Ficha 10
Resolução dos exercícios de auto-avaliação

III.1 - Determine as derivadas parciais de primeira ordem das seguintes funções:

a) $f(x,y,z)=\dfrac{xy+x^2z}{x+yz^2}$

Resolução:

- $$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial\left(\dfrac{xy+x^2z}{x+yz^2}\right)}{\partial x}=\frac{\dfrac{\partial(xy+x^2z)}{\partial x}(x+yz^2)-(xy+x^2z)\dfrac{\partial(x+yz^2)}{\partial x}}{(x+yz^2)^2}=\frac{(y+2xz)(x+yz^2)-(xy+x^2z)\cdot 1}{(x+yz^2)^2}$$
$$=\frac{yx+y^2z^2+2x^2z+2xyz^3-xy-x^2z}{(x+yz^2)^2}=\frac{y^2z^2+x^2z+2xyz^3}{(x+yz^2)^2}$$

- $$\frac{\partial f}{\partial y}(x,y,z)=\frac{\partial\left(\dfrac{xy+x^2z}{x+yz^2}\right)}{\partial y}=\frac{\dfrac{\partial(xy+x^2z)}{\partial y}(x+yz^2)-(xy+x^2z)\dfrac{\partial(x+yz^2)}{\partial y}}{(x+yz^2)^2}=\frac{(x+0)(x+yz^2)-(xy+x^2z)\cdot 0}{(x+yz^2)^2}$$
$$=\frac{x(x+yz^2)-0}{(x+yz^2)^2}=\frac{x(x+yz^2)}{(x+yz^2)^2}=\frac{x}{x+yz^2}$$

- $$\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial\left(\dfrac{xy+x^2z}{x+yz^2}\right)}{\partial z}=\frac{\dfrac{\partial(xy+x^2z)}{\partial z}(x+yz^2)-(xy+x^2z)\dfrac{\partial(x+yz^2)}{\partial z}}{(x+yz^2)^2}=\frac{(0+x^2)(x+yz^2)-(xy+x^2z)(0+2yz)}{(x+yz^2)^2}$$
$$=\frac{x^2(x+yz^2)-(xy+x^2z)2yz}{(x+yz^2)^2}=\frac{x^3+x^2yz^2-2xy^2z-2x^2yz^2}{(x+yz^2)^2}=\frac{x^3-x^2yz^2-2xy^2z}{(x+yz^2)^2}$$

b) $f(x,y,z)=\operatorname{arc\,sen}\left(\dfrac{xy}{z}\right)$

Resolução:

- $$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial\left(\operatorname{arc\,sen}\left(\dfrac{xy}{z}\right)\right)}{\partial x}=\frac{\dfrac{\partial\left(\dfrac{xy}{z}\right)}{\partial x}}{\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}=\frac{\dfrac{y}{z}}{\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}=\frac{y}{z\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}$$

- $$\frac{\partial f}{\partial y}(x,y,z)=\frac{\partial\left(\operatorname{arc\,sen}\left(\dfrac{xy}{z}\right)\right)}{\partial y}=\frac{\dfrac{\partial\left(\dfrac{xy}{z}\right)}{\partial y}}{\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}=\frac{\dfrac{x}{z}}{\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}=\frac{x}{z\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}$$

- $$\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial\left(\operatorname{arc\,sen}\left(\dfrac{xy}{z}\right)\right)}{\partial z}=\frac{\dfrac{\partial\left(\dfrac{xy}{z}\right)}{\partial z}}{\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}=\frac{xy\dfrac{\partial(z^{-1})}{\partial z}}{\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}=\frac{xy(-1)z^{-2}}{\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}=\frac{-\dfrac{xy}{z^2}}{\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}=\frac{-xy}{z^2\sqrt{1-\left(\dfrac{xy}{z}\right)^2}}$$

c) $f\left(x,y,z,v\right)=\frac{1}{2}\operatorname{tg}^2\left(x^2y^2+z^2v^2-xyzv\right)$

Resolução:

- $$\frac{\partial f}{\partial x}\left(x,y,z,v\right)=\frac{\partial\left(\frac{1}{2}\operatorname{tg}^2\left(x^2y^2+z^2v^2-xyzv\right)\right)}{\partial x}=\frac{1}{2}2\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\frac{\partial\left(\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\right)}{\partial x}$$
$$=\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\frac{\frac{\partial\left(x^2y^2+z^2v^2-xyzv\right)}{\partial x}}{\cos^2\left(x^2y^2+z^2v^2-xyzv\right)}=\frac{\operatorname{sen}\left(x^2y^2+z^2v^2-xyzv\right)}{\cos\left(x^2y^2+z^2v^2-xyzv\right)}\frac{2xy^2-yzv}{\cos^2\left(x^2y^2+z^2v^2-xyzv\right)}$$
$$=\left(2xy^2-yzv\right)\frac{\operatorname{sen}\left(x^2y^2+z^2v^2-xyzv\right)}{\cos^3\left(x^2y^2+z^2v^2-xyzv\right)}$$

- $$\frac{\partial f}{\partial y}\left(x,y,z,v\right)=\frac{\partial\left(\frac{1}{2}\operatorname{tg}^2\left(x^2y^2+z^2v^2-xyzv\right)\right)}{\partial y}=\frac{1}{2}2\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\frac{\partial\left(\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\right)}{\partial y}$$
$$=\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\frac{\frac{\partial\left(x^2y^2+z^2v^2-xyzv\right)}{\partial y}}{\cos^2\left(x^2y^2+z^2v^2-xyzv\right)}=\frac{\operatorname{sen}\left(x^2y^2+z^2v^2-xyzv\right)}{\cos\left(x^2y^2+z^2v^2-xyzv\right)}\frac{2x^2y-xzv}{\cos^2\left(x^2y^2+z^2v^2-xyzv\right)}$$
$$=\left(2x^2y-xzv\right)\frac{\operatorname{sen}\left(x^2y^2+z^2v^2-xyzv\right)}{\cos^3\left(x^2y^2+z^2v^2-xyzv\right)}$$

- $$\frac{\partial f}{\partial z}\left(x,y,z,v\right)=\frac{\partial\left(\frac{1}{2}\operatorname{tg}^2\left(x^2y^2+z^2v^2-xyzv\right)\right)}{\partial z}=\frac{1}{2}2\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\frac{\partial\left(\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\right)}{\partial z}$$
$$=\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\frac{\frac{\partial\left(x^2y^2+z^2v^2-xyzv\right)}{\partial z}}{\cos^2\left(x^2y^2+z^2v^2-xyzv\right)}=\frac{\operatorname{sen}\left(x^2y^2+z^2v^2-xyzv\right)}{\cos\left(x^2y^2+z^2v^2-xyzv\right)}\frac{2zv^2-xyv}{\cos^2\left(x^2y^2+z^2v^2-xyzv\right)}$$
$$=\left(2zv^2-xyv\right)\frac{\operatorname{sen}\left(x^2y^2+z^2v^2-xyzv\right)}{\cos^3\left(x^2y^2+z^2v^2-xyzv\right)}$$

- $$\frac{\partial f}{\partial v}\left(x,y,z,v\right)=\frac{\partial\left(\frac{1}{2}\operatorname{tg}^2\left(x^2y^2+z^2v^2-xyzv\right)\right)}{\partial v}=\frac{1}{2}2\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\frac{\partial\left(\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\right)}{\partial v}$$
$$=\operatorname{tg}\left(x^2y^2+z^2v^2-xyzv\right)\frac{\frac{\partial\left(x^2y^2+z^2v^2-xyzv\right)}{\partial v}}{\cos^2\left(x^2y^2+z^2v^2-xyzv\right)}=\frac{\operatorname{sen}\left(x^2y^2+z^2v^2-xyzv\right)}{\cos\left(x^2y^2+z^2v^2-xyzv\right)}\frac{2z^2v-xyz}{\cos^2\left(x^2y^2+z^2v^2-xyzv\right)}$$
$$=\left(2z^2v-xyz\right)\frac{\operatorname{sen}\left(x^2y^2+z^2v^2-xyzv\right)}{\cos^3\left(x^2y^2+z^2v^2-xyzv\right)}$$

d) $f(x,y,z,v)=\ln\cos\left(x^2y+z^2v^2\right)$

Resolução:

- $$\frac{\partial f}{\partial x}(x,y,z,v)=\frac{\partial\left(\ln\cos\left(x^2y+z^2v^2\right)\right)}{\partial x}=\frac{\frac{\partial\left(\cos\left(x^2y+z^2v^2\right)\right)}{\partial x}}{\cos\left(x^2y+z^2v^2\right)}=\frac{-\frac{\partial\left(x^2y+z^2v^2\right)}{\partial x}\operatorname{sen}\left(x^2y+z^2v^2\right)}{\cos\left(x^2y+z^2v^2\right)}$$
$$=\frac{-2xy\operatorname{sen}\left(x^2y+z^2v^2\right)}{\cos\left(x^2y+z^2v^2\right)}=-2xy\operatorname{tg}\left(x^2y+z^2v^2\right)$$

- $$\frac{\partial f}{\partial y}(x,y,z,v)=\frac{\partial\left(\ln\cos\left(x^2y+z^2v^2\right)\right)}{\partial y}=\frac{\frac{\partial\left(\cos\left(x^2y+z^2v^2\right)\right)}{\partial y}}{\cos\left(x^2y+z^2v^2\right)}=\frac{-\frac{\partial\left(x^2y+z^2v^2\right)}{\partial y}\operatorname{sen}\left(x^2y+z^2v^2\right)}{\cos\left(x^2y+z^2v^2\right)}$$
$$=\frac{-x^2\operatorname{sen}\left(x^2y+z^2v^2\right)}{\cos\left(x^2y+z^2v^2\right)}=-x^2\operatorname{tg}\left(x^2y+z^2v^2\right)$$

- $$\frac{\partial f}{\partial z}(x,y,z,v)=\frac{\partial\left(\ln\cos\left(x^2y+z^2v^2\right)\right)}{\partial z}=\frac{\frac{\partial\left(\cos\left(x^2y+z^2v^2\right)\right)}{\partial z}}{\cos\left(x^2y+z^2v^2\right)}=\frac{-\frac{\partial\left(x^2y+z^2v^2\right)}{\partial z}\operatorname{sen}\left(x^2y+z^2v^2\right)}{\cos\left(x^2y+z^2v^2\right)}$$
$$=\frac{-2zv^2\operatorname{sen}\left(x^2y+z^2v^2\right)}{\cos\left(x^2y+z^2v^2\right)}=-2zv^2\operatorname{tg}\left(x^2y+z^2v^2\right)$$

- $$\frac{\partial f}{\partial v}(x,y,z,v)=\frac{\partial\left(\ln\cos\left(x^2y+z^2v^2\right)\right)}{\partial v}=\frac{\frac{\partial\left(\cos\left(x^2y+z^2v^2\right)\right)}{\partial v}}{\cos\left(x^2y+z^2v^2\right)}=\frac{-\frac{\partial\left(x^2y+z^2v^2\right)}{\partial v}\operatorname{sen}\left(x^2y+z^2v^2\right)}{\cos\left(x^2y+z^2v^2\right)}$$
$$=\frac{-2vz^2\operatorname{sen}\left(x^2y+z^2v^2\right)}{\cos\left(x^2y+z^2v^2\right)}=-2vz^2\operatorname{tg}\left(x^2y+z^2v^2\right)$$

III.2 Calcule o gradiente das seguintes funções nos pontos onde estiver definido:

a) $f(x,y,z)=\ln\left(x^2+y^2\right)+z$

Resolução:

$$\operatorname{grad} f(x,y,z)=\nabla f(x,y,z)=\left(\frac{\partial f}{\partial x}(x,y,z),\frac{\partial f}{\partial y}(x,y,z),\frac{\partial f}{\partial z}(x,y,z)\right)$$
$$=\left(\frac{\partial\left(\ln\left(x^2+y^2\right)+z\right)}{\partial x},\frac{\partial\left(\ln\left(x^2+y^2\right)+z\right)}{\partial y},\frac{\partial\left(\ln\left(x^2+y^2\right)+z\right)}{\partial z}\right)$$
$$=\left(\frac{\frac{\partial\left(x^2+y^2\right)}{\partial x}}{x^2+y^2},\frac{\frac{\partial\left(x^2+y^2\right)}{\partial y}}{x^2+y^2},1\right)=\left(\frac{2x}{x^2+y^2},\frac{2y}{x^2+y^2},1\right)\quad,(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$$

b) $f(x,y,z)=e^{-x}\left(x^2+y^2+z^2\right)$

Resolução:

$$\operatorname{grad} f(x,y,z)=\nabla f(x,y,z)=\left(\frac{\partial f}{\partial x}(x,y,z),\frac{\partial f}{\partial y}(x,y,z),\frac{\partial f}{\partial z}(x,y,z)\right)$$

$$=\left(\frac{\partial\left(e^{-x}\left(x^2+y^2+z^2\right)\right)}{\partial x},\frac{\partial\left(e^{-x}\left(x^2+y^2+z^2\right)\right)}{\partial y},\frac{\partial\left(e^{-x}\left(x^2+y^2+z^2\right)\right)}{\partial z}\right)$$

$$=\left(\frac{\partial\left(e^{-x}\right)}{\partial x}\left(x^2+y^2+z^2\right)+e^{-x}\frac{\partial\left(x^2+y^2+z^2\right)}{\partial x},2ye^{-x},2ze^{-x}\right)$$

$$=\left(-e^{-x}\left(x^2+y^2+z^2\right)+e^{-x}2x,2ye^{-x},2ze^{-x}\right)$$

$$=e^{-x}\left(2x-x^2-y^2-z^2,2y,2z\right)\quad ,(x,y,z)\in\mathbb{R}^3$$

c) $f(x,y,z)=\dfrac{x}{x^2+y^2}+\dfrac{z}{x^2+y^2}$

Resolução:

$$\operatorname{grad} f(x,y,z)=\nabla f(x,y,z)=\left(\frac{\partial f}{\partial x}(x,y,z),\frac{\partial f}{\partial y}(x,y,z),\frac{\partial f}{\partial z}(x,y,z)\right)$$

$$=\left(\frac{\partial\left(\frac{x}{x^2+y^2}+\frac{z}{x^2+y^2}\right)}{\partial x},\frac{\partial\left(\frac{x}{x^2+y^2}+\frac{z}{x^2+y^2}\right)}{\partial y},\frac{\partial\left(\frac{x}{x^2+y^2}+\frac{z}{x^2+y^2}\right)}{\partial z}\right)$$

$$=\left(\frac{\partial\left(\frac{x}{x^2+y^2}\right)}{\partial x}+\frac{\partial\left(\frac{z}{x^2+y^2}\right)}{\partial x},\frac{\partial\left(\frac{x}{x^2+y^2}\right)}{\partial y}+\frac{\partial\left(\frac{z}{x^2+y^2}\right)}{\partial y},\frac{1}{x^2+y^2}\right)$$

$$=\left(\frac{\frac{\partial x}{\partial x}\left(x^2+y^2\right)-x\frac{\partial\left(x^2+y^2\right)}{\partial x}}{\left(x^2+y^2\right)^2}+\frac{\frac{\partial z}{\partial x}\left(x^2+y^2\right)-z\frac{\partial\left(x^2+y^2\right)}{\partial x}}{\left(x^2+y^2\right)^2},\frac{\frac{\partial x}{\partial y}\left(x^2+y^2\right)-x\frac{\partial\left(x^2+y^2\right)}{\partial y}}{\left(x^2+y^2\right)^2}+\frac{\frac{\partial z}{\partial y}\left(x^2+y^2\right)-z\frac{\partial\left(x^2+y^2\right)}{\partial y}}{\left(x^2+y^2\right)^2},\frac{1}{x^2+y^2}\right)$$

$$=\left(\frac{1\left(x^2+y^2\right)-x(2x)}{\left(x^2+y^2\right)^2}+\frac{0\left(x^2+y^2\right)-z(2x)}{\left(x^2+y^2\right)^2},\frac{0\left(x^2+y^2\right)-x(2y)}{\left(x^2+y^2\right)^2}+\frac{0\left(x^2+y^2\right)-z(2y)}{\left(x^2+y^2\right)^2},\frac{1}{x^2+y^2}\right)$$

$$=\left(\frac{-x^2+y^2-2xz}{\left(x^2+y^2\right)^2},\frac{-2xy-2yz}{\left(x^2+y^2\right)^2},\frac{1}{x^2+y^2}\right),\quad (x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$$

III.3 – Determine as equações dos planos tangentes aos gráficos das seguintes funções nos pontos indicados:

a) $f(x,y)=\ln\left(x^2+y^2\right)^{\frac{1}{2}}$, $P=(1,1)$

Resolução:

A equação do plano tangente à função no ponto $P=(1,1)$ é dada por:

$$z=f(1,1)+\nabla f(1,1)\cdot(x-1,y-1)\Leftrightarrow z-f(1,1)=\left(\frac{\partial f}{\partial x}(1,1),\frac{\partial f}{\partial y}(1,1)\right)\cdot(x-1,y-1)$$

$$\underset{(*)}{\Leftrightarrow} z-\frac{\ln 2}{2}=\left(\frac{1}{2},\frac{1}{2}\right)\cdot(x-1,y-1)\Leftrightarrow z-\frac{\ln 2}{2}=\frac{1}{2}(x-1)+\frac{1}{2}(y-1)$$

$$\Leftrightarrow -x-y+2z-\ln 2+2=0$$

***Cálculos Auxiliares*:** $(*)$

- $f(1,1)=\ln\left(1^2+1^2\right)^{\frac{1}{2}}=\dfrac{\ln 2}{2}$

- $\dfrac{\partial f}{\partial x}(x,y)=\dfrac{\partial\left(\ln\left(x^2+y^2\right)^{\frac{1}{2}}\right)}{\partial x}=\dfrac{\dfrac{\partial\left(\left(x^2+y^2\right)^{\frac{1}{2}}\right)}{\partial x}}{\left(x^2+y^2\right)^{\frac{1}{2}}}=\dfrac{\frac{1}{2}\left(x^2+y^2\right)^{\frac{1}{2}-1}2x}{\left(x^2+y^2\right)^{\frac{1}{2}}}=\dfrac{\left(x^2+y^2\right)^{-\frac{1}{2}}x}{\left(x^2+y^2\right)^{\frac{1}{2}}}=\dfrac{x}{\left(x^2+y^2\right)^{\frac{1}{2}+\frac{1}{2}}}=\dfrac{x}{x^2+y^2}$

$\Rightarrow \dfrac{\partial f}{\partial x}(1,1)=\dfrac{1}{1^2+1^2}=\dfrac{1}{2}$

- $\dfrac{\partial f}{\partial y}(x,y)=\dfrac{\partial\left(\ln\left(x^2+y^2\right)^{\frac{1}{2}}\right)}{\partial y}=\dfrac{\dfrac{\partial\left(\left(x^2+y^2\right)^{\frac{1}{2}}\right)}{\partial y}}{\left(x^2+y^2\right)^{\frac{1}{2}}}=\dfrac{\frac{1}{2}\left(x^2+y^2\right)^{\frac{1}{2}-1}2y}{\left(x^2+y^2\right)^{\frac{1}{2}}}=\dfrac{\left(x^2+y^2\right)^{-\frac{1}{2}}y}{\left(x^2+y^2\right)^{\frac{1}{2}}}=\dfrac{y}{\left(x^2+y^2\right)^{\frac{1}{2}+\frac{1}{2}}}=\dfrac{y}{x^2+y^2}$

$\Rightarrow \dfrac{\partial f}{\partial y}(1,1)=\dfrac{1}{1^2+1^2}=\dfrac{1}{2}$

b) $f(x,y,z)=z-e^x\sin y,\ P=\left(\ln 3,\dfrac{3\pi}{2},-3\right)$

<u>Resolução:</u>

A equação do plano tangente à função no ponto $P=\left(\ln 3,\dfrac{3\pi}{2},-3\right)$ é dada por:

$$w=f\left(\ln 3,\frac{3\pi}{2},-3\right)+\nabla f\left(\ln 3,\frac{3\pi}{2},-3\right)\cdot\left(x-\ln 3,y-\frac{3\pi}{2},z-(-3)\right)$$

$$\Leftrightarrow w-f\left(\ln 3,\frac{3\pi}{2},-3\right)=\left(\frac{\partial f}{\partial x}\left(\ln 3,\frac{3\pi}{2},-3\right),\frac{\partial f}{\partial y}\left(\ln 3,\frac{3\pi}{2},-3\right),\frac{\partial f}{\partial z}\left(\ln 3,\frac{3\pi}{2},-3\right)\right)\cdot\left(x-\ln 3,y-\frac{3\pi}{2},z+3\right)$$

$$\underset{(*)}{\Leftrightarrow} w-0=(3,0,1)\cdot\left(x-\ln 3,y-\frac{3\pi}{2},z+3\right)\Leftrightarrow w=3(x-\ln 3)+0\left(y-\frac{3\pi}{2}\right)+1(z+3)$$

$$\Leftrightarrow w=3x-3\ln 3+z+3\Leftrightarrow -3x-z+w+3\ln 3-3=0$$

***Cálculos Auxiliares*:** $(*)$

- $f\left(\ln 3,\dfrac{3\pi}{2},-3\right)=-3-e^{\ln 3}\sin\dfrac{3\pi}{2}=-3-3\cdot(-1)=0$

- $\dfrac{\partial f}{\partial x}(x,y,z)=\dfrac{\partial\left(z-e^x\sin y\right)}{\partial x}=-e^x\sin y\Rightarrow\dfrac{\partial f}{\partial x}\left(\ln 3,\dfrac{3\pi}{2},-3\right)=-e^{\ln 3}\sin\dfrac{3\pi}{2}=-e^{\ln 3}(-1)=3$

- $\dfrac{\partial f}{\partial y}(x,y,z)=\dfrac{\partial\left(z-e^x\sin y\right)}{\partial y}=-e^x\cos y\Rightarrow\dfrac{\partial f}{\partial x}\left(\ln 3,\dfrac{3\pi}{2},-3\right)=-e^{\ln 3}\cos\dfrac{3\pi}{2}=-e^{\ln 3}0=0$

- $\dfrac{\partial f}{\partial z}(x,y,z)=\dfrac{\partial\left(z-e^x\sin y\right)}{\partial z}=1\Rightarrow\dfrac{\partial f}{\partial x}\left(\ln 3,\dfrac{3\pi}{2},-3\right)=1$

III.4 – Seja $f(x,y)=\frac{x^3}{3}+\frac{y^3}{3}-\frac{x^2}{2}-2x-y+1$.

a) Determine o gradiente de $f(x,y)$.

Resolução:

$$\operatorname{grad} f(x,y)=\nabla f(x,y)=\left(\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y)\right)$$

$$=\left(\frac{\partial\left(\frac{x^3}{3}+\frac{y^3}{3}-\frac{x^2}{2}-2x-y+1\right)}{\partial x},\frac{\partial\left(\frac{x^3}{3}+\frac{y^3}{3}-\frac{x^2}{2}-2x-y+1\right)}{\partial y}\right)$$

$$=\left(\frac{3x^2}{3}-\frac{2x}{2}-2,\frac{3y^2}{3}-1\right)=\left(x^2-x-2,y^2-1\right),\ (x,y)\in\mathbb{R}^2$$

b) Obtenha a equação do plano tangente a f no ponto $(1,2)$.

Resolução:

A equação do plano tangente à função no ponto $(1,2)$ é dada por:

$$z=f(1,2)+\nabla f(1,2)\cdot(x-1,y-2)$$

Substituindo na função $f(x,y)$ a variável x por 1 e a variável y por 2, vem

$$f(1,2)=\frac{1^3}{3}+\frac{2^3}{3}-\frac{1^2}{2}-2\cdot1-2+1=-\frac{1}{2}.$$

Na alínea anterior viu-se que,

$$\nabla f(x,y)=\left(x^2-x-2,y^2-1\right),\ (x,y)\in\mathbb{R}^2.$$

Então,

$$\nabla f(1,2)=\left(1^2-1-2,2^2-1\right)=(-2,3).$$

Assim,

$$z=f(1,2)+\nabla f(1,2)\cdot(x-1,y-2)\Leftrightarrow z=f(1,2)+(-2,3)\cdot(x-1,y-2)$$

$$\Leftrightarrow z=-\frac{1}{2}-2(x-1)+3(y-2)\Leftrightarrow 2z=-1-4x+4+6y-12$$

$$\Leftrightarrow 4x-6y+2z+9=0$$

III.5 – Seja $f(x,y)=\frac{x^3}{3}-\frac{x^2}{2}-2x+\frac{y^3}{3}-y^2+\frac{3}{2}$.

a) Determine o gradiente de $f(x,y)$.

Resolução:

$$\operatorname{grad} f(x,y)=\nabla f(x,y)=\left(\frac{\partial f}{\partial x}(x,y),\frac{\partial f}{\partial y}(x,y)\right)$$

$$=\left(\frac{\partial\left(\frac{x^3}{3}-\frac{x^2}{2}-2x+\frac{y^3}{3}-y^2+\frac{3}{2}\right)}{\partial x},\frac{\partial\left(\frac{x^3}{3}-\frac{x^2}{2}-2x+\frac{y^3}{3}-y^2+\frac{3}{2}\right)}{\partial y}\right)$$

$$=\left(x^2-x-2,y^2-2y\right),\ (x,y)\in\mathbb{R}^2$$

b) Obtenha a equação do plano tangente a f no ponto $(1,-1)$.

Resolução:

A equação do plano tangente à função no ponto $(1,-1)$ é dada por:

$$z = f(1,-1) + \nabla f(1,-1) \cdot (x-1, y-(-1))$$

Substituindo na função $f(x,y)$ as variáveis x e y por 2 e 0, vem

$$f(1,-1) = \frac{1^3}{3} - \frac{1^2}{2} - 2\cdot 1 + \frac{(-1)^3}{3} - (-1)^2 + \frac{3}{2} = \frac{1}{3} - \frac{1}{2} - 2 - \frac{1}{3} - 1 + \frac{3}{2} = -2 .$$

Na alínea anterior viu-se que,

$$\nabla f(x,y) = \left(x^2 - x - 2, y^2 - 2y\right),\ (x,y) \in \mathbb{R}^2 .$$

Então,

$$\nabla f(1,-1) = \left(1^2 - 1 - 2, (-1)^2 - 2(-1)\right) = (-2,3).$$

Assim,

$$\begin{aligned} & z = f(1,-1) + \nabla f(1,-1) \cdot (x-1, y+1) \Leftrightarrow z = -2 + (-2,3)(x-1, y+1) \\ & \Leftrightarrow z = -2 + (-2)(x-1) + 3(y+1) \Leftrightarrow z = -2 - 2x + 2 + 3y + 3 \\ & \Leftrightarrow z = -2x + 3y + 3 \end{aligned}$$