CÁLCULO DIFERENCIAL E INTEGRAL III

TRANSFORMAÇÃO DE LAPLACE

EXERCÍCIOS

1. Através da definição, determine a transformada de Laplace das funções

(a)

$$f(t) = \begin{cases} 0 & \text{se } t \in [0,1[, \\ 1 & \text{se } t \geq 1. \end{cases}$$

(b)

$$g(t) = \begin{cases} t & \text{se } t \in [0,1], \\ 1 & \text{se } t > 1. \end{cases}$$

(c)

$$h(t) = \begin{cases} \operatorname{sen} t & \text{se } t \in [0,\pi], \\ 0 & \text{se } t > \pi. \end{cases}$$

2. Através das suas propriedades, determine a transformada de Laplace das funções a seguir indicadas, onde por H se designa a função de Heaviside e por a e b números reais.

(a) $\cosh(bt)$.

(b) $\operatorname{senh}(bt)$.

(c) $e^{at}\cosh(bt)$.

(d) $e^{at}\operatorname{senh}(bt)$.

(e) $e^{at}\operatorname{sen}(bt)$.

(f) $e^{at}\cos(bt)$.

(g) $t\operatorname{senh}(bt)$.

(h) $t^2\operatorname{senh}(bt)$.

(i) $t\operatorname{sen}(bt)$.

(j) $t^2\operatorname{sen}(bt)$.

(k) $H_1(t) + 3e^{-(t+6)}$.

(l) $H_1(t)\,\mathrm{sen}(t-1)$.

3. Com $n = 0, 1, 2, ...$, seja

$$f(t) = \begin{cases} 1 & \text{se } t \in [2n, 2n+1], \\ 0 & \text{se } t \in]2n+1, 2n+2[. \end{cases}$$

Mostre que, para $\mathrm{Re}(s) > 0$,

$$\mathcal{L}[f](s) = \frac{1}{s(1+e^{-s})}$$

4. Utilize a transformada de Laplace para resolver os seguintes problemas de valor inicial:

(a) $y' + y = \cos t, \quad y(0) = 1$.

(b) $y'' + 2y' + 5y = 0, \; y(0) = 2, \quad y'(0) = -1$.

5. Indique as inversas da transformada de Laplace das seguintes funções:

(a) $F(s) = \dfrac{s}{(s+1)(s+2)}$.

(b) $F(s) = \dfrac{s^2}{s^3 - 1}$.

(c) $F(s) = \dfrac{s+1}{s\,(s+3)^2}$.

(d) $F(s) = \dfrac{1}{(s+1)^3}$.

(e) $F(s) = \dfrac{e^{-s}}{s^2+1}$.

(f) $F(s) = \dfrac{e^{-3s}}{(s^2+1)(s^2+4)}$.

(g) $F(s) = \dfrac{e^{-s}}{s} + \dfrac{s}{(s+1)^2}$.

6. Resolva os seguintes problemas de valor inicial:

(a) $y'' + y = H_1(t), \quad y(0) = 0, \quad y'(0) = 0$.

(b) $y'' + y = \mathrm{sen}\, t, \quad y(0) = 0, \quad y'(0) = 1$.

(c) $y'' - 2y' + 2y = \cos t, \quad y(0) = 1, \quad y'(0) = 0$.

(d) $y'' + 2y' + y = 4e^{-t}, \quad y(0) = 2, \quad y'(0) = -1$.

(e) $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0, \quad y(0) = y''(0) = 0, \quad y'(0) = y'''(0) = 1$.

7. Resolva os problemas de valor inicial indicados a seguir, onde δ designa o delta de Dirac.

(a) $y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0.$

(b) $y'' + y = \delta(t - \pi) + \cos t, \quad y(0) = 0, \quad y'(0) = 1.$

RESPOSTAS

1. **(a)** $\mathcal{L}[f](s) = \dfrac{e^{-s}}{s}, \quad (\operatorname{Re} s > 0).$

(b) $\mathcal{L}[g](s) = \dfrac{1 - e^{-s}}{s^2}, \quad (\operatorname{Re} s > 0).$

(c) $$\mathcal{L}[h](s) = \begin{cases} \dfrac{1 + e^{-\pi s}}{s^2 + 1} & \text{se } s \neq \pm i, \\[2ex] \dfrac{\pi\, i}{2} & \text{se } s = -i, \\[2ex] -\dfrac{\pi\, i}{2} & \text{se } s = i. \end{cases}$$

2. **(a)** $\dfrac{s}{s^2 - b^2}, \quad (\operatorname{Re} s > |b|).$

(b) $\dfrac{b}{s^2 - b^2}, \quad (\operatorname{Re} s > |b|).$

(c) $\dfrac{s - a}{(s - a)^2 - b^2}, \quad (\operatorname{Re} s > |b| + a).$

(d) $\dfrac{b}{(s - a)^2 - b^2}, \quad (\operatorname{Re} s > |b| + a).$

(e) $\dfrac{b}{(s - a)^2 + b^2}, \quad (\operatorname{Re} s > a).$

(f) $\dfrac{s - a}{(s - a)^2 + b^2}, \quad (\operatorname{Re} s > a).$

(g) $\dfrac{2bs}{(s^2 - b^2)^2}, \quad (\operatorname{Re} s > |b|).$

(h) $\dfrac{6bs^2 + 2b^3}{(s^2 - b^2)^3}, \quad (\operatorname{Re} s > |b|).$

(i) $\dfrac{2bs}{(s^2 + b^2)^2}, \quad (\operatorname{Re} s > 0).$

(j) $\dfrac{6bs^2 - 2b^3}{(s^2 + b^2)^3}, \quad (\operatorname{Re} s > 0).$

(k) $\dfrac{e^{-s}}{s} + \dfrac{3}{e^6\,(s + 1)}, \quad (\operatorname{Re} s > 0).$

(l) $\dfrac{e^{-s}}{s^2 + 1}, \quad (\operatorname{Re} s > 0).$

4. **(a)** $y(t) = \frac{1}{2}\,(e^{-t} + \cos t + \operatorname{sen} t)$.

(b) $y(t) = 2\,e^{-t}\cos(2t) + \frac{1}{2}\,e^{-t}\operatorname{sen}(2t)$.

5. **(a)** $f(t) = 2\,e^{-2t} - e^{-t}$.

(b) $f(t) = \dfrac{1}{3}\Big(e^{t} + 2\,e^{-t/2}\cos\big(\tfrac{\sqrt{3}}{2}\,t\big)\Big)$.

(c) $f(t) = \dfrac{1}{9} + \dfrac{e^{-3t}}{9}\,(6t-1)$.

(d) $f(t) = \frac{1}{2}\,t^2\,e^{-t}$.

(e) $f(t) = \mathrm{H}_1(t)\operatorname{sen}(t-1)$.

(f) $f(t) = \frac{1}{3}\,\mathrm{H}_3(t)\operatorname{sen}(t-3) - \frac{1}{6}\,\mathrm{H}_3(t)\operatorname{sen}(2t-6)$.

(g) $f(t) = \mathrm{H}_1(t) + e^{-t}\,(1-t)$.

6. **(a)** $y(t) = \mathrm{H}_1(t)\,(1 - \cos(t-1))$.

(b) $y(t) = \frac{3}{2}\operatorname{sen} t - \frac{1}{2}\,t\cos t$.

(c) $y(t) = \frac{1}{5}\,(4\,e^{t}\cos t - 2\,e^{t}\operatorname{sen} t - 2\operatorname{sen} t + \cos t)$.

(d) $y(t) = (2\,t^2 + t + 2)\,e^{-t}$.

(e) $y(t) = (t - t^2 + \frac{2}{3}\,t^3)\,e^{t}$.

7. **(a)** $y(t) = e^{-t}\,(\cos t + \operatorname{sen} t) + \mathrm{H}_\pi(t)\,e^{-(t-\pi)}\operatorname{sen}(t-\pi)$.

(b) $y(t) = \mathrm{H}_\pi(t)\operatorname{sen}(t-\pi) + \frac{1}{2}\,t\operatorname{sen} t + \operatorname{sen} t$.