

12.9.7.1.1

EE24BTECH11030 - KEDARANANDA

Question:

Solve the differential equation:

$$\frac{d^2y}{dx^2} + 5x \left(\frac{dy}{dx} \right)^2 - 6y = \ln(x), \quad (1)$$

Solution:

Theoretical solution:

The given differential equation is a second-order nonlinear ordinary differential equation and cannot be theoretically solved using known methods.

Computational Solution: Euler's method

By the first principle of derivative,

$$y'(x) = \lim_{h \rightarrow 0} \frac{y(x+h) - y(x)}{h} \quad (2)$$

$$y(x+h) = y(x) + h(y'(x)), h \rightarrow 0 \quad (3)$$

For a m^{th} order differential equation,

Let

$$y_1 = y, y_2 = y', y_3 = y'', \dots, y_m = y^{m-1} \quad (4)$$

then we obtain the system

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_{m-1}' \\ y_m' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_m \\ f(x, y_1, y_2, \dots, y_m) \end{pmatrix} \quad (5)$$

Here, f is described by the given differential equation. The initial conditions $y_1(x_0) = K_1$, $y_2(x_0) = K_2$, \dots , $y_m(x_0) = K_m$.

Representing the system in Euler's form (using first principle of derivative),

$$\begin{pmatrix} y_1(x+h) \\ y_2(x+h) \\ \vdots \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) + hy_2(x) \\ y_2(x) + hy_3(x) \\ \vdots \\ y_m(x) + hf(x, y_1, y_2, \dots, y_m) \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} y_1(x+h) \\ \vdots \\ y_{m-1}(x+h) \\ y_m(x+h) \end{pmatrix} = \begin{pmatrix} y_1(x) \\ \vdots \\ y_{m-1}(x) \\ y_m(x) \end{pmatrix} + h \begin{pmatrix} y_2(x) \\ \vdots \\ y_m(x) \\ f(x, y_1, y_2, \dots, y_m) \end{pmatrix} \quad (7)$$

$$\mathbf{y}(x+h) = \mathbf{y}(x) + h \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{f(x, y_1, y_2, \dots, y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \quad (8)$$

$$\mathbf{y}(x+h) = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x, y_1, y_2, \dots, y_m)}{y_m} \end{pmatrix} \mathbf{y}(x) \quad (9)$$

Generalizing the system into an iterative format for plotting $y(x)$,

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \\ \vdots \\ y_{m,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \\ \vdots \\ y_{m,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ y_{3,n} \\ \vdots \\ f(x_n, y_{1,n}, y_{2,n}, \dots, y_{m,n}) \end{pmatrix} \quad (10)$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{f(x_n, y_{1,n}, y_{2,n}, \dots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y}_n, \text{ where } \mathbf{y}_n = \begin{pmatrix} y_{1,n}(x_n) \\ y_{2,n}(x_n) \\ \vdots \\ y_{m,n}(x_n) \end{pmatrix} \quad (11)$$

$$\mathbf{y}_{n+1} = \begin{pmatrix} 1 & h & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 + \frac{f(x_n, y_{1,n}, y_{2,n}, \dots, y_{m,n})}{y_{m,n}} \end{pmatrix} \mathbf{y}_n \quad (12)$$

$$x_{n+1} = x_n + h \quad (13)$$

Here, the vector \mathbf{y}_n is not to be confused with y_k which is the $(k-1)^{\text{th}}$ derivative of $y(x)$. The given differential equation can be represented as,

$$y'' + 5x(y')^2 - 6y = \ln(x) \quad (14)$$

$$y'' = \ln(x) - 5x(y')^2 + 6y \quad (15)$$

We see that $m = 2$, thus,

$$y_3 = y'' = \ln(x) - 5x(y')^2 + 6y \quad (16)$$

$$\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \ln(x) - 5x(y')^2 + 6y \end{pmatrix} \quad (17)$$

$$\begin{pmatrix} y_{1,n+1} \\ y_{2,n+1} \end{pmatrix} = \begin{pmatrix} y_{1,n} \\ y_{2,n} \end{pmatrix} + h \begin{pmatrix} y_{2,n} \\ \ln(x_n) - 5x_n(y_{2,n})^2 + 6y_{1,n} \end{pmatrix} \quad (18)$$

$$\mathbf{y}_{n+1} = \begin{pmatrix} 1 & h \\ 0 & 1 + \ln(x_n) - 5x_n(y_{2,n})^2 + 6y_{1,n} \end{pmatrix} \mathbf{y}_n \quad (19)$$

Iteratively plotting the above system taking initial conditions as

$$x_0 = 0, y_{1,0} = 0.01, y_{2,0} = 1 \quad (20)$$

we get the following plot.

