GATE 2024 MA(40-52)

EE24BTECH11030 - J.KEDARANANDA

1) Consider the following statements.
   I. There exists a proper subgroup G of $(\mathbb{Q}, +)$ such that $\mathbb{Q}/G$ is a finite group.
   II. There exists a subgroup G of $(\mathbb{Q}, +)$ such that $\mathbb{Q}/G$ is isomorphic to $(\mathbb{Z}, +)$.
   Which one of the following is correct?
   a) Both I and II are TRUE
   b) I is TRUE and II is FALSE
   c) I is FALSE and II is TRUE
   d) Both I and II are FALSE

2) Let X be the space $\mathbb{R}/\mathbb{Z}$ with the quotient topology induced from the usual topology on $\mathbb{R}$. Consider the following statements.
   I. X is compact.
   II. $X \setminus \{x\}$ is connected for any $x \in X$.
   Which one of the following is correct?
   a) Both I and II are TRUE
   b) I is TRUE and II is FALSE
   c) I is FALSE and II is TRUE
   d) Both I and II are FALSE

3) Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^7$. Let $\Sigma = \{v_1, \ldots, v_5\} \subset \mathbb{R}^7$ be a set of unit vectors such that $\langle v_i, v_j \rangle$ is a non-positive integer for all $1 \leq i \neq j \leq 5$. Define $N(\Sigma)$ to be the number of pairs (r, s), $1 \leq r, s \leq 5$, such that $\langle v_r, v_s \rangle \neq 0$. The maximum possible value of $N(\Sigma)$ is equal to
   a) 9
   b) 10
   c) 14
   d) 5

4) Let $f(x) = |x| + |x-1| + |x-2|, x \in [-1, 2]$. Which one of the following numerical integration rules gives the exact value of the integral

$$\int_{-1}^{2} f(x)\, dx$$

   a) The Simpson's rule
   b) The trapezoidal rule
   c) The composite Simpson's rule by dividing $[-1, 2]$ into 4 equal subintervals
   d) The composite trapezoidal rule by dividing $[-1, 2]$ into 3 equal subintervals

5) Consider the initial value problem (IVP)

$$y' = e^{-y^2} + 1, \quad y(0) = 0.$$

I. IVP has a unique solution on $\mathbb{R}$.
II. Every solution of IVP is bounded on its maximal interval of existence.

Which one of the following is correct?

a) Both I and II are TRUE
b) I is TRUE and II is FALSE
c) I is FALSE and II is TRUE
d) Both I and II are FALSE

6) Let A be a 2×2 non-diagonalizable real matrix with a real eigenvalue λ and v be an eigenvector of A corresponding to λ. Which one of the following is the general solution of the system $y' = Ay$ of first-order linear differential equations?

a) $c_1 e^{\lambda t} + c_2 t e^{\lambda t}$, where $c_1, c_2 \in \mathbb{R}$
b) $c_1 e^{\lambda t} v + c_2 t e^{\lambda t} v$, where $c_1, c_2 \in \mathbb{R}$
c) $c_1 e^{\lambda t} v + c_2 e^{\lambda t}(tv + u)$, where $c_1, c_2 \in \mathbb{R}$ and u is a 2×1 real column vector such that $(A - \lambda I)u = v$
d) $c_1 e^{\lambda t} v + c_2 t e^{\lambda t}(v + u)$, where $c_1, c_2 \in \mathbb{R}$ and u is a 2×1 real column vector such that $(A - \lambda I)u = v$

7) Let $D = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$. If the following second-order linear partial differential equation

$$y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} + y \frac{\partial u}{\partial y} = 0 \text{ on } D$$

is transformed to

$$\left(\frac{\partial^2 u}{\partial \eta^2} - \frac{\partial^2 u}{\partial \xi^2} \right) + \left(a \frac{\partial u}{\partial \eta} + \frac{\partial u}{\partial \xi} \right) \frac{1}{2\eta} + \left(a \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) \frac{1}{2\xi} = 0 \text{ on } D$$

for some $a, b \in \mathbb{R}$, via the coordinate transform $\eta = \frac{x^2}{2}$ and $\xi = \frac{y^2}{2}$, then which one of the following is correct?

a) $a = 2, b = 0$
b) $a = 0, b = -1$
c) $a = 1, b = -1$
d) $a = 1, b = 0$

8) Let $\ell^p = \left\{ x = (x_n)_{n \geq 1} : x_n \in \mathbb{R}, \|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty \right\}$ for $p = 1, 2$. Let

$$c_{00} = \{(x_n)_{n \geq 1} : x_n = 0 \text{ for all but finitely many } n \geq 1\}.$$

For $x = (x_n)_{n \geq 1} \in c_{00}$, define $f(x) = \sum_{n=1}^{\infty} \frac{x_n}{\sqrt{n}}$. Consider the following statements:

I. There exists a continuous linear functional F on $\ell^1, \|\cdot\|_1$ such that $F = f$ on c_{00}.
II. There exists a continuous linear functional G on $\ell^2, \|\cdot\|_2$ such that $G = f$ on c_{00}.

Which one of the following is correct?

a) Both I and II are TRUE

b) I is TRUE and II is FALSE
c) I is FALSE and II is TRUE
d) Both I and II are FALSE

9) Let $\ell_2^Z = \left\{(x_j)_{j\in\mathbb{Z}} : x_j \in \mathbb{R} \text{ and } \sum_{j=-\infty}^{\infty} x_j^2 < \infty\right\}$ endowed with the inner product

$$\langle x, y\rangle = \sum_{j=-\infty}^{\infty} x_j y_j, \quad x = (x_j)_{j\in\mathbb{Z}},\ y = (y_j)_{j\in\mathbb{Z}} \in \ell_2^Z.$$

Let $T : \ell_2^Z \to \ell_2^Z$ be given by $T((x_j)_{j\in\mathbb{Z}}) = (y_j)_{j\in\mathbb{Z}}$, where

$$y_j = \frac{x_j + x_{-j}}{2}, \quad j \in \mathbb{Z}.$$

Which of the following is/are correct?

(A) T is a compact operator
(B) The operator norm of T is 1
(C) T is a self-adjoint operator
(D) Range(T) is closed

10) Let X be the normed space $(\mathbb{R}^2, \|\cdot\|)$, where

$$\|(x, y)\| = |x| + |y|, \quad (x, y) \in \mathbb{R}^2.$$

Let $S = \{(x, 0) : x \in \mathbb{R}\}$ and $f : S \to \mathbb{R}$ be given by $f((x, 0)) = 2x$ for all $x \in \mathbb{R}$. Recall that a Hahn-Banach extension of f to X is a continuous linear functional F on X such that $F|_S = f$ and $\|F\| = \|f\|$, where $\|F\|$ and $\|f\|$ are the norms of F and f on X and S, respectively. Which of the following is/are true?

a) $F(x, y) = 2x + 3y$ is a Hahn-Banach extension of f to X
b) $F(x, y) = 2x + y$ is a Hahn-Banach extension of f to X
c) f admits infinitely many Hahn-Banach extensions to X
d) f admits exactly two distinct Hahn-Banach extensions to X

11) Let $\{[a, b) : a, b \in \mathbb{R}, a < b\}$ be a basis for a topology τ on $\mathbb{R}$. Which of the following is/are correct?

(A) Every (a, b) with $a < b$ is an open set in $(\mathbb{R}, \tau)$
(B) Every $[a, b]$ with $a < b$ is a compact set in $(\mathbb{R}, \tau)$
(C) $(\mathbb{R}, \tau)$ is a first-countable space
(D) $(\mathbb{R}, \tau)$ is a second-countable space

12) Let $T, S : \mathbb{R}^4 \to \mathbb{R}^4$ be two non-zero, non-identity $\mathbb{R}$-linear transformations. Assume $T^2 = T$. Which of the following is/are TRUE?

(A) T is necessarily invertible
(B) T and S are similar if $S^2 = S$ and Rank(T) = Rank(S)
(C) T and S are similar if S has only 0 and 1 as eigenvalues
(D) T is necessarily diagonalizable

13) Let $p_1 < p_2$ be the two fixed points of the function $g(x) = e^x - 2$, where $x \in \mathbb{R}$. For $x_0 \in \mathbb{R}$, let the sequence $(x_n)_{n\geq 1}$ be generated by the fixed point iteration

$$x_n = g(x_{n-1}), \quad n \geq 1.$$

Which one of the following is/are correct?

(A) $(x_n)_{n\geq 0}$ converges to p_1 for any $x_0 \in (p_1, p_2)$
(B) $(x_n)_{n\geq 0}$ converges to p_2 for any $x_0 \in (p_1, p_2)$
(C) $(x_n)_{n\geq 0}$ converges to p_2 for any $x_0 > p_2$
(D) $(x_n)_{n\geq 0}$ converges to p_1 for any $x_0 < p_1$