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ML and UMVU estimation in the M/D/1 queuing system

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ABSTRACT
In the imbedded Markov chain (IMC) analysis of M/G/1 queuing system, $X_1, X_2, \ldots, X_n, \ldots$ form a sequence of i.i.d random variables. where X_n denotes the number of customer arrivals during the service time of nth customer. In the M/D/1 queue, the distribution of common random variable X is the Poisson distribution with mean ρ, the traffic intensity. This fact is utilized for maximum likelihood (ML) and uniformly minimum variance unbiased (UMVU) estimation of traffic intensity, performance measures, transition probabilities of IMC, and correlation functions of departure process, based on a sample of fixed size n from $P(\rho)$ distribution. Also, consistent asymptotic normality (CAN) property of ML estimators (MLEs) is established. The MLEs and UMVUEs are compared.

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1. Problem, literature, and plan

Statisticians have dealt with parametric estimation in Poisson distribution due to many applications in various disciplines like ophthalmology, sociology, engineering, and biological science. In this article, the objective is parametric estimation in Poisson distribution arising in the imbedded Markov chain (IMC) analysis of the M/D/1 queuing system. A case for M/D/1 queuing system is: patients arrive for a certain test according to a Poisson process with rate λ, which is not known and are asked to drink a cup of mixture dispensed by an automatic machine which requires a constant time. The focus will be on estimation of traffic intensity (ρ), measures of effectiveness, and correlation functions in the stable-M/D/1 queuing system. The M/D/1 queuing system will be stable (or in steady state) iff the traffic intensity, ρ, is constrained to the parameter space $\Theta = \{\rho : o < \rho < 1\}$. Thus, the problem is that of parametric estimation under a constrained parameter space in Poisson distribution. However, the interest is also in transition probabilities of the IMC of M/D/1 queuing system, which are not necessarily steady-state measures. The sample information is $(X_1, \ldots, X_n)$, where X_n denotes the number of arrivals during the service time of nth customer. This aspect is further explained at the beginning of next section. The motivation for estimation emanates from mainly queuing performance measures. However, the motivation is also due to the fact that we are addressing the problem of estimation in a standard probability distribution namely Poisson distribution.

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Clarke (1957) is among the early researchers on estimation in queuing systems. Bhat and Subba Rao (1987) and Bhat et al. (1997) have reviewed this and related aspects. However, the literature on estimation in queuing systems in the post review period has prospered due to contributions from Conti (1999), Rio Insua et al. (1998), Armero and Bayarri (1999), Sharma and Kumar (1999), Zheng and Seila (2000), Armero and Conesa (1998, 2000), Butler and Huzurbazaar (2000), Huang and Brill (2001), Ausin et al. (2004), Mukherjee and Chowdhury (2005), Chu and Ke (2006), Chu and Ke (2007), Ausin et al. (2008), Choudhury and Borthakur (2008), Ramirez et al. (2008, 2008a), Kiessler and Lund (2009), Jayakrishna Udupa (2009), Xu et al. (2011), Srinivas et al. (2011), and Chowdhury and Mukherjee (2013).

The following is the layout of this article. In Section 2, the maximum likelihood estimator (MLE) of the traffic intensity ρ is obtained in the M/D/1 queuing system and MLE of traffic intensity is derived in the stable-M/D/1 queuing system. The MLE and uniformly minimum variance unbiased estimators (UMVUEs) of performance measures in the stable-M/D/1 queuing system are derived in Section 3. Estimation of correlation functions of Orders 1 and 2 of the departure process of the stable-M/D/1 queuing system is the objective of Section 4. In Section 5, the MLEs and UMVUEs of transition probabilities of the Markov chain imbedded in the M/D/1 queuing system are derived. In Section 6, consistent asymptotic normality (CAN) estimators of various measures are derived. A comparison of MLEs and UMVUEs derived in Sections 3 and 4 is the objective of Section 7, using asymptotic expected deficiency (AED) criterion.

2. Maximum likelihood estimation of traffic intensity

In the IMC analysis of M/G/1 queuing system, $X_1, X_2, \ldots, X_n, \ldots$ forms a sequence of i.i.d random variables. The probability distribution of common random variable, denoted by X, in M/D/1 queuing system is the well-known Poisson distribution with mean ρ. That is

$$P(X = x) = \frac{e^{-\rho}\rho^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where the parameter space is $\Theta = \{\rho : \rho > 0\}$. As mentioned earlier, the estimation procedures in this article are based on the random sample $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$, where X_j denotes the number of customer arrivals during the service time of jth customer. Clearly, $\sum_{i=1}^{n} X_i$ is a minimal complete sufficient statistic for Poisson family of distributions. The sample mean is an intuitive estimator of ρ, which turns out to be the MLE, UMVUE, and method of moments (MOM) estimator of ρ in M/D/1 queuing system. However, our special interest is in estimation of ρ in stable-M/D/1 queuing system as we wish to estimate measures of effectiveness and correlation functions of departure process in stable-M/D/1 queuing system. The ML estimation problem of ρ in the stable-M/D/1 queuing system can be formulated as

$$\max \mathcal{L}(\rho; \underset{\sim}{x}) = -n\rho + \sum_{i=1}^{n} x_i \cdot \log\rho - \log\prod_{i=1}^{n} x_i!$$

subject to $0 < \rho < 1$. The constraint $0 < \rho < 1$ is required as the M/D/1 queuing system is in steady state. However, the MLE may not exist for this problem. To circumvent this problem, we replace $0 < \rho < 1$ by $0 < \rho \leq 1^-$, where 1^- is a number less than 1 but close to 1 and is provided by other sources of information. A source is the knowledge of the queuing analyst and this analyst should be able to provide this number to the required accuracy like 0.9999 or

0.999999. Now the problem of MLE of ρ can be reformulated as

$$\max \mathcal{L}(\rho; \underset{\sim}{x}) = -n\rho + \sum_{i=1}^{n} x_i \cdot \log\rho - \log\prod_{i=1}^{n} x_i!$$
$$\text{s.t } \rho \leq 1^-$$
$$\rho \geq 0$$

where $\mathcal{L}(\rho; \underset{\sim}{x})$ denotes the log-likelihood function and $\mathcal{L}(\rho; \underset{\sim}{x})$ is a strictly concave function over the interval $(0, 1^-]$. Solving the above problem we get the ML estimate of ρ. The ML estimate is given by

$$\hat{\rho}_c = \begin{cases} \bar{x}_n, & \text{if} \quad \bar{x}_n \in [0, 1) \\ 1^-, & \text{if} \quad \bar{x}_n \notin [0, 1), \end{cases} \tag{2.1}$$

where $\bar{x}_n$ is the sample mean estimate given by $\bar{x}_n = \frac{\sum_{i=1}^{n} x_i}{n}$. Thus, the MLE is the sample mean estimate if $\bar{x}_n \leq 1^-$. However, if the sample mean estimate is greater than 1^-, then the MLE of ρ is taken to be 1^-.

3. Estimation of performance measures

The interest in this section is to estimate measures of system performance in the stable-M/D/1 queuing system. The measures are given by

$$L_q = \frac{1}{2} \cdot \frac{\rho^2}{1-\rho}, \tag{3.1}$$

and

$$L = \rho + \frac{1}{2}\frac{\rho^2}{1-\rho}, \tag{3.2}$$

following Gross and Harris (1985), where L_q is the expected number of customers in the queue while L is the expected number of customers in the system. The MLEs of L_q and L are easily obtained by the application of invariance property of MLEs. That is the MLEs are obtained by plugging $\hat{\rho}_c$, the MLE of ρ, for ρ in (3.1) and (3.2). Thus, the MLEs are

$$\hat{L}_q = \frac{1}{2} \cdot \frac{\hat{\rho}_c^2}{1-\hat{\rho}_c} \tag{3.3}$$

and

$$\hat{L} = \hat{\rho}_c + \frac{1}{2}\frac{\hat{\rho}_c^2}{1-\hat{\rho}_c}. \tag{3.4}$$

For UMVU estimation of L_q and L, we make the direct application of Lehmann–Scheffe theorem. Poisson family of densities induced by $T = \sum_{i=1}^{n} X_i$ is complete and thus $\sum_{i=1}^{n} X_i$ is a complete statistic. Also $\sum_{i=1}^{n} X_i$ is sufficient for Poisson family of densities. Thus, $T = \sum_{i=1}^{n} X_i$ is the complete sufficient statistic. By Lehmann–Scheffe theorem, $\phi(T)$ is the UMVUE of L_q if

$$E_\rho\{\phi(T)\} = L_q \quad \forall \quad \rho \in \Theta = \{\rho : 0 < \rho < 1\}.$$

That is

$$\sum_{t=0}^{\infty} \phi(t) \frac{e^{-n\rho}(n\rho)^t}{t!} = \frac{1}{2}\frac{\rho^2}{1-\rho}, \tag{3.5}$$

where the parameter space is restricted to $\Theta = \{\rho : 0 < \rho < 1\}$ as L_q and L are steady-state measures. Clearly, (3.5) is equivalent to

$$\sum_{t=0}^{\infty} \left\{ \frac{2\phi(t)n^t}{t!} \right\} \rho^t = \sum_{t=2}^{\infty} \left\{ \sum_{r=0}^{t-2} \frac{n^r}{r!} \right\} \rho^t,$$

where the condition $\rho < 1$ is used for expansion of $(1-\rho)^{-1}$ and $e^{n\rho}$ and thus the UMVUE to be derived holds only when the M/D/1 queuing system is in steady state, that is the stable-M/D/1 queuing system. The left hand side and right hand side of the above equation are both power series and thus we obtain the UMVUE of L_q by equating the coefficient of ρ^t, $t = 0, 1, 2, \ldots$. Thus, the UMVUE is given by

$$\tilde{L}_q = \phi(t) = \begin{cases} \frac{1}{2}\left\{ \sum_{r=0}^{t-2} \frac{n^r}{r!} \right\} \cdot \frac{t!}{n^t}, & t \geq 2 \\ 0, & t = 0, 1 \end{cases}$$

To obtain UMVUE of L, we note that $L = \rho + L_q$, the queuing relationship, and use the following well-known theorem mentioned in Patel et al. (1976) and Kale (2005).

Theorem 3.1. *If $T_1, T_2, \ldots, T_k$ are UMVU estimators of parametric functions $g_1(\theta), \ldots, g_k(\theta)$ respectively, then $\sum_{i=1}^{k} c_i T_i$ is the UMVU estimator of $\sum_{i=1}^{k} c_i g_i(\theta)$, where c_i, $i = 1, 2, \ldots, k$ are known constants.*

In terms of notations of the above theorem, $k = 2$, $c_1 = 1$, $c_2 = 1$, $\theta = \rho$, $g_1(\rho) = \rho$, and $g_2(\rho) = L_q$. Thus, the UMVUE of L is obtained by the application of Theorem 3.1. The estimator is given by

$$\begin{aligned} \tilde{L} &= \tilde{\rho} + \tilde{L}_q \\ &= \bar{X}_n + \tilde{L}_q \\ &= \begin{cases} \bar{X}_n + \frac{1}{2}\left\{ \sum_{r=0}^{t-2} \frac{n^r}{r!} \right\} \frac{t!}{n^t}, & t \geq 2 \\ \bar{X}_n, & t = 0, 1 \end{cases}. \end{aligned}$$

We now turn to estimation of correlation functions in the following section.

4. Estimation of correlation functions

The study of correlation structure of queuing processes has received attention because of its implication in estimation as well as in its own right. In this context, it is pertinent to recall the review of literature by Reynolds (1975). Jenkins (1966) derived the correlation functions of Orders 1 and 2 of the departure process in the stable-M/D/1 queuing system and showed that they are given by

$$\gamma_1 = \mathrm{Corr}(d_{n-1}, d_n) = \frac{e^{-\rho} + \rho - 1}{\rho + 1} \tag{4.1}$$

$$\gamma_2 = \mathrm{Corr}(d_{n-1}, d_{n+1}) = \frac{(\rho + 1)e^{-2\rho} + \rho - 1}{\rho + 1} \tag{4.2}$$

The MLEs of γ_1 and γ_2, denoted by $\hat{\gamma}_1$ and $\hat{\gamma}_2$, are easily obtained by plugging $\hat{\rho}_c$, in (2.1), for ρ in (4.1) and (4.2). We now use the method used in the preceding section to obtain UMVUEs of γ_1 and γ_2. As such the UMVUE of γ_1 is the solution to ψ of the following equation:

$$E_\rho\{\psi(T)\} = \gamma_1$$

which can be written as

$$\sum_{t=0}^{\infty} \psi(t) \cdot \frac{e^{-n\rho}(n\rho)^t}{t!} = \gamma_1,$$

and this is equivalent to

$$\sum_{t=0}^{\infty} \left\{ \frac{\psi(t)n^t}{t!} \right\} \rho^t = (1+\rho)^{-1} e^{n\rho} [e^{-\rho} + \rho - 1].$$

After expanding the R.H.S and collecting terms involving ρ^t, we get

$$\psi(t) = \begin{cases} \sum_{r=2}^{t} (-1)^{r+1} \left[\frac{(n-1)^r}{r!} + \frac{n^{r-1}}{(r-1)!} - \frac{n^r}{r!} \right] \frac{t!}{n^t}, & \text{if } t \text{ is an odd number} \\ \sum_{r=2}^{t} (-1)^{r} \left[\frac{(n-1)^r}{r!} + \frac{n^{r-1}}{(r-1)!} - \frac{n^r}{r!} \right] \frac{t!}{n^t}, & \text{if } t \text{ is an even number,} \end{cases}$$

the UMVUE of γ_1. Similarly, the UMVUE of γ_2 can be obtained. However, we derive UMVUEs of each of the three terms on right hand side of (4.2) and then use the Theorem 3.1 to obtain the UMVUE of γ_2. By Lehmann–Scheffe theorem, $\psi_1(T)$ is UMVUE of $e^{-2\rho}$ if ψ_1 is the solution of

$$\sum_{t=0}^{\infty} \psi_1(t) \frac{e^{-n\rho}(n\rho)^t}{t!} = e^{-2\rho} \tag{4.3}$$

for all $\rho \in \Theta = \{0 < \rho < 1\}$. That is (4.3) is equivalent to

$$\sum_{t=0}^{\infty} \left\{ \frac{\psi_1(t)n^t}{t!} \right\} \rho^t = 1 + (n-2)\rho + \frac{(n-2)^2}{2!}\rho^2 + \cdots,$$

which yields

$$\psi_1(t) = \begin{cases} \left(\frac{n-2}{n}\right)^t, & t = 0, 1, 2, \ldots (n \geq 2) \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Now, $\psi_2(t)$ is UMVUE of $\frac{\rho}{1+\rho}$ if it is the solution of

$$\sum_{t=0}^{\infty} \psi_2(t) \frac{e^{-n\rho}(n\rho)^t}{t!} = \frac{\rho}{1+\rho}.$$

This equation can be rewritten as

$$\sum_{t=0}^{\infty}\left\{\psi_2(t)\frac{n^t}{t!}\right\}\rho^t = e^{n\rho}\cdot\rho(1+\rho)^{-1}. \tag{4.5}$$

Expanding the R.H.S of (4.5) and equating the coefficients of ρ^t, we get

$$\psi_2(t) = \begin{cases} \dfrac{t!}{n^t}\displaystyle\sum_{r=0}^{t-1}\frac{n^r}{r!}(-1)^r, & \text{if } t \text{ is an odd number} \\ \dfrac{t!}{n^t}\displaystyle\sum_{r=0}^{t-1}\frac{n^r}{r!}(-1)^{r+1}, & \text{if } t \text{ is an even number} \\ \quad 0 \quad , & \text{if } t=0. \end{cases}$$

Similarly, UMVUE of $\frac{1}{1+\rho}$ is obtained and the UMVUE is given by

$$\psi_3(t) = \begin{cases} \quad 1 \quad , & \text{if } t=0 \\ \dfrac{t!}{n^t}\displaystyle\sum_{r=0}^{t}\frac{n^r}{r!}(-1)^{r+1}, & \text{if } t \text{ is anoddnumber} \\ \dfrac{t!}{n^t}\displaystyle\sum_{r=0}^{t}\frac{n^r}{r!}(-1)^r, & \text{if } t \text{ is anevennumber.} \end{cases}$$

Thus, the UMVUE of γ_2 is obtained using Theorem 3.1 and is given by

$$\tilde{\gamma}_2 = \begin{cases} \quad 0 & \text{, if } t=0 \\ \left(\dfrac{n-2}{n}\right)^t + \dfrac{t!}{n^t}\displaystyle\sum_{r=0}^{t-1}\frac{n^r}{r!}(-1)^r - \frac{t!}{n^t}\sum_{r=0}^{t}\frac{n^r}{r!}(-1)^{r+1}, & \text{if } t \text{ is an odd number} \\ \left(\dfrac{n-2}{n}\right)^t + \dfrac{t!}{n^t}\displaystyle\sum_{r=0}^{t-1}\frac{n^r}{r!}(-1)^{r+1} - \frac{t!}{n^t}\sum_{r=0}^{t}\frac{n^r}{r!}(-1)^r, & \text{if } t \text{ is an even number.} \end{cases}$$

5. Estimation of transition probabilities

The transition probabilities of IMC in the M/D/1 queuing system are given by

$$p_{ij} = \begin{cases} \dfrac{e^{-\rho}\rho^j}{j!}, & i=0,\ j\geq 0 \\ \dfrac{e^{-\rho}\rho^{j-i+1}}{(j-i+1)!}, & i>0,\ j\geq i-1 \\ 0, & \text{otherwise.} \end{cases}$$

On inspection we see that p_{ij} for all $\{i=0,\ j\geq 0\}$ and $\{i>0,\ j\geq i-1\}$ are the Poisson probabilities. Thus, the UMVU estimation of these p_{ij}'s is the classical problem of UMVU estimation of Poisson probabilities based on a random sample from Poisson distribution with mean ρ. It is well known that UMVUEs of Poisson probabilities are binomial probabilities with number of trials equal to t and success probability equal to $\frac{1}{n}$ (see, e.g., Lehmann and Casella, 1998). Hence, the UMVUEs are given by

$$\tilde{p}_{ij} = \begin{cases} \binom{t}{j} \left(\frac{1}{n}\right)^j \left(1-\frac{1}{n}\right)^{t-j}, & (i=0,\ j\geq 0) \\ \binom{t}{j-i+1} \left(\frac{1}{n}\right)^{j-i+1} \left(1-\frac{1}{n}\right)^{t-j+i-1}, & (i>0,\ j\geq i-1) \\ 0, & \text{otherwise} \end{cases}$$

MLEs of p_{ij}'s for all $\{i=0,\ j\geq i\}$ and $\{i>0,\ j\geq i-1\}$ are obtained by substituting $\bar{X}_n$ for ρ in p_{ij}'s. Having proposed MLEs and UMVUEs it is natural to compare them. However, there are countably infinite number of transition probabilities and thus we do not intend to compare the two classical estimators.

6. Consistent asymptotic normality

We discussed classical estimation of traffic intensity, measures of system performance, and correlation functions of departure process in the stable M/D/1 queuing system as well as that of transition probabilities in M/D/1 queuing system. In this section we concentrate on CAN property (see Kale, 2005). For this we start by noting that ρ is the Poisson mean and thus by weak law of large numbers (WLLN) $\bar{X}_n \xrightarrow{P} \rho$. Further, $\bar{X}_n$ has an asymptotic distribution which is a normal distribution with asymptotic mean ρ and asymptotic variance ρ/n. Thus $\bar{X}_n$ is a CAN estimator of ρ. To obtain CAN estimators of performance measures, we use the invariance property of CAN estimators under differentiable transformation. As a prerequisite for this we note that $\frac{dL_q}{d\rho} = \frac{1}{2}\cdot\frac{\rho(2-\rho)}{(1-\rho)^2} \neq 0$ if $\rho \neq 2$. Naturally, ρ must not be equal to 1 as it renders the derivative to be equal to infinity. All these conditions are satisfied as $\rho < 1$. Thus $\frac{1}{2}\cdot\frac{\bar{X}_n^2}{1-\bar{X}_n}$ is CAN for L_q with asymptotic variance $\frac{\rho^3(2-\rho)^2}{4n(1-\rho)^2}$. Furthermore, $\bar{X}_n + \frac{1}{2}\cdot\frac{\bar{X}_n^2}{1-\bar{X}_n}$ is CAN for L with asymptotic variance

$$\frac{\rho}{n}\left\{1++\frac{1}{2}\frac{\rho(2-\rho)}{(1-\rho)}\right\}^2 \quad \text{as} \quad \frac{dL}{d\rho} = 1+\frac{\rho(2-\rho)}{2(1-\rho)} \neq 0$$

for $\rho \neq 2$ (and of course $\rho \neq 1$), which are satisfied as $\rho < 1$.

The CAN estimator of γ_1 is $\frac{e^{-\bar{X}_n}+\bar{X}_n-1}{1+\bar{X}_n}$, as $\gamma_1' \neq 0$, with asymptotic variance

$$\frac{\rho}{n}\left\{\frac{(1+\rho)(1-e^{-\rho})-(e^{-\rho}+\rho-1)}{(1+\rho)^2}\right\}^2.$$

Similarly, as $\gamma_2' \neq 0$, $\frac{(\bar{X}_n+1)e^{-2\bar{X}_n}+\bar{X}_n-1}{1+\bar{X}_n}$ is CAN estimator of γ_2 with asymptotic variance

$$\frac{\rho}{n}\left\{\frac{(\rho+1)[(\rho+1)-e^{-2\rho}+1]-[(\rho+1)e^{-2\rho}+\rho-1]}{(1+\rho)^2}\right\}^2.$$

Thus, the CAN property of MLEs is established.

7. Comparison and recommendation

The MLEs and UMVUEs of system size probabilities and measures of system performance in M/M/1 queuing system were compared by Srinivas et al. (2011) using asymptotic expected deficiency (AED) criterion of Hodges and Lehmann (1970). The comparison was based on the following theorem due to Hwang and Hu (1990).

Table 1. Results based on AED computation.

Parametric function	Space of ρ	Performing estimator
L	$0 < \rho < 1$	UMVUE
L_q	$0 < \rho < 1$	UMVUE
γ_1	$0 < \rho < 1$	MLE
γ_2	$(0, 0.5435)$	UMVUE
	$(0.5455, 0.9900)$	MLE

Theorem 7.1. *Under certain regularity conditions, the AED of MLE $g(Z)$ of $\varphi(\theta)$ relative to the UMVUE $U(Z)$ for the exponential family with density*

$$f(x;\theta) = \exp\{\Phi_1(\theta)T(x) + \Phi_2(\theta) + d(x)\}, x \in S, \theta \in \Theta$$

is given by

$$AED(g(\mathbf{Z}), U(\mathbf{Z})) = V(\theta)\left\{\frac{\varphi'''(\theta)}{\varphi'(\theta)} + \frac{1}{4}\left(\frac{\varphi''(\theta)}{\varphi'(\theta)}\right)^2\right\} + V'(\theta)\frac{\varphi''(\theta)}{\varphi'(\theta)},$$

where $V(\theta) = \left\{\Phi_1'(\theta)\right\}^{-1}$.

The AED criterion is used to compare MLEs and UMVUEs of performance measures and correlation functions of Orders 1 and 2 in the stable-M/D/1 queuing system. The AEDs were computed for $\rho \in (0, 1)$ to compare MLEs with UMVUEs of Sections 3 and 4. The results are captured in Table 1. Clearly, based on AED computations and the corresponding results in Table 1, we recommend UMVUE over MLE for estimation of L and L_q while MLE is recommended over UMVUE for estimation of γ_1. However, such a categorical recommendation in case of γ_2 is not possible. For $\rho \in (0, 0.5435)$ UMVUE is recommended while MLE is recommended for $\rho \in (0.5435, 1)$ in the case of classical estimation of γ_2.

Bayesian estimation of ρ and other measures relative to balanced and LINEX loss functions have been considered and dissemination of information on this and related aspects will be in different communications. Also, classical estimation results on steady-state probability distribution of system size and results on testing of hypotheses will be conveyed separately.

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