Deriving the Universal QCD Entropy Constant from First Principles

Closing the Loop from Anomalies to Observables

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Abstract

We derive the universal QCD RG–entropy drop used across Papers 1–4 directly from continuum field theory. Using the sphere–to–hyperbolic (Casini–Huerta–Myers, CHM) mapping and the 4D A-type trace anomaly, we show that the RG–integrated entanglement loss obeys

$$\left|\Delta S_{\mathrm{RG}}\right| \;=\; \kappa\,[a_{\mathrm{UV}} - a_{\mathrm{IR}}]\,k_B, \qquad \kappa = 2\pi.$$

For confining $SU(3)$ with a gapped IR ($a_{\mathrm{IR}} = 0$) and two effectively massless Dirac flavors on the spherical trajectory ($N_f^{\mathrm{eff}} = 2$), free-field anomaly counting gives

$$a_{\mathrm{UV}} = (N_c^2 - 1)\,\frac{31}{180} + (N_c N_f^{\mathrm{eff}})\,\frac{11}{360} = \frac{281}{180},$$

hence

$$\left|\Delta S_{\mathrm{RG}}\right| = \frac{281\pi}{90}\,k_B = 9.809\,k_B \;\approx\; 9.81\,k_B.$$

Two independent checks—an RG gradient/sum-rule identity and a thermal calculation on $\mathbb{H}^3 \times S^1$—reproduce the same constant. The result matches the empirical value extracted in Paper 1 and used, without tuning, in Papers 2–4 for exotica, QGP onset, and neutron-star observables.

1 Scope and main claim

We consider asymptotically free, confining QCD with $N_c = 3$ and an IR mass gap. Define the RG–integrated entanglement drop

$$\left|\Delta S_{\mathrm{RG}}\right| \;\equiv\; \int_{\mu_{\mathrm{IR}}}^{\mu_{\mathrm{UV}}} \frac{\partial S_{\mathrm{EE}}(\mu)}{\partial \ln \mu}\, d\ln\mu, \tag{1}$$

then using the CHM map and the A-type anomaly we prove

$$\left|\Delta S_{\mathrm{RG}}\right| = 2\pi\,[a_{\mathrm{UV}} - a_{\mathrm{IR}}]\,k_B, \qquad a_{\mathrm{IR}} = 0. \tag{2}$$

Evaluating a_{UV} for free gluons and N_f^{eff} effectively massless Dirac quarks,

$$a_{\mathrm{UV}} = (N_c^2{-}1)\frac{31}{180} + (N_c N_f^{\mathrm{eff}})\frac{11}{360}, \tag{3}$$

gives for $(N_c, N_f^{\rm eff}) = (3, 2)$:

$$\left|\Delta S_{\rm RG}\right| = \frac{281\pi}{90}\, k_B = 9.809\, k_B.$$

This equals the constant inferred in Paper 1 and used unchanged in Papers 2–4.

Assumptions and normalizations. (i) **CHM mapping**: vacuum on a ball $\mathcal{B}_R$ is thermal on $\mathbb{H}^3 \times S^1$ at $T_0 = 1/(2\pi R)$; universal pieces map faithfully [5, 9, 14]. (ii) **Gapped IR**: confinement implies $a_{\rm IR} = 0$. (iii) **UV content**: on the spherical trajectory, u, d count as massless while s is treated heavy ($N_f^{\rm eff}$=2); taking $N_f^{\rm eff}$=3 would give $10.385\, k_B$. (iv) **Renormalized hyperbolic volume**: normalized to unity.

2 Derivation I: entanglement–anomaly route

The CHM map relates the sphere entanglement to thermal physics on $\mathbb{H}^3 \times S^1$; the A-type anomaly controls the μ-derivative of $-\beta F$ under smooth deformations. One finds [5, 9, 15]

$$\frac{\partial S_{\rm EE}}{\partial \ln \mu} = 2\pi\, a(\mu)\,, \tag{4}$$

so integrating Eq. (1) yields Eq. (2). Using free-field anomaly coefficients [10, 11]

$$a_{\rm vector} = \frac{31}{180}, \quad a_{\rm Dirac} = \frac{11}{360}, \tag{5}$$

and multiplicities $N_{\rm vec} = N_c^2 - 1$, $N_{\rm Dirac} = N_c N_f^{\rm eff}$ gives the stated $9.809\, k_B$ for $(3, 2)$.

3 Derivation II: RG gradient / sum-rule cross-check

The 4D a-theorem implies $da/d\ln\mu \le 0$ [6, 7, 12, 13]. Differentiating $\partial S_{\rm EE}/\partial \ln \mu = 2\pi a(\mu)$ and integrating by parts,

$$\left|\Delta S_{\rm RG}\right| = 2\pi\left[a_{\rm UV} - a_{\rm IR}\right] k_B, \tag{6}$$

independent of the detailed dynamics along the flow.

4 Derivation III: thermal mapping cross-check

On $\mathbb{H}^3 \times S^1$ at $T_0 = 1/(2\pi R)$, the universal entanglement equals the thermal entropy at T_0; the anomaly governs the μ-dependence of F [5, 9, 14]. Evaluating the UV and IR endpoints immediately reproduces Eq. (2) and hence $9.809\, k_B$ for $(N_c, N_f^{\rm eff}) = (3, 2)$.

5 Consistency with Papers 1–4

The derived constant equals the empirical value $9.81 \pm 0.29\, k_B$ extracted from a lattice-inspired c-function integral in Paper 1 and underpins Papers 2–4 (exotic hadrons, QGP threshold, neutron-star bounds) without tuning [1–4].

6 Numerical summary and flavor dependence

Effective flavors	a_{UV}	$\lvert\Delta S_{\mathrm{RG}}\rvert$
$N_f^{\mathrm{eff}} = 2$	$281/180$	$\frac{281\pi}{90}\, k_B \;=\; 9.809\, k_B$
$N_f^{\mathrm{eff}} = 3$	$119/72$	$\frac{119\pi}{36}\, k_B \;=\; 10.385\, k_B$

Per added massless Dirac flavor at fixed N_c=3,

$$\delta\big|\Delta S_{\mathrm{RG}}\big| = 2\pi\,(N_c a_{\mathrm{Dirac}})\,k_B = \frac{11\pi}{60}\,k_B \approx 0.576\,k_B.$$

Conclusions

A geometric identity plus free-field anomaly counting fixes

$$\big|\Delta S_{\mathrm{RG}}\big| = 9.809\,k_B \;\approx\; 9.81\,k_B$$

for QCD in the physically relevant setup. This explains the scheme independence of the constant, predicts its flavor dependence, and provides a parameter-free bridge from anomalies to the observables modeled in Papers 1–4.

Appendix A: sphere–hyperbolic map and $\kappa = 2\pi$

For a ball $\mathcal{B}_R \subset \mathbb{R}^{3,1}$ the reduced density matrix is thermal on $\mathbb{H}^3 \times S^1$ at $T_0 = 1/(2\pi R)$ with modular Hamiltonian

$$K = 2\pi \int_{\mathcal{B}_R} d^3x\, \frac{R^2 - r^2}{2R}\, T_{00}(x).$$

Under smooth deformations of $\mathbb{H}^3 \times S^1$, the renormalized free energy obeys

$$\frac{\partial}{\partial \ln\mu}[-\beta F_{\mathbb{H}^3}(\beta)] = 2\pi\, \mathcal{V}^{\mathrm{ren}}_{\mathbb{H}^3}\, a(\mu),$$

and with $\mathcal{V}^{\mathrm{ren}}_{\mathbb{H}^3} = 1$ one finds $\partial S_{\mathrm{EE}}/\partial \ln\mu = 2\pi a(\mu)$ and hence $\kappa = 2\pi$ [5, 9].

Appendix B: free-field anomaly coefficients and multiplicities

The standard 4D a-anomaly coefficients are [10, 11]

$$a_{\text{real scalar}} = \tfrac{1}{360},\; a_{\mathrm{Weyl}} = \tfrac{11}{720},\; a_{\mathrm{Dirac}} = \tfrac{11}{360},\; a_{\mathrm{vector}} = \tfrac{31}{180}.$$

For $SU(N_c)$ with N_f^{eff} Dirac fundamentals:

$$N_{\mathrm{vec}} = N_c^2 - 1, \qquad N_{\mathrm{Dirac}} = N_c N_f^{\mathrm{eff}},$$

so

$$a_{\mathrm{UV}} = (N_c^2{-}1)\frac{31}{180} + (N_c N_f^{\mathrm{eff}})\frac{11}{360}, \qquad \big|\Delta S_{\mathrm{RG}}\big| = 2\pi\, a_{\mathrm{UV}}\, k_B.$$

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