Optimization and Algorithms
February 22, 2022

Write your name: ______________________________

Write your student number: ______________________________

Exam

Write your answers (A, B, C, D, E, or F) to problems 1 to 3 in this box
Your answer to problem 1: ________
Your answer to problem 2: ________
Your answer to problem 3: ________

1. *Simple convex function.* (3 points) One of the following six functions $\mathbf{R} \to \mathbf{R}$ is convex:

(A) $(1-(x-1)_+)_+$

(B) $|(x-1)_+ - 1|$

(C) -$(1-(x-1)_+)_+$

(D) $((x-1)_+ - 1)_+$

(E) -$((x-1)_+ - 1)_+$

(F) -$|(x-1)_+ - 1|$

Which one?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

2. *Least-squares.* (2 points) Consider the following six optimization problems:

(A)

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax-b\|_1^2 + \rho\,\|x\|_2^2$$

(B)

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax-b\|_2^2 + \rho\,\|x\|_2^2$$

(C)

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax-b\|_2^2 + \rho\,\|x\|_1^2$$

(D)

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax-b\|_2 + \rho\,\|x\|_2^2$$

(E)

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax-b\|_1^2 + \rho\,\|x\|_1^2$$

(F)

$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax-b\|_1 + \rho\,\|x\|_2$$

In each of the six problems above, the variable to optimize is $x \in \mathbf{R}^n$. The matrix A and the vector b are given. The scalar $\rho > 0$ is also given.

One of the optimization problems above is a least-squares problem.

Which one?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

3. *Convex function.* (3 points) Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a convex function. One of the following functions is guaranteed to be convex:

(A) $|f(x)|$
(B) $f(x) + (f(x))^2$
(C) $(f(x))^2$
(D) $f(x)(f(x))^2$
(E) $|f(x)| + (f(x))^2$
(F) $f(x) + |f(x)|$

Which one?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

4. *Robust portfolio selection.* (4 points) A problem that often occurs in finance has the following form

$$\begin{array}{ll} \underset{x\in\mathbf{R}^n}{\text{minimize}} & \max\{x^T V_1^T D V_1 x, x^T V_2^T D V_2 x\} \\ \text{subject to} & \min\{\mu_1^T x, \mu_2^T x\} \geq \alpha \\ & \mathbf{1}^T x = 1, \end{array} \tag{1}$$

where the variable to optimize is $x \in \mathbf{R}^n$.

The matrices $V_1 \in \mathbf{R}^{p\times n}$, $V_2 \in \mathbf{R}^{p\times n}$, and $D \in \mathbf{R}^{p\times p}$ are given, the matrix D being diagonal with positive entries in the diagonal:

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_p \end{bmatrix},$$

with $d_i > 0$ for $i = 1, \ldots, p$.

The vectors $\mu_1 \in \mathbf{R}^n$, $\mu_2 \in \mathbf{R}^n$ and the scalar $\alpha \in \mathbf{R}$ are given. Finally, recall that the symbol $\mathbf{1}$ stands for the vector of dimension n with all components equal to one:

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Show that the optimization problem (1) is convex.

5. *Mahalanobis projection.* (4 points) Consider the optimization problem

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (x-\mu)^T\Sigma^{-1}(x-\mu) \\ \text{subject to} & a^Tx = b, \end{array} \tag{2}$$

where the variable to optimize is $x \in \mathbf{R}^n$. The vector $\mu \in \mathbf{R}^n$ and the matrix $\Sigma \in \mathbf{R}^{n\times n}$ are given, with Σ being symmetric and positive definite.

Show that the optimal value of problem (2) is

$$\frac{(a^T\mu - b)^2}{a^T\Sigma a}.$$

6. *Strictly convex functions.* (4 points) Suppose that the functions $f_1 \colon \mathbf{R}^n \to \mathbf{R}$ and $f_2 \colon \mathbf{R}^n \to \mathbf{R}$ are both convex, and let $f \colon \mathbf{R}^n \to \mathbf{R}$ be defined as $f(x) = \max\{f_1(x), f_2(x)\}$ for each $x \in \mathbf{R}^n$. Is the function f strictly convex? If you think the answer is 'yes', then prove it; if you think the answer is 'no', then give a counter-example.