# CSCB63 Assignment 1

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## Q1

### Q1-a

Q: $If\ f(n) \in \mathcal{O}(g)\ and\ g \in \mathcal{O}(h)\ then\ f \in \mathcal{O}(h), for\ all\ f, g, h\ in\ \mathbb{N} \to \mathbb{R}^{+}$

*Proof.*

1) From $f \in$ O(g) $we\ have:$

$$\exists c_1, n_1 \in \mathbb{R} \text{ s.t. } f(n) \leq c_1 \cdot g(n) \forall n \geq n_1$$

2) From $g \in$ O(h) $we\ have:$

$$\exists c_2, n_2 \in \mathbb{R} \text{ s.t. } g(n) \leq c_2 \cdot h(n) \forall n \geq n_2$$

3) From 1) and 2) we have:

$$f(x) = \begin{cases} f(n) \leq c_1 \cdot g(n) \forall n \geq n_0, \text{for some } c_1 \text{ and } n_1 \\ g(n) \leq c_2 \cdot h(n) \forall n \geq n_1, \text{for some } c_2 \text{ and } n_2 \end{cases} \Rightarrow f(n) \leq c_1(c_2 \cdot h(n)) \forall s.t. \begin{cases} n \geq n_1 \\ n \geq n_2 \end{cases}$$

4) From 3) we have:

$$f(n) \leq c_1 \cdot c_2 \cdot h(n)\ \forall n \geq max(n_1, n_2)$$

$$\Rightarrow f \in \mathcal{O}(h)$$

□

## Q1-b

Q: $If\ f \in \Omega(g)\ and\ g \in \Omega(h)\ then\ f \in \Omega(h), for\ all\ f, g, h\ in\ \mathbb{N} \rightarrow \mathbb{R}^+$

*Proof.*

1) From $f \in \Omega(g)$ we have:

$$\exists c_1, n_1 \in \mathbb{R} \text{ s.t. } f(n) \geq c_1 \cdot g(n) \forall n \geq n_1$$

2) From $g \in \Omega(h)$ we have:

$$\exists c_2, n_2 \in \mathbb{R} \text{ s.t. } g(n) \geq c_2 \cdot h(n) \forall n \geq n_2$$

3) From 1) and 2) we have:

$$f(x) = \begin{cases} f(n) \geq c_1 \cdot g(n) \forall n \geq n_0, \text{for some } c_1 \text{ and } n_1 \\ g(n) \geq c_2 \cdot h(n) \forall n \geq n_1, \text{for some } c_2 \text{ and } n_2 \end{cases} \Rightarrow f(n) \geq c_1(c_2 \cdot h(n)) \forall s.t. \begin{cases} n \geq n_0 \\ n \geq n_1 \end{cases}$$

4) From 3) we have:

$$f(n) \geq c_1 \cdot c_2 \cdot h(n)\ \forall n \geq max(n_0, n_1)$$

$$\Rightarrow f \in \Omega(h)$$

□

## Q1-c

Q: $\log_\phi(\sqrt{5}(n+2)) - 2 \in \mathcal{O}(\log_2(n))$ where $\phi$ is the golden ratio.

*Proof.* Let $f(n) = \log_\phi(\sqrt{5}(n+2)) - 2, \forall n \geq 0$

Then it follows that:

$$f(n) = \log_\phi(\sqrt{5}(n+2)) - 2 = \frac{\log_2(\sqrt{5}(n+2))}{\log_2(\phi)} - 2$$

$$f(n) = \frac{\log_\phi \sqrt{5}}{\log_2(\phi)} \cdot \log_2(n+2) - 2$$

$$f(n) < \frac{\log_\phi \sqrt{5}}{\log_2(\phi)} \cdot \log_2(n+2) = k \cdot \log_2(n+2) \ \forall n \geq 0$$

where $k = \frac{\log_\phi \sqrt{5}}{\log_2(\phi)} \in \mathbb{R}$

$$\Rightarrow f(n) < k \cdot \log_2(n+2) \leq k \cdot \log_2(n^2) = k \cdot 2 \cdot \log_2(n) \ \forall n \geq 2$$

which is equivalent to:

$$f(n) < 2k \cdot \log_2(n) \ \forall n \geq 2$$

$$\Rightarrow f(n) \in \mathcal{O}(\log_2(n))$$

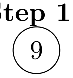
□

# Q2

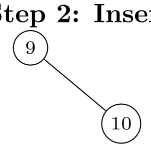
## Q2-a

Q: On an initially empty tree, show each step of inserting the keys 9, 10, 12, 14, 3, 34, 19, 37, 20.

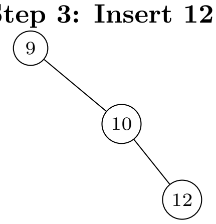
**Step 1: Insert 9**




**Step 2: Insert 10**



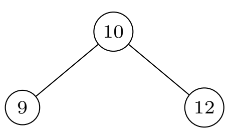

**Step 3: Insert 12**



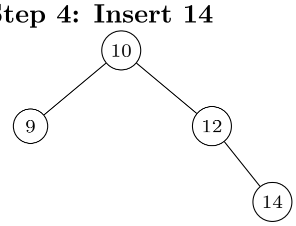

Imbalanced. Right heavy.

Need a left rotation.

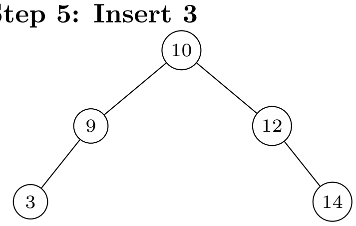
Left rotation:



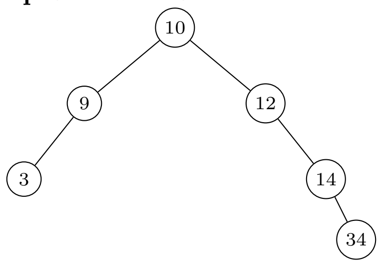

**Step 4: Insert 14**




**Step 5: Insert 3**



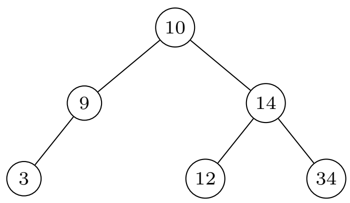
**Step 6: Insert 34**



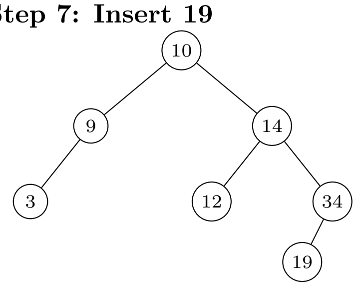
Imbalanced. Right heavy.

Need a left rotation.

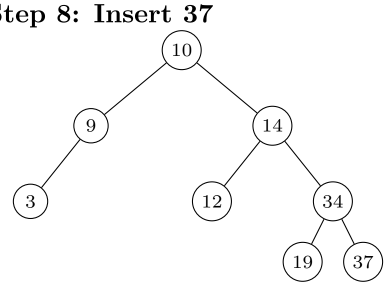
Left rotation:



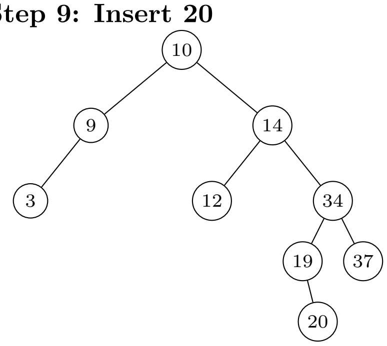
**Step 7: Insert 19**



**Step 8: Insert 37**



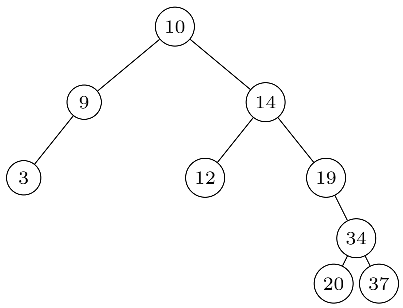
**Step 9: Insert 20**



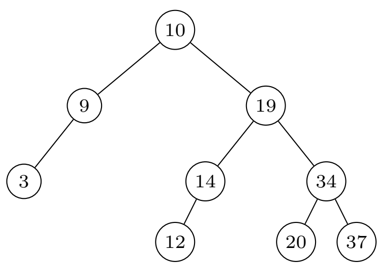
Imbalanced. Right heavy.

Need a double rotation.

1. Right rotation:




2. Left rotation:

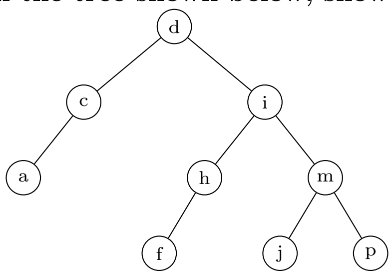



$\blacksquare$

## Q2-b-1

Q:

**On the tree shown below, show each step of deleting the keys p, d, h.**




**Step 1: Delete p**

Hold on, Latex is not a good fit for this question.

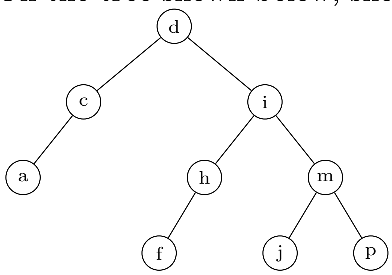
I will leave some space here for the hand-drawn tree.

■

## Q2-b-2

Q:

**On the tree shown below, show each step of deleting the keys p, d, h.**




**Step 2: Delete d**

Hold on, Latex is not a good fit for this question.

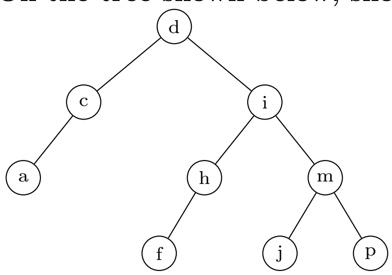
I will leave some space here for the hand-drawn tree.

■

## Q2-b-3

Q:

**On the tree shown below, show each step of deleting the keys p, d, h.**




**Step 3: Delete h**

Hold on, Latex is not a good fit for this question.

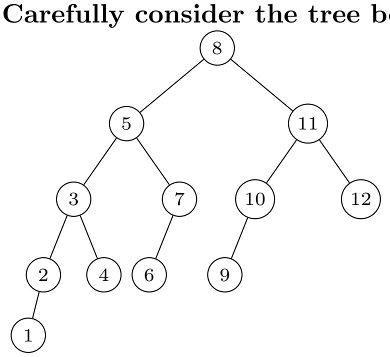
I will leave some space here for the hand-drawn tree.

■

## Q2-c

Q:

**Carefully consider the tree below.**




- If we start with an empty AVL tree, what sequence of insertions would result in this tree?

- Show each step of deleting key 11.

As usual, I will leave some space here for the hand-drawn tree.

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## Q4

Q: Consider an ADT consisting of a set S of distinct integers and the following operations:

- search(S, x): Return true if x is in S and false otherwise.
- insert(S, x): Insert the element x into the set S. This operation has no effect if x is already in S.
- delete(S, x): Delete the element x from the set S. This operation has no effect if x is not in S.
- min difference(S): Given a set S with size of at least 2, return a pair of distinct integers (x, y), with x $\in$ S, y $\in$ S, with the minimum absolute difference, i.e.
  $\forall x', \forall y', (x' \neq y' \rightarrow |x - y| \leq |x' - y'|)$

Your task is to design a data structure to implement this ADT, such that all operations are performed in $\mathcal{O}(\log\ n)$ time, where $n = |S|$. You will do so by augmenting our familiar AVL tree.

1. Describe all information that will be stored in the nodes.
2. Provide pseudo-code for each required operation.
3. Justify why your algorithms are correct and why they achieve the required time bound.

■