Technical University of Denmark
Department of Applied Mathematics and Computer Science
02458 Cognitive Modeling

Assignment 2 Report

Homework 2

Jan Heimes

Examiner:
Tobias Andersen

processing period: 4 weeks   date of submission: 29.10.2020

Technical University
of Denmark

I. Table of contents

processing period: 3 weeks

date of submission: 24.09.2020

II. Nomenclature

Acronyms

CDF	Cumulative distribution function
PPF	Percent Point Function
SVD	Singular-Value Decomposition
wl	wavelength
S	Structure
I	Image
m	Mongo
ju	Juju
Ck	Chakava
SD	Standart deviation

III. List of Figures

IV. List of Tables

1 Necker

In the following exercises it is solved a perceptual problem using the Bayesian approach. The problem is to infer a 3-dimensional structure from a 2-dimensional image. The structure is called S and the image is I. For the Bayesian approach it is to maximise the posterior probability P(S|I). By maximising the posterior probability of a shape, we are trying to find the real world shape which best resembles our spatial perception of the image. However, in the exercises we'll minimize the negative logarithm of the posterior probability.

1.1 Cube

The function used to find the arg max of the posterior probability is:

$$arg\ min\ [\frac{1}{\sigma^2_{noise}}(I_{perceived} - I_{projected})^2 + \frac{1}{\sigma^2_{prior}}(S - \hat{S})^2]$$

We will start by assuming a uniform prior probability, which allows us to remove
the prior as a parameter for the model. The equation then becomes: P(S|I) = P(I|S). Which means that the posterior probability is equal to the likelihood.
Below are the results of only using the likelihood.

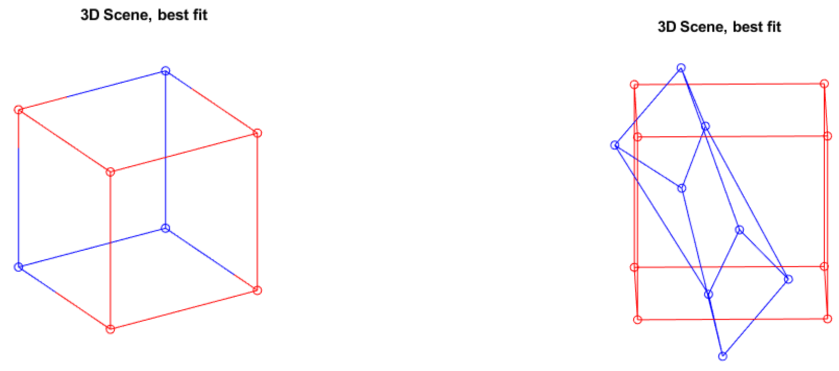
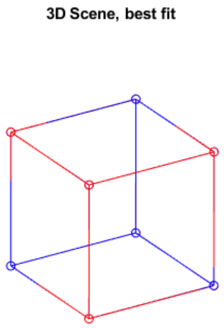


Figure 1 Picture of the best fit (Blue) and the real cube (Red) as well as a random rotation to check the correctness of the fit

One could easily be fooled in to believing that the solution found a correct
fit, however upon further inspection that is not the case.
From the initial perceived rotation, the fit is "correct". However rotating the image shows that the gitt is "squashed" and doesn't resemble the cube at all. This is caused by the assumption that the prior is uniform. This means that each point on the cube has an equal probability of

existing and therefore allowing our minimization problem to be dependant on the σ_{noise}, which is the noise of the visual perception of our image. However, using a prior will most likely give us better results. We'll do this by utilizing that we know that angles in a cube are 90 degree, and can therefore use it in our prior. By also taking into consideration the importance of angles and edges, we'll get a much better result. By selecting a σ_{prior} = 100000, we get the following results:



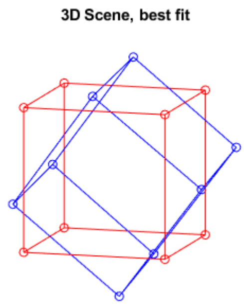


Figure 2 Picture of the best fit (Blue) and the real cube (Red) as well as a random rotation to check the correctness of the fit σ_{prior} = 100000 was used

1.2 Hexagonal cylinder

Now we'll do the same for a hexagonal cylinder instead of a cube, we'll use both a uniform prior, the prior used before and a new prior, which works better for this problem than a cube.

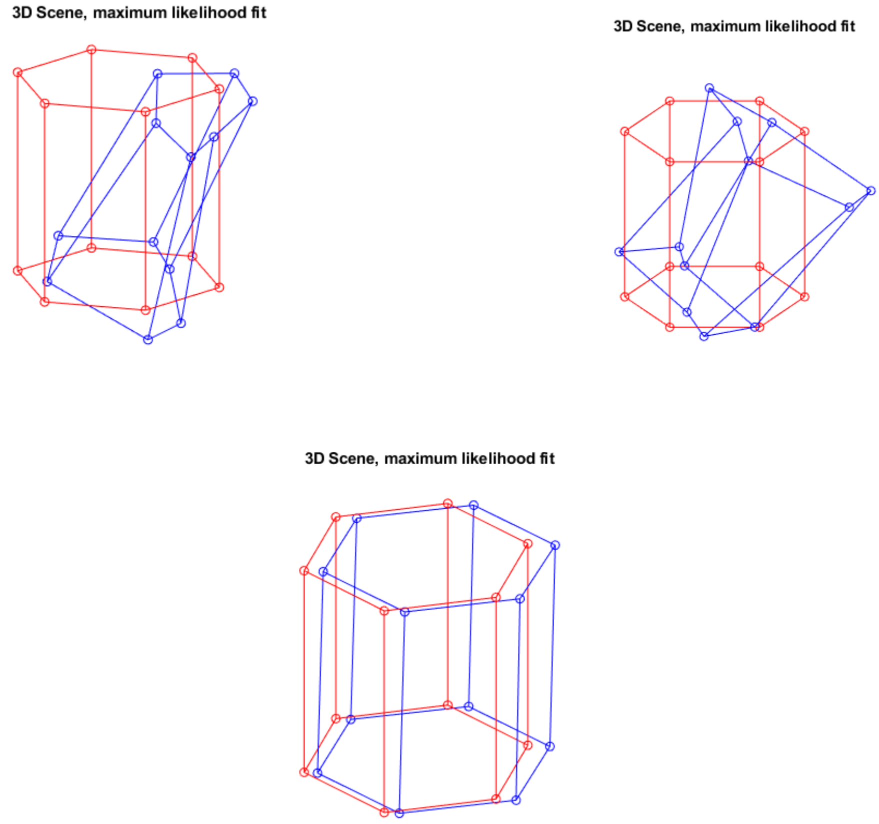


Figure 3 Picture of the best fit (Blue) and the real hexagonal cylinder (Red), The first figure shows a uniform prior, the second shows the prior used for the cube and the third shows a prior specific for this problem. A σ_{prior} of 100000 was used

Using a uniform and cube prior, won't work in this scenario as they don't correspond to the properties of the hexagonal cylinder. Using a prior similar tothe cube, but using angles which better describe a hexagonal cylinder, i.e 120 degree, gives us a much better representation of this shape. Although not a perfectvmodel, it works well enough to give us a good fit. In our brains we'd most likely do something similar, however also taking into consideration that the angles from the base up the height of the cylinder are 90 degree.

1.3 Explanation of unnecessary prior

When we view a projection of an object that we look on, one can imagine that to find and understand the correct form, we would have to perceive the object from two different viewpoints. One separate image is projected on each of our eyes. We now introduce an extra structure S2 and the posterior probability is thus now:

$$P(S_1, S_2|I) = \frac{P(I|S_1, S_2)P(S_1, S_2)}{P(I)}$$

As we are only interested in the presence of 1 actual shape of an object and not how the viewpoints perceive it, we will assume that we have a generic viewpoint independent of binocular disparity.
Since a prior would give us similar information to having multiple viewpoints, like the perception of angles, length of sides etc. we will assume a uniform prior distribution and we can now link the posterior and the likelihood.

$$P(S|I) = \sum_{i=0}^{S} P(I|S_i)$$

To conclude, we don't need a prior as an extra viewpoint makes the prior needless, because it gives a better understanding of the shape and angles of the object we are looking at and we can use that as a 'prior'. See changes to matlab file in attachments.

2 Bayesian Modeling of Vision

2.1 Juju fruit is ripe

It is to determine the probability of a fruit to be ripe which reflects light with a wavelength in the spectrum 540-550 nm. Juju-fruits, which are ripe on average reflect light with a wavelength of about 600 nm. On the other hand, unripe fruits are green and to reflect light with a wavelength of about 500 nm. A standard deviation (SD) of 50 nm is given. 15% of all fruits are ripe [P(ripe)=0,15]. Hence, we can find the probability that a fruit is ripe given the wavelength is in between 540 and 550 nm (wl = 540 < wavelength <550).

$$P(\text{ripe}|\text{wl}) = \frac{P(wl|ripe) * P(ripe)}{P(wl)}$$

With the aid of Bayes rule:

$$P(\text{ripe}|\text{wl}) = \frac{P(wl|ripe) * P(ripe)}{P(wl|ripe) * P(ripe) + P(wl|unripe) * P(unripe)}$$

The wl given that a fruit is ripe and unripe:

$$P(wl|ripe) = P\left(\frac{550-600}{50}\right) - P\left(\frac{540-600}{50}\right)$$

$$P(wl|unripe) = P\left(\frac{550-500}{50}\right) - P\left(\frac{540-500}{50}\right)$$

With the z-table of the Gaussian distribution the Probabilities for both cases above are calculated.

- $P(wl|ripe) = 0{,}15866 - 0{,}11507 = 0{,}0435$
- $P(wl|unripe) = 0{,}84134 - 0{,}78814 = 0{,}0532$

Placing those two values inside the function above for the probability that a Juju fruit is ripe given the wavelength *P(ripe|wl)*. The probability for a fruit reflecting light with a wl between 540 and 550 whiles being ripe is **12,6 %.**

2.2 Random fruit is ripe

The second question inherits from the first question. Now several fruits are taken into account. Table 1 shows an overview of the information given about the fruits, including their relative distribution in the jungle, state distributions, wavelengths in nm and standard deviations in nm.

Fruit	%Appearance (J)	State	State	Wl(nm)	SD(nm)
Juju-fruit(ju)	10	P(ripe)	0,15	600	50
		P(unripe)	0,85	500	50
Mango(m)	40	P(ripe)	0,80	580	20
		P(unripe)	0,20	520	20
Chakava(ck)	50	P(ripe)	0,10	400	100
		P(unripe)	0,90	550	100

Table 1 Fruits in the Jungle and their attributes

Now it is needed to calculate the probability of the wavelength given that a fruit is ripe or unripe like it is done as for the Juju fruit in Question 1, but the denominator is the sum over all fruits. Where the sum goes from the first until the last fruit. In this case 1 to 3. (Juju-fruit, Mango, Chakava)

$$denominator = \sum_{n=1}^{3} P(wl|ripe,n) * P(ripe,n) + P(wl|unripe,n) * P(unripe,n)$$

The wl given that a Mongo fruit is ripe in the spectrum of 550-540 nm wavelength:

$$P(wl|ripe,m) = P\left(\frac{550-580}{20}\right) - P\left(\frac{540-580}{20}\right) = 0{,}06681 - 0{,}02275 = 0{,}04406$$

The wl given that a Mongo fruit is unripe in the spectrum of 550-540 nm wavelength:

$$P(wl|unripe,m) = P\left(\frac{550-520}{20}\right) - P\left(\frac{540-520}{20}\right) = 0{,}93319 - 0{,}84132 = 0{,}09185$$

$$P(ripe,m) = 0{,}8 \qquad 1-P(ripe,m) = P(unripe,m) = 0{,}2$$

$$P(\text{ripe,m|wl}) = \frac{P(wl|ripe,m) * P(ripe,m)}{P(wl)} = \frac{P(wl|ripe,m) * P(ripe,m)}{\sum P(wl|ripe,n) * P(ripe,n) + P(wl|unripe,n) * P(unripe,n)}$$

The same procedure is done for the Chakava fruit leads to $P$and the juju fruit. This leads to:

- *P(ripe,m|wl)=**37,6%.***
- *P(ripe,ck|wl)= **1,12%***
- *P(ripe,ju|wl)= **1,4%***

Hence, the probability a chakava fruit is picked and reflecting light with a wavelength in the spectrum of 540-550 nm being ripe is 1,12% for a Mango 37,6% and for the Juju-fruit 1,4%.

The probability that a random fruit is ripe given the wavelength of 550-540 nm is

$$\sum P(ripe, fruit|wl) = 40{,}2\%$$

So, the chance is lower than 50%. But if a monkey sees another monkey eating already the fruit and enjoying it the possibility of the fruit being unripe falls apart.

2.3 Separation of ripe and unripe fruits

In the implementation, the maximum posterior decision rule has been implemented as an evaluation of the probability that a given fruit reflecting light in the seen (from the monkey's perspective) spectrum is ripe. If that probability is 50% or larger (inclusive), the posterior for the fruit being ripe is larger than that of the posterior for the fruit being unripe.

➔ The monkey decides to pick the fruit. Please refer to appendix for the *Python* code.

The monkey is able separating ripe from unripe fruits (Figure 4). It is observable that on average the monkey picks around 90% of the ripe fruits and 10% of the unripe fruits, leaving 10% of the ripe fruits and 90% of the unripe fruits. There is some error, but assuming that the cost of picking an unripe fruit is the same as leaving a ripe fruit, then the monkey's performance seems good.

```
There were 455 ripe fruits

The monkey picked 400 ripe fruits (87.91208791208791 percent of ripe fruits)

The monkey left 55 ripe fruits (12.087912087912088 percent of ripe fruits)

There were 545 unripe fruits

The monkey picked 67 unripe fruits (12.293577981651376 percent of unripe fruits)

The monkey left 478 unripe fruits (87.70642201834863 percent of unripe fruits)
```

Figure 4 Result of simulating the monkey picking fruits probability with Python

3 Signal detection

This exercise is about applying a Bayesian approach to signal detection theory. The question poses a situation in which an observer with d’ = 1.5 does signal detection in three different conditions with each condition having a different probability for trials containing a signal.

- Condition 1: 50%
- Condition 2: 95%
- Condition 3: 15%

An equal variance SDT is assumed. *d’* can be interpreted as the sensitivity of the observer. The 2nd lecture states an equation for the relation between the signal probability, d’ and the criterion c in the case of the max a posteriori rule. Therefore, the last slide of the second lecture is of great importance and the following calculation is based on it. A maximum a posteriori probability (MAP) estimate is an estimate of an unknown quantity, that equals the mode of the posterior distribution. MAP can be used to obtain a point estimate of an unobserved quantity on the basis of empirical data.
d’ and the signal probability in each condition is already given.
To obtain the criterion c in each condition, it is needed to plug in the signal probability and *d’* and then isolate it.

Starting with the MAP decision rule

$$P(signal\ |x) > P(noise\ |x)$$

Insert Bayes to introduce the prior:

$$\frac{P(x \mid signal)P(signal)}{P(x)} > \frac{P(x \mid noise)P(noise)}{P(x)}$$

using: $P(signal) = 1 - P(noise)$

$$\frac{P(x \mid signal)}{P(x \mid noise)} > \frac{1 - P(signal)}{P(signal)}$$

Insert Gaussian probability density function (σ = 1) take natural logarithm and replace the equation so that P(signal) stands alone.

$$x^2 - (x - d')^2 > \ln\left(\frac{1 - P(signal)}{P(signal)}\right)$$

$$\mathrm{x} > \frac{\ln\left(\frac{1 - P(signal)}{P(signal)}\right) + \frac{d'^2}{2}}{d'} = \frac{\ln\left(\frac{1 - P(signal)}{P(signal)}\right)}{d'} + \frac{d'}{2} = c$$

$$\mathrm{P(signal)} = \frac{1 - \exp\left(-\frac{(c - d')^2}{2}\right)^{\frac{1}{\sqrt{2\pi}}} - \left(-\frac{c^2}{2}\right)^{\frac{1}{\sqrt{2\pi}}}}{\exp\left(-\frac{(c - d')^2}{2}\right)^{\frac{1}{\sqrt{2\pi}}} - \left(-\frac{c^2}{2}\right)^{\frac{1}{\sqrt{2\pi}}}}$$

The equations have been solved via an programmable Calculator. Hence the following results are obtained.

- Condition 1 (50%): -1,29
- Condition 2 (95%): -1,38
- Condition 3 (15%): -0,96

Finally, higher signal probability creates a lower criterion value (more leftward criterion). This can be explained through the fact, that the observer says more times yes.

4 Bayesian Multisensory integration

In this exercise an observer has completed 50 trials in an auditory, a visual and an audio-visual spatial localisation task. The goal of this experiment is to show, that human are generally better in a perceptual task if they use more than one sensory modality to perform it. Hence human are much better at localizing a sound if it is accompanied by a visual stimulus, than audio stimulus alone. First of all, it is to estimate the free parameters.
Using the gaussian distribution relation described in the exercise:

$$\frac{\phi(x_A|\mu_A,\sigma_A)\phi(x_V|\mu_V,\sigma_V)}{P(x)} = \phi(x|\mu_{AV},\sigma_{AV})$$

Where,

$$\mu_{AV} = w\mu_A + (1 - w)\,\mu_V$$

$$w = \frac{\sigma_V^2}{\sigma_A^2 + \sigma_V^2}$$

$$\sigma_{AV}^2 = \frac{\sigma_A^2\sigma_V^2}{\sigma_A^2 + \sigma_V^2}$$

We use a optimizer function in python finding the minimum values of the free parameters:

$$\mu_A = -15.67$$

$$\sigma_A = 1.92$$

$$\mu_V = 22.82$$

$$\sigma_V = 2.91$$

$$\mu_{AV} = -4.01$$

$$\sigma_{AV} = 1.60$$

With the free optimized parameters, we find the negative log likelihood:

$$negLL \approx 322{,}55$$

We use the new optimized free parameters to make a Gaussian distribution for each of the localisation tasks and plot it against the original data.

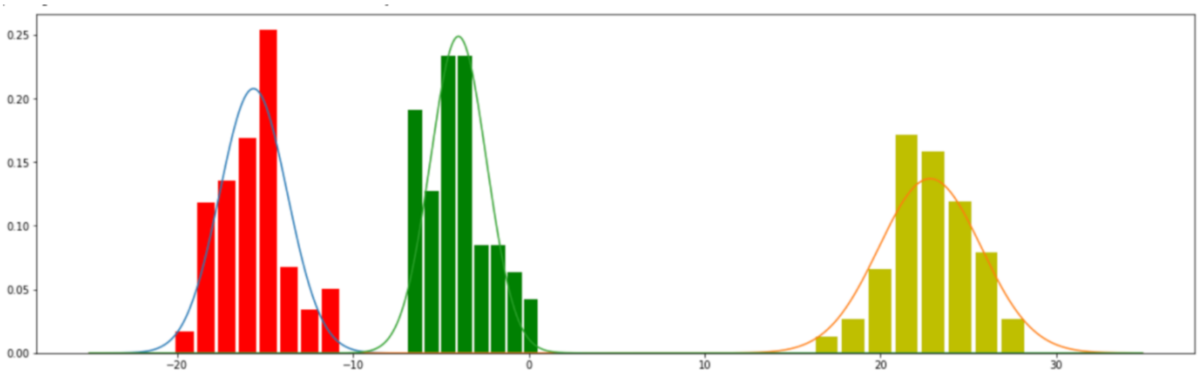


Figure 5 Plot of the data from an audio, visual and audiovisual trial, with Gaussian distributions over each of them using optimized parameters.

As we see in 4 the probability densities fit the histograms in a somewhat good fit. It is not perfect, but it also looks like the data is non uniform which can be due to human 'error'. Making the same experiment using data from a student which tends to zone out. We get the free optimized parameters to be

$$\mu_A = -16.13$$
$$\sigma_A = 3.89$$
$$\mu_V = 22.16$$
$$\sigma_V = 5.47$$
$$\mu_{AV} = -3.28$$
$$\sigma_{AV} = 3.17$$

With the free optimized parameters, we find the negative log likelihood:

$$negLL \approx 423{,}42$$

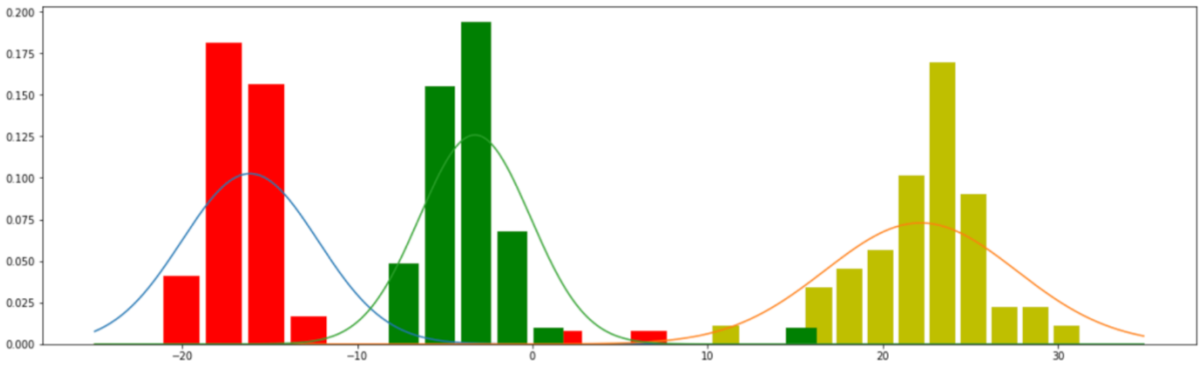


Figure 6 Plot of the data from an audio, visual and audiovisual trial, with Gaussian distribution over each of the, using optimized parameters. This time 4% of the data being random

This fit is not as correct as the fit above in Figure 5. As it is possible to observe there are several outliers in each dataset thus creating higher variance, and the fit worse.

See attached code for further explanations.