EXERCÍCIOS DE CÁLCULO DIFERENCIAL E INTEGRAL 1

Ficha 1

1

a) $\dfrac{\frac{x}{2}}{\frac{2}{x}} = \dfrac{x^2}{4} = \left(\dfrac{x}{2}\right)^2$ // , $x \neq 0$

b) $\dfrac{x+1}{\frac{1}{x}+\frac{1}{1}} = \dfrac{x+1}{\frac{x+1}{x}} = \dfrac{x\cancel{(x+1)}}{\cancel{x+1}} = x$ // $x \neq 0$

c) $\dfrac{1}{1+x} + \dfrac{1}{x^2+x} = \dfrac{x \cdot 1}{x(1+x)} + \dfrac{1}{x^2+x} = \dfrac{\cancel{x+1}}{x(\cancel{x+1})} = \dfrac{1}{x}$, $x \neq 0$ //

d) $\sqrt{x^2} = \pm x$, $x \neq 0$ //

e) $(\sqrt{x})^2 = x$, $x \geqslant 0$ //

f) $4^x \cdot \dfrac{4}{2^x} = \dfrac{4^{(x+1)}}{2^x} = \dfrac{2^{2(x+1)}}{2^x} = 2^{2x+2-x} = 2^{x+2}$ //

g) $2^{x^2} \cdot (2^x)^2 = 2^{x^2} \cdot 2^{2x} = 2^{2x+x^2}$ //

h) $\dfrac{\sqrt[3]{x^2}}{\sqrt[6]{x}} = \dfrac{(x^2)^{1/3}}{(x)^{1/6}} = \dfrac{x^{2/3}}{x^{1/6}} = x^{\frac{2}{3}-\frac{1}{6}} = x^{\frac{3}{6}} = x^{\frac{1}{2}} = \sqrt{x}$ //

i) $\sqrt{x-2} \cdot \sqrt{x+2} = \sqrt{(x-2)\cdot(x+2)} = \sqrt{x^2+2x-2x-4} =$

$= \sqrt{x^2-4}$ // $\quad x^2-4 \geqslant 0 \Leftrightarrow x \geq \pm 2$

j) $\dfrac{\sqrt{x}}{\sqrt{x+1}-\sqrt{x}} = \dfrac{\sqrt{x} \cdot (\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})} = \dfrac{\sqrt{x(x+1)}+x}{x+1+x} =$

$= \sqrt{x^2+x}$ //

$x \geq 0$

$x+1 \geqslant 0 \Leftrightarrow x \geq -1$

$\sqrt{x}+1 \neq \sqrt{x}$

k) $\log\left(\frac{1}{x}\right) + \log(x^2) = \log(x^{-1} \cdot x^2) = \log x$ //

l) $\log(2x^2 + 2x^{-2}) + \log\left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right) =$

$= \log(2(x^2 + x^{-2})) + \log\left(\frac{1}{2}(x^2 + x^{-2})\right) =$

$= \log 2 + \log(x^2 + x^{-2}) + \log 2^{-1} + \log(x^2 + x^{-2}) =$

$= \cancel{\log 2} - \cancel{\log 2} + 2\log(x^2 + x^{-2}) = 2\log(x^2 + x^{-2})$

2

a) $(x^2 - 3x + 2)(x - 1) \geq 0$

$x - 1 = 0 \Leftrightarrow x = 1$ //

$x^2 - 3x + 2 = 0 \Leftrightarrow x = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \Leftrightarrow x = \frac{3 \pm \sqrt{9 - 8}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{3 \pm \sqrt{1}}{2} \Leftrightarrow x = \frac{3+1}{2} \vee x = \frac{3-1}{2} \Leftrightarrow x = 2 \vee x = 1$ //

$-\infty$	1		2	$+\infty$	
$-$	0	$+$	$+$	$+$	$x - 1$
$+$	0	$-$	0	$+$	$x^2 - 3x + 2$
$-$	0	$-$	0	$+$	

$\Omega = [2, +\infty[\cup \{1\}$

b) $x \leq 2 - x^2 \Leftrightarrow 0 \leq -x^2 - x + 2 \Leftrightarrow -x^2 - x + 2 \geq 0$ //

$-x^2 - x + 2 = 0 \Leftrightarrow x = \frac{1 \pm \sqrt{1 - 4 \cdot (-1) \cdot 2}}{-2} \Leftrightarrow$

$\Leftrightarrow x = \frac{1 \pm \sqrt{9}}{-2} \Leftrightarrow x = \frac{1 \pm 3}{-2} \Leftrightarrow x = \frac{1+3}{-2} \vee x = \frac{1-3}{-2} \Leftrightarrow$

$\Leftrightarrow x = -\frac{4}{2} \vee x = \frac{2}{2} \Leftrightarrow x = -2 \vee x = 1$ //

$-\infty$	-2		1	$+\infty$
$-$	0	$+$	0	$+$

$\Omega = [-2, 1]$

c) $x^2 \leq 2 - x^4 \Leftrightarrow 0 \leq -x^4 - x^2 + 2$ $\quad x^2 = y$

$-y^2 - y + 2$ $\quad y = \frac{1 \pm \sqrt{1 - 4.(-2).2}}{-2} \Leftrightarrow y = \frac{1 \pm 3}{-2} \Leftrightarrow$

$x^2 = \cancel{-2} \vee x^2 = 1$ $\quad \Leftrightarrow y = \frac{-2}{-2} \vee y = \frac{4}{-2} \Leftrightarrow y = 1 \vee y = -2$

d) $x^3 + x \leq 2x^2 \Leftrightarrow x^3 - 2x^2 + x \leq 0$

$\Leftrightarrow x(x^2 - 2x + 2) \leq 0$

$x = 0$ //

$x^2 - 2x + 2 = 0 \Leftrightarrow x = \frac{2 \pm \sqrt{4 - 4.1.1}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-2 \pm \sqrt{4-4}}{2} \Leftrightarrow x = 1$ //

	$-\infty$	0		1	$+\infty$
x	$-$	0	$+$	$+$	$+$
$x^2 - 2x - 2$	$+$	$+$	$+$	0	$+$
	$-$	0	$+$	0	$+$

$\Omega =]-\infty, 0] \cup 1$

e) $\sqrt[3]{x^2 + 2x} = 2 \Leftrightarrow x^2 + 2x = 2^3 \Leftrightarrow x^2 + 2x - 8 = 0 \Leftrightarrow$

$\Leftrightarrow x = \frac{-2 \pm \sqrt{4 - 4.1.(-8)}}{2.1} \Leftrightarrow x = \frac{-2 \pm \sqrt{4 + 32}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-2 \pm \sqrt{36}}{2} \Leftrightarrow x = \frac{-2 + 6}{2} \vee x = \frac{-2 - 6}{2} \Leftrightarrow x = 2 \vee x = -4$ //

f) $\sqrt[3]{x-1} = \sqrt{x-1} \Leftrightarrow (x-2)^{1/3} = (x-2)^{2/2} \Leftrightarrow \frac{1}{3} = \frac{1}{2}$ // P.F.

Esta equação não tem soluções $\Omega = \emptyset$

g) $\frac{x-1}{x^2-1} \leq 1 \Leftrightarrow \frac{x-1}{(x+1)(x-1)} - 1 \leq 0 \Leftrightarrow \frac{1}{x+1} - \frac{x+1}{x+1} \leq 0 \Leftrightarrow$

$\Leftrightarrow \frac{1-(x+1)}{x+1} \leq 0 \Leftrightarrow \frac{x}{x+1} \leq 0$ //

$\Omega =]-1, 0]$

	$-\infty$		-1		0		$+\infty$
x	$-$	$-$	$-$	$-$	0	$+$	$+$
$x+1$	$-$	$-$	0	$+$	$+$	$+$	$+$
	$+$	$+$	ND	$-$	0	$+$	$+$

h) $x = \frac{1}{x} \Leftrightarrow {}^2x = 1 \Leftrightarrow x = 1 \vee x = -1$ //

i) $x < \frac{1}{x} \Leftrightarrow x - \frac{1}{x} < 0 \Leftrightarrow \frac{x^2-1}{x} < 0$ //

	$-\infty$		-1			0		1	$+\infty$
x	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
x^2-1	$+$	$+$	0	$-$	$-$	$-$	$-$	0	$+$
	$-$	$-$	0	$+$	$+$	ND	$-$	0	$+$

$x^2 - 1 = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$ // $\Omega =]-\infty, 0[\cup]0, 1[$

j) $x < |x| \Leftrightarrow$

$\Leftrightarrow x < x \vee x < -x \Leftrightarrow 2x < 0 \Leftrightarrow x < 0$

P.F. $\Omega =]-\infty, 0[$

K) $|x| \geq \frac{x}{2} + 1 \Leftrightarrow |x| \geq \frac{x}{2} + \frac{2}{2} \Leftrightarrow |x| \geq \frac{x+2}{2} \Leftrightarrow$

$\Leftrightarrow x \geq \frac{x+2}{2} \vee -x \geq \frac{x+2}{2} \Leftrightarrow 2x \geq x+2 \vee -2x \geq x+2 \Leftrightarrow$

$\Leftrightarrow 2x - x \geq 2 \vee -2x - x \geq 2 \Leftrightarrow x \geq 2 \vee -3x \geq 2 \Leftrightarrow$

$\Leftrightarrow x \geq 2 \vee x \leq -\frac{2}{3}$ // $\Omega =]-\infty, -\frac{2}{3}] \cup [2, +\infty[$

l) $|x| \leq |x-2| \Leftrightarrow x \leq |x-2| \vee -x \leq |x-2| \Leftrightarrow$

$\Leftrightarrow \cancel{x} \leq \cancel{x} - 2 \vee x \leq -x+2 \vee -x \leq x-2 \vee -\cancel{x} \leq -\cancel{x}+2 \Leftrightarrow$

$\Leftrightarrow 0 \leq -2 \vee 2x \leq 2 \vee -2x \leq -2 \vee 0 \leq 2 \Leftrightarrow$

P.F. P.V

$\Leftrightarrow x \leq 1$ // $\Omega =]-\infty, 1]$

m) $|x^2-2| \leq 2 \Leftrightarrow x^2-2 \leq 2 \vee -(x^2-2) \leq 2 \Leftrightarrow$

$\Leftrightarrow x^2 \leq 2+2 \vee -x^2 \leq 2-2 \Leftrightarrow$

$\Leftrightarrow x^2 \leq 4 \vee \underbrace{-x^2 \leq 0}_{P.V.} \Leftrightarrow x \leq \pm\sqrt{4} \Leftrightarrow x \leq 2 \vee x \leq -2$ //

$\Omega =]-\infty, 2] \cup]-\infty, -2]$

n) $\dfrac{x^4-16}{|x-1|} \leq 0 \Leftrightarrow \dfrac{x^4-16}{x-1} \leq 0 \vee \dfrac{x^4-16}{-x+2} \leq 0 \Leftrightarrow$

$\Leftrightarrow \dfrac{(x^2-4)(x^2+4)}{x-1} \vee \dfrac{(x^2-4)(x^2+4)}{1-x}$ $\rightarrow x=2 \vee x=-2$

$\hookrightarrow x=1$ $\hookrightarrow x=-1$

$\Omega =]-\infty, -2] \cup]1, 2]$

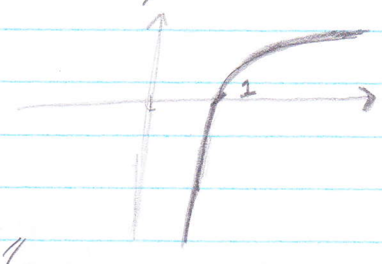
o) $e^{x^3} < 1 \Leftrightarrow e^{x^3} < e^0 \Leftrightarrow x^3 < 0$ // $\Omega = \mathbb{R}^-$

p) $e^{-2x} - 2e^{-x} \leq -1 \Leftrightarrow e^{-2x} - 2e^{-x} + 1 \leq 0$

$e^{-x} = y \Rightarrow y^2 - 2y + 1 = 0 \Leftrightarrow y = \dfrac{2 \pm \sqrt{4-4.1.1}}{2} \Leftrightarrow$

$\Leftrightarrow y = \dfrac{2}{2} \Leftrightarrow y = 1$

$\rightarrow e^{-x} = 1 \Leftrightarrow e^{-x} = e^0 \Leftrightarrow -x = 0$ //



q) $\log\left(\dfrac{1}{x}\right) \geq 0 \Leftrightarrow$

$\Leftrightarrow \cancel{\log 1} - \log x \geq 0 \Leftrightarrow -\log x \geq 0$ //

$\Omega =]0, 1]$

γ) $\log(x^2-3) \geq 0 \Leftrightarrow x^2-3 \geq 1 \Leftrightarrow x^2 \geq 4 \Leftrightarrow$

$\Leftrightarrow x \geq \pm 2$

$-\infty$		-2		2		$+\infty$
$+$	$+$	0	$-$	0	$+$	$+$

$\Omega =]-\infty, -2] \cup [2, +\infty[$

3

a) $\{x: \frac{x-1}{x+1} \leq 1\} \Leftrightarrow x \in \mathbb{R} \setminus \{-1\}$

b) $\{x: \underline{\frac{x^4-1}{x^3} \leq x}\} \overset{x\in}{\Leftrightarrow} \mathbb{R} \setminus \{0\}$

$x^4-1 \leq x.x^3 \Leftrightarrow x^4-1 \leq x^4 \to$ P.V

c) $\{x: |3x-4| \geq x^2\} \overset{x\in}{\Leftrightarrow} [-4, 1]$ //

$3x-4 \geq x^2 \vee -3x+4 \geq x^2 \Leftrightarrow -x^2-3x-4 \geq 0 \vee -x^2-3x+4 \geq 0$

$-x^2+3x-4=0 \Leftrightarrow$

$x = \frac{-3 \pm \sqrt{9-(4.(-1).(-4))}}{-2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-3 \pm \sqrt{-5}}{-2}$ impossível

$-x^2-3x+4=0 \Leftrightarrow$

$\Leftrightarrow x = \frac{3 \pm \sqrt{9-(4.(-1).4)}}{-2} \Leftrightarrow$

$\Leftrightarrow x = \frac{3 \pm \sqrt{9+16}}{-2} \Leftrightarrow x = \frac{3 \pm \sqrt{25}}{-2} \Leftrightarrow$

$\Leftrightarrow x = \frac{3+5}{-2} \vee x = \frac{3-5}{-2} \Leftrightarrow$

$\Leftrightarrow x = \frac{8}{-2} \vee x = \frac{-2}{-2} \Leftrightarrow x = -4 \vee$

$\vee x = 1$ //

d) $\{x: |x-1|(x^2-4) \geq 0\} \Leftrightarrow x \in]-\infty, -2] \cup \{1\} \cup [2, +\infty[$

$(x-1)(x^2-4) \geq 0 \vee (x+1)(x^2-4) \geq 0$

$x-1=0 \vee x^2-4=0 \vee x+1=0 \vee x^2-4=0 \Leftrightarrow$

$\Leftrightarrow x=1 \vee x=\pm\sqrt{4} \vee x=-2 \vee x=\pm\sqrt{4}$

	$-\infty$		-2		-1		1		2	$+\infty$
$\lvert x-1\rvert$	+	+	+	+	+	+	0	+	+	+
x^2-4	+	+	0	−	−	−	−	−	0	+
	+	+	0	−	−	−	0	−	0	+

$\Omega =]-\infty, -2] \cup \{1\} \cup [2, +\infty[$

e) $\{x : (|x|-1)(x^2-4) \leq 0\} \Leftrightarrow x \in]-\infty, -2] \cup [-1, 2]$

C.A. $(|x|-1)(x^2-4) \leq 0 \Leftrightarrow (x-1)(x^2-4) \leq 0 \lor$

$\lor (-x-1)(x^2-4) \leq 0$

$(x-1)=0 \lor x^2-4=0 \lor -x-1=0 \lor x^2-4=0 \Leftrightarrow$

$\Leftrightarrow x=1 \lor x=\pm 2 \lor x=-1 \lor x=\pm 2$ //

$\Omega =]-\infty, -2] \cup [-1, 2]$

F) $\{x : |x^2-1| \leq |x+1|\} \Leftrightarrow x \in [-1, 2]$

$x^2-1 \leq x+1 \lor x^2-\cancel{1} \leq -x-\cancel{1} \lor -x^2+\cancel{1} \leq x+\cancel{1} \lor -x^2+1 \leq -x-1 \Leftrightarrow$

$\Leftrightarrow x^2-x-2 \leq 0 \lor x^2+x \leq 0 \lor -x^2-x \leq 0 \lor -x^2+x+2 \leq 0$

$x^2+x=0 \Leftrightarrow x^2=-x \Leftrightarrow x=-1$ //

$-x^2-x=0 \Leftrightarrow -x^2=x \Leftrightarrow x=-1$ //

$x^2-x-2=0 \Leftrightarrow x = \frac{1 \pm \sqrt{1-(4.1.(-2))}}{2} \Leftrightarrow \frac{1 \pm \sqrt{9}}{2} = x \Leftrightarrow x = \frac{1+3}{2} \lor$

$\lor x = \frac{1-3}{2} \Leftrightarrow$

$\Leftrightarrow x = 2 \lor x = -1$ //

$-x^2+x+2=0 \Leftrightarrow x=\frac{-2\pm\sqrt{1-(4.(-2).2)}}{-2} \Leftrightarrow x=\frac{-1\pm\sqrt{9}}{-2} \Leftrightarrow$

$\Leftrightarrow x=\frac{-1+3}{-2} \vee x=\frac{-1-3}{-2} \Leftrightarrow x=-1 \vee x=2$ //

	$-\infty$		-1		2		$+\infty$
x^2+x	+	+	0	+	+	+	+
x^2-x-2	+	+	0	−	0	+	+
	+	+	0	−	0	+	+
$-x^2-x$	−	−	0	−	−	−	−
$-x^2+x+2$	−	−	0	+	0	−	−
	+	+	0	−	0	+	+

9) $\{x : x^2-|x|-2\leq 0\} \Leftrightarrow x\in[-2,2]$

$x^2-x-2\leq 0 \quad \vee \quad x^2+x-2\leq 0$

$x^2-x-2=0 \Leftrightarrow x=\frac{1\pm\sqrt{1-(4.1.(-2))}}{2} \Leftrightarrow x=\frac{1\pm\sqrt{1+8}}{2} \Leftrightarrow$

$\Leftrightarrow x=\frac{1\pm 3}{2} \Leftrightarrow x=2 \vee x=-1$ // $\quad x>0$

$x^2+x-2=0 \Leftrightarrow x=\frac{-1\pm\sqrt{1-(4.1.(-2))}}{2} \Leftrightarrow x=\frac{-1\pm 3}{2} \Leftrightarrow$

$\Leftrightarrow x=-2 \vee x=1$ // $\quad x<0$

	$-\infty$		-2		-1		1		2		$+\infty$
x^2-x-2	+	+	+	+	0	−	−	−	0	+	+
x^2+x-2	+	+	0	−	−	−	0	+	+	+	+

h) $\left\{x : \dfrac{x}{|x|-1} \geq 0\right\} \Leftrightarrow x \in \mathbb{R}_0^+ \setminus \{1\}$

$\dfrac{x}{|x|-1} \geq 0 \quad , \quad x \neq 1 \text{ e } x \neq -1$

i) $\left\{x : \dfrac{x^2-|x|}{x-3} \leq 0\right\} \Leftrightarrow x \in \left]-\infty, -1\right] \cup \{0\} \cup \left[1, 3\right[$

$x-3=0 \Leftrightarrow x=3$

$x^2-|x|=0 \Leftrightarrow x^2-x=0 \wedge x>0 \vee x^2+x=0 \wedge x<0 \Leftrightarrow$

$\Leftrightarrow x^2=x \wedge x>0 \vee x^2=-x \wedge x<0 \Leftrightarrow x=1 \vee x=-1$

	$-\infty$		-1		0	1		3		$+\infty$
$x^2-\|x\|$	+	+	0	-	0	0	+	+	+	+
$x-3$	-	-	-	-	-	-	-	0	+	+
	-	-	0	+	0	0	-	ND	+	+

Ficha 2

1.

a) $\{1\} \subset \{1,\{2,3\}\}$ falso

b) $\{1\} \in \{1,\{2,3\}\}$ Falso

c) $2 \in \{1,\{2,3\}\}$ falso

d) $1 \in \{\mathbb{R}\}$ falso 1 não é elemento de $\{\mathbb{R}\}$

e) $\emptyset = \{x \in \mathbb{N} : x = x+1\}$ Verdadeira - $x = x+1$ nunca se verifica

f) $\emptyset \in \{0\}$ falso

g) $\emptyset \subset \{0\}$ Verdadeiro

h) $\forall_{x \in \mathbb{R}}\ x > 0 \Leftrightarrow x^{-1} > 0$ Verdadeira

i) $\forall_{x \in \mathbb{R}}\ x > 1 \Leftrightarrow x^{-1} < 1$ verdadeira - se $x > 1$ então $\frac{1}{x} < 1$

j) $\forall_{x,y \in \mathbb{R}}\ x < y \Rightarrow y^{-1} < x^{-1}$ verdadeira

k) $\forall_{x \neq 0}\ x^2 > 0$ verdadeira - a função x^2 nunca é negativa

l) $\forall_{x,y \in \mathbb{R}}\ x < y \Rightarrow x^2 < y^2$ falso

m) $\forall_{x,y \in \mathbb{R}}\ x < y < 0 \Rightarrow x^2 > y^2$ Verdadeira

2

$\forall_{a>0}\ a+\frac{1}{a} \geq 1$

~~se $a \geq 1$~~

$\boxed{\text{se } a=1}$ $1+1 \geq 1$

$1 \leq \frac{1}{a} < 2$, $a>0$

então

para qualquer $a>0$

$a+\frac{1}{a} \geq$ é verdade

se $a<1 \Rightarrow \frac{1}{a} > 1$

então $a+\frac{1}{a} \geq 1$ cqd

3

a) $1+3+\ldots+(2n-1)=n^2$, $\forall n \in \mathbb{N}_1$ $=P(n)$

1) provar que é verdade para $n=1$

$(2\times 1-1)=1=1^2$, como queríamos demonstrar

2) $P(n) \Rightarrow P(n+1)$

$\overbrace{1+3+\ldots+(2n-1)}^{n^2}+(2(n+1)-1)=(n+1)^2 \Leftrightarrow$

$\Leftrightarrow n^2+(2n+2-1)=n^2+2n+1 \Leftrightarrow$

$\Leftrightarrow n^2+2n+1=n^2+2n+1$, cqd

A partir de 1) e 2) conclui-se que $P(n)$ é verdadeira para todo o $n \in \mathbb{N}_1$

b) $\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$, $\forall n \in \mathbb{N}_1$

1) $\boxed{n=1}$

$\frac{1}{1(1+1)}=\frac{1}{1+1} \Leftrightarrow \frac{1}{2}=\frac{1}{2} \rightarrow$ cqd

2) se $p(n) \Rightarrow p(n+1)$

$$\overbrace{\frac{1}{1\cdot 2} + \frac{1}{2\,3} + \ldots + \frac{1}{n(n+1)}} + \frac{1}{(n+1)(n+1+1)} = \frac{n+1}{n+1+1} \Leftrightarrow$$

$$\Leftrightarrow \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2} \Leftrightarrow$$

$$\Leftrightarrow \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)(n+1)}{(n+2)(n+1)} \Leftrightarrow$$

$$\Leftrightarrow \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} \quad // \quad \text{cqd}$$

apartir de 1) e 2) conclui-se que a proposição é verdadeira para todo o $n \in \mathbb{N}_1$

c) $(n!)^2 > 2^n n^2, \ \forall n \geq 4 \in \mathbb{N}$

1) $\boxed{n=4}$

$$(4!)^2 > 2^4 \cdot 4^2 \Leftrightarrow 48^2 > 16 \cdot 16 \Leftrightarrow$$

$$\Leftrightarrow 48^2 > 16^2 \quad \text{cqd}$$

2) $p(n) \Rightarrow p(n+1)$

$$((n+1)!)^2 > 2^{n+1} \cdot (n+1)^2$$

Pela hipotese:

$$(n!)^2 > 2^n n^2 \Leftrightarrow (n!)^2 (n+1)^2 > 2^n n^2 (n+1)^2 \Leftrightarrow$$

$$\Leftrightarrow (n!)^2 (n+1)^2 > 2^{n+1} n^2 (n+1)^2 \cdot \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow (n!)^2 (n+1)^2 > 2^{n+1} (n+1)^2 \cdot \underbrace{\frac{n^2}{2}} \qquad \frac{n^2}{2} \geq 1 \Leftrightarrow$$

$$\Leftrightarrow n^2 \geq 2 \Leftrightarrow n \geq \sqrt{2} \quad n \geq 4 \quad \text{cqd}$$

d) $n! \geq 2^{n-1}, \forall n \in \mathbb{N}_1$

1) $\boxed{n=1}$

$1! \geq 2^{1-1} \Leftrightarrow 1 \geq 2^0 \Leftrightarrow 1 \geq 1$ cqd

2) se $p(n) \Rightarrow p(n+1)$

$(n+1)! \geq 2^{(n+1)-1} = 2^n$

Pela hipotese

$(n!)(n+1) \geq 2^{n-1}(n+1) \Leftrightarrow (n!)(n+1) \geq 2^{n-1} \cdot 2 \cdot \frac{(n+1)}{2}$

$\Leftrightarrow (n!)(n+1) \geq 2^n \frac{n+1}{2} \rightarrow \geq 1$ (V) $\quad \frac{n+1}{2} \geq 1 \Leftrightarrow n+1 \geq 2$

$\Leftrightarrow n \geq \frac{2}{1} \Leftrightarrow n \geq 1$ cqd

4

a) $1+2+3+\ldots+n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}_1$

$\boxed{n=1}$

$1 = \frac{1(1+1)}{2} \Leftrightarrow 1 = 1$ cqd

$\boxed{P(n) \Rightarrow P(n+1)}$

$\overline{1+2+3+\ldots+n}^{\,n} + (n+1) = \frac{(n+1)((n+1)+1)}{2} \Leftrightarrow$

$\Leftrightarrow \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} \Leftrightarrow$

$\Leftrightarrow \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2} \Leftrightarrow$

$\Leftrightarrow \frac{n^2+n+2n+2}{2} = \frac{n^2+2n+n+2}{2}$ // cqd

b) $(a-1)(1+a+\dots+a^n) = a^{n+1}-1 \quad \forall a \in \mathbb{R}$
$\forall n \in \mathbb{N}$

$\boxed{n=1}$

$(a-1)(a+1) = a^{1+1}-1 \Leftrightarrow a^2+\cancel{a}-\cancel{a}-1 = a^2-1$ // cqd

$\boxed{\text{se } p(n) \Rightarrow p(n+1)}$

$\overbrace{(a-1)(1+a+\dots+a^n}+a^{n+1}) = a^{n+2}-1 \Leftrightarrow$

$\overset{h}{\Leftrightarrow} a^{n+1}-1+(a-1).a^{n+1} = a^{n+2}-1 \Leftrightarrow$

$\Leftrightarrow \cancel{a^{n+1}}-1+a^{n+2}-\cancel{a^{n+1}} = a^{n+2}-1$ cqd

hipotese de indução:

c) $\frac{1}{2!}+\frac{2}{3!}+\dots+\frac{n}{(n+1)!} = 1-\frac{1}{(n+1)!} \quad \forall n \in \mathbb{N}$

$\boxed{n=1} \qquad \sum_{i=1}^{n} \frac{i}{(i+1)!}$

$\frac{1}{2!} = 1-\frac{1}{(1+1)!} \Leftrightarrow \frac{1}{2} = 1-\frac{1}{2} \Leftrightarrow \frac{1}{2} = \frac{2}{2}-\frac{1}{2} \Leftrightarrow$

$\Leftrightarrow \frac{1}{2} = \frac{1}{2}$ cqd

$\boxed{\text{se } p(n) \Rightarrow p(n+1)}$

$\underbrace{\frac{1}{2!}+\frac{2}{3!}+\dots+\frac{n}{(n+1)!}}+\frac{n+1}{(n+2)!} = 1-\frac{1}{(n+2)!} \Leftrightarrow$

$\overset{h}{\Leftrightarrow} 1-\frac{1}{(n+1)!}+\frac{n+1}{(n+2)!} = 1-\frac{1}{(n+2)!} \Leftrightarrow$

$\Leftrightarrow 1-\frac{1}{(n+1)!}+\frac{n+1}{(n+1)!(n+2)} = 1-\frac{1}{(n+2)!} \Leftrightarrow$

$\Leftrightarrow 1 \frac{-(n+2)+n+2}{(n+1)!\,(n+2)} = 1 - \frac{1}{(n+2)!} \Leftrightarrow$

$\Leftrightarrow 1 + \frac{-\cancel{n}-2+\cancel{n}+1}{(n+2)!} = 1 - \frac{1}{(n+2)!} \Leftrightarrow$

$\Leftrightarrow 1 - \frac{1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$ cqd

d) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad n \in \mathbb{N}_1$

$\boxed{n=1}$

$1^3 = \left(\frac{1(1+1)}{2}\right)^2 \Leftrightarrow 1 = 1^2$ cqd

$\boxed{\text{se } p(n) \Rightarrow p(n+1)}$

$\underbrace{1^3 + 2^3 + 3^3 + \ldots + n^3}_{h} + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right) \Leftrightarrow$

$\Leftrightarrow \left(\frac{n(\cancel{n+1})}{2}\right)^2 + (n+1)^{\cancel{3}} = \left(\frac{(\cancel{n+1})(n+2)}{2}\right)^2 \Leftrightarrow$

$\Leftrightarrow \frac{n^2}{4} + n + 1 = \frac{(n+2)^2}{4} \Leftrightarrow \frac{n^2+4n+4}{4} = \frac{n^2+4n+4}{4}$ cqd

e) $1 + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \geq \sqrt{n} \quad \forall n \in \mathbb{N}_1$

$\boxed{n=1}$

$1 \geq \sqrt{1} \Leftrightarrow 1 \geq 1$ cqd

$\boxed{\text{se } p(n) \Rightarrow p(n+1)}$ $\frac{1}{\sqrt{n+1}}\left(\sqrt{n} + \frac{1}{\sqrt{n+1}}\right) = \frac{\sqrt{n}}{\sqrt{n+1}} + \frac{1}{n+2} =$

$\underbrace{1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}}_{\geq \sqrt{n}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}}$ $= \frac{\sqrt{n}\sqrt{n+1}+1}{n+2} \geq 1$

$\times\sqrt{n+1}$
$\geq\sqrt{n+1}$

$\sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}+\sqrt{n+1}+1}{\sqrt{n+1}} \geq \frac{\sqrt{n}\,\sqrt{n+1}}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$

$1 + \sqrt{n}\sqrt{n+1} \geq \sqrt{n}\sqrt{n+1} + 1$

$\boxed{\begin{array}{l} a \geq b,\ c > 0 \\ a.c \geq b.c \end{array}}$

5

a) $(n+2)! \geq 2^{2n}$ $\quad \forall n \in \mathbb{N}_1$

$\boxed{n=1}$

$(1+2)! \geq 2^{2.1} \Leftrightarrow 3! \geq 2^2 \Leftrightarrow 6 \geq 4$ // cqd

$\boxed{\text{se } p(n) \Leftrightarrow p(n+1)}$

$(n+1+2)! \geq 2^{2(n+1)} \Leftrightarrow (n+3)! \geq 2^{2n+2} \Leftrightarrow$

$\Leftrightarrow (n+2)!.(n+3) \geq 2^{2n+2}$ //

pela hipotese: $(n+2)! \geq 2^{2n} \Leftrightarrow (n+2)!(n+3) \geq 2^{2n}(n+3) \Leftrightarrow$

$\Leftrightarrow (n+2)!(n+3) \geq 2^{2n}.2^2.\frac{(n+3)}{2^2} \Leftrightarrow (n+2)!(n+3) \geq 2^{n+2}.\frac{n+3}{4}$

$\frac{n+3}{4} \geq 1 \Leftrightarrow n+3 \geq 4 \Leftrightarrow n \geq 4-3 \Leftrightarrow n \geq 1$ cqd

b) $2n-3 < 2^{n-2}$ $\quad n \in \mathbb{N} \wedge n \geq 5$

$\boxed{n=5}$

$2.5-3 < 2^{5-2} \Leftrightarrow 7 < 2^3 \Leftrightarrow 7 < 8$ // cqd

$\boxed{\text{Se } p(n) \Leftrightarrow p(n+1)}$

$2(n+1)-3 < 2^{n-1} \Leftrightarrow \boxed{2n-1 < 2^{n-1}}$

$2n-3 < 2^{n-2} \Leftrightarrow 2n-3+2 < 2^{n-1}+2 \Leftrightarrow 2n-1 < (2^{n-1}+2)\,\frac{1}{2}$
$\Leftrightarrow 2n-1 < 2^{n-2}+1$ cqd

c) $7^n - 1$ é multiplo de 6 $\forall n \in \mathbb{N}_1$

$\hookrightarrow 6K$

$\boxed{n=1}$

$7^1 - 1 = 6 = 6.1 \Rightarrow$ cqd

$\boxed{\text{se } p(n) \Leftrightarrow p(n+1)}$

$7^{n+1} - 1 = 7 . 7^n - 1$

d) $2^{2n}+2$ é multiplo de 3 $\forall n \in \mathbb{N}_1$

$\boxed{n=1}$

$2^{2\times 1}+2 = 2^2+2 \Leftrightarrow 4+2 = 6$ cqd

$\hookrightarrow$ multiplo de 3

$\boxed{\text{se } p(n) \Leftrightarrow p(n+1)}$

6 $a > -1$, $n \in \mathbb{N}$

$$(1+a)^n \geq 1 + n.a$$

$\boxed{n=1}$

$1+a \geq 1+a$ cqd

$\boxed{\text{se } p(n) \Rightarrow p(n+1)}$

$$(1+a)^{n+1} \geq 1 + an + a$$

$(1+a)^n \geq 1+n.a \iff (1+a)^n(1+a) \geq (1+na)(1+a)$

$\iff (1+a)^{n+1} \geq 1 + an + a + \underbrace{na^2}_{?}$

7. $P(n)$: "n^2+3n+1 é par"

a) $p(n+1)$: $(n+1)^2 + 3(n+1) + 1 = n^2 + 1 + 3n + 3 + 1$

$= n^2+3n+1+4$

pela hipotese "n^2+3n+1 é par"

e 4 é par, logo $n^2+3n+1+4$ é par

b) $\boxed{n=1}$ $1^2+3.1+1=5$

não, $\exists n$ que torna falsa a condição $p(n)$

c) n^2+3n+1 é impar $\forall n \in \mathbb{N}$

$\boxed{n=1}$

$1^2+3.1+1=5$ 5 é impar cqd

$\boxed{P(n) \Rightarrow P(n+1)}$

$(n+1)^2+3(n+1)+1 = n^2+2n+1+3n+3+1 =$
$= n^2+3n+1+2n+4$

Pela hipotese n^2+3n+1 é impar

$2n+4$ é par, logo $n^2+3n+1+2n+4$ é impar

8. $f: \mathbb{N} \longmapsto \mathbb{N}$

$f(0)=1$
$f(n+1)=(2n+2)(2n+1)f(n)$

Mostre que $f(n)=(2n)! \Rightarrow f(n+1)=(2(n+1))!=(2n+2)!$

$f(n+1)=(2n+2)(2n+1)(2n)!=(2n+2)!$ cqd

9.

$$(u_n) = \begin{cases} u_1 = 3 \\ u_{n+1} = \dfrac{3u_n}{(n+1)^2} \end{cases}$$

Mostre que $u_n = \dfrac{3^n}{(n!)^2} \quad \forall n \in \mathbb{N}_1$

$$u(n+1) = \frac{3 \cdot \frac{3^n}{(n!)^2}}{(n+1)^2} = \frac{3 \cdot 3^n}{((n+1)!)^2} = \frac{3^{n+1}}{((n+1)!)^2} \quad \text{cqd}$$

10.

$$u_n \begin{cases} u_1 = 1 \\ u_{n+1} = \sqrt{2u_n^2 + 1} \end{cases}$$

Mostre que $u_n = \sqrt{2^n - 1}$

$$\Downarrow$$

$$u_{n+1} = \sqrt{2^{n+1} - 1}$$

$$u_{n+1} = \sqrt{2(\cancel{\sqrt{2^n - 1}})^{\cancel{2}} + 1} = \sqrt{2(2^n - 1) + 1} =$$

$$= \sqrt{2 \cdot 2^n - 2 + 1} = \sqrt{2^{n+1} - 1} \quad \text{cqd}$$

11.

n impar $\to$ $2n+1$

$$(2n+1)^2 = \underbrace{2n^2 + 4n}_{\text{par}} + 1$$

$n^2 \to 2n$

12.

Ficha 3

1.

$A = \{ x \in \mathbb{R} : |x| \geq \frac{x}{2} + 2 \}$

$B = [-3, 4]$ $\qquad$ $C = \mathbb{R} \setminus \mathbb{Q}$

a)

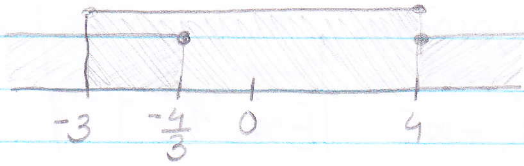
$$|x| \geq \frac{x}{2} + 2 \Leftrightarrow \underset{x \geq 0}{x \geq \frac{x}{2} + 2} \lor \underset{x < 0}{x \leq -\frac{x}{2} - 2} \Leftrightarrow$$

$$\Leftrightarrow x - \frac{x}{2} \geq 2 \lor x + \frac{x}{2} \leq -2 \Leftrightarrow \frac{2x - x}{2} \geq 2 \lor \frac{2x + x}{2} \leq -2 \Leftrightarrow$$

$$\Leftrightarrow \frac{x}{2} \geq 2 \lor \frac{3x}{2} \leq -2 \Leftrightarrow x \geq 4 \lor 3x \leq -4 \Leftrightarrow x \geq 4 \lor x \leq -\frac{4}{3}$$

$A = \{ x : x \in]-\infty, -\frac{4}{3}] \cup [4, +\infty[$

$A \cap B = [-3, -\frac{4}{3}] \cup \{4\}$



b)

$\sup A \rightarrow$ não existe
$\min(A \cap B) \rightarrow -3$
$\max(A \cap B) \rightarrow 4$
$\inf(A \cap B \cap C) \rightarrow -3$ $\qquad$ $]-3, -\frac{4}{3}[\cap \mathbb{R} \setminus \mathbb{Q}$
$\sup(A \cap B \cap C) \rightarrow -4/3$
$\min(A \cap B \cap C) \rightarrow$ Não existe

2.

$A = \{ x \in \mathbb{R} : \frac{x - 1}{x \log x} > 0 \} = x \in \mathbb{R}^+ \setminus \{1\}$

$x \log x = 0 \Leftrightarrow x = 0 \lor \log x = 0 \Leftrightarrow x = 0 \lor x = 1$

$B = \{ x \in \mathbb{R} : x = -\frac{1}{n}, n \in \mathbb{N}_1 \} = -1 \geq x > 0$

	A	A∪B	
sup	NE	NE	$\mathbb{R}^+$ não tem supremo
inf	1	-1	
max	NE	NE	$\mathbb{R}^+$ não tem máximo
min	NE	-1	

3.

$[1/2, 2] \setminus 1$

$A = \{x \in \mathbb{R}: \frac{1}{\log x} \geq 1\}$ $\quad B = \{1 - \frac{(-1)^n}{n}, n \in \mathbb{N}_1\}$

$\frac{1}{\log x} \geq 1 \Leftrightarrow$

$\frac{1}{\log x} - 1 \geq 0 \Leftrightarrow \frac{1 - \log x}{\log x} \geq 0$ | $1 - \log x = 0 \Leftrightarrow 1 = \log x \Leftrightarrow x = e$
$\log x = 0 \Leftrightarrow x = 1$

	0		1		e		$+\infty$
$1-\log x$	ND	+	+	+	0	-	
$\log x$	ND	-	0	+	+	+	
		-	ND	+	0	-	

$]1, e] = A$

	majorantes	minorantes	sup	inf	max	min
A	$[e, +\infty[$	$]-\infty, 1]$	e	1	e	—
B	$[2, +\infty[$	$]-\infty, 1/2]$	2	1/2	2	1/2

4.

$A = \{x \in \mathbb{R}: x^2 + 2|x| > 3\} = x \in]-\infty, -1[\cup]1, +\infty[$

a)

$x^2 + 2|x| > 3 \Leftrightarrow \underset{x \geq 0}{x^2 + 2.x > 3} \vee \underset{x<0}{x^2 + 2.(-x) > 3}$

$\Leftrightarrow x^2 + 2x - 3 > 0 \vee x^2 - 2x - 3 > 0$ //

$\underline{x > 1 \vee x < -3}$ $\quad$ $\underline{x < -1 \vee x > 3}$

$x^2 + 2x - 3 = 0 \Leftrightarrow$

$\Leftrightarrow x = \frac{-2 \pm \sqrt{4 - (4.1.(-3))}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-2 \pm \sqrt{4+12}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-2 \pm 4}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-2+4}{2} \vee x = \frac{-2-4}{2} \Leftrightarrow$

$\Leftrightarrow x = 1 \vee x = -3$ //

$x^2 - 2x - 3 = 0 \Leftrightarrow$

$\Leftrightarrow x = \frac{2 \pm \sqrt{4 - (4.1.(-3))}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{2 \pm 4}{2} \Leftrightarrow x = \frac{2+4}{2} \vee x = \frac{2-4}{2} \Leftrightarrow$

$\Leftrightarrow x = 3 \vee x = -1$ //

	$-\infty$	-3		-1		0		1		3	$+\infty$
x^2+2x-3	+	0	-	-	-	-	-	0	+	+	+
x^2-2x-3	+	+	+	0	-	-	-	-	-	0	+

b) $\inf A =$ NE $]1, \sqrt{2}[$

$\min A \cap B =$ NE

$\max A \cap B =$ NE

$\max A \cap B \cap \mathbb{Q} =$ NE

$\inf A \cap B \cap \mathbb{Q} = 1$

$\max C =$ NE

$\max B \setminus C =$ NE

$B =]0, \sqrt{2}[$

$C = \{ \sqrt{2} - \frac{1}{n} : n \in \mathbb{N}_1 \}$

$0 < C < \sqrt{2}$

$\hookrightarrow \sup C = \sqrt{2}$

5

a) $A = \{ x \in \mathbb{R} : |x-1| \leq x^2 - 1 \} =]-\infty, -1]$ $\quad B = [-2, 2]$

$|x-1| \leq x^2-1 \Leftrightarrow x - 1 \leq x^2 - 1 \Leftrightarrow x - 1 \geq 1 - x^2 \Leftrightarrow$

$\Leftrightarrow x \leq x^2 \vee x^2 + x \geq 1+1 \Leftrightarrow -x^2 + x \leq 0 \vee x^2 + x - 2 \geq 0$

$-x^2 + x = 0 \Leftrightarrow x^2 = x \Leftrightarrow x = \pm\sqrt{x} \Leftrightarrow x = \pm 1$ //

	$-\infty$	-1		1		2	$+\infty$
$-x^2+x$	$-$	0	$+$	0	$-$	$-$	$-$
x^2-x-2	$+$	0	$-$	$-$	$-$	0	$+$

$x^2 - x - 2 = 0 \Leftrightarrow x = \frac{1 \pm \sqrt{1-(4.1.(-2))}}{2} \Leftrightarrow x = \frac{1 \pm 3}{2} \Leftrightarrow x = \frac{4}{2} \vee x = \frac{-2}{2}$

$\Leftrightarrow x = 2 \vee x = -1$ //

b) $\operatorname{Max} A \cap B = -2$

$\operatorname{Min} A \cap B = 2$

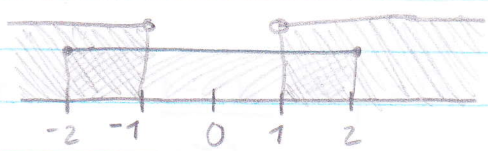
$\sup (A \cap B) \setminus \mathbb{Q} = 2$

$\inf (A \cap B) \setminus \mathbb{Q} = -2$

$\max (A \cap B) \setminus \mathbb{Q} = 2$

$\min (A \cap B) \setminus \mathbb{Q} = -2$

$A \cap B =$



6

a) $A = \{x : |x^2-2| \le 2x+1\}$ $\quad B = \mathbb{Q}$ $\quad C = \{\frac{1}{k^2} : k \in \mathbb{N}\}$

$|x^2-2| \le 2x+1 \Leftrightarrow x^2-2 \le 2x+1 \vee x^2-2 \ge -2x-1 \Leftrightarrow$

$\Leftrightarrow x^2-2x-2-1 \le 0 \vee x^2+2x-2+1 \ge 0 \Leftrightarrow$

$\Leftrightarrow x^2-2x-3 \le 0 \vee x^2+2x-1 \ge 0$ //

$x^2-2x-3=0 \Leftrightarrow x = \frac{2 \pm \sqrt{4-(4.1.(-3))}}{2} \Leftrightarrow x = \frac{2 \pm \sqrt{4+12}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{2 \pm \sqrt{16}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{2+4}{2} \vee x = \frac{2-4}{2} \Leftrightarrow x = 3 \vee x = -2$ //

$x^2+2x-1=0 \Leftrightarrow x = \frac{-2 \pm \sqrt{4-(4.1.(-1))}}{2} = x = \frac{-2 \pm \sqrt{8}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-2 \pm 2\sqrt{2}}{2} \Leftrightarrow x = -1 \pm \sqrt{2}$

	$-\infty$	$-1-\sqrt{2}$		-2		$-1\pm\sqrt{2}$		3	$+\infty$
x^2-2x-3	+	+	+	0	−	−	−	0	+
x^2+2x-1	+	0	−	−	−	0	+	+	+

b)

	$A \cap B$	C
sup	NE	1
inf	$-1\pm\sqrt{2}$	NE
max	NE	1
min	$-1\pm\sqrt{2}$	NE

7

a) $A = \{x : \frac{x^2-1}{x} \ge |x-1|\}$ $\quad B = \{x : \sin x = 0\}$ $\quad C = \mathbb{Q}$

$\frac{x^2-1}{x} \ge |x-1| \Leftrightarrow \frac{x^2-1}{x} \ge x-1 \vee -\frac{(x^2-1)}{x} \le x-1 \Leftrightarrow$

$\Leftrightarrow x^2-1 \ge x(x-1) \vee -x^2+1 \le x(x-1) \Leftrightarrow$

$\Leftrightarrow \cancel{x^2}-1 \ge \cancel{x^2}-x \vee -x^2+1 \le x^2-x \Leftrightarrow$

$\Leftrightarrow x-1 \ge 0 \vee -2x^2+x+1 \le 0$ //

$x-1=0 \Leftrightarrow x=1 \qquad -2x^2+x+1=0 \Leftrightarrow x = \frac{-1\sqrt{1-(4.(-2).1)}}{-4} \Leftrightarrow$

$\Leftrightarrow x = \frac{-1 \pm 3}{-4} \Leftrightarrow x = \frac{2}{-4} \vee x = \frac{-4}{-4} \Leftrightarrow x = -\frac{1}{2} \vee x = 1$ //

	$-\infty$	$-1/2$		1	$+\infty$
$\lvert x-1 \rvert$	$+$	$+$	$+$	0	$+$
$-2x^2+x+1$	$-$	0	$+$	0	$-$
	$-$	0	$+$	0	$-$

b)

	majorantes	minorantes
$A \cap C$	NE	$]-\infty, -1/2]$
$B \cap C$		

$\sup A = NE$
$\inf A \cap C = -1/2$
$\min A \cap C = -1/2$
$\min B = NE$
$\sup B \cap C = NE$

8

a) $A = \{ x : x \geq 0 \wedge \frac{x^4-4}{|x-1|} \leq 0 \} \Leftrightarrow x = [0, \sqrt{2}] \setminus \{1\}$

$$\frac{x^4-4}{|x-1|} \leq 0 \Leftrightarrow \frac{x^4-4}{x-1} \leq 0 \vee \frac{x^4-4}{1-x} \leq 0$$

$x^4-4=0 \Leftrightarrow (x^2-2)^2=0 \Leftrightarrow x^2-2=0 \Leftrightarrow x = \pm\sqrt{2}$

	$-\infty$	$-\sqrt{2}$		0		1		$\sqrt{2}$	$+\infty$
x^4-4	$+$	0	$-$	$-$	$-$	$-$	$-$	0	$+$
$\lvert x-1 \rvert$	$+$	$+$	$+$	$+$	$+$	0	$+$	$+$	$+$
	$+$	0	$-$	$-$	$-$	NE	$-$	0	$+$
x	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$

b) $B = \{ x : x \geq 0 \wedge \exists_{k \in \mathbb{N}}\, kx \notin \mathbb{Q} \}$

	A	$A \setminus B$
sup	$\sqrt{2}$	NE
inf	0	0
max	$\sqrt{2}$	NE
min	0	0

9

a) $A = \left\{ x : \dfrac{x^2-2}{|x|-1} \leq 0 \right\}$

$\dfrac{x^2-2}{|x|-1} \leq 0 \iff \dfrac{x^2-2}{x-1} \leq 0 \vee \dfrac{x^2-2}{-x-1} \leq 0$

$x^2-2=0 \iff x^2=2 \iff x=\pm\sqrt{2}$ //

$x-1=0 \iff x=1$ //

$-x-1=0 \iff -x=1 \iff x=-1$ //

	$-\infty$	$-\sqrt{2}$		-1		1		$\sqrt{2}$	$+\infty$
x^2-2	+	0	−	−	−	−	−	0	+
$\|x\|-1$	+	+	+	0	−	0	+	+	+
	+	0	−	NE	+	NE	−	0	+

$\Omega = [-\sqrt{2}, -1[\cup]1, \sqrt{2}]$

b) $B = \{ 2^{n/2} : n \in \mathbb{N}_1 \}$

	$A \cap \mathbb{Q}$	$B \cap \mathbb{Q}$
sup	$\sqrt{2}$	NE
max	$\sqrt{2}$	NE
inf	$-\sqrt{2}$	$\sqrt{2}$
min	$-\sqrt{2}$	$\sqrt{2}$

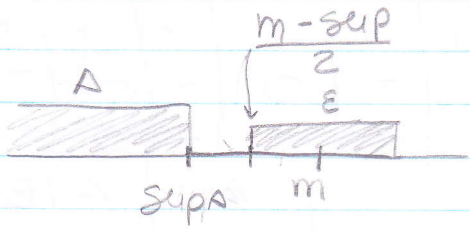
10. $A \subset \mathbb{R}$

A é majorado

$A \neq \emptyset$

$m \in$ majorantes de A

$m \neq \sup A$



$V_\varepsilon(m) = \{ x \in \mathbb{R} : |x-m| < \varepsilon \}$

Se $\varepsilon = \dfrac{m - \sup A}{2}$, temos que $\forall x \in A \; x \leq \sup(A)$

portanto $x < m - \varepsilon$ e $x \in V_\varepsilon(m)$ $\quad \sup(A) < m - \varepsilon \iff$

$\iff \sup(A) < m - \dfrac{(m - \sup A)}{2} \iff$

$\iff \dfrac{m - \sup A}{2} < m - \sup A$

11.

12. $U, V \neq \emptyset$

$U, V \in \mathbb{R}$ $\quad\quad$ $\sup U < \sup V$

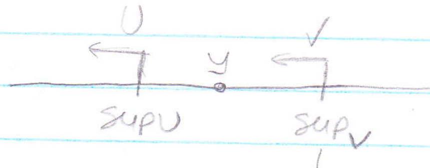
U, V majorados

a) $\forall x \in U \Rightarrow x < \sup V$

$\forall x \in U \leq \sup U < \sup V \Rightarrow \forall x \in U < \sup V$

b) $\exists y \in V : y > \sup U$

$y \leq \sup V$ $\quad$ não é majorante de V



$\Downarrow$

$\exists y \in V > \sup V$

ñ há majorantes menores

Ficha 4

1e2	majorada	minorada	limitada	monótona
a)	— SIM	— SIM	— SIM	decrescente
b)	sim	sim (0)	sim	não
c)	não	não	não	não
d)	não	sim(0)	não	não
e)	não	sim (1)	não	sim crescente
f)	não	não	não	não
g)	— Não	— sim (0)	— NÃO	sim crescente

b) $\frac{n+(-1)^n}{n}$ $\{0, \frac{3}{2}, \frac{2}{3}, \ldots\}$

min: $\frac{1+(-1)^1}{1} = 0$ sup = 2

a) $\frac{1}{\sqrt{n+1}}$

c) $(-1)^n n^2$ $\{-1, 2, -3, 4 \ldots\}$

$\sqrt{n+1} \rightarrow$ aumenta

então $\frac{1}{\sqrt{n+1}}$ diminui

$min = (-1)^1 1 = -1$

d) $n^{(-1)^n}$

g) $u_1 = 0$

$u_{n+1} = \frac{2u_n + 1}{3}$

min: $1^{(-1)^1} = -1$

$\{0, \frac{1}{3}, \frac{5}{6} \ldots\}$

$u_{2n} = 2n^{-1^{2n}} = 2n^1 = 2n$

$u_{2n+1} = (2n+1)^{-2} = 1/2n+2$

e) $1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$

$min = \frac{1}{2^0} = \frac{1}{1} = 1$

F) $u_1 = 1$, $u_{n+2} = -2u_n$ $\{1, -2, 4, -8$

$\lim_{x\to a} f(x) = b$

$f(x) \longrightarrow b$

3

a) $\frac{1}{\sqrt{n+1}} \longrightarrow 0$

$\forall \varepsilon > 0 \; \exists p \in \mathbb{N} \; \forall n \in \mathbb{N} : n \geq p \Rightarrow |u_n - 0| < \varepsilon$

Dado $\varepsilon > 0$: $\left| \frac{1}{\sqrt{n+1}} - 0 \right| < \varepsilon \Leftrightarrow \frac{1}{\sqrt{n+1}} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < \sqrt{n+1} \Leftrightarrow$

$\Leftrightarrow \left(\frac{1}{\varepsilon}\right)^2 < n+1 \Leftrightarrow n > \underbrace{\frac{1}{\varepsilon^2} - 1}_{= p}$

cqd

b) $\frac{n^2}{n^2+1} \longrightarrow 1$

$\forall \varepsilon > 0 \; \exists p \in \mathbb{N} \; \forall n \in \mathbb{N} : n \geq p \Rightarrow |u_n - 1| < \varepsilon$

dado um $\varepsilon > 0$: $\left| \frac{n^2}{n^2+1} - 1 \right| < \varepsilon \Leftrightarrow \frac{\cancel{n^2} - \cancel{n^2} + 1}{n^2+1} < \varepsilon \Leftrightarrow$

$\Leftrightarrow \frac{1}{n^2+1} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < n^2+1 \Leftrightarrow \frac{1}{\varepsilon} - 1 < n^2 \Leftrightarrow$

$\Leftrightarrow \underbrace{\sqrt{\frac{1}{\varepsilon} - 1}}_{p} < n$

cqd

c) $u_n = n^2$ é divergente $\rightarrow$ não tende para um numero real a

$\forall \varepsilon > 0 \; \exists p \in \mathbb{N} \; \forall n \in \mathbb{N} : n \geq p \Rightarrow |n^2 - a| < \varepsilon$

dado um $\varepsilon > 0$:

$|n^2 - a| < \varepsilon \Leftrightarrow n^2 < \varepsilon - a \Leftrightarrow n < \sqrt{\varepsilon - a}$

$\rightarrow \nexists p \in \mathbb{N} \; \forall n \in \mathbb{N} : n \geq p$

logo é divergente

4 a) $\frac{2n-1}{n+} \to 2$

$\forall \varepsilon>0 \ \exists p \in \mathbb{N} \ \forall n \in \mathbb{N}: \underline{n \geq p} \Rightarrow |u_n - 2| < \varepsilon$

Dado $\varepsilon>0$:

$$\left|\frac{2n-1}{n+1} - \frac{2}{1}\right| < \varepsilon \Leftrightarrow \left|\frac{2n-1-2(n+1)}{n+1}\right| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \left|\frac{\cancel{2n}-2-\cancel{2n}-2}{n+1}\right| < \varepsilon \Leftrightarrow \left|\frac{-3}{n+1}\right| < \varepsilon \Leftrightarrow \frac{3}{\varepsilon} < n+1 \Leftrightarrow$$

$$\Leftrightarrow \frac{3}{\varepsilon} - 1 < n \quad , \Rightarrow \frac{3}{\varepsilon} - 1 = p \quad \text{cqd}$$

b) $\frac{\sqrt{n^2-1}}{n} \to 1$

$\forall \varepsilon>0 \ \exists p \in \mathbb{N} \ \forall n \in \mathbb{N}: n \geq p \Rightarrow |u_n - 1| < \varepsilon$

Dado um $\varepsilon>0$:

$$\left|\frac{\sqrt{n^2-1}}{n} - 1\right| < \varepsilon \Leftrightarrow \left|\frac{\sqrt{n^2-1}-n}{n}\right| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \sqrt{n^2-1} - n < \varepsilon . n \Leftrightarrow \sqrt{n^2-1} < \varepsilon n + n \Leftrightarrow$$

$$\Leftrightarrow n^2 - 1 < (\varepsilon n + n)^2 \Leftrightarrow \cancel{n^2} - 1 < (\varepsilon n)^2 + 2n + \cancel{n^2} \Leftrightarrow$$

$$\Leftrightarrow 1 < n(\varepsilon n + 2) \Leftrightarrow n > 1 \vee \varepsilon n + 2 > 1 \Leftrightarrow$$

$$\Leftrightarrow n > 1 \vee \varepsilon n > 1-2 \Leftrightarrow n > 1 \vee n > -\frac{1}{\varepsilon} \ //$$

$$\Rightarrow p = 1 \quad \text{cqd} \qquad \to -\frac{1}{\varepsilon} < 0 \neq p$$

5.

a) $$\frac{(2n+1)^3+n}{n^3+1} = \left(\frac{+\infty}{+\infty}\right)$$

$$\lim \frac{\left(\frac{2n+1}{n}\right)^3+\frac{n}{n^3}}{\frac{n^3}{n^3}+\frac{1}{n^3}} = \lim \frac{\left(\frac{2n+1}{n}\right)^3+\frac{1}{n^2}}{1+\frac{1}{n^3}} = \frac{2^3+0}{1+0} = 8$$

c.a.

$$\lim \frac{2n+1}{n} = \lim \frac{2n}{n}+\frac{1}{n} = \lim 2+\frac{1}{n} = 2$$

b) $$\lim \frac{(2n+1)^3+n^2}{(n+1)^2(n+2)} = \lim \frac{\left(\frac{2n+1}{n}\right)^3+\frac{\cancel{n^2}}{n^{\cancel{3}}}}{\frac{n^3+4n^2+5n+2}{n^3}} =$$

$$\lim \frac{\left(\frac{2n+1}{n}\right)^3+\frac{1}{n}}{1+\frac{4}{n}+\frac{5}{n^2}+\frac{2}{n^3}} = \frac{2^3+0}{1+0} = 2^3 //$$

$$(n^2+2n+1)(n+2) = n^3+2n^2+2n^2+4n+n+2 =$$
$$= n^3+4n^2+5n+2$$

c) $$\lim \frac{(n+1)^2+2n^4}{(n+1)^4+2n^2} = \lim \frac{\left(\frac{n+1}{n^2}\right)^2+\frac{2n^4}{n^4}}{\left(\frac{n+1}{n}\right)^4+\frac{2n^2}{n^4}} =$$

$$= \lim \frac{\left(\frac{1}{n}+\frac{1}{n^2}\right)^2+2}{\left(1+\frac{1}{n}\right)^4+\frac{2}{n^2}} = \frac{0+2}{1+0} = 2$$

d) $$\lim \frac{\sqrt{n}-1}{\sqrt{n-1}} = \frac{1-1}{1} = 0$$

$$\lim \frac{n-1}{n} = \frac{1-0}{1} = 1$$

$$\frac{n}{n+1} = \frac{1}{1+0} = 1$$

e) $\lim \frac{1}{n}\left(2+\frac{1}{n}\right) = 0$

F) $\lim \frac{1}{n}(2n+\sqrt{n})^{0.\infty} = \lim \frac{2n}{n} + \frac{\sqrt{n}}{n} = \lim 2 + \underbrace{\frac{\sqrt{n}}{n}}_{\to 0} = 2$

g) NE $S = \left\{ \frac{-1}{n!}, \frac{1}{n!} \right\}$, logo é divergente

S: conjunto dos sublimites da sucessão

h) $\lim \frac{\sqrt{n} \to 1}{\sqrt[4]{4n^2+1}} = 1$

$\downarrow$

1

$\frac{x_{n+1}}{x_n} = \frac{n+2}{n} = \lim 1 + \frac{1}{n} = 1$

$\frac{4(n+2)^2+2}{4n^2+2} = \frac{4(n^2+2n+4)+2}{4n^2+2}$

i) $\lim \frac{\sqrt[3]{n+1}}{\sqrt[3]{n+2}} =$

$= \frac{\frac{4n^2}{n^2}+\frac{8n}{n^2}+\frac{9}{n^2}}{\frac{4n^2}{n^2}+\frac{2}{n^2}} = \frac{4+8/n+9/n^2}{4+2/n^2}$

$= \frac{1+1}{1} = 2$

$\frac{n+2}{n+2} = \frac{1+2/n}{1+2/n} = 1$

$= \frac{4+0+0}{4+0} = 1$

j) $\lim \frac{n+1}{n!} = \lim \frac{n}{n!} + \frac{1}{n!} = 0$

K) NE $S = \left\{ \frac{1}{\sqrt{n}}, 0 \right\}$

S: conjunto dos sublimites da sucessão

L) $\lim \dfrac{2^{n+1}+3^{n+1}}{2^n+3^n} = \lim \dfrac{2\left(\frac{2}{3}\right)^n + \frac{3^{\cancel{n+1}}}{\cancel{3^n}}}{\left(\frac{2}{3}\right)^n + 1} = \dfrac{2.0+3}{0+1} = 3$

$\left(\frac{2}{3}\right)^n \to 0$

m) $\lim \dfrac{\sqrt[n]{1000}+1000}{n} = \dfrac{1+1000}{+\infty} = 0$

$\sqrt[n]{1000} \to 1$, $n \to +\infty$

n) $\lim \dfrac{n^n}{n^n+1} = \lim \dfrac{n^n/n^n}{n^n/n^n + 1/n^n} = \dfrac{1}{1+0} = 1$

o) $\lim \dfrac{\sqrt[n]{3}}{\sqrt[3]{n}} = \dfrac{1}{1} = 1$

$\sqrt[n]{3} \to 1$, $\sqrt[3]{n} \to 1$

p) $\lim \dfrac{4^n}{1+4^{n^2}} = \lim \dfrac{\frac{4^n}{4^{n^2}}}{\frac{1}{4^{n^2}} + \cancel{\frac{4^{n^2}}{4^{n^2}}}} = \lim \dfrac{4^{n-n^2}}{\left(\frac{1}{4}\right)^{n^2}+1} = 0$

$n-n^2 \to -\infty$, $4^{n-n^2} \to 0$, $\left(\frac{1}{4}\right)^{n^2} \to 0$

q) $\lim \dfrac{(a^n)^2}{a^{n^2}} = \lim \dfrac{a^{2n}}{a^{n^2}} = \lim a^{2n-n^2} = 0$

6

$x_n = \dfrac{a^n}{2^{1+2n}}$

a) $\dfrac{a^n}{2^{1+2n}} = \dfrac{a^n}{2.2^{2n}} = \dfrac{1}{2}\left(\dfrac{a}{4}\right)^n$

$\lim a^n \begin{cases} +\infty, & a>1 \\ 1, & a=1 \\ 0, & -1<a<1 \\ NE, & a\leq -1 \end{cases}$

Convergente

$-1 < \dfrac{a}{4} < 1 \iff -4 < a < 4$

b) $\left(\dfrac{a}{4}\right) = -1 \iff a = -4$

7

a) $u_n = 1 - \frac{1}{n}$ tem termos em $]-\infty, 1[$ e é crescente

$u_n = -1$

b) monótona convergente

$u_n = \frac{(-1)^n}{n}$ $1 + \frac{(-1)^n}{n}$

c) $u_n = (-1)^n$ divergente, $|u_n|$ convergente

d) $u_n = (-1)^n$ limitada e divergente

e) $u_n = \frac{1}{n} \cdot (-2)^n$ $u_{2n} = 1/2$ e $u_{2n+1} = 1/3$ termos em $\frac{1}{n}$ e é divergente

f) $u_n = \frac{\sqrt{2}}{n}$ termos em $\mathbb{R} \setminus \mathbb{Q}$, $\to \mathbb{Q}$

8

$A = \{x \in \mathbb{R} : x^2 + 2|x| > 3\} =]-\infty, -1[\cup]1, +\infty[$

$B =]0, \sqrt{2}[$ $C = \{\sqrt{2} - \frac{1}{n} : n \in \mathbb{N}_1\}$

a)

b)

c)

d)

e)

f)

9)

$$\begin{cases} u_1 = a \\ u_{n+1} = r + u_n \end{cases} \qquad \begin{cases} v_1 = a \\ v_{n+1} = r v_n \end{cases}$$

a)

$u_n = a + (n-1)r$	$v_n = ar^{n-1}$
$u_1 = a + (1-1)r = a$	$v_1 = ar^{1-1} = a$
$u_{n+1} = a + (n+1-1)r = a + nr$	$v_{n+1} = ar^n$
$u_{n+1} = r + u_n = r + a + (n-1)r =$	$v_{n+1} = r v_n =$
$= \cancel{r} + a + rn - \cancel{r} =$	$= r . ar^{n-1} =$
$= a + nr$ cqd	$= \frac{\cancel{r} . a . r^n}{\cancel{r}} = ar^n$ cqd

b)

i) $\forall\, r, a > 0$

ii) $\forall a, \forall\, r < 0$

iii) $\forall a, r > 0$

iv) se a ou r forem negativos (e apenas um dos)

c) u_n não é limitada $\forall a \in \mathbb{R}, r \neq 0$

$$\begin{cases} u_1 = a \\ u_{n+1} = r + u_n \end{cases}$$

$u_2 = r + a$

$u_n = (n-1)r + a$ se $n > 1 \Rightarrow n-1 > 0$

se $r > 0$ $(n-2).r > 0 \to +\infty$

u_n não é limitada

se $r < 0$ $(n-2)\, r < 0 \to -\infty$

quando v_n é limitada? e convergente?

$v_n = a . r^{n-2}$

limitada	convergente
se a	se $a > 0$

10.

$$u_n \begin{cases} u_1 = 1 \\ u_{n+1} = 1 + \frac{u_n}{2} \end{cases}$$

a) $u_n \leq 2 \quad \forall n \in \mathbb{N}_1$

$n=1$

$u_n = 1 \leq 2$

$1 + \frac{u_n}{2} \leq 2 \Leftrightarrow \frac{u_n}{2} \leq 1 \Leftrightarrow u_n \leq 1.2 \Leftrightarrow u_n \leq 2$ cqd

b) $u_1 = 1$

$u_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$u_3 = 1 + \frac{3}{2} = \frac{5}{2}$

$u_1 < u_2 < u_3 < \ldots < u_n$

$u_4 = 1 + \frac{5}{2} = \frac{7}{2}$ ...

logo é crescente

c) Se un é crescente e limitada por $1 \leq u_n \leq 2$ então $\lim u_n \rightarrow 2$

11.

$$u_n \begin{cases} u_1 = \frac{3}{2} \\ u_{n+1} = \frac{u_n^2 + 2}{3} \end{cases}$$

$1 < u_n < 2$

a) $u_1 = \frac{3}{2} \qquad 1 < \frac{3}{2} < 2$

$P(n+1) \Rightarrow P(n): 1 < \frac{u_n^2+2}{3} < 2 \Leftrightarrow 3 < u_n^2 + 2 < 6 \Leftrightarrow$

$\Leftrightarrow 1 < u_n^2 < 4 \Leftrightarrow \sqrt{1} < u_n < \sqrt{4} \Leftrightarrow$

$\Leftrightarrow 1 < u_n < 2$ // cqd

b) u_n estritamente decrescente

$$u_{n+1} - u_n = \frac{u_n^2+2}{3} - u_n = \frac{\boxed{u_n^2+2-3u_n}}{3} < 0$$

≤0

<0

por a)

P.E. $x^2-3x+2 \leq 0 \Leftrightarrow x = \frac{3 \pm \sqrt{9-4.1.2}}{2} \Leftrightarrow x = \frac{3\pm\sqrt{9-8}}{2} \Leftrightarrow$
$\Leftrightarrow x = 3 \pm 1/2 \Leftrightarrow x = 2 \vee x = 1$ cqd

c) u_n é decrescente e limitada, logo u_n é convergente. u_{n+1} é subsuc de u_n, então $\lim u_{n+1} = \lim u_n$

$a = \frac{a^2+2}{3} \Leftrightarrow a = 1 \vee a = 2$ Como u_n decrescente $\lim u_n = 1$

12. $u_1 > 1$ $\quad u_{n+1} = 2 - \frac{1}{u_n}$, $n \in \mathbb{N}_1$

$u_2 = 2 - \frac{1}{u_1} \rightarrow u_2 > u_1$
$\quad \hookrightarrow u_1 > 1$

logo, u_n é crescente

quando $n \to +\infty$, $u_n \to +\infty$
logo
$2 - \frac{1}{+\infty} = 2 = \lim u_n$

13.

a) $u_n: \begin{cases} u_1 = 1 \\ u_{n+1} = \sqrt{2+u_n} \end{cases}$

PROVE:

$1 \leq u_n < 2 \quad \forall n \in \mathbb{N}.$

$\boxed{n=1}$ $\quad u_1 = 1 \quad 1 \leq u_1 < 2$ cqd

$\boxed{n+1}$ $1 \leq \sqrt{2+u_n} < 2$

$1 \leq u_n < 2 \Leftrightarrow 2+1 \leq u_n+2 < 2+2 \Leftrightarrow$
$\Leftrightarrow \sqrt{3} \leq \sqrt{u_n+2} < \sqrt{4} \Leftrightarrow \sqrt{3} \leq \sqrt{u_n+2} < 2$

$1 \leq \sqrt{3} < 2$ cqd

b)

$u_1 = 1$
$u_2 = \sqrt{2+1} = \sqrt{3}$ $u_1 < u_2 < u_3 \ldots < u_n$
$u_2 = \sqrt{2+\sqrt{3}}$
$\rightarrow u_n$ é crescente

$u_{n+2} - u_n = \dfrac{(2-u_n)(u_n+1)}{u_n + \sqrt{2+u_n}}$

$u_{n+1} - u_n = \sqrt{2+u_n} - u_n = \dfrac{(\sqrt{2+u_n}-u_n)(\sqrt{2+u_n}+u_n)}{u_n+\sqrt{2+u_n}}$

$= \dfrac{2+u_n+\cancel{u_n\sqrt{2+u_n}}-\cancel{u_n\sqrt{2+u_n}}-u_n^2}{u_n+\sqrt{2+u_n}} = \dfrac{-u_n^2+u_n+2}{u_n}$ cqd

c) u_n é crescente ✓ logo u_n é convergente
u_n é limitada ✓

14. $u_n > 0$ $\dfrac{u_{n+1}}{u_n} < 1 \Rightarrow u_n$ é descrescente
$\rightarrow u_{n+1} > u_n$
decrescente
logo é limitada

$\rightarrow$ logo é convergente

15. $x_n \rightarrow$ monótona | mesmo limite? $\forall n \in \mathbb{N}_1$ $|x_n - y_n| < \dfrac{1}{n}$
$y_n \rightarrow$ limitada
$\rightarrow$ limitada?

limitada

Mesmo grau $\Rightarrow$ limite é = ao produto dos coeficientes de maior grau

$\frac{(-1)^n \cdot 1}{|a|^n} \to 0$ tende p/ 0

ficha 5

1. $u_n = \frac{(-1)^{n+1}}{n}$

- limitada

$-\frac{1}{2} \leq u_n \leq 1$

$v_n = \frac{n^{n+1}}{n^n + 1}$

- não limitada
- divergente

$w_n = u_n \cdot v_n$

- divergente?
- limitada

$-1 < w_n < 1$

$$w_n = \frac{n^{n+1}}{n^n+1} \cdot \frac{(-1)^{n+1}}{n} =$$

$$= \frac{n^{n+1} \cdot (-1)^{n+1}}{n(n^n+1)} = \frac{(-n)^{n+1}}{n^{n+1}+n} \Rightarrow n^{n+1}+n > (-n)^{n+1}$$

2. $u_n = \cos(n!\pi)$

- limitada $[-1, 1]$

para $n \geq 2$, $n!$ par

1 k par

-1 k impar

$\Rightarrow \cos(n!\pi) = 1$

logo, $\lim u_n = 1$

$v_n = \frac{n \cos(n\pi)}{2n+1}$

$\frac{n}{2n+1} \cdot \cos(n\pi) \to$ 1 ou -1 $\to (-1)^n$

$\to \frac{1}{2+0} = \frac{1}{2}$

$v_{2n} \to 1/2$ $\quad v_{2n+1} \to -\frac{1}{2}$

não $\exists$ limite

$a \in \mathbb{R}$

$w_n = \frac{1+a^n}{1+a^{2n}}$

$\boxed{a > 1}$ $a^n \to +\infty$

$$\frac{\frac{1}{a^{2n}} + \frac{1}{a^n}}{\frac{1}{a^{2n}} + 1} \to 0$$

$\boxed{a < -1}$

$$\frac{1+(-1)^n|a|^n}{1+(-1)^{2n}|a|^{2n}} = \frac{1+(-1)^n|a|^n}{1+|a|^{2n}} = \frac{\frac{1}{|a|^{2n}} + \frac{(-1)^n}{|a|^n}}{\frac{1}{|a|^{2n}} + 1} \to 0$$

$\boxed{a=1} \to 1$ $\quad \boxed{a=-1}$ NE lim

3.

a) $\lim \frac{1}{(-1)^n n^2 + 2} = 0$

$\hookrightarrow$ não é limitada

$u_{2n} =$

$u_{2n+1} =$

b) $\lim \left(1+(-1)^n\right)\left(1+\frac{1}{n}\right)$

$\left(1+\frac{1}{n}\right) \to 1$

$(1+(-1)^n) \to$ não é convergente

$u_{2n} =$

$u_{2n+1} =$

c) $\lim \frac{n(1+(-1)^n)}{2}$ não é limitada, não é convergente

$u_{2n} =$ $\quad$ $u_{2n+1} =$

d) $\lim \frac{2n^2+(-1)^n}{n^2-1} \to u_{2n} =$

$u_{2n+1} =$

e) $\lim \frac{n+\cos n}{2n-1} = \lim \frac{1+\cos n/n}{2-\frac{1}{n}} = \frac{1+0}{2-0} = \frac{1}{2}$

$\underset{\substack{\uparrow\\0}}{-\frac{1}{n}} \leq \frac{\cos n}{n} \leq \underset{\substack{\uparrow\\0}}{\frac{1}{n}}$ por teorema das sucessões enquadradas

$\frac{\cos n}{n} \to$ limitada então $=0$

f) $\lim \left(-1-\frac{1}{n}\right)^n = -e$

4. u_n convergente

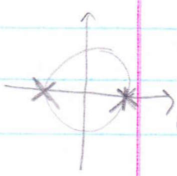
$u_{2n} \in]0,1[\qquad u_{2n+1} \in \mathbb{R} \setminus \{0,1\}$ PROVAR: $\lim u_n \in \{0,1\}$

5. $\begin{cases} u_1 = a \\ u_{n+1} = (-1)^n u_n + \frac{u_n}{n+2} \end{cases}$ se u_n convergente $\lim u_n = 0$

$\overline{\mathbb{R}} = \mathbb{R} \cup]-\infty, +\infty[$

6.

a) $u_n = \frac{1}{n} + 2\cos n\pi = \frac{1}{n} + 2(-1)^n$



$\cos(n\pi) = (-1)^n$ $\quad u_{2n} = \frac{1}{2n} + 2 \to 2 \quad u_{2n+1} = \frac{1}{2n+1} - 2 \to -2$

$S: \{-2, 2\}$

b) $u_{2n} = 0$ e $u_{2n+1} = n$

$S = \{0, +\infty\}$

c) $1, 2, 3, 4, 5, 6, 7, 8 \ldots$

$S = \{+\infty\}$

d) $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5 \ldots$

$S = \{\mathbb{N}, +\infty\}$

• Sim, como no caso da alínea b

• em $\overline{\mathbb{R}}$ o conjunto dos sublimites nunca é singular se a sucessão for divergente

7. $A = \{x \in \mathbb{R} : |2x+1| < |x|\}$, $B = \{\frac{(-1)^n}{n} : n \in \mathbb{N}_1\}$,

$C = [-1, +\infty[$

a) $|2x+1| < |x| \Leftrightarrow 2x+1 < |x| \vee -(2x+1) < |x| \Leftrightarrow$

$\Leftrightarrow 2x+1 < x \vee 2x+1 < -x \vee -2x-1 < x \vee -2x-1 < -x \Leftrightarrow$

$\Leftrightarrow 2x - x < -1 \vee 2x + x < -1 \vee -2x - x < 1 \vee -2x + x < 1 \Leftrightarrow$

$\Leftrightarrow \boxed{x < -1} \vee \underline{3x < -1} \vee \underline{-3x < 1} \vee \boxed{-x < 1} \Leftrightarrow$

$\Leftrightarrow x < -1 \vee x < -\frac{1}{3}$

$A =]-1, -\frac{1}{3}[$

b)

$\inf C = -1$ $\qquad$ $B = [-1, \frac{1}{2}]$

$\min (C/A) = -1$

$\sup (A \setminus \mathbb{Q}) = -\frac{1}{3}$

$\max (A \cup B) = 1/2$

$\inf B = -1$

$\max (B \setminus \mathbb{Q}) =$ não existe

c) i) sim. Toda a sucessão decrescente e limitada é convergente

ii)

iii) não. Pode convergir para um numero em C

iv) sim. S, U_n é limitada, pelo teorema de BW garante que U_n tem alguma SS convergente, portanto U_n tem pelo menos um sublimite.

v) sim.

8.

a) $A = \{ x \in \mathbb{R} : \frac{x^2+1}{x-2} \geq x \}$

$\frac{x^2+1}{x-2} - x \geq 0 \Leftrightarrow$

$\Leftrightarrow \frac{x^2+1-x(x-2)}{x-2} \geq 0 \Leftrightarrow \frac{\cancel{x^2}+1-\cancel{x^2}+2x}{x-2} \geq 0 \Leftrightarrow$

$\Leftrightarrow \frac{2x+1}{x-2} \geq 0$

$x-2 \neq 0 \Leftrightarrow x \neq 2$

$2x+1=0 \Leftrightarrow 2x=-1 \Leftrightarrow x = -\frac{1}{2}$ $\qquad$ $A =]-\infty, -\frac{1}{2}] \cup]2, +\infty[$

	$-\infty$	$-1/2$		2		$+\infty$
$2x+1$	$-$	0	$+$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	0	$+$	$+$
Q	$+$	0	$-$	ND	$+$	$+$

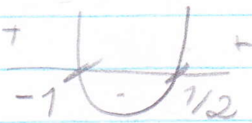
$B = \{x \in \mathbb{R} : \log(2x^2 + x) \geq 0\}$

$\log(2x^2+x) \geq 0 \Leftrightarrow 2x^2 + x \geq 1 \Leftrightarrow 2x^2 + x - 1 \geq 0$

$2x^2 + x - 1 = 0 \Leftrightarrow x = \frac{-1 \pm \sqrt{1 - 4.2.(-1)}}{2.2} \Leftrightarrow$

$\Leftrightarrow x = \frac{-1 \pm \sqrt{1+8}}{2.2} \Leftrightarrow x = \frac{-1 \pm 3}{4} \Leftrightarrow x = \frac{-1+3}{4} \vee x = \frac{-1-3}{4}$

$\Leftrightarrow x = \frac{1}{2} \vee x = -1$ //



$B =]-\infty, -1] \cup [1/2, +\infty[$

b)

min A = NE

sup $B \cap \mathbb{Q}$ = NE

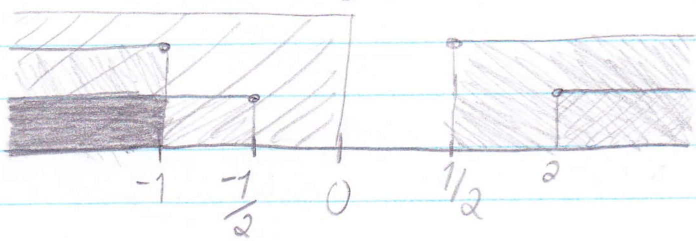
inf $A \cap B$ = NE

sup $B \cap \mathbb{R}^- \setminus \mathbb{Q} = -1$

max $A \cap \mathbb{Q}^- = -1/2$

c) se u_n é crescente em $A \cap B \cap \mathbb{R}^-$ $]-\infty, -1]$
então é convergente

tende para -1



d) se u_n é sucessão em $B \cap \mathbb{R}^+$
$\longrightarrow [1/2, +\infty[$

então $y_n = (-1)^n x_n$ é divergente

$\longrightarrow -x_n, x_{n+2}, -x_{n+2}, x_{n+3} \dots$

$y_{2n} = x_{2n}$

$y_{2n+1} = -x_{2n+2}$

e) $x_n =$

Tender para um n.º em $]-1/2, 2]$

Em $A \cap \mathbb{R} \setminus \mathbb{Q}$ $\quad]-\infty, -\frac{1}{2}] \cup]2, +\infty[$

9) $\quad A = \mathbb{R}^+ / \{1\} \quad B = \{-\frac{1}{n}, n \in \mathbb{N}_1\}$

a) falso

b) $\quad A \cap V_{1/2}(0)$

$\mathbb{R}^+ \setminus \{1\} \cap]-1/2, 1/2[=]-1/2, 1[\cup]1, 1/2[$

c)

10. $\quad A = [-1+\sqrt{2}, 3] \quad C = \{\frac{1}{K^2}, K \in \mathbb{N}_1$

i)

ii) $]-\infty, -1+\sqrt{2}[\cup]3, +\infty[\quad a_{n+1} a_n < 0$

iii) Verdadeiro $\quad a_n \to 0$

11.

a) $1 - \sqrt{n} \to -\infty$

$\lim \sqrt{n} = \lim \frac{n+1}{n} = \lim \frac{n/n + 1/n}{n/n}$
$= \lim 1 + \frac{1}{n} = 1 + 0 = 1$

$\lim 1 - \sqrt{n} = 1 - 1 = 0$

?

b) $\frac{n^2+1}{n} \to +\infty$

$\lim \frac{n^2+1}{n} = \lim \frac{n^2/n^2 + 1/n^2}{n/n^2} =$

$= \frac{1+0}{0} =$

?

$\boxed{n^p < a^n < n! < n^n}$

12.

a) $\lim \frac{n^n}{1000^n} = +\infty$

b) $\lim n^{n+1} - n^n = \lim n^n(n-1) = +\infty(+\infty - 1) = +\infty$

ou

$n^{n+1}\left(1 - \frac{1}{n}\right) = +\infty$

c) $\lim 3^n - (2n)! = -\infty$

d) $\lim (n! - n^{1000})^n = \left[n!\left(1 - \frac{n^{1000}}{n!}\right)\right]^n \geq \underbrace{n!\left(1 - \frac{n^{1000}}{n!}\right)}_{\to +\infty}$

$\frac{n^{1000}}{n!} \to 0$

e) $\lim \frac{(2n)!}{n!} = \frac{+\infty}{+\infty} = +\infty$

f) $\lim \sqrt[n]{\frac{n+2}{n+2}} = 1$

$\lim \frac{a_{n+1}}{a_n} = l$?

$\lim \frac{n+3}{n+2} \cdot \frac{n+1}{n+2} = 1 \cdot 1 = 1$

g) $\lim \frac{n^{1000}}{1{,}0002^n} = 0$ $\qquad \frac{n^p}{a^n} = 0$

h) $\lim \sqrt[n]{\frac{n}{n^2+1}} = 1$

$\lim \frac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} = \lim \frac{n+1}{n} \cdot \frac{n+1}{n^2+2n+2} = \lim 1 + \frac{1}{n} \cdot \frac{0+1}{1+0+0}$

$= 1 + 0 \cdot \frac{1}{1} = 1$

$$\lim \left(1+\frac{1}{n}\right)^n = e \qquad \left(1+\frac{a}{n}\right)^n \to e^a$$
$$\left(1-\frac{1}{n}\right)^n = e^{-1}$$

i) $\sqrt[n]{3^n+2} \to 3$

$$\lim \frac{3^{n+1}+2}{3^n+2} = \lim \frac{1+2/3^{n+1}}{\frac{1}{3}+2/3^{n+1}} = \frac{1+0}{1/3+0} = 3$$

j) $\lim \sqrt[n]{n!} = +\infty$

$$\lim \frac{(n+1)!}{n!} = \lim \frac{\cancel{n!}(n+1)}{\cancel{n!}} = \lim n+1 = +\infty$$

k) $\lim \left(2-\frac{1}{n}\right)^n \geq \lim \left(\frac{3}{2}\right)^n = +\infty$ $\quad 2-\frac{1}{n} \to 2 \quad \exists p \in \mathbb{N} \ \forall n > p \quad 2-1/n \geq \frac{3}{2}$ exemplo entre 1 e 2

l) $\lim \left(1-\frac{1}{2^{n-1}}\right)^{2^n} = \lim \left(\left(1-\frac{1}{2^{n-1}}\right)^{2^n \frac{1}{2}}\right)^2 = \lim \left(\left(1-\frac{1}{2^{n-1}}\right)^{2^{n-1}}\right)^2 =$

$= e^{-2}$

m) $\lim \left(1+\frac{1}{n}\right)^{n^2} = \lim \left(\left(1+\frac{1}{n}\right)^n\right)^n = e^n = +\infty$

1.3.

a) $\lim \frac{2n+3}{3n-1} = \frac{2}{3}$

b) $\lim \frac{n^2+1}{n^4+3} = \lim \frac{n^2/n^4+1/n^4}{1+3/n^4} = \frac{0+0}{1+0} = 0$

c) $\lim \frac{2^n+1}{2^{n+1}-1} = \lim \frac{2^n/2^{n+1}+1/2^{n+1}}{1-1/2^{n+1}} = \frac{0+0}{1-0} = 0$

d) $\lim \frac{n^3+1}{n^2+2n-1} = \lim \frac{n^2\left(n+\frac{1}{n^2}\right)}{n^2+2n-1} = \frac{1.(0+0)}{1+0+0} = 0$

e) $\lim \dfrac{(-1)^n n^3 + 1}{n^2 + 2} =$

$=$

f) $\lim \dfrac{n^p}{n!} = +\infty$

g) $\sqrt[n]{1 + \dfrac{1}{n}} = \underline{1}$ ou $\left(1 + \dfrac{1}{n}\right)^{1/n} = 1^0 = \underline{1}$

$\dfrac{a_{n+1}}{a_n} = \lim \dfrac{(n+1)+1}{n+1} \cdot \dfrac{n}{1+n} = \lim \dfrac{n\,(n+2)}{n+1} =$

$= \dfrac{1\,(1+0)}{1+0} = \underline{1}$

h) $\lim \dfrac{\left(\dfrac{1}{2}\right)^n}{n^3} = +\infty$

i) $\lim \dfrac{3^n}{n^2} = +\infty$

j) $\sqrt[n]{\dfrac{n^2+n-1}{n+3}}$

$\lim \dfrac{a_{n+1}}{a_n} = \lim \dfrac{(n+1)^2+n}{n+4} \cdot \dfrac{n+3}{n^2+n-1} = \lim \sqrt{\dfrac{n^2+3n+1}{n+4}} \cdot \dfrac{n+3}{n^2+n-1} =$

$= +\infty \cdot \dfrac{0+0}{1+0-0}$

k) $\sqrt[n]{2^n+1}$

$\lim \dfrac{2^{n+1}+1}{2^n+1} = +\infty$

l) $\sqrt[n]{(n+1)!-n!} \to +\infty$

$\dfrac{a_{n+1}}{a_n} \quad \dfrac{(n+2)!-(n+1)!}{(n+1)!-n!} = \dfrac{(n+1)!\cdot(n+2)-(n+1)!}{n!\cdot(n+1)!-n!} =$

$\lim \dfrac{1\cdot(n+2)-1}{1-n!/(n+1)!} = \dfrac{n+1}{1-0} = +\infty$

m) $\lim \left(1+\dfrac{1}{n^2}\right)^{n^3} = \lim \left(\left(1+\dfrac{1}{n^2}\right)^{n^2}\right)^n = e^n$

n) $\lim \left(1-\dfrac{1}{n!}\right)^{n!} = e^{-1}$

o) $\lim \left(1+\dfrac{1}{n^3}\right)^{n^2} = \lim \left(\left(1+\dfrac{1}{n^3}\right)^{n^3}\right)^{\frac{1}{n}} = e^{\frac{1}{n}}$

14.

a) $\lim \dfrac{n!}{n^{1000}} = +\infty$

b) $\lim \dfrac{(2n)!+2}{(3n)!+3} = \lim \dfrac{2+2/n!}{3+3/n!} = \dfrac{2}{3}$

c) $\lim \dfrac{(2n)!}{(2n)^n} = 0$

d) $\lim \dfrac{(n!)^2}{(2n)! + 2} =$

e) $\lim \dfrac{2^n \cdot n!}{n^n} = 0$

f) $\lim \dfrac{3^n n!}{n^n} = 0$

g) $\lim n^{1/n} = \sqrt{n} = 1$

h) $\lim \left(\dfrac{1}{n}\right)^{1/n} = \lim \sqrt[n]{1/n} = 0$

$\lim \dfrac{1}{n+1} \cdot \dfrac{1}{n} = \lim \dfrac{1}{n^2+n} = 0$

i) $\lim \left(\dfrac{1}{n}\right)^n = \lim \dfrac{1}{n^n} = 0$

j) $\lim \left(2 - \dfrac{1}{n}\right)^n = 2^{+\infty} = +\infty$

k) $\lim \left(\dfrac{n-1}{2n^2+1}\right)^{2/n} = \left(\lim \sqrt[n]{\dfrac{n-1}{2n^2+1}}\right)^2 = 1$

$\lim \dfrac{n}{2n^2+4n} \cdot \dfrac{2n^2+1}{n-1} = \lim \dfrac{n}{n-2} \cdot \dfrac{2n^2+1}{2n^2+4n} =$

$= 1 \cdot 1 = 1$

l) $\lim \frac{2^{(n^2)}}{75^n} = \lim \left(\frac{2^n}{75}\right)^n \geq \underbrace{\frac{2^n}{75}}_{\to +\infty}$

15.

a) PROVÁMOS NA TEÓRICA

i) se $u_n \to +\infty$ então $\frac{1}{u_n} \to 0$

ii) se $u_n > 0$ e $u_n \to 0$ então $\frac{1}{u_n} \to +\infty$

b) $u_n \to 0 \overset{?}{\Rightarrow} \left(\underbrace{\frac{1}{u_n} \to +\infty}_{\text{sse } u_n > 0} \vee \frac{1}{u_n} \to -\infty\right)$

Não é verdade

$u_n = -\frac{1}{n}$

Ficha 6

1. a) $y = e^{x^2-2}$ $\qquad x > 0$

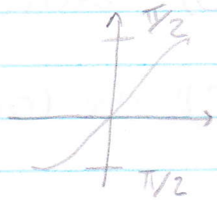
$\Leftrightarrow \log y = x^2 - 2 \Leftrightarrow \log y + 2 = x^2 \Leftrightarrow$

$\Leftrightarrow \sqrt{\log y + 2} = F^{-1}(y)$

$D' =]e^{-2}, +\infty[$

$D =]0, +\infty[$

b) $2\operatorname{sen} x$, $x \in]-\pi/2, \frac{\pi}{2}[$

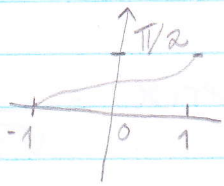


$D' =]-\pi/2, \pi/2[$ $\qquad 2\operatorname{sen} x = y \Leftrightarrow$

$D =]-1, 1[$ $\qquad \Leftrightarrow \operatorname{sen} x = y/2 \Leftrightarrow$

$\Leftrightarrow \dfrac{\underline{\text{arcoseno} x}}{2}$

c) $\cos(2x)$ $\qquad x \in]0, \frac{\pi}{2}[$

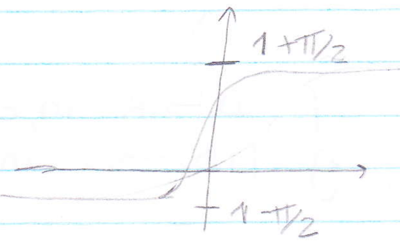


$D' =]0, \pi/2[$

$D =]-1, 1[$ $\qquad \text{arcocoseno}\left(\frac{1}{x}\right)$ $\qquad$?

ou arcocoseco$(2x)$

d) $tg(x-1)$, $x \in]1 - \frac{\pi}{2}, 1 + \pi/2[$



$D' =]1 - \pi/2, 1 + \pi/2[$

$D = \mathbb{R}$

arcotangente$(x-1)$

2. $\arccos 0 = k\pi + \frac{\pi}{2}$ $\qquad \arccos 1 = 2k\pi$

$\boxed{\arccos x = y \Rightarrow \operatorname{sen} y = x}$ $\qquad \arccos\left(-\frac{1}{2}\right) = \frac{4\pi}{3}$ e

$\operatorname{arcsen}\left(-\frac{1}{2}\right)$

3. $\arcsin a = x \Leftrightarrow \operatorname{sen} x = a$

$\arccos a = x \Leftrightarrow \cos x = a$

$\operatorname{arctg} a = x \Leftrightarrow \operatorname{tg} x = a$

4.

a) $\cos(\arccos x) = y \Leftrightarrow \arccos y = \arccos x$

b) $\operatorname{sen}(\operatorname{arcsen} x) = x \Leftrightarrow \operatorname{arcsen} x = \operatorname{arcsen} x$

c) $\cos(\operatorname{arcsen} x) = \sqrt{1-x^2} \Leftrightarrow \arccos(\sqrt{1-x^2}) = \operatorname{arcsen} x$

d) $\operatorname{sen}(\arccos x) = \sqrt{1-x^2} \Leftrightarrow \operatorname{arcsen}(\sqrt{1-x^2}) = \arccos x$

e) $\operatorname{tg}(\operatorname{arcsen} x) = \dfrac{x}{\sqrt{1-x^2}} \Leftrightarrow \operatorname{arctg}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \operatorname{arcsen} x$

f) $\operatorname{tg}(\arccos x) = \dfrac{\sqrt{1-x^2}}{x} \Leftrightarrow \operatorname{arctg}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \arccos x$

5. $f : D \to \mathbb{R}$ injectiva | $g(y) = x$ sse $f(x) = y$

$g : f(D) \to D$ inversa de f

a) f crescente : $\forall x_1, x_2 \in D : x_1 \leq x_2 \Leftrightarrow f(x_1) \leq f(x_2)$

(f decrescente é análoga)

$\Downarrow$ g é inversa f

$f(x_1) \leq f(x_2) \Leftrightarrow x_1 \leq x_2$

$\Downarrow$

$y_1 \leq y_2 \Leftrightarrow g(y_1) \leq g(y_2)$

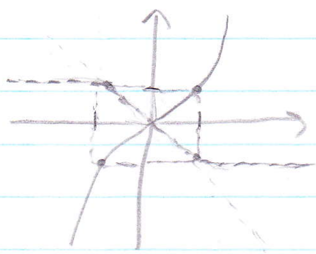
b) f ímpar: $\forall x \in D$

$f(x) = -f(-x)$

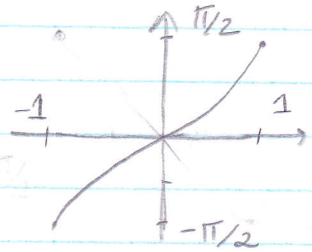
$\Downarrow$ g é inversa de f

$g(f(x)) = -g(f(-x))$

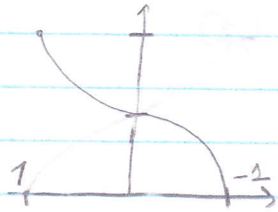
$\Leftrightarrow g(y) = -g(-y)$



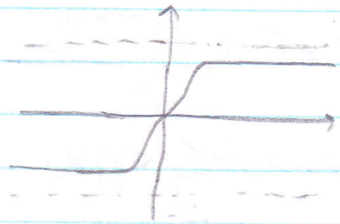
c) arcsen | arccos | arctg



$\forall x_1, x_2 \in [-1, 1], x_1 < x_2$
$\text{arcsen}\, x_1 < \text{arcsen}\, x_2$
e
$\forall x \in [-1, 1]$
$\text{arcsen}\, x = -\text{arcsen}(-x)$



$\forall x_1, x_2 \in [-1, 1]$
$x_1 < x_2$
$\text{arccos}\, x_1 > \text{arccos}\, x_2$
$\forall x \in [-1, 1]$
$\text{arccos}\, x = \text{arccos}(-x)$



$\forall x_1, x_2 \in \mathbb{R}$
$x_1 < x_2$
$\text{arctg}\, x_1 < \text{arctg}\, x_2$

$\forall x \in \mathbb{R}$
$\text{arctg}\, x = \text{arctg}(-x)$

6)

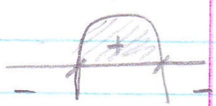
a) $f(x) = \dfrac{x}{\sqrt{4 - x^2}}$

$D = \{x : \sqrt{4-x^2} \neq 0 \wedge 4 - x^2 \geq 0\}$

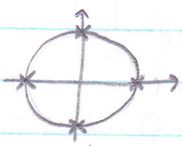
$= \{x : -x^2 + 4 > 0\}$

$=]-2, 2[$

$-x^2 + 4 = 0 \Leftrightarrow$
$\Leftrightarrow 4 = x^2 \Leftrightarrow x = \pm\sqrt{4}$
$\Leftrightarrow x = \pm 2$



b) $f(x) = \dfrac{1}{\cos^2 x} + \dfrac{1}{\text{sen}^2 x}$



$Df = \{x : \cos^2 x \neq 0 \wedge \text{sen}^2 x \neq 0\}$

$= \mathbb{R} \setminus \{k\pi, k \in \mathbb{Z}\} \cup \mathbb{R} \setminus \{k\pi + \frac{\pi}{2}, k \in \mathbb{Z}\}$

$= \mathbb{R} \setminus \{\frac{k\pi}{2} : k \in \mathbb{Z}\}$

c) $f(x) = tgx + cotgx$

$\hookrightarrow \mathbb{R}$ $\quad \hookrightarrow [-1,1]$

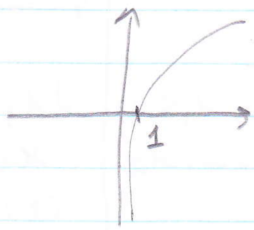
$Df = [-1,1]$

c) $f(x) = \log(\log x)$

$Df = \{x : \log x > 0 \wedge x > 0\}$

$= \{x : x > 1 \wedge x > 0\}$

$= \{x : x > 1\} =]1, +\infty[$



e) $f(x) = \log(1 - x^{3/2})$

$DF = \{x : 1 - x^{3/2} > 0\}$

$= \{x : x > 1\} =]1, +\infty[$

$1 - x^{3/2} > 0 \Leftrightarrow$

$\Leftrightarrow \sqrt{x^3} > 1 \Leftrightarrow$

$\Leftrightarrow x^3 > 1^2 \Leftrightarrow x > \sqrt[3]{1} \Leftrightarrow x > 1$

$(x^3)^{\frac{1}{2}} = \sqrt{x^3}$

f) $f(x) = arctg\left(\frac{1}{1-x^2}\right)$

$df(x) = \{x : 1 - x^2 \neq 0\}$

$= \mathbb{R} \setminus \{-1, 1\}$

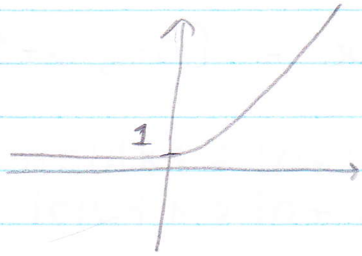
g) $f(x) = arcos\left(\frac{1}{x}\right)$

$Df(x) = \{x : x \neq 0 \wedge -1 \leq \frac{1}{x} \leq 1\}$

$= \{x : x > 0 \wedge x \geq 1 \vee x < 0 \wedge x \leq -1\}$

$= \mathbb{R} \setminus]-1, 0, 1[$

h) $f(x) = \arcsin(e^x)$



P.V.

$D = \{x : -1 \leq e^x \leq 1\}$

$= \{x : e^x \leq 1\} =]-\infty, 0]$

i) $f(x) = \log(1 - \arcsin x)$

$D = \{x : 1 - \arcsin x > 0 \wedge -1 \leq x \leq 1\}$

$= \{x : x < \text{sen}\, 1 \wedge -1 \leq x \leq 1\}$

$= \{x : -1 \leq x < \text{sen}\, 1\}$

$= [-1, \text{sen}\, 1[$

$1 - \arcsin x > 0 \Leftrightarrow$
$\Leftrightarrow \arcsin x < 1 \Leftrightarrow$
$\Leftrightarrow x < \text{sen}\, 1$

7. u_n é monótona $\Rightarrow \text{arctg}\, u_n$ é convergente

$-\pi/2 < \text{arctg}\, x < \pi/2$, limitada

a função $\text{arctg}\, u_n$ é composta por uma função limitada e uma sucessão monótona, logo é convergente

8. $f(x) = x^2 + 1$ $\qquad g(x) = |x|$

as funções polinomiais e módulo são contínuas em todo o seu domínio

$\forall \varepsilon > 0 \ \exists \delta > 0 : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$|1 + x^2| - |1 + a^2| < \varepsilon \Leftrightarrow |x^2 - a^2| < \varepsilon \Leftrightarrow$

$\Leftrightarrow |(x+a)(x-a)| < \varepsilon$ → encontrar um $\delta > 0$

se $x \in [a-1, a+1] \to \delta < 1$

$|x+a| \leq |x| + |a| \Leftrightarrow |x+a| \leq 1 + |a| + |a| \Leftrightarrow$
$\Leftrightarrow |x+a| \leq 1 + 2|a|$

se $|x-a| . (1+2|a|) < \varepsilon \Rightarrow |(x-a)(x+a)| < \varepsilon$

$|(x-a)(x+a)| < |(x-a)(1+2|a|)| < \varepsilon$

basta $\delta = \dfrac{\varepsilon}{1+2|a|}$, porque $|x-a| < \delta \Rightarrow |x-a| . (1-2|a|) <$
$< \delta . (1-2|a|) = \varepsilon$

9. $\lim x_n = 1$, $x_n > 1 \Rightarrow$ convergente

a) $f(x) = \dfrac{1}{x}$ $x \neq 0$
$\to$ limitada

$\lim f(x_n) = \lim \dfrac{1}{x_n} = \dfrac{1}{1} = 1$

b) $f(x) = \log x$, $x > 0$
$\to$ não é convergente

$\lim_{x \to +\infty} \log x = +\infty$

$f(x_n) = \log(x_n)$ $x_n > 1$

$\lim f(x_n) = \lim \log(1) = 0$

c) $f(x) = \dfrac{1}{x-1}$, $x \neq 1$

10. $\emptyset : [a,b] \to \mathbb{R}$ continua a e $b \in \mathbb{R}$ $a<b$

$\exists x_n$, termos em $[a,b]$

$\lim \emptyset(x_n) = 0$
$\quad \hookrightarrow$ sublimite de $\emptyset$

Se $\emptyset$ é continua em $[a,b]$ e $\lim \emptyset(x_n)=0$ então $x_n = 0$.

$\Downarrow$

Como $a<b$ $\boxed{\lim_{x\to a} f(x) = f(a)}$

$\exists$ ponto c entre $]a,b[$ em que $f(c)=0$

11. $g : [0,1] \to \mathbb{R}$ contínua

a)

b) se $\exists x_n$ em $[0,1]$ tq $g(x_n) = \frac{1}{n}$ $\forall n \in \mathbb{N}$

então $\exists c \in [0,1]$ tq $g(c)=0$

12.

a) $\dfrac{x+1}{x^3+x} = \dfrac{\overbrace{x+1}^{\text{continua}}}{\underbrace{x(x^2+1)}_{\text{continua}}}$

$x=0?$

$\lim_{x\to 0} \dfrac{x+1}{x(x^2+1)} = \dfrac{1}{0} = \underset{?}{\pm\infty}$ Ponto de descontinuidade

b) $\dfrac{x+1}{x^4+3x^3+2x^2} = \dfrac{x+1}{x^2(x+2)(x+1)}$ $D_f = \mathbb{R} \setminus \{-2,-1,0\}$

$\nearrow$ continua

$\hookrightarrow$ Continua

$x^4+3x^3+2x^2 = x^2(x^2+3x+2) = x^2 \cdot (x+2)(x+1)$

$x^2+3x+2 = 0 \Leftrightarrow x = \dfrac{-3 \pm \sqrt{9-4.1.2}}{2} \Leftrightarrow x = \dfrac{-3 \pm 1}{2} \Leftrightarrow$

$\Leftrightarrow x = \dfrac{-2}{2} \lor x = \dfrac{-4}{2} \Leftrightarrow x = -1 \lor x = -2$

$\lim\limits_{x \to 2^-} \dfrac{x+1}{\underbrace{x^2(x+2)(x+1)}_{\to 0^-}} = -\infty$

Descontinua em $x=-2$

$\lim\limits_{x \to -2^+} \dfrac{x+1}{\underbrace{x^2(x+2)(x+1)}_{\to 0^+}} = +\infty$

$\lim\limits_{x \to 0} \dfrac{x+1}{x^2(x+2)(x+1)} = \dfrac{1}{0.2.1} = \dfrac{1}{0} = +\infty$ Descontinua em $x=0$

$\lim\limits_{x \to -1} \dfrac{\cancel{x+1}}{x^2(x+2)(\cancel{x+1})} = \lim\limits_{x \to -1} \dfrac{1}{x^2(x+2)} = \dfrac{1}{1.1} = 1$

Prolongado a 1

c) $\sqrt{x} - \dfrac{1}{x^2+x} = \sqrt{x} - \dfrac{1}{x(x+1)}$

$\hookrightarrow$ continua

$\hookrightarrow$ Continua $(x \geq 0)$

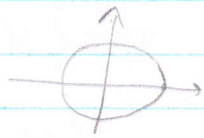
$\lim\limits_{x \to 0^+} \sqrt{x} - \dfrac{1}{x(x+1)} =$

$= 0^+ - \dfrac{1}{0} = -\infty$

d) $\operatorname{sen}(\cos\sqrt{1-x^2})$ $\longrightarrow$ contínua em todo o domínio

$D = \{x : 1-x^2 \geq 0\}$

$= \{x : -1 \leq x \leq 1\}$



$= [-1, 1]$

$\lim\limits_{x \to -1} \operatorname{sen}(\cos\sqrt{1-x^2}) = \operatorname{sen}(\cos 0) = \operatorname{sen} 1$

prolongado a sen1

$\lim\limits_{x \to 1} \operatorname{sen}(\cos\sqrt{1-x^2}) = \operatorname{sen} \underline{1}$

e) $\cos \dfrac{1}{\sqrt{1-x^2}}$ $\longrightarrow$ continua em todo o domínio

$D = \{x : \sqrt{1-x^2} \neq 0 \wedge 1-x^2 \geq 0\}$

$= \{x : -1 < x < 1\}$

$\lim\limits_{x \to 1^+} \dfrac{1}{\sqrt{1-1^2}} = \dfrac{1}{0} = +\infty = \lim\limits_{x \to 1} \dfrac{1}{\sqrt{1-x^2}}$

descontinua em 1 e $\underline{-1}$

f) $\sqrt[3]{tg\, 2x - cotg\, 2x}$

P. descontinuidade $\rightarrow$ $\lim_{x} f(x)$

e

$\lim_{x+-} f(x)$

g) $\dfrac{1}{\sqrt{x^2+1}} + \dfrac{1}{\sqrt[3]{x^3-1}}$

$D = \{x : \sqrt{x^2+1} \neq 0 \wedge x^2+1 \geq 0 \wedge \sqrt[3]{x^3-1} \neq 0\}$

$= \{x : x \neq 1\}$

$= \mathbb{R} \setminus \{1\}$

$x^3 - 1 = 0 \Leftrightarrow$

$\Leftrightarrow x^3 = 1 \Leftrightarrow x = 1$

$\lim_{x \to 1} f(x) = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt[3]{1^2-1}} = \dfrac{1}{2} + \infty = +\infty$

descontínuo em 0

h) $\dfrac{|x^2-1|}{x^2-1}$

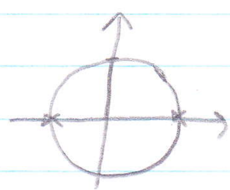
$D = \{x : x^2 - 1 \neq 0\}$

$= \{x \neq 1\} = \mathbb{R} \setminus \{1\}$

$\lim_{x \to 1^-} \dfrac{\overset{\to 0^-}{|x^2-1|}}{\underset{\to 0^-}{x^2-1}} = -\infty$

$\lim_{x \to 1^+} \dfrac{\overset{0^+}{|x^2-1|}}{\underset{0^+}{x^2-1}} = +\infty$

i) $\sqrt{-\text{sen}^2 x}$



$D = \{x : -\text{sen}^2 x \geq 0\}$

$-\text{sen}^2 x = 0$

13. $f: \mathbb{R} \to \mathbb{R}$ contínua em $x = 1$

onde é contínua $g(x) = f(\operatorname{sen} x)$

$$\begin{tikzcd} x \arrow[r, "f(x)"] \arrow[rr, bend right, "g(x)"'] & f(x) \arrow[r, "h(x)"] & F(\operatorname{sen} x) \end{tikzcd}$$

- $\operatorname{sen} x$ é contínua em todo o domínio $\hookrightarrow \mathbb{R}$
- contradomínio $= [-1, 1]$
- se $f(x)$ é contínua em $x = 1$, $g(x)$ é contínua quando $\operatorname{sen} x = 1 \iff x = \frac{\pi}{2} + 2k\pi$

14. $f: \mathbb{R} \to \mathbb{R}$ contínua em $x = 0$

$$g(x) = f(\operatorname{tg} x - \operatorname{cotg} x)$$

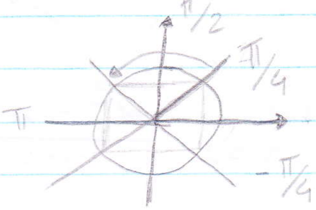
$g(x)$ é obrigatoriamente contínua quando $(\operatorname{tg} x - \operatorname{cotg} x) = 0$

$\operatorname{tg} x - \operatorname{cotg} x = 0 \iff \operatorname{cotg} x = \operatorname{tg} x \iff \frac{1}{\operatorname{tg} x} = \operatorname{tg} x \iff$

$\iff 1 = \operatorname{tg} x \cdot \operatorname{tg} x \iff 1 = (\operatorname{tg} x)^2 \iff \sqrt{1} = \operatorname{tg} x \iff$

$\iff \operatorname{tg} x = \pm 1 \iff x = \frac{\pi}{4} + \frac{k\pi}{2} \iff$

$\iff x = \frac{\pi + 2k\pi}{4}$



15. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x d(x)$

$d: \mathbb{R} \to \mathbb{R}$ dirichelet

$$d(x) = \begin{cases} 1 & \text{se } x \in \mathbb{Q} \\ 0 & \text{se } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \quad // \text{descontínua}$$

$$f(x) \begin{cases} 0 & \text{se } x \in \mathbb{R} \setminus \mathbb{Q} \\ x & \text{se } x \in \mathbb{Q} \end{cases}$$

$$\lim_{x \to 0} f(x) = 0 = f(0)$$

16.

a) $\lim_{x \to 0} \frac{1}{x^2} = +\infty$

$\forall \varepsilon > 0 \; \exists R > 0 \; \forall x \in D \; 0 < |x| < R \Rightarrow f(x) > \varepsilon$

$\frac{1}{x^2} > \varepsilon \quad 0 < |x| < R$

$\Leftrightarrow 1 > \varepsilon x^2 \Leftrightarrow \sqrt{\frac{1}{\varepsilon}} > x$

b) $\lim_{x \to +\infty} \frac{1}{x} = 0$

$\forall \varepsilon > 0 \; \exists R > 0 \; \forall x \in D \; x > R \Rightarrow f(x) < \varepsilon$

c) $\lim_{x \to +\infty} \sqrt{x} = +\infty$

$\forall \varepsilon > 0 \; \exists R > 0 \; \forall x \in D \; x > R \Rightarrow f(x) > \varepsilon$

17.

a) $\lim_{x \to 0} \frac{x^3 - x^2 + x - 1}{x^2 - 1} = \frac{-1}{-1} = 1$

b) $\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2+1)}{(x+1)\cancel{(x-1)}} = \frac{2}{2} = 1$

$$\begin{array}{l|l} x^3 - x^2 + x - 1 & x - 1 \\ \hline x^3 - x^2 & x^2 + 1 \\ \quad 0 \quad x - 1 & \\ \qquad 0 & \end{array}$$

c) $\lim_{x \to 0} \frac{e^{x^2}-1}{x} = \lim_{x \to 0} x\left(\frac{e^{x^2}-1}{x^2}\right) =$ $y = x^2$

$= \lim_{x \to 0} x \cdot \lim \frac{e^y-1}{y} = 0 \cdot 1$

d) $\lim_{x \to 0} \left[x^2\left(1-\cos\frac{1}{x}\right)\right]$

e) $\lim_{x \to 0} \text{sen}\frac{1}{x}$

$\lim_{x \to 0^+} \text{sen}(1/x) = \lim_{y \to +\infty} \text{sen}\, y$

pelo critério de Heine:

$y_k = 2\pi k \to +\infty \quad \text{sen}\, y \to 0$
$y_k = 2\pi k + \frac{\pi}{2} \to +\infty \quad \text{sen}\, y \to 1$

a função tem dois sublimites diferentes, logo $\lim_{x \to 0} \text{sen}\frac{1}{x}$ não existe

f) $\lim_{x \to +\infty} \text{sen}\underbrace{\frac{1}{x}}_{\to 0} = \text{sen}\, 0 = 1$

g) $\lim_{x \to 0} x\, \text{sen}\frac{1}{x} = 0$

Composta por x (tende p/ 0) e $\text{sen}\frac{1}{x}$ (limitada)

18.

a) $\lim_{x\to 1} \dfrac{x^2-x}{x^2-3x+2} = \lim_{x\to 1} \dfrac{x(\cancel{x-1})}{(\cancel{x-1})(x-2)} =$

$= \dfrac{1}{1-2} = -1$

$x = \dfrac{3 \pm \sqrt{9-4.1.2}}{2} \Leftrightarrow x = \dfrac{3\pm 1}{2} \Leftrightarrow$

$\Leftrightarrow x = 2 \vee x = 1$

b)

$\lim_{x\to 0} \dfrac{tg\, 5x}{x \arccos x} =$

f Louro : 2-N 4.18
15:30h-18h

Ficha 7

1. $f\left(-\frac{1}{n}\right) = 1 - f\left(\frac{1}{n}\right)$

$f(0^-) + f(0^+) = \lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)$

Pelo critério de Heine:

$\frac{1}{n} \to 0^+ \Rightarrow f\left(\frac{1}{n}\right) \to f(0^+)$

$-\frac{1}{n} \to 0^- \Rightarrow f\left(-\frac{1}{n}\right) \to f(0^-)$

$\lim_{x\to 0^-} f\left(-\frac{1}{n}\right) = \lim_{x\to 0^-} 1 - f\left(\frac{1}{n}\right) \Leftrightarrow f(0^-) = 1 - f(0^+) \Leftrightarrow$

$\Leftrightarrow f(0^-) + f(0^+) = 1$ //

---

se $\exists \lim_{x\to 0} f(x) = a$

então $f(0^-) = f(0^+) = a$ e $a + a = 1 \Leftrightarrow$

$\Leftrightarrow a = \frac{1}{2}$ //

2. $f(x) = \frac{x + |x|}{2} d(x)$ $\hookrightarrow$ dirichelt

a) $d(x) \begin{cases} 0 & \text{se } x \notin \mathbb{Q} \\ 1 & \text{se } x \in \mathbb{Q} \end{cases}$

minorada
min = 0

$f(x) = \begin{cases} 0 & \text{se } x \notin \mathbb{Q} \wedge x \leq 0 \\ x & \text{se } x \in \mathbb{Q} \end{cases}$

$D = \mathbb{Q}_0^+$

b) $\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x = +\infty$

c) Em que pontos f é contínua?

em $]-\infty, 0]$ $\quad \lim_{x \to a} f(x) = f(a) \quad a \in]-\infty, 0]$

3. $f: \mathbb{R} \to \mathbb{R}$ contínua em $x = 1$

$$f(x) = \begin{cases} 0 & , \text{se } x \leq -1 \\ \arcsin x & , \text{se } -1 < x < 1 \\ k \sin\left(\frac{\pi}{2} x\right) & , \text{se } x \geq 1 \end{cases}$$

a)

$\lim_{x \to 1^+} f(x) = f(1) = k \sin\left(\frac{\pi}{2} \cdot 1\right) = k$

$k = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \arcsin x = \arcsin 1 = \frac{1}{2}$

b) → Para $x \in \mathbb{R} \setminus \{-1, 1\}$ f é contínua, pois é dada por funções contínuas.

→ f é contínua em $x = 1$

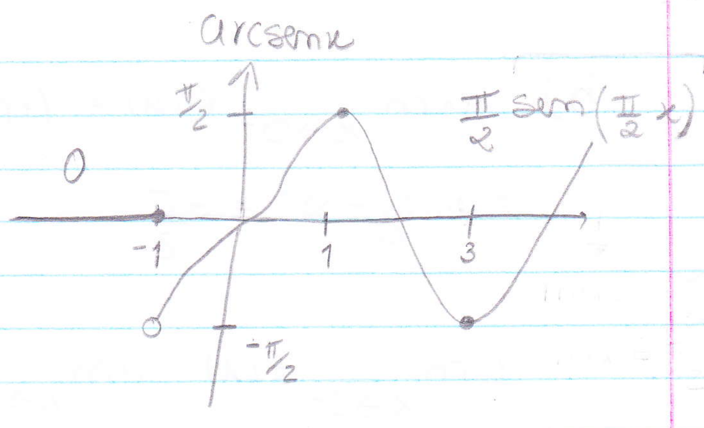
$\boxed{\text{f é contínua em } x = -1 \text{?}}$ $\quad f(-1) = 0 \quad \lim_{x \to -1^-} f(x) = 0$

$\neq$

não é contínua em $x = -1$ $\quad \lim_{x \to -1^+} f(x) = \arcsin(-1) = -\pi/2$

c)

- Não tem supremo
- não tem máximo

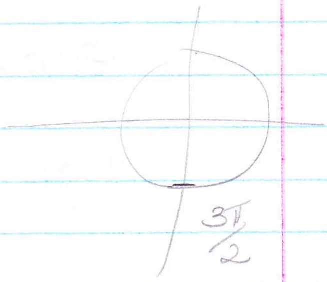


$\underbrace{\frac{\pi}{2}\operatorname{sen}\left(\frac{\pi}{2}x\right)}_{[-1,1]}$ $\begin{cases} max = \pi/2 \\ min = -\frac{\pi}{2} \end{cases}$

$\frac{\pi}{2}\operatorname{sen}\left(\frac{\pi}{2}x\right) = -\frac{\pi}{2} \Leftrightarrow$

$\Leftrightarrow \frac{\pi}{2}\cdot(-1) = -\frac{\pi}{2} \Leftrightarrow \frac{\pi}{2}x = \frac{3\pi}{2} \Leftrightarrow$

$\Leftrightarrow x = 3$

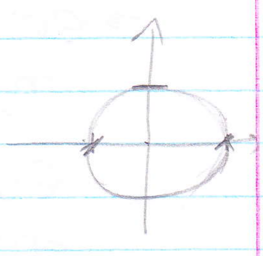


f é contínua em $[1,3]$, $f(1) = \pi/2$ e $f(3) = -\pi/2$
Pelo teorema do valor intermédio, existe pelo menos um $c \in [1,3]$ em que $f(c) = 0$

d) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 0 = 0$

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\pi}{2}\operatorname{sen}\left(\frac{\pi}{2}x\right)$

$u_k = \frac{\pi}{2} + 2k\pi \to 1$
$\neq$
$u_k = \pi k \to 0$ logo $\nexists \lim_{x\to+\infty} f(x)$



4. $\varphi : \mathbb{R} \to \mathbb{R}$ $\begin{cases} \operatorname{arctg}\frac{1}{x}, \text{ se } x<0 \\ 1 + e^{1-x}, \text{ se } x \geq 0 \end{cases}$

a) • a função arctg x é contínua em $\mathbb{R}$, logo $\operatorname{arctg}\frac{1}{x}$ é contínua em $\mathbb{R}\setminus\{0\}$

• $1 + e^{1-x}$ é contínua, porque é composta por uma constante e uma função exponencial (ambas contínuas)

b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \operatorname{arctg} \underbrace{\frac{1}{x}}_{\to -\infty} =$

$= \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 + e^{1-x} = 1 + e$

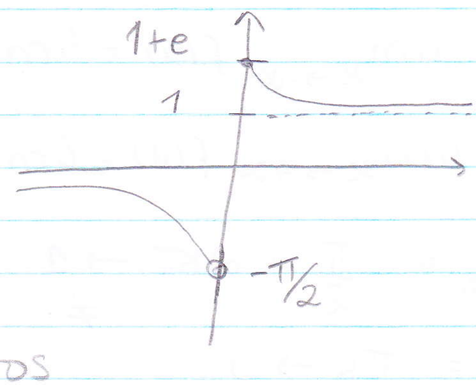
$f(0) = 1+e = \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

logo, $f(x)$ é contínua à direita no ponto $x=0$
$f(x)$ ñ é contínua à esq no ponto $x=0$,
não é contínua em $x=0$

c) $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} 1 + e^{1-x} = 1 + e^{-\infty} = 1 + 0 = 1$

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \operatorname{arctg} \frac{1}{x} = \operatorname{arctg} \frac{1}{+\infty} = \operatorname{arctg} 0 = 0$

d) $D' =]-\pi/2, 0[\cup]1, 1+e]$



Pelo T do Valor intermédio
→ $f(0) = 1+e$
$\lim_{x \to +\infty} f(x) = 1$

$f(x)$ atinge todos os pontos entre $]1, 1+e]$

→ $\lim_{x \to 0^-} f(x) = -\pi/2$
$\lim_{x \to -\infty} f(x) = 0$
} $f(x)$ atinge todos os pontos em $]-\pi/2, 0[$

$x \geq 0$

$0 < e^{-x} \leq 1 \Leftrightarrow 0 < e^{1-x} \leq e \Leftrightarrow 1 < 1 + e^{1-x} \leq e+1$

$x < 0$

$$\frac{1}{x} < 0 \iff -\pi/2 < \operatorname{arctg}\frac{1}{x} < 0$$

5.

a) $f(x) = e^{-\frac{1}{x^2}}$

é contínua em $\mathbb{R}\setminus\{0\}$ porque é composta por duas funções contínuas em $\mathbb{R}\setminus\{0\}$

$$-\frac{1}{x^2} = y \longmapsto e^y = f(x)$$

$g(x) = x \operatorname{sen}\frac{1}{x} - \cos\frac{1}{x}$

é contínua em $\mathbb{R}\setminus\{0\}$, é composta por funções trigonométricas, polinomiais e racionais.

b) $f(x) = e^{-1/x^2}$

$$\lim_{x\to 0} e^{-1/x^2} = e^{-1/+\infty} = e^0 = 1$$

É prolongável por continuidade no ponto 0

$$f(x)\begin{cases} 1 & \text{se } x = 0 \\ e^{-1/x^2} & \text{se } x \neq 0 \end{cases}$$ é contínua em $\mathbb{R}$

$g(x) = x \operatorname{sen}\frac{1}{x} - \cos\frac{1}{x}$

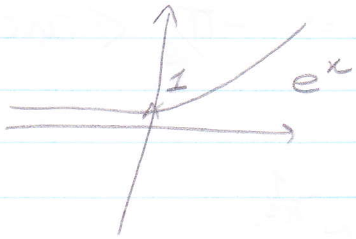
$$\lim_{x\to 0} x \cdot \operatorname{sen}\frac{1}{x} - \cos\frac{1}{x} = 0 \cdot \cancel{\operatorname{sen}(\pm\infty)} - \cos(\pm\infty)$$

$= -\cos(\pm\infty)$ $\nearrow -1$ $\searrow 0$ é descontínua

Posta

c) $\varphi(x)$ e $\psi(x)$ são limitadas

$\varphi(x) = e^{-1/x^2}$



$0 < e^{1/x} \leq e \Leftrightarrow$

$\Leftrightarrow 0 < e^{1/x \cdot -1/x} \leq e^{1/x} \Leftrightarrow$

$\Leftrightarrow 0 < e^{-1/x^2} \leq e^{-1/x} \leq e$

Logo $\varphi(x)$ é limitada

$\psi(x) = x \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x}$

limitada $[-1, 1]$

$-1 \leq \operatorname{sen} \frac{1}{x} \leq 1 \Leftrightarrow -x \leq x \operatorname{sen} \frac{1}{x} \leq x \Leftrightarrow$

$\Leftrightarrow -x - \cos \frac{1}{x} \leq x \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x} \leq x - \cos \frac{1}{x}$

6.

a) $f(x) = \frac{\sqrt{x}}{x-1}$

$D = \{x : x-1 \neq 0 \wedge x \geq 0\}$

$= \{x : x \neq 1 \wedge x \geq 0\}$

$= [0, 1[\cup]1, +\infty[= \mathbb{R}_0^+ \setminus \{1\}$

b) $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\sqrt{x}}{x-1} = \frac{1}{+\infty} = 0$

$x_n = x$

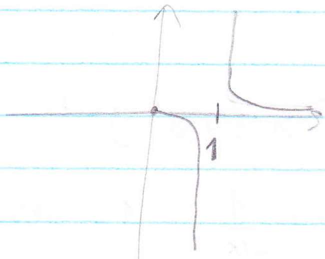
$\lim \frac{x}{x+1} = \frac{1}{1+\frac{1}{x}} = \frac{1}{1} = 1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{\sqrt{x}}{\underbrace{x-1}_{0^-}} = \frac{1}{0^-} = -\infty$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{\sqrt{x}}{\underbrace{x-1}_{0^+}} = \frac{1}{0^+} = +\infty$$

c)

$f(0) = 0$



$D' = \mathbb{R}$

→ porque quando $f(1^-) = -\infty$

e $f(1^+) = +\infty$

→ $\lim_{x \to +\infty} f(x) = 0$

mas $f(x)$ atinge 0 porque $f(0) = 0$

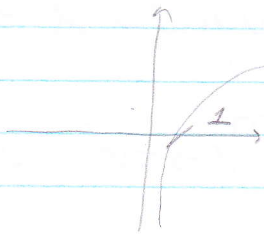
d)

7. $f(x) = \log \log (1+x)$ $\qquad g(x) = \sqrt{x} \cdot \operatorname{sen} \frac{1}{x^2}$

$D =]0, +\infty[$

a) $\log(1+x) > 0 \Leftrightarrow 1 + x > 1 \Leftrightarrow$

$\Leftrightarrow x > 0$



A função é contínua em todo o seu domínio, porque é composta por funções logarítmicas g(x) é composta por funções contínuas ($\sqrt{x}$) e sen $\frac{1}{x^2}$ logo é contínua no seu domínio $]0,+\infty[$

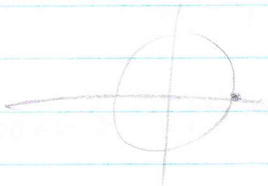
b)

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \log\log(1+x) =$$

$$= \log\log(+\infty) = \log(+\infty) = +\infty$$

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \sqrt{x}\cdot \text{sen}\,\frac{1}{x^2} =$$

$$= 1\cdot \text{sen}\,\frac{1}{+\infty} = 1\cdot \text{sen}(0) = 1\cdot 0 = 0$$



c) $\lim_{x\to 0} \log\log(1+x) =$

$= \log\log(1) = \log 0 = -\infty$ não é prolongável por continuidade a 0

$$\lim_{x\to 0} \sqrt{x}\cdot \text{sen}\,\frac{1}{x^2} = 0\cdot \overset{NE}{\text{sen}\,\frac{1}{0}}$$

d) Contradomínio $f(x) = \mathbb{R}$

quando $x\to 0$ $f(x)\to -\infty$

$$\lim_{x\to 0} \log\log(x+1) = \log\log 1 = \log 0 = -\infty$$

e quando $x\to+\infty$ $f(x)\to+\infty$

$$\lim_{x\to+\infty} \log\log(x+1) = \log\log(+\infty) = \log(+\infty) = +\infty$$

8. $f(x) = \begin{cases} -e^{1/x}, & x<0 \\ \log \frac{1}{1+x^2}, & se\ x>0 \end{cases}$

a)

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -e^{1/x} =$$

$$= -e^{1/-\infty} = -e^{0} = -1$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \log \frac{1}{1+x^2} =$$

$$= \log \frac{1}{+\infty} = \log 0 = -\infty$$

b) sim: no ramo $x<0$ a função é exponencial, logo é contínua.

no ramo $x>0$ a função é logarítmica, e também contínua

c) $\lim_{x \to 0^+} \log \frac{1}{1+x^2} = \log 1 = 0$

$$\lim_{x \to 0^-} -e^{1/x} = -e^{-\infty} =$$

d)

9.

a) $g: [0, +\infty[\longrightarrow \mathbb{R}$ contínua

$$e(x) = g(1-x^2)$$

$$1-x^2 \geq 0 \Leftrightarrow -x^2 \geq -1 \Leftrightarrow$$

$$\Leftrightarrow x^2 \leq 1 \Leftrightarrow -1 \leq x \leq 1$$

Weirstrass

$D\, e(x) = [-1, 1] \to$ contínua

$\min e(x) = e(-1) = e(1) = g(0)$

$\max e(x) = e(0) = g(1)$

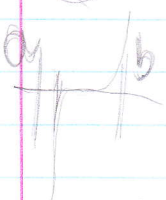
b) se $g(x)$ fosse $g:]0, +\infty[\longrightarrow \mathbb{R}$ contínua

$1 - x^2 > 0 \Leftrightarrow -1 < x < 1$

Logo não iria existir $e(-1) = e(1)$ que seria o máximo da função

Como $g(0)$ não fazia parte do domínio de $e(x)$, logo não ia existir mínimo de $e(x)$

10. $g:]a, b[\longrightarrow \mathbb{R}$ contínua



$$\lim_{x \to a} g(x) = -\lim_{x \to b} g(x) = -\infty$$

Como $-\infty < g < +\infty$, então tem pelo menos um zero ($\exists c \in]a,b[$ tq $g(c) = 0$)

então $h(c) = \text{arctg}(g(c)^2) =$

$= \text{arctg}\, 0 = 0$

Como $h(0) = \pi/2$ e h é continua em $[0, c]$,
logo h atinge todos os valores em $[0, \pi/2]$

portanto $h(x): [0, \pi/2] \to \mathbb{R}$

11)

$\sin^3 x + \cos^3 x = 0$ tem pelo menos uma raiz em $]0, \pi[$

→ a função é continua no seu domínio

12. f: continua em IR $\left.\begin{array}{l}\lim_{x \to +\infty} f(x) \\ \lim_{x \to -\infty} f(x)\end{array}\right\}$ são finitos

a) se $\lim_{x \to +\infty} f(x) = a$

$\lim_{x \to -\infty} f(x) = b$ $\qquad b < f(x) < a$

dominio da função é $]b, a[$

b)

$a . b < 0$ → Tem pelo menos um zero em $]a, b[$

$f(c) = 0$

$$g(x) = \frac{1}{1 + [f(x)]^2}$$

maximo $g(x) = g(\underbrace{f(c)}_{=0}) = \frac{1}{1} = 1$

Ficha 8

1.

a) $\log(x \operatorname{sh} x)$

$\hookrightarrow > 0$

b) $\operatorname{arcsen}(\operatorname{arctg} x)$

$\forall x \in]-1,1[\leftarrow$ $\hookrightarrow \mathbb{R} \setminus \{-1,1\}$

É diferenciável em $\operatorname{arctg} \in]-1,1[$ e $x \in \mathbb{R} \setminus \{-1,1\}$

$\operatorname{arctg} x = -1 \Leftrightarrow x = \operatorname{tg}(-1)$ $\qquad \to x \in]\operatorname{tg}(-1), \operatorname{tg}(1)[$

$\operatorname{arctg} x = 1 \Leftrightarrow x = \operatorname{tg}(1)$

$\operatorname{sen} 3\theta = x$

$\operatorname{arcsen} x = 3\theta$

$\operatorname{arcsen}' x = \dfrac{1}{\sqrt{1+x^2}}$

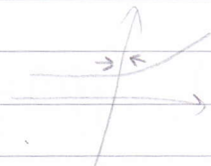
$(\operatorname{arcsen}(\operatorname{arctg} x))' = \dfrac{1}{\sqrt{1+(\operatorname{arctg} x)^2}}$

c) $\dfrac{e^x}{1+x}$ é diferenciável em $\mathbb{R} \setminus \{-1\}$

$$D' = \frac{e^x \cdot (1+x) - e^x \cdot 1}{(1+x)^2} = \frac{\cancel{e^x} + xe^x - \cancel{e^x}}{(x+1)^2}$$

2.

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$



$$\lim_{x \to 0^-} \frac{\frac{x}{1+e^{1/x}} - 0}{x - 0} = \lim_{x \to 0^-} \frac{1}{1+e^{1/x}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

$$\lim_{x \to 0^+} \frac{1}{1+e^{1/x}} = \frac{1}{1+e^{+\infty}} = \frac{1}{1+\infty} = 0$$

3. se $x \neq 0$ $\quad f(x) = x \dfrac{1+e^{1/x}}{2+e^{1/x}}$

$$\lim_{x\to 0^-} \frac{x\frac{1+e^{1/x}}{2+e^{1/x}} - 0}{x - 0} = \lim_{x\to 0^-} \frac{1+e^{1/x}}{2+e^{1/x}} =$$

$$= \frac{1+e^{1/0^-}}{2+e^{1/0^-}} = \frac{1+0}{2+0} = \frac{1}{2}$$

$$\lim_{x\to 0^+} \frac{1+e^{1/x}}{2+e^{1/x}} = \lim_{x\to 0^+} \frac{1/e^{1/x}+1}{2/e^{1/x}+1} = \frac{0+1}{0+1} = 1$$

4.

$$f(x) = \begin{cases} x^2 \operatorname{sen}\left(\frac{1}{x}\right) & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

a)

É contínua em $\mathbb{R} \setminus \{0\}$

$$f'(x) = 2x \operatorname{sen}\frac{1}{x} - x^2 \cos\frac{1}{x} =$$

$$= x\left(2\operatorname{sen}\frac{1}{x} - x\cos\frac{1}{x}\right) \quad \forall x \in \mathbb{R}\setminus\{0\}$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} x\Big(\underbrace{2\operatorname{sen}\frac{1}{x}} - \underbrace{x\cos\frac{1}{x}}\Big)$$

Não existem

b) $f(0) = 0$

$$f(0^-) = x\left(2\operatorname{sen}\frac{1}{x} - x\cos\frac{1}{x}\right) = f(0^+)$$

↳ tem derivada em 0

$f'(0) = 0$

5.

f diferenciável em $\mathbb{R}$

$f(0) = f(\pi) = 0$

$g = f(\operatorname{sen} x) + \operatorname{sen} f(x)$

$g(0) = f(\operatorname{sen} 0) + \operatorname{sen} f(0)$

$= f(0) + \operatorname{sen} 0 = 0 = f(0) = f(\pi)$

$g(\pi) = f(\operatorname{sen}\pi) + \operatorname{sen} f(\pi) =$

$= f(0) + \operatorname{sen} 0 = f(\pi) = f(0)$

$= 0$

então $g'(0) + g'(\pi) = 0 + 0 = f'(0) + f'(\pi)$

6. $f: \mathbb{R} \to \mathbb{R}$, diferenciável

$(\operatorname{arctg} f(x) + f(\operatorname{arctg} x))' = (\operatorname{arctg} f(x))' + (f(\operatorname{arctg} x))'$

$= \dfrac{1}{1-(f(x))^2} + (f(\operatorname{arctg} x))' \quad \forall x \in \mathbb{R}$

7. $g: \mathbb{R} \to \mathbb{R}$ duas vezes diferenciável

$\ell :]0, +\infty[\to \mathbb{R} \quad \ell(x) = e^{g(\log x)}$

g, g' e g''

$\ell'(1) \qquad \ell''(e)$

$\ell'(1) = e^{g(\overset{\to 0}{\log 1})} \cdot g(\overset{\nearrow 0}{\log 1})' = g(0)' \cdot e^{g(0)}$

$\ell''(e) = e^{g(\log e)} \cdot g(\log e)'' = g(e)'' \cdot e^{g(1)}$

8. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^4 e^{-x}$
$g: \mathbb{R} \to \mathbb{R}$ diferenciável

$(g \circ f)'(x) = g'(f(x)) \cdot g'(x) = g'(x^4 e^{-x}) \cdot g'(x)$

9. $g: \mathbb{R} \to \mathbb{R}$ Estritamente monótona $\quad f: [-1, 1] \to \mathbb{R}$
$g(0) = 2$ Diferenciável
$g'(0) = \frac{1}{2}$ $\quad f(x) = g(\text{arcsen}\, x)$

a) $f'(x) = g'(f(x)) \cdot g'(\text{arcsen}\, x)$

$= g'(\text{arcsen}\, x) \cdot \text{arcsen}\, x = g'(\text{arcsen}\, x) \cdot \frac{1}{\sqrt{1+x^2}}$
$\forall x \in [-1, 1]$

Como g é diferenciável, então f é diferenciável no domínio de arcsen x

$f'(0) = g'(\text{arcsen}\, 0) \cdot g'(0)$

$= g'(0) \cdot g'(0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ $\quad \text{arcsen}\, 0 = x$
$\Leftrightarrow \text{sen}\, x = 0$
$\Leftrightarrow x = k\pi = 0$

b) $f(x) = g(\text{arcsen}\, x)$ $\quad \forall x \in [-1, 1]$

a função arcsen x é injectiva no intervalo $[-1, 1]$
Como $g(x)$ é estritamente monótona, $g(\text{arcsen}\, x)$ é injectiva

como g é injectiva

$(f^{-1})'(2) =$ $\quad f(x) = 2$ $\quad f(0) = 2$

$g(\text{arcsen}\, x) = 2$

$= \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{\frac{1}{2}} = 2$ $\quad g(0) = 2$

10. $f: \mathbb{R} \longrightarrow]-1,1[$ diferenciável injectiva

$f(2) = 0$

$f'(2) = 2$

$g(x) = \arccos(f(x))$

a) $\arccos(x)$ é monotona e injectiva e $f(x)$ é injectiva.

logo $g(x)$ é injectiva

$\arccos(x)$ é diferenciável para $\forall x \in]-1,1[$

$f(x)$ é diferenciável e toma valores no intervalo $[-1, 1]$

logo $g(x)$ é diferenciável

$g'(2) = (\arccos(f(2)))'$

$= \frac{1}{\sqrt{1-(f(2))^2}} = \frac{1}{\sqrt{1}} = 1$

$(g^{-1})'(\frac{\pi}{2}) = \frac{1}{g'(g^{-1}(\frac{\pi}{2}))} = \frac{1}{2}$

|| $g^{-1}(\frac{\pi}{2}) \Leftrightarrow g(x) = \frac{\pi}{2} \Leftrightarrow f(x) = 0 \Leftrightarrow x = 2$

$g(x) = \frac{\pi}{2} \Leftrightarrow \arccos f(x) = \frac{\pi}{2} \Leftrightarrow \cos\frac{\pi}{2} = 0$

$f(x) = 0 \Leftrightarrow x = 2$

b)

(g^{-1})

$$\begin{tikzcd} & {]-1,1[} & {]0,\pi[} \\ x \arrow[r, bend right, "f(x)"'] & f(x) \arrow[r, bend right, "g(x)"'] & \arccos f(x) \arrow[l, bend right, "\cos f(x)"'] \end{tikzcd}$$

$(g^{-1}):]0,\pi[\longrightarrow]-1,1[$

Não tem máximo nem mínimo, mas é limitada

11. $sh(x) = \dfrac{e^{x} - e^{-x}}{2}$ $\qquad$ $ch(x) = \dfrac{e^{x} + e^{-x}}{2}$

a)

i) $ch^2 x - sh^2 x = 1$

$$\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2 =$$

$$\frac{(e^{x} - e^{-x})^2 - (e^{x} + e^{-x})^2}{4} = \frac{2e^0 - (-2e^0)}{2} =$$

$$= \frac{2+2}{4} = 1 \quad cqd$$

ii) $sh(x+y) = shx\, chy + chx\, shy$

$$sh(x+y) = \frac{e^{x+y} - e^{-(x+y)}}{2}$$

e

$$\frac{e^{x} - e^{-x}}{2} \cdot \frac{e^{y} + e^{-y}}{2} + \frac{e^{x} + e^{-x}}{2} \cdot \frac{e^{y} - e^{-y}}{x} =$$

$$\frac{(e^{x} - e^{-x})(e^{y} + e^{-y}) + (e^{x} + e^{-x})(e^{y} - e^{-y})}{4} = \frac{e^{x+y} + \cancel{e^{x-y}} - \cancel{e^{-x+y}} - e^{-y-x} + e^{x+y} - \cancel{e^{x-y}} + \cancel{e^{y-x}} - e^{-x-y}}{4}$$

$$= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} = \frac{e^{x+y} - e^{-x-y}}{2} \quad cqd$$

$$\frac{e^{x+y}+e^{-(x+y)}}{2}$$

iii) $ch(x+y) = chx\,chy + shx\,shy$

$$\frac{e^{x}+e^{-x}}{2}\cdot\frac{e^{y}+e^{-y}}{2}+\frac{e^{x}-e^{-x}}{2}\cdot\frac{e^{y}-e^{-y}}{2}=$$

$$=\frac{2e^{x+y}+2e^{-(x+y)}}{2}=\frac{e^{x+y}+e^{-(x+y)}}{2} \quad cqd$$

$\frac{e^{2x}-e^{-2x}}{2}$ iv) $sh(2x) = 2shx\,chx$

$$2\left(\frac{e^{x}-e^{-x}}{2}\cdot\frac{e^{x}+e^{-x}}{2}\right)=\frac{(e^{x}-e^{-x})(e^{x}+e^{-x})}{2}=$$

$$=\frac{e^{2x}-e^{-2x}}{2} \quad cqd$$

$\frac{e^{2x}+e^{-2x}}{2}$ v) $ch(2x) = ch^2x + sh^2x$

$$\left(\frac{e^{x}+e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)+\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}-e^{-x}}{2}\right)=$$

$$=\frac{e^{2x}+\cancel{e^{0}}+\cancel{e^{0}}+e^{-2x}+e^{2x}-\cancel{e^{0}}-\cancel{e^{0}}+e^{-2x}}{4}=$$

$$=\frac{\cancel{2}e^{2x}+\cancel{2}e^{-2x}}{\cancel{4}}=\frac{e^{2x}+e^{-2x}}{2} \quad cqd$$

b) sh é impar

$sh(x) = -sh(-x)$

$$-sh(-x) = -\left(\frac{e^{-x}-e^{x}}{2}\right)=$$

$$=\frac{-e^{-x}+e^{x}}{2}=\frac{e^{x}-e^{-x}}{2}$$

cqd

ch é par

$ch(x) = ch(-x)$

$$ch(-x) = \frac{e^{-x}+e^{x}}{2}=\frac{e^{x}+e^{-x}}{2}$$

cqd

d) São ambos contínuas em $\mathbb{R}$

$$D\,sh(x) = \frac{(e^x - e^{-x})' \cdot 2 - (e^x - e^{-x}) \cdot 2'}{4} =$$

$$= \frac{2(e^x + e^{-x}) - 0}{4} = \frac{e^x + e^{-x}}{2} = ch(x)$$

$$D\,ch(x) = \frac{(e^x + e^{-x})' 2 - (e^x + e^{-x}) \cdot 2'}{4} =$$

$$= \frac{2(e^x - e^{-x})}{4} = \frac{e^x - e^{-x}}{2} = sh(x)$$

e) sh(x)

É negativa quando $e^{-x} > e^x \Leftrightarrow$
$\Leftrightarrow -x > x \Leftrightarrow$
$\Leftrightarrow -x - x > 0 \Leftrightarrow$
$\Leftrightarrow -2x > 0 \Leftrightarrow x < 0$

Derivada decrescente

É positiva quando

$e^{-x} < e^x \Leftrightarrow$
$\Leftrightarrow x > 0$

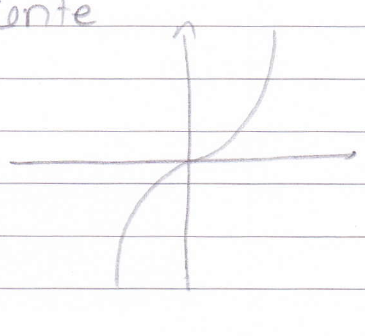
derivada crescente

$sh(0) = 0$

zero

impar

Como $D\,sh(x) = ch(x)$
então sh é estritamente crescente



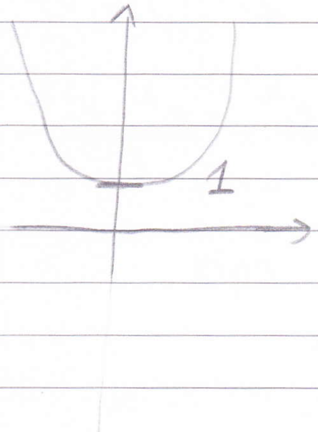
ch(x)

nunca é negativa,
não tem zeros

$D\,ch(x) = 0 \Leftrightarrow$
$sh(x) = 0 \Leftrightarrow$
$x = 0$

ch é estr. decrescente em $]-\infty, 0]$
ch é estritamente crescente em $]0, +\infty[$
tem extremo (mínimo) quando $x = 0$
Par

$$\min = \frac{e^0 + e^{-0}}{2} = 1$$



f) $(sh\,x^{-1}) = argsh(x)$ | $(ch\,x^{-1}) = argch$

$x = sh\,y$ sse $y = argsh(x)$ | $x = ch\,y$ sse $y = argch$

não percebi!

12. $f :]0,1[\to \mathbb{R}$ diferenciável

$f\left(\frac{1}{n+1}\right) = 0 \quad \forall n \in \mathbb{N}$

a) $\forall n \geq 2$ f tem max em $\left[1/_{n+1}, \frac{1}{n}\right]$

$\boxed{n=2}$ $F\left(\frac{1}{3}\right) = 0$ tem max em $\left[1/_3, \frac{1}{2}\right]$?

b) f é necessariamente limitada

c) f' tem necessariamente infinitos zeros

13. $f : \mathbb{R} \to \mathbb{R}$ duas vezes diferenciável
cruza a recta $x = y$ em 3 pontos

então f'' tem pelo menos um zero

f

$\exists \quad a < b < c$ tal que $f(a) = a$
$f(b) = b$ — 2 vezes dif.
$\rightarrow \exists\, z \in \mathbb{R} : f''(z) = 0 \qquad f(c) = c$

Aplicando o teorema de lagrange:

$\rightarrow$ a $[a, b]$ $\quad \exists\, d \in]a, b[$

$$tq \quad f'(d) = \frac{f(b) - f(a)}{b - a} = 1$$

$\rightarrow$ a $[b, c]$ $\quad \exists\, e \in]b, c[$

$$tq \quad f'(e) = \frac{f(c) - f(b)}{c - b} = 1$$

Aplicando o teorema de Rolle
a f' em $[d, e]$

$\exists\, z \in [d, e]$ tq $f'(z) = 0$

cqd

ou
$g(x) = f(x) - x$
$g(a) = g(b) = g(c)$
pelo T. de Rolle
$\exists\, c_1 \in]a, b[$ tq $g'(c_1) = 0$
$\exists\, c_2 \in]a, b[$ tq $g'(c_2) = 0$
$\exists\, c_3 \in]c_1, c_2[$ tq $g''(c_3) = 0$
$g'(x) = f'(x) - 1$
$g''(x) = f''(x)$

14. $3x^2 - e^x$ tem exactamente 3 zeros

$(3x^2 - e^x)' = 2x - e^x$

ficha 9

1. $f: \mathbb{R}_0^+ \to \mathbb{R}$ diferenciável e $f(n) = (-1)^n$ $\forall n \in \mathbb{N}$

-1 $\quad$ 1

$\Rightarrow$ a sua derivada não tem limite no infinito

$\lim f'(n)$ Não existe

$\hookrightarrow \lim ((-1)^n)'$

2. $f(0) = 0$ derivada é uma função crescente

$$\boxed{g(x) = \frac{f(x)}{x} \text{ é crescente em } \mathbb{R}^+}$$

Aplicando o T de Lagrange a $f(x)$ no intervalo $]0, x[$ $\forall x \in \mathbb{R}_+$

$$\exists c \in]0, x[\text{ tq } \frac{f(x) - f(0)}{x - 0} = f(c)$$

$$\Leftrightarrow \frac{f(x)}{x} = f(c)$$

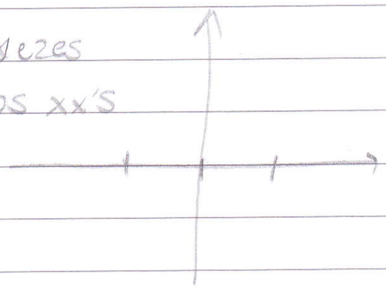
Como $c > 0$ então em $\mathbb{R}_+$ $f(c) = g(x)$

3. Se $f \in C^1(\mathbb{R})$ e $f(x) = x^2$ tem 3 soluções
(uma negativa, uma positiva, uma nula)

$\downarrow$

então $f'(x)$ tem pelo menos um zero

- Se é derivável uma vez
- Passa 3 vezes pelo ref. dos xx's



~~pelo~~ teorema de Rolle
para qualquer intervalo $[a, b]$ tq $f(a) = f(b) = 0$ $a < 0$, $b > 0$

$f(a) = f(b) = f(0) = 0$

então $\exists d \in]a, 0[: f'(d) = 0$

4.

a) $D \dfrac{x}{x^2+1} = \dfrac{1.(x^2+1) - x(2x+0)}{(x^2-1)^2} = \dfrac{x^2+1-2x^2}{(x^2-1)^2} =$

$= \dfrac{-(x^2-1)}{(x^2-1)^2} = -\dfrac{1}{x^2-1}$ $\quad x \neq 1$ e -1

→ Não tem extremos
→ A função é | decrescente em $]-\infty, -1[\cup]1, +\infty[$
| crescente em $]-1, 1[$

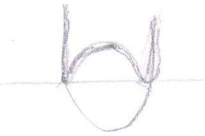
b) $\dfrac{1}{x} + \dfrac{1}{x^2} = \dfrac{1.1 - 0.x}{x^2} + \dfrac{1.2x - 0.x^2}{x^4} =$

$= \dfrac{1}{x^2} + \dfrac{2x}{x^4} = \dfrac{x^2+2x}{x^4} = \dfrac{x+2}{x^3}$ $\quad x \neq 0$

→ tem zero em $x = -2$

		-2		0	$+\infty$
$x+2$	$-$	0	$+$	$+$	$+$
x^3	$-$	$-$	$-$	0	$+$
	$+$	0	$-$	ND	$+$

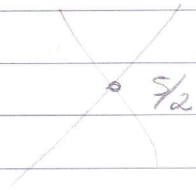
→ A função é
crescente de $]-\infty, -2[$
tem um máximo local em -2
é decrescente em $]-2, 0[$
e crescente em $]0, +\infty[$



c) $|x^2-5x+6|' =$

$(x^2-5x+6)' = 2x-5 \qquad x = \frac{5}{2}$

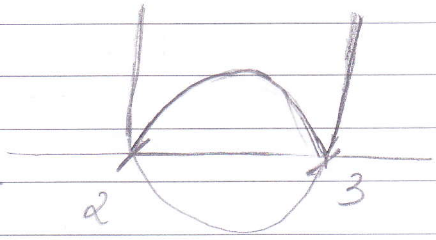
$(-x^2+5x-6)' = -2x+5 \qquad x = \frac{-5}{-2}$



$x^2-5x+6=0 \Leftrightarrow x = \frac{5 \pm \sqrt{25-4.1.6}}{2} \Leftrightarrow x = \frac{5 \pm \sqrt{1}}{2} \Leftrightarrow$

$\Leftrightarrow x = \frac{6}{2} \vee x = \frac{4}{2} \Leftrightarrow$

$\Leftrightarrow x = 3 \vee x = 2$

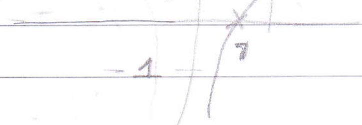


		$5/2$	
$2x-5$	$-$	0	$+$
$-2x+5$	$+$	0	$-$
	$-$	0	$-$

d) $(x\log x)' = \cancel{x} \cdot \frac{1}{\cancel{x}} + \log x = 1 + \log x$

$\forall x \in \mathbb{R}^+$

$1+\log x = 0 \Leftrightarrow -\log x = 1$



$\Leftrightarrow \log \frac{1}{x} = 1 \Leftrightarrow \frac{1}{x} = e \Leftrightarrow$

$\Leftrightarrow 1 = ex \Leftrightarrow \frac{1}{e} = x$

a função é | decrescente em $]0, \frac{1}{e}[$
crescente em $]\frac{1}{e}, +\infty[$

tem mínimo em $x = \frac{1}{e}$

e) $\Big(\underbrace{e^{\overbrace{-x^2}^{\text{sempre}<0}}}_{\text{sempre}\leq 1}\Big)' = -2x \cdot e^{-x^2}$

		0	
$-2x$	+	0	-
e^{-x^2}	-	-	-
	-	0	+

$-2x \cdot e^{-x^2} = 0 \Leftrightarrow -x = 0$

a função | tem máximo em $x = 0$
| é crescente em $]0, +\infty[$
| é decrescente em $]-\infty, 0[$

f) $\left(\frac{e^x}{x}\right)' = \frac{e^x - e^x \cdot x}{x^2}$ $\quad \forall x \neq 0$

$\frac{e^x - e^x \cdot x}{x^2} = 0 \Leftrightarrow e^x - e^x \cdot x = 0 \Leftrightarrow e^x = e^x \cdot x$

$\Leftrightarrow \frac{e^x}{e^x} = x \Leftrightarrow x = 1$

f é | decrescente em $]-\infty, 0[\cup]0, 1[$
| crescente em $]1, +\infty[$

g) $(xe^{-x})' = x \cdot (-e^{-x}) + e^{-x} = -xe^{-x} + e^{-x}$

$\Leftrightarrow e^{-x}(-x+1) = e^{-x}(1-x)$

$e^{-x}(1-x) = 0 \Leftrightarrow \underbrace{e^{-x} = 0}_{\text{PF}} \vee 1 - x = 0 \Leftrightarrow$

$\Leftrightarrow x = -1$

		-1	
e^{-x}	+	+	+
$1-x$	+	0	-
	+	[illegible]	-

f é | crescente em $]-\infty, -1[$
| decrescente em $]-1, +\infty[$

tem máximo em $x = -1$

h) $\left(\arctg x - \log\sqrt{1+x^2}\right) = \frac{1}{\sqrt{1+x^2}} -$

$\left(\log \frac{1}{\sqrt{1+x^2}}\right)' =$

$\left(\frac{1}{\sqrt{1+x^2}}\right)' = \frac{0 \cdot \sqrt{1+x^2} - 1 \cdot \frac{1}{2\sqrt{1+x^2}}}{1+x^2} = -\frac{1}{2\sqrt{1+x^2} \cdot (1+x^2)}$

$\left(\sqrt{1+x^2}\right)' = \frac{(1+x^2)^{-1/2}}{2} = \frac{1}{2\sqrt{1+x^2}}$

5. a) $f: \mathbb{R} \to \mathbb{R} \quad f(x) = |x| e^{-\frac{x^2}{2}}$

$\lim_{-\infty} \underbrace{|x|}_{\to +\infty} e^{-\frac{x^2}{2}} = +\infty \cdot e^{-\infty} = +\infty \cdot 0$

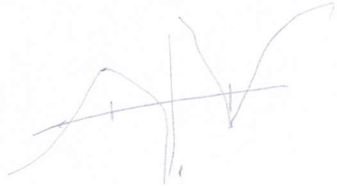
$\lim_{-\infty} \frac{|x|}{e^{x^2/2}}$ pela regra de cauchy $= \lim_{x \to -\infty} \frac{1}{\underbrace{x e^{x^2/2}}_{\to \infty}} = 0$

$\lim_{+\infty} \underbrace{|x|}_{\to +\infty} e^{-x^2/2} = +\infty \cdot e^{-\infty} = +\infty \cdot 0$

$\lim_{+\infty} \frac{|x|}{e^{x^2/2}}$ pela regra de cauchy $= \lim_{x \to +\infty} \frac{1}{\underbrace{x e^{x^2/2}}_{+\infty}} = 0$

b) em $\mathbb{R} \setminus \{0\}$

$$\left(|x| e^{-\frac{x^2}{2}}\right)' = \begin{cases} \frac{1 - x^2}{e^{x^2/2}} & \text{se } x > 0 \\ \frac{x^2 - 1}{e^{x^2/2}} & \text{se } x < 0 \end{cases}$$



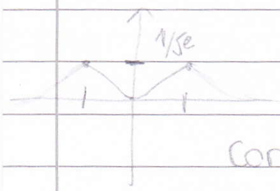
c) $\dfrac{1-x^2}{e^{x^2/2}}$

		-1	0	1	
$1-x^2$	$-$	0	$+$	0	$-$
$e^{x^2/2}$	$+$	$+$	$+$	$+$	$+$
φ	$-$	0	$+$	0	$-$
x^2-1	$+$	0	$-$	0	$+$
φ	$+$	0	$-$	0	$+$

crescente em $]-\infty,-1[$
tem um máximo em $x=-1$
decrescente em $]-1,0[$
crescente em $]0,1[$
tem um máximo em $x=1$
decrescente em $]1,+\infty[$

d) maximos $x=-1$ | $x=1$

$|-1|\,e^{-\frac{(-1)^2}{2}} = 1.e^{-\frac{1}{2}} = \dfrac{1}{\sqrt{e}}$ | $|1|\,e^{-1^2/2} = 1\cdot e^{-1/2} = \dfrac{1}{\sqrt{e}}$



Como $\lim\limits_{x\to-\infty} |x|e^{-x^2/2} = \lim\limits_{x\to+\infty} |x|e^{-x^2/2} = 0$

Contradomínio de $g(x) =]0, 1/\sqrt{e}]$

6. $g(x) = \begin{cases} e^x + \alpha x + \beta & \text{se } x \leq 0 \\ \operatorname{arctg}(e^x + e^{-x} - 1) & \text{se } x > 0 \end{cases}$

a)

$\lim\limits_{x\to0^+} \operatorname{arctg}(e^x + e^{-x} - 1) = \lim\limits_{x\to0^+} \operatorname{arctg}(\underset{>1}{e^{0^+}} + \underset{<1}{e^{0^-}} - 1) =$

$= \operatorname{arctg}(1) = \pi/4$

$e^0 + \alpha.0 + \beta = \pi/4 \Leftrightarrow 1 + \beta = \pi/4 \Leftrightarrow \beta = \pi/4 - 1 \Leftrightarrow \beta = \dfrac{\pi}{4} - \dfrac{4}{4}$

$\Leftrightarrow \beta = \dfrac{\pi - 4}{4}$

$\beta = 4 \qquad \alpha = -1$

b)

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \overset{\nearrow 0}{e^{x}} - \overset{\nearrow +\infty}{x} + 4 =$$

$$\lim_{x\to-\infty} e^{x} - e^{\log x} + 4$$

$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} \operatorname{arctg}(\overset{\nearrow +\infty}{e^{x}} + \overset{\nearrow 0}{e^{-x}} - 1)$$

c) $(e^{x} - x + 4)' = e^{x} - 1 \qquad x \leq 0$

$$(\operatorname{arctg}(e^{x}+e^{-x}-1))' = \frac{1}{1-(e^{x}+e^{-x}-1)^2} \cdot (e^{x}+e^{-x}-1)'$$

$$= \frac{1}{1-(e^{x}+e^{-x}-1)^2} \cdot (e^{x}-e^{-x})$$

d) $e^{x}-1$

7. $f: \mathbb{R} \to \mathbb{R} \qquad |x| e^{-|x-1|}$

a) $\lim_{x \to -\infty} |x| e^{-|x-1|} = \frac{|x|}{e^{|x-1|}}$ pela regra de Cauchy

$\lim_{x \to -\infty} = \frac{1}{e^{|x-1|}}$ se $x \neq 0$ $\Leftrightarrow$

$\Leftrightarrow \lim_{x \to -\infty} \frac{1}{e^{+\infty}} = \frac{1}{+\infty} = 0 = \lim_{x \to +\infty} |x| e^{-|x-1|}$

$$|x-1|' = \begin{cases} 1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \end{cases}$$

b) em $x \neq 0$

$$(|x| e^{-|x-1|})' = \begin{cases} -1 \cdot e^{-|x-1|} + |x| \cdot e^{-|x-1|} & x < 0 \\ 1 \cdot e^{-|x-1|} + |x| \cdot (-e^{-|x-1|}) & x > 0 \end{cases} =$$

$$= \begin{cases} -e^{-|x-1|} + |x| e^{-|x-1|} \\ e^{-|x-1|} - |x| e^{-|x-1|} \end{cases} = \begin{cases} e^{-|x-1|}(-1-x) & x < 0 \\ e^{-|x-1|}(1-x), & x > 0 \end{cases}$$

c)

		-1		0		1	
$e^{-\lvert x-1\rvert}$	+	+	+	+	+	+	+
$(-x-1)$	+	0	−	−	−	−	−
$(-x+1)$	+	+	+	+	+	0	−
f'	+	0	−	ND	−	0	+
	$\nearrow$	MAX	$\searrow$	MIN	$\searrow$	MAX	$\nearrow$

$x = -1$

$|-1|\, e^{-|-1-1|} = 1 \cdot e^{-2} = e^{-2}$

$x = 1$

$|1|\, e^{-|1-1|} = e^{0} = 1$ → máximo absoluto

$x = 0$

$0 \cdot e^{1} = 0$ → mínimo absoluto

d) $[0, 1]$

3. $f(x) = x + 2\,\text{arctg}|x| \quad \mathbb{R} \to \mathbb{R}$

a) $\lim_{x \to -\infty} x + 2\,\text{arctg}|x| = -\infty + 2 \cdot \frac{\pi}{2} = -\infty + \pi = -\infty$

$\lim_{x \to +\infty} x + 2\,\text{arctg}|x| = +\infty + \frac{2\pi}{2} = +\infty + \pi = +\infty$

b) $\mathbb{R} \setminus \{0\}$

$(x + 2\,\text{arctg}\,x) = 1 + 2 \cdot \frac{1}{1-x^2} = 1 + \frac{2}{1-x^2} \qquad x > 0$

$|x|' = 1$

$(x + 2\,\text{arctg}(-x))' = 1 + 2 \cdot \frac{1}{1-(-x)^2} \times (-x)' \qquad x < 0$

$= \frac{1-2}{1+x^2}$ $\qquad x = 0$

$|x| = -1 \qquad \lim_{x \to 0^+} f(x)' = 3 \neq \lim_{x \to 0^-} f(x)'$

$\boxed{x<0}$

c)

$$1-\frac{2}{1+x^2}=0 \Leftrightarrow \frac{2}{1+x^2}=1 \Leftrightarrow 2=1+x^2 \Leftrightarrow$$

$$\Leftrightarrow 2-1=x^2 \Leftrightarrow x^2=1 \Leftrightarrow x=1 \wedge x=-1$$

		-1	0	1	
$1-\frac{2}{1+x^2}$	+	0	-	0	+
$1+\frac{2}{1+x^2}$	+	+	+	+	+
$f'(x)$	+	0	- NE +	+	+

$$1+\frac{2}{1+x^2}=0 \Leftrightarrow \frac{2}{1+x^2}=-1 \Leftrightarrow 2=-1-x^2 \Leftrightarrow$$

$$\Leftrightarrow 3=-x^2 \Leftrightarrow x^2=-3 \quad PF$$

f é crescente em $]-\infty,-1[$
tem máximo em $x=-1$ → máximo local
é decrescente em $]-1,0[$
tem um vértice em $x=0$ → mínimo local
é crescente em $]0,+\infty[$

$$f(0)=0+2\operatorname{arctg}0=2.0=0$$

$$f(-1)=-1+2\operatorname{arctg}(-1)=\frac{\pi}{2}-1$$

d) $f:]-\infty,0] \rightarrow]-\infty, \pi/2 - 1$

9. $g(0) = g'(0) = 0$ $\quad$ g' é estritamente monótona

$e(x) = 2\,tg(g(x)) - g(x)$

então $e(0)$ é extremo local de e

$e(0)' = 0$

$e'(x) = (2tg(g(x)) - g(x))' = (2tg(g(x)))' - g(x)' =$

$= 2 \cdot (tg(g(x)))' - g(x)' = 2 \cdot tg'(g(x)) \cdot g(x)' - g(x)'$

no ponto $x = 0$

$2 \cdot tg'(0) \cdot 0 - 0 = 0$

Como $x=0$ é zero da derivada, 0 é extremo local de e

10. $f: V_\varepsilon(0)$ $\quad$ dif em $V_\varepsilon(0) \setminus \{0\}$

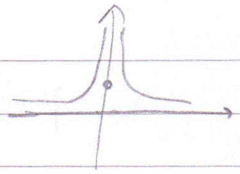
e $xf'(x) > 0 \quad \forall x \in V_\varepsilon(0) \setminus \{0\}$

a) f contínua em $x=0$

Prove que $f(0)$ é um extremo

$x > 0 \vee f'(x) > 0$	$x < 0 \vee f'(x) < 0$
a função é crescente à esquerda do 0	a função é decrescente à direita do zero

Logo 0 é um máximo local da função $\quad$ Se existir $f'(0) = 0$

b) por exemplo 

não é contínua em $x=0$, não se pode concluir que este é extremo

11.

a) $\lim_{x\to 0} \frac{a^x - b^x}{x}$ pela regra de Cauchy

$\lim_{x\to 0} \frac{x a^{x-1} - x b^{x-1}}{1} = \lim_{x\to 0} x(a^{x-1} - b^{x-1})$

$= 0(a^{-1} - b^{-1}) = 0$

b) $\lim_{x\to +\infty} \frac{\log(x+e^x)}{x}$ pela regra de Cauchy

$\lim_{x\to +\infty} \frac{(x+e^x)'}{(x+e^x)} = \lim_{x\to +\infty} \frac{1+e^x}{x+e^x}$ pela regra de Cauchy

$= \lim_{x\to +\infty} \frac{e^x}{1+e^x} = \lim_{x\to +\infty} \frac{1}{\frac{1}{e^x}+1} = \frac{1}{0+1} = 1$

c) $\lim_{x\to 1} \log x . \log\log x = \lim_{x\to 1} \frac{\log\log x}{\frac{1}{\log x}} =$

pela regra de Cauchy $= \lim_{x\to 1} \frac{\frac{1}{\log x} \cdot \cancel{\frac{1}{x}}}{-\frac{1}{(\log x)^2} \cdot \cancel{\frac{1}{x}}} =$

$= \lim_{x\to 1} -\frac{\log^2 x}{\log x} = \lim_{x\to 1} -\log x = 0$

$-\infty$

d) $\lim_{x\to 0^+} \frac{e^{-1/x}}{x}$ pela regra de Cauchy $=$

$= \lim_{x\to 0^+} \frac{1/x^2 \cdot e^{-1/x}}{1} =$

e) $\lim_{x\to 0^-} \frac{e^{-1/x}}{x}$

f) $\lim_{x\to 1^+} x^{\log\log x} = \lim_{x\to 1^+} (e^{\log x})^{\log\log x} =$

$= \lim_{x\to 1^+} e^{\log x \log\log x} = e^0 = 1$

g) $\lim_{x\to +\infty} x^{\frac{1}{x-1}} =$

12.

a) $\lim_{x\to 0} \frac{10^x - 5^x}{x}$ pela r. de Cauchy

$= \lim_{x\to 0} \frac{x10^{x-1} - x5^{x-1}}{1} = \frac{0.10^{-1} - 0.5^{-1}}{1} = 0$

b) $\lim_{x \to 0^+} \frac{x^2 \operatorname{sen} \frac{1}{x}}{\operatorname{sen} x}$ pela regra de cauchy

$\lim_{x \to 0^+} \frac{(x^2 \operatorname{sen} \frac{1}{x})'}{-\cos x} = \lim_{x \to 0^+} \frac{2x \operatorname{sen} \frac{1}{x} + \cos \frac{1}{x}}{-\cos x}$

C.A

$\left(x^2 \operatorname{sen} \frac{1}{x}\right)' = 2x . \operatorname{sen} \frac{1}{x} + x^2 . \frac{1}{x^2} \cos \frac{1}{x}$

$= 2x . \operatorname{sen} \frac{1}{x} + \cos \frac{1}{x}$

13.

a) $\lim_{x \to +\infty} \frac{2^{x \nearrow}}{x^2} = +\infty$

b) $\lim_{x \to -\infty} \frac{2^x}{x^2} = 0$

14.

a) $\lim_{x \to 0^+} (\operatorname{sen} x)^{\operatorname{sen} x} = \lim_{x \to 0^+} e^{\log \operatorname{sen} x^{\operatorname{sen} x}}$

$= \lim_{x \to 0^+} e^{\operatorname{sen} x . \log \operatorname{sen} x} = e^0 = 1$

$\lim_{x \to 0^+} \operatorname{sen} x . (\log(\operatorname{sen} x)) = \lim_{x \to 0^+} \frac{\log \operatorname{sen} x}{\frac{1}{\operatorname{sen} x}} = \frac{\frac{-\cos x}{\operatorname{sen} x}}{\frac{-\cos x}{\operatorname{sen} x^2}} =$

$= \frac{\operatorname{sen} x^{2} . (-\cos x)}{\operatorname{sen} x . (-\cos x)} = \lim_{x \to 0^+} \operatorname{sen} x = 0$

$(\log \operatorname{sen} x)' = \frac{-\cos x}{\operatorname{sen} x}$ $\left(\frac{1}{\operatorname{sen} x}\right)' = \frac{-\cos x}{\operatorname{sen} x^2}$

b) $\lim_{x \to +\infty} (\log x)^{1/x}$ $\Leftrightarrow \lim_{x\to+\infty} e^{\boxed{\frac{1}{x}\log(\log x)}}$ $= e^0 = \underline{1}$

$+\infty^0$ $\quad 0.\infty$

$\Leftrightarrow \lim_{x\to+\infty} \frac{1}{x}\log(\log x) = \lim_{x\to+\infty} \frac{\log(\log x)}{x} =$

$\boxed{\text{cauchy } \frac{\infty}{\infty} \text{ ou } \frac{0}{0}}$

$= \lim_{x\to+\infty} \frac{\frac{1}{\log x}\cdot\frac{1}{x}}{1} = \lim_{x\to+\infty} \frac{1}{x\log x} = \frac{1}{+\infty} = 0$

15.

a) $\lim_{x\to+\infty} x \operatorname{sen}\frac{1}{x} = \lim_{x\to+\infty} \frac{\operatorname{sen}\frac{1}{x}}{\frac{1}{x}} =$

$+\infty . 0$

R. cauchy $\frac{\cancel{\frac{1}{x^2}}\cdot\left(-\cos\frac{1}{x}\right)}{\cancel{\frac{1}{x^2}}} = \lim_{x\to+\infty} -\cos\frac{1}{x} = -1$

b) $\lim_{x\to\frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x} = \lim_{x\to\frac{\pi}{4}} e^{\log \operatorname{tg} x^{\operatorname{tg} 2x}} =$

$= \lim_{x\to\frac{\pi}{4}} e^{\operatorname{tg} 2x \,.\, \log \operatorname{tg} x}$

$\lim_{x\to\pi/4} \operatorname{tg} 2x \,.\, \log\operatorname{tg} x = \lim_{x\to\pi/4} \frac{\log\operatorname{tg} x}{\frac{1}{\operatorname{tg} 2x}}$ R. cauchy

$(\log \operatorname{tg} x)' = \frac{1}{\cos^2 x \operatorname{tg} x}$

$\lim_{x\to\pi/4}$

16. $\lim \left(\frac{1}{n}\right)^{\operatorname{sen}\frac{1}{n}}$

$$\lim_{x\to 0} x^{\operatorname{sen} x} \overset{0^0}{=} \lim_{x\to 0} e^{\log x^{\operatorname{sen} x}} = \lim_{x\to 0} e^{\operatorname{sen} x \log x}$$

C.A:

$$\lim_{x\to 0} \operatorname{sen} x \log x = \lim_{x\to 0} \frac{\log x}{\frac{1}{\operatorname{sen} x}} \quad \text{R. cauchy}$$

$$\lim_{x\to 0} \frac{1/x}{\frac{-\cos x}{\operatorname{sen} x^2}} = \lim_{x\to 0} -\frac{\operatorname{sen} x^2}{x.\cos x} \quad \text{R. cauchy}$$

$$\lim_{x\to 0} -\frac{1-\cos x^2}{x\cos x} = \lim_{x\to 0} \frac{-1}{x\cos x} - \frac{\cos x}{x}$$

$$= -\frac{1}{0^+} - \frac{1}{1}$$

?

17.

a) ordem 2, ponto 1

$f(x)=e^{2x}$ $\quad f'(x)=2e^{2x}$ $\quad f''(x)=2.2.e^{2x}=4e^{2x}$

Mclaurin $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$

$$= e^0 + 2e^0x + \frac{4e^0x^2}{2} = 1+2x+\frac{4x^2}{2} = 2x^2+2x+1$$

Taylor $f(x) = e^1 + 2e^1x + \frac{4e^1x^2}{2} = 2ex^2 + 2ex + e = e(2x^2+2x+1)$

$f(x) = \log(1+x) \quad f'(x) = \frac{1}{1+x} \quad f''(x) = \frac{1}{(1+x)^2}$

Mclaurin $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!}$

$= \log 1 + \frac{1}{1} x + \frac{1}{2} x^2$

$= x + \frac{1}{2} x^2$

taylor $f(x) = \log 2 + \frac{1}{1+1} x + \frac{\frac{1}{4}}{2} x^2 = \log 2 + \frac{1}{2} x + \frac{1}{8} x^2$

$f(x) = \cos(\pi x) \quad f'(x) = -\pi \operatorname{sen}(\pi x) \quad f''(x) = -2\pi \cos(\pi x)$

Mclaurin $f(x) = \cos 0 - \pi \operatorname{sen} 0 \, x \, \frac{-2\pi \cos 0 \, x^2}{2} = 1 - \pi x^2$

taylor $f(x) = \cos\pi + (-\pi \operatorname{sen}\pi) + \frac{(-2\pi\cos\pi)}{2} x^2 = -1 + \pi x^2$

b)

18.

19. $\forall x \in \mathbb{R} \; \exists \xi_x$ entre 0 e x

como $x \in [0,1]$ $\quad e^{\xi_x} \frac{x^3}{6} \leq \underbrace{e^{\xi_x}}_{\leq 1} \frac{1}{6} < \frac{1}{6}$

$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + f'''(\xi_x) \frac{x^3}{3!}$

$f(x) = e^{-x}$ $\quad \forall x \in \mathbb{R}$

$e^{-x} = 1 - x + \frac{x^2}{2} - e^{-\xi_x} \cdot \frac{x^3}{3!}$

$f'(x) = -e^{-x}$ $\quad$ como $x \in [0,1]$

$0 < \xi_x < x < 1$

$f''(x) = +e^{-x}$

$\left| e^{-x} - \left(1 - x + \frac{x^2}{2}\right) \right| = \left| -\underbrace{e^{\xi_x}}_{>0} \cdot \underbrace{\frac{x^3}{3!}}_{>0 \to [0,1]} \right| = e^{\xi_x} \frac{x^3}{6}$

$f'''(x) = -e^{-x}$

TU AMAS-ME ?!

20.

$$P_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2$$

$$g(0) = f(e^0) = f(1) = 2$$

$$\underset{2}{f(1)} + \underset{-1}{f'(1)}(x-1) + \underbrace{\frac{\overset{2.2!=4}{f''(1)}(x-1)^2}{2!}} = 3 - x \underbrace{+2(x-1)^2}$$

$$= 2 - (x-1) + 2(x-1)^2$$

$$g'(x) = e^x f'(e^x) \Rightarrow g'(0) = e^0 f'(e^0) = f'(1) = -1$$

$$g''(x) = e^x f''(e^x).e^x + e^x . f'(e^x)$$

$$\Downarrow$$

$$g''(0) = e^0 f''(e^0).e^0 + e^0 . f'(e^0)$$

$$= 1 . f''(1)\, 1 + 1 . f'(1)$$

$$= (-1) + 4 = 3$$

$$P_2(x) = 2 + (-1).x + \frac{3}{2!}x^2$$

$$= 2 - x + \frac{3}{2}x^2$$

21. Se $f: \mathbb{R} \to \mathbb{R}$ e $f^n(x) = 0 \quad \forall x \in \mathbb{R}$

então f tem grau inferior a n

$$P_n(x) = f(0) + \ldots + \frac{\overset{=0}{f^n(0)}\, x^n}{n!}$$

22. $I \in \mathbb{R}$ $]a,b[$
$f \in C^2(I)$
$\forall a \in I$

$f''(a) = \lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$

$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + R_3(x)$

$\Leftrightarrow \frac{f''(a)(x-a)^2}{2} = -f(a) - f'(a)(x-a) + f(x)$

$\Leftrightarrow f''(a)(x-a)^2 = 2(-f(a) - f'(a)(x-a) + f(x))$

$\Leftrightarrow f''(a) = \frac{2(-f(a) - f'(a)(x-a) + f(x))}{(x-a)^2}$

$x - a = h$
$\Leftrightarrow x = h + a$

$\Leftrightarrow f''(a) = \frac{-2f(a) - f'(a).h + f(h+a)}{h^2}$

? $\Leftrightarrow f''(a) = \frac{f(h+a) - 2f(a) - f'(a).\cancel{h}}{h^2}$ $\lim_{h \to 0} \frac{f(a+h) - f(a)}{\cancel{h}}$

23. $g(x) = x f(x)$ $g'(x) = f(x) + x f'(x)$

$g''(x) = f'(x) + f'(x) + x.f''(x) = 2f'(x) + xf''(x)$
crescente $g''(0) = 0$

$P_1(x) = g(0) + g'(0)x = f(0)x$

24. $f(x) = \operatorname{arctg}(x^2)$ $f'(x) = \dfrac{2x}{1-x^4}$ $\forall x \neq \{1, -1\}$

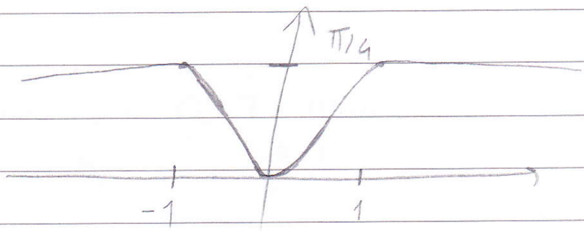
$\mathbb{R} \to]0, \pi/4[$

		-1		0		1	
$1-x^4$	-	0	+	+	+	0	-
$2x$	-	-	-	0	+	+	+
$2x/1-x^4$	+	NE	-	0	+	NE	-

$f(x)$ não tem derivada em $x = \{-1, 1\}$

crescente em $]-\infty, -1[$
tem maximo em $x=-1$ $f(x) = \pi/4$
decrescente de $]-1, 0[$
tem um minimo em $x = 0$
crescente em $]0, 1[$
tem maximo em $x=1$ $f(x) = \pi/4$
decrescente em $]-1, +\infty[$

$\boxed{\operatorname{arctg}(-1^2) = \operatorname{arctg} 1 = \pi/4}$



$f''(x) = \dfrac{2(1-x^4) - 2x(4x^3)}{(1-x^4)^2}$

$= \dfrac{2 - 2x^4 + 8x^4}{(1-x^4)^2} = \dfrac{6x^4+2}{(1-x^4)^2}$

$f''(x) > 0$
$f(x)$ é convexa
não tem pontos de inflexão

25. $f\ \mathbb{R} \to \mathbb{R}$ $f(x) = x^4 e^{-x}$

$f'(x) = 4x^3 . e^{-x} + x^4 . (-e^{-x})$
$= 4x^3 . e^{-x} - x^4 e^{-x}$
$= e^{-x}(4x^3 - x^4)$
$= e^{-x} . x^3(4-x)$

$f'(x) = 0$, $x = 0$, $x = 4$

		0		4	
$e^{-x} . x^3$	-	0	+	+	+
$4-x$	+	+	+	0	-
⊕	-	0	+	0	-

$f(x)$ decrescente em $]-\infty, 0[$
minimo $x = 0$ $\boxed{f(x) = 0}$
crescente em $]0, 4[$
maximo $x = 4$ $\boxed{f(x) = 16 . e^{-4}}$
decrescente em $]-\infty, +\infty[$

$$\lim_{x \to -\infty} \overset{+\infty}{x^4} \, \overset{+\infty}{e^{-x}} = \lim_{x \to -\infty} +\infty$$

$$\lim_{x \to +\infty} \overset{+\infty \cdot 0}{x^4 e^{-x}} = \lim_{x \to +\infty} \frac{e^{-x}}{\frac{1}{x^4}} \quad \text{R Cauchy}$$

$$= \lim_{x \to +\infty} \frac{-e^{-x}}{\frac{3x}{x^8}} = \lim_{x \to +\infty} \frac{-e^{-x} \cdot x^8}{3x}$$

26. $f(x) = \dfrac{\operatorname{sen} x}{1 - \operatorname{sen} x}$

$$f'(x) = \frac{\cos x (1 - \operatorname{sen} x) - \operatorname{sen} x (-\cos x)}{(1 - \operatorname{sen} x)^2}$$

$$= \frac{\cos x - \cos x \cdot \cancel{\operatorname{sen} x} + \cancel{\operatorname{sen} x} \cos x}{(1 - \operatorname{sen} x)^2}$$

$(\operatorname{sen}x^2)' = 2\operatorname{sen}x \cdot \cos x$

ficha 10

1.

a) $P\,2x^2 + 3x^3 = 2P\,\cancel{2}x^2 + 3P\,\cancel{3}x^3 = \frac{2x^3}{3} + \frac{3x^4}{4}$

b) $\frac{x^{-1/2+1}}{-\frac{1}{2}+1} + \log|x| + \frac{x^{-2+1}}{-2+1} = 2\sqrt{x} + \log|x| - \frac{1}{x}$

c) $P\left(\frac{x^2-x+1}{\sqrt{x}}\right) = P\left(\underbrace{x\sqrt{x}}_{x^{3/2}} - \sqrt{x} + \frac{1}{\sqrt{x}}\right) =$

$= \frac{x^{3/2+1}}{3/2+1} - \frac{x^{1/2+1}}{1/2+1} + \frac{x^{-1/2+1}}{-1/2+2}$

d) $P\sqrt[3]{1-x} = P(1-x)^{1/3} = -\frac{(1-x)^{1/3+1}}{1/3+1} = -\frac{(1-x)^{4/3}}{4/3}$

$\frac{2}{3} - \frac{3}{3} = \frac{1}{3}$

e) $P\,\frac{\sqrt[3]{x^2} + \sqrt{x^3}}{x} = P\,\frac{(x^2)^{1/3} + (x^3)^{1/2}}{x} = P\,\frac{x^{2/3}}{x} + \frac{x^{3/2}}{x} =$

$\frac{3}{2} - \frac{2}{2} = \frac{1}{2}$

$= P(x^{-1/3} + x^{1/2}) = \frac{3}{2}\cdot x^{2/3} + \frac{2}{3}x^{3/2}$

f) $P\,2x\sqrt[5]{x^2-1} = P\,(x^2-1)'(x^2-1)^{1/5} = \frac{(x^2-1)^{1/5+1}}{1/5+1}$

$-\frac{1}{3} + \frac{3}{3} = \frac{2}{3}$

$= \frac{5}{6}\sqrt[5]{(x^2-1)^6} = \frac{5}{6}(x^2-1)\sqrt[5]{(x^2-1)}$

$\frac{1}{2} + \frac{2}{2} = \frac{3}{2}$

g) $\frac{1}{4}P\,\frac{4x^3}{3+x^4} = \frac{1}{4}\log|3+x^4| = \frac{1}{4}\log(3+x^4)$

h) $\frac{1}{2}P\,\frac{2e^x}{1+2e^x} = \frac{1}{2}\log|1+2e^x|$

i) $P\,\frac{\cos x}{1+\operatorname{sen}x} = \log|1+\operatorname{sen}x|$

j) $P\,\operatorname{sen}2x = \frac{-\cos 2x}{2}$

k) $P \dfrac{\text{sen} 2x}{1+\text{sen}^2 x} = P \dfrac{2 \text{sen} x \cos x}{1+\text{sen}^2 x} = \log|1+\text{sen}^2 x|$

l) $P \cos^2 x = \dfrac{\cos x^3}{-3\,\text{sen} x}$

m) $P \dfrac{1}{\cos^2 x} = P \dfrac{1}{\frac{1+\cos 2x}{2}} = P \dfrac{2}{1+\cos 2x}$ $\quad \cos^2 x = \dfrac{1+\cos 2x}{2}$

n) $P \dfrac{e^{tg x}}{\cos^2 x} =$

o) $\dfrac{1}{2} P\, 2x \cos(x^2+2) = \dfrac{\text{sen}(x^2+2)}{2}$

p) $P\, e^x \text{sen}(e^x) = -\cos(e^x)$

q) $P\, x^2 \sqrt[3]{1+x^3} = P x^2 (1+x^3)^{1/3} = \dfrac{(1+x^3)^{1/3+1}}{1/3+1} =$

$= \dfrac{(1+x^3)^{4/3}}{4/3} = \dfrac{3(1+x^3)^{4/3}}{4}$

r) $P \dfrac{e^x}{(1+e^x)^2} = \dfrac{\log(1+e^x)^2}{2}$

s) $P \dfrac{\text{sen} x}{1+\cos^2 x} = \text{arctg}(\cos x)$

t) $\frac{1}{2} P \frac{2 \cdot 1}{\sqrt{1-4x^2}} = \text{arcoseno}(2x)$

u) $P \frac{x+1}{\sqrt{1-x^2}} = P \frac{x}{\sqrt{1-x^2}} + \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\text{arcos } x} = P \frac{x}{(1-x^2)^{1/2}} + (\text{arcos} x)'$

$= \frac{1}{2} P\text{-}2x \cdot (1-x^2)^{-1/2} + (\text{arcos} x)' = \frac{\frac{(1-x^2)^{1/2}}{1/2}}{-\frac{2}{1}} + \text{arcos} x =$

$= -\sqrt{1-x^2} + \text{arcos} x$

v) $\frac{1}{6} P \frac{6x^3}{(1+x^4)^2} = \frac{\log|(1+x^4)^2|}{6}$

w) $P \cos^3 x \sqrt{\text{sen} x} = P \cos^3 x \cdot (\text{sen} x)^{1/2}$

x) $P\ tg^2 x = \frac{tg x^3}{2}$

~~z~~ k') $P \frac{\cos x}{1+\text{sen}^2 x} = \text{arctg}(\text{sen} x)$

2.

a) $P (x^2+1)^3 = \frac{(x^2+1)^4}{4}$ $2x$

b) $P e^{x+3} = P e^x \cdot e^{x+2} =$

c) $P 2^{x-1}$

d) $P \dfrac{1}{\sqrt[5]{1-2x}} = P \dfrac{1}{(1-2x)^{1/5}} = -\dfrac{1}{2} P -2(1-2x)^{-1/5} = \dfrac{(1-2x)^{-\frac{1}{5}+1}}{-1/5+1} \cdot -\dfrac{1}{2}$

$= \dfrac{(1-2x)^{4/5}}{4/5} \cdot \left(-\dfrac{1}{2}\right) = \dfrac{5(1-2x)^{4/5}}{-8}$

e) $\dfrac{1}{2} P \dfrac{2x}{1+x^2} = \dfrac{\log|1+x^2|}{2}$

f) $P \dfrac{x^3}{x^8+1}$

g) $P \operatorname{cotg} x = P \dfrac{\cos x}{\operatorname{sen} x} = \log|\operatorname{sen} x|$

h) $P\, 3^{\operatorname{sen}^2 x} \operatorname{sen} 2x = P\, \underline{3^{\operatorname{sen}^2 x}} \cdot 2 \operatorname{sen} x \cdot \cos x$

$= P\, 3^{\operatorname{sen}^2 x} \cdot (\operatorname{sen} x^2)'$

i) $P \dfrac{\operatorname{tg}\sqrt{x}}{\sqrt{x}} = P \operatorname{tg}(x^{1/2}) \cdot x^{-1/2} = P \dfrac{\operatorname{sen}(x^{1/2})}{\cos(x^{-1/2})} \cdot x^{-1/2}$

$= \log|-\cos\sqrt{x}|$

j) $P \dfrac{e^x}{\sqrt{1-e^{2x}}} = -\arccos e^x$

k) $P \dfrac{x}{(1+x^2)^\alpha} = \dfrac{1}{2} P\, 2x \cdot (1+x^2)^{-\alpha} = \dfrac{(1+x^2)^{-\alpha+1}}{2 \cdot (-\alpha+1)}$

l) $P \cos x \cos 2x = P \cos x . (\cos^2 x + \text{sen}^2 x) =$

$= P \cos x . \cos^2 x - \cos x \, \text{sen}^2 x = P \cos x (1 - \text{sen}^2 x) - \cos x \, \text{sen}^2 x$

$= P \cos x - \cos x \, \text{sen}^2 x - \cos x \, \text{sen}^2 x =$

$= P \cos x - P \cos x \, \text{sen}^2 x - P \cos x \, \text{sen}^2 x =$

$= \text{sen} x - \frac{2 \, \text{sen}^3 x}{3}$

m) $P \, \text{sen}^3 x \cos^4 x = P \, \text{sen} x (1 - \cos^2 x) \cos^4 x =$

$= P(\text{sen} x - \cos^2 x \, \text{sen} x) \cos^4 x = P \, \text{sen} x \cos^4 x - \cos^6 x \, \text{sen} x$

$= -\frac{\cos^5}{5} + \frac{\cos^7 x}{7}$

n) $P \, tg^3 x + tg^4 x = P \left(\frac{\text{sen} x}{\cos x}\right)^3 + \left(\frac{\text{sen} x}{\cos x}\right)^4$

$= P \frac{\text{sen}^3 x}{\cos^3 x} + \frac{\text{sen}^4 x}{\cos^4 x} = P \frac{\text{sen} x . (1 - \cos^2 x)}{\cos^3 x} +$

$+ P \frac{\text{sen}^2 x (1 - \cos^2 x)}{\cos^4 x}$

3. a) $P \sqrt{2x} + \sqrt{\frac{x}{2}} = 2P(\cancel{2}x)^{1/2} + \frac{1}{2} P \left(\frac{x}{2}\right)^{1/2} =$

$= \frac{4}{3} x^{3/2} + \frac{1}{2} \cdot \frac{2}{3} x^{3/2} = \frac{4}{3} x^{3/2} + \frac{2}{6} x^{3/2}$

$\frac{2}{2} + \frac{1}{2} = \frac{3}{2}$

$x^a =$

3b) $P\ 3\operatorname{sen}x + 2x^2 = 3P\operatorname{sen}x + \frac{2}{3}P3x^2$

$= 3.\cos x + \frac{2}{3}.x^3$

c) $\frac{1}{3}P\ \frac{3x^2}{1+x^3} = \frac{\log|1+x^3|}{3}$

d) $\frac{1}{-2}P-2xe^{-x^2} = \frac{e^{-x^2}}{-2}$

e) $P\ \frac{3\operatorname{sen}x}{(1+\cos x)^2}$

f) $P\ x\sqrt{1+x^2} = \frac{1}{2}P2x.(1+x^2)^{1/2} = \frac{(1+x^2)^{1/2+1}}{1/2+1} \cdot \frac{1}{2}$

g) $\frac{1}{2}P\ e^{2\operatorname{sen}x}2\cos x = \frac{e^{2\operatorname{sen}x}}{2}$

h) $P\ \frac{1}{1+e^x} =$

i) $P\operatorname{tg}x = P\ \frac{\operatorname{sen}x}{\cos x} = -\log|\cos x|$

j) $P\ \frac{1}{2+x^2} =$

k) $P \; tgx \cdot sec^3 x = P \dfrac{senx}{cosx} \cdot \dfrac{1}{cosx^3} =$

$= P \dfrac{senx}{cosx} \cdot \dfrac{1}{cosx(1-senx^2)} =$

l) $P \; cos^3 x \cdot sen^3 x = P \; cos^3 x \cdot senx(1-cos^2 x)$

$= P \; senx \, cosx^3 - senx cosx = -\dfrac{cosx^4}{4} + cosx$

m) $P \dfrac{1}{(1+x^2)arctgx} = P \left(\dfrac{1}{1+x^2} \cdot \dfrac{1}{arctgx} \right) = log|arctgx|$

n) $P \dfrac{x}{1+x^4} = \dfrac{arctg(x^2)}{2}$

o) $P \dfrac{1}{\sqrt{x}(1+x)} = P \dfrac{1}{x^{1/2}(1+x)}$

5.

a) $\frac{1}{2} P\, 2x \operatorname{sen}(x^2) = -\frac{1}{2} \cdot \cos x^2 + c \qquad c \in \mathbb{R}$

$f(0) = 0$

$-\frac{1}{2} \cdot \cos 0^2 + c = 0 \Leftrightarrow c = \frac{1}{2}$

b) $\lim_{x \to +\infty} f(x) = 0 \qquad$ NE

cos x periódico

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a) $P \frac{e^x}{2+e^x} = \log|2+e^x| + c$

$\log|2+e^0| + c = 0 \Leftrightarrow c = \log 3$

b) $\lim_{x \to +\infty} f(x) = 0 \qquad$ NE

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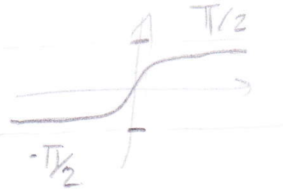
a) $P \frac{1}{(1+x^2)(1+\operatorname{arctg}^2 x)} = P \frac{1}{1+x^2} \cdot \frac{1}{1+\operatorname{arctg}^2 x} =$

$= \operatorname{arctg}(\operatorname{arctg} x) + c$

$\operatorname{arctg}(\operatorname{arctg} 0) + c = 0 \Leftrightarrow 0 + c = 0 \Leftrightarrow c = 0$

b) $\lim_{x \to +\infty} f(x)\ \operatorname{arctg}(\operatorname{arctg} x) = \lim_{y \to \pi/2} \operatorname{arctg} x = \operatorname{arctg} \frac{\pi}{2} \neq 0$

$x = \operatorname{arctg} \frac{\pi}{2} \Leftrightarrow \operatorname{tg} x = \frac{\pi}{2}$



$$\lim_{x \to +\infty} f(x) = 0 \Leftrightarrow \lim_{x \to +\infty} \operatorname{arctg} \frac{\pi}{2} + C = 0 \Leftrightarrow$$

$$\Leftrightarrow C = -\operatorname{arctg} \frac{\pi}{2}$$