Optimization and Algorithms
October 19, 2022

Write your name: _____

Write your student number: _____

Write your answers (A, B, C, D, E, or F) to problems 1 to 4 in this box

Your answer to problem 1: _____

Your answer to problem 2: _____

Your answer to problem 3: _____

Your answer to problem 4: _____

Quiz

1. *Moving as far away as possible from a point.* Consider a vehicle that moves in the plane $\mathbf{R}^2$. The state of the vehicle at time t is denoted by $x(t) \in \mathbf{R}^4$. The first two components of $x(t)$ correspond to the position of the vehicle at time t; the last two components of $x(t)$ correspond to the velocity of the vehicle at time t.

   The initial state of the vehicle (at time $t = 1$) is given and is denoted by x_{init}. Assume that the state of the vehicle evolves as

   $$x(t + 1) = Ax(t) + Bu(t), \quad t = 1, 2, \ldots, T - 1,$$

   where the matrices $A \in \mathbf{R}^{4 \times 4}$ and $B \in \mathbf{R}^{4 \times 2}$ are given, and $u(t)$ represents a control signal.

   We wish to design the control signal $u(t)$, for $t = 1, 2, \ldots, T-1$, such that the vehicle is located as far away as possible from a given point $b \in \mathbf{R}^2$ at time $t = T$ (that is, at the end of the time-horizon $\{1, 2, \ldots, T\}$).

   Also, as the vehicle is moving (from $t = 1$ to $t = T$), it should never enter a given dangerous disk $D = \{p \in \mathbf{R}^2 \colon \|p - c\|_2 < R\}$, where the center c and the radius R of the disk are given.

   Finally, there is an upper bound on the magnitude of the control signal (as measured by the Euclidean norm $\|\cdot\|_2$). Specifically, the magnitude of $u(t)$ should never exceed a given limit $U > 0$, for $t = 0$ to $t = T - 1$.

   Let

   $$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

   and consider the following problem formulations:

(A)

$$\begin{array}{ll}
\underset{x,u}{\text{minimize}} & \|Ex(T) - b\|_2 + \sum_{t=1}^{T-1} (\|u(t)\|_2 - U)^2 \\
\text{subject to} & x(1) = x_{\text{init}} \\
& x(t+1) = Ax(t) + Bu(t), \quad t = 1, \ldots, T-1 \\
& \|Ex(t) - c\|_2 \geq R, \quad t = 1, \ldots, T
\end{array}$$

(B)

$$\begin{array}{ll}
\underset{x,u}{\text{minimize}} & \|Ex(T) - b\|_2 \\
\text{subject to} & x(1) = x_{\text{init}} \\
& x(t+1) = Ax(t) + Bu(t), \quad t = 1, \ldots, T-1 \\
& \|Ex(t) - c\|_2 < R, \quad t = 1, \ldots, T \\
& \|u(t)\|_2 \leq U, \quad t = 1, \ldots, T-1
\end{array}$$

(C)

$$\begin{array}{ll}
\underset{x,u}{\text{minimize}} & -\|Ex(T) - b\|_2 \\
\text{subject to} & x(1) = x_{\text{init}} \\
& x(t+1) = Ax(t) + Bu(t), \quad t = 1, \ldots, T-1 \\
& \|Ex(t) - c\|_2 \geq R, \quad t = 1, \ldots, T \\
& \|u(t)\|_2 \leq U, \quad t = 1, \ldots, T-1
\end{array}$$

(D)

$$\begin{array}{ll}
\underset{x,u}{\text{minimize}} & -\|Ex(T) - b\|_2 \\
\text{subject to} & x(1) = x_{\text{init}} \\
& x(t+1) = Ax(t) + Bu(t), \quad t = 1, \ldots, T-1 \\
& \|Ex(t) - c\|_2 \geq R, \quad t = 1, \ldots, T \\
& \|u(t)\|_2 > U, \quad t = 1, \ldots, T-1
\end{array}$$

(E)

$$\begin{array}{ll}
\underset{x,u}{\text{minimize}} & -\|Ex(T) - b\|_2 + \sum_{t=1}^{T-1} (\|u(t)\|_2 - U)^2 \\
\text{subject to} & x(1) = x_{\text{init}} \\
& x(t+1) = Ax(t) + Bu(t), \quad t = 1, \ldots, T-1 \\
& \|Ex(t) - c\|_2 \geq R, \quad t = 1, \ldots, T
\end{array}$$

(F)

$$\begin{array}{ll}
\underset{x,u}{\text{minimize}} & \|Ex(T) - b\|_2 \\
\text{subject to} & x(1) = x_{\text{init}} \\
& x(t+1) = Ax(t) + Bu(t), \quad t = 1, \ldots, T-1 \\
& \|Ex(t) - c\|_2 \geq R, \quad t = 1, \ldots, T \\
& \|u(t)\|_2 > U, \quad t = 1, \ldots, T-1
\end{array}$$

For the given context, one of the formulations above is appropriate. Which one?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

2. *Unconstrained optimization.* Consider the following optimization problems:

(A)

$$\underset{x \in \mathbf{R}}{\text{minimize}} \quad e^x + \tfrac{1}{2}|x|$$

(B)
$$\underset{x \in \mathbf{R}}{\text{minimize}} \quad e^{2|x|} - \tfrac{1}{2}|x|$$

(C)
$$\underset{x \in \mathbf{R}}{\text{minimize}} \quad e^{2|x|} - x$$

(D)
$$\underset{x \in \mathbf{R}}{\text{minimize}} \quad e^{2|x|} + x$$

(E)
$$\underset{x \in \mathbf{R}}{\text{minimize}} \quad e^{|x|} + \tfrac{1}{2}|x|$$

(F)
$$\underset{x \in \mathbf{R}}{\text{minimize}} \quad \tfrac{1}{2}e^{x} + |x|$$

The point $x^\star = 0$ is not a global minimizer for one of the problems above. Which one?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

3. *Least-squares.* For a vector $v = (v_1, v_2, \ldots, v_n) \in \mathbf{R}^n$, the symbol $C(v)$ denotes the $n \times n$ (circulant) matrix obtained by letting the vector v become its first column and then rotating v, one component at a time, to get the remaining columns; for example, for $v = (v_1, v_2, v_3, v_4) \in \mathbf{R}^4$, we have

$$C(v) = \begin{bmatrix} v_1 & v_4 & v_3 & v_2 \\ v_2 & v_1 & v_4 & v_3 \\ v_3 & v_2 & v_1 & v_4 \\ v_4 & v_3 & v_2 & v_1 \end{bmatrix}.$$

Consider the following optimization problem

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad \|x - r\|_2^2 + \rho \sum_{k=1}^{K} \|C(x)a_k - b_k\|_2^2 \ , \tag{1}$$

where the vectors $a_k \in \mathbf{R}^n$ and $b_k \in \mathbf{R}^n$ are given for $1 \le k \le K$. The vector $r \in \mathbf{R}^n$ and the constant $\rho > 0$ are also given.

Problem (1) can be rewritten as a least-squares problem, that is, as a problem of the form

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad \|\mathcal{A}x - \beta\|_2^2 \ . \tag{2}$$

Consider the following pairs $\mathcal{A}$ and β:

(A)
$$\mathcal{A} = \begin{bmatrix} \rho a_1^T \\ \rho a_2^T \\ \vdots \\ \rho a_K^T \\ I_n \end{bmatrix} \qquad \beta = \begin{bmatrix} \|b_1\|_2^2 \\ \|b_2\|_2^2 \\ \vdots \\ \|b_K\|_2^2 \\ r \end{bmatrix}$$

3

(B)

$$\mathcal{A} = \begin{bmatrix} \sqrt{\rho}C(a_1 - b_1) \\ \sqrt{\rho}C(a_2 - b_2) \\ \vdots \\ \sqrt{\rho}C(a_K - b_K) \\ I_n \end{bmatrix} \qquad \beta = \begin{bmatrix} 0_n \\ 0_n \\ \vdots \\ 0_n \\ r \end{bmatrix}.$$

(C)

$$\mathcal{A} = \begin{bmatrix} \sqrt{\rho}C(a_1) \\ \sqrt{\rho}C(a_2) \\ \vdots \\ \sqrt{\rho}C(a_K) \\ I_n \end{bmatrix} \qquad \beta = \begin{bmatrix} \sqrt{\rho}b_1 \\ \sqrt{\rho}b_2 \\ \vdots \\ \sqrt{\rho}b_K \\ r \end{bmatrix}$$

(D)

$$\mathcal{A} = \sqrt{\rho}\begin{bmatrix} C(a_1) \\ C(a_2) \\ \vdots \\ C(a_K) \\ I_n \end{bmatrix} \qquad \beta = \sqrt{\rho}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_K \\ r \end{bmatrix}$$

(E)

$$\mathcal{A} = \begin{bmatrix} \sqrt{\rho}a_1^T \\ \sqrt{\rho}a_2^T \\ \vdots \\ \sqrt{\rho}a_K^T \\ I_n \end{bmatrix} \qquad \beta = \begin{bmatrix} \|b_1\|_2^2 \\ \|b_2\|_2^2 \\ \vdots \\ \|b_K\|_2^2 \\ r \end{bmatrix}$$

(F)

$$\mathcal{A} = \begin{bmatrix} \rho C(a_1) \\ \rho C(a_2) \\ \vdots \\ \rho C(a_K) \\ I_n \end{bmatrix} \qquad \beta = \begin{bmatrix} \rho b_1 \\ \rho b_2 \\ \vdots \\ \rho b_K \\ r \end{bmatrix}$$

(The symbol I_n denotes the $n \times n$ identity matrix, and the symbol 0_n denotes the n-dimensional vector whose components are all zero.)

Which of the pairs $\mathcal{A}$ and β above makes problem (1) equivalent to the least-squares (2)?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1

4. *Closed-form solution.* For a vector $v = (v_1, v_2, \ldots, v_n) \in \mathbf{R}^n$, the symbol $D(v)$ denotes the $n \times n$ (diagonal) matrix obtained by displaying the vector v along its

diagonal; for example, for $v = (v_1, v_2, v_3, v_4) \in \mathbf{R}^4$, we have

$$D(v) = \begin{bmatrix} v_1 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 \\ 0 & 0 & v_3 & 0 \\ 0 & 0 & 0 & v_4 \end{bmatrix}.$$

Consider the optimization problem

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad \|(A + RD(x))\,\theta + b\|_2^2, \tag{3}$$

where the matrices $A \in \mathbf{R}^{m \times n}$ and $R \in \mathbf{R}^{m \times n}$, and the vectors $\theta \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$ are given. Assume that the columns of R are linearly independent and all components of the vector θ are nonzero.

One of the following vectors is a global minimizer of (3):

(A)
$$x^\star = -D(\theta)\left(R^T R\right) R^T (A\theta + b)$$

(B)
$$x^\star = D(\theta)\left(R^T R\right)^{-1} R^T (A\theta - b)$$

(C)
$$x^\star = -D(\theta)^{-1}\left(R^T R\right)^{-1} R^T (A\theta + b)$$

(D)
$$x^\star = D(\theta)^{-1}\left(R^T R\right)^{-1} R^T (A\theta - b)$$

(E)
$$x^\star = D(\theta)\left(R^T R\right)^{-1} R^T (A\theta + b)$$

(F)
$$x^\star = D(\theta)^{-1}\left(R^T R\right) R^T (A\theta + b)$$

Which one?

Write your answer (A, B, C, D, E, or F) in the box at the top of page 1