Resolução do 1^o TESTE DE ÁLGEBRA LINEAR

Licenciatura em Engenharia Informática e de Computadores - Alameda

1) $[A \mid B] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ \alpha & 0 & -1 & -2 \\ 0 & \alpha & -1 & -1 \end{array}\right] \underset{-\alpha L_1 + L_2 \to L_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & \alpha & -1 & \alpha - 2 \\ 0 & \alpha & -1 & -1 \end{array}\right] \underset{-L_2 + L_3 \to L_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & \alpha & -1 & \alpha - 2 \\ 0 & 0 & 0 & 1 - \alpha \end{array}\right]$

O sistema é possível e indeterminado se e só se car $A =$ car $[A \mid B] < 3$ ($= n^o$ de colunas de A) se e só se $\alpha = 1$. Se $\alpha = 1$ então a solução geral é: $\{(1+s, 2+s, 3+s) : s \in \mathbb{R}\}$.

2) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}^{-1} (2I - A) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \Leftrightarrow A = 2I - \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

3) $E_{23}(1)E_{12}(1)A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow A = (E_{12}(1))^{-1} (E_{23}(1))^{-1} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ou seja $A = LU$ com $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ e $U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

4) Atendendo a $\left[\begin{array}{cc|c} 2 & 0 & a_0 \\ 1 & 1 & a_1 \\ 1 & 1 & a_2 \end{array}\right] \underset{\cdots}{\rightarrow} \left[\begin{array}{cc|c} 2 & 0 & a_0 \\ 1 & 1 & a_1 \\ 0 & 0 & -a_1 + a_2 \end{array}\right]$, $a_0 + a_1 t + a_2 t^2 \in U \Leftrightarrow -a_1 + a_2 = 0$.

Atendendo a $\left[\begin{array}{cc|c} 1 & 1 & a_0 \\ 1 & 0 & a_1 \\ 1 & 1 & a_2 \end{array}\right] \underset{\cdots}{\rightarrow} \left[\begin{array}{cc|c} 1 & 1 & a_0 \\ 1 & 0 & a_1 \\ 0 & 0 & -a_0 + a_2 \end{array}\right]$, $a_0 + a_1 t + a_2 t^2 \in V \Leftrightarrow -a_0 + a_2 = 0$. Logo

$$\begin{aligned} U \cap V &= \{a_0 + a_1 t + a_2 t^2 : -a_1 + a_2 = 0 \text{ e } -a_0 + a_2 = 0\} = \\ \underset{a_1 = a_2 \ \ a_0 = a_2}{=} & \{a_2 + a_2 t + a_2 t^2 : a_2 \in \mathbb{R}\} = \{a_2 \left(1 + t + t^2\right) : a_2 \in \mathbb{R}\} = L(\{1 + t + t^2\}). \end{aligned}$$

5) Atendendo a $\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 1 & 0 & 0 & c \\ 1 & 1 & 1 & d \end{array}\right] \underset{\cdots}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 0 & b \\ 1 & 0 & 0 & c \\ 0 & 0 & 0 & -a + d \end{array}\right]$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W \Leftrightarrow -a + d = 0$.

Logo $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R}) : -a + d = 0 \right\}$. Considerando $A \notin W$ por exemplo $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ tem-se

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 0 & 0 & b \\ 1 & 0 & 0 & 0 & c \\ 1 & 1 & 1 & 0 & d \end{array}\right] \underset{\cdots}{\rightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & -1 & -1 & -a + b + c \\ 0 & 0 & 0 & -1 & -a + d \end{array}\right]$$

logo $\mathcal{M}_{2\times 2}(\mathbb{R}) \subset W + L(\{A\})$. A inclusão $W + L(\{A\}) \subset \mathcal{M}_{2\times 2}(\mathbb{R})$ decorre de se ter $W \subset \mathcal{M}_{2\times 2}(\mathbb{R})$ e $L(\{A\}) \subset \mathcal{M}_{2\times 2}(\mathbb{R})$. Assim: $W + L(\{A\}) = \mathcal{M}_{2\times 2}(\mathbb{R})$.

6) Seja $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ tal que $\mathcal{N}(A) = \mathcal{N}(A^2)$. Vejamos que $\mathcal{N}(A) \cap \mathcal{C}(A) \subset \{\mathbf{0}\}$ uma vez que a inclusão $\{\mathbf{0}\} \subset \mathcal{N}(A) \cap \mathcal{C}(A)$ é óbvia. Seja $u \in \mathcal{N}(A) \cap \mathcal{C}(A)$. Então $Au = \mathbf{0}$ e existe v tal que $u = Av$. Mas $A^2 v = A(Av) = Au = \mathbf{0}$ ou seja $v \in \mathcal{N}(A^2)$ e como $\mathcal{N}(A) = \mathcal{N}(A^2)$ então $v \in \mathcal{N}(A)$, isto é, $u = Av = \mathbf{0}$. Logo $\mathcal{N}(A) \cap \mathcal{C}(A) \subset \{\mathbf{0}\}$ e assim

$$\mathcal{N}(A) \cap \mathcal{C}(A) = \{\mathbf{0}\}.$$