matemática discreta
2° teste

18
19

TESTE MODELO
TESTE 2019

GRUPO 1

DERIVADA FINITA DE UMA SUCESSÃO

$$u_k' = u_{k+1} - u_k$$

Nota:

$$\sum_{k=0}^{n-1} a = a \times ((n-1) + 0 + 1)$$

TEOREMA FUNDAMENTAL DO CÁLCULO FINITO

$$\sum_{k=p}^{n} (u_k)' = [\, u_k \,]_p^{n+1} = u_{n+1} - u_p$$

1 (A) Começando por calcular a derivada finita da sucessão de termo geral $u_k = \frac{2^k}{k}$, e recorrendo depois ao teorema fundamental do cálculo finito, obtenha uma forma fechada para:

$$\sum_{k=1}^{n-1} \frac{(k-1)2^k}{k^2+k}$$

DERIVADA FINITA:

$$\left(\frac{2^k}{k}\right)' = \frac{2^{k+1}}{k+1} - \frac{2^k}{k} = \frac{k2^{k+1} - (k+1)2^k}{k(k+1)} = \frac{2^k(2k-k-1)}{k^2+k} = \frac{(k-1)2^k}{k^2+k}$$

TFCF:

$$\sum_{k=1}^{n-1} \frac{(k-1)2^k}{k^2+k} = \sum_{k=1}^{n-1} u_k' = [u_k]_1^n = \left[\frac{2^k}{k}\right]_1^n = \frac{2^n}{n} - \frac{2^1}{1} = \frac{2^n}{n} - 2$$

FORMAS FECHADAS POLINÓMICAS

$u_k^{\underline{R}}$, $R \in \mathbb{N} \rightarrow u_k^{\underline{0}} = 1$, $u_k^{\underline{R}} = u_k \times u_{k-1} \times u_{k-2} \times \ldots \times u_{k-(R-1)}$

$$(k^{\underline{R}})' = Rk^{\underline{(R-1)}}$$

$$\sum_{k=p}^{n} k^{\underline{R}} = \left[\frac{k^{\underline{R+1}}}{R+1}\right]_p^{n+1}$$

$$u_k^{\underline{-R}} = \frac{1}{u_{k+1} \times u_{k+2} \times \ldots \times u_{k+R}}$$

$$\sum_{k=p}^{n} k^{\underline{-R}} = \left[\frac{k^{\underline{-R+1}}}{-R+1}\right]_p^{n+1}$$

CASO PARTICULAR

$$u_k = ax+b$$

$$\sum_{k=0}^{n} (ak+b)^{\underline{-R}} = \left[\frac{ak+b^{\underline{-R+1}}}{a \times (-R+1)}\right]_p^{n+1}$$

2 (M) Calcule uma forma fechada para $\sum_{k=0}^{n-1} (k^3 + 7k + 1)$

1. Dividir e fatorizar

$$\sum_{k=0}^{n-1} (k^3+7k+1) = \sum_{k=0}^{n-1} k^3 + 7k + \sum_{k=0}^{n-1} 1 \longrightarrow 1 \times (n-1) + 0 + 1) = n$$

$$k^3 + 7k \downarrow k(k^2+7)$$

2. Regra de Ruffini:

1	1	0	7
	1	1	8
2		2	
	1	3	

$$k^{\underline{3}} + 3k^{\underline{2}} + 8k^{\underline{1}}$$

3. Aplicar formas fechadas

$$k^3 + 3k^2 + 8k^1 = \left[\frac{k^4}{4}\right]_0^n + 3\left[\frac{k^3}{3}\right]_0^n + 8\left[\frac{k^2}{2}\right]_0^n =$$

$$= \frac{1}{4}n(n-1)(n-2)(n-3) + n(n-1)(n-2) + 4n(n-1) + n\ (1)$$

TABELA FORMAS FECHADAS

termo geral	forma fechada
$u_1, u_2, \dots, u_n$	$u_0 + u_1 z + u_2 z^2 + \dots + u_n z^n$
$1, a, a^2, \dots$ $u_n = a^k$	$\frac{1}{1-az}$
$1, 0, 0, 0 \dots 1, 0, 0, 0$ $u_n = \begin{cases} 1, & \text{se } k = p \\ 0, & \text{caso contrário} \end{cases}$	$\frac{1}{1-z^p}$
$1, 2, 3, \dots$ $u_k = k+1$	$\frac{1}{(1-z)^2}$
$0, 1, 2, \dots$ $u_k = k$	$\frac{z}{(1-z)^2}$

3 [M] Calcule uma forma fechada para a soma das n primeiras parcelas da soma seguinte:

$$\frac{3}{5\times 8\times 11} + \frac{6}{8\times 11\times 14} + \frac{9}{11\times 14\times 17} + \dots$$

1. Descobrir o somatório correspondente

$$\sum_{k=1}^{n} \frac{3k}{(3k+2)(3k+5)(3k+8)}$$

2. Encontrar forma fechada

$$\sum_{k=1}^{n} \frac{3k+8-8}{(3k+2)(3k+5)(3k+8)} = \sum_{k=1}^{n} \frac{1}{(3k+2)(3k+5)} - 8\sum_{k=1}^{n} \frac{1}{(3k+2)(3k+5)(3k+8)} =$$

$$= \sum_{k=1}^{n} (3k-1)^{\underline{-2}} - 8\sum_{k=1}^{n} (3k-1)^{\underline{-3}} = \left[\frac{(3k-1)^{\underline{-1}}}{-1}\right]_1^{n+1} - 8\left[\frac{(3k-1)^{\underline{-2}}}{-2}\right]_1^{n+1} =$$

$$= -\left[\frac{1}{3k+2}\right]_1^{n+1} + \frac{8}{2}\left[\frac{1}{(3k+2)(3k+5)}\right]_1^{n+1}$$ desenvolver...

GRUPO 2 19

Use funções geradoras para resolver a recorrência

$$s_0 = -2,\ s_1 = -1 \quad e \quad s_k = 8s_{k-1} - 7s_{k-2} + 3 \ (k \in \mathbb{N}_2)$$

1. $G_s(z) = \sum_{k=0}^{\infty} s_k z^k$ 2. separar termos conhecidos

$$G_s(z) = \sum_{k=0}^{+\infty} s_k z^k = s_0 + s_1 z + \sum_{k=2}^{+\infty} s_k z^k = -2 - z + \sum_{k=2}^{+\infty} (8s_{k-1} - 7s_{k-2} + 3) z^k =$$

3. Dividir somatórios e passar para fora

$$= -2 - z + 8z^1 \sum_{k=2}^{+\infty} s_{k-1} z^{k-1} - 7z^2 \sum_{k=2}^{+\infty} s_{k-2} z^{k-2} + 3\left(\sum_{k=0}^{+\infty} z^k \underbrace{- z^0 - z^1}_{\text{para começar no } 0}\right) =$$

4. Começar somatórios no 0

$$= -2 - z + 8z\left(\sum_{k=0}^{+\infty} s_k z^k - s_0\right) - 7z^2 \sum_{k=0}^{+\infty} s_k z^k + \underbrace{\frac{3}{1-z}}_{\text{tabela}} - 3 - 3z =$$

5. Substituir por $G_s(z)$

$$-2-z+8z\left(G_s(z)+2\right)-7z^2G_s(z)+\frac{3}{1-z}-3-3z=$$

$$=-5-4z+8zG_s(z)+16z-7z^2G_s(z)+\frac{3}{1-z}=$$

6. Isolar termos $G_s(z)$

$$\Leftrightarrow \underline{\underline{G_s(z)}}-8G_s(z)+7z^2G_s(z)=-5+12z+\frac{3}{1-z}\Leftrightarrow$$

$$\Leftrightarrow (1-8z+7z^2)G_s(z)=\frac{(-5+12z)(1-z)+3}{1-z}\Leftrightarrow$$

$$\Leftrightarrow G_s(z)=\frac{(-5+12z)(1-z)+3}{(1-8z+7z^2)(1-z)}$$

7. Fatorizar:

Raízes de $1-8z+7z^2$ são $z=1 \vee z=\frac{1}{7}$, fica:

$$G_s(z)=\frac{(-5+12z)(1-z)+3}{7(z-1)(z-\frac{1}{7})(1-z)}=\frac{(-5+12z)(1-z)+3}{(1-7z)(1-z)^2}=$$

8. Pôr na forma: $\frac{A}{exp}+\frac{B}{exp}\dots$

$$=\frac{A}{1-z}+\frac{B}{(1-z)^2}+\frac{C}{1-7z}$$

$$A(1-7z)(1-z)+B(1-7z)+C(1-z)^2=(-5+12z)(1-z)+3$$

9. Atribuir valores a z que permitam resolver o sistema:

eliminar o A:

$z=1$

$$B(1-7)=(-5+12)\times 0+3 \Leftrightarrow -6B=3\Leftrightarrow B=-\frac{1}{2}$$

eliminar o B:

$z=\frac{1}{7}$

$$C\left(1-\frac{1}{7}\right)^2=\left(-5+\frac{12}{7}\right)\left(1-\frac{1}{7}\right)+3\Leftrightarrow C\left(\frac{6}{7}\right)^2=\left(-\frac{23}{7}\right)\times\left(\frac{6}{7}\right)+3\Leftrightarrow$$

$$\Leftrightarrow \frac{36}{49}C=-\frac{138}{49}+\frac{147}{49}\Leftrightarrow\frac{36}{49}C=\frac{9}{49}\Leftrightarrow C=\frac{9}{36}\Leftrightarrow C=\frac{1}{4}$$

eliminar o C:

$z=0$

$$A+B+C=-2\Leftrightarrow A-\frac{1}{2}+\frac{1}{4}=-2\Leftrightarrow A=-\frac{8}{4}+\frac{2}{4}-\frac{1}{4}\Leftrightarrow A=-\frac{7}{4}$$

10. Substituir

$$-\frac{1}{1-z}\times\frac{7}{4}\times\left(-\frac{1}{(1-z)^2}\times\frac{1}{2}\right)+\frac{1}{1-7z}\times\frac{1}{4}=$$

$$= -\frac{7}{4}\sum_{k=0}^{+\infty} z^k - \frac{1}{2}\sum_{k=0}^{+\infty}(k+1)z^k + \frac{1}{4}\sum_{k=0}^{+\infty} 7^k z^k =$$

$$= \sum_{k=0}^{+\infty}\left(-\frac{9}{4} - \frac{1}{2}\times k + \frac{1}{4}\times 7^k\right) z^k$$

Termo geral $-\frac{9}{4} - \frac{k}{2} + \frac{7^k}{4} \quad (k \in \mathbb{N})$

Grupo 3 19

Usando o princípio da indução matemática demonstre que para qualquer $n \in \mathbb{N}$ se verifica

$$\sum_{k=0}^{n}((k(k+4)+5)\times(k+2)!) = (n+2)\times(n+3)!-2$$

1. Provar para o 1º termo:

Base 1º termo é 0

$\rightarrow ((0\times(0+4)+5)\times(0+2)! = (0+2)\times(0+3)!-2 \Leftrightarrow$

$\Leftrightarrow 5\times 2! = 2\times 3!-2 \Leftrightarrow 10 = 10$

2. Hipótese: copiar o que se quer demonstrar

$$\sum_{k=0}^{n}((k(k+4)+5)\times(k+2)!) = (n+2)\times(n+3)!-2$$

3. Tese $\rightarrow$ substituir n por $n+1$

$$\sum_{k=0}^{n+1}((k(k+4)+5)\times(k+2)!) = (n+3)\times(n+4)!-2$$

4. Pegar no lado esquerdo, e passando pela hip., chegar ao lado direito

$$\sum_{k=0}^{n+1}((k(k+4)+5)\times(k+2)!) = \sum_{k=0}^{n}((k(k+4)+5)\times(k+2)!) + ((n+1)(n+5)+5\times(n+3)! =$$

$$\overset{HIP}{=} (n+2)\times(n+3)!-2+(n^2+6n+5+5)\times(n+3)! =$$

$$= (n+2+n^2+6n+10)\times(n+3)!-2 =$$

$$= (n^2+7n+12)\times(n+3)!-2 =$$

$$= (n+3)(n+4)\times(n+3)!-2 =$$

$$= (n+3)(n+4)!-2 \quad \text{c.q.d.}$$

C. Aux:

$x = \frac{-7\pm\sqrt{49-4\times1\times12}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-7\pm\sqrt{1}}{2} \Leftrightarrow$

$\Leftrightarrow x=-3 \vee x=-4$