12.9.ex.19

EE24BTECH11030 - KEDARANANDA

Question:
Solve the differential equation:
$y' = y + \cos x$
Solution:
Theoretical solution:
The given differential equation is a first-order linear ordinary differential equation. Let $y(0) = c_1$. By the definition of the Laplace transform,
$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$
Some used properties of the Laplace transform include:

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) \tag{1}$$
$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{2}$$
$$\mathcal{L}(e^{at} f(t)) = F(s-a), \quad \text{where } F(s) = \mathcal{L}(f(t)) \tag{3}$$
$$\mathcal{L}(\cos x) = \frac{s}{s^2+1} \tag{4}$$

Applying the Laplace transform to the given differential equation:
$y' - y = \cos x$
Take the Laplace transform on both sides:

$$\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(\cos x) \tag{5}$$

Using the properties of the Laplace transform:

$$(s\mathcal{L}(y) - y(0)) - \mathcal{L}(y) = \frac{s}{s^2+1} \tag{6}$$

Let $\mathcal{L}(y) = Y(s)$. Substituting $y(0) = c_1$, we get:

$$sY(s) - c_1 - Y(s) = \frac{s}{s^2+1} \tag{7}$$

Simplify:

$$(s-1)Y(s) = c_1 + \frac{s}{s^2+1} \tag{8}$$
$$Y(s) = \frac{c_1}{s-1} + \frac{s}{(s^2+1)(s-1)} \tag{9}$$

Partial fraction decomposition:
For $\frac{s}{(s^2+1)(s-1)}$, decompose into:

$$\frac{s}{(s^2+1)(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \tag{10}$$

Solve for A, B, and C by equating numerators:

$$s = A(s^2+1) + (Bs+C)(s-1) \tag{11}$$

Equating coefficients:

$$A + B = 0 \quad \text{(coefficient of } s^2\text{)} \tag{12}$$
$$-B + C = 1 \quad \text{(coefficient of } s\text{)} \tag{13}$$
$$A - C = 0 \quad \text{(constant term)} \tag{14}$$

The partial fraction decomposition becomes:

$$\frac{s}{(s^2+1)(s-1)} = \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1} \tag{15}$$

Rewrite $Y(s)$:

$$Y(s) = \frac{c_1}{s-1} + \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1} \tag{16}$$

Combine terms:

$$Y(s) = \frac{c_1 + \frac{1}{2}}{s-1} - \frac{\frac{1}{2}s}{s^2+1} + \frac{\frac{1}{2}}{s^2+1} \tag{17}$$

Take the inverse Laplace transform:

Using the properties of the Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t \tag{18}$$
$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t \tag{19}$$
$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t \tag{20}$$

Apply the inverse transform to each term:

$$y(t) = \left(c_1 + \frac{1}{2}\right)e^t - \frac{1}{2}\cos t + \frac{1}{2}\sin t \tag{21}$$

Final solution:

$$y(x) = \left(c_1 + \frac{1}{2}\right)e^x - \frac{1}{2}\cos x + \frac{1}{2}\sin x \tag{22}$$

We use the bilinear z-transform:

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}. \tag{23}$$

Step 1: Substitute and Simplify Each Term

First Term: $\frac{c_1 + \frac{1}{2}}{s-1}$ Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$:

$$\frac{c_1 + \frac{1}{2}}{s - 1} = \frac{c_1 + \frac{1}{2}}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} - 1}. \tag{24}$$

Simplify the denominator:

$$\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} - 1 = \frac{\frac{2}{T}(1 - z^{-1}) - (1 + z^{-1})}{1 + z^{-1}}. \tag{25}$$

Thus:

$$\frac{c_1 + \frac{1}{2}}{s - 1} = (c_1 + \frac{1}{2}) \cdot \frac{1 + z^{-1}}{\frac{2}{T} - 1 - \left(\frac{2}{T} + 1\right)z^{-1}}. \tag{26}$$

Second Term: $-\frac{\frac{1}{2}s}{s^2+1}$ Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$:

$$-\frac{\frac{1}{2}s}{s^2 + 1} = -\frac{\frac{1}{2} \cdot \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1}. \tag{27}$$

Simplify:

$$-\frac{\frac{1}{2}s}{s^2 + 1} = -\frac{\frac{1}{2} \cdot \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1} \tag{28}$$

$$= -\frac{\frac{1}{T}(1 - z^{-1})}{\frac{\left(\frac{2}{T}\right)^2(1-z^{-1})^2}{(1+z^{-1})^2} + 1} \tag{29}$$

$$= -\frac{\frac{1}{T}(1 - z^{-1})(1 + z^{-1})^2}{\left(\frac{2}{T}\right)^2(1 - z^{-1})^2 + (1 + z^{-1})^2}. \tag{30}$$

Third Term: $\frac{\frac{1}{2}}{s^2+1}$ Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$:

$$\frac{\frac{1}{2}}{s^2+1} = \frac{\frac{1}{2}}{\frac{\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2+1}{(1+z^{-1})^2}} \tag{31}$$

$$= \frac{\frac{1}{2}(1+z^{-1})^2}{\left(\frac{2}{T}\right)^2(1-z^{-1})^2+(1+z^{-1})^2}. \tag{32}$$

Step 2: Combine Terms The total $Y(z)$ is:

$$(c_1+\frac{1}{2})\cdot\frac{1+z^{-1}}{\frac{2}{T}-1-\left(\frac{2}{T}+1\right)z^{-1}} - \frac{\frac{1}{T}(1-z^{-1})}{\left(\frac{2}{T}\right)^2(1-z^{-1})^2+(1+z^{-1})^2} + \frac{\frac{1}{2}(1+z^{-1})^2}{\left(\frac{2}{T}\right)^2(1-z^{-1})^2+(1+z^{-1})^2} \tag{33}$$

Computational Solution Bilinear:(sim2)

$$\left(\left(\frac{2}{T}\right)^2\left(1-z^{-1}\right)^2+\left(1+z^{-1}\right)^2\right)^{-1}\left[\left(\frac{2}{T}-1\right)-\left(\frac{2}{T}+1\right)z^{-1}\right]Y(z) = f(z^{-1},z^{-2},z^{-3})+c \tag{34}$$

$$\left(\left(\frac{4}{T^2}+1\right)+\left(2-\frac{8}{T^2}\right)z^{-1}+\left(\frac{4}{T^2}+1\right)z^{-2}\right)\left(\frac{2}{T}-1-\left(\frac{2}{T}+1\right)z^{-1}\right)Y(z) = f(z^{-1},z^{-2},z^{-3})+c \tag{35}$$

$$\left(a+bz^{-1}+cz^{-2}\right)\left(d+ez^{-1}\right)Y(z) = f(z^{-1},z^{-2},z^{-3})+c \tag{36}$$

$$adY(z)+(ae+bd)z^{-1}Y(z)+(cd+be)z^{-2}Y(z)+cez^{-3}Y(z) = f(z^{-1},z^{-2},z^{-3})+c \tag{37}$$

where $a=\left(\frac{4}{T^2}+1\right)$, $b=\left(2-\frac{8}{T^2}\right)$, $c=\left(\frac{4}{T^2}+1\right)$, $d=\frac{2}{T}-1$, $e=-\left(\frac{2}{T}+1\right)$

$$\begin{aligned}&\rightarrow adz^3Y(z)+(ae+bd)z^2Y(z)+(cd+be)z^1Y(z)+ceY(z)\\&= f'(z^1,z^2,z^3)+c\end{aligned} \tag{38}$$

$$\begin{aligned}&\rightarrow ady[n+3]+(ae+bd)y[n+2]+(cd+be)y[n+1]+cey[n]\\&= f''(\delta[n],\delta[n+1],\delta[n+2],\delta[n+3])\end{aligned} \tag{39}$$

$$\text{Given: } y[0] = c_1,\ y[1] = y[0] + hy'[0] \tag{40}$$

$$y[1] = c_1 + h(1 + a) \tag{41}$$

$$y[2] = y[1] + h(y'[1]) \tag{42}$$

$$y[2] = y[1] + h(y[1] + \cos(h)) \tag{43}$$

$$y[3] = y[2] + h(y[2] + \cos(2h)) \tag{44}$$

$$n \geq 1 \rightarrow \delta[n], \delta[n+1], \delta[n+2], \delta[n+3] = 0 \tag{45}$$

$$\text{Difference equation: } y[n+3] = -\frac{1}{ad}\left[(ae + bd)y[n+2] + (cd + be)y[n+1] + cey[n]\right] \tag{46}$$

Computational Solution: Trapezoid Method:(sim1)

Step 1: Transform the DE into a First-Order System

The given differential equation is already in first-order form: $\frac{dy}{dx} = y + \cos x$.
Let: $y = y_1$, so $\frac{dy_1}{dx} = y_1 + \cos x$.

Step 2: Apply the Trapezoidal Rule

Using the Trapezoidal Rule, the equation can be written as:

$$y_{n+1} - y_n = \frac{h}{2}((y_n + \cos x_n) + (y_{n+1} + \cos x_{n+1})), \tag{47}$$

where h is the step size, y_n is the value of y at x_n, and y_{n+1} is the value of y at $x_{n+1} = x_n + h$.

Step 3: Solve for y_{n+1}

Rearranging Equation (1) to isolate y_{n+1}:

$$y_{n+1} - \frac{h}{2}y_{n+1} = y_n + \frac{h}{2}(y_n + \cos x_n + \cos x_{n+1}), \tag{48}$$

$$y_{n+1}(1 - \frac{h}{2}) = y_n(1 + \frac{h}{2}) + \frac{h}{2}(\cos x_n + \cos x_{n+1}), \tag{49}$$

$$y_{n+1} = \frac{y_n(1 + \frac{h}{2}) + \frac{h}{2}(\cos x_n + \cos x_{n+1})}{1 - \frac{h}{2}}. \tag{50}$$

Step 4: Iterative Scheme

The final iterative formula is:

$$y_{n+1} = \frac{y_n(1 + \frac{h}{2}) + \frac{h}{2}(\cos x_n + \cos x_{n+1})}{1 - \frac{h}{2}}. \tag{51}$$

Step 5: Initial Conditions and Computation

Given the initial condition: $x_0 = 0$, $y_0 = 0$, and a chosen step size h, the values of y_n can be iteratively computed for subsequent x_n.

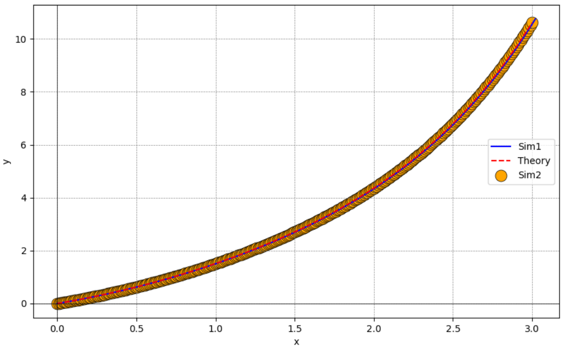

Fig. 0: Here Sim-1 plot represents the plot given by Trapezoid Method, and Sim-2 which is given by Bilinear transform using the same value of h. This plot clearly shows the accuracy of the Bilinear transform method.