10.3.2.3.1

EE24BTECH11030 - KEDARANANDA

Question:
On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following pair of linear equations are consistent, or inconsistent .

$$3x + 2y = 5; 2x - 3y = 7 \tag{1}$$

Theoritical Solution: To determine whether the given pair of linear equations is consistent or inconsistent, we compare the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, and $\frac{c_1}{c_2}$, where:

$$a_1x + b_1y = c_1 \quad \text{and} \quad a_2x + b_2y = c_2 \tag{2}$$

From the equations:

$$3x + 2y = 5 \quad \text{and} \quad 2x - 3y = 7, \tag{3}$$

we identify:

$$a_1 = 3, \ b_1 = 2, \ c_1 = 5, \ a_2 = 2, \ b_2 = -3, \ c_2 = 7. \tag{4}$$

Now calculate the ratios:

$$\frac{a_1}{a_2} = \frac{3}{2}, \quad \frac{b_1}{b_2} = \frac{2}{-3}, \quad \frac{c_1}{c_2} = \frac{5}{7}. \tag{5}$$

Since:

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}, \tag{6}$$

the given pair of equations is **consistent** and the lines represented by the equations are **intersecting**. Therefore, the system of equations has a unique solution.

Computational Solution:

Solution using LU Factorization

Given the system of linear equations:

$$3x + 2y = 5, \tag{7}$$
$$2x - 3y = 7. \tag{8}$$

We rewrite the equations as:

$$x_1 = x, \tag{9}$$
$$x_2 = y, \tag{10}$$

giving the system:

$$3x_1 + 2x_2 = 5, \tag{11}$$
$$2x_1 - 3x_2 = 7. \tag{12}$$

Step 1: Convert to Matrix Form

We write the system as:

$$A\mathbf{x} = \mathbf{b}, \tag{13}$$

where:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix}, \tag{14}$$
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{15}$$
$$\mathbf{b} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}. \tag{16}$$

Step 2: LU Factorization of Matrix A

We decompose A as:

$$A = LU, \tag{17}$$

where L is a lower triangular matrix and U is an upper triangular matrix. Let:

$$L = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix}, \tag{18}$$
$$U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}. \tag{19}$$

First, we compute U:

$$u_{11} = 3, \tag{20}$$
$$u_{12} = 2, \tag{21}$$
$$\ell_{21} = \frac{2}{3}, \tag{22}$$
$$u_{22} = -3 - \left(\frac{2}{3} \cdot 2\right) = -\frac{13}{3}. \tag{23}$$

Thus, the LU decomposition is:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}, \tag{24}$$
$$U = \begin{bmatrix} 3 & 2 \\ 0 & -\frac{13}{3} \end{bmatrix}. \tag{25}$$

Step 3: Solve $L\mathbf{y} = \mathbf{b}$ (Forward Substitution)

We solve:

$$L\mathbf{y} = \mathbf{b} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}. \tag{26}$$

From the first row:

$$y_1 = 5. \tag{27}$$

From the second row:

$$\frac{2}{3}y_1 + y_2 = 7 \quad \Longrightarrow \quad \frac{2}{3} \cdot 5 + y_2 = 7 \quad \Longrightarrow \quad y_2 = \frac{11}{3}. \tag{28}$$

Thus:

$$\mathbf{y} = \begin{bmatrix} 5 \\ \frac{11}{3} \end{bmatrix}. \tag{29}$$

Step 4: Solve $U\mathbf{x} = \mathbf{y}$ (Backward Substitution)

We solve:

$$U\mathbf{x} = \mathbf{y} \quad \text{or} \quad \begin{bmatrix} 3 & 2 \\ 0 & -\frac{13}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{11}{3} \end{bmatrix}. \tag{30}$$

From the second row:

$$-\frac{13}{3}x_2 = \frac{11}{3} \quad \Longrightarrow \quad x_2 = -\frac{11}{13}. \tag{31}$$

From the first row:

$$3x_1 + 2x_2 = 5 \quad \Longrightarrow \quad 3x_1 + 2\left(-\frac{11}{13}\right) = 5, \tag{32}$$

$$3x_1 - \frac{22}{13} = 5 \quad \Longrightarrow \quad 3x_1 = 5 + \frac{22}{13} = \frac{65}{13} + \frac{22}{13} = \frac{87}{13}, \tag{33}$$

$$x_1 = \frac{87}{39} = \frac{29}{13}. \tag{34}$$

Thus:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{29}{13} \\ -\frac{11}{13} \end{bmatrix}. \tag{35}$$

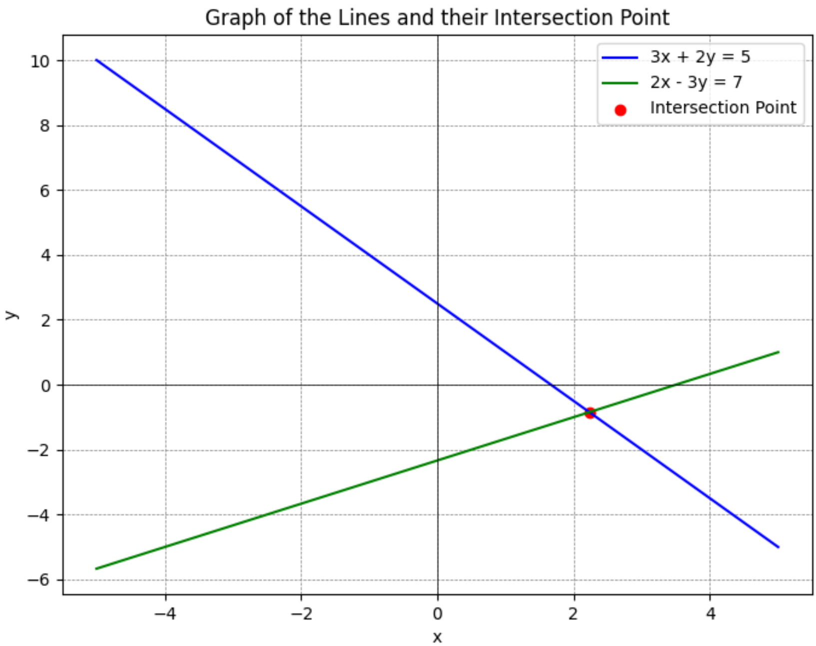
Final Solution

The solution is:

$$x = \frac{29}{13}, \tag{36}$$

$$y = -\frac{11}{13}. \tag{37}$$

As we can clearly see that there is solution for the given lines these are consistent


Graph of the Lines and their Intersection Point
3x + 2y = 5
2x - 3y = 7
Intersection Point
10
8
6
4
2
0
−2
−4
−6
y
−4
−2
0
2
4
x