Communications in Statistics - Theory and Methods

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lsta20

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To cite this article: P. Chandrasekhar, V. S. Vaidyanathan, T. M. Durairajan & V. S. S. Yadavalli (2021) Classical and Bayes estimation in the $M|D|1$ queueing system, Communications in Statistics - Theory and Methods, 50:22, 5411-5421, DOI: 10.1080/03610926.2020.1734833

To link to this article: https://doi.org/10.1080/03610926.2020.1734833

Published online: 09 Mar 2020.

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
2021, VOL. 50, NO. 22, 5411–5421
https://doi.org/10.1080/03610926.2020.1734833

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Classical and Bayes estimation in the $M|D|1$ queueing system

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ABSTRACT

By considering an $M|D|1$ queueing system, the maximum likelihood and consistent estimators of traffic intensity are derived by observing the number of entity arrivals during the service time of an entity. Uniform minimum variance unbiased estimators for the expected waiting times per entity in the system and queue are obtained. Further, Bayes estimators of traffic intensity, measures of system performance, minimum posterior risk, and minimum Bayes risk associated with these estimators are derived. Also, the Bayes estimator of traffic intensity and its risk function are derived under LINEX loss function.

ARTICLE HISTORY
Received 23 August 2019
Accepted 8 February 2020

KEYWORDS
Bayes estimator; consistent estimator; $M|D|1$ queueing system; minimum posterior risk; uniform minimum variance unbiased estimator

AMS 2010 SUBJECT CLASSIFICATION:
Primary 60K25;
Secondary 90B22

1. Introduction

Queueing theory is often applied in the design and analysis of telecommunication systems, traffic networks, service systems, and so on. Most of the studies on queueing models are confined to only obtaining expressions for transient or stationary (steady state) solutions and do not consider the associated statistical inference problems. Queueing theory has attracted researchers all over the world from various disciplines due to many stochastic, operations research, and statistical problems that arise naturally in the study of several complex queues. Statistical analysis of queueing systems has a number of interesting and challenging problems. In particular, Bayesian methods are well suited to deal with them as they incorporate prior information and importantly, they can handle in a natural way frequently occurring issues such as restrictions in the parameter space and prediction problems.

Some of the performance measures in queueing theory are traffic intensity ρ, expected queue size or expected number of entities in the queue L_Q, expected system size or expected number of entities in the system L_S, expected waiting time per entity in the system W_S and expected waiting time per entity in the queue W_Q. It may be noted that all these measures of system performance are functions of queueing parameters

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namely arrival rate λ and service rate μ. Usually, we assume that these queueing parameters are known constants. However, in real life situations, it may not be so. If one can combine the prior information about the variations in λ and μ with the observed data on queueing systems, one may obtain better estimates of the measures of system performance, which is obtained by Bayesian analysis.

An excellent exposition of general Bayesian analysis as applied to queueing theory can be found in McGrath, Gross, and Singpurwalla (1987). However, much of the work till date has been on the usual Markovian arrivals and services and hence there is a need to analyze queues with non Markovian arrivals or services. See Gross and Harris (2010).

Chowdhury and Maiti (2014) have derived the Bayes estimator of traffic intensity in an $M|Er|1$ queueing model based on the number of arrivals during the service times of customers under squared error loss function (SELF) and precautionary loss function (PLF) along with Bayes estimator of mean queue length. Yadavalli, Vaidyanathan, and Chandrasekhar (2017) have analyzed in detail a Markovian queueing model with balking and derived uniformly minimum variance unbiased estimator (UMVUE) and consistent asymptotic normal (CAN) estimator for the expected number of entities in the system. By considering an $M|Er|1$ queueing model, Vaidyanathan and Chandrasekhar (2018) have obtained Bayes estimators of arrival and service parameters both under SELF and ELF along with minimum posterior risk and minimum Bayes risk associated with the estimators under SELF.

Our interest in this article is on inferential analysis of an $M|D|1$ queueing model. $M|D|1$ queueing models can be applied in situations in which the service time is constant, say b. Brun and Garcia (2000) have studied in detail an $M|D|1|N$ queueing model and derived closed form formulae for the distribution of the number of customers in the system along with an explicit solution for the mean queue length and average waiting time. Garcia, Brun, and Gauchard (2002) have derived an analytical expression of the time dependent probability distribution of $M|D|1|N$ queue initialized in an arbitrary deterministic state. Srinivas and Kale (2016) have obtained the maximum likelihood estimator (MLE) and UMVUE of the traffic intensity, transition probabilities of imbedded Markov chain and correlation functions of departure process for $M|D|1$ queueing model. While making statistical inference on the $M|D|1$ steady state queues, restrictions on the parameter space are to be taken into account. In such situations, Bayesian procedures handle the restricted parameter spaces in a natural way. See Mukherjee and Chowdhury (2010).

In our $M|D|1$ queueing model, we assume the queue is in steady state and $\rho = \lambda b < 1$. For the system under consideration, we have discussed in detail the maximum likelihood and Bayesian procedures. We use Beta distribution of first kind as prior for the traffic intensity. This prior distribution is used to evaluate the posterior distribution along with Bayes estimators of traffic intensity, measures of system performance, minimum posterior risks, and minimum Bayes risks of the estimators. In Section 2, the model, MLEs of traffic intensity, other measures of system performance and UMVUEs of the expected waiting time per entity in the queue as well as system are derived. Bayesian analysis is carried out in detail in Section 3. Concluding remarks are given in Section 4.

2. System description

2.1. $M|D|1$ Queueing system

An $M|D|1$ queue is a stochastic process with the state space $\{0,1,2,\dots\}$, where the state value corresponds to the number of entities in the system, including any currently in service. $M|D|1$ queueing model is a special case of a general non Poisson model $(M|G|1):(GD|\infty|\infty)$. The concept of imbedded Markov chains was introduced by Kendall (1953) and it implies that $M|G|1$ queues can also be studied by observing the state of the system (number of entities in the system) at epochs of departure and the transition probability matrix (tpm) is determined by the probability that certain numbers of entity arrivals will occur during a service time. The advantage of this analysis is that the problem reduces to a discrete time Markov chain even though the original system is non Markovian. $M|D|1$ queueing system is a single channel system with Poisson arrivals at a mean rate λ and constant service time equal to b. Suppose $p_n(t)$ is the probability that there are n entities in the system at time t. Assume that the steady state probabilities $p_n = \lim_{t\to\infty} p_n(t)$ exist. Using the probability generating function (pgf)

$$P(z) = \sum_{n=0}^{\infty} p_n z^n, \quad |z| < 1$$

it can be shown that

$$P(z) = \frac{(1-\rho)(1-z)}{\left[1 - ze^{\rho(1-z)}\right]} \tag{1}$$

where $\rho = \lambda b < 1$. See Kashyap and Chaudhry (1988).

From Equation (1), the following measures of system performance can be obtained.

1. The expected number of entities in the system denoted by L_S is given by $L_S = \rho + \frac{\rho^2}{2(1-\rho)}$.
2. The expected number of entities in the queue denoted by L_Q is given by $L_Q = \frac{\rho^2}{2(1-\rho)}$.
3. The expected waiting time per entity in the system denoted by W_S is given by $W_S = b[1 + \frac{\rho}{2(1-\rho)}]$.
4. The expected waiting time per entity in the queue denoted by W_Q is given by $W_Q = \frac{b\rho}{2(1-\rho)}$.
5. The expected number of entities in the queue (assuming that N_Q, the number of entities in the queue in steady state is at least one) denoted by ${L^1}_Q$ is given by

$$\begin{aligned} {L^1}_Q &= E\left[N_Q | N_Q \neq 0\right] \\ &= \sum_{n=1}^{\infty} (n-1){p_n}^1 \end{aligned} \tag{2}$$

where ${p_n}^1 = \frac{p_n}{\sum_{n=2}^{\infty} p_n}$. See Gross and Harris (2010). After simplification, Equation (2) can be written as

$${L^1}_Q = \frac{L_S - 1 + p_0}{(1 - p_0 - p_1)} \tag{3}$$

Expanding the right hand side of Equation (1) in powers of z and after some algebraic simplification, we get p_x, the coefficient of z^x, $x = 0$, 1 namely $p_0 = (1-\rho)$ and $p_1 = (e^\rho - 1)(1-\rho)$. Substituting for p_0, p_1, and L_S in Equation (3), it is seen that

$$L^1{}_Q = \frac{\rho^2}{2(1-\rho)[1-(1-\rho)e^\rho]}$$

In the next section, the MLE and consistent estimator of traffic intensity in the $M|D|1$ queue based on the number of entity arrivals during the service time of an entity are obtained.

2.2. MLE of traffic intensity ρ and its asymptotic distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample of n observations drawn from Poisson distribution with probability mass function (pmf) given by

$$P[X = x] = \frac{e^{-\rho}\rho^x}{x!}, \quad x = 0, 1, 2, \ldots \tag{4}$$

where X_j denotes the number of entity arrivals during the service time of the jth entity. Further, it may be noted that the underlying $M|D|1$ queueing system will be in steady state if the parameter space of ρ is constrained by the interval (0,1). It can be seen that the MLE of ρ, namely $\hat{\rho}$ is $\hat{\rho} = \frac{y}{n}$, where $y = \sum_{i=1}^{n} x_i$. Clearly, Y is distributed as Poisson with parameter $n\rho$ with the pmf given by

$$\begin{aligned} P[Y = y] &= \frac{e^{-n\rho}(n\rho)^y}{y!} \\ &= f(y;\rho)(\text{say}), \quad y = 0, 1, 2, \ldots \end{aligned}$$

Thus the pmf of $\hat{\rho}$ is given by

$$\begin{aligned} P[\hat{\rho} = u] &= P[Y = nu] \\ &= \frac{e^{-n\rho}(n\rho)^{nu}}{(nu)!}, \quad u = \frac{y}{n} \end{aligned}$$

yielding $E(\hat{\rho}) = \sum_{y=0}^{\infty} \frac{y}{n} \frac{e^{-n\rho}(n\rho)^{nu}}{(nu)!}$. For large n, it is clear that $E(\hat{\rho}) = \rho$ and $\mathrm{Var}(\hat{\rho}) = \frac{\rho}{n} \to 0$ as $n \to \infty$, i.e., $\hat{\rho}$ is a consistent estimator of ρ. Since a consistent solution of the likelihood equation is normally distributed about the true value of the parameter, we have $\frac{\hat{\rho} - E(\hat{\rho})}{\sqrt{\mathrm{Var}(\hat{\rho})}} = \frac{\sqrt{n}(\hat{\rho} - \rho)}{\sqrt{\rho}} \xrightarrow{d} N(0,1)$ for large n.

2.3. MLEs of measures of system performance

In the previous section, the MLE of traffic intensity parameter ρ is shown to be $\hat{\rho} = \frac{y}{n}$, where $y = \sum_{i=1}^{n} x_i$. By the application of invariance property of maximum likelihood estimators, it is readily seen that the MLEs of various measures of system performance can be obtained by replacing ρ by $\hat{\rho}$.

2.4. UMVUE of the expected waiting time per entity in the queue

Consider the Poisson probability law given in Equation (4) and the corresponding parameter space $\Theta = \{\rho | 0 < \rho < 1\}$. In this section, we derive the UMVUE of the expected waiting time per entity in the queue given by $W_Q = \frac{b\rho}{2(1-\rho)}$. Since $Y = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for the Poisson family of probability mass functions induced by $X_1, X_2, \ldots, X_n$, the UMVUE of $\frac{b\rho}{2(1-\rho)}$ is derived by the application of Lehmann–Scheffe theorem. Hence, $\phi(y)$ is the UMVUE of $\frac{b\rho}{2(1-\rho)}$ provided $E_\rho[\phi(Y)] = \frac{b\rho}{2(1-\rho)}, 0 < \rho < 1$, which implies that

$$\begin{aligned}\sum_{y=0}^{\infty} \phi(y) \frac{e^{-n\rho}(n\rho)^y}{y!} &= \frac{b}{2}\rho(1-\rho)^{-1} \\ \Rightarrow \sum_{y=0}^{\infty} \phi(y) \frac{(n\rho)^y}{y!} &= \frac{b}{2}\rho\left(\sum_{r=0}^{\infty} \frac{(n\rho)^r}{r!}\right)\left(\sum_{y=0}^{\infty} \rho^y\right) \\ &= \frac{b}{2}\sum_{r=0}^{\infty}\sum_{y=0}^{\infty} \frac{n^r}{r!}\rho^{(r+y+1)}\end{aligned} \tag{5}$$

Let $k = r + y + 1$. It may be noted that $k \geq 1$ and hence for a fixed k, $(y, r) \in \{(0, k-1), (1, k-2), \ldots, (k-1, 0)\}$. In view of this, Equation (5) can be written as

$$\sum_{y=0}^{\infty} \phi(y) \frac{(n\rho)^y}{y!} = \frac{b}{2}\sum_{k=1}^{\infty}\sum_{r=0}^{k-1} \frac{n^r}{r!}\rho^k \tag{6}$$

The UMVUE of W_Q is obtained by equating the coefficient of $\rho^k, k = 0, 1, 2, \ldots$ in Equation (6). Thus the UMVUE of W_Q denoted by $\widehat{W_Q}$ is given by

$$\widehat{W_Q} = \phi(k) = \begin{cases} \frac{b}{2}\left(\sum_{r=0}^{k-1} \frac{n^r}{r!}\right)\frac{k!}{n^k}, & k = 1, 2, 3, \ldots \\ 0, & k = 0 \end{cases} \tag{7}$$

Remark: It may be noted that $W_S = b[1 + \frac{\rho}{2(1-\rho)}] = b + W_Q$. Hence, the UMVUE of W_S is given by $\widehat{W_S} = b + \widehat{W_Q}$, where $\widehat{W_Q}$ is given by Equation (7).

3. Bayesian analysis of $M|D|1$ queueing system

In this section, we derive the Bayes estimator for traffic intensity and various measures of system performance. Beta distribution of first kind with parameters (α, β) is taken as the prior for traffic intensity ρ. The prior distribution of ρ has the density function

$$\tau(\rho|\alpha, \beta) = \frac{1}{\beta_1(\alpha, \beta)}\rho^{\alpha-1}(1-\rho)^{\beta-1}, \quad 0 < \rho < 1 \tag{8}$$

We know that the $M|D|1$ queueing system will be in steady state, if the traffic intensity ρ is constrained by the parameter space $\Theta = \{\rho | 0 < \rho < 1\}$. Thus, the problem reduces to that of Bayes estimation under a constrained parameter space in Poisson distribution. Hence, it is appropriate to assume Beta distribution of first kind as the prior for ρ.

3.1. Bayes estimator of ρ

Assume that the traffic intensity ρ has prior distribution with the density given in Equation (8). The marginal pdf of $Y = \sum_{i=1}^{n} X_i$, which is called the predictive pdf is given by

$$\begin{aligned} f^*(y) &= \int_0^1 f(y;\rho)\tau(\rho|\alpha,\beta)d\rho \\ &= \frac{1}{\beta_1(\alpha,\beta)}\int_0^1 \frac{e^{-n\rho}(n\rho)^y}{y!}\rho^{\alpha-1}(1-\rho)^{\beta-1}d\rho \\ &= \frac{n^y}{y!\beta_1(\alpha,\beta)}\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j,\beta) \end{aligned}$$

Hence, the posterior distribution of ρ given the sample data $X_1, X_2, \ldots, X_n$ is given by

$$\begin{aligned} q(\rho|x_1,x_2,\ldots,x_n) &= \frac{f(y;\rho)\tau(\rho|\alpha,\beta)}{\int_0^1 f(y;\rho)\tau(\rho|\alpha,\beta)d\rho} \\ &= ke^{-n\rho}\rho^{(y+\alpha)-1}(1-\rho)^{\beta-1}, \quad 0<\rho<1 \end{aligned}$$

where $k = \left[\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j,\beta)\right]^{-1}$. It may be noted that the posterior distribution of ρ reflects both the prior information (α,β) and the sample information $y = \sum_{i=1}^{n} x_i$. Thus the Bayes estimator of ρ, say ρ^* given the sample data is given by

$$\begin{aligned} \rho^* &= E[\rho|X_1,X_2,\ldots,X_n] \\ &= \int_0^1 \rho q(\rho|x_1,x_2,\ldots,x_n)d\rho \\ &= k\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+1,\beta) \end{aligned} \tag{9}$$

3.2. Minimum posterior risk associated with Bayes estimator ρ^* of ρ

The minimum posterior risk associated with Bayes estimator ρ^* is given by

$$\begin{aligned} V_\rho(\rho^*|X_1,X_2,\ldots,X_n) &= E[\rho^*-\rho]^2 \\ &= E\left[\left(k\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+1,\beta)\right)-\rho\right]^2 \\ &= k^2\left\{\left(\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+2,\beta)\right)\left(\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j,\beta)\right)\right. \\ &\quad \left. -\left(\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+1,\beta)\right)^2\right\} \end{aligned}$$

3.3. Minimum Bayes risk of the estimator ρ^*

The marginal distribution $h(x_1, x_2, \ldots, x_n)$ of the data $X_1, X_2, \ldots, X_n$ is given by

$$\begin{aligned} h(x_1, x_2, \ldots, x_n) &= \int_0^1 L(\rho | x_1, x_2, \ldots, x_n) \tau(\rho | \alpha, \beta) d\rho \\ &= \frac{1}{\beta_1(\alpha, \beta) \prod_{i=1}^{n} x_i!} \sum_{j=0}^{\infty} \frac{(-n)^j}{j!} \beta_1(y + \alpha + j, \beta) \\ &= \frac{1}{k\beta_1(\alpha, \beta) \prod_{i=1}^{n} x_i!} \end{aligned}$$

Now, the minimum Bayes risk ν_{τ, ρ^*} of ρ^* is given by

$$\begin{aligned} \nu_{\tau, \rho^*} &= E\left[V_\rho(\rho^* | X_1, X_2, \ldots, X_n)\right] \\ &= \sum_{y=0}^{\infty} V_\rho(\rho^* | x_1, x_2, \ldots, x_n) h(x_1, x_2, \ldots, x_n) \\ &= \frac{k}{\beta_1(\alpha, \beta) \prod_{i=1}^{n} x_i!} \sum_{y=0}^{\infty} \left\{ \left(\sum_{j=0}^{\infty} \frac{(-n)^j}{j!} \beta_1(y + \alpha + j, \beta) \right) \left(\sum_{j=0}^{\infty} \frac{(-n)^j}{j!} \beta_1(y + \alpha + j + 2, \beta) \right) \right. \\ &\quad \left. - \left(\sum_{j=0}^{\infty} \frac{(-n)^j}{j!} \beta_1(y + \alpha + j + 1, \beta) \right)^2 \right\} \end{aligned}$$

3.4. Bayes estimator of L_S

The Bayes estimator of L_S, say L_S^* given the data $X_1, X_2, \ldots, X_n$ is given by

$$\begin{aligned} L_S^* &= E[L_S | X_1, X_2, \ldots, X_n] \\ &= E\left[(\rho + \frac{\rho^2}{2(1-\rho)}) | X_1, X_2, \ldots, X_n \right] \\ &= \int_0^1 \left(\rho + \frac{\rho^2}{2(1-\rho)} \right) q(\rho | x_1, x_2, \ldots, x_n) d\rho \\ &= k \sum_{j=0}^{\infty} \frac{(-n)^j}{j!} \beta_1(y + \alpha + j + 1, \beta) + \frac{k}{2} \sum_{j=0}^{\infty} \frac{(-n)^j}{j!} \beta_1(y + \alpha + j + 2, \beta - 1) \end{aligned}$$

3.5. Bayes estimator of L_Q

The Bayes estimator of L_Q, say L_Q^* given the data $X_1, X_2, \ldots, X_n$ is given by

$$\begin{aligned}
L_Q^* &= E\left[L_Q|X_1, X_2, \ldots, X_n\right] \\
&= E\left[\left(\frac{\rho^2}{2(1-\rho)}\right)|X_1, X_2, \ldots, X_n\right] \\
&= \frac{1}{2}\int_0^1 \left(\frac{\rho^2}{(1-\rho)}\right) q(\rho|x_1, x_2, \ldots, x_n) d\rho \\
&= \frac{k}{2}\sum_{j=0}^{\infty} \frac{(-n)^j}{j!}\beta_1(y+\alpha+j+2, \beta-1)
\end{aligned}$$

3.6. Minimum posterior risk associated with Bayes estimator L_Q^* of L_Q

The minimum posterior risk associated with Bayes estimator L_Q^* of L_Q is given by

$$\begin{aligned}
V_{L_Q}(L_Q^*|X_1, X_2, \ldots, X_n) &= E\left[L_Q^* - L_Q\right]^2 \\
&= \frac{k}{4}\left[\left(\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+4, \beta-2)\right)\right. \\
&\quad \left. - k\left(\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+2, \beta-1)\right)^2\right]
\end{aligned}$$

3.7. Minimum Bayes risk of the estimator L_Q^* of L_Q

The minimum Bayes risk ν_{τ, L_Q^*} of L_Q^* is given by

$$\begin{aligned}
\nu_{\tau, L_Q^*} &= \sum_{y=0}^{\infty} V_{L_Q}(L_Q^*|x_1, x_2, \ldots, x_n) h(x_1, x_2, \ldots, x_n) \\
&= \frac{k}{4\beta_1(\alpha, \beta)\prod_{i=1}^{n} x_i!}\sum_{y=0}^{\infty}\left\{\left(\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j, \beta)\right)\left(\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+4, \beta-2)\right)\right. \\
&\quad \left. - \left(\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+2, \beta-1)\right)^2\right\}
\end{aligned}$$

3.8. Bayes estimator of W_Q

The Bayes estimator of W_Q denoted by W_Q^* given the data $X_1, X_2, \ldots, X_n$ is given by

$$\begin{aligned}
W_Q^* &= E\left[W_Q|X_1, X_2, \ldots, X_n\right] \\
&= \frac{b}{2}\int_0^1 \frac{\rho}{(1-\rho)} q(\rho|x_1, x_2, \ldots, x_n) d\rho \\
&= \frac{kb}{2}\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+1, \beta-1) \qquad (10)
\end{aligned}$$

3.9. Bayes estimator of W_S

The Bayes estimator of W_S denoted by W_S^* given the data $X_1, X_2, ..., X_n$ is given by

$$\begin{aligned} W_S^* &= E[W_S|X_1, X_2, ..., X_n] \\ &= b\int_0^1 \left[1 + \frac{\rho}{2(1-\rho)}\right] q(\rho|x_1, x_2, ..., x_n)d\rho \\ &= b\left[1 + \frac{k}{2}\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j+1, \beta-1)\right] \end{aligned} \tag{11}$$

3.10. Estimation of ρ under LINEX loss function

The squared error loss function also known as quadratic loss function assumes the loss to be symmetric thereby treating overestimation and underestimation to be of equal magnitude. However this may not be true in real life situations as the consequences of overestimation and underestimation are not always same. Hence, it would be appropriate to use asymmetric loss functions to estimate the parameters. One such asymmetric loss function that is widely used in the context of Bayesian estimation is the LINEX loss function introduced by Varian (1975) and Zellner (1986). It is defined as

$$\mathcal{L}_L(\rho, \rho_L^*) = e^{h(\rho_L^*-\rho)} - h(\rho_L^* - \rho) - 1, \quad h \neq 0 \tag{12}$$

where ρ_L^* is the Bayes estimator of ρ under LINEX loss function and is given by

$$\rho_L^* = \frac{-1}{h}\log E(e^{-h\rho}|X_1, X_2, ..., X_n) \tag{13}$$

Now,

$$\begin{aligned} E(e^{-h\rho}|X_1, X_2, ..., X_n) &= \int_0^{\infty} e^{-h\rho}q(\rho|x_1, x_2, ..., x_n)d\rho \\ &= \frac{1}{\sum_{j=0}^{\infty}\frac{(-n)^j}{j!}\beta_1(y+\alpha+j, \beta)}\int_0^1 e^{-(h+n)\rho}\rho^{(y+\alpha)-1}(1-\rho)^{\beta-1}d\rho \\ &= \frac{\sum_{j=0}^{\infty}\frac{[-(h+n)]^j\beta_1(y+\alpha+j, \beta)}{j!}}{\sum_{j=0}^{\infty}\frac{(-n)^j\beta_1(y+\alpha+j, \beta)}{j!}} \end{aligned} \tag{14}$$

Substituting Equation (14) in Equation (13), the Bayes estimator of ρ under LINEX loss function is obtained.

3.11. Risk function of Bayes estimator ρ_L^* under LINEX loss

The risk function of Bayes estimator ρ_L^* under LINEX loss is given by

$$
\begin{aligned}
R_L(\rho,\rho_L^*) &= E\left[\mathcal{L}_L(\rho,\rho_L^*)\right] \\
&= \sum_{y=0}^{\infty}\left[e^{h(\rho_L^*-\rho)} - h(\rho_L^*-\rho) - 1\right]P(Y=y) \\
&= e^{-(h+n)\rho}\sum_{y=0}^{\infty}\frac{\sum_{j=0}^{\infty}\frac{[-(h+n)]^j\beta_1(y+\alpha+j,\beta)}{j!}}{\sum_{j=0}^{\infty}\frac{(-n)^j\beta_1(y+\alpha+j,\beta)}{j!}}\frac{(n\rho)^y}{y!} \\
&\quad + e^{-n\rho}\sum_{y=0}^{\infty}\log\left(\frac{\sum_{j=0}^{\infty}\frac{[-(h+n)]^j\beta_1(y+\alpha+j,\beta)}{j!}}{\sum_{j=0}^{\infty}\frac{(-n)^j\beta_1(y+\alpha+j,\beta)}{j!}}\right)\frac{(n\rho)^y}{y!} \\
&\quad + h\rho - 1
\end{aligned}
$$

4. Conclusion

Maximum likelihood and consistent estimators of traffic intensity in an $M|D|1$ queue are obtained based on the number of entity arrivals during service time of an entity by assuming the underlying stationary distribution to be Poisson. It is observed that the average number of entity arrivals in the system during the service time is the maximum likelihood estimator of traffic intensity. In addition, UMVU estimators for expected waiting time per entity in the queue and expected waiting time per entity in the system are obtained. Further, Bayes estimators of measures of system performance along with their minimum posterior risk and minimum Bayes risk are derived.

Acknowledgments

The authors thank the referee for useful comments and suggestions, which have significantly helped in the presentation of the article.

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