11.16.3.3.3

EE24BTECH11030 - KEDARANANDA

Question:
A die is rolled. Find the probability that a number greater than or equal to one will appear.
Solution:
Theoretical solution:

$$\text{Total outcomes} = 6.$$

$$\text{Favorable outcomes} = 6.$$

$$P(\text{Number} \geq 1) = \frac{\text{Favorable outcomes}}{\text{Total outcomes}} = \frac{6}{6} = 1.$$

Computational solution:

PMF for a Single Die Roll

For a single die roll, the probability mass function (PMF) is:

$$P_X(k) = \begin{cases} \frac{1}{6}, & \text{if } 1 \leq x \leq 6 \\ 0, & \text{if } x > 6 \\ 0, & \text{if } x < 1 \end{cases}$$

Conditions for PMF

A valid PMF must satisfy the following conditions:
1) Non-negativity: $\forall k, P_X(k) \geq 0$.
2) Normalization: $\sum_k P_X(k) = 1$.

For a single die roll:

$$P_X(1) + P_X(2) + P_X(3) + P_X(4) + P_X(5) + P_X(6) = 1. \tag{2.1}$$

Substituting $P_X(k) = \frac{1}{6}$ for all valid outcomes:

$$6 \times \frac{1}{6} = 1. \tag{2.2}$$

Computational Steps

For each $k \in \{1, 2, \ldots, 6\}$:
1) Compute the binomial coefficient (trivial for single outcomes).
2) Multiply by $\frac{1}{6}$ to compute $P_X(k)$.

Result for Single Roll

The PMF values are:

$$P_X(1) = \frac{1}{6}, \quad P_X(2) = \frac{1}{6}, \quad \ldots, \quad P_X(6) = \frac{1}{6}. \tag{2.1}$$

Conclusion

The probability of rolling a number greater than or equal to one is the sum of all PMF values:

$$P(X \geq 1) = \sum_{k=1}^{6} P_X(k) = \frac{6}{6} = 1. \tag{2.2}$$

Z-Transform Expansion

$$P(z) = \sum_{n=0}^{\infty} p(n) z^{-n}. \tag{2.3}$$

$$P(z) = \sum_{n=1}^{6} \frac{1}{6} z^{-n}. \tag{2.4}$$

The Z-transform for the number rolled is given by:

$$T(z) = \left(\frac{1}{6}z^{-1} + \frac{1}{6}z^{-2} + \frac{1}{6}z^{-3} + \frac{1}{6}z^{-4} + \frac{1}{6}z^{-5} + \frac{1}{6}z^{-6}\right) \tag{2.5}$$

where:

- Each term represents the probability of a specific outcome multiplied by its corresponding power of z.

Expansion of $T(z)$

Simplify the expression:

$$T(z) = \frac{1}{6}(z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}). \tag{2.6}$$

The coefficient of z^{-k} in $T(z)$ gives the probability $P_X(k)$, where X is the outcome of the die roll.

Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) $F(x)$ of a discrete random variable X, representing the outcome of a die roll, is defined as:

$$F(x) = P(X \leq x)$$

For a die roll with outcomes $1, 2, 3, 4, 5, 6$, the CDF is:

$$F(x) = \begin{cases} 0, & \text{if } x < 1 \\ \frac{1}{6}, & \text{if } 1 \le x < 2 \\ \frac{2}{6}, & \text{if } 2 \le x < 3 \\ \frac{3}{6}, & \text{if } 3 \le x < 4 \\ \frac{4}{6}, & \text{if } 4 \le x < 5 \\ \frac{5}{6}, & \text{if } 5 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases}$$