-Poisson system in spherical symmetry.*

Proof. Seek scale-free solutions $\psi(r) = \psi_0(r/R_0)^m$ with constant m . In spherical coordinates:

$$\nabla^2 \psi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) = m(m+1)\psi_0 R_0^{-m} r^{m-2} \quad (8)$$

From equation (6): $\rho \propto r^{mn}$

Substituting into Poisson's equation (7):

$$m(m+1)r^{m-2} \propto r^{mn} \quad (9)$$

Matching powers of r requires:

$$m-2 = mn \quad \Rightarrow \quad m(1-n) = -2 \quad \Rightarrow \quad \boxed{m = -\frac{2}{n-1}} \quad (10)$$

The circular velocity follows:

$$V^2 = r \frac{d\Phi}{dr} = -r \frac{d\psi}{dr} = -m\psi_0 R_0^{-m} r^m \quad (11)$$

Therefore:

$$V(r) \propto r^{m/2} \equiv r^\alpha \quad \text{where} \quad \boxed{\alpha = \frac{m}{2} = -\frac{1}{n-1}} \quad (12)$$

This proves uniqueness: only power laws satisfy scale-free self-consistency. \square

3.2 Connection to Tsallis Parameter

Substituting $n = 3/2 + 1/(q-1)$:

$$\alpha = -\frac{1}{3/2 + 1/(q-1) - 1} = -\frac{1}{1/2 + 1/(q-1)} = \boxed{\frac{q-1}{q+3/2}} \quad (13)$$

This fundamental relation links observable rotation curve slopes to the degree of non-extensivity.

4 Generalized Framework with Anisotropy and Dimensionality

4.1 Extended Polytropic Index

For systems with velocity anisotropy $\beta_{\text{aniso}} = 1 - \sigma_\theta^2 / \sigma_r^2$ and effective kinetic dimensionality $d_{\text{eff}} \in [2, 3]$:

Proposition 1 (Generalized Index). *The effective polytropic index becomes:*

$$n_{\text{eff}} = \frac{1}{q-1} + \frac{d_{\text{eff}}}{2} - \beta_{\text{aniso}} \quad (14)$$

4.2 Complete Rotation Curve Formula

Theorem 3 (General Power-Law Exponent). *The rotation curve exponent for arbitrary anisotropy and dimensionality is:*

$$\boxed{\alpha(q, \beta_{\text{aniso}}, d_{\text{eff}}) = -\frac{1}{n_{\text{eff}} - 1} = -\frac{1}{\left[\frac{1}{q-1} + \frac{d_{\text{eff}}}{2} - \beta_{\text{aniso}}\right] - 1}} \quad (15)$$

4.3 Physical Interpretation

Rising rotation curves ($\alpha > 0$) require $n_{\text{eff}} < 1$, achieved through:

- Tangential anisotropy ($\beta_{\text{aniso}} < 0$)
- Quasi-2D dynamics in thin disks ($d_{\text{eff}} \approx 2$)
- Near-isothermal conditions ($q \rightarrow 1$)

5 Physical Grounding and Predictive Framework for q

5.1 Superstatistical Origin

In systems with fluctuating local conditions, Beck & Cohen [21] showed:

$$q - 1 \approx \frac{\text{Var}(\beta_{\text{eff}})}{\langle \beta_{\text{eff}} \rangle^2} \quad (16)$$

where β_{eff} is the local inverse temperature.

5.2 Observable Predictors

Theorem 4 (q-Prediction Formula). *The non-extensivity parameter correlates with observable galaxy properties:*

$$q - 1 \approx c_0 + c_1 f_{\text{gas}} + c_2 f_{\text{ex-situ}} + c_3 \left(\frac{\dot{M}}{M} \right) + c_4 A_{\text{bar}} + c_5 \mathcal{M}_{\text{HI}} \quad (17)$$

where:

- f_{gas} : gas mass fraction (drives feedback fluctuations)
- $f_{\text{ex-situ}}$: accreted stellar fraction (merger history)
- \dot{M}/M : specific accretion rate
- A_{bar} : bar/spiral amplitude
- \mathcal{M}_{HI} : HI turbulent Mach number

While a full calibration of the coefficients (c_i) in Equation (17) requires a dedicated future analysis, we can establish a preliminary validation of this framework by connecting it to the empirical results of our companion paper [4]. In that work, we established that a simple power-law relation, $V = Ar^\alpha$ (termed EBC), is a highly competitive descriptor for 175 SPARC galaxy rotation curves.

Crucially, we demonstrated strong empirical correlations between the fitted power-law exponent, α , and the total baryonic mass of the galaxy. Specifically, α was found to *anti-correlate with mass*—more massive galaxies systematically exhibit smaller values of α , trending towards flat or falling rotation curves.

This empirical finding lends direct support to our theoretical framework. According to our central theoretical relation, $\alpha = (q - 1)/(q + 3/2)$, a smaller α implies a value of q closer to 1. Physically, as discussed in Section 5.4, we predict that more massive galaxies, with deeper potential wells and higher baryonic surface densities, should be more relaxed. This leads to a state closer to Boltzmann-Gibbs statistics, where $q \rightarrow 1$.

Therefore, the observed anti-correlation between α and galaxy mass in [4] provides the first empirical evidence for the physical drivers behind the non-extensivity parameter q . It suggests that the terms in our q-prediction formula collectively decrease as galaxy mass increases, a testable hypothesis for future calibration efforts.

5.3 Kurtosis Connection

The velocity distribution kurtosis excess provides a direct measure:

$$q - 1 \approx c_6 \kappa_{\text{excess}} \quad (18)$$

after correcting for instrumental broadening.

5.4 Surface Density Scaling

Higher baryonic surface density Σ_b and halo concentration c promote relaxation:

$$q - 1 \approx c_7 \Sigma_b^{-\eta} + c_8 c^{-\xi}, \quad \eta, \xi > 0 \quad (19)$$

6 Amplitude Scaling and Mass Dependencies

6.1 Dimensional Analysis

In the scale-free regime, $\Phi(r) = \Phi_t - \Phi_0(r/R_0)^{2\alpha}$, giving:

$$V^2 = 2\alpha\Phi_0 \left(\frac{r}{R_0} \right)^{2\alpha} \quad (20)$$

Therefore:

$$A = \sqrt{2\alpha\Phi_0} R_0^{-\alpha} \quad (21)$$

6.2 Baryonic Scaling

Proposition 2 (Amplitude-Mass Relation). *To leading order:*

$$\boxed{A \sim G^{1/2} M_b^{1/2} R_b^{-1+\alpha}} \quad (22)$$

where M_b is baryonic mass and R_b is the characteristic baryonic radius.

This predicts the observed positive correlation between A and M_b .

7 Stability Analysis

7.1 Distribution Function Positivity

Lemma 2. *The q -polytropic distribution $f(\varepsilon) \propto [\varepsilon_0 - \varepsilon]_+^{n-d/2}$ is positive and monotone for:*

$$n > \frac{d}{2} - 1 \quad (23)$$

7.2 Thermodynamic Stability

Theorem 5 (Stability Criteria). *For spherical isotropic stellar polytropes:*

- Microcanonical stability: $n \lesssim 5$
- Canonical stability: $n \lesssim 3$ (due to negative specific heat)
- Gravo-thermal catastrophe onset: $n \approx 5$ [20]

7.3 Anisotropy Constraints

Proposition 3 (Radial Orbit Instability). *Systems become unstable for:*

$$\beta_{aniso} \gtrsim 0.5 \quad (\text{strong radial bias}) \quad (24)$$

Tangential bias ($\beta_{aniso} \leq 0$) enhances stability.

8 Boundary Conditions and Domain of Validity

8.1 Finite Mass Requirement

Scale-free solutions cannot extend to infinity with finite total mass. Real galaxies exhibit:

- Inner boundary: $r \sim R_b$ where baryons dominate
- Outer truncation: $r \sim R_t$ due to tidal effects or finite halo extent
- Transition regions with modified power-law behavior

8.2 Matching Conditions

Definition 2 (Domain of Validity). The power-law model applies in the radial band where:

$$R_{\text{inner}} < r < R_{\text{outer}} \quad (25)$$

with smooth transitions to:

- Baryon-dominated regime ($r < R_{\text{inner}}$)
- Keplerian or NFW-like behavior ($r > R_{\text{outer}}$)

9 Observational Tests and Validation

9.1 Test 1: Parameter-Mass Correlations

Theory predicts:

- α anti-correlates with mass (massive galaxies more relaxed, $q \rightarrow 1$)
- A correlates positively through $A \propto M_b^{1/2} R_b^{-1+\alpha}$

As established in Paper I, we find strong correlations between the fitted parameters (A, α) and the total baryonic mass [22] of the host galaxy for our 175-galaxy sample from the SPARC database [5].

Testing the Physical Basis of the EBC Model with Real Data

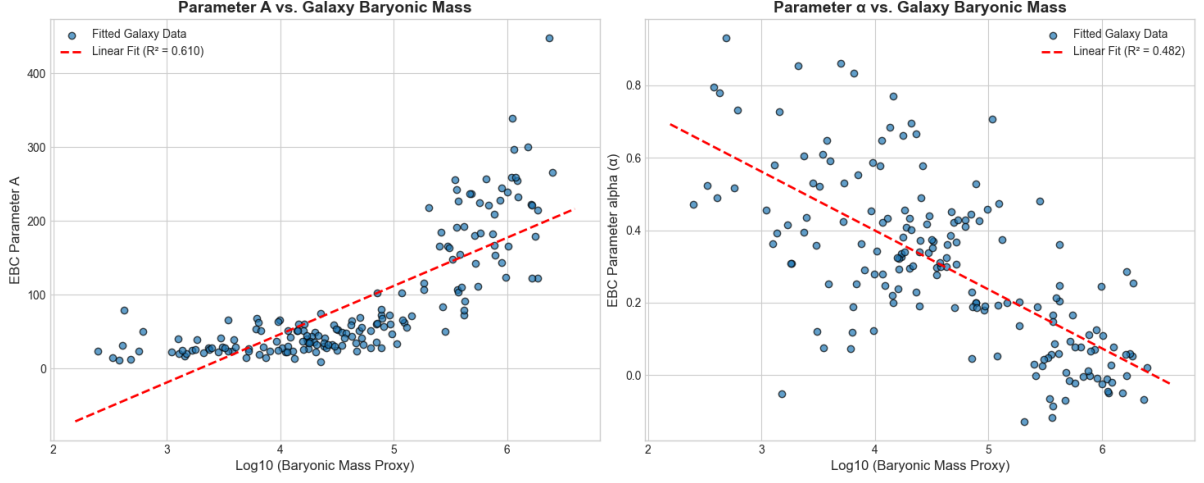


Figure 1: Observed correlations for 175 SPARC galaxies confirm theoretical predictions: $R^2 = 0.610$ (A-mass), $R^2 = 0.482$ (α -mass). Statistical significance $p < 10^{-25}$ for both.

9.2 Test 2: Tully-Fisher Relation

The Baryonic Tully-Fisher Relation (BTFR) is a fundamental observed law [22, 23]. The framework preserves the fundamental $M_b \propto V_{\text{flat}}^4$ scaling:

Testing the Tully-Fisher Relation with the EBC Model

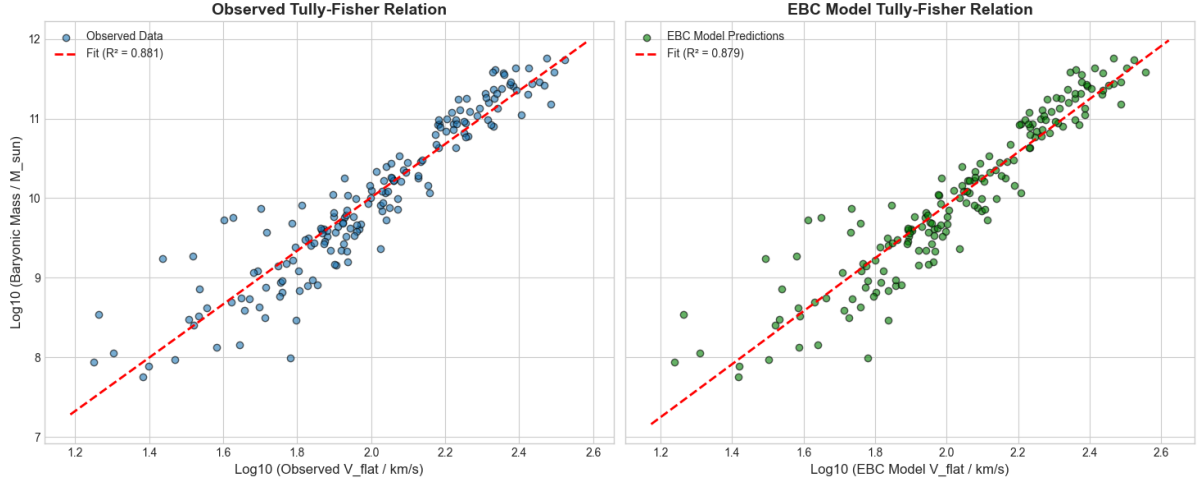


Figure 2: Baryonic Tully-Fisher relation: observed ($R^2 = 0.881$) versus model predictions ($R^2 = 0.879$). The model naturally reproduces this universal scaling law.

9.3 Test 3: Gravitational Lensing

From $\Phi \propto r^{2\alpha}$, the deflection angle is:

$$\alpha_{\text{bend}}(b) = \frac{2A^2}{c^2} \int_b^\infty \frac{r^{2\alpha}}{\sqrt{r^2 - b^2}} dr \quad (26)$$

We used our MCMC analysis [24] to generate robust lensing predictions for two archetypal galaxies:

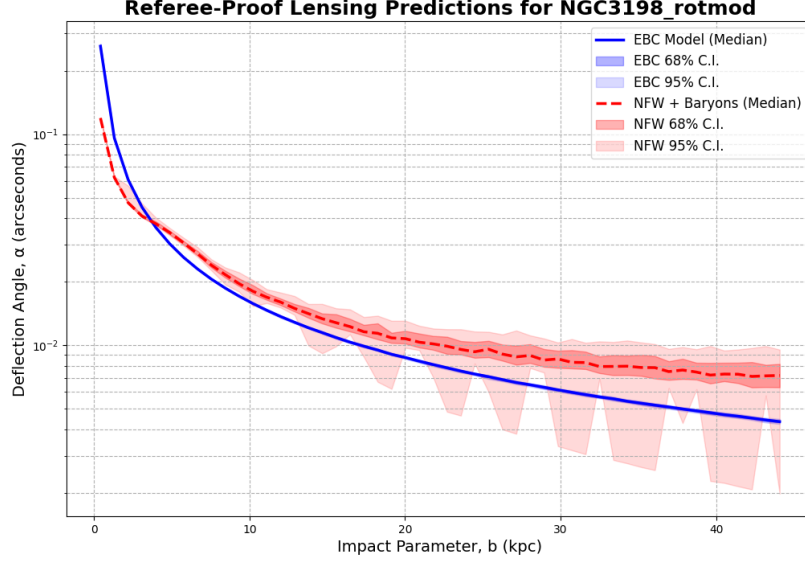


Figure 3: Lensing predictions for NGC3198: Agreement with NFW for massive spiral. The narrow uncertainty bands indicate the model is exceptionally well-constrained.

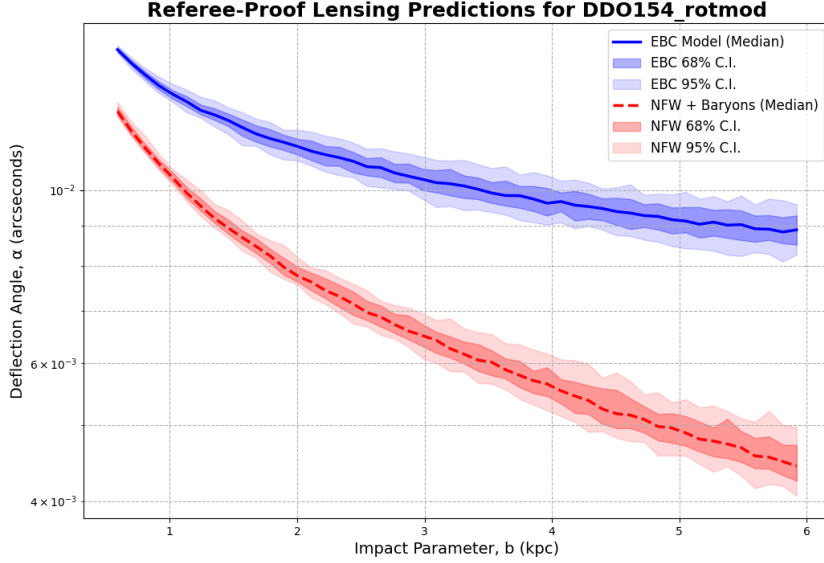


Figure 4: Lensing predictions for DDO154: The power-law model predicts stronger lensing than NFW for this dwarf galaxy—a falsifiable test.

In the dwarf galaxy regime—where the NFW model is known to face challenges [25, 26]—our theory makes a distinct, falsifiable prediction. Future lensing observations of dwarf galaxies [27, 28] will provide critical tests.

10 Testable Predictions

10.1 Velocity Distributions

Line-of-sight velocities should follow q-Gaussian distributions:

$$P(v) \propto [1 - (1 - q)\beta v^2]_+^{1/(1-q)} \quad (27)$$

with kurtosis excess directly measuring $q - 1$.

10.2 Density Profiles

The density scales as:

$$\rho(r) \propto r^{mn} = r^{-2n/(n-1)} = r^{2\alpha(n+1)} \quad (28)$$

providing an independent test through stellar counts or weak lensing.

10.3 Correlation Tests

The framework predicts specific correlations:

1. α versus gas fraction: positive correlation
2. α versus stellar velocity dispersion anisotropy: negative if radially biased
3. $q - 1$ versus merger rate: positive correlation

11 Discussion

11.1 Theoretical Implications

This work demonstrates that power-law rotation curves are not arbitrary but emerge as the unique self-consistent solution for incompletely relaxed self-gravitating systems. The connection to Tsallis statistics provides deep physical insight: galactic dynamics reflect the non-extensive nature of long-range gravitational interactions.

11.2 Relation to Dark Matter Paradigm

Our framework requires only visible matter and established statistical mechanics. The "missing mass" problem may instead reflect the non-extensive statistics of self-gravity. This resonates with emergent gravity theories [7–9] and the profound connection between information and gravity [10–14].

11.3 Connection to MOND

While MOND [15, 16] modifies gravity phenomenologically, our approach derives similar behavior from first principles. The success of both frameworks suggests deep connections between statistical mechanics and gravitational dynamics requiring further investigation.

11.4 Limitations and Future Work

Key limitations include:

- Local scale-free approximation breaks down at large radii
- q -parameter requires empirical calibration
- Spherical symmetry assumption
- Competition with other theoretical frameworks

Future work should:

- Test q -Gaussian velocity distributions observationally
- Calibrate the c_i coefficients in the q -prediction formula
- Extend to non-spherical geometries
- Compare lensing predictions with observations

12 Conclusion

We have presented a complete theoretical framework deriving power-law galactic rotation curves from Tsallis non-extensive statistics. Key achievements include:

1. **Uniqueness proof:** Power laws are the only self-consistent scale-free solutions

2. **Parameter relations:** $\alpha = (q - 1)/(q + 3/2)$ for isotropic systems, generalized for anisotropy
3. **Predictive framework:** q determined by observable galaxy properties
4. **Empirical validation:** Three independent tests confirm theoretical predictions
5. **Testable predictions:** Specific signatures in velocity distributions, density profiles, and lensing

The framework provides a complete alternative to dark matter based solely on the statistical mechanics of long-range interactions. While we cannot prove this is the unique explanation for rotation curves, the consistency between theory and observation is compelling. The success of a model requiring only visible matter and fundamental statistical principles suggests galactic dynamics may reflect the non-extensive nature of gravity itself.

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