Solution of Exam 2

Problem 1 Answer: E

Problem 2 Answer: B

Details ① note that $x^* = 0$ is a global minimizer of a function $\phi : \mathbb{R} \to \mathbb{R}$ of the form $\phi(x) = f(x) + x_+$ (where $f : \mathbb{R} \to \mathbb{R}$ is a convex, continuously differentiable function) only if

$$\dot{\phi}(0^-) = \lim_{x \to 0^-} \phi(x) \leq 0 \quad \text{and} \quad \dot{\phi}(0^+) = \lim_{x \to 0^+} \phi(x) \geq 0$$

To see why, suppose, for example, that $\dot{\phi}(0^-) > 0$; then ϕ would be increasing when approaching $x^* = 0$ from the left — that invalidates $x^* = 0$ being a minimum. Similarly, suppose $\dot{\phi}(0^+) < 0$, then ϕ would be decreasing when departing from $x^* = 0$ to the right — that invalidates $x^* = 0$ being a minimum.

② In our case, $\phi(x) = \underbrace{e^{x-a} + e^{-x} + x^2 - 2x}_{f(x)} + x_+$.

Thus, $x^* = 0$ minimizes ϕ when

$$\begin{aligned}
\dot{\phi}(0^-) \leq 0 \text{ and } \dot{\phi}(0^+) \geq 0 &\iff \dot{f}(0) \leq 0 \text{ and } \dot{f}(0) + 1 \geq 0 \\
&\iff e^{-a} - 3 \leq 0 \text{ and } e^{-a} - 2 \geq 0 \\
&\iff 2 \leq e^{-a} \leq 3 \\
&\iff \underbrace{-\log 3}_{\approx -1.1} \leq a \leq \underbrace{-\log 2}_{\approx -0.7}
\end{aligned}$$

The only number a in $\{-2,-1,0,1,2,3\}$ that satisfies the inequalities above is $a=-1$

Problem 3 Answer: C

<u>Details</u> ① The gradient descent algorithm is $x_{k+1} = x_k + \alpha_k d_k$, where $\alpha_k > 0$ is the stepsize and $d_k = -\nabla f(x_k)$

② For $x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have $\nabla f(x_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, for the stepsize $\alpha_0 = 1$, we have $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Problem 4

First, note that $f(r) = r^T D r = \| D^{1/2} r \|_2^2$, where $D^{1/2} = \begin{bmatrix} d_1^{1/2} & & & \\ & d_2^{1/2} & & \\ & & \ddots & \\ & & & d_n^{1/2} \end{bmatrix}$

Now:

$$\begin{aligned} \min_{s,v} \quad & \|s - \bar{s}\|_2^2 + f(v) \\ \text{s.t.} \quad & y = As + v \end{aligned} \quad \Leftrightarrow \quad \min_{s} \; \|s - \bar{s}\|^2 + f(y - As)$$

use the constraint to eliminate the variable v: $v = y - As$

$$\Leftrightarrow \quad \min_{s} \; \|s - \bar{s}\|^2 + \| D^{1/2}(As - y)\|_2^2$$

use the identity: $\|A_1 s - b_1\|_2^2 + \|A_2 s - b_2\|_2^2 = \left\| \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} s - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right\|_2^2$

$$\Leftrightarrow \quad \min_{s} \; \Big\| \underbrace{\begin{bmatrix} I \\ D^{1/2} A \end{bmatrix}}_{\mathcal{A}} s - \underbrace{\begin{bmatrix} \bar{s} \\ D^{1/2} y \end{bmatrix}}_{\mathcal{B}} \Big\|_2^2$$

Problem 5

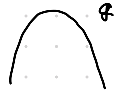
$$\max_{x_1, x_2} \; f(x_1, x_2) \quad \text{s.t.} \quad x_1 + x_2 = 1 \quad \iff \quad \max_{x_1} \; f(x_1, 1-x_1)$$

use the constraint to eliminate the variable x_2: $x_2 = 1 - x_1$

$$\iff \max_{x_1} \; \underbrace{\tfrac{1}{2} a x_1^2 + b x_1 (1-x_1) + \tfrac{1}{2} a (1-x_1)^2}_{\text{“}}$$

$$g(x_1) = (a-b) x_1^2 + (b-a) x_1 + \tfrac{1}{2} a$$

because $b > a$, this is a parabola that looks like this



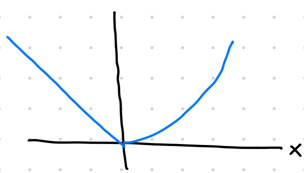
thus, the global maximizer occurs at the stationary point: $\dot{g}(x_1^*) = 0 \iff x_1^* = 1/2$

Because $x_2^* = 1 - x_1^* = 1/2$, the global maximizer of f (under the constraint $1^T x = 1$) is

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

Problem 6

① First, note that $g : \mathbb{R} \to \mathbb{R}$ is convex: this can be obtained directly from its graph:



② The given problem is

$$\min_{x} \quad \underbrace{\underbrace{g(x_1-c_1)}_{f_1(x)} + \cdots + \underbrace{g_n(x_n-c_n)}_{f_n(x)}}_{f(x)}$$

$$\text{s.t.} \quad Ax = b,$$

where

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

③ The constraints are linear. Thus, the problem is convex if f is convex

④ $f = f_1 + f_2 + \cdots + f_n$ where $f_i : \mathbb{R}^n \to \mathbb{R}$, $f_i(x) = g(x_i - c_i)$, for $1 \leq i \leq n$

<u>f_i is convex</u> because $f_i = g \circ h_i$, where $h_i(x) = [0 \cdots \underset{i}{1} \cdots 0]\, x - c_i$ is an affine map, and g is convex

<u>f is convex</u> because it is the sum of convex functions

Problem 7

① We start by obtaining a closed-form expression for $f(x)$, for fixed $x \in \mathbb{R}$:

$$f(x) = \max \{ \| (a+u)x - b \|_2 \ : \ \|u\|_2 = r \}$$

$$= \max \{ (\| ax-b + ux \|_2^2)^{1/2} : \|u\|_2 = r \}$$

$$= \max \{ (\|ax-b\|_2^2 + \|u\|_2^2 x^2 + 2u^T (ax-b) x)^{1/2} : \|u\|_2 = r \}$$

$$= \max \{ (\|ax-b\|_2^2 + r^2 x^2 + 2u^T (ax-b) x)^{1/2} : \|u\|_2 = r \}$$

$$= \left(\|ax-b\|_2^2 + r^2x^2 + 2 \max \{ u^T (ax-b) x : \|u\|_2 = r \} \right)^{1/2}$$

$$= \left(\|ax-b\|_2^2 + r^2x^2 + 2r|x| \|ax-b\|_2 \right)^{1/2}$$

$$= \left((\|ax-b\|_2 + r|x|)^2 \right)^{1/2}$$

$$= \underbrace{\|ax - b\|_2}_{f_1(x)} + r\underbrace{|x|}_{f_2(x)}$$

(2) f_1 is convex because $f_1 = g \circ h$, where $h(x) = ax - b$ is an affine map and $g = \|\cdot\|_2$ is convex

f_2 is convex because $f_2 = |\cdot|$

f is convex because $f = f_1 + r f_2$ is a nonnegative combination of convex functions