

Question-12.9.ex.19

EE24BTECH11030 - J.KEDARANANDA

Question

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

Theoretical Solution

Theoretical solution:

Consider the isosceles triangle ABC:

$A(a, 0)$, $B(a \cos \theta, b \sin \theta)$, and $C(a \cos \theta, -b \sin \theta)$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times BC \times \text{height of } \triangle ABC \quad (1)$$

$$\text{Height of } \triangle ABC = a(1 - \cos \theta) \quad (2)$$

$$BC = 2b \sin \theta \quad (3)$$

$$\Delta = \frac{1}{2} \times 2b \sin \theta \times a(1 - \cos \theta) \quad (4)$$

$$= ab \sin \theta (1 - \cos \theta) \quad (5)$$

Theoretical Solution: Maximum Area

For the maximum area of the triangle:

$$\frac{d\Delta}{d\theta} = ab \cos \theta (1 - \cos \theta) + ab \sin^2 \theta = 0 \quad (6)$$

$$\cos \theta (1 - \cos \theta) + \sin^2 \theta = 0 \quad (7)$$

$$\cos \theta = \cos 2\theta \quad (8)$$

$$\cos \theta = -\frac{1}{2} \quad (9)$$

For the maximum value:

$$\cos \theta = -\frac{1}{2}, \quad \sin \theta = \frac{\sqrt{3}}{2} \quad (10)$$

$$\text{Maximum area} = ab \times \frac{\sqrt{3}}{2} \left(1 + \frac{1}{2}\right) \quad (11)$$

$$= \frac{3\sqrt{3}}{4} ab \quad (12)$$

Computational Solution

Computational solution:

By the gradient descent algorithm, the difference equation is given by,

$$f(x) = ab \sin x(1 - \cos x) \quad (13)$$

$$(14)$$

to find the value of x_n for the maximum area:

$$x_{n+1} = x_n + \mu f'(x_n) \quad (15)$$

$$x_{n+1} = x_n + \mu (ab \cos x_n(1 - \cos x_n) + ab \sin^2 x_n) \quad (16)$$

$$x_{n+1} = x_n + \mu ab \cos x_n(1 - \cos x_n) + \mu ab \sin^2 x_n$$

Taking the Z-transform on both sides:

$$zX(z) - zx_0 = X(z) + \mu ab \cos(X(z))(1 - \cos(X(z))) + \mu ab \sin^2(X(z)).$$

$$(Z - 1)X(Z) - Zx_0 = \mu ab \left(\frac{1 - Z^{-1} \cos 1}{1 - 2Z^{-1} \cos 1 + Z^{-2}} - \frac{1 - Z^{-1} \cos 2}{1 - Z^{-1} \cos 2 + Z^{-2}} \right) \quad (17)$$

Convergence of the Sequence

If the sequence x_n has to converge:

$$\lim_{n \rightarrow \infty} x_{n+1} - x_n = 0 \quad (18)$$

$$\implies \lim_{n \rightarrow \infty} \mu ab(\cos(2x_n) - \cos(x_n)) = 0 \quad (19)$$

$$\implies \mu \lim_{n \rightarrow \infty} \mu ab(\cos(2x_n) - \cos(x_n)) = 0, \mu > 0 \quad (20)$$

$$\implies \lim_{n \rightarrow \infty} \cos(2x_n) - \cos(x_n) = 0 \quad (21)$$

$$\implies \lim_{n \rightarrow \infty} x_n = \frac{2\pi}{3} \quad (22)$$

Computational Solution for Minimum Area

To find the value of x_n for the minimum area:

$$x_{n+1} = x_n - \mu f'(x_n) \quad (23)$$

$$x_{n+1} = x_n - \mu (ab \cos x_n (1 - \cos x_n) + ab \sin^2 x_n) \quad (24)$$

$$x_{n+1} = x_n - \mu ab \cos x_n (1 - \cos x_n) + \mu ab \sin^2 x_n$$

Taking the Z-transform on both sides:

$$zX(z) - zx_0 = X(z) - \mu ab \cos(X(z))(1 - \cos(X(z))) + \mu ab \sin^2(X(z)).$$

$$(Z - 1)X(Z) - Zx_0 = -\mu ab \left(\frac{1 - Z^{-1} \cos 1}{1 - 2Z^{-1} \cos 1 + Z^{-2}} - \frac{1 - Z^{-1} \cos 2}{1 - Z^{-1} \cos 2 + Z^{-2}} \right) \quad (25)$$

Computational Solution for Minimum Area (Convergence Condition)

If the sequence x_n has to converge, we have:

$$\lim_{n \rightarrow \infty} x_{n+1} - x_n = 0 \quad (26)$$

$$\implies \lim_{n \rightarrow \infty} -\mu ab(\cos(2x_n) - \cos(x_n)) = 0 \quad (27)$$

$$\implies \mu \lim_{n \rightarrow \infty} -\mu ab(\cos(2x_n) - \cos(x_n)) = 0, \mu > 0 \quad (28)$$

$$\implies \lim_{n \rightarrow \infty} \cos(2x_n) - \cos(x_n) = 0 \quad (29)$$

$$\implies \lim_{n \rightarrow \infty} x_n = 0 \quad (30)$$

Diagram

