CÁLCULO DIFERENCIAL E INTEGRAL III

SISTEMAS DE EQUAÇÕES DIFERENCIAIS

EXERCÍCIOS

1. Considere as seguintes matrizes

$$\text{i)}\quad A=\begin{bmatrix} 0&0&0&1&0\\ 0&0&0&1&0\\ 0&0&0&1&0\\ 0&0&0&1&0\\ 0&0&0&1&0 \end{bmatrix};\quad \text{ii)}\quad A=\begin{bmatrix} 2&0&0\\ 0&3&0\\ 0&1&3 \end{bmatrix};\quad \text{iii)}\quad A=\begin{bmatrix} 2&-2&2\\ 0&0&1\\ 0&0&1 \end{bmatrix}.$$

Para cada uma das matrizes A acima indicadas, responda às seguintes questões:

(a) Calcule A^2, A^3 e A^4.

(b) Conjecture e demonstre uma fórmula para A^n, com $n \in \mathbb{N}$.

(c) Calcule e^{At} pela definição.

2. Explicite e^{At} para cada uma das matrizes.

(a) $A=\begin{bmatrix} 2&0&0\\ 0&-1&0\\ 0&0&2 \end{bmatrix}$ $\begin{bmatrix} 2-\lambda & & \\ & 1-\lambda & \\ & & 2-\lambda \end{bmatrix}$

(b) $A=\begin{bmatrix} 0&0&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{bmatrix}$

(c) $A=\begin{bmatrix} 3&0&0\\ 1&3&0\\ 0&1&3 \end{bmatrix}$

(d) $A=\begin{bmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 0&1&1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&1&-1&0\\ 0&0&0&0&0&\frac{1}{2} \end{bmatrix}$

(e) $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

3. Relativamente à matriz

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

calcule e^{At} e resolva o problema de valor inicial $X' = AX, \quad X(0) = [1 \;\; 2]^T$.

4. Com respeito a cada uma das matrizes A a seguir indicadas, calcule e^{At} e resolva o problema de valor inicial $X' = AX$, $X(0) = [1 \;\; 1 \;\; 1]^T$.

(a) $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

5. Seja

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

Determine e^{At} e resolva o problema de valor inicial $X' = AX$, $X(\pi/2) = [1 \;\; 0 \;\; 1]^T$.

6. Para a matriz

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

calcule e^{At} e resolva o problema de valor inicial $X' = AX$, $X(1) = [1 \;\; 0 \;\; 1]^T$.

7. Considere as seguintes matrizes:

i) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$; ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$; iii) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

Para cada uma delas, calcule e^{At} e resolva o problema de valor inicial $X' = AX$, $X(0) = [1 \;\; 0 \;\; 1]^T$.

8. Resolva o problema de valor inicial

$$\begin{cases} x' = -2x + 4y + t^6 \\ y' = -x + 2y + t^5 \\ x(0) = y(0) = 1 \end{cases}$$

RESPOSTAS

1. i)

(a) $A^2 = A^3 = A^4 = A.$

(b) $A^n = A, \quad n \in \mathbb{N}.$

(c) $e^{At} = I + (e^t - 1)\, A.$

ii)

(a) $A^2 = \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 2 \cdot 3 & 3^2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 2^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 3 \cdot 3^2 & 3^3 \end{bmatrix},$

$A^4 = \begin{bmatrix} 2^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 4 \cdot 3^3 & 3^4 \end{bmatrix}.$

(b) $A^n = \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & n \cdot 3^{n-1} & 3^n \end{bmatrix}.$

(c) $e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & t\, e^{3t} & e^{3t} \end{bmatrix}.$

iii)

(a) $A^2 = \begin{bmatrix} 2^2 & -2^2 & 2^2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 2^3 & -2^3 & 2^3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 2^4 & -2^4 & 2^4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$

(b) $A^n = \begin{bmatrix} 2^n & -2^n & 2^n \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$

(c) $e^{At} = \begin{bmatrix} e^{2t} & 1 - e^{2t} & e^{2t} - 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}.$

2. **(a)** $e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$

(b) $$e^{At} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 \\ \frac{t^2}{2!} & t & 1 & 0 & 0 \\ \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 & 0 \\ \frac{t^4}{4!} & \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 \end{bmatrix}$$

(c) $$e^{At} = \begin{bmatrix} e^{3t} & 0 & 0 \\ te^{3t} & e^{3t} & 0 \\ \frac{t^2}{2}e^{3t} & te^{3t} & e^{3t} \end{bmatrix}$$

(d) $$e^{At} = \begin{bmatrix} e^t & 0 & 0 & 0 & 0 & 0 \\ te^t & e^t & 0 & 0 & 0 & 0 \\ \frac{t^2}{2}e^t & te^t & e^t & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & t\,e^{-t} & e^{-t} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{t/2} \end{bmatrix}$$

(e) $$A = \begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

3. $$e^{At} = \begin{bmatrix} \dfrac{e^{3t}+e^{-t}}{2} & \dfrac{e^{3t}-e^{-t}}{4} \\ e^{3t}-e^{-t} & \dfrac{e^{3t}+e^{-t}}{2} \end{bmatrix}, \quad X(t) = \begin{bmatrix} e^{3t} \\ 2\,e^{3t} \end{bmatrix}.$$

4. (a) $$e^{At} = \begin{bmatrix} e^t & \dfrac{e^{3t}-e^t}{2} & \dfrac{e^t-e^{3t}}{2} \\ 0 & e^{3t} & e^{2t}-e^{3t} \\ 0 & 0 & e^{2t} \end{bmatrix}, \quad X(t) = \begin{bmatrix} e^t \\ e^{2t} \\ e^{2t} \end{bmatrix}.$$

(b) $$e^{At} = \begin{bmatrix} \dfrac{e^t}{6}+\dfrac{e^{-2t}}{3}+\dfrac{e^{3t}}{2} & -\dfrac{e^t}{3}+\dfrac{e^{-2t}}{3} & \dfrac{e^t}{2}-e^{-2t}+\dfrac{e^{3t}}{2} \\ -\dfrac{2e^t}{3}-\dfrac{e^{-2t}}{3}+e^{3t} & \dfrac{4e^t}{3}-\dfrac{e^{-2t}}{3} & -2e^t+e^{-2t}+e^{3t} \\ -\dfrac{e^t}{6}-\dfrac{e^{-2t}}{3}+\dfrac{e^{3t}}{2} & \dfrac{e^t}{3}-\dfrac{e^{-2t}}{3} & -\dfrac{e^t}{2}+e^{-2t}+\dfrac{e^{3t}}{2} \end{bmatrix},$$

$$X(t)=\begin{bmatrix} \dfrac{e^{t}}{3}-\dfrac{e^{-2t}}{3}+e^{3t} \\[2ex] -\dfrac{4e^{t}}{3}+\dfrac{e^{-2t}}{3}+2e^{3t} \\[2ex] -\dfrac{e^{t}}{3}+\dfrac{e^{-2t}}{3}+e^{3t} \end{bmatrix}$$

5. $$e^{At}=\begin{bmatrix} e^{-2t} & 0 & 0 \\[1ex] 0 & e^{-2t}\cos t & e^{-2t}\operatorname{sen} t \\[1ex] 0 & -e^{-2t}\operatorname{sen} t & e^{-2t}\cos t \end{bmatrix}, \quad X(t)=\begin{bmatrix} e^{\pi-2t} \\[1ex] -e^{\pi-2t}\cos t \\[1ex] e^{\pi-2t}\operatorname{sen} t \end{bmatrix}.$$

6. $$e^{At}=\begin{bmatrix} e^{2t} & e^{2t}-e^{t} & 0 \\[1ex] 0 & e^{t} & 0 \\[1ex] 0 & 2\,e^{t}-2\,e^{2t} & e^{2t} \end{bmatrix}, \quad X(t)=e^{2(t-1)}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

7. i) $$e^{At}=\begin{bmatrix} e^{2t} & t\,e^{2t} & 0 \\[1ex] 0 & e^{2t} & 0 \\[1ex] e^{2t}-e^{3t} & e^{2t}+t\,e^{2t}-e^{3t} & e^{3t} \end{bmatrix}, \quad X(t)=\begin{bmatrix} e^{2t} \\[1ex] 0 \\[1ex] e^{2t} \end{bmatrix}.$$

ii) $$e^{At}=\begin{bmatrix} (1-t)\,e^{2t} & t\,e^{2t} & t\,e^{2t} \\[1ex] 0 & e^{2t} & 0 \\[1ex] -t\,e^{2t} & t\,e^{2t} & (1+t)\,e^{2t} \end{bmatrix}, \quad X(t)=\begin{bmatrix} e^{2t} \\[1ex] 0 \\[1ex] e^{2t} \end{bmatrix}.$$

iii) $$e^{At}=\begin{bmatrix} e^{2t} & \dfrac{t^2\,e^{2t}}{2} & \left(t-\dfrac{t^2}{2}\right)e^{2t} \\[1ex] 0 & (1+t)\,e^{2t} & -t\,e^{2t} \\[1ex] 0 & t\,e^{2t} & (1-t)\,e^{2t} \end{bmatrix}, \quad X(t)=\begin{bmatrix} \left(1+t-\dfrac{t^2}{2}\right)e^{2t} \\[1ex] -t\,e^{2t} \\[1ex] (1-t)\,e^{2t} \end{bmatrix}.$$

8. $$X(t)=\begin{bmatrix} -\dfrac{t^8}{28}+\dfrac{5t^7}{21}+2t+1 \\[2ex] -\dfrac{t^8}{56}+\dfrac{t^7}{21}+\dfrac{t^6}{6}+t+1 \end{bmatrix}.$$