12.6.6.8

EE24BTECH11030 - KEDARANANDA

Question: Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$with its vertex at one end of the major axis.

Solution:

Theoretical solution: Consider the isosceles triangle ABC:
$A(a, 0)$, $B(a\cos\theta, b\sin\theta)$, and $C(a\cos\theta, -b\sin\theta)$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times BC \times \text{height of } \triangle ABC \tag{1}$$

$$\text{Height of } \triangle ABC = a(1 - \cos\theta) \tag{2}$$

$$BC = 2b\sin\theta \tag{3}$$

$$\Delta = \frac{1}{2} \times 2b\sin\theta \times a(1 - \cos\theta) \tag{4}$$

$$= ab\sin\theta(1 - \cos\theta) \tag{5}$$

For the maximum area of the triangle:

$$\frac{d\Delta}{d\theta} = ab\cos\theta(1 - \cos\theta) + ab\sin^2\theta = 0 \tag{6}$$

$$\cos\theta(1 - \cos\theta) + \sin^2\theta = 0 \tag{7}$$

$$\cos\theta = \cos 2\theta \tag{8}$$

$$\tag{9}$$

$$\cos\theta = -\frac{1}{2} \tag{10}$$

$$\tag{11}$$

For the maximum value:

$$\cos\theta = -\frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2} \tag{12}$$

$$\text{Maximum area} = ab \times \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) \tag{13}$$

$$= \frac{3\sqrt{3}}{4}ab \tag{14}$$

Computational solution:
By the gradient descent algorithm, the difference equation is given by,

$$f(x) = ab \sin x(1 - \cos x) \tag{15}$$
$$\tag{16}$$

to find value of x_n for max area

$$\begin{aligned} x_{n+1} &= x_n + \mu f'(x_n) \quad (17)\\ x_{n+1} &= x_n + \mu \left(ab \cos x_n (1 - \cos x_n) + ab \sin^2 x_n \right) \quad (18)\\ x_{n+1} &= x_n + \mu ab \cos x_n (1 - \cos x_n) + \mu ab \sin^2 x_n \end{aligned}$$

Taking the Z-transform on both sides:

$$zX(z) - zx_0 = X(z) + \mu ab \cos(X(z))(1 - \cos(X(z))) + \mu ab \sin^2(X(z)).$$

$$(Z - 1)X(Z) - Zx_0 = \mu ab \left(\frac{1 - Z^{-1} \cos 1}{1 - 2Z^{-1} \cos 1 + Z^{-2}} - \frac{1 - Z^{-1} \cos 2}{1 - Z^{-1} \cos 2 + Z^{-2}} \right) \tag{19}$$

If the sequence x_n has to converge,

$$\begin{aligned} &\lim_{n\to\infty} |x_{n+1} - x_n| = 0 \quad (20)\\ \implies &\lim_{n\to\infty} |\mu ab(\cos(2x_n) - \cos(x_n))| = 0 \quad (21)\\ \implies &\mu \lim_{n\to\infty} |\mu ab(\cos(2x_n) - \cos(x_n))| = 0, \ \mu > 0 \quad (22)\\ \implies &\lim_{n\to\infty} \cos(2x_n) - \cos(x_n) = 0 \quad (23)\\ \implies &\lim_{n\to\infty} x_n = \frac{2\pi}{3} \quad (24) \end{aligned}$$

to find value of x_n for min area

$$\begin{aligned} x_{n+1} &= x_n - \mu f'(x_n) \quad (25)\\ x_{n+1} &= x_n - \mu \left(ab \cos x_n (1 - \cos x_n) + ab \sin^2 x_n \right) \quad (26)\\ x_{n+1} &= x_n - \mu ab \cos x_n (1 - \cos x_n) + \mu ab \sin^2 x_n \end{aligned}$$

Taking the Z-transform on both sides:

$$zX(z) - zx_0 = X(z) - \mu ab \cos(X(z))(1 - \cos(X(z))) + \mu ab \sin^2(X(z)).$$

$$(Z - 1)X(Z) - Zx_0 = -\mu ab \left(\frac{1 - Z^{-1} \cos 1}{1 - 2Z^{-1} \cos 1 + Z^{-2}} - \frac{1 - Z^{-1} \cos 2}{1 - Z^{-1} \cos 2 + Z^{-2}} \right) \tag{27}$$

If the sequence x_n has to converge,

$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0 \tag{28}$$

$$\implies \lim_{n\to\infty} |-\mu ab(\cos(2x_n) - \cos(x_n))| = 0 \tag{29}$$

$$\implies \mu \lim_{n\to\infty} |-\mu ab(\cos(2x_n) - \cos(x_n))| = 0, \ \mu > 0 \tag{30}$$

$$\implies \lim_{n\to\infty} \cos(2x_n) - \cos(x_n) = 0 \tag{31}$$

$$\implies \lim_{n\to\infty} x_n = 0 \tag{32}$$