Question-12.9.ex.19

EE24BTECH11030 - J.KEDARANANDA

Question

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$with its vertex at one end of the major axis.

Theoretical solution:
Consider the isosceles triangle ABC:
$A(a,0)$, $B(a\cos\theta, b\sin\theta)$, and $C(a\cos\theta, -b\sin\theta)$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times BC \times \text{height of } \triangle ABC \tag{1}$$
$$\text{Height of } \triangle ABC = a(1-\cos\theta) \tag{2}$$
$$BC = 2b\sin\theta \tag{3}$$
$$\Delta = \frac{1}{2} \times 2b\sin\theta \times a(1-\cos\theta) \tag{4}$$
$$= ab\sin\theta(1-\cos\theta) \tag{5}$$

Theoretical Solution: Maximum Area

For the maximum area of the triangle:

$$\frac{d\Delta}{d\theta} = ab\cos\theta(1-\cos\theta) + ab\sin^2\theta = 0 \tag{6}$$

$$\cos\theta(1-\cos\theta) + \sin^2\theta = 0 \tag{7}$$

$$\cos\theta = \cos 2\theta \tag{8}$$

$$\cos\theta = -\frac{1}{2} \tag{9}$$

For the maximum value:

$$\cos\theta = -\frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2} \tag{10}$$

$$\text{Maximum area} = ab \times \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) \tag{11}$$

$$= \frac{3\sqrt{3}}{4}ab \tag{12}$$

Computational solution:

By the gradient descent algorithm, the difference equation is given by,

$$f(x) = ab\sin x(1-\cos x) \tag{13}$$

$$\tag{14}$$

to find the value of x_n for the maximum area:

$$x_{n+1} = x_n + \mu f'(x_n) \tag{15}$$

$$x_{n+1} = x_n + \mu\left(ab\cos x_n(1-\cos x_n) + ab\sin^2 x_n\right) \tag{16}$$

$$x_{n+1} = x_n + \mu ab\cos x_n(1-\cos x_n) + \mu ab\sin^2 x_n$$

Taking the Z-transform on both sides:

$$zX(z) - zx_0 = X(z) + \mu ab\cos(X(z))(1-\cos(X(z))) + \mu ab\sin^2(X(z)).$$

$$(Z-1)X(Z) - Zx_0 = \mu ab\left(\frac{1-Z^{-1}\cos 1}{1-2Z^{-1}\cos 1 + Z^{-2}} - \frac{1-Z^{-1}\cos 2}{1-2Z^{-1}\cos 2 + Z^{-2}}\right) \tag{17}$$

Convergence of the Sequence

If the sequence x_n has to converge:

$$\lim_{n\to\infty} x_{n+1} - x_n = 0 \tag{18}$$

$$\implies \lim_{n\to\infty} \mu ab(\cos(2x_n) - \cos(x_n)) = 0 \tag{19}$$

$$\implies \mu \lim_{n\to\infty} \mu ab(\cos(2x_n) - \cos(x_n)) = 0,\ \mu > 0 \tag{20}$$

$$\implies \lim_{n\to\infty} \cos(2x_n) - \cos(x_n) = 0 \tag{21}$$

$$\implies \lim_{n\to\infty} x_n = \frac{2\pi}{3} \tag{22}$$

Computational Solution for Minimum Area

To find the value of x_n for the minimum area:

$$x_{n+1} = x_n - \mu f'(x_n) \tag{23}$$

$$x_{n+1} = x_n - \mu \left(ab \cos x_n (1 - \cos x_n) + ab \sin^2 x_n \right) \tag{24}$$

$$x_{n+1} = x_n - \mu ab \cos x_n (1 - \cos x_n) + \mu ab \sin^2 x_n$$

Taking the Z-transform on both sides:

$$zX(z) - zx_0 = X(z) - \mu ab \cos(X(z))(1 - \cos(X(z))) + \mu ab \sin^2(X(z)).$$

$$(Z-1)X(Z) - Zx_0 = -\mu ab \left(\frac{1 - Z^{-1}\cos 1}{1 - 2Z^{-1}\cos 1 + Z^{-2}} - \frac{1 - Z^{-1}\cos 2}{1 - 2Z^{-1}\cos 2 + Z^{-} \tag{25}$$

Computational Solution for Minimum Area (Convergence Condition)

If the sequence x_n has to converge, we have:

$$\lim_{n\to\infty} x_{n+1} - x_n = 0 \tag{26}$$

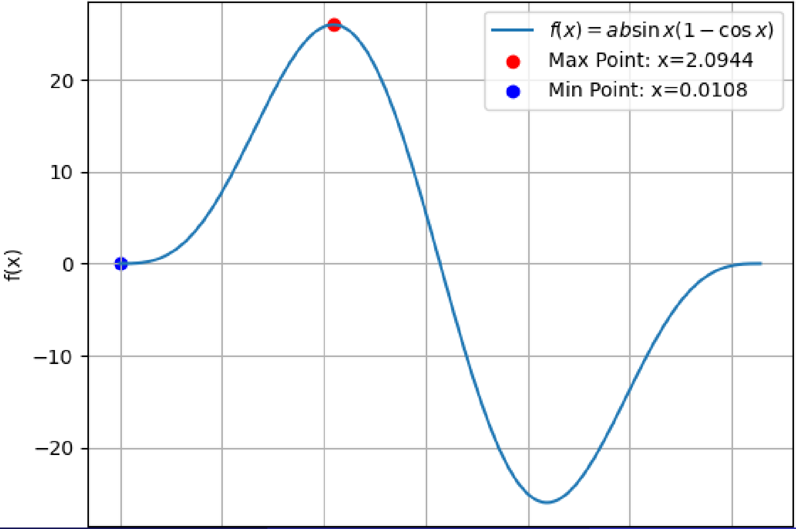
$$\implies \lim_{n\to\infty} -\mu ab(\cos(2x_n) - \cos(x_n)) = 0 \tag{27}$$

$$\implies \mu \lim_{n\to\infty} -\mu ab(\cos(2x_n) - \cos(x_n)) = 0,\ \mu > 0 \tag{28}$$

$$\implies \lim_{n\to\infty} \cos(2x_n) - \cos(x_n) = 0 \tag{29}$$

$$\implies \lim_{n\to\infty} x_n = 0 \tag{30}$$

Diagram



$f(x) = ab\sin x(1-\cos x)$
Max Point: x=2.0944
Min Point: x=0.0108
f(x)
20
10
0
−10
−20