12.8.1.12

EE24BTECH11030 - KEDARANANDA

Question: Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines.x = 0 and x = 2 is

Solution:

Variable	Description	values
$\mathbf{V}$	Quadratic form of the matrix	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$
$\mathbf{u}$	Linear coefficient vector	$\begin{pmatrix}0\\0\end{pmatrix}$
f	constant term	-4
$\mathbf{m}$	The direction vector of line	$\begin{pmatrix}1\\0\end{pmatrix}$
$\mathbf{h}$	Point on line	$\begin{pmatrix}2\\0\end{pmatrix}$

TABLE 0: Variables used

The point of intersection of the line with the circle is

$$x_i = h + k_i m \tag{1}$$

where, k_i is a constant and is calculated as follows:

$$k_i = \frac{1}{m^\top Vm}\left(-m^\top (Vh+u) \pm \sqrt{[m^\top (Vh+u)]^2 - g(h)(m^\top Vm)}\right). \tag{2}$$

Substituting the input parameters into k_i,

$$k_i = \frac{1}{\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}\left(-\begin{pmatrix}1 & 0\end{pmatrix}\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}\right)\right) \pm$$

$$\sqrt{\left[\begin{pmatrix}1 & 0\end{pmatrix}\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}\right)\right]^2 - g(h)\left(\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)}. \tag{3}$$

We get,
$k_i = 0, -4.$

Substituting k_i into $x_i = h + k_i m$, we get

$$x_1 = \begin{pmatrix}2\\0\end{pmatrix} + (0)\begin{pmatrix}1\\0\end{pmatrix} \tag{4}$$

$$\implies x_1 = \begin{pmatrix}2\\0\end{pmatrix}, \tag{5}$$

$$x_2 = \begin{pmatrix}2\\0\end{pmatrix} + (-4)\begin{pmatrix}1\\0\end{pmatrix} \tag{6}$$

$$\implies x_2 = \begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}-4\\0\end{pmatrix} \tag{7}$$

$$\implies x_2 = \begin{pmatrix}-2\\0\end{pmatrix}. \tag{8}$$

The area of the region bounded by the circle $x^2 + y^2 = 4$, the line $x = 0$, and $x = 2$ in the first quadrant is: Area $= \int_0^2 \sqrt{4 - x^2}\, dx$.

Using trigonometric substitution, we calculate:

$$\text{Area} = \int_0^2 \sqrt{4 - x^2}\, dx \tag{9}$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 \tag{10}$$

$$= \left(\frac{2}{2}\sqrt{4 - 2^2} + 2\sin^{-1}\left(\frac{2}{2}\right)\right) - \left(\frac{0}{2}\sqrt{4 - 0^2} + 2\sin^{-1}\left(\frac{0}{2}\right)\right) \tag{11}$$

$$= \left(0 + 2 \cdot \frac{\pi}{2}\right) - (0 + 0) \tag{12}$$

$$= \pi. \tag{13}$$

Thus, the area of the region is: π.

Computational Solution:

Taking trapezoid-shaped strips of small area and adding them all up. Say we have to find the area of y_x from $x = x_0$ to $x = x_n$, discretize the points on the x axis $x_0, x_1, x_2, \dots, x_n$ such that they are equally spaced with the step size h.

Sum of all trapezoidal areas is given by,

$$A = \frac{1}{2}h\left(y(x_1) + y(x_0)\right) + \frac{1}{2}h\left(y(x_2) + y(x_1)\right) + \cdots + \frac{1}{2}h\left(y(x_n) + y(x_{n-1})\right) \tag{14}$$

$$= h\left[\frac{1}{2}\left(y(x_0) + y(x_n)\right) + y(x_1) + \cdots + y(x_{n-1})\right] \tag{15}$$

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \dots x_n)$ be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2}h\left(y(x_n + h) + y(x_n)\right) \tag{16}$$

We can repeat this till we get required area.

Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2}h(y_{n+1} + y_n) \tag{17}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2}h((y_n + hy'_n) + y_n) \tag{18}$$

$$A_{n+1} = A_n + \frac{1}{2}h(2y_n + hy'_n) \tag{19}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{20}$$

$$x_{n+1} = x_n + h \tag{21}$$

In the given question, $y_n = \sqrt{4 - x_n^2}$ and $y'_n = \frac{-x_n}{\left(\sqrt{4-x_n^2}\right)}$

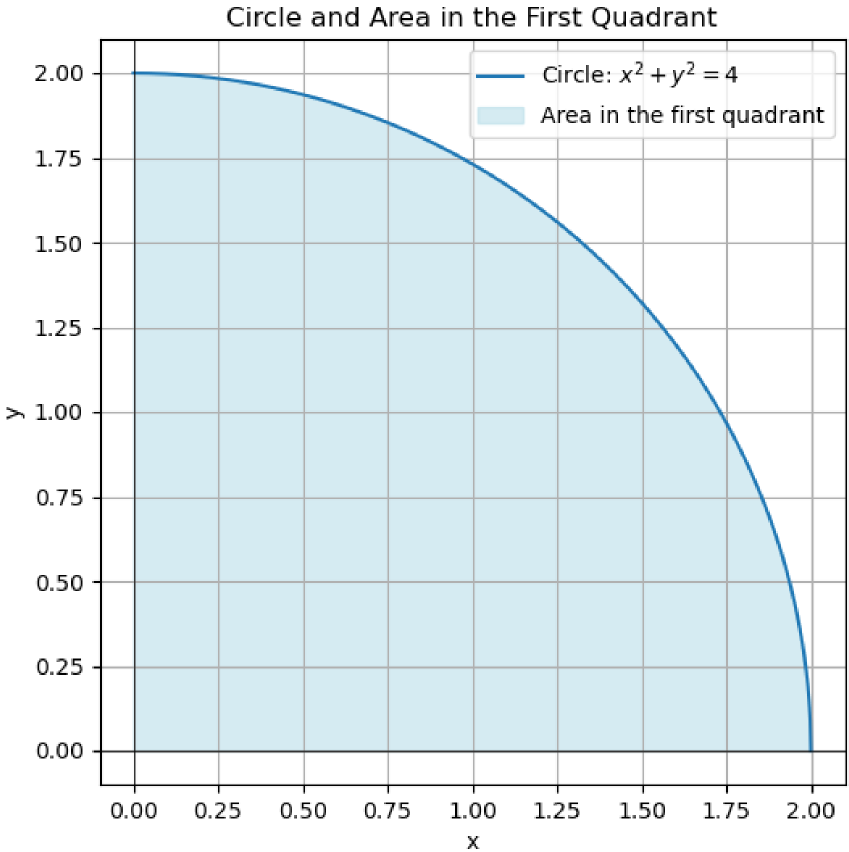
The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2y'_n \tag{22}$$

$$A_{n+1} = A_n + h\left(\sqrt{4 - x_n^2}\right) + \frac{1}{2}h^2\left(\frac{-x_n}{\left(\sqrt{4 - x_n^2}\right)}\right) \tag{23}$$

$$x_{n+1} = x_n + h \tag{24}$$

Iterating till we reach $x_n = 2$ will return required area.
Area obtained computationally: 3.1416 sq. units
Area obtained theoretically: π sq. units = 3.1416 sq.unis


Circle and Area in the First Quadrant
Circle: $x^2 + y^2 = 4$
Area in the first quadrant
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x