

EXERCÍCIOS DE CÁLCULO DIFERENCIAL E INTEGRAL 1

Ficha 1

1

a) $\frac{x}{\frac{2}{x}} = \frac{x^2}{4} = \left(\frac{x}{2}\right)^2, x \neq 0$

b) $\frac{x+1}{\frac{1+x}{x}} = \frac{x+1}{\frac{x+1}{x}} = \frac{x(x+1)}{x+1} = x, x \neq 0$

c) $\frac{1}{1+x} + \frac{1}{x^2+x} = \frac{x \cdot 1}{x(1+x)} + \frac{1}{x^2+x} = \frac{x+1}{x(x+1)} = \frac{1}{x}, x \neq 0$

d) $\sqrt{x^2} = \pm x, x \neq 0$

e) $(\sqrt{x})^2 = x, x \geq 0$

f) $4^x \cdot \frac{4}{2^x} = \frac{4^{(x+1)}}{2^x} = \frac{2^{2(x+1)}}{2^x} = 2^{2x+2-x} = 2^{x+2}$

g) $2^{x^2} \cdot (2^x)^2 = 2^{x^2} \cdot 2^{2x} = 2^{2x+x^2}$

h) $\frac{\sqrt[3]{x^2}}{6\sqrt{x}} = \frac{(x^2)^{\frac{1}{3}}}{(x)^{\frac{1}{6}}} = \frac{x^{\frac{2}{3}}}{x^{\frac{1}{6}}} = x^{\frac{2}{3}-\frac{1}{6}} = x^{\frac{3}{6}} = x^{\frac{1}{2}} = \sqrt{x}$

i) $\sqrt{x-2} \cdot \sqrt{x+2} = \sqrt{(x-2)(x+2)} = \sqrt{x^2+2x-2x-4} =$

$= \sqrt{x^2-4}$ // $x^2-4 \geq 0 \Leftrightarrow x \geq \pm 2$

j) $\frac{\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} = \frac{\sqrt{x} \cdot (\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})} = \frac{\sqrt{x(x+1)}+x}{x+1+x} =$

$= \sqrt{x^2+x}$ // $x \geq 0$

$x+1 \geq 0 \Leftrightarrow x \geq -1$

$\sqrt{x+1} \neq \sqrt{x}$

$$K) \log\left(\frac{1}{x}\right) + \log(x^2) = \log(x^{-2} \cdot x^2) = \log x, //$$

$$L) \log(2x^2 + 2x^{-2}) + \log\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right) =$$

$$= \log(2(x^2 + x^{-2})) + \log\left(\frac{1}{2}(x^2 + x^{-2})\right) =$$

$$= \log 2 + \log(x^2 + x^{-2}) + \log 2^{-1} + \log(x^2 + x^{-2}) =$$

$$= \log 2 - \log 2 + 2 \log(x^2 + x^{-2}) = 2 \log(x^2 + x^{-2})$$

2

$$a) (x^2 - 3x + 2)(x - 2) \geq 0$$

$$x - 1 = 0 \Leftrightarrow x = 1 //$$

$$x^2 - 3x + 2 = 0 \Leftrightarrow x = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \Leftrightarrow x = \frac{3 \pm \sqrt{9 - 8}}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{3 \pm \sqrt{1}}{2} \Leftrightarrow x = \frac{3+1}{2} \vee x = \frac{3-1}{2} \Leftrightarrow x = 2 \vee x = 1 //$$

$-\infty$	1		2	$+\infty$	
-	0	+	+	+	$x - 1$
+	0	-	0	+	$x^2 - 3x + 2$
-	0	-	0	+	

$$\Omega = [2; +\infty[\cup \{1\}$$

$$b) x \leq 2 - x^2 \Leftrightarrow 0 \leq -x^2 - x + 2 \Leftrightarrow -x^2 - x + 2 \geq 0 //$$

$$-x^2 - x + 2 = 0 \Leftrightarrow x = \frac{1 \pm \sqrt{1 - 4 \cdot (-1) \cdot 2}}{-2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{1 \pm \sqrt{9}}{-2} \Leftrightarrow x = \frac{1 \pm 3}{-2} \Leftrightarrow x = \frac{1+3}{-2} \vee x = \frac{1-3}{-2} \Leftrightarrow$$

$$\Leftrightarrow x = -\frac{4}{2} \vee x = \frac{2}{2} \Leftrightarrow x = -2 \vee x = 1 //$$

$$-\infty \left| \begin{matrix} -2 \\ 0 \end{matrix} \right| + \left| \begin{matrix} 1 \\ 0 \end{matrix} \right| + \infty$$

$\Omega = [-2, 1]$

c) $x^2 \leq 2 - x^4 \Leftrightarrow 0 \leq -x^4 - x^2 + 2 \Leftrightarrow x^2 = y$

$$-y^2 - y + 2 \quad y = \frac{1 \pm \sqrt{1 - 4 \cdot (-1) \cdot 2}}{-2} \Leftrightarrow y = \frac{1 \pm 3}{-2} \Leftrightarrow$$

$$x^2 = \cancel{x^2} \sqrt{x^2 - 1} \Leftrightarrow y = \frac{-2}{-2} \vee y = \frac{4}{-2} \Leftrightarrow y = 1 \vee y = -2$$

d) $x^3 + x \leq 2x^2 \Leftrightarrow x^3 - 2x^2 + x \leq 0$

$$\Leftrightarrow x(x^2 - 2x + 2) \leq 0$$

$$x = 0,$$

$$x^2 - 2x + 2 = 0 \Leftrightarrow x = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 1}}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{-2 \pm \sqrt{4 - 4}}{2} \Leftrightarrow x = 1 //$$

	$-\infty$	0	1	∞
x	-	0	+	+
$x^2 - 2x - 2$	+	+	+	0
	-	0	+	+

$$\Omega = [-\infty, 0] \cup 1$$

e) $\sqrt[3]{x^2 + 2x} = 2 \Leftrightarrow x^2 + 2x = 2^3 \Leftrightarrow x^2 + 2x - 8 = 0 \Leftrightarrow$
 $\Leftrightarrow x = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-8)}}{2 \cdot 1} \Leftrightarrow x = \frac{-2 \pm \sqrt{4 + 32}}{2} \Leftrightarrow$

$$\Leftrightarrow x = \frac{-2 \pm \sqrt{36}}{2} \Leftrightarrow x = \frac{-2 + 6}{2} \vee x = \frac{-2 - 6}{2} \Leftrightarrow x = 2 \vee x = -4 //$$

$$F) \sqrt[3]{x-1} = \sqrt{x-1} \Leftrightarrow (x-1)^{1/3} = (x-1)^{1/2} \Leftrightarrow \frac{1}{3} = \frac{1}{2} \text{ // P.f.}$$

Esta equação não tem soluções $\mathcal{R} = \emptyset$

$$g) \frac{x-1}{x^2-1} \leq 1 \Leftrightarrow \frac{x-1}{(x+1)(x-1)} - 1 \leq 0 \Leftrightarrow \frac{1}{x+1} - \frac{x+1}{x+1} \leq 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{1-(x+1)}{x+1} \leq 0 \Leftrightarrow \frac{-x}{x+1} \leq 0, \quad x+1$$

$$\mathcal{R} =]-1, 0]$$

x	$-$	$-$	-1	0	$+$	$+\infty$
$x+1$	$-$	$-$	0	$+$	$+$	$+$
$+/-$	$+$	$+$	ND	$-$	0	$+$

$$h) x = \frac{1}{x} \Leftrightarrow x^2 = 1 \Leftrightarrow x = 1 \vee x = -1 //$$

$$i) x < \frac{1}{x} \Leftrightarrow x-1 < 0 \Leftrightarrow \frac{x^2-1}{x} < 0 //, x^2-1$$

$$x^2-1=0 \Leftrightarrow x^2=1 \Leftrightarrow x=\pm 1 \quad \mathcal{R} =]-\infty, 0] \cup [0, 1[$$

$$j) x < |x| \Leftrightarrow$$

$$\Leftrightarrow x < x \vee x < -x \Leftrightarrow 2x < 0 \Leftrightarrow x < 0$$

P.F.

$$\mathcal{R} =]-\infty, 0[$$

$$K) |x| \geq \frac{x}{2} + 1 \Leftrightarrow |x| \geq \frac{x+2}{2} \Leftrightarrow |x| \geq \frac{x+2}{2} \Leftrightarrow$$

$$\Leftrightarrow x \geq \frac{x+2}{2} \vee -x \geq \frac{x+2}{2} \Leftrightarrow 2x \geq x+2 \vee -2x \geq x+2 \Leftrightarrow$$

$$\Leftrightarrow 2x - x \geq 2 \vee -2x - x \geq 2 \Leftrightarrow x \geq 2 \vee -3x \geq 2 \Leftrightarrow$$

$$\Leftrightarrow x \geq 2 \vee x \leq -\frac{2}{3} // \quad \mathcal{R} =]-\infty, -\frac{2}{3}] \cup [2, +\infty[$$

$$l) |x| \leq |x-2| \Leftrightarrow x \leq |x-2| \vee -x \leq |x-2| \Leftrightarrow$$

$$\Leftrightarrow x \leq x-2 \vee x \leq -x+2 \vee -x \leq x-2 \vee -x \leq -x+2 \Leftrightarrow$$

$$\Leftrightarrow 0 \leq -2 \vee 2x \leq 2 \vee -2x \leq -2 \vee 0 \leq 2 \Leftrightarrow$$

P.F

P.V

$$\Leftrightarrow x \leq 1, \quad \mathbb{R} \setminus [-\infty, 1]$$

m) $|x^2 - 2| \leq 2 \Leftrightarrow x^2 - 2 \leq 2 \vee -(x^2 - 2) \leq 2 \Leftrightarrow$
 $x^2 \leq 2+2 \vee -x^2 \leq 2-2 \Leftrightarrow$
 $x^2 \leq 4 \vee -x^2 \leq 0 \Leftrightarrow x \leq \pm\sqrt{4} \Leftrightarrow x \leq 2 \vee x \leq -2,$
 P.V.
 $\mathbb{R} =]-\infty, 2] \cup]-\infty, -2]$

n) $\frac{x^4 - 76}{|x-1|} \leq 0 \Leftrightarrow \frac{x^4 - 76}{x-1} \leq 0 \vee \frac{x^4 - 76}{-x+2} \leq 0 \Leftrightarrow$
 $\Leftrightarrow \frac{(x^2-4)(x^2+4)}{x-1} \vee \frac{(x^2-4)(x^2+4)}{1-x} \Leftrightarrow x=2 \vee x=-2$
 $\hookrightarrow x=1 \quad \hookrightarrow x=-1 \quad \mathbb{R} =]-\infty, -2] \cup [1, 2]$

o) $e^{x^3} < 1 \Leftrightarrow e^{x^3} < e^0 \Leftrightarrow x^3 < 0, \quad \mathbb{R} = \mathbb{R}^+$

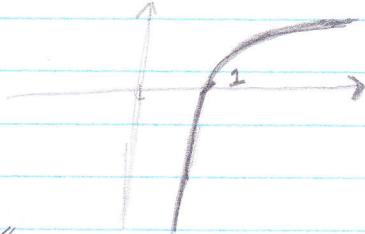
p) $e^{-2x} - 2e^{-x} \leq -1 \Leftrightarrow e^{-2x} - 2e^{-x} + 1 \leq 0$

$e^{-x} = y \Rightarrow y^2 - 2y + 1 = 0 \Leftrightarrow y = \frac{2 \pm \sqrt{4-4 \cdot 1 \cdot 1}}{2} \Leftrightarrow$
 $\Leftrightarrow y = \frac{2}{2} \Leftrightarrow y = 1$
 $\Rightarrow e^{-x} = 1 \Leftrightarrow e^{-x} = e^0 \Leftrightarrow -x = 0, \quad \mathbb{R}$

q) $\log\left(\frac{1}{x}\right) \geq 0 \Leftrightarrow$

$\Leftrightarrow \log 1 - \log x \geq 0 \Leftrightarrow -\log x \geq 0, \quad \mathbb{R}$

$\mathbb{R} =]0, 1]$



$$① \log(x^2 - 3) \geq 0 \Leftrightarrow x^2 - 3 \geq 1 \Leftrightarrow x^2 \geq 4 \Leftrightarrow$$

$$\Leftrightarrow x \geq \pm 2 \quad -\infty \quad | \quad -2 \quad | \quad 2 \quad | \quad +\infty$$

+ + | 0 | - | 0 | + +

$$S =]-\infty, -2] \cup [2, +\infty[$$

3

$$a) \{x : \frac{x-1}{x+1} < 1\} \Leftrightarrow x \in \mathbb{R} \setminus \{-1\}$$

$$b) \{x : \frac{x^4 - 1}{x^3} \leq x\} \stackrel{x \neq}{\Leftrightarrow} \mathbb{R} \setminus \{0\}$$

$$x^4 - 1 \leq x \cdot x^3 \Leftrightarrow x^4 - 1 \leq x^4 \rightarrow \text{p.v}$$

$$c) \{x : |3x-9| \geq x^2\} \stackrel{x \neq}{\Leftrightarrow} [-9, 1],$$

$$3x-9 \geq x^2 \vee -3x+9 \geq x^2 \Leftrightarrow -x^2 - 3x + 9 \geq 0 \vee -x^2 + 3x + 9 \geq 0$$

$$\begin{aligned} &= x^2 + 3x - 9 = 0 \Leftrightarrow \\ x &= \frac{-3 \pm \sqrt{9 - 4 \cdot (-1) \cdot (-9)}}{-2} \Leftrightarrow \end{aligned}$$

$$\Leftrightarrow x = \frac{-3 \pm \sqrt{5}}{-2} \text{ impossible!}$$

$$\begin{cases} -x^2 + 3x + 9 = 0 \Leftrightarrow \\ \Leftrightarrow x = \frac{3 \pm \sqrt{9 + 76}}{-2} \Leftrightarrow x = \frac{3 \pm \sqrt{25}}{-2} \Leftrightarrow \\ \Leftrightarrow x = \frac{3+5}{-2} \vee x = \frac{3-5}{-2} \Leftrightarrow \\ \Leftrightarrow x = \frac{8}{-2} \vee x = \frac{-2}{-2} \Leftrightarrow x = -4 \vee x = 1 \end{cases}$$

$$d) \{x : |x-1| (x^2 - 4) \geq 0\} \stackrel{x^2 - 4}{\Leftrightarrow} x \in]-\infty, -2] \cup [1, +\infty[$$

$$(x-2)(x^2 - 4) \geq 0 \quad \vee (x+1)(x^2 - 4) \geq 0$$

$$\begin{aligned} &x-2 = 0 \vee x^2 - 4 = 0 \vee x+1 = 0 \vee x^2 - 4 = 0 \Leftrightarrow \\ \Rightarrow x &= 1 \vee x = \pm \sqrt{4} \vee x = -2 \vee x = \pm \sqrt{4} \end{aligned}$$

	$-\infty$	-2	-1	1	2	$+\infty$
$ x-1 $	+	+	+	+	+	+
x^2-4	+	0	-	-	0	+
	+	0	-	-	0	+

$$n =]-\infty, -2] \cup \{1\} \cup [2, +\infty[$$

$$\text{e)} \quad \{x : (|x|-1)(x^2-4) \leq 0\} \Leftrightarrow x \in]-\infty, -2] \cup [-1, 2]$$

C.A. $(|x|-1)(x^2-4) \leq 0 \Leftrightarrow (x-1)(x^2-4) \leq 0 \vee (-x-1)(x^2-4) \leq 0$

$$(x-1)=0 \vee x^2-4=0 \vee -x-1=0 \vee x^2-4=0 \Leftrightarrow$$

$$\Leftrightarrow x=1 \vee x=\pm 2 \vee x=-1 \vee x=\pm 2$$

$$n =]-\infty, -2] \cup [-1, 2]$$

$$\text{f)} \quad \{x : |x^2-1| \leq |x+1|\} \Leftrightarrow x \in [-1, 2]$$

$$x^2-1 \leq x+1 \vee x^2-1 \geq -x-1 \Leftrightarrow -x^2+x+1 \leq 0 \vee -x^2+x+2 \leq 0 \Leftrightarrow$$

$$\Leftrightarrow x^2-x-2 \leq 0 \vee x^2+x \leq 0 \vee -x^2+x \leq 0 \Leftrightarrow -x^2+x+2 \leq 0 \Leftrightarrow$$

$$x^2+x=0 \Leftrightarrow x^2=-x \Leftrightarrow x=-1$$

$$-x^2-x=0 \Leftrightarrow -x^2=x \Leftrightarrow x=-1$$

$$x^2-x-2=0 \Leftrightarrow x = \frac{1 \pm \sqrt{1-(4 \cdot 1 \cdot (-2))}}{2} \Leftrightarrow \frac{1 \pm \sqrt{9}}{2} = x \Leftrightarrow x = \frac{1+3}{2} \vee$$

$$\vee x = \frac{1-3}{2} \Leftrightarrow$$

$$\Leftrightarrow x = 2 \vee x = -1$$

$$-x^2 + x + 2 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1 - (4 \cdot (-2)) \cdot 2}}{-2} \Leftrightarrow x = \frac{-1 \pm \sqrt{9}}{-2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{-1+3}{-2} \vee x = \frac{-1-3}{-2} \Leftrightarrow x = -1 \vee x = 2 //$$

	$-\infty$	-1	2	$+\infty$
$x^2 + x$	+	0	+	+
$x^2 - x - 2$	+	0	0	+
$-x^2 - x$	-	0	-	-
$-x^2 + x + 2$	-	0	0	-
	+	0	0	+

9) $\left\{ x : x^2 - |x| - 2 \leq 0 \right\} \Leftrightarrow x \in [-2, 2]$

$$x^2 - x - 2 \leq 0 \vee x^2 + x - 2 \leq 0$$

$$x^2 - x - 2 = 0 \Leftrightarrow x = \frac{1 \pm \sqrt{1 - (4 \cdot 1 \cdot (-2))}}{2} \Leftrightarrow x = \frac{1 \pm \sqrt{1+8}}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{1 \pm 3}{2} \Leftrightarrow x = 2 \vee x = -1 // \quad x > 0$$

$$x^2 + x - 2 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1 - (4 \cdot 1 \cdot (-2))}}{-2} \Leftrightarrow x = \frac{-1 \pm 3}{2} \Leftrightarrow$$

$$\Leftrightarrow x = -2 \vee x = 1 // \quad x < 0$$

	$-\infty$	-2	-1	0	1	2	$+\infty$
$x^2 - x - 2$	+	+	+	0	-	0	+
$x^2 + x - 2$	+	0	-	-	0	+	+

b) $\{x : \frac{x}{|x|-1} \geq 0\} \Leftrightarrow x \in \mathbb{R}_0^+ \setminus \{1\}$

$$\frac{x}{|x|-1} \geq 0, \quad x \neq 1 \text{ e } x \neq -1$$

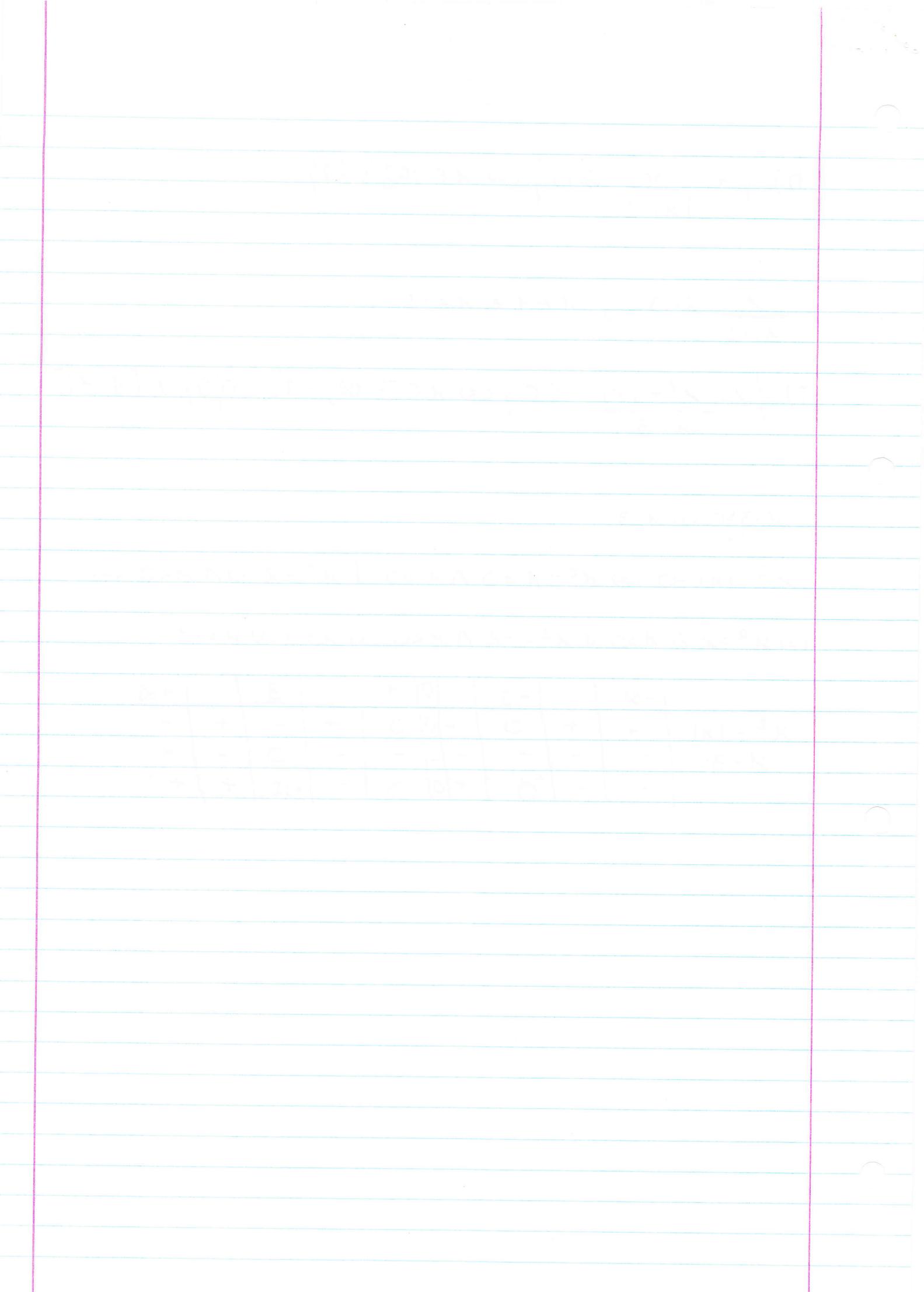
i) $\{x : \frac{x^2 - |x|}{x-3} \leq 0\} \Leftrightarrow x \in]-\infty, -1] \cup \{0\} \cup [1, 3[$

$$x-3=0 \Leftrightarrow x=3$$

$$x^2 - |x| = 0 \Leftrightarrow x^2 - x = 0 \wedge x > 0 \vee x^2 + x = 0 \wedge x < 0 \Leftrightarrow$$

$$(\Rightarrow x^2 = x \wedge x > 0 \vee x^2 = -x \wedge x < 0 \Leftrightarrow x = 1 \vee x = -1)$$

	$-\infty$	-1	0	1	3	$+\infty$
$x^2 - x $	+	0	-0	0	+	+
$x-3$	-	-	-	-	0	+
	-	0	+0	0	ND	+



Ficha z

1.

a) $\{1\} \subset \{1, \{2, 3\}\}$ false

b) $\{1\} \in \{1, \{2, 3\}\}$ false

c) $2 \in \{1, \{2, 3\}\}$ false

d) $1 \in \{\mathbb{R}\}$ false - 1 não é elemento de $\{\mathbb{R}\}$

e) $\emptyset = \{x \in \mathbb{N} : x = x + 1\}$ verdadeira - $x = x + 1$ nunca se verifica

f) $\emptyset \in \{0\}$ false

g) $\emptyset \subset \{0\}$ verdadeiro

h) $\forall_{x \in \mathbb{R}} x > 0 \Rightarrow x^{-1} > 0$ verdadeira

i) $\forall_{x \in \mathbb{R}} x > 1 \Rightarrow x^{-1} < 1$ verdadeira - se $x > 1$ então $\frac{1}{x} < 1$

j) $\forall_{x, y \in \mathbb{R}} x < y \Rightarrow y^{-1} < x^{-1}$ verdadeira

k) $\forall_{x \neq 0} x^2 > 0$ verdadeira - a função x^2 nunca é negativa

l) $\forall_{x, y \in \mathbb{R}} x < y \Rightarrow x^2 < y^2$ false

m) $\forall_{x, y \in \mathbb{R}} x < y < 0 \Rightarrow x^2 > y^2$ verdadeira

2

$$\forall a > 0 \quad a + \frac{1}{a} \geq 1$$

Se $a \geq 1$

$\boxed{\text{Se } a=1}$	$1+1 \geq 1$
$1 \leq \frac{1}{a} < 2, a > 0$	$\text{então } a + \frac{1}{a} \geq 1 \text{ cqd}$
então	$\text{se } a < 1 \Rightarrow \frac{1}{a} > 1$
$\text{para qualquer } a > 0$	$a + \frac{1}{a} \geq 1$
$a + \frac{1}{a} \geq 1$ é verdade	

$$\text{se } a < 1 \Rightarrow \frac{1}{a} > 1$$

$$\text{então } a + \frac{1}{a} \geq 1 \text{ cqd}$$

3

a) $1+3+\dots+(2n-1)=n^2, \forall n \in \mathbb{N}_1 = P(n)$

1) Provar que é verdade para $n=1$

$$(2 \times 1 - 1) = 1 = 1^2, \text{ como queríamos demonstrar}$$

2) $P(n) \Rightarrow P(n+1)$

$$\begin{aligned} & \overbrace{1+3+\dots+(2n-1)}^{n^2} + (2(n+1)-1) = (n+1)^2, \Leftrightarrow \\ & \Leftrightarrow n^2 + (2n+2-1) = n^2 + 2n + 1 \Leftrightarrow \\ & \Leftrightarrow n^2 + 2n + 1 = n^2 + 2n + 1, \text{ cqd} \end{aligned}$$

Apartir de 1) e 2) conclui-se que $P(n)$ é verdadeira para todo o $n \in \mathbb{N}_1$.

b) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}, \forall n \in \mathbb{N}_1$

1) $\boxed{n=1}$

$$\frac{1}{1(1+1)} = \frac{1}{1+1} \Leftrightarrow \frac{1}{2} = \frac{1}{2} \rightarrow \text{cqd}$$

2) Se $p(n) \Rightarrow p(n+1)$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+1+1)} = \frac{n+2}{n+1+1} \Leftrightarrow$$

$$\Leftrightarrow \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+2}{n+2} \Leftrightarrow$$

$$\Leftrightarrow \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)(n+2)}{(n+2)(n+2)} \Leftrightarrow$$

$$\Leftrightarrow \frac{n^2+2n+1}{(n+2)(n+2)} = \frac{n^2+2n+2}{(n+1)(n+2)} // \text{ cqd}$$

Apartir de 1) e 2) conclui-se que a proposição é verdadeira para todo $n \in \mathbb{N}_1$

c) $(n!)^2 > 2^n n^2, \forall n \geq 4 \in \mathbb{N}$

1) $|n=4|$

$$(4!)^2 > 2^4 \cdot 4^2 \Leftrightarrow 48^2 > 16 \cdot 16 \Leftrightarrow$$

$$\Leftrightarrow 48^2 > 16^2 \text{ cqd}$$

2) $p(n) \Rightarrow p(n+1)$

$$((n+2)!)^2 > 2^{n+1} \cdot (n+1)^2$$

Pela hipótese: $(n!)^2 > 2^n n^2 \Leftrightarrow (n!)^2 (n+1)^2 > 2^n n^2 (n+1)^2 \Leftrightarrow$

$$\Leftrightarrow (n!)^2 (n+1)^2 > 2^{n+1} n^2 (n+1)^2 \cdot \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow (n!)^2 (n+1)^2 > 2^{n+1} (n+1)^2 \cdot \frac{n^2}{2}, \frac{n^2}{2} \geq 1 \Leftrightarrow$$

$n \geq \sqrt{2}$ (cqd)

d) $n! \geq 2^{n-1}$, $\forall n \in \mathbb{N}_1$

1) $n=1$

$$1! \geq 2^{1-1} \Leftrightarrow 1 \geq 2^0 \Leftrightarrow 1 \geq 1 \text{ cqd}$$

2) se $P(n) \Rightarrow P(n+1)$

$$(n+1)! \geq 2^{(n+1)-1} = 2^n$$

Pela hipótese

$$\begin{aligned} & (n!) \cdot (n+1) \geq 2^{n-1} \cdot (n+2) \Leftrightarrow (n!) \cdot (n+1) \geq 2^{n-1} \cdot \frac{2}{2} \cdot (n+2) \\ & \Leftrightarrow (n!) \cdot (n+1) \geq 2^n \quad \text{---} \quad \text{V} \quad \frac{n+1}{2} \geq 1 \Leftrightarrow n+1 \geq 2 \\ & \Leftrightarrow n \geq \frac{2}{1} \Leftrightarrow n \geq 1 \text{ cqd} \end{aligned}$$

a) $1+2+3+\dots+n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}_1$

$n=1$

$$1 = \frac{1(1+1)}{2} \Leftrightarrow 1 = 1 \text{ cqd}$$

$P(n) \Rightarrow P(n+1)$

$$1+2+3+\dots+n+(n+1) = \frac{(n+1)(n+2)+1}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{n(n+1)+(n+2)}{2} = \frac{(n+2)(n+2)}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{n(n+2)+2(n+1)}{2} = \frac{(n+2)(n+2)}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{n^2+n+2n+2}{2} = \frac{n^2+2n+n+2}{2} \quad \text{cqd} \quad //$$

$$b) (a-1)(1+a+\dots+a^n) = a^{n+1}-1 \quad \forall a \in \mathbb{R} \quad \forall n \in \mathbb{N}$$

|n=1|

$$(a-1)(a+1) = a^{1+1}-1 \Leftrightarrow a^2 + a - a - 1 = a^2 - 1 \quad // \text{cqd}$$

|Se p(n) \Rightarrow p(n+1)|

$$(a-1)(1+a+\dots+a^n+a^{n+1}) = a^{n+2}-1 \Leftrightarrow$$

$$\Leftrightarrow a^{n+1}-1 + (a-1) \cdot a^{n+1} = a^{n+2}-1 \Leftrightarrow$$

$$\Leftrightarrow \cancel{a^{n+1}-1} + a^{n+2} - \cancel{a^{n+1}} = a^{n+2}-1 \quad \text{cqd}$$

hipótese de indução:

$$c) \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \quad \forall n \in \mathbb{N}$$

|n=1| $\sum_{i=1}^n \frac{i}{(i+1)!}$

$$\frac{1}{2!} = 1 - \frac{1}{(1+1)!} \Leftrightarrow \frac{1}{2} = 1 - \frac{1}{2} \Leftrightarrow \frac{1}{2} - \frac{2}{2} - \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2} = \frac{1}{2} \quad \text{cqd}$$

|Se p(n) \Rightarrow p(n+2)|

$$\underbrace{\frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n}{(n+1)!}}_{\text{h}} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!} \Leftrightarrow$$

$$\Leftrightarrow 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!} \Leftrightarrow 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} - \frac{1}{(n+2)!} = 0$$

$$\Leftrightarrow 1 - \frac{1}{(n+1)!} + \frac{n+1-1}{(n+2)!} = 1 - \frac{1}{(n+2)!} \Leftrightarrow$$

$$\Leftrightarrow 1 - \frac{(n+2) + n+2}{(n+1)! (n+2)} = 1 - \frac{1}{(n+2)!} \Leftrightarrow$$

$$\Leftrightarrow 1 + \frac{-n-2+n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!} \Leftrightarrow$$

$$\Leftrightarrow 1 - \frac{1}{(n+2)!} = 1 - \frac{1}{(n+2)!} \text{ cqd}$$

d) $1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad n \in \mathbb{N}_1$

$$\boxed{n=1}$$

$$1^3 = \left(\frac{1(1+1)}{2}\right)^2 \Leftrightarrow 1 = 1^2 \text{ cqd}$$

$$\boxed{\text{Se } p(n) \Rightarrow p(n+1)}$$

$$\underbrace{1^3 + 2^3 + 3^3 + \dots + n^3}_{\text{h}} + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{n^2 + n+1}{4} = \frac{(n+2)^2}{4} \Leftrightarrow \frac{n^2 + 4n + 4}{4} = \frac{n^2 + 4n + 4}{4} \text{ cqd}$$

e) $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} \geq \sqrt{n} \quad \forall n \in \mathbb{N}_1$

$$\boxed{n=1}$$

$$1 \geq \sqrt{1} \Leftrightarrow 1 \geq 1 \text{ cqd}$$

$$\boxed{\text{Se } p(n) \Rightarrow p(n+1)} \quad \frac{1}{\sqrt{n+1}} \left(\sqrt{n} + \frac{1}{\sqrt{n+1}} \right) = \frac{\sqrt{n}}{\sqrt{n+1}} + \frac{1}{n+1} =$$

$$\underbrace{1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}}_{\geq \sqrt{n}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1} + 1}{\sqrt{n+1}} \geq 1$$

$$\frac{\sqrt{n} + 1}{\sqrt{n+1}} = \frac{\sqrt{n} + \sqrt{n+1} + 1}{\sqrt{n+1}} \geq \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \frac{\sqrt{n+1}}{\sqrt{n+1}} = 1$$

$$1 + \sqrt{n}\sqrt{n+1} \geq \sqrt{n}\sqrt{n+1} + 1$$

$$\begin{cases} a \geq b, c > 0 \\ a.c \geq b.c \end{cases}$$

5.

$$\text{a)} \quad (n+2)! \geq 2^{2n} \quad \forall n \in \mathbb{N},$$

$$\boxed{n=1}$$

$$(1+2)! \geq 2^{2 \cdot 1} \Leftrightarrow 3! \geq 2^2 \Leftrightarrow 6 \geq 4 // \text{ cqd}$$

$$\boxed{\text{Se } p(n) \Leftrightarrow p(n+1)}$$

$$(n+1+2)! \geq 2^{2(n+1)} \Leftrightarrow (n+3)! \geq 2^{2n+2} \Leftrightarrow$$

$$\Leftrightarrow (n+2)! \cdot (n+3) \geq 2^{2n+2} //$$

$$\text{pela hipótese: } (n+2)! \geq 2^{2n} \Leftrightarrow (n+2)! \cdot (n+3) \geq 2^{2n} \cdot (n+3) \Leftrightarrow$$

$$\Leftrightarrow (n+2)! \cdot (n+3) \geq 2^{2n} \cdot 2^2 \cdot \frac{(n+3)}{2^2} \Leftrightarrow (n+2)! \cdot (n+3) \geq 2^{n+2} \cdot \frac{n+3}{4}$$

$$\frac{n+3}{4} \geq 1 \Leftrightarrow n+3 \geq 4 \Leftrightarrow n \geq 4-3 \Leftrightarrow n \geq 1 \text{ cqd}$$

$$\text{b)} \quad 2^{n-3} < 2^{n-2} \quad n \in \mathbb{N} \wedge n \geq 5$$

$$\boxed{n=5}$$

$$2^{5-3} < 2^{5-2} \Leftrightarrow 2^2 < 2^3 \Leftrightarrow 4 < 8 // \text{ cqd}$$

| Se $p(n) \Leftrightarrow p(n+1)$

$$2(n+1)-3 < 2^{n-1} \Leftrightarrow 2n-1 < 2^{n-1}$$

$$\begin{aligned} 2n-3 &< 2^{n-2} \Leftrightarrow 2n-3+2 < 2^{n-2}+2 \Leftrightarrow 2n-1 < (2^{n-2}+2) \frac{1}{2} \\ \Leftrightarrow 2n-1 &< 2^{n-2} + \underline{1} \quad \text{cqdl} \end{aligned}$$

c) $7^n - 1$ é múltiplo de 6 $\forall n \in \mathbb{N}_1$
 $\hookrightarrow 6k$

| $n=1$

$$7^1 - 1 = 6 = 6 \cdot 1 \Rightarrow \text{cqdl}$$

| Se $p(n) \Leftrightarrow p(n+1)$

$$7^{n+1} - 1 = 7 \cdot 7^n - 1$$

d) $2^{2n} + 2$ é múltiplo de 3 $\forall n \in \mathbb{N}_1$

| $n=1$

$$2^{2 \times 1} + 2 = 2^2 + 2 \Leftrightarrow 4 + 2 = 6 \quad \text{cqdl}$$

\hookrightarrow múltiplo de 3

| Se $p(n) \Leftrightarrow p(n+2)$

6

 $a > -1, n \in \mathbb{N}$

$$(1+a)^n \geq 1 + n.a$$

$$\boxed{n=1}$$

$$1+a \geq 1+a \quad \text{cqd}$$

$$\boxed{\text{se } p(n) \Rightarrow p(n+1)}$$

$$(1+a)^{n+1} \geq 1 + an + a$$

$$(1+a)^n \geq 1 + n.a \quad (\Rightarrow (1+a)^n(1+a) \geq (1+n)a(1+a))$$

$$\Leftrightarrow (1+a)^{n+1} \geq 1 + an + a + \underline{na^2}$$

?

7. $P(n)$: " $n^2 + 3n + 1$ é par"

$$\text{a)} \quad p(n+1): (n+1)^2 + 3(n+1) + 1 = n^2 + 1 + 3n + 3 + 1$$

$$= n^2 + 3n + 1 + 4 \quad ?$$

pela hipótese " $n^2 + 3n + 1$ é par"é 4 é par, logo $n^2 + 3n + 1 + 4$ é par

b) $\boxed{I_{n=1}}$ $1^2 + 3 \cdot 1 + 1 = 5$

não, $\exists n$ que torna falsa a condição $P(n)$

c) $n^2 + 3n + 1$ é ímpar $\forall n \in \mathbb{N}$

$\boxed{I_{n=1}}$

$$1^2 + 3 \cdot 1 + 1 = 5 \quad \text{é ímpar} \quad \text{cqd}$$

$\boxed{P(n) \Rightarrow P(n+1)}$

$$\begin{aligned} (n+1)^2 + 3(n+1) + 1 &= n^2 + 2n + 1 + 3n + 3 + 1 = \\ &= n^2 + 3n + 1 + 2n + 4 \end{aligned}$$

Pela hipótese $n^2 + 3n + 1$ é ímpar

$2n+4$ é par, logo $n^2 + 3n + 1 + 2n + 4$ é ímpar

8. $f: \mathbb{N} \rightarrow \mathbb{N}$

$$f(0) = 1$$

$$f(n+1) = (2n+2)(2n+1)f(n)$$

Mostrar que $f(n) = (2n)!$ $\Rightarrow f(n+1) = (2(n+1))! = (2n+2)!$

$$f(n+1) = (2n+2)(2n+1)(2n)! = (2n+2)! \quad \text{cqd}$$

9.

$$(u_n) = \begin{cases} u_1 = 3 \\ u_{n+1} = \frac{3u_n}{(n+1)^2} \end{cases}$$

Muestra que $u_n = \frac{3^n}{(n!)^2} \quad \forall n \in \mathbb{N}_1$

$$u_{n+1} = \frac{3 \cdot \frac{3^n}{(n!)^2}}{(n+1)^2} = \frac{3 \cdot 3^n}{((n+1)!)^2} = \frac{3^{n+1}}{((n+1)!)^2} \text{ cqd}$$

10.

$$u_n \begin{cases} u_1 = 1 \\ u_{n+1} = \sqrt{2u_n^2 + 1} \end{cases}$$

Muestra que $u_n = \sqrt{2^n - 1}$

$$\Downarrow$$

$$u_{n+1} = \sqrt{2^{n+1} - 1}$$

$$u_{n+1} = \sqrt{2(\sqrt{2^n - 1})^2 + 1} = \sqrt{2(2^n - 1) + 1} = \\ = \sqrt{2 \cdot 2^n - 2 + 1} = \sqrt{2^{n+1} - 1} \text{ cqd}$$

11.

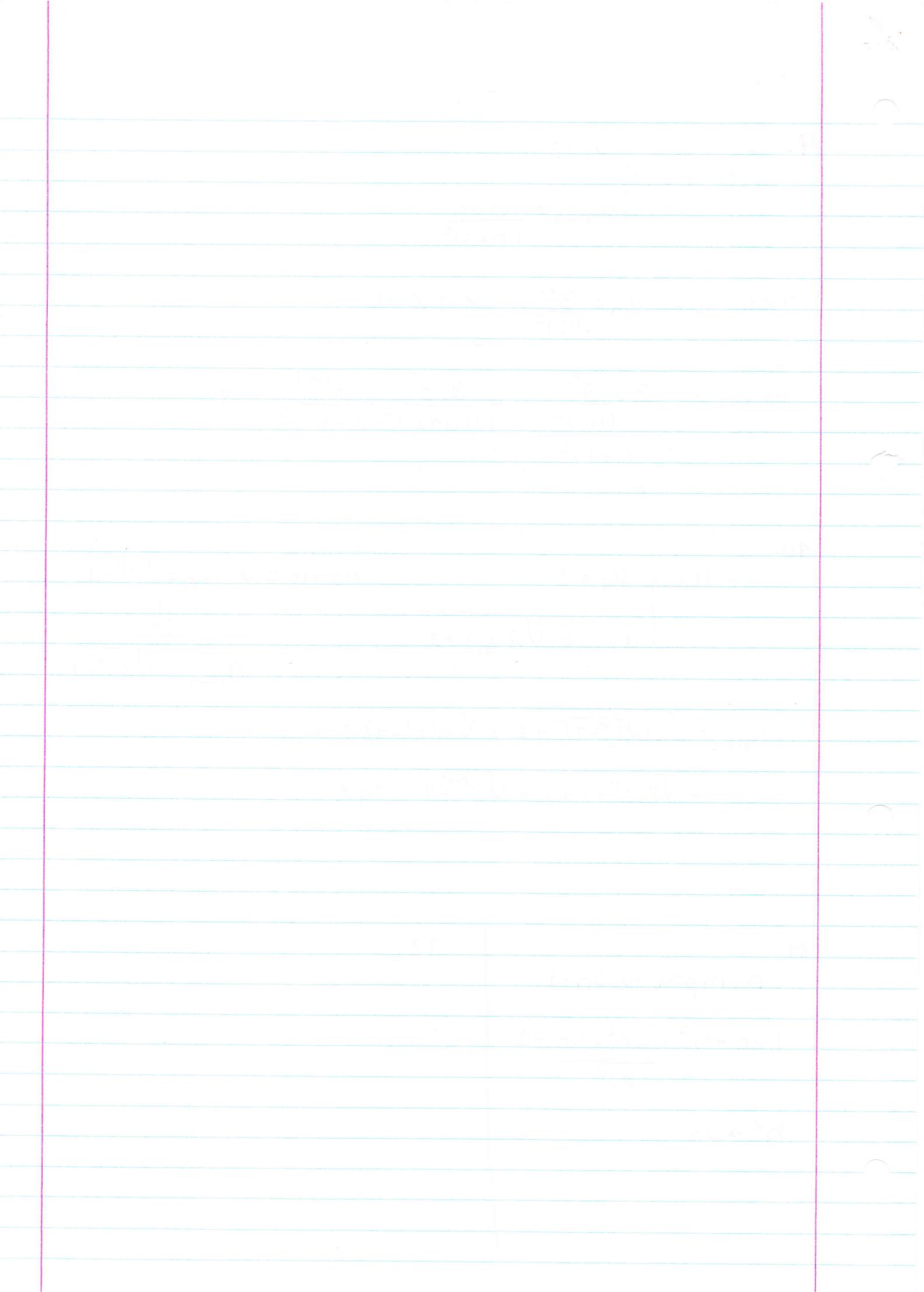
n impar $\rightarrow 2n+1$

$$(2n+1)^2 = 2n^2 + 4n + 1$$

par

$$n^2 \rightarrow 2n$$

12.



Ficha 3

1.

$$A = \{x \in \mathbb{R} : |x| \geq \frac{x}{2} + 2\}$$

$$B = [-3, 4] \quad C = \mathbb{R} \setminus \mathbb{Q}$$

a)

$$|x| \geq \frac{x}{2} + 2 \Leftrightarrow x \geq \frac{x}{2} + 2 \vee x \leq -\frac{x}{2} - 2 \Leftrightarrow$$

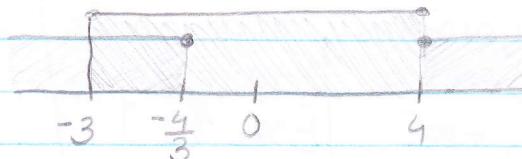
$$\Leftrightarrow x - \frac{x}{2} \geq 2 \vee x + \frac{x}{2} \leq -2 \Leftrightarrow \frac{x}{2} \geq 2 \vee \frac{3x}{2} \leq -2 \Leftrightarrow$$

$$\Leftrightarrow \frac{x}{2} \geq 2 \vee \frac{3x}{2} \leq -2 \Leftrightarrow x \geq 4 \vee 3x \leq -4 \Leftrightarrow x \geq 4 \vee x \leq -\frac{4}{3}$$

$$A = \{x : x \in]-\infty, -\frac{4}{3}\} \cup [4, +\infty[$$

$$A \cap B = [-3, -\frac{4}{3}]$$

$$\cup \{4\}$$



b)

$\sup A \rightarrow$ não existe

$\min(A \cap B) \rightarrow -3$

$\max(A \cap B) \rightarrow 4$

$\inf(A \cap B^c) \rightarrow -3$

$\sup(A \cap B^c) \rightarrow -\frac{4}{3}$

$\min(A \cap B^c) \rightarrow$ não existe

2.

$$A = \{x \in \mathbb{R} : \frac{x-1}{x \log x} > 0\} = x \in \mathbb{R}^+ \setminus \{1\}$$

$$x \log x = 0 \Leftrightarrow x = 0 \vee \log x = 0 \Leftrightarrow x = 1 \vee x = 1$$

$$B = \{x \in \mathbb{R} : x = -\frac{1}{n}, n \in \mathbb{N}_1\} = -1 \geq x > 0$$

	A	$\sim \cup B$
--	---	---------------

sup	NE	NE \mathbb{R}^+ não tem supremo
inf	1	-1
max	NE	NE \mathbb{R}^+ não tem máximo
min	NE	-1

3.

$$A = \left\{ x \in \mathbb{R} : \frac{1}{\log x} \geq 1 \right\} \quad B = \left\{ 1 - \frac{(-1)^n}{n}, n \in \mathbb{N} \right\}$$

$$\frac{1}{\log x} \geq 1 \Leftrightarrow$$

$$\frac{1}{\log x} - 1 \geq 0 \Leftrightarrow \frac{1 - \log x}{\log x} \geq 0 \quad \begin{cases} 1 - \log x = 0 \Leftrightarrow 1 = \log x \Leftrightarrow x = e \\ \log x = 0 \Leftrightarrow x = 1 \end{cases}$$

$$\begin{array}{c|c|c|c|c|c} 1 & 0 & 1 & e & +\infty \\ \hline 1 - \log x & ND & + & 0 & - \\ \hline \log x & ND & - & 0 & + \\ \hline & - & ND & 0 & - \end{array} \quad [1, e] = A$$

	majorantes	minorantes	sup	inf	max	min
A	[e, +\infty[]-\infty, 1]	e	1	e	-
B	[2, +\infty[]-\infty, 1/2]	2	1/2	2	1/2

4.

$$A = \left\{ x \in \mathbb{R} : x^2 + 2|x| \geq 3 \right\} = x \in]-\infty, -1] \cup [1, +\infty[$$

a)

$$x^2 + 2|x| \geq 3 \Leftrightarrow x^2 + 2x \geq 3 \vee x^2 + 2(-x) \geq 3$$

$$\Leftrightarrow x^2 + 2x - 3 \geq 0 \vee x^2 - 2x - 3 \geq 0 //$$

$$\underline{x \geq 1} \vee \underline{x \leq -3}$$

$$\underline{x \leq -1} \vee \underline{x \geq 3}$$

$$x^2 + 2x - 3 = 0 \Leftrightarrow$$

$$x^2 - 2x - 3 = 0 \Leftrightarrow$$

$$\Leftrightarrow x = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-3)}}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{2 \pm \sqrt{4 - (4 \cdot 1 \cdot (-3))}}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{-2 \pm \sqrt{4 + 12}}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{2 \pm 4}{2} \Leftrightarrow x = \frac{2 + 4}{2} \vee x = \frac{2 - 4}{2} \Leftrightarrow$$

$$\Leftrightarrow x = 3 \vee x = -1 //$$

$$\Leftrightarrow x = \frac{-2 + 4}{2} \vee x = \frac{-2 - 4}{2} \Leftrightarrow$$

$$\Leftrightarrow x = 1 \vee x = -3 //$$

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c} & +\infty & -3 & -1 & 0 & 1 & 3 & +\infty \\ \hline x^2 + 2x - 3 & + & 0 & - & - & - & 0 & + & + & + \\ \hline x^2 - 2x - 3 & + & + & 0 & - & - & - & - & - & 0 & + \end{array}$$

$$\begin{aligned}
 b) \quad & \text{inf } A = NE \xrightarrow{\exists j_1, \sqrt{2}} \\
 & \min A \cap B = NE \\
 & \max A \cap B = NE \\
 & \max A \cap B \cap \mathbb{Q} = NE \\
 & \inf A \cap B \cap \mathbb{Q} = 1 \\
 & \max C = NE \\
 & \max B \setminus C = NE
 \end{aligned}$$

$$B = \mathbb{J} 0, \sqrt{2} \mathbb{L}$$

$$C = \left\{ \sqrt{2} - \frac{1}{n} : n \in \mathbb{N}, \right\}$$

$$0 < c < \sqrt{2}$$

$$\sup c = \sqrt{2}$$

5

$$a) \quad A = \{x \in \mathbb{R} : |x-1| \leq x^2 - 1\} = [-\infty, -1] \quad B = [-2, 2]$$

$$\begin{aligned}
 |x-1| \leq x^2 - 1 &\Leftrightarrow x-1 \leq x^2 - 1 \Leftrightarrow x-1 \geq 1-x^2 \Leftrightarrow \\
 &\Leftrightarrow x \leq x^2 \vee x^2 + x \geq 1+1 \Leftrightarrow -x^2 + x \leq 0 \vee x^2 + x - 2 \geq 0
 \end{aligned}$$

$$\begin{array}{c}
 -x^2 + x = 0 \Leftrightarrow x^2 = x \Leftrightarrow x = \pm \sqrt{x} \Leftrightarrow x = \pm 1 \\
 \begin{array}{c|c|c|c|c|c}
 -\infty & -1 & 1 & 2 & +\infty \\
 \hline
 - & 0 & + & 0 & - & - \\
 -x^2 + x & + & 0 & - & - & 0 \\
 \hline
 x^2 - x - 2 & + & 0 & - & - & +
 \end{array}
 \end{array}$$

Erro na passagem para a tabela:
 x^2+x-2 passa a x^2-2

$$x^2 - x - 2 = 0 \Leftrightarrow x = \frac{1 \pm \sqrt{1-(4 \cdot 1 \cdot (-2))}}{2} \Leftrightarrow x = \frac{1 \pm 3}{2} \Leftrightarrow x = \frac{3}{2} \vee x = -\frac{1}{2}$$

$$\Leftrightarrow x = 2 \vee x = -1$$

$$b) \quad \max A \cap B = -2$$

$$A \cap B =$$

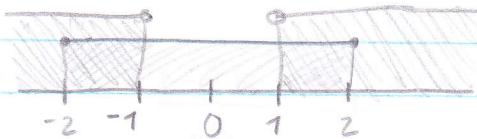
$$\min A \cap B = 2$$

$$\sup(A \cap B) \setminus \mathbb{Q} = 2$$

$$\inf(A \cap B) \setminus \mathbb{Q} = -2$$

$$\max(A \cap B) \setminus \mathbb{Q} = 2$$

$$\min(A \cap B) \setminus \mathbb{Q} = -2$$



6

$$\text{a) } A = \{x : |x^2 - 2| \leq 2x + 1\} \quad B = \mathbb{Q} \quad C = \left\{ \frac{1}{k^2} : k \in \mathbb{N} \right\}$$

$$\begin{aligned} |x^2 - 2| \leq 2x + 1 &\Leftrightarrow x^2 - 2 \leq 2x + 1 \quad \text{and} \quad x^2 - 2 \geq -2x - 1 \\ &\Leftrightarrow x^2 - 2x - 1 \leq 0 \vee x^2 + 2x - 2 + 1 \geq 0 \Leftrightarrow \\ &\Leftrightarrow x^2 - 2x - 3 \leq 0 \vee x^2 + 2x - 1 \geq 0 // \end{aligned}$$

$$\begin{aligned} x^2 - 2x - 3 = 0 &\Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - (4 \cdot 1 \cdot (-3))}}{2} = \frac{2 \pm \sqrt{4 + 12}}{2} \\ &\Leftrightarrow x = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} \Leftrightarrow x = 3 \vee x = -1 \end{aligned}$$

$$\begin{aligned} x^2 + 2x - 1 = 0 &\Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - (4 \cdot 1 \cdot (-1))}}{2} = \frac{-2 \pm \sqrt{8}}{2} \\ &\Leftrightarrow x = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} \end{aligned}$$

$$\begin{array}{c|ccccc|c} & -\infty & -1-\sqrt{2} & -2 & -1+\sqrt{2} & 3 & +\infty \\ \hline x^2 - 2x - 3 & + & + & + & 0 & - & - & 0 & + \\ x^2 + 2x - 1 & + & 0 & - & - & - & 0 & + & + \end{array}$$

	$A \cap B$	C
\sup	NE	1
\inf	$-1 \pm \sqrt{2}$	NE
\max	NE	1
\min	$-1 \pm \sqrt{2}$	NE

7

$$\text{a) } A = \left\{ x : \frac{x^2 - 1}{x} \geq |x - 1| \right\} \quad B = \{x : \sin x = 0\} \quad C = \mathbb{Q}$$

$$\begin{aligned} \frac{x^2 - 1}{x} \geq |x - 1| &\Leftrightarrow \frac{x^2 - 1}{x} \geq x - 1 \vee -\frac{x^2 - 1}{x} \leq x - 1 \Leftrightarrow \\ &\Leftrightarrow x^2 - 1 \geq x(x - 1) \vee -x^2 + 1 \leq x(x - 1) \Leftrightarrow \\ &\Leftrightarrow x^2 - 1 \geq x^2 - x \vee -x^2 + 1 \leq x^2 - x \Leftrightarrow \\ &\Leftrightarrow x - 1 \geq 0 \vee -2x^2 + x + 1 \leq 0 // \end{aligned}$$

$$x - 1 = 0 \Leftrightarrow x = 1 \quad -2x^2 + x + 1 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1 - (4 \cdot (-2) \cdot 1)}}{-4} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{-1 \pm 3}{-4} \Leftrightarrow x = \frac{2}{-4} \vee x = \frac{-4}{-4} \Leftrightarrow x = -\frac{1}{2} \vee x = 1 //$$

$(x-1)$	-	-	-	+	1	+	+
$-2x^2+x+1$	-	+	0	+	0	-	-
	-	0	+	0	0	-	-

b)

	majorantes	minorantes
$A \cap C$	NE	$[-\infty, -1/2]$
$B \cap C$		

$$\sup A = NE$$

$$\inf A \cap C = -1/2$$

$$\min A \cap C = -1/2$$

$$\min B = NE$$

$$\sup B \cap C = NE$$

8

a) $A = \{x : x \geq 0 \wedge \frac{x^4 - 4}{|x-1|} \leq 0\} \Leftrightarrow x \in [0, \sqrt{2}] \setminus \{1\}$

$$\frac{x^4 - 4}{|x-1|} \leq 0 \Leftrightarrow \frac{x^4 - 4}{x-1} \leq 0 \vee \frac{x^4 - 4}{1-x} \leq 0$$

$$x^4 - 4 = 0 \Leftrightarrow (x^2 - 2)^2 = 0 \Leftrightarrow x^2 - 2 = 0 \Leftrightarrow x = \pm\sqrt{2}$$

$x^4 - 4$	-	-	0	-	1	-	$\sqrt{2}$	+
$ x-1 $	+	0	-	-	0	-	0	+
	+	+	+	+	0	+	+	+
x	+	0	-	-	NE	-	0	+
	-	-	-	0	+	+	+	+

b) $A = A \cap B \quad B = \{x : x \geq 0 \wedge \exists_{k \in \mathbb{N}} kx \notin \mathbb{Q}\}$

Sup	$\sqrt{2}$	NE
inf	0	0
max	$\sqrt{2}$	NE
min	0	0

9

$$a) A = \left\{ x : \frac{x^2 - 2}{|x| - 1} \leq 0 \right\}$$

$$\frac{x^2 - 2}{|x| - 1} \leq 0 \Leftrightarrow \frac{x^2 - 2}{x - 1} \leq 0 \vee \frac{x^2 - 2}{-x - 1} \leq 0$$

$$x^2 - 2 = 0 \Leftrightarrow x^2 = 2 \Leftrightarrow x = \pm \sqrt{2},$$

$$x - 1 = 0 \Leftrightarrow x = 1,$$

$$-x - 1 = 0 \Leftrightarrow -x = 1 \Leftrightarrow x = -1,$$

	$-\infty$	$-\sqrt{2}$	-1	1	$\sqrt{2}$	$+\infty$
$x^2 - 2$	+	0	-	-	0	+
$ x - 1$	+	+	+	0	+	+
	+	0	-	NE	+	+

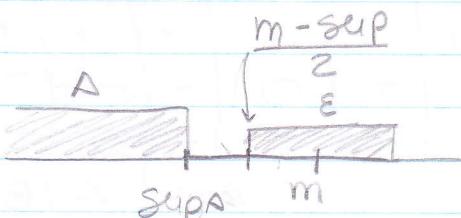
$$x = [-\sqrt{2}, -1] \cup [1, \sqrt{2}]$$

b)

	$A \cap Q$	$B \cap Q$
Sup	$\sqrt{2}$	NE
max	$\sqrt{2}$	NE
inf	$-\sqrt{2}$	$\sqrt{2}$
min	$-\sqrt{2}$	$\sqrt{2}$

$$B = \left\{ 2^{\frac{n}{2}} : n \in \mathbb{N}_1 \right\}$$

10. ACIR

 A é majorado $A \neq \emptyset$ m é majorantes de A $m \neq \sup_A$ 

$$V_\varepsilon(m) = \left\{ x \in \mathbb{R} : |x - m| < \varepsilon \right\}$$

Se $\varepsilon = \frac{m - \sup_A}{2}$, temos que $\forall x \in A \quad x \leq \sup_A$

portanto $x < m - \varepsilon$ e $x \in V_\varepsilon(m)$

$$\begin{aligned} &\sup_A < m - \frac{m - \sup_A}{2} \Leftrightarrow \\ &\Leftrightarrow \sup_A < \frac{m + \sup_A}{2} \Leftrightarrow \\ &\Leftrightarrow \frac{m - \sup_A}{2} < m - \frac{m - \sup_A}{2} \end{aligned}$$

11.

\cup e \vee limitados

12.

$$U, V \neq \emptyset$$

$$U, V \in \mathbb{R}$$

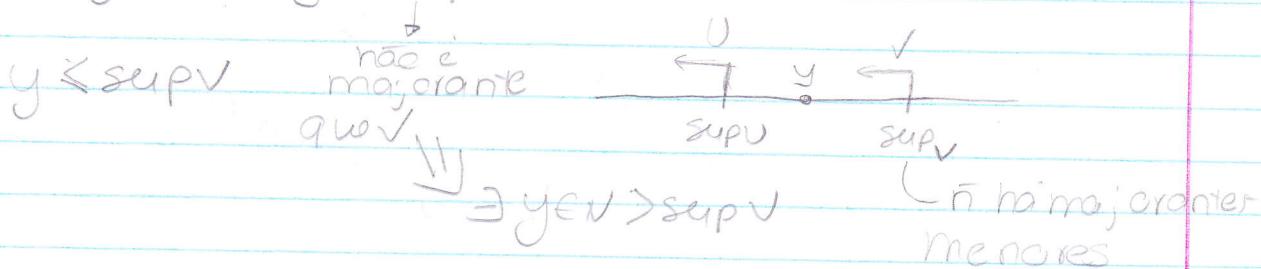
U, V majorados

$$\sup U < \sup V$$

a) $\forall x \in U \Rightarrow x < \sup V$

$$\forall x \in U \leq \sup U < \sup V \Rightarrow \forall x \in U < \sup V$$

b) $\exists y \in V : y > \sup U$



Ficha 4

1 e 2	majorada	minorada	(imitada)	monotona
a)	— sim	— sim	— sim	descrescente
b)	sim	sim (0)	sim	não
c)	não	não	não	não
d)	não	sim (0)	não	não
e)	não	sim (1)	não	sim crescente
f)	não	não	não	não
g)	— não	— sim (0)	— não	sim crescente

b) $\frac{n + (-1)^n}{n}$ $\left\{ 0, \frac{3}{2}, \frac{2}{3}, \dots \right\}$

min: $\frac{1 + (-2)^1}{1} = 0$ sup = 2

a) $\frac{1}{\sqrt{n+1}}$

c) $(-1)^n n^2$ $\left\{ -1, 2, -3, 4, \dots \right\}$ $\sqrt{n+1} \rightarrow$ aumenta
então $\frac{1}{\sqrt{n+1}}$ diminui

min = $(-1)^1 1 = -1$

d) $n^{(-1)^n}$

min: $1^{(-1)^1} = -1$

$u_{2n} = 2n^{-2^n} = 2n^{\frac{1}{2^n}} = 2n$

$u_{2n+1} = (2n+2)^{-2} = \frac{1}{2n+2}$

e) $1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$

min = $\frac{1}{2^0} = \frac{1}{1} = 1$

g) $u_1 = 0$

$u_{n+1} = \frac{2u_n + 1}{3}$

$\left\{ 0, \frac{1}{3}, \frac{5}{6}, \dots \right\}$

f) $u_1 = 1, u_{n+2} = -2u_n$ $\left\{ 1, -2, 4, -8, \dots \right\}$

$$\lim_{n \rightarrow a} f(n) = b$$

$$f(n) \rightarrow b$$

3

a) $\frac{1}{\sqrt{n+1}} \rightarrow 0$

$\forall \varepsilon > 0 \exists p \in \mathbb{N} \forall n \in \mathbb{N}: n \geq p \Rightarrow |un - 0| < \varepsilon$

Dado $\varepsilon > 0$: $\left| \frac{1}{\sqrt{n+1}} - 0 \right| < \varepsilon \Leftrightarrow \frac{1}{\sqrt{n+1}} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < \sqrt{n+1} \Leftrightarrow$
 $\left(\frac{1}{\varepsilon} \right)^2 < n+1 \Leftrightarrow n > \frac{1}{\varepsilon^2} - 1$

$\rightarrow p = \frac{1}{\varepsilon^2} - 1$ cqd

b) $\frac{n^2}{n^2+1} \rightarrow 1$

$\forall \varepsilon > 0 \exists p \in \mathbb{N} \forall n \in \mathbb{N}: n \geq p \Rightarrow |un - 1| < \varepsilon$

dado um $\varepsilon > 0$: $\left| \frac{n^2}{n^2+1} - 1 \right| < \varepsilon \Leftrightarrow \frac{n^2 - n^2 + 1}{n^2 + 1} < \varepsilon \Leftrightarrow$
 $\Leftrightarrow \frac{1}{n^2 + 1} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n^2 + 1 \Leftrightarrow \frac{1}{\varepsilon} - 1 < n^2 \Leftrightarrow$
 $\Leftrightarrow \sqrt{\frac{1}{\varepsilon} - 1} < n$

$\rightarrow p = \sqrt{\frac{1}{\varepsilon} - 1}$ cqd

c) $u_n = n^2$ é divergente \rightarrow não tende para um número real a

$\forall \varepsilon > 0 \exists p \in \mathbb{N} \forall n \in \mathbb{N}: n \geq p \Rightarrow |n^2 - a| < \varepsilon$

dado um $\varepsilon > 0$:

$$|n^2 - a| < \varepsilon \Leftrightarrow n^2 < \varepsilon + a \Leftrightarrow n < \sqrt{\varepsilon + a}$$

$\rightarrow \nexists p \in \mathbb{N} \forall n \in \mathbb{N}: n \geq p$

logo é divergente

$$9 \text{ a) } \frac{2n-1}{n+1} \rightarrow 2$$

$\forall \varepsilon > 0 \exists p \in \mathbb{N} \forall n \in \mathbb{N}: n \geq p \Rightarrow |u_n - 2| < \varepsilon$

Dado $\varepsilon > 0$:

$$\left| \frac{2n-1 - 2}{n+1} \right| < \varepsilon \Leftrightarrow \left| \frac{2n-1 - 2(n+1)}{n+1} \right| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \left| \frac{2n-2 - 2n-2}{n+1} \right| < \varepsilon \Leftrightarrow \left| \frac{-4}{n+1} \right| < \varepsilon \Leftrightarrow \frac{4}{n+1} < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \frac{3}{\varepsilon} + 1 < n \quad , \quad \Rightarrow \frac{3}{\varepsilon} + 1 = p \quad \text{cqdl}$$

$$\text{b) } \frac{\sqrt{n^2-1}}{n} \rightarrow 1$$

$\forall \varepsilon > 0 \exists p \in \mathbb{N} \forall n \in \mathbb{N}: n \geq p \Rightarrow |u_n - 1| < \varepsilon$

Dado um $\varepsilon > 0$:

$$\left| \frac{\sqrt{n^2-1} - 1}{n} \right| < \varepsilon \Leftrightarrow \left| \frac{\sqrt{n^2-1} - n}{n} \right| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \sqrt{n^2-1} - n < \varepsilon \cdot n \Leftrightarrow \sqrt{n^2-1} < \varepsilon n + n \Leftrightarrow$$

$$\Leftrightarrow n^2-1 < (\varepsilon n + n)^2 \Leftrightarrow n^2-1 < (\varepsilon n)^2 + 2n + n^2 \Leftrightarrow$$

$$\Leftrightarrow 1 < n(\varepsilon n + 2) \Leftrightarrow n > 1 \vee \varepsilon n + 2 > 1 \Leftrightarrow$$

$$\Leftrightarrow n > 1 \vee \varepsilon n > 1 - 2 \Leftrightarrow n > 1 \vee n > -\frac{1}{\varepsilon} \quad //$$

$$\Rightarrow p = 1 \quad \text{cqdl}$$

$$\Leftrightarrow -\frac{1}{\varepsilon} < 0 \neq p$$

5.

$$\text{a) } \frac{(2n+1)^3 + n}{n^3 + 1} = \left(\frac{+\infty}{+\infty} \right)$$

$$\lim \frac{\left(\frac{(2n+1)^3 + n}{n^3 + 1}\right)}{\frac{n^3 + 1}{n^3}} = \lim \frac{\left(\frac{(2n+1)^3 + \frac{1}{n^2}}{1 + \frac{1}{n^3}}\right)}{\frac{1^3 + 0}{1 + 0}} = \frac{2^3 + 0}{1 + 0} = 8$$

c.n.

$$\lim \frac{2n+1}{n} = \lim \frac{2n+1}{n} = \lim 2 + \frac{1}{n} = 2$$

$$\text{b) } \lim \frac{(2n+1)^3 + n^2}{(n+1)^2(n+2)} = \lim \frac{\frac{(2n+1)^3 + \frac{n^2}{n^2}}{n^3 + 4n^2 + 5n + 2}}{n^3} =$$

$$\lim \frac{\frac{(2n+1)^3 + 1}{1 + \frac{4}{n} + \frac{5}{n^2} + \frac{2}{n^3}}}{1 + 0} = \frac{2^3 + 0}{1 + 0} = 2^3 / 1$$

$$(n^3 + 2n^2)(n+2) = n^3 + 2n^2 + 2n^2 + 4n + n+2 = \\ = n^3 + 4n^2 + 5n + 2$$

$$\text{c) } \lim \frac{(n+1)^2 + 2n^3}{(n+1)^4 + 2n^2} = \lim \frac{\frac{(n+1)^2 + \frac{2n^3}{n^4}}{(n+1)^4 + \frac{2n^2}{n^4}}}{\frac{(n+1)^4 + 2n^2}{n^4}} =$$

$$= \lim \frac{\frac{\left(\frac{1}{n} + \frac{1}{n^2}\right)^2 + 2}{\left(1 + \frac{1}{n}\right)^4 + \frac{2}{n^2}}}{1 + 0} = \frac{0+2}{1+0} = 2$$

$$\text{d) } \lim \frac{\sqrt{n}-1}{\sqrt{n+1}} = \frac{1-1}{1} = 0$$

$$\lim \frac{n-1}{n} = \frac{1-0}{1} = 1$$

$$\frac{n}{n+1} = \frac{1}{1+0} = 1$$

$$e) \lim \frac{1}{n} \left(2 + \frac{1}{n} \right) = 0$$

$$f) \lim \frac{1}{n} (2n + \sqrt{n}) \stackrel{0.00}{=} \left(\lim \frac{2n}{n} + \lim \frac{\sqrt{n}}{n} \right) = \lim 2 + \lim \frac{\sqrt{n}}{n} = 2 \downarrow 0$$

$$g) \text{NE } S = \left\{ -\frac{1}{n!}, \frac{1}{n!} \right\}, \text{ logo é divergente}$$

S: conjunto dos sublimites da sucessão

$$h) \lim \frac{\sqrt{n}}{\sqrt[4]{9n^2+1}} = \underset{1}{\cancel{-1}}$$

$$\frac{x_{n+1}}{x_n} = \frac{n+2}{n} = \underset{\text{lim}}{1 + \frac{1}{n}} = 1$$

$$\frac{9(n+2)^2 + 2}{9n^2 + 2} = \frac{9(n^2 + 4n + 4) + 2}{9n^2 + 2}$$

$$i) \lim \frac{\sqrt[3]{n+1}}{\sqrt[3]{n+2}} = \frac{\frac{1}{n^2} + \frac{8}{n^2} + \frac{9}{n^2}}{\frac{4}{n^2} + \frac{2}{n^2}} = \frac{1 + 8/n + 9/n^2}{4 + 2/n^2}$$

$$= \frac{-1 + 1}{1} = 0$$

$$\frac{n+2}{n+2} = \frac{1 + 1/n}{1 + 2/n} = \frac{1 + 0}{1 + 0} = 1$$

$$j) \lim \frac{n+1}{n!} = \lim \frac{n}{n!} + \frac{1}{n!} = 0$$

$$k) \text{NE } S = \left\{ \frac{1}{\sqrt{n}}, 0 \right\}$$

S: conjunto dos sublimites da sucessão

$$L) \lim_{n \rightarrow \infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n} = \lim_{n \rightarrow \infty} \frac{2 \left(\frac{2}{3}\right)^n + \frac{3^{n+1}}{3^n}}{\left(\frac{2}{3}\right)^n + 1} = \frac{2 \cdot 0 + 3}{0 + 1} = 3$$

$$m) \lim_{n \rightarrow \infty} \sqrt[n]{1000 + 10000} = \lim_{n \rightarrow \infty} \frac{1 + 1000}{n} = 0$$

$$n) \lim_{n \rightarrow \infty} \frac{n^n}{n^{n+1}} = \lim_{n \rightarrow \infty} \frac{n^n}{n^n + n^{n+1}} = \frac{1}{1+0} = 1$$

$$o) \lim_{n \rightarrow \infty} \sqrt[3]{\frac{3}{5n}} = \frac{1}{1} = 1$$

$$p) \lim_{n \rightarrow \infty} \frac{q^n}{1+q^{n^2}} = \lim_{n \rightarrow \infty} \frac{\frac{q^n}{q^{n^2}}}{\frac{1}{q^{n^2}} + \frac{q^{n^2}}{q^{n^2}}} = \lim_{n \rightarrow \infty} \frac{q^{n-n^2}/0}{\left(\frac{1}{q^n}\right)+1} = 0$$

$$q) \lim_{n \rightarrow \infty} \frac{(a^n)^2}{a^{n^2}} = \lim_{n \rightarrow \infty} \frac{a^{2n}}{a^{n^2}} = \lim_{n \rightarrow \infty} a^{2n-n^2} = 0$$

6

$$x_n = \frac{a^n}{2^{1+2n}}$$

$$a) \frac{a^n}{2^{1+2n}} = \frac{a^n}{2 \cdot 2^{2n}} = \frac{1}{2} \left(\frac{a}{4}\right)^n$$

$$-1 < \frac{a}{4} < 1 \Leftrightarrow -4 < a < 4$$

$$\lim a^n \begin{cases} +\infty, & a > 1 \\ 1, & a = 1 \\ 0, & -1 < a < 1 \\ \text{N.E.}, & a \leq -1 \end{cases}$$

Convergente

$$b) \left(\frac{a}{4}\right) = -1 \Leftrightarrow a = -4$$

7

a) $u_n = \frac{1-1}{n}$

tem termos em $]-\infty, 1]$ e é crescente

$u_n = -1$

b)

$u_n = \frac{(-1)^n}{n}$

mórbida convergente

$$1 + \frac{(-1)^n}{n}$$

c) $u_n = (-1)^n$

divergente, $|u_n|$ convergente

d) $u_n = (-1)^n$

limiteida e divergente

e) $u_n = \frac{1}{n} \cdot (-2)^n$

$$u_{2n} = \frac{1}{2} \text{ e } u_{2n+2} = \frac{1}{3}$$

termos em $\frac{1}{n}$ e é divergente

f) $u_n = \frac{\sqrt{2}}{n}$

termos em $\mathbb{R} \setminus \mathbb{Q}, \rightarrow \mathbb{Q}$

8

$$A = \{x \in \mathbb{R} : x^2 + 2|x| > 3\} =]-\infty, -1] \cup [1, +\infty)$$

B = $]0, \sqrt{2}[$

$$C = \left\{ \sqrt{2} - \frac{1}{n} : n \in \mathbb{N}_1 \right\}$$

a)

b)

c)

d)

e)

f)

9)

$$\left\{ \begin{array}{l} u_1 = a \\ u_{n+1} = r + u_n \end{array} \right.$$

$$\left\{ \begin{array}{l} v_1 = a \\ v_{n+1} = rv_n \end{array} \right.$$

a)

$$u_n = a + (n-1)r$$

$$u_1 = a + (1-1)r = a$$

$$u_{n+1} = a + (n+1-1)r = a + nr$$

$$u_{n+1} = r + u_n = r + a + (n-1)r =$$

$$= x + a + rn - x =$$

$$= a + nr \text{ qd}$$

$$v_n = ar^{n-1}$$

$$v_1 = a \cdot 1^{-1} = a$$

$$v_{n+1} = ar^n$$

$$v_{n+1} = r v_n = \frac{r}{n} =$$

$$= r \cdot ar^n = ar^n$$

X qd

b)

$$i) \forall r, a > 0$$

$$ii) \forall a, \forall r < 0$$

$$iii) \forall a, r > 0$$

iv) Se a ou r forem negativos (e apenas um deles)

c) u_n não é limitada $\forall a \in \mathbb{R}, r \neq 0$

$$\left\{ \begin{array}{l} u_1 = a \\ u_2 = r+a \\ u_{n+1} = r + u_n \end{array} \right. \quad u_n = (n-1)r + a \quad \begin{array}{l} \text{se } n > 1 \Rightarrow n-1 > 0 \\ \text{se } r > 0 \text{ e } n-1 > 0 \end{array} \quad \begin{array}{l} \text{un não é limitada} \\ \text{se } r < 0 \text{ e } n-1 > 0 \end{array}$$

$\rightarrow +\infty$

$\text{se } r > 0 \text{ e } n-1 > 0$

un não é limitada

$\text{se } r < 0 \text{ e } n-1 > 0$

$\rightarrow -\infty$

O quando v_n é limitada? é convergente?

$$v_n = a \cdot r^{n-2}$$

limitada

convergente

Se a

Se $a > 0$

10.

$$u_n \begin{cases} u_1 = 1 \\ u_{n+1} = 1 + \frac{u_n}{2} \end{cases}$$

a) $u_n \leq 2 \quad \forall n \in \mathbb{N}$

$n=1$

$$u_1 = 1 \leq 2$$

$$1 + \frac{u_n}{2} \leq 2 \quad (\Rightarrow \frac{u_n}{2} \leq 1 \Rightarrow u_n \leq 1 \cdot 2 \Rightarrow u_n \leq 2 \text{ qd})$$

b)

$$u_1 = 1$$

$$u_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$u_3 = 1 + \frac{3}{2} = \frac{5}{2}$$

$$u_1 < u_2 < u_3 < \dots < u_n$$

$$u_4 = 1 + \frac{5}{2} = \frac{7}{2}$$

Logo é crescente

c) Se u_n é crescente e limitada por $1 \leq u_n \leq 2$
então $\lim u_n \rightarrow 2$

11.

$$u_n \begin{cases} u_1 = \frac{3}{2} \\ \quad \quad \quad 1 < u_n < 2 \end{cases}$$

$$u_{n+1} = \frac{u_n^2 + 2}{3}$$

$$1 < u_n < 2$$

a) $u_1 = \frac{3}{2} \quad 1 < \frac{3}{2} < 2$

$$P(n+2) \Rightarrow P(n) : 1 < \frac{u_n^2 + 2}{3} < 2 \quad (\Rightarrow 3 < u_n^2 + 2 < 6 \Rightarrow)$$

$$\Leftrightarrow 1 < u_n^2 < 4 \quad (\Rightarrow \sqrt{1} < u_n < \sqrt{4} \Rightarrow)$$

$$\Leftrightarrow 1 < u_n < 2, \text{ qd}$$

b) u_n descrecente estritamente

$$u_{n+2} - u_n = \frac{u_n^2 + 2}{3} - u_n = \frac{u_n^2 + 2 - 3u_n}{3} \leq 0$$

$$\leq 0$$

então

$$u_{n+2} - u_n = \frac{u_n^2 + 2 - 3u_n}{3} \leq 0$$

$$\begin{aligned} & x^2 - 3x + 2 \leq 0 \Leftrightarrow x = \frac{3 \pm \sqrt{9 - 4 \cdot 2}}{2} \Leftrightarrow x = \frac{3 \pm \sqrt{9 - 8}}{2} \Leftrightarrow \\ & \Leftrightarrow x = 3 \pm 1/2 \Leftrightarrow x = 2 \vee x = 1 \text{ e qd} \end{aligned}$$

c) u_n é descrecente e limitada, logo u_n é convergente. u_{n+2} é subsequência de u_n , então $\lim u_{n+2} = \lim u_n$

$$a = \frac{a^2 + 2}{3} \Leftrightarrow a = 1 \vee a = 2 \text{ Como } u_n \text{ descrecente, } \lim u_n = 1$$

$$12. \quad u_1 > 1 \quad u_{n+1} = 2 - \frac{1}{u_n}, \quad n \in \mathbb{N}_1$$

5

$$u_2 = 2 - \frac{1}{u_1} \Rightarrow u_2 > u_1$$

logo, u_n é crescente

quando $n \rightarrow \infty$, $u_n \rightarrow \infty$

logo

$$2 - \frac{1}{\infty} = 2 = \lim u_n$$

13.

$$a) \quad u_n = \begin{cases} u_1 = 1 \\ u_{n+1} = \sqrt{2 + u_n} \end{cases}$$

PROVE:

$$1 \leq u_n \leq 2 \quad \forall n \in \mathbb{N}.$$

$n=1$	$u_1 = 1$	$1 \leq u_1 \leq 2$ qd
-------	-----------	------------------------

1_{n+1}

$$1 \leq \sqrt{2+u_n} < 2$$

$$\begin{aligned} 1 \leq u_n < 2 &\Rightarrow 2+1 \leq u_{n+2} < 2+2 \Rightarrow \\ \Rightarrow \sqrt{3} \leq \sqrt{u_{n+2}} &< \sqrt{4} \Rightarrow \sqrt{3} \leq \sqrt{u_{n+2}} < 2 \end{aligned}$$

$$1 \leq \sqrt{3} < 2$$

qd

b)

$$u_1 = 1$$

$$u_2 = \sqrt{2+1} = \sqrt{3} \quad u_1 < u_2 < u_3 \dots < u_n$$

$$u_2 = \sqrt{2+\sqrt{3}}$$

$\rightarrow u_n$ é crescente

$$u_{n+2} - u_n = \frac{(2-u_n)(u_{n+1})}{u_n + \sqrt{2+u_n}}$$

$$\begin{aligned} u_{n+1} - u_n &= \sqrt{2+u_n} - u_n = \frac{(\sqrt{2+u_n} - u_n)(\sqrt{2+u_n} + u_n)}{u_n + \sqrt{2+u_n}} \\ &= \frac{2+u_n + u_n\sqrt{2+u_n} - u_n\sqrt{2+u_n} - u_n^2}{u_n + \sqrt{2+u_n}} = \frac{-u_n^2 + u_n + 2}{u_n} \text{ qd} \end{aligned}$$

c) u_n é crescente ✓ \rightarrow u_n é convergente
 u_n é limitada ✓

14. $u_n > 0$ $\frac{u_{n+1}}{u_n} < 1 \rightarrow u_n$ é descrescente

↓
descrescente

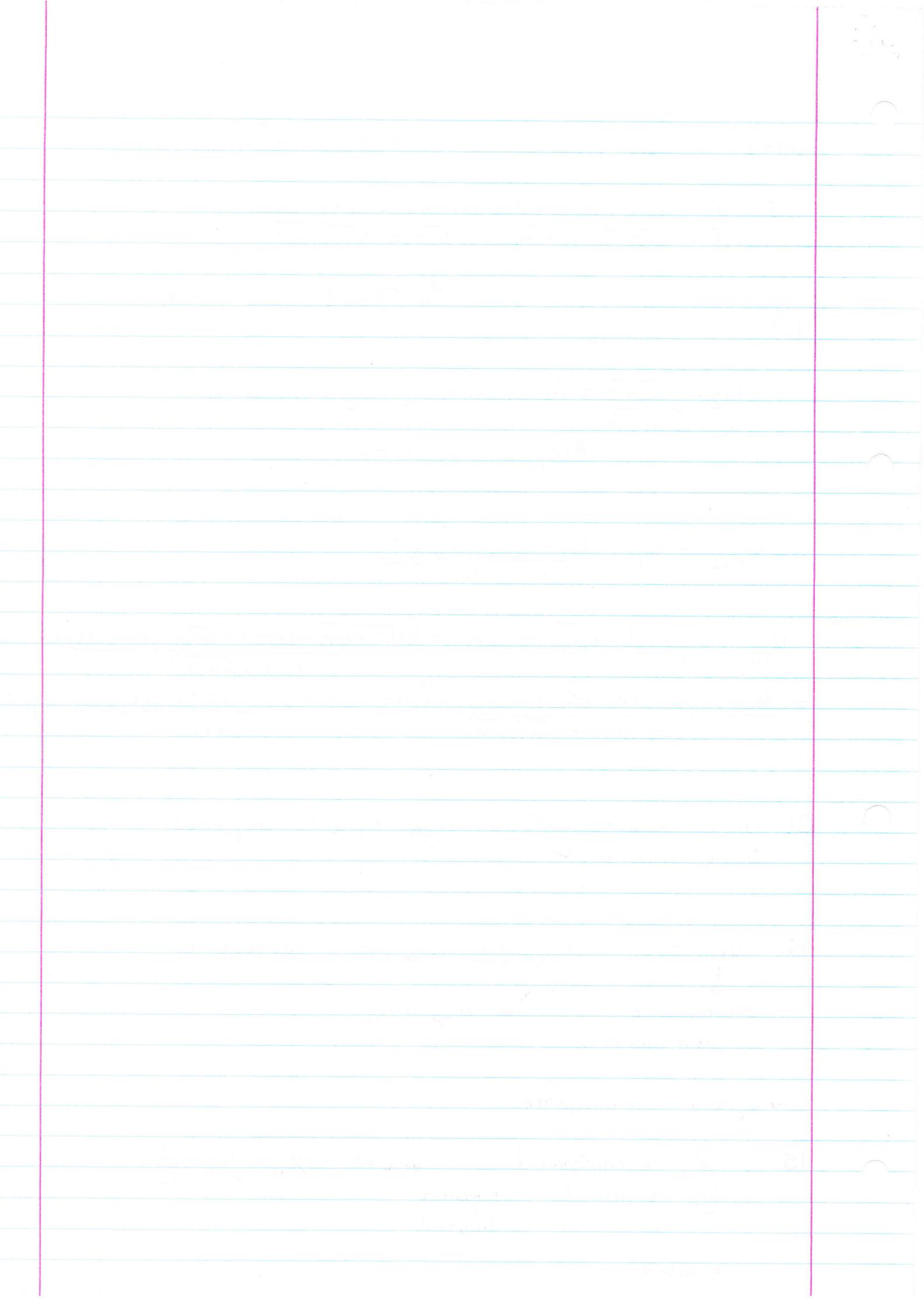
$$\frac{u_{n+1}}{u_n} < 1$$

$$\rightarrow u_{n+1} > u_n$$

Logo é limitada

\rightarrow u_n é convergente

15. $x_n \rightarrow$ monótona | $\forall n \in \mathbb{N}, |x_n - y_n| < \frac{1}{n}$
 $y_n \rightarrow$ limitada | mesmo limite?
limitada?



limítada Mesmo grau \Rightarrow limite é = ao produto das coeficientes de maior grau

$$(-1)^n \xrightarrow{n \rightarrow \infty} 0$$

$|a|^n \rightarrow$ tende p/0

fichas

1. $u_n = \frac{(-1)^{n+1}}{n}$

• limitada

$$-\frac{1}{2} \leq u_n \leq 1$$

$$v_n = \frac{n^{n+1}}{n^n + 1}$$

- não limitada
- divergente

$$w_n = u_n \cdot v_n$$

- divergente
- limitada
- $-1 < w_n < 1$

$$w_n = \frac{n^{n+1}}{n^n + 2} \cdot \frac{(-1)^{n+1}}{n} =$$

$$= \frac{n^{n+2} \cdot (-1)^{n+2}}{n(n^n + 1)} = \frac{(-n)^{n+1}}{n^{n+2} + n} \rightarrow n^{n+2} + n > (-n)^{n+1}$$

2. $u_n = \cos(n/\pi)$

• limitada $[-1, 1]$

para $n \geq 2$, $n!$ par

$$\Rightarrow \cos(n/\pi) = 1$$

logo, $\lim u_n = 1$

$$v_n = \frac{n \cos(n\pi)}{2n+1}$$

$$\frac{n}{2n+1} \cdot \cos(n\pi) \xrightarrow[n \rightarrow \infty]{1 \text{ ou } -1} (-1)^n$$

$$\frac{1}{2+0} = \frac{1}{2}$$

$a \in \mathbb{R}$

$$w_n = \frac{1+a^n}{1+a^{2n}}$$

$$a > 1 \quad a^n \rightarrow +\infty$$

$$\frac{\frac{1}{a^{2n}} + \frac{1}{a^n}}{\frac{1}{a^{2n}} + 1} \rightarrow 0$$

$$a < -1$$

3.

a) $\lim \frac{1}{(-1)^n n^2 + 2} = 0$

L, não é limitada

$$u_{2n} =$$

$$u_{2n+1} =$$

$$\frac{1 + (-1)^n |a|^n}{1 + (-1)^{2n} |a|^{2n}} =$$

$$= \frac{1 + (-1)^n |a|^n}{1 + |a|^{2n}} =$$

$$= \frac{1}{|a|^{2n}} + \frac{(-1)^n}{|a|^n} \xrightarrow[|a|=1]{} 0$$

$$\frac{1}{|a|^{2n}} + 1 \xrightarrow[|a|=-1]{} \infty$$

$$|a|=1 \rightarrow 1 \quad |a|=-1, \text{im}$$

b)

$$\lim \left(1 + (-1)^n \right) \left(1 + \frac{1}{n} \right)$$

$\xrightarrow{n \rightarrow \infty}$ não é convergente

$u_{2n} =$
 $u_{2n+1} =$

c)

$$\lim \frac{n(1 + (-1)^n)}{2}$$

$\xrightarrow{n \rightarrow \infty}$ não é limitada, não é convergente

$u_{2n} =$
 $u_{2n+1} =$

d)

$$\lim \frac{2n^2 + (-1)^n}{n^2 - 1}$$

$\xrightarrow{n \rightarrow \infty}$ não é convergente

$u_{2n} =$
 $u_{2n+1} =$

e) $\lim \frac{n + \cos n}{2n-1} = \lim \frac{\frac{1+\cos n}{n}}{\frac{2-1}{n}} = \frac{1+0}{2-0} = \frac{1}{2}$

$\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ por teorema dos sucessos enquadados

$\frac{\cos n}{n} \rightarrow 0$ então $\frac{1+\cos n}{n} \rightarrow 1$

f) $\lim \left(-1 - \frac{1}{n} \right)^n = -e$

4.

un convergente

$$u_{2n} \in [0, 1] \quad u_{2n+1} \in \mathbb{R} \setminus \{0, 1\} \quad \text{PROVAR: } \lim_{n \rightarrow \infty} u_n \in [0, 1]$$

5.

$$u_1 = a$$

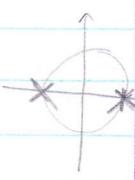
$$u_{n+1} = (-1)^n u_n + \frac{u_n}{n+1}$$

Se un convergente
 $\lim_{n \rightarrow \infty} u_n = 0$

$$\overline{\mathbb{R}} = \mathbb{R} \cup]-\infty, +\infty[$$

6.

a) $u_n = \frac{1}{n} + 2 \cos n\pi = \frac{1}{n} + 2(-1)^n$



$\cos(n\pi) = (-1)^n$

$$u_{2n} = \frac{1}{2n} + 2 \rightarrow 2 \quad u_{2n+1} = \frac{1}{2n+1} - 2 \rightarrow -2$$

$$S: \{-2, 2\}$$

b) $u_{2n} = 0 \quad e \quad u_{2n+1} = n$

$$S = \{0, +\infty\}$$

c) $1, 2, 3, 4, 5, 6, 7, 8, \dots$

$$S = \{+\infty\}$$

d) $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots$

$$S = \{\mathbb{N}, +\infty\}$$

* Sim, como no caso da clínica b

* em $\overline{\mathbb{R}}$ o conjunto das sublimites nunca é singular se a sequência for divergente

7. $A = \{x \in \mathbb{R} : |2x+1| < |x|\}$, $B = \left\{ \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$,

$$C = [-1, +\infty[$$

a) $|2x+1| < |x| \Leftrightarrow 2x+1 < |x| \vee (2x+1) < -|x| \Leftrightarrow$
 $\Leftrightarrow 2x+1 < x \vee 2x+1 < -x \Leftrightarrow 2x-x < -1 \vee 2x+x < 1 \Leftrightarrow$
 $\Leftrightarrow \boxed{2x < -1} \vee \boxed{3x < 1} \Leftrightarrow \boxed{x < -\frac{1}{2}} \vee \boxed{x < \frac{1}{3}} \Leftrightarrow$
 $\Leftrightarrow x < -1 \vee x < -\frac{1}{2}$

$$A =]-1, -\frac{1}{2}[$$

b)

$$\inf C = -1$$

$$\min(c/A) = -1$$

$$\sup(A \setminus Q) = -\frac{1}{3}$$

$$\max(A \cup B) = \frac{1}{2}$$

$$\inf B = -1$$

$$\max(B \setminus Q) = \text{não existe}$$

$$B = [-1, \frac{1}{2}]$$

c) i) sim. Toda a sequência decrescente e limitada é convergente
ii)

iii) não. Pode收敛ir para um número em C.

iv) sim. S, u_n é limitada, pelo teorema de BW garante

v) sim. que u_n tem alguma ss convergente, portanto u_n tem pelo menos um subsequente.

8.

a) $A = \{x \in \mathbb{R} : \frac{x^2+1}{x-2} \geq x\}$

$$\frac{x^2+1-x}{x-2} \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{x^2+x-1}{x-2} \geq 0 \Leftrightarrow \frac{x^2+x-2x-2}{x-2} \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{2x+1}{x-2} \geq 0$$

$$x-2 \neq 0 \Leftrightarrow x \neq 2$$

$$2x+1=0 \Leftrightarrow 2x=-1 \Leftrightarrow x=-\frac{1}{2}$$

$$A =]-\infty, -\frac{1}{2}] \cup]2, +\infty[$$

$x < -\infty$	-	$-\infty$	$-\frac{1}{2}$	+	2	+	+	$+\infty$
$2x+1$	-	0	+	+	+	+	+	+
$x-2$	-	-	-	0	+	+	-	-
Q	+	0	-	ND	+	+	+	+

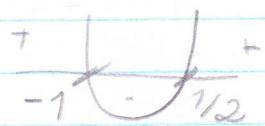
$$B = \{x \in \mathbb{R} : \log(2x^2 + x) \geq 0\}$$

$$\log(2x^2 + x) \geq 0 \Leftrightarrow 2x^2 + x \geq 1 \Leftrightarrow 2x^2 + x - 1 \geq 0$$

$$2x^2 + x - 1 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1+4 \cdot 2 \cdot (-1)}}{2 \cdot 2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{-1 \pm \sqrt{1+8}}{2 \cdot 2} \Leftrightarrow x = \frac{-1 \pm 3}{4} \Leftrightarrow x: \frac{-1+3}{4} \cup x = \frac{-1-3}{4}$$

$$\Leftrightarrow x = \frac{1}{2} \vee x = -1$$



$$B =]-\infty, -1] \cup [\frac{1}{2}, +\infty[$$

b)

$$\min A = \text{NE}$$

$$\sup B \cap \mathbb{Q} = \text{NE}$$

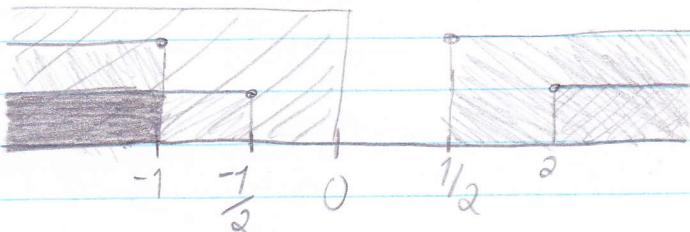
$$\inf A \cap B = \text{NE}$$

$$\sup B \cap \mathbb{R} \setminus \mathbb{Q} = -1$$

$$\max A \cap \mathbb{Q} = -\frac{1}{2}$$

c) Se u_n é crescente em $A \cap B \cap \mathbb{R} =]-\infty, -1]$
então é convergente

tende para -1



d) Se u_n é sucessão em $B \cap \mathbb{R}$

então $y_n = (-1)^n u_n$ é divergente

$\hookrightarrow -u_n, u_{n+1}, -u_{n+2}, u_{n+3}, \dots$

$$y_{2n} = u_{2n}$$

$$y_{2n+1} = -u_{2n+1}$$

(e) $\Delta_n =$

Tender para um não
em $[-\frac{1}{2}, 2]$

Em $A \cap (\mathbb{R} \setminus \mathbb{Q})$ $[-\infty, -\frac{1}{2}] \cup [2, +\infty]$

9) $A = \mathbb{R}^+ \setminus \{1\}$ $B = \{-\frac{1}{n}, n \in \mathbb{N}_1\}$

a) Falso

b)

$$A \cap V_{\frac{1}{12}}(0)$$

$$B^+ \cap \left(-\frac{1}{2}, \frac{1}{2}\right] = [-\frac{1}{2}, 1] \cup [1, \frac{1}{2}]$$

c)

10.

$$A = [-1 + \sqrt{2}, 3] \quad C = \left\{ \frac{1}{k^2}, k \in \mathbb{N}_1 \right\}$$

i)

$$[-\infty, -1 + \sqrt{2}] \cup [3, +\infty] \quad a_{n+2} a_n < 0$$

iii) Verdadeiro $a_n \rightarrow 0$

11.

a) $1 - \sqrt{n} \rightarrow -\infty$

b) $\frac{n^2 + 1}{n} \rightarrow +\infty$

$$\lim \sqrt{n} = \lim \frac{n+1}{\sqrt{n}} = \lim \frac{\cancel{n} + \frac{1}{n}}{\sqrt{n}} = \lim 1 + \frac{1}{n} = 1 + 0 = 1$$

$$\lim 1 - \sqrt{n} = 1 - 1 = 0$$

$$\lim \frac{n^2 + 1}{n} = \lim \frac{\frac{n^2}{n^2} + \frac{1}{n^2}}{\frac{n}{n^2}} = \lim \frac{1 + \frac{1}{n^2}}{\frac{1}{n^2}} =$$

$$\boxed{n^p < a^n < n! < n^n}$$

12.

a) $\lim \frac{n^n}{7000^n} = +\infty$

b) $\lim n^{n+1} - n^n = \lim n^n(n-1) = +\infty (+\infty - 1) = +\infty$
oder

c) $\lim 3^n - (2n)! = -\infty$ $n^{n+2} \left(1 - \frac{1}{n}\right) = +\infty$

d) $\lim (n! - n^{1000})^n = \left[n! \left(1 - \frac{n^{1000}}{n!}\right)\right]^n \geq n! \left(1 - \frac{n^{1000}}{n!}\right)$

e) $\lim \frac{(2n)!}{n!} = \frac{+\infty}{+\infty} = +\infty$ $\rightarrow 0 // \rightarrow +\infty$

f) $\lim_n \sqrt[n]{\frac{n+2}{n+1}} = 1$

$\lim \frac{a_{n+2}}{a_n} = ?$

$\lim \frac{n+3}{n+2} \cdot \frac{n+1}{n+2} = 1 \cdot 1 = 1$

g) $\lim \frac{n^{1000}}{10002^n} = 0$ $\frac{n^p}{a^n} = 0$

h) $\lim \sqrt[n]{\frac{n}{n^2+1}} = 1$

$\lim \frac{n+1}{\frac{(n+2)^2+1}{n}} = \lim \frac{n+1}{n} \cdot \frac{n+1}{n^2+2n+2} = \lim 1 + \frac{1}{n} \cdot \frac{0+1}{1+0+0}$

$= 1 + 0 \cdot \frac{1}{1} = 1$

$$\lim \left(1 + \frac{1}{n}\right)^n = e \quad \left(1 - \frac{1}{n}\right)^n = e^{-1}$$

$$i) \sqrt[n]{3^n + 2} \rightarrow 3$$

$$\lim \frac{3^{n+2}}{3^n + 2} = \lim \frac{1 + 2/3^{n+1}}{\frac{1}{3} + 2/3^{n+1}} = \frac{1+0}{\frac{1}{3}+0} = 3$$

$$j) \lim \sqrt[n]{n!} = +\infty$$

$$\lim \frac{(n+2)!}{n!} = \lim \frac{n!(n+1)}{n!} = \lim n+1 = +\infty$$

$$k) \lim \left(2 \cdot \frac{1}{n}\right)^n \geq \lim \left(\frac{3}{2}\right)^n = +\infty \quad \text{exemplo} \quad \exists p \in \mathbb{N} \quad \forall n > p \quad \frac{2-1}{n} \rightarrow 2 \quad \frac{2-1}{n} \geq \frac{3}{2}$$

$$l) \lim \left(1 - \frac{1}{2^{n-1}}\right)^{2^n} = \lim \left(\left(1 - \frac{1}{2^{n-1}}\right)^{\frac{2^n}{2}}\right)^2 = \lim \left(\left(1 - \frac{1}{2^{n-1}}\right)^{2^{n-1}}\right)^2 =$$

$$= e^{-2}$$

$$m) \lim \left(1 + \frac{1}{n}\right)^{n^2} = \lim \left(\left(1 + \frac{1}{n}\right)^n\right)^n = e^n = +\infty$$

13.

$$a) \lim \frac{2n+3}{3n-1} = \frac{2}{3}$$

$$b) \lim \frac{n^2+1}{n^4+3} = \lim \frac{n^2/n^4 + 1/n^4}{1 + 3/n^4} = \frac{0+0}{1+0} = 0$$

$$c) \lim \frac{2^n+1}{2^{n+1}-1} = \lim \frac{2^n/2^{n+1} + 1/2^{n+1}}{1 - 1/2^{n+1}} = \frac{0+0}{1-0} = 0$$

$$d) \lim \frac{n^3+1}{n^2+2n-1} = \lim \frac{n^2(n+\frac{1}{n^2})}{n^2+2n-1} = \frac{1.(0+0)}{1+0+0} = 0$$

$$\textcircled{e}) \lim \frac{(-1)^n n^3 + 1}{n^2 + 2} =$$

=

$$f) \lim \frac{n^p}{n!} = +\infty$$

$$g) \sqrt[n]{1 + \frac{1}{n}} = 1 \quad \text{ou} \quad \left(1 + \frac{1}{n}\right)^{\frac{1}{n}} = 1^0 = 1$$

$$\frac{a_{n+1}}{a_n} = \lim \frac{(n+1)+1}{n+1} \cdot \frac{n}{1+n} = \lim \frac{n(n+2)}{n+1} = \\ = \frac{1 \cdot (1+0)}{1+0} = 1$$

$$h) \lim \frac{\left(\frac{1}{2}\right)^n}{n^3} = +\infty$$

$$i) \lim \frac{3^n}{n^2} = +\infty$$

$$\textcircled{j}) \sqrt[n]{\frac{n^2 + n - 1}{n+3}}$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{(n+1)^2 + n}{n+4} \cdot \frac{n+3}{n^2 + n - 1} = \lim \frac{\sqrt[n]{n^2 + 3n + 1} \cdot \frac{n+3}{n^2 + n - 1}}{\sqrt[n]{n+4}} = \\ = +\infty \cdot \frac{0+0}{1+0-0}$$

K) $\sqrt[n]{2^n + 1}$

$$\lim \frac{2^{n+1} + 1}{2^n + 1} = +\infty$$

L) $\sqrt[n]{(n+2)! - n!} \rightarrow +\infty$

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)! - (n+1)!}{(n+1)! - n!} = \frac{(n+1)! \cdot (n+2) - (n+1)!}{n! \cdot (n+1)! - n!} =$$

$$\lim \frac{1 \cdot (n+2) - 1}{1 - n!/ (n+1)!} = \frac{n+1}{1-0} = +\infty$$

m) $\lim \left(1 + \frac{1}{n^2}\right)^{n^3} = \lim \left(\left(1 + \frac{1}{n^2}\right)^{n^2}\right)^n = e^n$

n) $\lim \left(1 - \frac{1}{n!}\right)^{n!} = e^{-1}$

a) $\lim \left(1 + \frac{1}{n^3}\right)^{n^2} = \lim \left(\left(1 + \frac{1}{n^3}\right)^{n^3}\right)^{\frac{1}{n}} = e^{\frac{1}{n}}$

14.

a) $\lim \frac{n!}{n^{7000}} = +\infty$

b) $\lim \frac{(2n)! + 2}{(3n)! + 3} = \lim \frac{2 + 2/n!}{3 + 3/n!} = \frac{2}{3}$

$$c) \lim \frac{(2n)!}{(2n)^n} = 0$$

$$d) \lim \frac{(n!)^2}{(2n)! + 2} =$$

$$e) \lim \frac{2^n \cdot n!}{n^n} = 0$$

$$f) \lim \frac{3^n n!}{n^n} = 0$$

$$g) \lim n^{1/n} = \sqrt[n]{n} = 1$$

$$h) \lim \left(\frac{1}{n}\right)^{1/n} = \lim \sqrt[n]{\frac{1}{n}} = 0$$

$$\lim \frac{1}{n+1} \cdot \frac{1}{n} = \lim \frac{1}{n^2+n} = 0$$

$$i) \lim \left(\frac{1}{n}\right)^n = \lim \frac{1}{n^n} = 0$$

$$j) \lim \left(2 - \frac{1}{n}\right)^n = 2^{+\infty} = +\infty$$

$$k) \lim \left(\frac{n-1}{2n^2+1}\right)^{1/n} = \left(\lim \sqrt[n]{\frac{n-1}{2n^2+1}}\right)^2 = 1$$

$$\begin{aligned} \lim \frac{n}{2n^2+4n} \cdot \frac{2n^2+1}{n-1} &= \lim \frac{n}{n-2} \cdot \frac{2n^2+1}{2n^2+4n} = \\ &= 1 \cdot 1 = 1 \end{aligned}$$

$$8) \lim \frac{2^{(n^2)}}{75^n} = \lim \left(\frac{2^n}{75}\right)^n \geq \frac{2^n}{75} \xrightarrow{n \rightarrow +\infty}$$

15.

a) PROVAMOS

PROVAMOS
TEOREMA
se $u_n \rightarrow +\infty$
então $\frac{1}{u_n} \rightarrow 0$

ii) se $u_n > 0$ e $u_n \rightarrow 0$
então $\frac{1}{u_n} \rightarrow +\infty$

$$b) u_n \rightarrow 0 \stackrel{?}{\Rightarrow} \left(\underbrace{\frac{1}{u_n} \rightarrow +\infty}_{\text{Se } u_n > 0} \vee \frac{1}{u_n} \rightarrow -\infty \right)$$

Não é verdade

$$u_n = -\frac{1}{n}$$

Ficha 6

1. a) $y = e^{x^2-2} \quad x > 0$

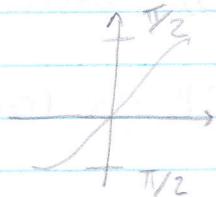
$$\Leftrightarrow \log y = 2x^2 - 2 \Leftrightarrow \log y + 2 = 2x^2 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{\log y + 2} = F^{-1}(y)$$

$$D' = [e^{-2}, +\infty[$$

$$D =]0, +\infty[$$

b) $\arcsen x, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

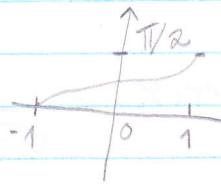


$$D' = [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$D = [-1, 1] \quad (\Leftrightarrow \sin x = y/2 \Leftrightarrow)$$

$$\frac{\arcsen x}{2}$$

(c) $\cos(2x) \quad x \in [0, \frac{\pi}{2}]$

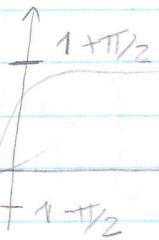


$$D' = [0, \frac{\pi}{2}]$$

$$D = [-1, 1] \quad \text{arcocoseno}(\frac{1}{x})$$

ou arcocoseco(2x)

d) $\operatorname{tg}(x-1), x \in [1-\frac{\pi}{2}, 1+\frac{\pi}{2}]$



$$D' = [1-\frac{\pi}{2}, 1+\frac{\pi}{2}]$$

$$D = \mathbb{R}$$

arcotangente(x-1)

2. $\arccos 0 = k\pi + \frac{\pi}{2}$

$$\arccos 1 = 2k\pi$$

$\arccos x = y \Rightarrow \cos y = x$

$$\arccos(-\frac{1}{2}) = \frac{4\pi}{3} \in$$

$\arcsen(-\frac{1}{2})$

3. $\arcsen x = \alpha \Leftrightarrow \sin \alpha = x$
 $\arccos x = \alpha \Leftrightarrow \cos \alpha = x$
 $\operatorname{arctg} x = \alpha \Leftrightarrow \operatorname{tg} \alpha = x$

4.

a) $\cos(\arccos x) = y \Leftrightarrow \arccos y = \arccos x$

b) $\sin(\arcsen x) = x \Leftrightarrow \arcsen x = \arcsen x$

c) $\cos(\arcsen x) = \sqrt{1-x^2} \Leftrightarrow \arccos(\sqrt{1-x^2}) = \arcsen x$

d) $\sin(\arccos x) = \sqrt{1-x^2} \Leftrightarrow \arcsen(\sqrt{1-x^2}) = \arccos x$

e) $\operatorname{tg}(\arccos x) = \frac{x}{\sqrt{1-x^2}} \Leftrightarrow \operatorname{arctg}\left(\frac{x}{\sqrt{1-x^2}}\right) = \arccos x$

f) $\operatorname{tg}(\arccos x) = \frac{\sqrt{1-x^2}}{x} \Leftrightarrow \operatorname{arctg}\left(\frac{\sqrt{1-x^2}}{x}\right) = \arccos x$

5. $f: D \rightarrow \mathbb{R}$ injectiva
 $g: f(D) \rightarrow D$ inversa de f

a) f crescente: $\forall x_1, x_2 \in D: x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$

(f decrescente é análoga)

\Downarrow g é inversa de f
 $f(x_1) < f(x_2) \Leftrightarrow x_1 < x_2$

$y_1 < y_2 \Leftrightarrow g(y_1) < g(y_2)$

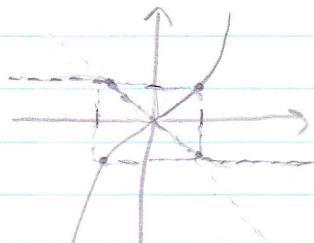
b) f impar: $\forall x \in D$

$f(x) = -f(-x)$

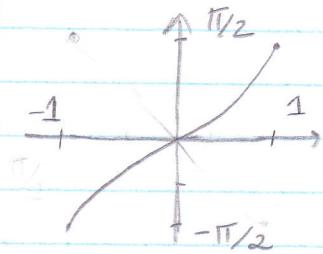
\Downarrow g é inversa de f

$g(f(x)) = -g(f(-x))$

$\Leftrightarrow g(y) = -g(-y)$



c) \arcsen

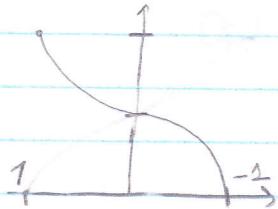


$$\forall x_1, x_2 \in [-1, 1], x_1 < x_2 \quad \arcsen x_1 < \arcsen x_2$$

$$\forall x \in [-1, 1]$$

$$\arcsen x = -\arcsen(-x)$$

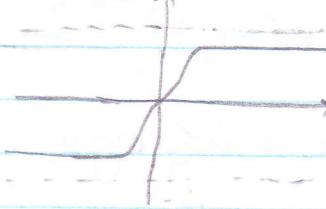
\arccos



$$\forall x_1, x_2 \in [-1, 1], x_1 < x_2 \quad \arccos x_1 > \arccos x_2$$

$$\forall x \in [-1, 1] \quad \arccos x = \arccos(-x)$$

\arctg



$$\forall x_1, x_2 \in \mathbb{R}, x_1 < x_2 \quad \arctg x_1 < \arctg x_2$$

$$\forall x \in \mathbb{R}$$

$$\arctg x = -\arctg(-x)$$

6)

a) $f(x) = \frac{x}{\sqrt{9-x^2}}$

$$D = \{x : \sqrt{9-x^2} \neq 0 \wedge 9-x^2 \geq 0\}$$

$$= \{x : -x^2 + 9 > 0\}$$

$$= [-2, 2]$$

$$-x^2 + 9 = 0 \Leftrightarrow$$

$$\Leftrightarrow 9 = x^2 \Leftrightarrow x = \pm \sqrt{9}$$

$$\Leftrightarrow x = \pm 3$$



b) $f(x) = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}$



$$DF = \{x : \cos^2 x \neq 0 \wedge \sin^2 x \neq 0\}$$

$$= \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\} \cup \mathbb{R} \setminus \left\{k\pi + \frac{\pi}{2}, k \in \mathbb{Z}\right\}$$

$$= \mathbb{R} \setminus \left\{ \frac{k\pi}{2} : k \in \mathbb{Z} \right\}$$

$$c) f(x) = \operatorname{tg}x + \operatorname{cotg}x$$

$\hookrightarrow \mathbb{R}$ $\hookrightarrow [-1, 1]$

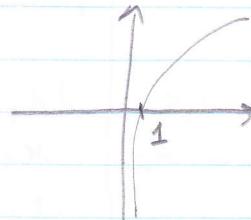
$$\mathcal{D}f = [-1, 1]$$

$$d) f(x) = \log(\log x)$$

$$\mathcal{D}f = \{x : \log x > 0 \wedge x > 0\}$$

$$= \{x : x > 1 \wedge x > 0\}$$

$$= \{x : x > 1\} =]1, +\infty[$$



$$e) f(x) = \log(1 - x^{3/2})$$

$$\mathcal{D}f = \{x : 1 - x^{3/2} > 0\}$$

$$= \{x : x > 1\} =]1, +\infty[$$

$$\begin{aligned} 1 - x^{3/2} &> 0 \Leftrightarrow \\ \sqrt{x^3} &> 1 \Leftrightarrow \\ x^3 &> 1^2 \Leftrightarrow x > \sqrt[3]{1} \Leftrightarrow x > 1 \end{aligned}$$

$$f) f(x) = \arctan\left(\frac{1}{1-x^2}\right) \quad \left(x^3\right)^{\frac{1}{2}} = \sqrt{x^3}$$

$$\mathcal{D}f(x) = \{x : 1 - x^2 \neq 0\}$$

$$= \mathbb{R} \setminus \{-1, 1\},$$

$$g) f(x) = \arccos\left(\frac{1}{x}\right)$$

$$\mathcal{D}f(x) = \{x : x \neq 0 \wedge -1 \leq \frac{1}{x} \leq 1\}$$

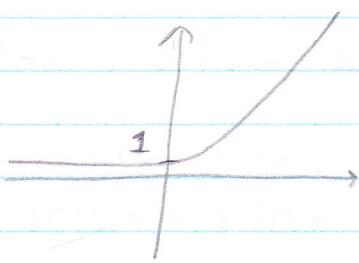
$$= \{x : x > 0 \wedge x \geq 1 \vee x < 0 \wedge x \leq -1\}$$

$$= \mathbb{R} \setminus [-1, 0, 1]$$

$$h) f(x) = \arcsen(e^x)$$

$$D = \{x : -1 \leq e^x \leq 1\}$$

$$\Rightarrow \{x : e^x \leq 1\} = [-\infty, 0]$$



$$i) f(x) = \log(1 - \arcsen x)$$

$$D = \{x : 1 - \arcsen x > 0 \wedge -1 \leq x \leq 1\}$$

$$= \{x : x < \operatorname{sen} 1 \wedge -1 \leq x \leq 1\}$$

$$= \{x : -1 \leq x < \operatorname{sen} 1\}$$

$$= [-1, \operatorname{sen} 1]$$

$$1 - \arcsen x > 0 \Leftrightarrow$$

$$\arcsen x < 1 \Leftrightarrow$$

$$x < \operatorname{sen} 1$$

7. u_n é monótona $\Rightarrow \arctg u_n$ é convergente

$$-\frac{\pi}{2} < \arctg u < \frac{\pi}{2}, \text{ limitada}$$

a função $\arctg u_n$ é composta por uma função limitada e uma sucessão monótona, logo é convergente

$$8) f(x) = x^2 + 1 \quad g(x) = |x|$$

as funções polinomiais e módulo são contínuas em todo o seu domínio

$$\forall \varepsilon > 0 \ \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

$$|1+x^2| - |1+a^2| < \varepsilon \Leftrightarrow |x^2 - a^2| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow |(x+a)(x-a)| < \varepsilon \rightarrow \text{encontrar um } \delta > 0$$

Se $x \in [a-1, a+1] \Rightarrow \delta < 1$

$$|x+a| \leq |x| + |a| \Rightarrow |x+a| \leq 1 + |a| + |a| \Rightarrow \\ \Rightarrow |x+a| \leq 1 + 2|a|$$

$$\text{Se } |x-a| \cdot (1+2|a|) < \varepsilon \Rightarrow |(x-a)(x+a)| < \varepsilon$$

$$|(x-a)(x+a)| < |(x-a)(1+2|a|)| < \varepsilon$$

$$\text{basta } \delta = \frac{\varepsilon}{1+2|a|}, \text{ porque } |x-a| < \delta \Rightarrow |x-a| \cdot (1+2|a|) < \\ < \delta \cdot (1+2|a|) = \varepsilon$$

9. $\lim x_n = 1, x_n > 1 \Rightarrow \text{Convergente}$

a) $f(x) = \frac{1}{x}, x \neq 0$ \hookrightarrow limitada

$$\lim f(x_n) = \lim \frac{1}{x_n} = \frac{1}{1} = 1$$

b) $f(x) = \log x, x > 0$ \hookrightarrow não é convergente

$$\lim_{x \rightarrow +\infty} \log x = +\infty$$

$$f(x_n) = \log(x_n), x_n > 1$$

$$\lim f(x_n) = \lim \log(1) = 0$$

c) $f(x) = \frac{1}{x-1}, x \neq 1$

10. $\phi : [a, b] \rightarrow \mathbb{R}$ continua $a, b \in \mathbb{R}$ $a < b$

$\exists x_n$, termos em $[a, b]$

$$\lim \phi(x_n) = 0$$

↳ sublimite de ϕ

Se ϕ é contínua em $[a, b]$ e $\lim \phi(x_n) = 0$
então $x_n = 0$.



Como $a < b$

\exists ponto c entre a, b
em que $f(c) = 0$

$$\boxed{\lim_{x \rightarrow a} f(x) = f(a)}$$

11. $g : [0, 1] \rightarrow \mathbb{R}$ contínua

a)

b) se $\exists x_n$ im $[0, 1]$ tq $g(x_n) = \frac{1}{n} \quad \forall n \in \mathbb{N}$

então $\exists c \in [0, 1]$ tq $g(c) = 0$

12.

a) $\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)}$

continua

continua

$x=0$?

$$\lim_{x \rightarrow 0} \frac{x+1}{x(x^2+1)} = \frac{1}{0} = \pm\infty$$

Ponto de descontinuidade

$$b) \frac{x+1}{x^4 + 3x^3 + 2x^2} = \frac{x+1}{x^2(x+2)(x+1)} \quad \begin{matrix} \nearrow \text{continua} \\ DF = \mathbb{R} \setminus \{-2, -1, 0\} \end{matrix}$$

\hookrightarrow continua

$$x^4 + 3x^3 + 2x^2 = x^2(x^2 + 3x + 2) = x^2(x+2)(x+1)$$

$$x^2 + 3x + 2 = 0 \Leftrightarrow x = \frac{-3 \pm \sqrt{9-4 \cdot 1 \cdot 2}}{2} \Leftrightarrow x = \frac{-3 \pm 1}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{-2}{2} \vee x = \frac{-4}{2} \Leftrightarrow x = -1 \vee x = -2$$

$$\lim_{x \rightarrow 2^-} \frac{x+1}{x^2(x+2)(x+1)} = -\infty \quad \begin{matrix} \hookrightarrow 0^- \\ \text{Descontinua em } x = -2 \end{matrix}$$

$$\lim_{x \rightarrow -2^+} \frac{x+1}{x^2(x+2)(x+1)} = +\infty \quad \begin{matrix} \hookrightarrow 0^+ \\ \text{Descontinua em } x = -2 \end{matrix}$$

$$\lim_{x \rightarrow 0} \frac{x+1}{x^2(x+2)(x+1)} = \frac{1}{0 \cdot 2 \cdot 1} = \frac{1}{0} = +\infty \quad \begin{matrix} \text{Descontinua em} \\ x = 0 \end{matrix}$$

$$\lim_{x \rightarrow -1} \frac{x+1}{x^2(x+2)(x+1)} = \lim_{x \rightarrow -1} \frac{1}{x^2(x+2)} = \frac{1}{1 \cdot 1} = 1$$

$$(c) \sqrt{x} - \frac{1}{x^2+x} = \sqrt{x} - \frac{1}{x(x+1)} \quad \begin{matrix} \hookrightarrow \text{continua} \\ \text{Prolongado a 1} \end{matrix}$$

$$\lim_{x \rightarrow 0^+} \sqrt{x} - \frac{1}{x(x+1)} = \begin{matrix} \hookrightarrow \text{continua } (x > 0) \end{matrix}$$

$$= 0^+ - \frac{1}{0} = -\infty$$

d) $\operatorname{sem}(\cos \sqrt{1-x^2})$ \rightarrow contínua em todo o domínio

$$D = \{x : 1-x^2 \geq 0\}$$

$$= \{x : -1 < x < 1\}$$

$$= [-1, 1]$$



$$\lim_{x \rightarrow -1} \operatorname{sem}(\cos \sqrt{1-x^2}) = \operatorname{sem}(\cos 0) = \operatorname{sem} 1$$

$$\lim_{x \rightarrow 1} \operatorname{sem}(\cos \sqrt{1-x^2}) = \operatorname{sem} 1$$

prolongando a sem 1

e) $\cos \frac{1}{\sqrt{1-x^2}}$ \rightarrow contínua em todo o domínio

$$D = \{x : \sqrt{1-x^2} \neq 0 \wedge 1-x^2 \geq 0\}$$

$$= \{x : -1 < x < 1\}$$

$$\lim_{x \rightarrow 1^+} \frac{1}{\sqrt{1-x^2}} = \frac{1}{0} = +\infty = \lim_{x \rightarrow 1^+} \frac{1}{1+\sqrt{1-x^2}}$$

$$f) \sqrt[3]{\operatorname{tg} 2x - \operatorname{cotg} 2x}$$

descontínua
em 1 e -1

P. descontinuidade $\rightarrow \lim_{x \rightarrow c} f(x)$

$$\lim_{x \rightarrow c^-} f(x)$$

g) $\frac{1}{\sqrt{x^2+1}} + \frac{1}{\sqrt[3]{x^3-1}}$

$$D = \{x : \sqrt{x^2+1} \neq 0 \wedge x^2+1 \geq 0 \wedge \sqrt[3]{x^3-1} \neq 0\}$$

$$\Rightarrow x : x \neq 1$$

$$= \mathbb{R} \setminus \{1\}$$

$$x^3-1=0 \Leftrightarrow$$

$$\Leftrightarrow x^3=1 \Leftrightarrow x=1$$

$$\lim_{x \rightarrow 1} \frac{f(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{1^3-1}}}{2} = \frac{1}{2} + \infty = +\infty$$

descontínuo
em 0

h) $\frac{|x^2-1|}{x^2-1}$

$$D = \{x : x^2-1 \neq 0\}$$

$$\Rightarrow x \neq 1 \quad = \mathbb{R} \setminus \{1\}$$

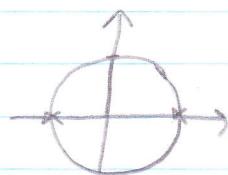
$$\lim_{x \rightarrow 1^-} \frac{|x^2-1|}{x^2-1} \stackrel{0^-}{=} -\infty$$

$$\lim_{x \rightarrow 1^+} \frac{|x^2-1|}{x^2-1} \stackrel{0^+}{=} \stackrel{0^-}{=} +\infty$$

i) $\sqrt{-\sin^2 x}$

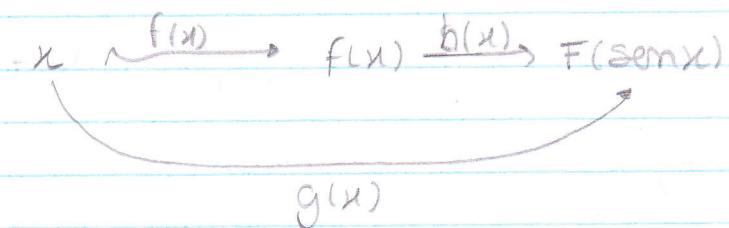
$$D = \{x : -\sin^2 x \geq 0\}$$

$$-\sin^2 x = 0$$



13. $f: \mathbb{R} \rightarrow \mathbb{R}$ continua em $x=1$

Onde é
continua
 $g(x) = f(\operatorname{sen}x)$



- $\operatorname{sen}x$ é continua em todo o domínio
- domínio = $[-1, 1] \hookrightarrow \mathbb{R}$
- se $f(x)$ é continua em $x=1$, $g(x)$ é continua quando $\operatorname{sen}x=1 \Leftrightarrow x = \frac{\pi}{2} + 2k\pi$

14. $f: \mathbb{R} \rightarrow \mathbb{R}$ continua em $x=0$

$$g(x) = f(\operatorname{tg}x - \operatorname{cotg}x)$$

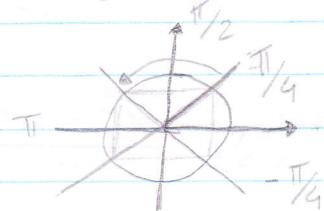
$g(x)$ é obrigatoriamente continua quando $(\operatorname{tg}x - \operatorname{cotg}x) = 0$

$$\operatorname{tg}x - \operatorname{cotg}x = 0 \Leftrightarrow \operatorname{cotg}x = \operatorname{tg}x \Leftrightarrow \frac{1}{\operatorname{tg}x} = \operatorname{tg}x \Leftrightarrow$$

$$\Leftrightarrow 1 = \operatorname{tg}x \cdot \operatorname{tg}x \Leftrightarrow 1 = (\operatorname{tg}x)^2 \Leftrightarrow \sqrt{1} = \operatorname{tg}x \Leftrightarrow$$

$$\Leftrightarrow \operatorname{tg}x = \pm 1 \Leftrightarrow x = \frac{\pi}{4} + \frac{k\pi}{2} \Leftrightarrow$$

$$\Leftrightarrow x = \frac{\pi}{4} + 2k\pi$$



15. $f: \mathbb{R} \rightarrow \mathbb{R}$ $f(x) = xd(x)$

$d: \mathbb{R} \rightarrow \mathbb{R}$ dinihelet

$$d(x) = \begin{cases} 1 & \text{se } x \in \mathbb{Q} \\ 0 & \text{se } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

// descontínua

$$f(x) = \begin{cases} 0 & \text{se } x \in \mathbb{R} \setminus \mathbb{Q} \\ x & \text{se } x \in \mathbb{Q} \end{cases}$$

$$\lim_{x \rightarrow 0} f(x) = 0 = f(0)$$

16.

$$\text{a) } \lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$$

$\forall \varepsilon > 0 \exists R > 0 \forall x \in D \quad 0 < |x| < R \Rightarrow f(x) > \varepsilon$

$$\frac{1}{x^2} > \varepsilon \quad 0 < |x| < R$$

$$\Leftrightarrow 1 > \varepsilon x^2 \Leftrightarrow \sqrt{\frac{1}{\varepsilon}} > |x|$$

$$\text{b) } \lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

$\forall \varepsilon > 0 \exists R > 0 \forall x \in D \quad n > R \Rightarrow f(x) < \varepsilon$

$$\text{c) } \lim_{x \rightarrow +\infty} \sqrt{x} = +\infty$$

$\forall \varepsilon > 0 \exists R > 0 \forall x \in D \quad x > R \Rightarrow f(x) > \varepsilon$

17.

$$\text{a) } \lim_{x \rightarrow 0} \frac{x^3 - x^2 + x - 1}{x^2 - 1} = \lim_{x \rightarrow 0} \frac{-1}{-1} = 1$$

$$\text{b) } \lim_{x \rightarrow 1} \frac{x^3 - x^2 + x - 1}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x^2+1)}{(x+1)(x-1)} = \lim_{x \rightarrow 1} \frac{x^2+1}{x+1} = \frac{2}{2} = 1$$

$$\frac{x^3 - x^2 + x - 1}{x^2 - 1} \quad \begin{array}{|l} x-1 \\ \hline x^2+1 \end{array}$$

$$\textcircled{O} -x-1$$

$$\textcircled{O}$$

$$c) \lim_{x \rightarrow 0} \frac{e^{x^2}-1}{x} = \lim_{x \rightarrow 0} x \left(\frac{e^{x^2}-1}{x^2} \right) =$$

$$= \lim_{x \rightarrow 0} x \cdot \lim_{y \rightarrow 0} \frac{e^y-1}{y} = 0 \cdot 1$$

$$d) \lim_{x \rightarrow 0} \left[x^2 \left(1 - \cos \frac{1}{x} \right) \right]$$

$$e) \lim_{x \rightarrow 0} \frac{\sin \frac{1}{x}}{x}$$

$$\lim_{x \rightarrow 0^+} \sin(\frac{1}{x}) = \lim_{y \rightarrow +\infty} \sin y$$

pelo critério de Heine:

$$y_k = 2\pi k \rightarrow +\infty \quad \sin y \rightarrow 0$$

$$y_k = 2\pi k + \frac{\pi}{2} \rightarrow +\infty \quad \sin y \rightarrow 1$$

A função tem dois sublimites diferentes,
logo $\lim_{x \rightarrow 0} \frac{\sin \frac{1}{x}}{x}$ não existe

$$f) \lim_{x \rightarrow +\infty} \frac{\sin \frac{1}{x}}{x} = \sin 0 = 1$$

$$g) \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

(composta por x (tende p/0) e $\sin \frac{1}{x}$ (limitada))

18.

a) $\lim_{x \rightarrow 1} \frac{x^2 - x}{x^2 - 3x + 2} = \lim_{x \rightarrow 1} \frac{x(x-1)}{(x-1)(x-2)} =$

$\frac{1 \cdot 0}{0 \cdot 0} = \frac{0}{0}$

$\therefore x = \frac{3 \pm \sqrt{9-4 \cdot 1 \cdot 2}}{2} \Rightarrow x = \frac{3 \pm 1}{2} \Rightarrow$

$\Rightarrow x = 2 \vee x = 1$

b) $\lim_{x \rightarrow 0} \frac{\tan x}{x \cos x} =$

Fauro : 2-N 4.78

15:30h - 18h

Ficha 7

$$1. \quad f\left(-\frac{1}{n}\right) = 1 - f\left(\frac{1}{n}\right)$$

$$f(0^-) + f(0^+) = \lim_{x \rightarrow 0^-} f(x) + \lim_{x \rightarrow 0^+} f(x)$$

Pelo critério de Heine:

$$\frac{1}{n} \rightarrow 0^+ \Rightarrow f\left(\frac{1}{n}\right) \rightarrow f(0^+)$$

$$-\frac{1}{n} \rightarrow 0^- \Rightarrow f\left(-\frac{1}{n}\right) \rightarrow f(0^-)$$

$$\lim_{x \rightarrow 0^-} f\left(-\frac{1}{n}\right) = \lim_{n \rightarrow \infty} 1 - f\left(\frac{1}{n}\right) \Leftrightarrow f(0^-) = 1 - f(0^+) \Leftrightarrow$$

$$\Leftrightarrow f(0^-) + f(0^+) = 1$$

$$\text{Se } \exists \lim_{x \rightarrow 0} f(x) = a$$

$$\text{então } f(0^-) = f(0^+) = a \text{ e } a + a = 1 \Leftrightarrow \\ \Leftrightarrow a = \frac{1}{2}$$

$$2. \quad f(x) = \frac{x + |x|}{2} \quad \hookrightarrow \text{dinheirito}$$

$$\text{a) } d(x) \begin{cases} 0 & \text{se } x \notin \mathbb{Q} \\ 1 & \text{se } x \in \mathbb{Q} \end{cases}$$

minorada
min = 0

$$f(x) = \begin{cases} 0 & \text{se } x \notin \mathbb{Q} \wedge x \leq 0 \\ x & \text{se } x \in \mathbb{Q} \end{cases}$$

$$\mathbb{D} = \mathbb{Q}_0^+$$

b) $\lim_{x \rightarrow -\infty} f(x) = 0$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} x = +\infty$$

c) Em quais pontos f é contínua?

em $[-\infty, 0]$ $\lim_{x \rightarrow a} f(x) = f(a) \quad a \in [-\infty, 0]$

3. $f: \mathbb{R} \rightarrow \mathbb{R}$ contínua em $x=1$

$$f(x) = \begin{cases} 0 & \text{se } x \leq -1 \\ \arcsen x & \text{se } -1 < x < 1 \\ k \operatorname{sen}\left(\frac{\pi}{2}x\right) & \text{se } x \geq 1 \end{cases}$$

a)

$$\lim_{x \rightarrow 1^+} f(x) = f(1) = k \operatorname{sen}\left(\frac{\pi}{2} \cdot 1\right) = k$$

$$k = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \arcsen x = \arcsen 1 = \frac{\pi}{2}$$

b) \rightarrow Para $x \in \mathbb{R} \setminus \{-1, 1\}$ f é contínua, pois é dada por funções contínuas.

$\rightarrow f$ é contínua em $x=1$

f é contínua em
 $x=-1$?

não é contínua

em $x=-1$

$$f(-1) = 0 \quad \lim_{x \rightarrow -1^-} f(x) = 0 \neq$$

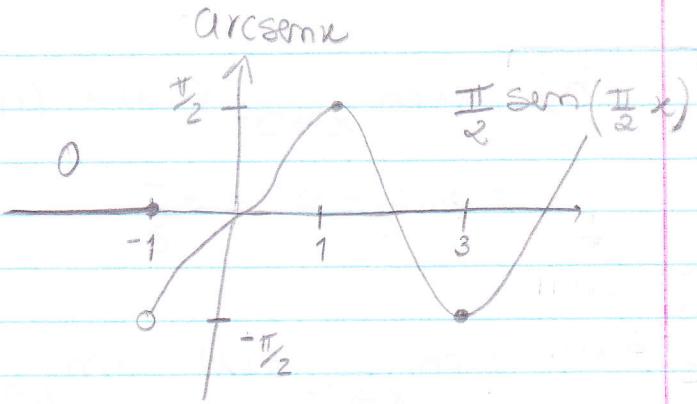
$$\lim_{x \rightarrow -1^+} f(x) = \operatorname{arc}\operatorname{sen}(-1) = -\frac{\pi}{2}$$

c)

- Não tem supremo
- não tem máximo

$$\frac{\pi}{2} \operatorname{sen}\left(\frac{\pi}{2}x\right), \quad \begin{cases} \max = \frac{\pi}{2} \\ \min = -\frac{\pi}{2} \end{cases}$$

$\boxed{[-1, 1]}$



$$\frac{\pi}{2} \operatorname{sen}\left(\frac{\pi}{2}x\right) = -\frac{\pi}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\pi}{2} \cdot (-1) = -\frac{\pi}{2} \Leftrightarrow \frac{\pi}{2}x = \frac{3\pi}{2} \Leftrightarrow$$

$$\Leftrightarrow x = 3$$



f é contínua em $[1, 3]$, $f(1) = \frac{\pi}{2}$ e $f(3) = -\frac{\pi}{2}$. Pelo teorema do valor intermediário, existe pelo menos um $c \in (1, 3)$ em que $f(c) = 0$.

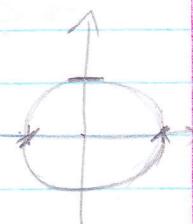
d) $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} 0 = 0$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{\pi}{2} \operatorname{sen}\left(\frac{\pi}{2}x\right) =$$

$$y_K = \frac{\pi}{2} + 2K\pi \rightarrow 1 \neq$$

$$y_K = \pi K \rightarrow 0$$

Logo $\lim_{x \rightarrow +\infty} f(x)$



4. $\rho : \mathbb{R} \rightarrow \mathbb{R}$ $\begin{cases} \operatorname{arctg} \frac{1}{x}, & \text{se } x < 0 \\ 1 + e^{1-x}, & \text{se } x \geq 0 \end{cases}$

a) - a função $\operatorname{arctg} x$ é contínua em \mathbb{R} , logo $\operatorname{arctg} \frac{1}{x}$ é contínua em $\mathbb{R} \setminus \{0\}$.

• $1 + e^{1-x}$ é contínua, porque é composta por uma constante e uma função exponencial (ambas contínuas).

$$\text{b)} \lim_{x \rightarrow 0^-} \varrho(x) = \lim_{x \rightarrow 0^-} \operatorname{arctg} \frac{1}{x} = \\ = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$$

$$\lim_{x \rightarrow 0^+} \varrho(x) = \lim_{x \rightarrow 0^+} 1 + e^{1-x} = 1 + e$$

$$\varrho(0) = 1 + e = \lim_{x \rightarrow 0^+} \varrho(x) \neq \lim_{x \rightarrow 0^-} \varrho(x)$$

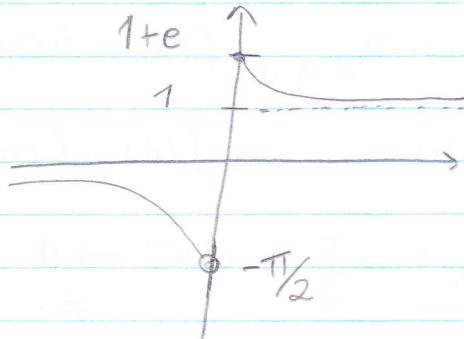
Logo, $\varrho(x)$ é continua à direita no ponto $x=0$

$\varrho(x)$ não é continua à esq no ponto $x=0$,
não é continua em $x=0$

$$\text{c)} \lim_{x \rightarrow +\infty} \varrho(x) = \lim_{x \rightarrow +\infty} 1 + e^{1-x} = 1 + e^{-\infty} = 1 + 0 = 1$$

$$\lim_{x \rightarrow -\infty} \varrho(x) = \lim_{x \rightarrow -\infty} \operatorname{arctg} \frac{1}{x} = \operatorname{arctg} \frac{1}{+\infty} = \operatorname{arctg} 0 = 0$$

$$\text{d)} D = [-\frac{\pi}{2}, 0] \cup [1, 1+e]$$



Pelo T do valor intermédio

$$\Rightarrow \varrho(0) = 1 + e$$

$$\lim_{x \rightarrow +\infty} \varrho(x) = 1$$

$\varrho(x)$ atinge todos os pontos entre $[1, 1+e]$

$$\Rightarrow \lim_{x \rightarrow 0^-} \varrho(x) = -\frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \varrho(x) = 0$$

$\varrho(x)$ atinge todos

os pontos em $[-\frac{\pi}{2}, 0]$

$$x \geq 0$$

$$0 < e^{-x} \leq 1 \Leftrightarrow 0 < e^{1-x} \leq e \Leftrightarrow 1 < 1 + e^{1-x} \leq e + 1$$

$$x < 0$$

$$\frac{1}{x} < 0 \Leftrightarrow -\frac{\pi}{2} < \arctan \frac{1}{x} < 0$$

5.

a) $\rho(x) = e^{-\frac{1}{x^2}}$

é contínua em $\mathbb{R} \setminus \{0\}$ porque é composta por duas funções contínuas em $\mathbb{R} \setminus \{0\}$

$$\begin{array}{c} -\frac{1}{x^2} = y \\ \downarrow \\ e^y = \rho(x) \end{array}$$

$$\chi(x) = x \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x}$$

é contínua em $\mathbb{R} \setminus \{0\}$, é composta por funções trigonométricas, polinomiais e racionais.

b) $\rho(x) = e^{-\frac{1}{x^2}}$

$$\lim_{x \rightarrow 0} e^{-\frac{1}{x^2}} = e^{-1+\infty} = e^0 = 1$$

é prolongável por continuidade no ponto 0

$$\rho(x) \begin{cases} 1 & \text{se } x = 0 \\ e^{-\frac{1}{x^2}} & \text{se } x \neq 0 \end{cases}$$

é contínua em \mathbb{R}

$$\chi(x) = x \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x}$$

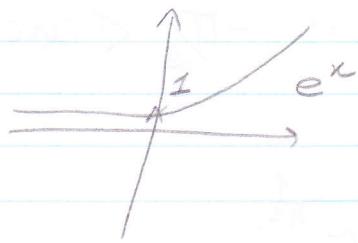
$$\begin{aligned} \lim_{x \rightarrow 0} x \cdot \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x} &= 0 \cdot \operatorname{sen}(\pm \infty) - \cos(\pm \infty) \\ &= -\cos(\pm \infty) \end{aligned}$$

$\left. \begin{matrix} -1 \\ 0 \end{matrix} \right\rangle$ é descontínua

posto

c) $\varphi(x)$ e $\chi(x)$ são limitadas

$$\varphi(x) = e^{-\frac{1}{x^2}}$$



$$0 < e^{\frac{1}{x}} \leq e \Leftrightarrow$$
$$\Leftrightarrow 0 < e^{\frac{1}{x} + 1/x} \leq e^{1/x} \Leftrightarrow$$
$$\Leftrightarrow 0 < e^{-\frac{1}{x^2}} \leq e^{1/x} \leq e$$

Logo $\varphi(x)$ é limitada

$$\chi(x) = x \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x}$$

limitada $[-1, 1]$

$$-1 \leq \operatorname{sen} \frac{1}{x} \leq 1 \Leftrightarrow -x \leq x \operatorname{sen} \frac{1}{x} \leq x \Leftrightarrow$$

$$\Leftrightarrow -x - \cos \frac{1}{x} \leq x \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x} \leq x - \cos \frac{1}{x}$$

6.

a) $f(x) = \frac{\sqrt{x}}{x-1}$

$$D = \{x : x-1 \neq 0 \wedge x \geq 0\}$$

$$= \{x : x \neq 1 \wedge x \geq 0\}$$

$$= [0, 1] \cup]1, +\infty[= \mathbb{R}_0 \setminus \{1\}$$

b) $\lim_{\substack{x \rightarrow +\infty \\ x_n = x}} f(x) = \lim_{x \rightarrow +\infty} \frac{\sqrt{x}}{x-1} = \frac{1}{+\infty} = 0$

$$\lim_{x \rightarrow +\infty} \frac{x}{x+1} = \frac{1}{1+\frac{1}{x}} = \frac{1}{1} = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{\sqrt{x}}{x-1} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{\sqrt{x}}{x-1} = \frac{1}{0^+} = +\infty$$

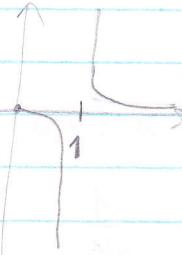
c)

$$f(0) = 0$$

$$D' = \mathbb{R}$$

- > porque quando $f(1^-) = -\infty$
e $f(1^+) = +\infty$
- $\rightarrow \lim_{x \rightarrow +\infty} f(x) = 0$

mas $f(x)$ atinge 0 porque $f(0) = 0$



d)

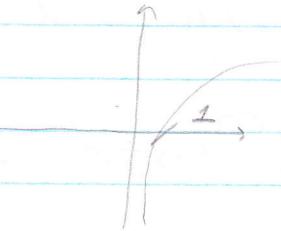
$$7. \quad f(x) = \log \log(1+x) \quad g(x) = \sqrt{x} \cdot \operatorname{sen} \frac{1}{x^2}$$

$$D =]0, +\infty[$$

$$a) \log(1+x) > 0 \Leftrightarrow 1+x > 1 \Leftrightarrow x > 0$$

$$\Leftrightarrow x > 0$$

A função é contínua em todo o domínio, porque é composta por funções logarítmicas.



$g(x)$ é composta por funções contínuas (\sqrt{x}) e $\operatorname{sen} \frac{1}{x^2}$,
logo é contínua no seu domínio $[0, +\infty]$

b)

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \log \log(1+x) = \\ = \log \log(+\infty) = \log(+\infty) = +\infty$$

$$\lim_{x \rightarrow +\infty} g(x) = \lim_{x \rightarrow +\infty} \sqrt{x} \cdot \operatorname{sen} \frac{1}{x^2} =$$

$$= 1 \cdot \operatorname{sen} \frac{1}{+\infty} = 1 \cdot \operatorname{sen}(0) = 1 \cdot 0 = 0$$



c) $\lim_{x \rightarrow 0} \log \log(1+x) =$

$$= \log \log(1) = \log 0 = -\infty \quad \text{não é prolongável} \xrightarrow{\text{por continuidade}} 0$$

$$\lim_{x \rightarrow 0} \sqrt{x} \cdot \operatorname{sen} \frac{1}{x^2} = 0 \cdot \operatorname{sen} \frac{1}{0} = 0 \quad \text{PNE}$$

d) contra domínio $f(x) = \mathbb{R}$

quando $x \rightarrow 0$ $f(x) \rightarrow -\infty$

$$\lim_{x \rightarrow 0} \log \log(x+1) = \log \log 1 = \log 0 = -\infty$$

e quando $x \rightarrow +\infty$ $f(x) \rightarrow +\infty$

$$\lim_{x \rightarrow +\infty} \log \log(x+1) = \log \log(+\infty) = \log(+\infty) = +\infty$$

$$8. \quad f(x) = \begin{cases} -e^{1/x}, & x < 0 \\ \log \frac{1}{1+x^2}, & \text{se } x \geq 0 \end{cases}$$

a)

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} -e^{1/x} = -e^{1/(-\infty)} = -e^0 = -1$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \log \frac{1}{1+x^2} = \log \frac{1}{+\infty} = \log 0 = -\infty$$

b) sim: no ramo $x < 0$ a função é exponencial, logo é contínua.

No ramo $x > 0$ a função é logarítmica, e também continua.

c) $\lim_{x \rightarrow 0^+} \log \frac{1}{1+x^2} = \log 1 = 0$

$$\lim_{x \rightarrow 0^+} -e^{1/x} = -e^{-\infty} =$$

0)

q.

a) $g: [0, +\infty] \rightarrow \mathbb{R}$ continua

$$e(x) = g(1-x^2)$$

$$1-x^2 \geq 0 \Leftrightarrow -x^2 \geq -1 \Leftrightarrow$$

$$\Leftrightarrow x^2 \leq 1 \Leftrightarrow -1 \leq x \leq 1$$

~~Weitstrass~~ ① $e(x) = [-1, 1] \rightarrow$ continua

$$\min e(x) = e(-1) = e(1) = g(0)$$

$$\max e(x) = e(0) = g(1)$$

b) se $g(x)$ fosse $g: \mathbb{R} \rightarrow \mathbb{R}$ continua

$$1-x^2 > 0 \Leftrightarrow -1 < x < 1$$

Logo não iria existir $e(-1) = e(1)$ que seria o máximo da função

Como $g(c)$ não faria parte do domínio de $e(x)$, logo não iria existir mínimo de $e(x)$

10. $g:]a, b[\rightarrow \mathbb{R}$ continua

$$\lim_{x \rightarrow a^-} g(x) = -\lim_{x \rightarrow b^+} g(x) = -\infty$$

Como $-\infty < g < +\infty$, então tem pelo menos um zero ($\exists c \in]a, b[$ tq $g(c) = 0$)

$$\text{então } h(c) = \arctg(g(c)^2) =$$

$$= \arctg 0 = 0$$

Como $h(0) = \frac{\pi}{2}$ e h é contínua em $[0, \frac{\pi}{2}]$,
logo h atinge todos os valores em $[0, \frac{\pi}{2}]$

Portanto $h(u): [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$

11)

$\sin^3 x + \cos^3 x = 0$ tem pelo menos uma raiz em $[0, \pi]$

→ A função é contínua no seu domínio

12. f: continua em \mathbb{R} $\lim_{x \rightarrow +\infty} f(x)$ $\lim_{x \rightarrow -\infty} f(x)$ são finitos

a) se $\lim_{x \rightarrow +\infty} f(x) = a$

$$\lim_{x \rightarrow -\infty} f(x) = b \quad b < f(x) < a$$

domínio da função é $[b, a]$

b)

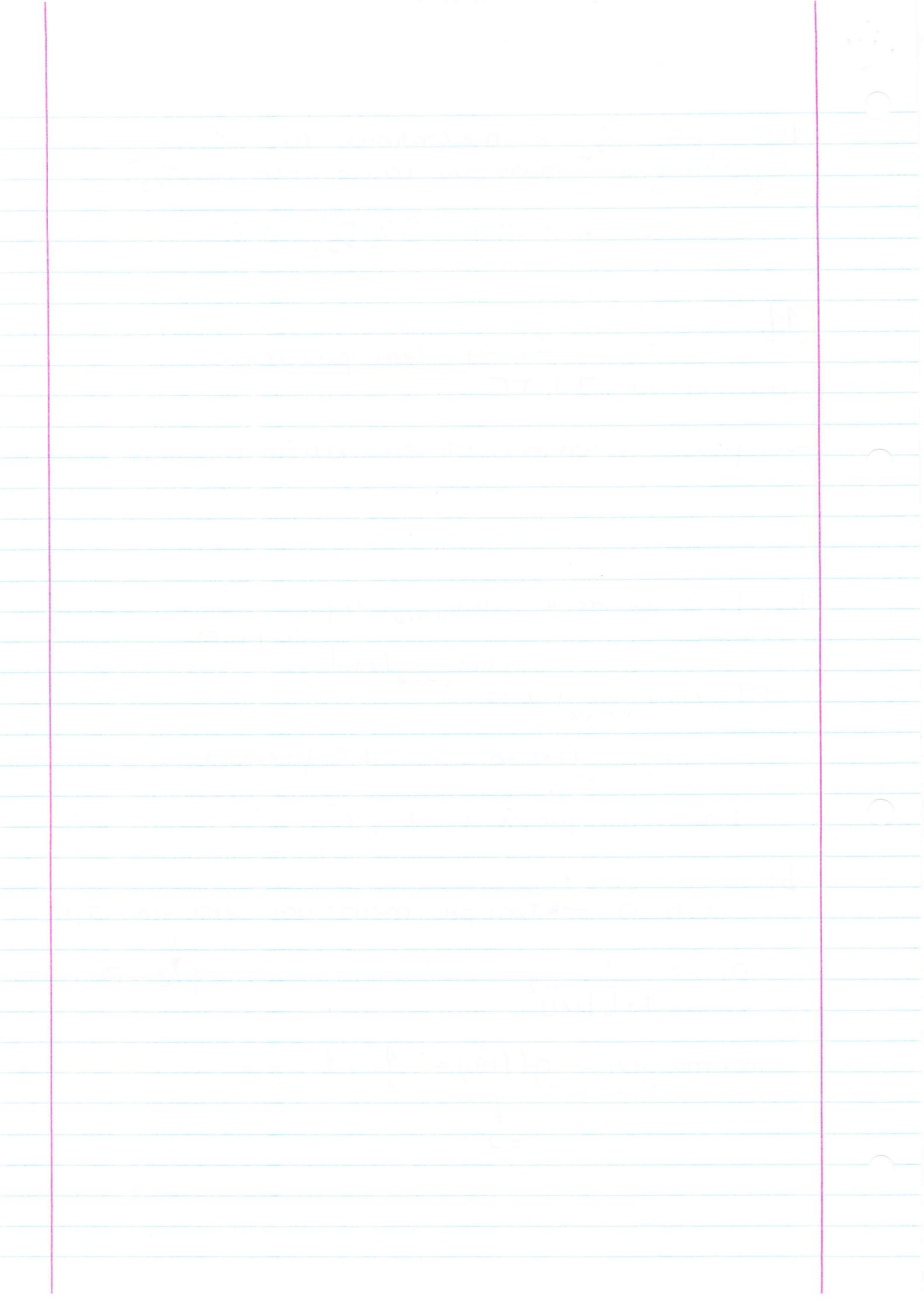
$a \cdot b < 0 \rightarrow$ Tem pelo menos um zero em $[a, b]$

$$g(u) = \frac{1}{1 + [f(u)]^2}$$

$$f(c) = 0$$

$$\text{Máximo } g(u) = g(f(c)) = \frac{1}{1} = 1$$

\downarrow
 $= 0$



Ficha 8

1.

a) $\log(\sinh x)$
 $\hookrightarrow x > 0$

b) $\arcsen(\operatorname{arctg} x)$
 $\forall x \in]-1, 1[\hookrightarrow \mathbb{R} \setminus \{-1, 1\}$

É diferenciável em $\operatorname{arctg} \in]-1, 1[\text{ e } x \in \mathbb{R} \setminus \{-1, 1\}$

$$\operatorname{arctg} x = -1 \Leftrightarrow x = \operatorname{tg}(-1)$$

$$\operatorname{arctg} x = 1 \Leftrightarrow x = \operatorname{tg}(1)$$

$$\operatorname{sen} 3x = x$$

$$\operatorname{arcsen} x = 3x$$

$$\operatorname{arcsen} x = \frac{1}{\sqrt{1+x^2}}$$

$$(\operatorname{arcsen}(\operatorname{arctg} x))' = \frac{1}{\sqrt{1+(\operatorname{arctg} x)^2}}$$

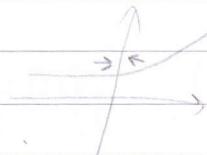
c) $\frac{e^x}{1+x}$

é diferenciável em $\mathbb{R} \setminus \{-1\}$

$$D' = \frac{e^x \cdot (1+x) - e^x \cdot 1}{(1+x)^2} = \frac{e^x + xe^x - e^x}{(1+x)^2}$$

2.

$$f(x) = \begin{cases} \frac{x}{1+e^{1/x}} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$



$$\lim_{x \rightarrow 0^-} \frac{\frac{x}{1+e^{1/x}} - 0^-}{\frac{x}{1} - 0^-} = \lim_{x \rightarrow 0^-} \frac{1}{1+e^{1/x}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$$\lim_{x \rightarrow 0^+} \frac{1}{1+e^{1/x}} = \frac{1}{1+e^{+\infty}} = \frac{1}{1+\infty} = 0$$

3. se $x \neq 0$ $f(x) = x \frac{1+e^{1/x}}{2+e^{1/x}}$

$$\lim_{x \rightarrow 0^-} \frac{x \frac{1+e^{1/x}}{2+e^{1/x}} - 0}{x - 0} = \lim_{x \rightarrow 0^-} \frac{1+e^{1/x}}{2+e^{1/x}} =$$

$$= \frac{1+e^{1/0^-}}{2+e^{1/0^-}} = \frac{1+0}{2+0} = \frac{1}{2}$$

$$\lim_{x \rightarrow 0^+} \frac{1+e^{1/x}}{2+e^{1/x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}e^{1/x} + 1}{2e^{1/x} + 1} = \frac{0+1}{0+1} = 1$$

9.

$$f(x) = \begin{cases} x^2 \operatorname{sen}\left(\frac{1}{x}\right) & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

a)

é contínua em $\mathbb{R} \setminus \{0\}$

$$\begin{aligned} f'(x) &= 2x \operatorname{sen}\frac{1}{x} - x^2 \cos\frac{1}{x} = \\ &= x \left(2 \operatorname{sen}\frac{1}{x} - x \cos\frac{1}{x} \right) \quad \forall x \in \mathbb{R} \setminus \{0\} \end{aligned}$$

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} x \left(2 \operatorname{sen}\frac{1}{x} - x \cos\frac{1}{x} \right)$$

não existem

b) $f(0) = 0$

$$f(0^-) = x \left(2 \operatorname{sen}\frac{1}{x} - x \cos\frac{1}{x} \right) = f(0^+)$$

tem derivada em 0

$$F'(0) = 0$$

5.

$$f \text{ diferenciável em } \mathbb{R}$$
$$f(0) = f(\pi) = 0$$

$$g = f(\operatorname{sen}x) + \operatorname{sen}f(x)$$

$$\begin{aligned} g(0) &= f(\operatorname{sen}0) + \operatorname{sen}f(0) \\ &= f(0) + \operatorname{sen}0 = 0 = f(0) = f(\pi) \\ g(\pi) &= f(\operatorname{sen}\pi) + \operatorname{sen}f(\pi) = \\ &= f(0) + \operatorname{sen}0 = f(\pi) = f(0) \\ &= 0 \end{aligned}$$

$$\text{Então } g'(0) + g'(\pi) = 0 + 0 = f'(0) + f'(\pi)$$

6. $f: \mathbb{R} \rightarrow \mathbb{R}$, diferenciável

$$\begin{aligned} (\arctg f(x) + f(\arctg x))' &= (\arctg f(x))' + (f(\arctg x))' \\ &= \frac{1}{1-(f(x))^2} + (f(\arctg x))' \quad \forall x \in \mathbb{R} \end{aligned}$$

7. $g: \mathbb{R} \rightarrow \mathbb{R}$ duas vezes diferenciável

$$e: [0, +\infty] \rightarrow \mathbb{R} \quad e(x) = e^{g(\log x)}$$

$$g, g' \circ g''$$

$$e'(1) \quad e''(e)$$

$$\begin{aligned} e'(1) &= e^{g(\log 1)} \cdot g(\log 1)' = g(0) \cdot e^{g(0)} \\ e''(e) &= e^{g(\log e)} \cdot g(\log e)'' = g(e)'' \cdot e^{g(1)} \end{aligned}$$

8. $f: \mathbb{R} \rightarrow \mathbb{R}$ $f(u) = u^4 e^{-u}$

$g: \mathbb{R} \rightarrow \mathbb{R}$ diferenciável

$$(g \circ f)'(u) = g'(f(u)) \cdot g(u) = g'(u^4 e^{-u}) \cdot g(u)$$

9. $g: \mathbb{R} \rightarrow \mathbb{R}$ estreitamente monótona $f: [-1, 1] \rightarrow \mathbb{R}$

$g(0) = 2$ Diferenciável

$$g'(0) = \frac{1}{2}$$

$$F(u) = g(\arcsen u)$$

a) $f'(u) = g'(F(x)) \cdot g'(\arcsen x)$

$$= g'(\arcsen x) \cdot \arcsen x = g'(\arcsen x) \cdot \frac{1}{\sqrt{1+x^2}}$$

Como g é diferenciável, então f é diferenciável no domínio de $\arcsen x$

$$f'(0) = g'(\arcsen 0) \cdot g'(0)$$

$$= g'(0) \cdot g'(0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$\arcsen 0 = x$
 $\Rightarrow \sin x = 0$
 $\Rightarrow x = k\pi = 0$

b) $f(u) = g(\arcsen u) \quad \forall u \in [-1, 1]$

a função $\arcsen x$ é injetiva no intervalo $[-1, 1]$

Como $g(u)$ é estreitamente monótona,
 $g(\arcsen u)$ é injetiva

Como g é injetiva

$$\begin{aligned} (F^{-1})'(2) &= \frac{1}{f'(2)} & f(2) &= 2 \\ &= \frac{1}{g'(\arcsen 2)} & g(\arcsen 2) &= 2 \\ &= \frac{1}{g'(0)} & g(0) &= 2 \end{aligned}$$

10. $f: \mathbb{R} \rightarrow]-1, 1[$ diferenciável
injetiva

$$f(2) = 0$$

$$f'(2) = 2$$

$$g(u) = \arccos(f(u))$$

a) $\arccos(u)$ é monótona e injetiva
e $f(u)$ é injetiva.

Logo $g(u)$ é injetiva

$\arccos(u)$ é diferenciável para $\forall u \in]-1, 1[$
 $f(u)$ é diferenciável e toma valores
no intervalo $[-1, 1]$

Logo $g(u)$ é diferenciável

$$\begin{aligned} g'(2) &= (\arccos(f(2)))' \\ &= \frac{1}{\sqrt{1-(f(2))^2}} = \frac{1}{\sqrt{1}} = 1 \end{aligned}$$

$$(g^{-1})'\left(\frac{\pi}{2}\right) = \frac{1}{g'(g^{-1}\left(\frac{\pi}{2}\right))} = \frac{1}{2}$$

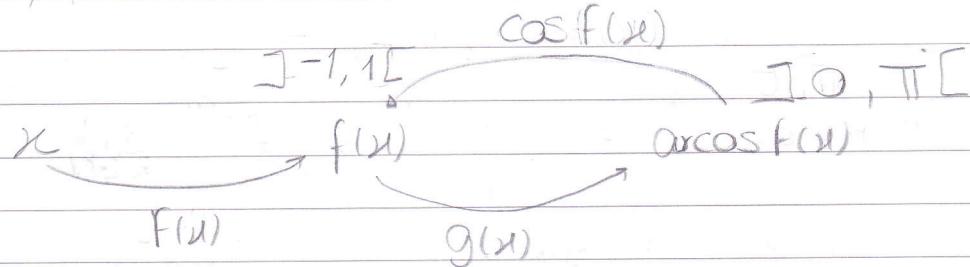
$$\| g^{-1}\left(\frac{\pi}{2}\right) \Leftrightarrow g(u) = \frac{\pi}{2} \Leftrightarrow f(u) = 0 \Leftrightarrow u = 2$$

$$g(u) = \frac{\pi}{2} \Leftrightarrow \arccos f(u) = \frac{\pi}{2} \Leftrightarrow \cos \frac{\pi}{2} = 0$$

$$f(u) = 0 \Leftrightarrow u = 2$$

b)

$$(g^{-1})$$



$$(g^{-1}): [0, \pi] \rightarrow [-1, 1]$$

Não tem máximo nem mínima, mas é limitada

$$11. \operatorname{sh}(u) = \frac{e^u - e^{-u}}{2} \quad \operatorname{ch}(u) = \frac{e^u + e^{-u}}{2}$$

a)

$$\text{i)} \operatorname{ch}^2 u - \operatorname{sh}^2 u = 1$$

$$\left(\frac{e^u + e^{-u}}{2}\right)^2 - \left(\frac{e^u - e^{-u}}{2}\right)^2 =$$

$$\frac{(e^u - e^{-u})^2 - (e^u + e^{-u})^2}{4} = \frac{2e^0 - 2e^0}{2} =$$

$$= \frac{2+2}{4} = 1 \quad \text{cqd}$$

$$\text{ii)} \operatorname{sh}(u+y) = \operatorname{sh}u \operatorname{chy} + \operatorname{ch}u \operatorname{shy}$$

$$\operatorname{sh}(u+y) = \frac{e^{u+y} - e^{-(u+y)}}{2} =$$

$$\frac{e^u - e^{-u}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^u + e^{-u}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$(e^u - e^{-u})(e^y + e^{-y}) + (e^u + e^{-u})(e^y - e^{-y}) = e^{u+y} + e^{\frac{u-y}{2}} - e^{\frac{-u+y}{2}} - e^{-u-y}$$

$$+ e^{u+y} - e^{x+y} + e^{u-x-y} - e^{-x-y}$$

$$\frac{e^{x+y} + e^{-(x+y)}}{2}$$

iii) $\text{ch}(x+y) = \text{ch}x\text{ch}y + \text{sh}x\text{sh}y$

$$\frac{e^x + e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \\ = \frac{2e^{x+y} + 2e^{-(x+y)}}{4} = \frac{e^{x+y} + e^{-(x+y)}}{2} \quad \text{cqd}$$

$\frac{e^{2x} - e^{-2x}}{2}$ iv) $\text{sh}(2x) = 2\text{sh}x\text{ch}x$

$$2 \left(\frac{e^x - e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2} \right) = 2 \cdot \frac{(e^x - e^{-x})(e^x + e^{-x})}{4} = \\ = \frac{e^{2x} - e^{-2x}}{2} \quad \text{cqd}$$

v) $\text{ch}(2x) = \text{ch}^2 x + \text{sh}^2 x$

$$\frac{e^{2x} + e^{-2x}}{2} \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) + \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x - e^{-x}}{2} \right) = \\ = \frac{e^{2x} + e^{-2x} + e^0 + e^{-2x} + e^{2x} - e^0 - e^{-2x} + e^{-2x}}{4} = \\ = \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2} \quad \text{cqd}$$

b) sh é ímpar

$$\text{sh}(x) = -\text{sh}(-x)$$

$$-\text{sh}(-x) = \left(\frac{e^{-x} - e^x}{2} \right) =$$

$$= -\frac{e^{-x} + e^x}{2} = \frac{e^x - e^{-x}}{2}$$

ch é par

$$\text{ch}(x) = \text{ch}(-x)$$

$$\text{ch}(-x) = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2}$$

cqd

cqd

d) São ambas contínuas em \mathbb{R}

$$D \operatorname{sh}(u) = \frac{(e^u - e^{-u})' \cdot 2 - (e^u - e^{-u}) \cdot 2'}{2} = \\ = \frac{2(e^u + e^{-u}) - 0}{2} = \frac{e^u + e^{-u}}{2} = \operatorname{ch}(u)$$

$$D \operatorname{ch}(u) = \frac{(e^u + e^{-u})' \cdot 2 - (e^u + e^{-u}) \cdot 2'}{2} =$$

$$= \frac{2(e^u - e^{-u})}{2} = \frac{e^u - e^{-u}}{2} = \operatorname{sh}(u)$$

e) $\operatorname{sh}(u)$

$\operatorname{sh}(u)$ é negativa quando
 $e^{-u} > e^u \Leftrightarrow u < 0$

$$\Leftrightarrow -u > 0 \Leftrightarrow u < 0$$

$$\Leftrightarrow -u - u > 0 \Leftrightarrow u < 0$$

$$\Leftrightarrow -2u > 0 \Leftrightarrow u < 0$$

$\operatorname{sh}(u)$ é positiva quando

$$e^{-u} < e^u \Leftrightarrow u > 0$$

$$\Leftrightarrow u > 0$$

$$\operatorname{sh}(0) = 0$$

zero ímpar

Como $D\operatorname{sh}(u) = \operatorname{ch}(u)$

Então sh é estritamente crescente

$\operatorname{ch}(u)$

nunca é negativa,
não tem zeros

$$D \operatorname{ch}(u) = 0 \Leftrightarrow$$

$$\operatorname{sh}(u) = 0 \Leftrightarrow$$

$$u = 0$$

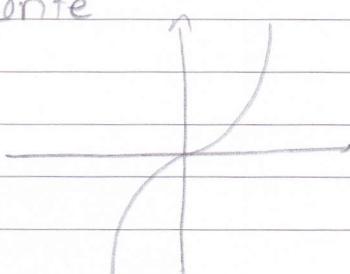
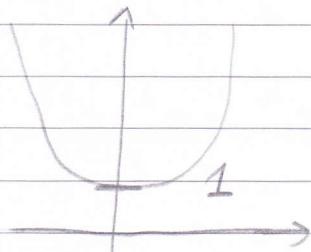
ch é decrescente em $]-\infty, 0]$

ch é estritamente crescente em $[0, +\infty[$

tem extremo (mínimo) quando $u = 0$

Par

$$\min = \frac{e^0 + e^{-0}}{2} = 1$$



F)

$$(\sinh^{-1}) = \operatorname{argsh}(x)$$

$$(\cosh^{-1}) = \operatorname{argch}(x)$$

$$x = \sinh y \text{ sse } y = \operatorname{argsh}(x)$$

$$x = \cosh y \text{ sse } y = \operatorname{argch}(x)$$

não percebo

12.

$f: [0, 1] \rightarrow \mathbb{R}$ diferenciável

$$f\left(\frac{1}{n+1}\right) = 0 \quad \forall n \in \mathbb{N}$$

a) $\forall n \geq 2$ f tem max em $\left[\frac{1}{n+1}, \frac{1}{n}\right]$

| $n=2$ | $f\left(\frac{1}{3}\right) = 0$ tem max em $\left[\frac{1}{3}, \frac{1}{2}\right]$?

b) f é necessariamente limitada

c) f' tem necessariamente infinitos zeros

13.. $f: \mathbb{R} \rightarrow \mathbb{R}$ duas vezes diferenciável

cruza a recta $x=y$ em 3 pontos

Então f'' tem pelo menos um zero



$\exists a < b < c$ tal que $f(a) = a$

$$f(b) = b$$

$\rightarrow \exists z \in \mathbb{R} : f''(z) = 0$

$$f(c) = c$$

2 zeros
dit.

ou $g(x) = f(x)$

$g(a) = g(b) = g(c)$

peot. derolle

$\Rightarrow a, b, c$ zeros

Aplicando o Teorema de Lagrange:

$\rightarrow a \in [a, b] \quad \exists d \in]a, b[$

$$\text{tg } f'(d) = \frac{f(b) - f(a)}{b - a} = 1$$

$\rightarrow a \in [b, c] \quad \exists e \in]b, c[$

$$\text{tg } f'(e) = \frac{f(c) - f(b)}{c - b} = 1$$

Aplicando o teorema de Rolle

af' em $[d, e]$

$\exists z \in [d, e] \quad \text{tg } f'(z) = 0$

cqd

14.

$3x^2 - e^x$ tem exatamente 3 zeros

$$(3x^2 - e^x)' = 6x - e^x$$

ficha 9

$$\begin{matrix} -1 & \nearrow & \nearrow \\ & & 1 \end{matrix}$$

1. $f: \mathbb{R}_+^* \rightarrow \mathbb{R}$ diferenciável e $f(n) = (-1)^n \forall n \in \mathbb{N}$

→ A sua derivada não tem limite no infinito

$\lim f'(n)$ não existe

$$\Rightarrow \lim ((-1)^n)'$$

2. $f(0) = 0$ derivada é uma função crescente

$$\left| g(x) = \frac{f(x)}{x} \text{ é crescente em } \mathbb{R}_+^* \right.$$

Aplicando o T de lagrange a $f(x)$ no intervalo $[0, x] \forall x \in \mathbb{R}_+$

$$\exists c \in [0, x] \text{ tq } \frac{f(x) - f(0)}{x - 0} = f'(c)$$

$$\Leftrightarrow \frac{f(x)}{x} = f'(c)$$

Como $c > 0$ enteo em \mathbb{R}_+ $F(c) = g(u)$

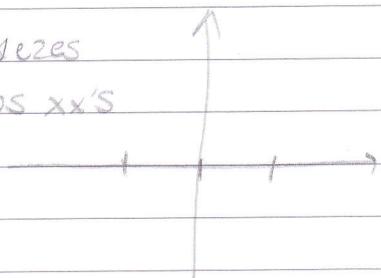
3. Se $f \in C^1(\mathbb{R})$ e $f(x) = x^2$ tem 3 soluções
(uma negativa, uma positiva, uma nula)

- Se é derivável

uma vez

- Passa 3 vezes

pelo ref. dos xx's



↓
então $f'(x)$ tem pelo
menos um zero

~~prova~~ teorema de rolle

para qualquer intervalo $[a, b]$ tq $f(a) = f(b) = 0$ e $a < b$
 $f'(c) = 0$

então $\exists d \in [a, b] : f'(d) = 0$

4

a) $D \frac{x}{x^2+1} = \frac{1.(x^2+1) - x(2x+0)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} =$

$$= \frac{-(x^2-1)}{(x^2+1)^2} = \frac{1}{x^2-1} \quad x \neq 1 \text{ e } -1$$

→ Não tem extremos

→ A função é

decrecente em $]-\infty, -1]$	\cup	$[1, +\infty[$
crescente em $[-1, 1]$	\cup	$+\infty$

b) $\frac{1}{x} + \frac{1}{x^2} = \frac{1 \cdot 1 - 0 \cdot x}{x^2} + \frac{1 \cdot 2x - 0 \cdot x^2}{x^3} =$

$$= \frac{1}{x^2} + \frac{2x}{x^3} = \frac{x^2+2x}{x^3} = \frac{x+2}{x^2} \quad x \neq 0$$

→ tem zero em $x=-2$

→ A função é
crescente de $]-\infty, -2[$
tem um máximo local em -2
é decrescente em $]-2, 0[$
e crescente em $[0, +\infty[$

$x+2$	-	0	+	+	+
x^3	-	-	-	0	+
+	0	-	ND	+	+

b)

(c) $|x^2 - 5x + 6| = 3$

$$(x^2 - 5x + 6)' = 2x - 5 \quad x = \frac{5}{2}$$

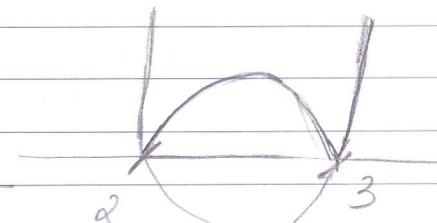
$$(-x^2 + 5x - 6)' = -2x + 5 \quad x = \frac{5}{2}$$

$$x^2 - 5x + 6 = 0 \quad (\Rightarrow) \quad x = \frac{5 \pm \sqrt{25 - 4 \cdot 1 \cdot 6}}{2} \quad (\Rightarrow) \quad x = \frac{5 \pm \sqrt{1}}{2}$$

$$(\Rightarrow) \quad x = \frac{6}{2} \quad \vee \quad x = \frac{4}{2} \quad (\Rightarrow)$$

$$(\Rightarrow) \quad x = 3 \quad \vee \quad x = 2$$

		$\frac{5}{2}$	
$2x - 5$	-	0	+
$-2x + 5$	+	0	-
	-	0	-



01) $(x \log x)' = x \cdot \frac{1}{x} + \log x = 1 + \log x$
 $\forall x \in \mathbb{R}^+$

$$1 + \log x = 0 \Leftrightarrow \log x = -1$$

$$\Leftrightarrow \log \frac{1}{x} = 1 \Leftrightarrow \frac{1}{x} = e \Leftrightarrow$$

$$\Leftrightarrow 1 = ex \Leftrightarrow \frac{1}{e} = x$$

A função é descrente em $[0, \frac{1}{e}]$ com um mínimo
crescente em $[\frac{1}{e}, +\infty]$ em $x = \frac{1}{e}$

e) $\left(e^{-x^2}\right)' \overset{\text{sempre} < 0}{=} -2x \cdot e^{-x^2}$

sempre < 1

$-2x$	+	0	-
e^{-x^2}	-	-	-
-	0	+	

$$-2x \cdot e^{-x^2} = 0 \Leftrightarrow -2x = 0$$

a função tem máximos em $x = 0$
 crescente em $]0, +\infty[$
 decrescente em $]-\infty, 0[$

f) $\left(\frac{e^x}{x}\right)' = \frac{e^x - e^x \cdot x}{x^2} = \frac{e^x(1-x)}{x^2} \quad \forall x \neq 0$

$$\frac{e^x - e^x \cdot x}{x^2} = 0 \Leftrightarrow e^x - e^x \cdot x = 0 \Leftrightarrow e^x = e^x \cdot x$$

$$\Leftrightarrow \frac{e^x}{e^x} = x \Leftrightarrow x = 1$$

f é crescente em $]-\infty, 0[\cup]0, 1[$
 decrescente em $]1, +\infty[$

g) $(xe^{-x})' = x \cdot (-e^{-x}) + e^{-x} \Leftrightarrow -xe^{-x} + e^{-x}$

$$\Leftrightarrow e^{-x}(-x+1) = e^{-x}(1-x)$$

$$e^{-x}(1-x) = 0 \Leftrightarrow e^{-x} = 0 \vee 1-x=0 \Leftrightarrow$$

PF

$$\Leftrightarrow x = -1$$

e^{-x}	+	-1	
$-$	+	+	+
$1-x$	+	0	-
$-$	+	0	-

f é crescente em $]-\infty, -1[$
 decrescente em $]-1, +\infty[$
 tem máximos em $x = -1$

h) $\arctan x - \log \sqrt{1+x^2} = \frac{1}{\sqrt{1+x^2}} -$

$$\left(\log \frac{1}{\sqrt{1+x^2}} \right)' =$$

$$\left(\frac{1}{\sqrt{1+x^2}} \right)' = 0 \cdot \sqrt{1+x^2} - 1 \cdot \frac{-1}{2\sqrt{1+x^2}} = -\frac{1}{2\sqrt{1+x^2}(1+x^2)}$$

$$\left(\sqrt{1+x^2} \right)' = \frac{(1+x^2)^{-1/2}}{2} = \frac{1}{2\sqrt{1+x^2}}$$

5. a) $f: \mathbb{R} \rightarrow \mathbb{R}$ $f(x) = |x| e^{-\frac{x^2}{2}}$

$$\lim_{x \rightarrow +\infty} |x| e^{-\frac{x^2}{2}} = +\infty \cdot e^{-\infty} = +\infty \cdot 0$$

$$\lim_{x \rightarrow -\infty} \frac{|x|}{e^{\frac{x^2}{2}}} \text{ pelo negra de Cauchy} = \lim_{x \rightarrow -\infty} \frac{1}{xe^{\frac{x^2}{2}}} = 0$$

$$\lim_{x \rightarrow +\infty} |x| e^{-\frac{x^2}{2}} = +\infty \cdot e^{-\infty} = +\infty \cdot 0$$

$$\lim_{x \rightarrow +\infty} \frac{|x|}{e^{\frac{x^2}{2}}} \text{ pelo negra de Cauchy} = \lim_{x \rightarrow +\infty} \frac{1}{xe^{\frac{x^2}{2}}} = 0$$

b) $\lim_{x \rightarrow 0}$

$$\left(|x| e^{-\frac{x^2}{2}} \right)' = \begin{cases} \frac{1 - x^2}{e^{\frac{x^2}{2}}} & \text{se } x > 0 \\ \frac{x^2 - 1}{e^{\frac{x^2}{2}}} & \text{se } x < 0 \end{cases}$$

$$c) \frac{1-x^2}{e^{x^2}}$$

	-1	0	1	
$1-x^2$	-	0	0	-
e^{x^2}	+	+	+	+

crescente em $]-\infty, -1[$

tem um máximo em $x = -1$

decrescente em $[-1, 0[$

crescente em $]0, 1[$

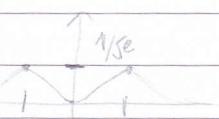
tem um máximo em $x = 1$

decrescente em $]1, +\infty[$

+	-	0	+	0	-	
x^2-1	+	0	-	0	+	
-	0	+	0	-	0	+

$$d) \text{ máximos } x = -1 \quad | \quad x = 1$$

$$\left| -1 \right| e^{\frac{-1^2}{2}} = 1 \cdot e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} \quad | \quad \left| 1 \right| e^{-\frac{1^2}{2}} = 1 \cdot e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$



$$\text{Como } \lim_{x \rightarrow -\infty} |x| e^{-\frac{x^2}{2}} = \lim_{x \rightarrow +\infty} |x| e^{-\frac{x^2}{2}} = 0$$

[contradomínio de $g(x) =]0, 1/\sqrt{e}]$

$$6. \quad g(x) = \begin{cases} e^x + \alpha x + \beta & \text{se } x < 0 \\ \arctg(e^x + e^{-x} - 1) & \text{se } x > 0 \end{cases}$$

(a)

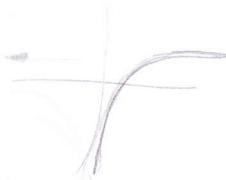
$$\lim_{x \rightarrow 0^+} \arctg(e^x + e^{-x} - 1) = \lim_{x \rightarrow 0^+} \arctg(e^{0^+} + e^{0^-} - 1) =$$

$$= \arctg(1) = \frac{\pi}{4}$$

$$e^0 + \alpha \cdot 0 + \beta = \frac{\pi}{4} \quad (\Rightarrow 1 + \beta = \frac{\pi}{4} \Rightarrow \beta = \frac{\pi}{4} - 1 \Rightarrow \beta = \frac{\pi}{4} - \frac{4}{4})$$

$$\Rightarrow \beta = \frac{\pi - 4}{4}$$

$$\beta = 4 \quad \alpha = -1$$



b)

$$\lim_{x \rightarrow -\infty} g(x) = \lim_{x \rightarrow -\infty} e^x - x + 4 = e^{-\infty} + \cancel{x} + 4 = 4$$

$$\lim_{x \rightarrow \infty} e^x - e^{\log x} + 4$$

$$\lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} \arctg(e^x + e^{-x} - 1)$$

c) $(e^x - x + 4)' = e^x - 1 \quad x < 0$

$$(\arctg(e^x + e^{-x} - 1))' = \frac{1}{1 - (e^x + e^{-x} - 1)^2} \cdot (e^x + e^{-x} - 1)'$$

$$= \frac{1}{1 - (e^x + e^{-x} - 1)^2} \cdot (e^x - e^{-x})$$

d) $e^x - 1$

7. $f: \mathbb{R} \rightarrow \mathbb{R}$ $|x| e^{-|x-1|}$

a) $\lim_{x \rightarrow -\infty} |x| e^{-|x-1|} = \frac{|x|}{e^{|x-1|}}$ per le regole
di Cauchy

$$\lim_{x \rightarrow -\infty} = \frac{1}{e^{|x-1|}} \quad \text{se } x \neq 0 \quad (\Rightarrow)$$

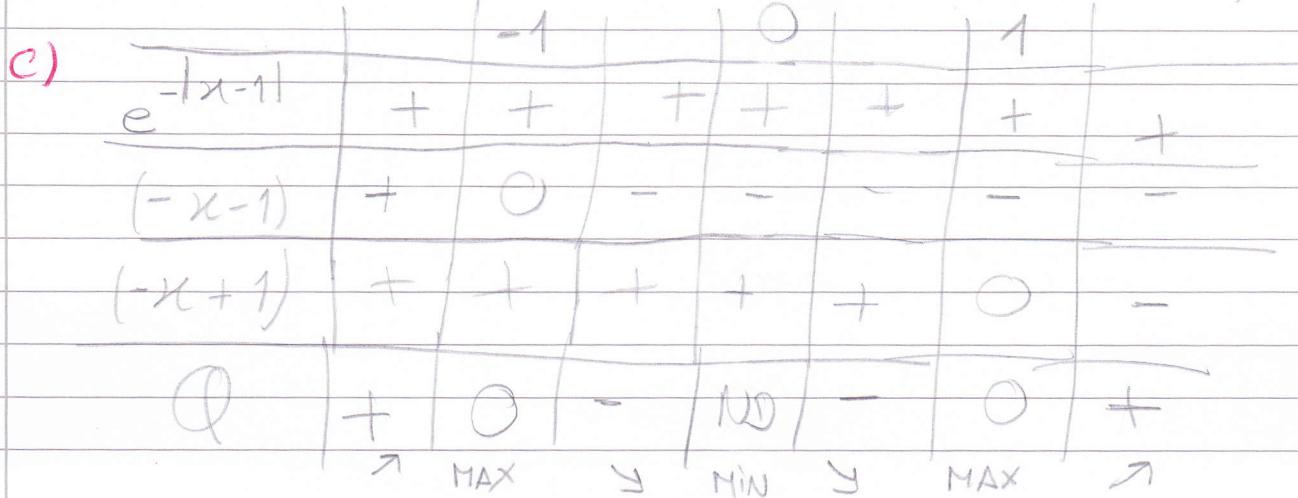
$$\Rightarrow \lim_{x \rightarrow -\infty} \frac{1}{e^{+x}} = \frac{1}{+\infty} = 0 = \lim_{x \rightarrow +\infty} |x| e^{-|x-1|}$$

$$|x-1|' = \begin{cases} 1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \end{cases}$$

b) $\lim_{x \neq 0}$

$$(|x| e^{-|x-1|})' = \begin{cases} -1 \cdot e^{-|x-1|} + |x| e^{-|x-1|} & x < 0 \\ 1 \cdot e^{-|x-1|} + |x| \cdot (-e^{-|x-1|}) & x > 0 \end{cases}$$

$$= \begin{cases} -e^{-|x-1|} + |x| e^{-|x-1|} & x < 0 \\ e^{-|x-1|} - |x| e^{-|x-1|} & x > 0 \end{cases} = \begin{cases} e^{-|x-1|}(-1 - x) & x < 0 \\ e^{-|x-1|}(1 - x), & x > 0 \end{cases}$$



$$\lim_{x \rightarrow 0^+} f(x) = 3 \neq \lim_{x \rightarrow 0^-} f(x)$$

$$(x + 2 \operatorname{arctg} x) = 1 + 2 \cdot \frac{1 - x^2}{1 + x^2}, \quad x < 0$$

b) $|x| = 1$

$$\lim_{x \rightarrow 1^+} x + 2 \operatorname{arctg} x = +\infty + \frac{\pi}{2} = +\infty + \pi = +\infty$$

$$x = -1 \Rightarrow x + 2 \operatorname{arctg} x = -\infty + \frac{\pi}{2} = -\infty + \pi = -\infty$$

$$f(x) = x + 2 \operatorname{arctg} x \quad \text{IR} \rightarrow \text{IR}$$

a) $[0, 1]$

$x_0 = 0 \Rightarrow$ minimo absoluto

$$x = 0$$

$$|1|e^{-|1-x|} = e_0 = 1 \Rightarrow$$
 maximo absoluto

$$x = 1$$

$$|-1|e^{-|1-x|} = 1 \cdot e^{-2} = e^{-2}$$

$$x = -1$$

$y < 0$

C)

$$1 - \frac{2}{1+x^2} = 0 \Leftrightarrow \frac{2}{1+x^2} = 1 \Leftrightarrow 2 = 1+x^2 \Leftrightarrow$$

$$\Leftrightarrow 2-1 = x^2 \Leftrightarrow x^2 = 1 \Leftrightarrow x = 1 \wedge x = -1$$

$$\begin{array}{c}) \\ \hline 1 - \frac{2}{1+x^2} + 0 - 0 + \\ \hline 1 + \frac{2}{1+x^2} + + + + + \\ \hline f'(x) + 0 - \text{NE} + + + \end{array}$$

$$1 + \frac{2}{1+x^2} = 0 \Leftrightarrow \frac{2}{1+x^2} = -1 \Leftrightarrow 2 = -1-x^2 \Leftrightarrow$$

$$\Leftrightarrow 3 = -x^2 \Leftrightarrow x^2 = -3 \quad \text{P.F.}$$

f é crescente em $]-\infty, -1[$

tem máx. em $x = -1 \rightarrow$ max. local

é decrescente em $] -1, 0 [$

tem um vértice em $x = 0 \rightarrow$ min. local

é crescente em $] 0, +\infty [$

$$f(0) = 0 + 2 \arctg 0 = 2 \cdot 0 = 0$$

$$f(-1) = -1 + 2 \arctg (-1) = \frac{\pi}{2} - 1$$

d) $f:]-\infty, 0] \rightarrow]-\infty, \frac{\pi}{2} - 1]$

9. $g(0) = g'(0) = 0$ g' é estritamente monótona

$$e(u) = 2 \operatorname{tg}(g(u)) - g(u)$$

Então $e(0)$ é extremo local de e

$$\rightarrow e'(0) = 0$$

$$e'(x) = (2 \operatorname{tg}(g(x)) - g(x))' = (2 \operatorname{tg}(g(x)))' - g(x)' =$$

$$= 2 \cdot (\operatorname{tg}(g(x)))' - g(x)' = 2 \cdot \operatorname{tg}'(g(x)) \cdot g(x)' - g(x)'$$

No ponto $x=0$

$$2 \cdot \operatorname{tg}'(0) \cdot 0 - 0 = 0$$

Como $x=0$ é zero da derivada, 0 é extremo local de e

10. $f: V_\varepsilon(0)$ dif em $V_\varepsilon(0) \setminus \{0\}$

$$\text{e } \forall x \in V_\varepsilon(0) \setminus \{0\}$$

a) f continua em $x=0$

Prove que $f(0)$ é um extremo

$$x > 0 \vee f'(u) > 0$$

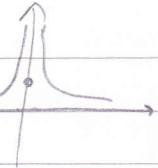
$$x < 0 \vee f'(u) < 0$$

a função é crescente
à esquerda do 0

a função é decrescente
à direita do zero

Logo 0 é um máximo
local da função

Se existir $f'(0) = 0$

b) por exemplo 

não é contínua em $x=0$, não se pode concluir que este é extremo

11.

a) $\lim_{x \rightarrow 0} \frac{a^x - b^x}{x}$ pela regra de Cauchy

$$\lim_{x \rightarrow 0} \frac{x a^{x-1} - x b^{x-1}}{1} = \lim_{x \rightarrow 0} x(a^{x-1} - b^{x-1})$$
$$= 0(a^{-1} - b^{-1}) = 0$$

b) $\lim_{x \rightarrow +\infty} \frac{\log(x + e^x)}{x}$ pela regra de Cauchy

$$\lim_{x \rightarrow +\infty} \frac{(x + e^x)'}{(x + e^x)} = \lim_{x \rightarrow +\infty} \frac{1 + e^x}{x + e^x} \text{ pela regra de Cauchy}$$

$$= \lim_{x \rightarrow +\infty} \frac{e^x}{1 + e^x} = \lim_{x \rightarrow +\infty} \frac{1}{\frac{1 + e^x}{e^x}} = \frac{1}{0+1} = 1$$

c) $\lim_{x \rightarrow 1} \log x \cdot \log \log x = \lim_{x \rightarrow 1} \frac{\log \log x}{\frac{1}{\log x}} =$

pela regra
do Cauchy $= \lim_{x \rightarrow 1} \frac{\frac{1}{\log x} \cdot \frac{1}{h}}{-\frac{1}{(\log x)^2} \cdot \frac{1}{h}} =$

$$= \lim_{x \rightarrow 1} -\frac{\log^2 x}{\log x} = \lim_{x \rightarrow 1} -\log x = 0$$

(d) $\lim_{n \rightarrow 0^+} \frac{e^{-1/n}}{n}$ pella regola de Cauchy

$$= \lim_{n \rightarrow 0^+} \frac{n^2 \cdot e^{-1/n}}{1} =$$

$$= \lim_{n \rightarrow 0^+} n^2 e^{-1/n}$$

(e) $\lim_{x \rightarrow 0^-} \frac{e^{-1/x}}{x}$

F) $\lim_{x \rightarrow 1^+} x^{\log \log x} = \lim_{x \rightarrow 1^+} (e^{\log x})^{\log \log x} =$

$$= \lim_{x \rightarrow 1^+} e^{\log x \log \log x} = e^0 = 1$$

(g) $\lim_{n \rightarrow +\infty} \frac{1}{n!} =$

12.

a) $\lim_{n \rightarrow 0} \frac{10^n - 5^n}{n}$ pella r de Cauchy

$$= \lim_{n \rightarrow 0} \frac{x^{10^{n-1}} - x^{5^{n-1}}}{1} = \frac{0.10^{-1} - 0.5^{-1}}{1} = 0$$

b) $\lim_{n \rightarrow 0^+} \frac{n^2 \operatorname{sen} \frac{1}{n}}{\operatorname{sen} n}$ pela regra de Cauchy

$$\lim_{n \rightarrow 0^+} \frac{(n^2 \operatorname{sen} \frac{1}{n})'}{-\cos n} = \lim_{n \rightarrow 0^+} \frac{2n \operatorname{sen} \frac{1}{n} + n^2 \cdot \frac{1}{n} \cos \frac{1}{n}}{-\cos n}$$

c.a

$$\begin{aligned} \left(n^2 \operatorname{sen} \frac{1}{n}\right)' &= 2n \cdot \operatorname{sen} \frac{1}{n} + n^2 \cdot \frac{1}{n} \cos \frac{1}{n} \\ &= 2n \cdot \operatorname{sen} \frac{1}{n} + \cos \frac{1}{n} \end{aligned}$$

13.

a) $\lim_{n \rightarrow +\infty} \frac{2^n}{n^2} = +\infty$

b) $\lim_{n \rightarrow -\infty} \frac{2^n}{n^2} = 0$

14.

a) $\lim_{x \rightarrow 0^+} (\operatorname{sen} x)^{\operatorname{sen} x} = \lim_{x \rightarrow 0^+} e^{\log(\operatorname{sen} x)}$

$$= \lim_{x \rightarrow 0^+} e^{\operatorname{sen} x \cdot \log \operatorname{sen} x} = e^0 = 1$$

$$\lim_{x \rightarrow 0^+} \operatorname{sen} x \cdot (\log \operatorname{sen} x) = \lim_{x \rightarrow 0^+} \frac{\log \operatorname{sen} x}{\frac{1}{\operatorname{sen} x}} = \frac{-\cos x}{\operatorname{sen} x} =$$

$$= \frac{\operatorname{sen} x \cdot (-\cos x)}{\operatorname{sen} x \cdot \frac{1}{\operatorname{sen} x}} = \lim_{x \rightarrow 0^+} \frac{\operatorname{sen} x \cdot (-\cos x)}{\operatorname{sen} x^2} = 0$$

$$(\log \operatorname{sen} x)' = -\frac{\cos x}{\operatorname{sen} x} \quad \left(\frac{1}{\operatorname{sen} x}\right)' = \frac{-\cos x}{\operatorname{sen} x^2}$$

$$b) \lim_{x \rightarrow +\infty} (\log x)^{\frac{1}{\ln x}} \underset{x \rightarrow +\infty}{\underset{\substack{\nearrow \\ +\infty^0}}{\Rightarrow}} \lim_{x \rightarrow +\infty} e^{\frac{0}{\ln(\log x)}} = e^0 = 1$$

$$\underset{x \rightarrow +\infty}{\underset{\substack{\nearrow \\ +\infty^0}}{\Rightarrow}} \lim_{x \rightarrow +\infty} \frac{1}{x} \log(\log x) = \lim_{x \rightarrow +\infty} \frac{\log(\log x)}{x} =$$

Cauchy $\frac{\infty}{\infty}$ ou $\frac{0}{0}$

$$= \lim_{x \rightarrow +\infty} \frac{\frac{1}{\log x} \cdot \frac{1}{x}}{1} = \lim_{x \rightarrow +\infty} \frac{1}{x \log x} = \frac{1}{+\infty} = 0$$

15.

$$a) \lim_{x \rightarrow +\infty} \frac{\sin \frac{1}{x}}{x} = \lim_{x \rightarrow +\infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} =$$

$$\text{R. Cauchy. } \frac{\frac{1}{x^2} \cdot (-\cos \frac{1}{x})}{-\frac{1}{x^2}} = \lim_{x \rightarrow +\infty} -\frac{\cos \frac{1}{x}}{x} = -1$$

b)

$$\lim_{x \rightarrow \frac{\pi}{2}} (\tan x)^{\frac{\tan 2x}{\tan x}} = \lim_{x \rightarrow \frac{\pi}{2}} e^{\frac{\log \tan x}{\frac{1}{\tan 2x}}} =$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} e^{\frac{\tan 2x \cdot \log \tan x}{\tan x}}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \tan 2x \cdot \log \tan x = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\log \tan x}{\frac{1}{\tan 2x}} \text{ R. Cauchy}$$

$$= \frac{(\log \tan x)'}{\frac{1}{\tan 2x}} = \frac{-\frac{1}{\tan x} \cdot \frac{1}{\cos^2 x}}{\frac{1}{\tan 2x}} = \frac{\frac{1}{\tan x} \cdot \frac{1}{\cos^2 x}}{\frac{2}{\sin 2x}} = \frac{\frac{1}{\sin x \cos x}}{\frac{2}{\sin 2x}} = \frac{1}{2}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\sin x \cos x}$$

$$16. \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right)^{\sin \frac{1}{n}}$$

$$\lim_{n \rightarrow \infty} x^{\sin x} = \lim_{n \rightarrow \infty} e^{\log x^{\sin x}} = \lim_{n \rightarrow \infty} e^{\sin x \log x}$$

D.

$$\lim_{n \rightarrow \infty} \frac{\sin x \log x}{1} \stackrel{n \rightarrow \infty}{\text{Cauchy}}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{-\cos x}{\sin x^2}} = \lim_{n \rightarrow \infty} \frac{\sin x^2}{n \cos x} \stackrel{n \rightarrow \infty}{\text{Cauchy}}$$

$$\lim_{n \rightarrow \infty} \frac{-1 - \cos x^2}{n \cos x} = \lim_{n \rightarrow \infty} \frac{-1}{n \cos x} - \frac{\cos x}{n}$$

$$= -\frac{1}{0^+} - \frac{1}{1}$$

17.

a) Andem 2, pentru 1

$$f(x) = e^{2x} \quad f'(x) = 2e^{2x} \quad f''(x) = 2 \cdot 2 \cdot e^{2x} = 4e^{2x}$$

MCV

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$$

$$= e^0 + 2e^0 x + \frac{4e^0 x^2}{2} = 1 + 2x + \frac{4x^2}{2} = 2x^2 + 2x + 1$$

AN

$$f(x) = e^x + 2e^x x + \frac{4e^x x^2}{2} = 2e^x x^2 + 2e^x x + e = e(2x^2 + 2x + 1)$$

$$f(u) = \log(1+u) \quad f'(u) = \frac{1}{1+u} \quad f''(u) = \frac{1}{(1+u)^2}$$

Naylor's formula: $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$

$$= \log 1 + \frac{1}{1}x + \frac{1}{2}x^2$$

$$= x + \frac{1}{2}x^2$$

Taylor: $F(u) = \log 2 + \frac{1}{1+1}x + \frac{1}{2!}x^2 = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2$

$$f(u) = \cos(\pi u) \quad f'(u) = -\pi \sin(\pi u) \quad f''(u) = -2\pi \cos(\pi u)$$

Naylor's formula: $F(u) = \cos 0 - \pi \sin 0 \cdot u - \frac{2\pi \cos 0 \cdot u^2}{2} = 1 - \pi u^2$

Taylor: $F(u) = \cos \pi + (-\pi \sin \pi)u + \frac{(-2\pi \cos \pi)u^2}{2!} = -1 + \pi u^2$

b)

18.

$$19. \forall x \in \mathbb{R} \exists \xi_x \text{ entre } 0 \text{ et } x$$

$$\text{comme } x \in [0, 1] \quad e^{\xi_x} \leq 1 \quad \frac{x^3}{6} < e^{\xi_x} \frac{1}{6} < \frac{1}{6}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(\xi_x)x^3}{3!}$$

$$f(u) = e^{-u} \quad \forall x \in \mathbb{R} \quad e^{-u} = \sqrt{1-u + \frac{u^2}{2}} - e^{-\xi_x} \cdot \frac{u^3}{3!}$$

$$f'(u) = -e^{-u} \quad \text{comme } u \in [0, 1] \quad \frac{u^3}{3!}$$

$$f''(u) = +e^{-u} \quad 0 < \xi_x < u < 1 \quad > 0$$

$$f'''(u) = -e^{-u} \quad \left| e^{-u} - \left(1-u + \frac{u^2}{2} \right) \right| = \left| -e^{-\xi_x} \cdot \frac{u^3}{3!} \right| = e^{-\xi_x} \frac{u^3}{6}$$

tu m'as - me ?!

20.

$$P_2(x) = g(0) + \frac{g'(0)x}{1!} + \frac{g''(0)x^2}{2!}$$

$$g(0) = f(e^0) = f(1) = 2$$

$$f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} = 2 - x + 2(x-1)^2$$
$$= 2 - x + 2(x-1)^2$$

$$g'(x) = e^x f'(e^x) \Rightarrow g'(0) = e^0 f'(e^0) = f'(1) = -1$$

$$g''(x) = e^x f''(e^x) \cdot e^x + e^x \cdot f'(e^x)$$

$$g''(0) = e^0 f''(e^0) \cdot e^0 + e^0 \cdot f'(e^0)$$

$$= 1 \cdot f''(1) \cdot 1 + 1 \cdot f'(1)$$

$$= (-1) + 9 = 8$$

$$P_2(x) = 2 + (-1) \cdot x + \frac{8}{2!} x^2$$
$$= 2 - x + \frac{8}{2} x^2$$

21. Se $f: \mathbb{R} \rightarrow \mathbb{R}$ e $f^n(x) = 0 \quad \forall x \in \mathbb{R}$
então f tem grau inferior a n

$$P_n(x) = f(0) + \dots + \frac{f^n(0) x^n}{n!}$$

22. $I \in \mathbb{R}$ $\exists a, b \in$

$f \in C^2(I)$

$\forall x \in I$

$$f''(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + R_3(x) \quad (\Rightarrow)$$

$$\Leftrightarrow f''(a)(x-a)^2 = -f(a) - f'(a)(x-a) + f(x)$$

$$\Leftrightarrow f''(a)(x-a)^2 = 2(-f(a) - f'(a)(x-a) + f(x))$$

$$\Leftrightarrow f''(a) = \frac{2(-f(a) - f'(a)(x-a) + f(x))}{(x-a)^2}$$

$$\Leftrightarrow f''(a) = \frac{-2f(a) - f'(a).h + f(h+a)}{h^2} \quad \begin{matrix} x-a=h \\ \Leftrightarrow x=h+a \end{matrix}$$

$$\Leftrightarrow f''(a) = \frac{f(h+a) - 2f(a) - f'(a).h}{h^2} \quad \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

23.

$$g(x) = xf(x) \quad g'(x) = f(x) + xf'(x)$$

$$g''(x) = f'(x) + f'(x) + x \cdot f''(x) = 2f'(x) + xf''(x)$$

nesunte $g''(0) = 0$

$$P_1(x) = g(0) + g'(0)x = f(0)x$$

$$24. f(x) = \operatorname{arctg}(x^2)$$

$$f'(x) = \frac{2x}{1-x^4} \quad \forall x \neq \{1, -1\}$$

$$\mathbb{R} \rightarrow [0, \frac{\pi}{4}]$$

f'	-1	0	1
$1-x^4$	-	0+	+0-

f''	-	0	+	+	+
$2x/1-x^4$	+	NE	-	0+	NE-

$f(x)$ não tem derivada em $x = \{-1, 1\}$

$$\begin{array}{c|ccccc}
2x & - & - & 0 & + & + \\
\hline
2x/1-x^4 & + & NE & - & 0 & + NE -
\end{array}$$

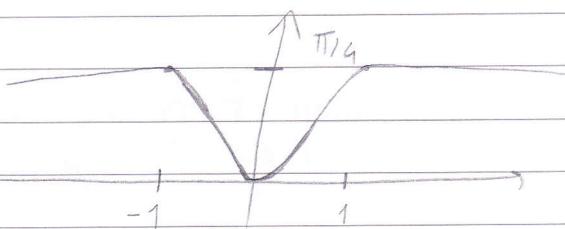
crescente em $]-\infty, -1[$

tem máximo em $x = -1$ $f(-1) = \frac{\pi}{4}$ $\operatorname{arctg}(-1^2) = \operatorname{arctg}1 = \frac{\pi}{4}$
decrescente de $]-1, 0[$

tem um mínimo em $x = 0$

crescente em $[0, 1[$

tem máximo em $x = 1$ $f(1) = \frac{\pi}{4}$
decrescente em $]-1, +\infty[$



$$f''(x) = \frac{2(1-x^4)-2x(-4x^3)}{(1-x^4)^2}$$

$$= \frac{2-2x^4+8x^4}{(1-x^4)^2} = \frac{6x^4+2}{(1-x^4)^2}$$

$$f''(x) > 0$$

$f(x)$ é convexa

não tem pontos de inflexão

$$25. f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = x^4 e^{-x}$$

$$\begin{aligned} f'(x) &= 4x^3 \cdot e^{-x} + x^4 \cdot (-e^{-x}) \\ &= 4x^3 e^{-x} - x^4 e^{-x} \\ &= e^{-x}(4x^3 - x^4) \\ &= e^{-x} \cdot x^3(4-x) \end{aligned}$$

$e^{-x} \cdot x^3$	-	0	+	+	+
$4-x$	+	+	+	0	-

$$f'(x) = 0, x = 0, x = 4$$

$4-x$	+	+	+	0	-
0	-	0	+	0	=

$f(x)$ decrescente em $]-\infty, 0[$

mínimo $x = 0$ $f(x) = 0$

crescente em $[0, 4[$

máximo $x = 4$ $f(x) = 16 \cdot e^{-4}$

decrescente em $]4, +\infty[$

$$\lim_{n \rightarrow -\infty} n^4 e^{-n} = \lim_{n \rightarrow -\infty} +\infty$$

$$\lim_{n \rightarrow +\infty} n^4 e^{-n} = \lim_{n \rightarrow +\infty} \frac{e^{-n}}{\frac{1}{n^4}}$$

R. Cauchy

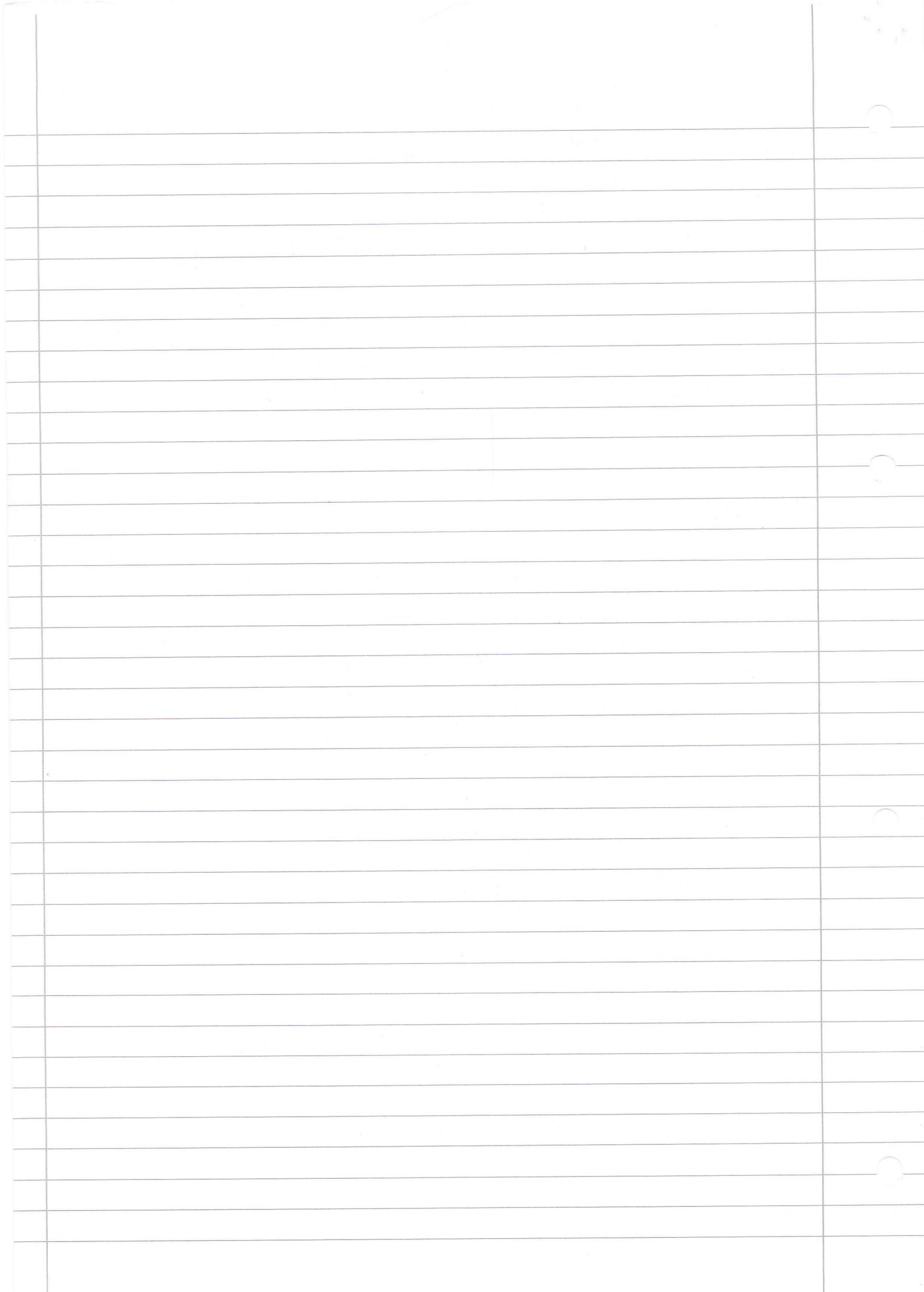
$$= \lim_{n \rightarrow +\infty} \frac{-e^{-n}}{\frac{3n}{n^8}} = \lim_{n \rightarrow +\infty} \frac{-e^{-n} \cdot n^8}{3n}$$

$\approx \lim_{n \rightarrow +\infty}$

26. $f(u) = \frac{\sin u}{1 - \sin u}$

$$f'(u) = \frac{\cos u (1 - \sin u) - \sin u (-\cos u)}{(1 - \sin u)^2}$$

$$= \frac{\cos u - \cos u \sin u + \sin u \cos u}{(1 - \sin u)^2}$$



$$(\operatorname{sen} x)^2 = 2 \operatorname{sen} x \cdot \cos x$$

ficha 10

1:

$$\text{a) } P_{2x^2 + 3x^3} = 2P_{2x^2} + 3P_{3x^3} = 2\frac{x^3}{3} + 3\frac{x^4}{9}$$

$$\text{b) } \frac{x^{-1/2} + 1}{-1/2 + 1} + \log|x| + \frac{x^{-2+1}}{-2+1} = 2\sqrt{x} + \log|x| - \frac{1}{x}$$

$$\text{c) } P\left(\frac{x^2 - x + 1}{1 + \sqrt{x}}\right) = P\left(x\sqrt{x} - \sqrt{x} + \frac{1}{\sqrt{x}}\right) =$$
$$= \frac{x^{3/2+1}}{3/2+1} - \frac{x^{1/2+1}}{1/2+1} + \frac{x^{-1/2+1}}{-1/2+2}$$

$$\text{d) } P^3 \sqrt[3]{1-x} = P(1-x)^{1/3} = (1-x)^{1/3+1} = -(1-x)^{4/3}$$

$$\text{e) } P^3 \sqrt[3]{x^2 + \sqrt{x^3}} = P(x^2)^{1/3} + (x^3)^{1/2} = P \frac{x^{2/3}}{x} + \frac{x^{3/2}}{x} = (x^{1/3} + x^{1/2})$$
$$= P(x^{-1/3} + x^{1/2}) = \frac{3}{2} \cdot x^{2/3} + \frac{2}{3} x^{3/2}$$

$$\text{f) } P_{2x^5 \sqrt[5]{x^2-1}} = P(x^2-1)^1 (x^2-1)^{1/5} = (x^2-1)^{1/5+1}$$
$$= -\frac{5}{6} \sqrt[5]{(x^2-1)^6} = -\frac{5}{6} (x^2-1) \sqrt[5]{(x^2-1)}$$

$$\text{g) } \frac{1}{4} P \frac{4x^3}{3+x^4} = \frac{1}{4} \log|3+x^4| = \frac{1}{4} \log(3+x^4)$$

$$\text{h) } \frac{1}{2} P \frac{2e^x}{1+2e^x} = \frac{1}{2} \log|1+2e^x|$$

$$\text{i) } P \frac{\cos x}{1+\operatorname{sen} x} = \log|1+\operatorname{sen} x|$$

$$\text{j) } P \operatorname{sen} 2x = -\frac{\cos 2x}{2}$$

$$k) P \frac{\sin 2x}{1 + \sin^2 x} = P - \frac{2 \sin x \cos x}{1 + \sin^2 x} = \log(1 + \sin^2 x)$$

$$l) P \cos^2 x = \frac{\cos^2 x}{-\frac{1}{3} \sin x}$$

$$m) P \frac{1}{\cos^2 x} = P \frac{1}{\frac{1 + \cos 2x}{2}} = P \frac{2}{1 + \cos 2x}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$n) P \frac{e^{\tan x}}{\cos^3 x} =$$

$$o) P \frac{2x \cos(x^2 + 2)}{x^2} = \frac{\sin(x^2 + 2)}{2}$$

$$p) P e^x \sin(e^x) = -\cos(e^x)$$

$$q) P \sqrt[3]{1+x^3} = P x^2 (1+x^3)^{1/3} = \frac{(1+x^3)^{1/3+1}}{1/3+1} =$$

$$= (1+x^3)^{4/3} = 3(1+x^3)^{4/3}$$

$$r) P \frac{e^x}{(1+e^x)^2} = \frac{\log((1+e^x)^2)}{2}$$

$$s) P \frac{\sin x}{1 + \cos^2 x} = \arctan(\cos x) + C$$

$$t) \frac{1}{2} P \frac{2 \cdot 1}{\sqrt{1-x^2}} = \arccos \text{sem}(2x)$$

$$u) P \frac{x+1}{\sqrt{1-x^2}} = P \frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} = P \frac{x}{\sqrt{1-x^2}} + (\arccos x)' \quad \nearrow \arccos x$$

$$\frac{1}{2} P \cdot 2x \cdot (1-x^2)^{-1/2} + (\arccos x)' = \frac{(1-x^2)^{1/2} + \arccos x}{1/2} =$$

$$= \sqrt{1-x^2} + \arccos x$$

$$\sqrt{1} P \frac{6x^3}{6(1+x^4)^2} = \frac{\log(1+x^4)^2}{6}$$

$$(w) P \cos^3 x \sqrt{\text{sem} x} = P \cos^3 x (\text{sem} x)^{1/2}$$

$$x) P \operatorname{tg}^2 x = \frac{\operatorname{tg} x^3}{2}$$

$$2. K') P \frac{\cos x}{1+\text{sem}^2 x} = \arctg(\text{sem} x)$$

$$a) P \overset{2x}{(x^2+1)^3} = \frac{(x^2+1)^4}{4}$$

$$b) P e^{x+3} = P e^x \cdot e^{x+2} =$$

$$c) P 2^{x-1} =$$

$$d) P \frac{1}{\sqrt[5]{1-2x}} = P \frac{1}{(1-2x)^{1/5}} = P -2(1-2x)^{-1/5} = \frac{(1-2x)^{-1/5+1}}{-1/5+1} = \frac{1}{2}$$

$$= \frac{(1-2x)^{4/5}}{4/5} \cdot \left(-\frac{1}{2}\right) = \frac{5(1-2x)^{4/5}}{-8}$$

$$e) P \frac{2x}{1+x^2} = \frac{\log|1+x^2|}{2}$$

$$f) P \frac{x^3}{x^8+1}$$

$$g) P \cotgx = P \frac{\cos x}{\sin x} = \log|\operatorname{sepx}|$$

$$h) P \frac{\sin^2 x}{\operatorname{sepx}} = P \frac{\sin^2 x}{\cancel{2\sin x}} \cdot \cos x \\ = P \frac{\sin^2 x}{2} \cdot (\operatorname{sepx})^2$$

$$i) P \frac{\operatorname{tg} \sqrt{x}}{\sqrt{x}} = P \operatorname{tg}(x^{1/2}) \cdot x^{-1/2} = P \frac{\operatorname{sepx}^{1/2}}{\cos(x^{1/2})} \cdot x^{-1/2} \\ = \log|\operatorname{cas}\sqrt{x}|$$

$$j) P \frac{e^x}{\sqrt{1-e^{2x}}} = -\operatorname{arcos} e^x$$

$$k) P \frac{x}{(1+x^2)^\alpha} = P 2x \cdot (1+x^2)^{-\alpha} = \frac{(1+x^2)^{-\alpha+1}}{2 \cdot (-\alpha+1)}$$

$$\begin{aligned}
 l) P \cos x \cos 2x &= P \cos x (\cos^2 x + \sin^2 x) = \\
 &= P \cos x \cdot \cos^2 x - \cos x \sin x \cdot \sin x = P \cos x (1 - \sin^2 x) - \cos x \\
 &= P \cos x - \cos x \sin^2 x - \cos x \sin^2 x = \\
 &= P \cos x - P \cos x \sin^2 x - P \cos x \sin^2 x = \\
 &= \sin x - \frac{2 \sin^3 x}{3}
 \end{aligned}$$

$$\begin{aligned}
 m) P \sin^3 x \cos^4 x &= P \sin x (1 - \cos^2 x) \cos^4 x = \\
 &= P(\sin x - \cos^2 x \sin x) \cos^4 x = P \sin x \cos^4 x - \cos^6 x \sin x \\
 &= -\frac{\cos^5}{5} + \frac{\cos^7}{7}
 \end{aligned}$$

$$n) P \tan^3 x + \tan^4 x = P \left(\frac{\sin x}{\cos x} \right)^3 + \left(\frac{\sin x}{\cos x} \right)^4$$

$$= P \frac{\sin^3 x}{\cos^3 x} + \frac{\sin^4 x}{\cos^4 x} = P \frac{\sin x \cdot (1 - \cos^2 x)}{\cos^3 x} +$$

$$\frac{\sin^2 x (1 - \cos^2 x)}{\cos^4 x}$$

$$3. a) P \sqrt{2x} + \sqrt{\frac{x}{2}} = 2P(x)^{1/2} + \left(\frac{x}{2}\right)^{1/2} =$$

$$= \frac{4}{3} x^{3/2} + \frac{1}{2} \cdot \frac{2}{3} x^{3/2} = \frac{4}{3} x^{3/2} + \frac{1}{3} x^{3/2}$$

$$x^a =$$

$$3b) P \sin x + 2x^2 = 3P \sin x + \frac{2}{3} P_3 x^2$$

$$= 3 \cos x + \frac{2}{3} x^3$$

$$c) \frac{1}{3} P \frac{3x^2}{1+x^3} = \frac{\log|1+x^3|}{3}$$

$$d) \frac{1}{2} P_2 x e^{-x^2} = \frac{e^{-x^2}}{2}$$

$$e) P \frac{3 \sin x}{(1+\cos x)^2}$$

$$f) P x \sqrt{1+x^2} = \frac{1}{2} P_2 x (1+x^2)^{1/2} = \frac{(1+x^2)^{1/2+1}}{1/2+1} \cdot \frac{1}{2}$$

$$g) \frac{1}{2} P e^{2 \sin x} \frac{2 \cos x}{2 \cos x} = \frac{e^{2 \sin x}}{2}$$

$$h) P \frac{1}{1+e^x} =$$

$$i) P \tan x = P \frac{\sin x}{\cos x} = -\log|\cos x|$$

$$j) P \frac{1}{1+x^2} =$$

$$\textcircled{k}) P \operatorname{tg} x \cdot \sec^3 x = P \cdot \frac{\operatorname{sen} x}{\cos x} \cdot \frac{1}{\cos^3 x} = \\ = P \frac{\operatorname{sen} x}{\cos x} \cdot \frac{1}{\cos x (1 - \operatorname{sen}^2 x)} =$$

$$\textcircled{l}) P \cos^3 x \cdot \operatorname{sen}^3 x = P \cos^3 x \cdot \operatorname{sen} x (1 - \cos^2 x) \\ = P \operatorname{sen} x \cos^3 x - \operatorname{sen} x \cos x = -\frac{\cos x^4}{4} + \cos x$$

$$\textcircled{m}) P \frac{1}{(1+n^2) \operatorname{arctg} x} = P \left(\frac{1}{1+n^2} \cdot \frac{1}{\operatorname{arctg} x} \right) = \log \left| \operatorname{arctg} x \right|$$

$$\textcircled{n}) P \frac{x}{1+n^2} = \frac{\operatorname{arctg}(x^2)}{2}$$

$$\textcircled{o}) P \frac{1}{\sqrt{n}(1+x)} = P \frac{1}{n^{1/2}(1+n)}$$



5.

a) $\frac{1}{2} \int 2x \sin(x^2) dx = -\frac{1}{2} \cos x^2 + C \quad C \in \mathbb{R}$

$f(c) = 0$

$$-\frac{1}{2} \cdot \cos x^2 + C = 0 \Leftrightarrow C = \frac{1}{2}$$

b) $\lim_{x \rightarrow +\infty} f(x) = 0 \quad \text{NE}$
 $\cos x$ periódico

a) $P \frac{e^x}{2+e^x} = \log|2+e^x| + C$

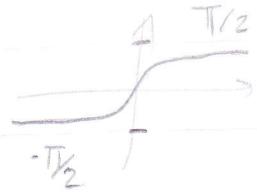
$$\log|2+e^0| + C = 0 \Leftrightarrow C = \log 3$$

b) $\lim_{x \rightarrow +\infty} f(x) = 0 \quad \text{NE}$

a) $P \frac{1}{(1+x^2)(1+\arctg^2 x)} = P \frac{1}{1+x^2} \cdot \frac{1}{1+\arctg^2 x} =$
 $= \arctg(\arctg x) + C$

$$\arctg(\arctg 0) + C = 0 \Leftrightarrow 0 + C = 0 \Leftrightarrow C = 0$$

b) $\lim_{x \rightarrow +\infty} f(x) = \arctg(\arctg x) = \lim_{y \rightarrow \pi/2} \arctg y = \arctg \frac{\pi}{2} \neq 0$
 $x = \arctg \frac{\pi}{2} \Leftrightarrow \tan x = \frac{\pi}{2}$



$$\lim_{x \rightarrow \infty} f(x) = 0 \Leftrightarrow \lim_{x \rightarrow \infty} \operatorname{arctg} \frac{\pi}{2} + c = 0 \Leftrightarrow$$

$$\Leftrightarrow c = -\operatorname{arctg} \frac{\pi}{2}$$