# Question-12.9.ex.19

EE24BTECH11030 - J.KEDARANANDA

# Question

Solve the differential equation:
$y' = y + \cos x$

# Theoritical Solution :

The given differential equation is a first-order linear ordinary differential equation. Let $y(0) = c_1$. By the definition of the Laplace transform,

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$$

Some used properties of the Laplace transform include:

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0) \tag{1}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{2}$$

$$\mathcal{L}(e^{at}f(t)) = F(s-a), \quad \text{where } F(s) = \mathcal{L}(f(t)) \tag{3}$$

$$\mathcal{L}(\cos x) = \frac{s}{s^2+1} \tag{4}$$

**Applying the Laplace transform to the given differential equation:**

$$y' - y = \cos x$$

Take the Laplace transform on both sides:

$$\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(\cos x) \tag{5}$$

Using the properties of the Laplace transform:

$$(s\mathcal{L}(y) - y(0)) - \mathcal{L}(y) = \frac{s}{s^2+1} \tag{6}$$

Let $\mathcal{L}(y) = Y(s)$. Substituting $y(0) = c_1$, we get:

$$sY(s) - c_1 - Y(s) = \frac{s}{s^2+1} \tag{7}$$

Simplify:

$$(s-1)Y(s) = c_1 + \frac{s}{s^2+1} \tag{8}$$

$$Y(s) = \frac{c_1}{s-1} + \frac{s}{(s^2+1)(s-1)} \tag{9}$$

**Partial fraction decomposition:**

For $\frac{s}{(s^2+1)(s-1)}$, decompose into:

$$\frac{s}{(s^2+1)(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \tag{10}$$

Solve for $A$, $B$, and $C$ by equating numerators:

$$s = A(s^2+1) + (Bs+C)(s-1) \tag{11}$$

# Partial fraction decomposition:

Equating coefficients:

$$A + B = 0 \quad \text{(coefficient of } s^2\text{)} \tag{12}$$
$$-B + C = 1 \quad \text{(coefficient of } s\text{)} \tag{13}$$
$$A - C = 0 \quad \text{(constant term)} \tag{14}$$

The partial fraction decomposition becomes:

$$\frac{s}{(s^2+1)(s-1)} = \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1} \tag{15}$$

**Rewrite $Y(s)$:**

$$Y(s) = \frac{c_1}{s-1} + \frac{\frac{1}{2}}{s-1} + \frac{-\frac{1}{2}s + \frac{1}{2}}{s^2+1} \tag{16}$$

Combine terms:

$$Y(s) = \frac{c_1 + \frac{1}{2}}{s-1} - \frac{\frac{1}{2}s}{s^2+1} + \frac{\frac{1}{2}}{s^2+1} \tag{17}$$

**Final solution:**

$$y(x) = \left(c_1 + \frac{1}{2}\right) e^x - \frac{1}{2}\cos x + \frac{1}{2}\sin x \tag{18}$$

**Computational Solution Bilinear (sim2):**
We use the bilinear $z$-transform:

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}. \tag{19}$$

The total $Y(z)$ is:

$$Y(z) = \left(c_1 + \frac{1}{2}\right) \cdot \frac{1+z^{-1}}{\frac{2}{T} - 1 - \left(\frac{2}{T}+1\right) z^{-1}} \tag{20}$$

$$- \frac{\frac{1}{T}(1-z^{-1})}{\left(\frac{2}{T}\right)^2 (1-z^{-1})^2 + (1+z^{-1})^2} \tag{21}$$

$$+ \frac{\frac{1}{2}(1+z^{-1})^2}{\left(\frac{2}{T}\right)^2 (1-z^{-1})^2 + (1+z^{-1})^2} \tag{22}$$

# Computational Solution: Extended Steps

$$\left( \left(\frac{2}{T}\right)^2 \left(1 - z^{-1}\right)^2 + \left(1 + z^{-1}\right)^2 \right)^{-1} \left[ \left(\frac{2}{T} - 1\right) - \left(\frac{2}{T} + 1\right) z^{-1} \right] Y(z)$$
$$= f\left(z^{-1}, z^{-2}, z^{-3}\right) + c \qquad (23)$$

$$\left( \left(\frac{4}{T^2} + 1\right) + \left(2 - \frac{8}{T^2}\right) z^{-1} + \left(\frac{4}{T^2} + 1\right) z^{-2} \right) \left( \frac{2}{T} - 1 - \left(\frac{2}{T} + 1\right) z^{-1}\right.$$
$$= f\left(z^{-1}, z^{-2}, z^{-3}\right) + c \qquad (24)$$

$$\left(a + bz^{-1} + cz^{-2}\right)\left(d + ez^{-1}\right) Y(z) = f\left(z^{-1}, z^{-2}, z^{-3}\right) + c \qquad (25)$$

$$adY(z) + (ae + bd)z^{-1}Y(z) + (cd + be)z^{-2}Y(z) + cez^{-3}Y(z)$$
$$= f(z^{-1}, z^{-2}, z^{-3}) + c \quad (26)$$

where $a = \left(\frac{4}{T^2} + 1\right)$, $b = \left(2 - \frac{8}{T^2}\right)$, $c = \left(\frac{4}{T^2} + 1\right)$, $d = \frac{2}{T} - 1$, $e = -$

$$\to adz^3Y(z) + (ae + bd)z^2Y(z) + (cd + be)z^1Y(z) + ceY(z)$$
$$= f'(z^1, z^2, z^3) + c \quad (27)$$

$$\to ady[n+3] + (ae + bd)y[n+2] + (cd + be)y[n+1] + cey[n]$$
$$= f''(\delta[n], \delta[n+1], \delta[n+2], \delta[n+3]) \quad (28)$$

# Computational Solution: Extended Steps

$$\text{Given: } y[0] = c_1,\ y[1] = y[0] + hy'[0] \tag{29}$$

$$y[1] = c_1 + h(1 + a) \tag{30}$$

$$y[2] = y[1] + h(y'[1]) \tag{31}$$

$$y[2] = y[1] + h(y[1] + \cos(h)) \tag{32}$$

$$y[3] = y[2] + h(y[2] + \cos(2h)) \tag{33}$$

$$n \geq 1 \rightarrow \delta[n], \delta[n+1], \delta[n+2], \delta[n+3] = 0 \tag{34}$$

# Difference Equation

Difference equation: $$y[n+3] = -\frac{1}{ad}\left[(ae+bd)y[n+2] + (cd+be)y[n+1]\right. \tag{36}$$

**Computational Solution: Trapezoid Method: (sim1)**
The given differential equation is already in first-order form:
$\frac{dy}{dx} = y + \cos x$.
Let: $y = y_1$, so $\frac{dy_1}{dx} = y_1 + \cos x$.
Using the Trapezoidal Rule, the equation can be written as:

$$y_{n+1} - y_n = \frac{h}{2}\Big((y_n + \cos x_n) + (y_{n+1} + \cos x_{n+1})\Big), \tag{37}$$

where $h$ is the step size, $y_n$ is the value of $y$ at $x_n$, and $y_{n+1}$ is the value of $y$ at $x_{n+1} = x_n + h$.
Rearranging Equation (1) to isolate $y_{n+1}$:

$$y_{n+1} - \frac{h}{2}y_{n+1} = y_n + \frac{h}{2}(y_n + \cos x_n + \cos x_{n+1}), \tag{38}$$

$$y_{n+1}\Big(1 - \frac{h}{2}\Big) = y_n\Big(1 + \frac{h}{2}\Big) + \frac{h}{2}(\cos x_n + \cos x_{n+1}), \tag{39}$$

# Computational Solution: Trapezoid Method (Continued)

$$y_{n+1} = \frac{y_n\left(1 + \frac{h}{2}\right) + \frac{h}{2}(\cos x_n + \cos x_{n+1})}{1 - \frac{h}{2}}. \tag{40}$$

The final iterative formula is:

$$y_{n+1} = \frac{y_n\left(1 + \frac{h}{2}\right) + \frac{h}{2}(\cos x_n + \cos x_{n+1})}{1 - \frac{h}{2}}. \tag{41}$$

Given the initial condition: $x_0 = 0$, $y_0 = 0$, and a chosen step size $h$, the values of $y_n$ can be iteratively computed for subsequent $x_n$.

# Diagram