12.9.ex.19

EE24BTECH11030 - KEDARANANDA

Question:
Solve the differential equation:
$y' = y + \cos x$
Solution:
Theoretical solution:
The given differential equation is a first-order linear ordinary differential equation. Let $y(0) = c_1$. By the definition of the Laplace transform,
$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$
Some used properties of the Laplace transform include:

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) \tag{1}$$
$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{2}$$
$$\mathcal{L}(e^{at} f(t)) = F(s-a), \quad \text{where } F(s) = \mathcal{L}(f(t)) \tag{3}$$
$$\mathcal{L}(\cos x) = \frac{s}{s^2+1} \tag{4}$$

Applying the Laplace transform to the given differential equation:
$y' - y = \cos x$
Take the Laplace transform on both sides:

$$\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(\cos x) \tag{5}$$

Using the properties of the Laplace transform:

$$(s\mathcal{L}(y) - y(0)) - \mathcal{L}(y) = \frac{s}{s^2+1} \tag{6}$$

Let $\mathcal{L}(y) = Y(s)$. Substituting $y(0) = c_1$, we get:

$$sY(s) - c_1 - Y(s) = \frac{s}{s^2+1} \tag{7}$$

Simplify:

$$(s-1)Y(s) = c_1 + \frac{s}{s^2+1} \tag{8}$$
$$Y(s) = \frac{c_1}{s-1} + \frac{s}{(s^2+1)(s-1)} \tag{9}$$

Partial fraction decomposition:
For $\frac{s}{(s^2+1)(s-1)}$, decompose into:

$$\frac{s}{(s^2+1)(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \tag{10}$$

Solve for A, B, and C by equating numerators:

$$s = A(s^2+1) + (Bs+C)(s-1) \tag{11}$$

Expand and collect terms:

$$s = A(s^2) + A + Bs^2 - Bs + Cs - C \tag{12}$$

$$s = (A+B)s^2 + (-B+C)s + (A-C) \tag{13}$$

Equating coefficients:

$$A + B = 0 \quad \text{(coefficient of } s^2\text{)} \tag{14}$$

$$-B + C = 1 \quad \text{(coefficient of } s\text{)} \tag{15}$$

$$A - C = 0 \quad \text{(constant term)} \tag{16}$$

Solve this system:

$$B = -A \tag{17}$$

$$C = A \tag{18}$$

$$-(-A) + A = 1 \implies 2A = 1 \implies A = \frac{1}{2} \tag{19}$$

Thus:

$$A = \frac{1}{2}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{2} \tag{20}$$

The partial fraction decomposition becomes:

$$\frac{s}{(s^2+1)(s-1)} = \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1} \tag{21}$$

Rewrite $Y(s)$:

$$Y(s) = \frac{c_1}{s-1} + \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1} \tag{22}$$

Combine terms:

$$Y(s) = \frac{c_1 + \frac{1}{2}}{s-1} - \frac{\frac{1}{2}s}{s^2+1} + \frac{\frac{1}{2}}{s^2+1} \tag{23}$$

Take the inverse Laplace transform:

Using the properties of the Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t \tag{24}$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t \tag{25}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t \tag{26}$$

Apply the inverse transform to each term:

$$y(t) = \left(c_1 + \frac{1}{2}\right)e^t - \frac{1}{2}\cos t + \frac{1}{2}\sin t \tag{27}$$

Final solution:

$$y(x) = \left(c_1 + \frac{1}{2}\right)e^x - \frac{1}{2}\cos x + \frac{1}{2}\sin x \tag{28}$$

Computational Solution: Trapezoid Method

Step 1: Transform the DE into a First-Order System

The given differential equation is already in first-order form: $\frac{dy}{dx} = y + \cos x$.
Let: $y = y_1$, so $\frac{dy_1}{dx} = y_1 + \cos x$.

Step 2: Apply the Trapezoidal Rule

Using the Trapezoidal Rule, the equation can be written as:

$$y_{n+1} - y_n = \frac{h}{2}((y_n + \cos x_n) + (y_{n+1} + \cos x_{n+1})), \tag{29}$$

where h is the step size, y_n is the value of y at x_n, and y_{n+1} is the value of y at $x_{n+1} = x_n + h$.

Step 3: Solve for y_{n+1}

Rearranging Equation (1) to isolate y_{n+1}:

$$y_{n+1} - \frac{h}{2}y_{n+1} = y_n + \frac{h}{2}(y_n + \cos x_n + \cos x_{n+1}), \tag{30}$$

$$y_{n+1}(1 - \frac{h}{2}) = y_n(1 + \frac{h}{2}) + \frac{h}{2}(\cos x_n + \cos x_{n+1}), \tag{31}$$

$$y_{n+1} = \frac{y_n(1 + \frac{h}{2}) + \frac{h}{2}(\cos x_n + \cos x_{n+1})}{1 - \frac{h}{2}}. \tag{32}$$

Step 4: Iterative Scheme

The final iterative formula is:

$$y_{n+1} = \frac{y_n(1 + \frac{h}{2}) + \frac{h}{2}(\cos x_n + \cos x_{n+1})}{1 - \frac{h}{2}}. \tag{33}$$

Step 5: Initial Conditions and Computation

Given the initial condition: $x_0 = 0$, $y_0 = 0$, and a chosen step size h, the values of y_n can be iteratively computed for subsequent x_n.