Ficha 10
Resolução dos exercícios propostos

I. 1 - Determine as derivadas parciais de primeira ordem das seguintes funções:

a) $f(x,y)=\frac{3}{4}xy-1$

Resolução:

- $$\frac{\partial f}{\partial x}(x,y)=\frac{\partial}{\partial x}\left(\frac{3}{4}xy-1\right)\underset{\substack{\text{y é uma constante}\\\text{na derivação em}\\\text{ordem a x}}}{=}\frac{3}{4}y\frac{\partial(x)}{\partial x}-0=\frac{3}{4}y.1=\frac{3}{4}y$$

- $$\frac{\partial f}{\partial y}(x,y)=\frac{\partial}{\partial y}\left(\frac{3}{4}xy-1\right)\underset{\substack{\text{x é uma constante}\\\text{na derivação em}\\\text{ordem a y}}}{=}\frac{3}{4}x\frac{\partial}{\partial y}(y)=\frac{3}{4}x.1=\frac{3}{4}x$$

b) $f(x,y,z)=5\cdot\sin(2xy+z)+2x^2-3xy+6z^2$

Resolução:

- $$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial\left(5\cdot\sin(2xy+z)+2x^2-3xy+6z^2\right)}{\partial x}=\frac{\partial\left(5\cdot\sin(2xy+z)\right)}{\partial x}+\frac{\partial\left(2x^2\right)}{\partial x}+\frac{\partial(-3xy)}{\partial x}+\frac{\partial\left(6z^2\right)}{\partial x}$$
$$=5\frac{\partial\left(\sin(2xy+z)\right)}{\partial x}+\frac{2\partial\left(x^2\right)}{\partial x}-3y\frac{\partial(x)}{\partial x}+0=5\cdot 2y\cdot\cos(2xy+z)+4x-3y=10y\cos(2xy+z)+4x-3y$$

- $$\frac{\partial f}{\partial y}(x,y,z)=\frac{\partial\left(5\cdot\sin(2xy+z)\right)}{\partial y}+\frac{\partial\left(2x^2\right)}{\partial y}+\frac{\partial(-3xy)}{\partial y}+\frac{\partial\left(6z^2\right)}{\partial y}=5\frac{\partial(2xy+z)}{\partial y}\cdot\cos(2xy+z)-3x\frac{\partial(y)}{\partial y}+0$$
$$=5\cdot 2x\cdot\cos(2xy+z)-3x=10x\cos(2xy+z)-3x$$

- $$\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial\left(5\cdot\sin(2xy+z)\right)}{\partial z}+\frac{\partial\left(2x^2\right)}{\partial z}+\frac{\partial(-3xy)}{\partial z}+\frac{\partial\left(6z^2\right)}{\partial z}=5\frac{\partial(2xy+z)}{\partial z}\cos(2xy+z)+0+6\frac{\partial\left(z^2\right)}{\partial z}$$
$$=5\cos(2xy+z)+6\cdot 2z=5\cos(2xy+z)+12z$$

c) $f(x,y,z)=xy+x^2z+e$

Resolução:

- $$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial\left(xy+x^2z\right)}{\partial x}\underset{\text{derivada da soma}}{=}\frac{\partial(xy)}{\partial x}+\frac{\partial\left(x^2z\right)}{\partial x}=y+2xz$$

- $$\frac{\partial f}{\partial y}(x,y,z)=\frac{\partial\left(xy+x^2z\right)}{\partial y}\underset{\text{derivada da soma}}{=}\frac{\partial(xy)}{\partial y}+\frac{\partial\left(x^2z\right)}{\partial y}=x+0=x$$

- $$\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial\left(xy+x^2z\right)}{\partial z}\underset{\text{derivada da soma}}{=}\frac{\partial(xy)}{\partial z}+\frac{\partial\left(x^2z\right)}{\partial z}=0+x^2=x^2$$

d) $f(x,y,z)=\pi+\dfrac{xy}{2z}$

Resolução:

- $\dfrac{\partial f}{\partial x}(x,y,z)=\dfrac{\partial\left(\dfrac{xy}{2z}\right)}{\partial x}=\dfrac{y}{2z}$

- $\dfrac{\partial f}{\partial y}(x,y,z)=\dfrac{\partial\left(\dfrac{xy}{2z}\right)}{\partial y}=\dfrac{x}{2z}$

- $\dfrac{\partial f}{\partial z}(x,y,z)=\dfrac{\partial\left(\dfrac{xy}{2z}\right)}{\partial z}=xy\dfrac{1}{2}\dfrac{\partial\left(z^{-1}\right)}{\partial z}=\dfrac{1}{2}xy(-1)z^{-2}=-\dfrac{xy}{2z^2}$

e) $f(x,y)=2+\dfrac{x}{y}-4+\dfrac{y}{x}$

Resolução:

- $\dfrac{\partial f}{\partial x}(x,y)=\dfrac{\partial\left(\dfrac{x}{y}+\dfrac{y}{x}\right)}{\partial x}=\dfrac{\partial\left(\dfrac{x}{y}\right)}{\partial x}+\dfrac{\partial\left(\dfrac{y}{x}\right)}{\partial x}=\dfrac{1}{y}+y\dfrac{\partial\left(\dfrac{1}{x}\right)}{\partial x}=\dfrac{1}{y}+y\dfrac{\partial\left(x^{-1}\right)}{\partial x}=\dfrac{1}{y}+y(-1)x^{-1-1}=\dfrac{1}{y}-yx^{-2}=\dfrac{1}{y}-\dfrac{y}{x^2}$

- $\dfrac{\partial f}{\partial y}(x,y)=\dfrac{\partial\left(\dfrac{x}{y}+\dfrac{y}{x}\right)}{\partial y}=\dfrac{\partial\left(\dfrac{x}{y}\right)}{\partial y}+\dfrac{\partial\left(\dfrac{y}{x}\right)}{\partial y}=x\dfrac{\partial\left(\dfrac{1}{y}\right)}{\partial y}+\dfrac{1}{x}=x\dfrac{\partial\left(y^{-1}\right)}{\partial y}+\dfrac{1}{x}=x(-1)y^{-1-1}+\dfrac{1}{x}=-xy^{-2}+\dfrac{1}{x}=-\dfrac{x}{y^2}+\dfrac{1}{x}$

f) $f(x,y)=\dfrac{x-y}{x+y}+\sqrt[3]{3}$

Resolução:

- $\dfrac{\partial f}{\partial x}(x,y)=\dfrac{\partial\left(\dfrac{x-y}{x+y}\right)}{\partial x}=\dfrac{\dfrac{\partial(x-y)}{\partial x}(x+y)-(x-y)\dfrac{\partial(x+y)}{\partial x}}{(x+y)^2}=\dfrac{1(x+y)-(x-y)1}{(x+y)^2}=\dfrac{2y}{(x+y)^2}$

- $\dfrac{\partial f}{\partial y}(x,y)=\dfrac{\partial\left(\dfrac{x-y}{x+y}\right)}{\partial y}=\dfrac{\dfrac{\partial(x-y)}{\partial y}(x+y)-(x-y)\dfrac{\partial(x+y)}{\partial y}}{(x+y)^2}=\dfrac{-1(x+y)-(x-y)1}{(x+y)^2}=\dfrac{-x-y-x+y}{(x+y)^2}=\dfrac{-2x}{(x+y)^2}$

g) $f(x,y)=5e^{2xy}$

Resolução:

- $\dfrac{\partial f}{\partial x}(x,y)=\dfrac{\partial\left(5e^{2xy}\right)}{\partial x}=5\dfrac{\partial(2xy)}{\partial x}e^{2xy}=10ye^{xy}$

- $\dfrac{\partial f}{\partial y}(x,y)=\dfrac{\partial\left(5e^{2xy}\right)}{\partial y}=5\dfrac{\partial(2xy)}{\partial y}e^{2xy}=10xe^{xy}$

h) $f(x,y)=20+xe^{xy}$

Resolução:

- $\dfrac{\partial f}{\partial x}(x,y)=\dfrac{\partial\left(xe^{xy}\right)}{\partial x}\underset{\text{derivada do produto}}{=}\dfrac{\partial(x)}{\partial x}e^{xy}+x\dfrac{\partial\left(e^{xy}\right)}{\partial x}=1e^{xy}+x\dfrac{\partial(xy)}{\partial x}e^{xy}=e^{xy}+xye^{xy}$

- $\dfrac{\partial f}{\partial y}(x,y)=\dfrac{\partial\left(xe^{xy}\right)}{\partial y}\underset{\text{derivada do produto}}{=}\dfrac{\partial(x)}{\partial y}e^{xy}+x\dfrac{\partial\left(e^{xy}\right)}{\partial y}=0\cdot e^{xy}+x\dfrac{\partial(xy)}{\partial y}e^{xy}=x^2e^{xy}$

Derivadas parciais pela definição

I.2 Considere a função definida por,

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & \text{se } (x,y)\neq(0,0)\\ 0 \quad , & \text{se } (x,y)=(0,0)\end{cases}.$$

a) Determine as derivadas parciais da função em $(0,0)$.

Resolução:

Atendendo a que a função $f(x,y)$ está definida por ramos, as derivadas parciais no ponto $(0,0)$, ponto de ligação, são calculadas pela definição. Deste modo, temos:

- $\dfrac{\partial f}{\partial x}(0,0)=\lim\limits_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{2\cdot h\cdot 0}{h^2+0^2}-0}{h}=\lim\limits_{h\to 0}\dfrac{0}{h}=\lim\limits_{h\to 0}0=0$

- $\dfrac{\partial f}{\partial y}(0,0)=\lim\limits_{k\to 0}\dfrac{f(0,0+k)-f(0,0)}{k}=\lim\limits_{k\to 0}\dfrac{f(0,k)-f(0,0)}{k}=\lim\limits_{k\to 0}\dfrac{\dfrac{2\cdot 0\cdot k}{0^2+k^2}-0}{k}=\lim\limits_{k\to 0}\dfrac{0}{k}=\lim\limits_{k\to 0}0=0$

b) Prove que f não é contínua na origem.

Resolução:

Para provar que a função $f(x,y)$ não é contínua no ponto $(0,0)$ tem de se verificar uma das seguintes condições:

- ❖ não existe $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ ou
- ❖ se existir $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ então $\lim\limits_{(x,y)\to(0,0)} f(x,y)\neq f(0,0)$.

Temos que, $\lim\limits_{(x,y)\to(0,0)}\dfrac{2xy}{x^2+y^2}=\dfrac{2\cdot 0\cdot 0}{0^2+0^2}=\dfrac{0}{0}$ (indeterminação).

Como o $\lim\limits_{(x,y)\to(0,0)}\dfrac{2xy}{x^2+y^2}=\dfrac{0}{0}$, temos de recorrer aos limites direccionais na vizinhança do ponto (0,0).

Calculemos o limite quando (x,y) se aproxima de $(0,0)$ segundo a direcção da recta $y = x$:

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0}\frac{2xx}{x^2+x^2} = \lim_{x\to 0}\frac{2x^2}{2x^2} = \lim_{x\to 0} 1 = 1 .$$

Assim, se o $\lim_{(x,y)\to(0,0)} f(x,y)$ existir tem que ser 1, devido a unicidade do limite.

Como $f(0,0) = 0 \neq \lim_{(x,y)\to(0,0)} f(x,y)$, podemos concluir que f não verifica a condição ii), logo não é contínua no ponto $(0,0)$.

Gradiente e Plano Tangente

I.3 Seja $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - \frac{x^2}{2} - \frac{y^2}{2} - 2y + 1$.

a) Determine o gradiente de $f(x,y)$.

Resolução:

$$\begin{aligned} \operatorname{grad} f(x,y) = \nabla f(x,y) &= \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) \\ &= \left(\frac{\partial\left(\frac{x^3}{3}+\frac{y^3}{3}-\frac{x^2}{2}-\frac{y^2}{2}-2y+1\right)}{\partial x}, \frac{\partial\left(\frac{x^3}{3}+\frac{y^3}{3}-\frac{x^2}{2}-\frac{y^2}{2}-2y+1\right)}{\partial y}\right) \\ &= \left(\frac{3x^2}{3}-\frac{2x}{2}, \frac{3y^2}{3}-\frac{2y}{2}-2\right) = \left(x^2-x, y^2-y-2\right),\ (x,y)\in\mathbb{R}^2 \end{aligned}$$

b) Obtenha a equação do plano tangente ao gráfico de f no ponto $(2,0)$.

Resolução:

A equação do plano tangente à função no ponto $(2,0)$ é dada por:

$$z = f(2,0) + \nabla f(2,0)\cdot(x-2, y-0) .$$

Substituindo na função $f(x,y)$ as variáveis x e y por 2 e 0, vem

$$f(2,0) = \frac{2^3}{3} + \frac{0^3}{3} - \frac{2^2}{2} - \frac{0^2}{2} - 2\cdot 0 + 1 = \frac{5}{3} .$$

Na alínea anterior viu-se que,

$$\nabla f(x,y) = \left(x^2 - x, y^2 - y - 2\right),\ (x,y)\in\mathbb{R}^2 .$$

Então,

$$\nabla f(2,0) = \left(2^2 - 2, 0^2 - 0 - 2\right) = (2,-2).$$

Assim,

$$\begin{aligned} &z = f(2,0) + \nabla f(2,0)\cdot(x-2, y-0) \Leftrightarrow z = f(2,0) + (2,-2)\cdot(x-2, y) \\ &\Leftrightarrow z = \frac{5}{3} + 2(x-2) - 2y \Leftrightarrow -2x + 2y + z + \frac{7}{3} = 0 \end{aligned}$$