Optimization and Algorithms
February 6, 2023

Write your name: ____________________________

Write your student number: ____________________________

Write your answers (A, B, C, D, E, or F) to problems 1 to 3 in this box

Your answer to problem 1: __________

Your answer to problem 2: __________

Your answer to problem 3: __________

Exam

1. *Deconflicted trajectories.* A trajectory $\mathcal{T}$ of duration T in $\mathbf{R}^d$ is a sequence of T points in $\mathbf{R}^d$, denoted as $\mathcal{T} = \{x(1), x(2), \ldots, x(T)\}$, with $x(t) \in \mathbf{R}^d$ for $1 \leq t \leq T$. Note that t denotes discrete-time; thus t is an integer (such as $t = 0, 1, 2, 3. \ldots$).

Let $\mathcal{T}_1 = \{x_1(1), x_1(2), \ldots, x_1(T)\}$ and $\mathcal{T}_2 = \{x_2(1), x_2(2), \ldots, x_2(T)\}$ be two trajectories of duration T in $\mathbf{R}^d$. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are space-deconflicted if $\|x_1(t) - x_2(s)\|_2 > \epsilon$ for $1 \leq t, s \leq T$, where ϵ is a given positive number. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are time-deconflicted if $\|x_1(t) - x_2(t)\|_2 > \epsilon$ for $1 \leq t \leq T$.

Consider the following two controlled dynamic linear systems. The state of system 1 at time t is denoted by $x_1(t) \in \mathbf{R}^d$, for $1 \leq t \leq T$ and obeys the recursion

$$x_1(t+1) = A_1 x_1(t) + B_1 u_1(t), \quad 0 \leq t \leq T-1,$$

where $A_1 \in \mathbf{R}^{d \times d}$ and $B_1 \in \mathbf{R}^{d \times p}$ are given matrices, $x_1(0) \in \mathbf{R}^d$ is a given initial state and $u_1(t) \in \mathbf{R}^p$ is the control input of system 1 at time t, for $0 \leq t \leq T-1$. Note that the trajectory $\mathcal{T}_1$ depends on the inputs $u_1(t)$, $0 \leq t \leq T-1$.

Similarly, for system 2 we have

$$x_2(t+1) = A_2 x_2(t) + B_2 u_2(t), \quad 0 \leq t \leq T-1.$$

Note that the trajectory $\mathcal{T}_2$ depends on the inputs $u_2(t)$, $0 \leq t \leq T-1$.

Finally, let $\mathcal{T}_{\text{ref}} = \{r(1), r(2), \ldots, r(T)\}$ be a given, fixed reference trajectory of duration T in $\mathbf{R}^d$.

We want to design the control inputs $u_1(t)$ $(0 \leq t \leq T-1)$ and $u_2(t)$ $(0 \leq t \leq T-1)$ so that:

- the final state $x_1(T)$ of system 1 is as close as possible to a given, desired state $p_1 \in \mathbf{R}^d$;

- the final state $x_2(T)$ of system 2 is as close as possible to a given, desired state $p_2 \in \mathbf{R}^d$;
- the trajectories $\mathcal{T}_1$ and $\mathcal{T}_2$ are time-deconflicted;
- the trajectories $\mathcal{T}_1$ and $\mathcal{T}_{\text{ref}}$ are space-deconflicted;
- the trajectories $\mathcal{T}_2$ and $\mathcal{T}_{\text{ref}}$ are space-deconflicted.

One of the following problem formulations is suitable for the given context.

(A)

$$\begin{array}{ll} \underset{\{x_1(t),u_1(t),x_2(t),u_2(t)\}}{\text{minimize}} & \|x_1(T)-p_1\|_2^2 + \|x_2(T)-p_2\|_2^2 \\ \text{subject to} & x_1(t+1) = A_1x_1(t) + B_1u_1(t), \quad 0 \le t \le T-1 \\ & x_2(t+1) = A_2x_2(t) + B_2u_2(t), \quad 0 \le t \le T-1 \\ & \min\{\|x_1(t)-x_2(s)\|_2 : 1, s \le t \le T\} < \epsilon \\ & \min\{\|x_1(t)-r(s)\|_2 : 1 \le t, s \le T\} > \epsilon \\ & \min\{\|x_2(t)-r(s)\|_2 : 1 \le t, s \le T\} > \epsilon \end{array} \tag{1}$$

(B)

$$\begin{array}{ll} \underset{\{x_1(t),u_1(t),x_2(t),u_2(t)\}}{\text{minimize}} & \|x_1(T)-p_1\|_2^2 + \|x_2(T)-p_2\|_2^2 \\ \text{subject to} & x_1(t+1) = A_1x_1(t) + B_1u_1(t), \quad 0 \le t \le T-1 \\ & x_2(t+1) = A_2x_2(t) + B_2u_2(t), \quad 0 \le t \le T-1 \\ & \max\{\|x_1(t)-x_2(t)\|_2 : 1 \le t \le T\} > \epsilon \\ & \max\{\|x_1(t)-r(s)\|_2 : 1 \le t, s \le T\} > \epsilon \\ & \max\{\|x_2(t)-r(s)\|_2 : 1 \le t, s \le T\} > \epsilon \end{array} \tag{2}$$

(C)

$$\begin{array}{ll} \underset{\{x_1(t),u_1(t),x_2(t),u_2(t)\}}{\text{minimize}} & \|x_1(T)-p_1\|_2^2 + \|x_2(T)-p_2\|_2^2 \\ \text{subject to} & x_1(t+1) = A_1x_1(t) + B_1u_1(t), \quad 0 \le t \le T-1 \\ & x_2(t+1) = A_2x_2(t) + B_2u_2(t), \quad 0 \le t \le T-1 \\ & \min\{\|x_1(t)-x_2(t)\|_2 : 1 \le t \le T\} < \epsilon \\ & \min\{\|x_1(t)-r(s)\|_2 : 1 \le t, s \le T\} < \epsilon \\ & \min\{\|x_2(t)-r(s)\|_2 : 1 \le t, s \le T\} < \epsilon \end{array} \tag{3}$$

(D)

$$\begin{array}{ll} \underset{\{x_1(t),u_1(t),x_2(t),u_2(t)\}}{\text{minimize}} & \|x_1(T)-p_1\|_2^2 + \|x_2(T)-p_2\|_2^2 \\ \text{subject to} & x_1(t+1) = A_1x_1(t) + B_1u_1(t), \quad 0 \le t \le T-1 \\ & x_2(t+1) = A_2x_2(t) + B_2u_2(t), \quad 0 \le t \le T-1 \\ & \max\{\|x_1(t)-x_2(t)\|_2 : 1 \le t \le T\} < \epsilon \\ & \max\{\|x_1(t)-r(s)\|_2 : 1 \le t, s \le T\} < \epsilon \\ & \max\{\|x_2(t)-r(s)\|_2 : 1 \le t, s \le T\} < \epsilon \end{array} \tag{4}$$

(E)

$$
\begin{array}{ll}
\underset{\{x_1(t),u_1(t),x_2(t),u_2(t)\}}{\text{minimize}} & \|x_1(T)-p_1\|_2^2+\|x_2(T)-p_2\|_2^2 \\
\text{subject to} & x_1(t+1)=A_1x_1(t)+B_1u_1(t), \quad 0\le t\le T-1 \\
& x_2(t+1)=A_2x_2(t)+B_2u_2(t), \quad 0\le t\le T-1 \\
& \min\{\|x_1(t)-x_2(t)\|_2 : 1\le t\le T\}>\epsilon \\
& \min\{\|x_1(t)-r(s)\|_2 : 1\le t,s\le T\}>\epsilon \\
& \min\{\|x_2(t)-r(s)\|_2 : 1\le t,s\le T\}>\epsilon
\end{array} \tag{5}
$$

(F)

$$
\begin{array}{ll}
\underset{\{x_1(t),u_1(t),x_2(t),u_2(t)\}}{\text{minimize}} & \|x_1(T)-p_1\|_2^2+\|x_2(T)-p_2\|_2^2 \\
\text{subject to} & x_1(t+1)=A_1x_1(t)+B_1u_1(t), \quad 0\le t\le T-1 \\
& x_2(t+1)=A_2x_2(t)+B_2u_2(t), \quad 0\le t\le T-1 \\
& \min\{\|x_1(t)-x_2(s)\|_2 : 1\le t,s\le T\}>\epsilon \\
& \min\{\|x_1(t)-r(s)\|_2 : 1\le t,s\le T\}>\epsilon \\
& \min\{\|x_2(t)-r(s)\|_2 : 1\le t,s\le T\}>\epsilon
\end{array} \tag{6}
$$

Which one?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

2. *Unconstrained optimization.* Consider the optimization problem

$$\underset{x\in\mathbf{R}}{\text{minimize}} \quad e^{x-a}+e^{-x}+x^2-2x+x_+. \tag{7}$$

The point $x^\star=0$ is a global minimizer of (7) for one of the following choices of a:

(A) $a=-2$

(B) $a=-1$

(C) $a=0$

(D) $a=1$

(E) $a=2$

(F) $a=3$

Which one?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

Hint: the numerical values $\log(2)\simeq 0.7$ and $\log(3)\simeq 1.1$ might be useful

3. *Gradient descent algorithm.* Consider the function $f\colon \mathbf{R}^2\to\mathbf{R}$ given by $f(a,b)=\frac{1}{2}a^2+(a-b)^2$. Suppose we do one iteration of the gradient descent algorithm (applied to f) starting from the point

$$x_0=\begin{bmatrix}1\\1\end{bmatrix},$$

and using the stepsize 1.

Which of the following points is the next iteration x_1?

(A)

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(B)

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(C)

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(D)

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(E)

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

(F)

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

4. *Signal-denoising as a least-squares problem.* Consider the function $f\colon \mathbf{R}^n \to \mathbf{R}$, $f(r) = r^T D r$, where D is a given $n \times n$ diagonal matrix with positive diagonal entries:

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix},$$

with $d_i > 0$ for $1 \leq i \leq n$.

Consider the following optimization problem

$$\begin{array}{ll} \underset{s,v}{\text{minimize}} & \|s - \overline{s}\|_2^2 + f(v) \\ \text{subject to} & y = As + v, \end{array} \tag{8}$$

where the variables to optimize are $s \in \mathbf{R}^p$ and $v \in \mathbf{R}^n$; the matrix $A \in \mathbf{R}^{n \times p}$ and the vectors $y \in \mathbf{R}^n$, and $\overline{s} \in \mathbf{R}^p$ are given. This problem can be interpreted as a signal-denoising problem: we observe y and want to decompose it as the sum of a signal of interest s and noise v; we know that s should be close to the nominal signal $\overline{s}$ and that v should be close to zero (the larger the d_i, the more confident we are that the component v_i should be close to zero).

Problem (8) can be reduced to a least-squares problem involving only the variable s, that is, it can be reduced to a problem of the form

$$\underset{s}{\text{minimize}} \quad \|\mathcal{A}s - \beta\|_2^2 \tag{9}$$

for some matrix $\mathcal{A}$ and vector β.

Give $\mathcal{A}$ and β in terms of the constants D, y, A, and $\overline{s}$.

5. *A simple optimization problem.* Consider the function $f\colon \mathbf{R}^2 \to \mathbf{R}$, $f(x) = \frac{1}{2}x^T M x$, where

$$M = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

The constants a and b satisfy $0 < a < b$.

Solve in closed-form the optimization problem

$$\begin{array}{ll} \underset{x}{\text{maximize}} & f(x) \\ \text{subject to} & \mathbf{1}^T x = 1, \end{array} \tag{10}$$

where $\mathbf{1}$ denotes the vector $\mathbf{1} = (1, 1)$.

6. *A convex optimization problem.* Consider the following optimization problem

$$\begin{array}{ll} \underset{x_1, x_2, \ldots, x_n}{\text{minimize}} & g(x_1 - c_1) + g(x_2 - c_2) + \cdots + g(x_n - c_n) \\ \text{subject to} & a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b, \end{array} \tag{11}$$

where the variables to optimize are $x_i \in \mathbf{R}$, for $1 \leq i \leq n$. The vectors $a_i \in \mathbf{R}^p$, $1 \leq i \leq n$ and $b \in \mathbf{R}^p$ are given. The constants c_i, $1 \leq i \leq n$, are also given. The function $g\colon \mathbf{R} \to \mathbf{R}$ is defined as follows:

$$g(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

Show that (11) is a convex optimization problem.

7. *A convex function based on a worst-case representation.* Show that the function $f\colon \mathbf{R} \to \mathbf{R}$,

$$f(x) = \max\left\{ \|(a+u)x - b\|_2 : \|u\|_2 = r \right\} \tag{12}$$

is convex, where the vectors $a, b \in \mathbf{R}^n$ and the constant $r > 0$ are given.

In words: f takes as input a number x and returns as output the largest value of the expression

$$\|(a+u)x - b\|_2$$

as u ranges over the sphere centered at the origin and with radius r.