Question-10.3.2.3.1

EE24BTECH11030 - J.KEDARANANDA

Question

On comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the following pair of linear equations are consistent, or inconsistent .

$$3x + 2y = 5; 2x - 3y = 7 \tag{1}$$

Theoretical Solution

Theoritical solution: To determine whether the given pair of linear equations is consistent or inconsistent, we compare the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, and $\frac{c_1}{c_2}$, where:

$$a_1x + b_1y = c_1 \quad \text{and} \quad a_2x + b_2y = c_2 \tag{2}$$

From the equations:

$$3x + 2y = 5 \quad \text{and} \quad 2x - 3y = 7, \tag{3}$$

we identify:

$$a_1 = 3,\ b_1 = 2,\ c_1 = 5,\ a_2 = 2,\ b_2 = -3,\ c_2 = 7. \tag{4}$$

Now calculate the ratios:

$$\frac{a_1}{a_2} = \frac{3}{2}, \quad \frac{b_1}{b_2} = \frac{2}{-3}, \quad \frac{c_1}{c_2} = \frac{5}{7}. \tag{5}$$

Since:

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}, \tag{6}$$

by the equations are **intersecting**. Therefore, the system of equations has a unique solution.

Solution using LU Factorization:

Given the system of linear equations:

$$3x + 2y = 5, \tag{7}$$
$$2x - 3y = 7. \tag{8}$$

We rewrite the equations as:

$$x_1 = x, \tag{9}$$
$$x_2 = y, \tag{10}$$

giving the system:

$$3x_1 + 2x_2 = 5, \tag{11}$$
$$2x_1 - 3x_2 = 7. \tag{12}$$

Step-by-Step Procedure for LU Factorization

Step-by-Step Procedure:

1. **Initialization:**
   - Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$.
2. **Iterative Update:**
   - For each pivot $k = 1, 2, \ldots, n$:
     - Compute the entries of $\mathbf{U}$ using the first update equation.
     - Compute the entries of $\mathbf{L}$ using the second update equation.
3. **Result:**
   - After completing the iterations, the matrix $\mathbf{A}$ is decomposed into $\mathbf{L} \cdot \mathbf{U}$, where $\mathbf{L}$ is a lower triangular matrix with ones on the diagonal, and $\mathbf{U}$ is an upper triangular matrix.

1. Update for $U_{k,j}$ (Entries of U)

For each column $j \geq k$, the entries of $\mathbf{U}$ in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j}, \quad \text{for } j \geq k.$$

This computes the elements of the upper triangular matrix $\mathbf{U}$ by eliminating the lower triangular portion.

2. Update for $L_{i,k}$ (Entries of L)

For each row $i > k$, the entries of $\mathbf{L}$ in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right), \quad \text{for } i > k.$$

This equation computes the elements of the lower triangular matrix $\mathbf{L}$, where each entry in the column is determined by the values in the rows above it

LU Factorization and Forward Substitution

We decompose A as:

$$A = LU, \tag{13}$$

where L is a lower triangular matrix and U is an upper triangular matrix. By running the iteration code, we get the L and U matrices:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}, \tag{14}$$

$$U = \begin{bmatrix} 3 & 2 \\ 0 & -\frac{13}{3} \end{bmatrix}. \tag{15}$$

We solve:

$$L\mathbf{y} = \mathbf{b} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}. \tag{16}$$

From the first row:

$$y_1 = 5. \tag{17}$$

Forward and Backward Substitution

From the second row:

$$\frac{2}{3}y_1 + y_2 = 7 \quad \Longrightarrow \quad \frac{2}{3} \cdot 5 + y_2 = 7 \quad \Longrightarrow \quad y_2 = \frac{11}{3}. \tag{18}$$

Thus:

$$\mathbf{y} = \begin{bmatrix} 5 \\ \frac{11}{3} \end{bmatrix}. \tag{19}$$

We solve:

$$U\mathbf{x} = \mathbf{y} \quad \text{or} \quad \begin{bmatrix} 3 & 2 \\ 0 & -\frac{13}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{11}{3} \end{bmatrix}. \tag{20}$$

From the second row:

$$-\frac{13}{3}x_2 = \frac{11}{3} \quad \Longrightarrow \quad x_2 = -\frac{11}{13}. \tag{21}$$

From the first row:

$$3x_1 + 2x_2 = 5 \quad \Longrightarrow \quad 3x_1 + 2\left(-\frac{11}{13}\right) = 5. \tag{22}$$

Solution Completion: Backward Substitution and Final Solution

From the first row:

$$3x_1 - \frac{22}{13} = 5 \quad \implies \quad 3x_1 = 5 + \frac{22}{13} = \frac{65}{13} + \frac{22}{13} = \frac{87}{13}, \tag{23}$$

$$x_1 = \frac{87}{39} = \frac{29}{13}. \tag{24}$$

Thus:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{29}{13} \\ -\frac{11}{13} \end{bmatrix}. \tag{25}$$

The solution is:

$$x = \frac{29}{13}, \tag{26}$$

$$y = -\frac{11}{13}. \tag{27}$$

As we can clearly see, there is a solution for the given lines, so the system of equations is **consistent**.

Diagram