5 Cálculo Diferencial — Primitivação (Soluções)

1.

a) $\dfrac{2}{3}x^3 + \dfrac{3}{4}x^4$, b) $2\sqrt{x} + \log x - \dfrac{1}{x}$, $x > 0$,

c) $P\left(\dfrac{x^2 - x + 1}{\sqrt{x}}\right) = P\left(x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) = \dfrac{2}{5}x^{\frac{5}{2}} - \dfrac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} =$

$\dfrac{2}{5}\sqrt{x^5} - \dfrac{2}{3}\sqrt{x^3} + 2\sqrt{x}$, d) $-\dfrac{3}{4}\sqrt[3]{(1-x)^4}$,

e) $P\left(\dfrac{\sqrt[3]{x^2} + \sqrt{x^3}}{x}\right) = P\left(x^{-\frac{1}{3}} + x^{\frac{1}{2}}\right) = \dfrac{3}{2}\sqrt[3]{x^2} + \dfrac{2}{3}\sqrt{x^3}$,

f) $\dfrac{5}{6}\sqrt[5]{(x^2-1)^6}$, g) $\dfrac{1}{4}\log(3 + x^4)$, h) $\dfrac{1}{2}\log(1 + 2e^x)$, i) $\log(1 + \operatorname{sen} x)$,

j) $-\dfrac{1}{2}\cos(2x)$, k) $P\left(\dfrac{\operatorname{sen}(2x)}{1 + \operatorname{sen}^2 x}\right) = P\left(\dfrac{2\operatorname{sen} x \cos x}{1 + \operatorname{sen}^2 x}\right) = \log(1 + \operatorname{sen}^2 x)$,

l) $P\left(\cos^2 x\right) = P\left(\dfrac{\cos(2x) + 1}{2}\right) = \dfrac{1}{4}\operatorname{sen}(2x) + \dfrac{x}{2}$,

m) $\operatorname{tg} x$, n) $e^{\operatorname{tg} x}$, o) $\dfrac{1}{2}\operatorname{sen}(x^2 + 2)$, p) $-\cos(e^x)$,

q) $\dfrac{1}{4}\sqrt[3]{(1 + x^3)^4}$, r) $-\dfrac{1}{1 + e^x}$, s) $-\operatorname{arctg}(\cos x)$,

t) $P\left(\dfrac{1}{\sqrt{1 - 4x^2}}\right) = P\left(\dfrac{1}{\sqrt{1 - (2x)^2}}\right) = \dfrac{1}{2}\operatorname{arcsen}(2x)$,

u) $P\left(\dfrac{x + 1}{\sqrt{1 - x^2}}\right) = P\left(\dfrac{x}{\sqrt{1 - x^2}}\right) + P\left(\dfrac{1}{\sqrt{1 - x^2}}\right) = -\sqrt{1 - x^2} + \operatorname{arcsen} x$,

v) $P\left(\dfrac{x^3}{(1 + x^4)^2}\right) = -\dfrac{1}{4(1 + x^4)}$, w) $P\left(\cos^3 x\sqrt{\operatorname{sen} x}\right) =$

$= P\left(\cos x(1 - \operatorname{sen}^2 x)\sqrt{\operatorname{sen} x}\right) = P\left(\cos x(\sqrt{\operatorname{sen} x} - \operatorname{sen}^{\frac{5}{2}} x)\right) = \dfrac{2}{3}\operatorname{sen}^{\frac{3}{2}} x - \dfrac{2}{7}\operatorname{sen}^{\frac{7}{2}} x$,

x) $P(\operatorname{tg}^2 x) = P(\sec^2 x - 1) = \operatorname{tg} x - x$. 102

2.

a) $\frac{1}{7}x^7+\frac{3}{5}x^5+x^3+x$; b) e^{x+3}; c) $\frac{1}{\log 2}2^{x-1}$;

d) $P\left(\frac{1}{\sqrt[5]{1-2x}}\right)=-\frac{1}{2}P\left(-2(1-2x)^{-\frac{1}{5}}\right)=-\frac{5}{8}(1-2x)^{\frac{4}{5}}$;

e) $P\left(\frac{x}{1+x^2}\right)=\frac{1}{2}P\left(\frac{2x}{1+x^2}\right)=\frac{1}{2}\log(1+x^2)=\log\sqrt{1+x^2}$;

f) $P\left(\frac{x^3}{x^8+1}\right)=\frac{1}{4}P\left(\frac{4x^3}{(x^4)^2+1}\right)=\frac{1}{4}\operatorname{arctg}(x^4)$;

g) $P(\operatorname{cotg}x)=P\left(\frac{\cos x}{\operatorname{sen}x}\right)=\log|\operatorname{sen}x|$;

h) $P\left(3^{\operatorname{sen}^2 x}\operatorname{sen}2x\right)=P\left(3^{\operatorname{sen}^2 x}2\operatorname{sen}x\cos x\right)=P\left(3^{\operatorname{sen}^2 x}(\operatorname{sen}^2 x)'\right)=\frac{1}{\log 3}3^{\operatorname{sen}^2 x}$;

i) $P\left(\frac{\operatorname{tg}\sqrt{x}}{\sqrt{x}}\right)=2P\left(\frac{1}{2\sqrt{x}}\operatorname{tg}\sqrt{x}\right)=2P\left(\left(\sqrt{x}\right)'\operatorname{tg}\sqrt{x}\right)=-2\log|\cos\sqrt{x}|$;

j) $\operatorname{arcsen}e^x$; k) $\frac{1}{2(1-\alpha)}\frac{1}{(1+x^2)^{\alpha-1}}$, se $\alpha\neq 1$, $\log\sqrt{1+x^2}$, se $\alpha=1$;

l) $P(\cos x\cos 2x)=P\left(\cos x(1-2\operatorname{sen}^2 x)\right)=P\left(\cos x-2\cos x\operatorname{sen}^2 x\right)=$

$=\operatorname{sen}x-\frac{2}{3}\operatorname{sen}^3 x$;

m) $P\left(\operatorname{sen}^3 x\cos^4 x\right)=P\left(\operatorname{sen}x(1-\cos^2 x)\cos^4 x\right)=P\left(\operatorname{sen}x(\cos^4 x-\cos^6 x)\right)=$

$=-\frac{1}{5}\cos^5 x+\frac{1}{7}\cos^7 x$;

n) $P(\operatorname{tg}^3 x+\operatorname{tg}^4 x)=P((\sec^2 x-1)\operatorname{tg}x)+P((\sec^2 x-1)\operatorname{tg}^2 x)=$

$P(\sec^2 x\operatorname{tg}x)-P(\operatorname{tg}x)+P(\sec^2 x\operatorname{tg}^2 x)-P(\operatorname{tg}^2 x)=$

$\frac{1}{2}\operatorname{tg}^2 x+\log|\cos x|+\frac{1}{3}\operatorname{tg}^3 x-\operatorname{tg}x+x$.

3.

a) $\sqrt{2x^3}$, b) $-3\cos x+\frac{2}{3}x^3$, c) $\frac{1}{3}\log\left|1+x^3\right|$,

d) $-\frac{1}{2}e^{-x^2}$, e) $\frac{3}{1+\cos x}$, f) $\frac{1}{3}\left(1+x^2\right)^{3/2}$,

g) $\frac{1}{2}e^{2\operatorname{sen}x}$, h) $-\log(1+e^{-x})$, i) $-\log|\cos x|$,

j) $\dfrac{1}{\sqrt{2}}\operatorname{arctg}\dfrac{x}{\sqrt{2}}$, k) $\dfrac{1}{3}\sec^3 x$, l) $\dfrac{1}{4}\operatorname{sen}^4 x - \dfrac{1}{6}\operatorname{sen}^6 x$,

m) $\log|\operatorname{arctg} x|$, n) $\dfrac{1}{2}\operatorname{arctg}\left(x^2\right)$, o) $2\operatorname{arctg}(\sqrt{x})$,

p) $\dfrac{\sqrt{3}}{3}\operatorname{arctg}\left(\sqrt{3}x\right)$, q) $\dfrac{1}{2}\operatorname{arctg}\left(\dfrac{1}{2}e^x\right)$, r) $\dfrac{2}{3}\sqrt{(\operatorname{arcsen} x)^3}$,

s) $\dfrac{1}{2\sqrt{2}}\operatorname{arcsen}\left(\sqrt{2}x^2\right)$, t) $\log\sqrt[3]{\left|\dfrac{x-2}{x+1}\right|}$, u) $-\dfrac{1}{x+1}$,

v) $\operatorname{sen}(\log x)$, w) $\log(\log x)$, x) $\operatorname{tg} x + \dfrac{1}{3}\operatorname{tg}^3 x$.

4. a) Calculamos primeiro uma primitiva de $\frac{1}{4+9x^2}$:

$$P\left(\frac{1}{4+9x^2}\right) = \frac{1}{4}P\left(\frac{1}{1+\left(\frac{3}{2}x\right)^2}\right) = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x.$$

Temos então, para $x \in \mathbb{R}$, $f(x) = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x + c$, com $c \in \mathbb{R}$. Para determinar c temos $f(0) = c = 1$, logo $f(x) = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x + 1$.

b) $P(\left(\frac{1}{x-1}\right) = \log|x-1|$, para $x \neq 1$. Temos então

$$g(x) = \begin{cases} \log(x-1) + c_1, & \text{se } x > 1 \\ \log(1-x) + c_2, & \text{se } x < 1. \end{cases}$$

com $c_1, c_2 \in \mathbb{R}$. Para determinar as constantes, temos $g(0) = \log 1 + c_2 = 0$, logo $c_2 = 0$, e $g(2) = \log 1 + c_1 = 3$, logo $c_2 = 3$.

c) O domínio da secante é $\mathbb{R}\setminus\{\frac{\pi}{2}+k\pi \ : \ k \in \mathbb{Z}\}$. Neste conjunto temos $P(\sec^2 x) = \operatorname{tg} x$, e portanto para $x \in \left]\frac{\pi}{2} + (k-1)\pi, \frac{\pi}{2} + k\pi\right[$, para cada $k \in \mathbb{Z}$, temos $h(x) = \operatorname{tg} x + c_k$. Como $k\pi \in \left]\frac{\pi}{2} + (k-1)\pi, \frac{\pi}{2} + k\pi\right[$, temos que $0 + c_k = k$, ou seja, $c_k = k$.

5.
- $P(x\operatorname{sen}(x^2)) = \frac{1}{2}\cos(x^2)$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \frac{1}{2}\cos(x^2) + C$, com $C \in \mathbb{R}$.
  a) $F(0) = 0 \Leftrightarrow \frac{1}{2} + C = 0$, logo $C = -\frac{1}{2}$.
  b) $\lim_{x\to+\infty} F(x)$ não existe, para qualquer $C \in \mathbb{R}$, logo não existe uma primitiva nas condições dadas.
- $P(\frac{e^x}{2+e^x}) = \log(2+e^x)$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \log(2+e^x)+C$, com $C \in \mathbb{R}$.
  a) $F(0) = 0 \Leftrightarrow \log 3 + C = 0$, logo $C = -\log 3$.

b) $\lim_{x\to+\infty} F(x) = +\infty$, para qualquer $C \in \mathbb{R}$, logo não existe uma primitiva nas condições dadas. .

- $P(\frac{1}{(1+x^2)(1+\operatorname{arctg}^2 x)}) = \operatorname{arctg}(\operatorname{arctg} x)$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \operatorname{arctg}(\operatorname{arctg} x) + C$, com $C \in \mathbb{R}$.

  a) $F(0) = 0 \Leftrightarrow C = 0$.

  b) $\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \operatorname{arctg}(\operatorname{arctg} x) + C = \operatorname{arctg} \frac{\pi}{2} + C$, logo $C = -\operatorname{arctg} \frac{\pi}{2}$.

6.

$$\text{a) } P\left(\frac{1}{1-x}\right) = -\log|1-x|, \qquad \text{b) } P\left(\frac{1}{(x-3)^3}\right) = -\frac{1}{2(x-3)^2},$$

$$\text{c) } P\left(\frac{x+1}{x^2+1}\right) = \frac{1}{2}\log(x^2+1) + \operatorname{arctg} x,$$

$$\text{d) } P\left(\frac{x}{1+(x-1)^2}\right) = \frac{1}{2}\log(1+(x-1)^2) + \operatorname{arctg}(x-1),$$

$$\text{e) } P\left(\frac{2x+1}{x^2+4}\right) = \log(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right), \qquad \text{f) } P\left(\frac{1}{x^2+2x+2}\right) = \operatorname{arctg}(x+1).$$

7. a) $P\left(\frac{1}{x^2+x}\right) = P\left(\frac{1}{x(x+1)}\right)$. Usando a decomposição em fracções simples $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$ temos

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{Ax + A + Bx}{x(x+1)} = \frac{(A+B)x + A}{x(x+1)}$$

logo $A + B = 0$ e $A = 1$, ou seja, $A = 1$ e $B = -1$. Temos então

$$P\left(\frac{1}{x^2+x}\right) = P\left(\frac{1}{x} - \frac{1}{x+1}\right) = \log|x| - \log|x+1| = \log\left|\frac{x}{x+1}\right|.$$

b) Usando a decomposição em fracções simples $\frac{x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$, temos

$$\begin{aligned}
\frac{x+1}{x(x-1)^2} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\
&= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2} \\
&= \frac{Ax^2 - 2Ax + A + Bx^2 - Bx + Cx}{x(x-1)^2} \\
&= \frac{(A+B)x^2 + (-2A - B + C)x + A}{x(x-1)^2}
\end{aligned}$$

logo $A + B = 0$, $-2A - B + C = 1$, $A = 1$, ou seja, $A = 1$, $B = -1$, $C = 2$. Temos então

$$P\left(\frac{x+1}{x(x-1)^2}\right) = P\left(\frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}\right) = \log|x| - \log|x-1| - \frac{2}{x-1}.$$

c) Usando a decomposição em fracções simples $\frac{x^2+x-4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$, temos

$$\begin{aligned}\frac{x^2+x-4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx+C}{x^2+4} \\ &= \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)} \\ &= \frac{(A+B)x^2+4A+Cx}{x(x^2+4)}\end{aligned}$$

logo $A+B=1$, $C=1$ e $4A=-4$, ou seja, $A=-1$, $B=2$, $C=1$. Temos então

$$P\left(\frac{x^2+x-4}{x(x^2+4)}\right) = P\left(-\frac{1}{x} + \frac{2x+1}{x^2+4}\right) = -\log|x| + \log(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right).$$

d) $2\log|x-1| - \log|x| + \dfrac{1}{x}$, e) $\dfrac{x^4}{4} + \dfrac{x^2}{2} + \dfrac{1}{2}\log|x^2-1|$,

f) $\log\left|\dfrac{x+2}{x+1}\right| - \dfrac{2}{x+2}$, g) $\dfrac{x^2}{2} + \log|x+1| + \dfrac{1}{x+1}$,

h) $x + \dfrac{1}{4}\log\left|\dfrac{x-1}{x+1}\right| - \dfrac{1}{2}\operatorname{arctg} x$, i) $\dfrac{1}{2}\log(x^2+4) + \operatorname{arctg}\left(\dfrac{x}{2}\right) + \dfrac{1}{2}\log\left|\dfrac{x-2}{x+2}\right|$.

8. a) O domínio de $\frac{1}{x^2+x}$ é $\mathbb{R}\setminus\{-1,0\}$. A forma geral das primitivas desta função é:

$$\begin{cases}\log x - \log(x+1) + C_1, & \text{se } x > 0,\\ \log(-x) - \log(x+1) + C_2, & \text{se } -1 < x < 0,\\ \log(-x) - \log(-x-1) + C_3, & \text{se } x < -1,\end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

b) O domínio de $\frac{x+1}{x(x-1)^2}$ é $\mathbb{R}\setminus\{0,1\}$. A forma geral das primitivas desta função é:

$$\begin{cases}\log x - \log(x-1) - \frac{2}{x-1} + C_1, & \text{se } x > 1,\\ \log x - \log(-x+1) - \frac{2}{x-1} + C_2, & \text{se } 0 < x < 1,\\ \log(-x) - \log(-x+1) - \frac{2}{x-1} + C_3, & \text{se } x < 0,\end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

c) O domínio de $\frac{x^2+x-4}{x(x^2+4)}$ é $\mathbb{R}\setminus\{0\}$. A forma geral das primitivas desta função é:

$$\begin{cases}-\log x + \log(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_1, & \text{se } x > 0,\\ -\log(-x) + \log(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_2, & \text{se } x < 0,\end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias.

d) O domínio de $\frac{x^2+1}{x^2(x-1)}$ é $\mathbb{R} \setminus \{0, 1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} 2\log(x-1) - \log x + 1/x + C_1, & \text{se } x > 1, \\ 2\log(1-x) - \log x + 1/x + C_2, & \text{se } 0 < x < 1, \\ 2\log(1-x) - \log(-x) + 1/x + C_3, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

9. a) $\dfrac{1}{2}e^{x^2+2x} + C$, com $C \in \mathbb{R}$.

b) $P\left(\dfrac{x+3}{x^4-x^2}\right) = P\left(\dfrac{x+3}{x^2(x-1)(x+1)}\right)$, para $x \in \mathbb{R} \setminus \{-1, 0, 1\}$. Escrevendo

$$\frac{x+3}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$$

tem-se $A = -1$, $B = -3$, $C = 2$, $D = -1$ (verifique). Logo,

$$P\left(\frac{x+3}{x^4-x^2}\right) = -\log|x| + \frac{3}{x} + 2\log|x-1| - \log|x+1| = \frac{3}{x} + \log\frac{(x-1)^2}{|x(x+1)|}.$$

A forma geral da primitiva em $]1, +\infty[$ é $G(x) = \frac{3}{x} + \log\frac{(x-1)^2}{x(x+1)} + K$, com $K \in \mathbb{R}$. Tem-se

$$\lim_{x\to+\infty} G(x) = \lim_{x\to+\infty} \frac{3}{x} + \log\frac{(x-1)^2}{x(x+1)} + K = \log(1) + K = K,$$

logo $\lim_{x\to+\infty} G(x) = 3 \Leftrightarrow K = 3$.

10. $P\left(\frac{1}{(x-1)^2}\right) = -\frac{1}{x-1}$, para $x \in \mathbb{R} \setminus \{1\}$. A forma geral das primitivas é:

$$\begin{cases} -\frac{1}{x-1} + C_1, & \text{se } x > 1, \\ -\frac{1}{x-1} + C_2, & \text{se } x < 1, \end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias. Como $F(2) = 0$, temos $-1 + C_1 = 0 \Leftrightarrow C_1 = 1$. Como $\lim_{x\to-\infty} -\frac{1}{x-1} = 0$, de $\lim_{x\to-\infty} F(x) = 10$ tem-se $C_2 = 10$.

11. Sendo $P\left(\frac{1}{1+x}\right) = \log(x+1)$, para todo o $x \in]-1, +\infty[$, temos

$$\psi'(x) = \log(x+1) + C_1.$$

A condição $\psi'(0) = 1$, resulta em $C_1 = 1$. Usando primitivação por partes (verifique!) temos

$$P(\log(x+1) + 1) = (x+1)\log(x+1),$$

ou seja $\psi(x) = (x+1)\log(x+1) + C_2$. Dado que $\psi(0) = 1$, obtém-se o resultado

$$\psi(x) = (x+1)\log(x+1) + 1.$$

12.

a) $P(xe^x) = xe^x - P(e^x) = (x-1)e^x$,

b) $P(x \operatorname{arctg} x) = \dfrac{x^2}{2} \operatorname{arctg} x - P\left(\dfrac{x^2}{2} \dfrac{1}{1+x^2}\right)$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} P\left(1 - \frac{1}{1+x^2}\right) = \frac{1}{2}\left(-x + (x^2+1) \operatorname{arctg} x\right),$$

c) $P(\operatorname{arcsen} x) = x \operatorname{arcsen} x - P\left(x \frac{1}{\sqrt{1-x^2}}\right) = x \operatorname{arcsen} x + \sqrt{1-x^2}$,

d) $P(x \operatorname{sen} x) = -x \cos x + P(\cos x) = -x \cos x + \operatorname{sen} x$,

e) $P(x^3 e^{x^2}) = P(x^2 \cdot x e^{x^2}) = x^2 \frac{e^{x^2}}{2} - P\left(2x \frac{e^{x^2}}{2}\right) = (x^2-1)\frac{e^{x^2}}{2}$,

f) $P(\log^3 x) = x \log^3 x - P(3 \log^2 x) = x(\log^3 x - 3 \log^2 x) + P(6 \log x) =$
$x(\log^3 x - 3\log^2 x + 6 \log x) - P(6) = x(\log^3 x - 3 \log^2 x + 6 \log x - 6)$,

g) $P(x^n \log x) = \frac{1}{n+1} x^{n+1} \log x - P\left(\frac{1}{n+1} x^{n+1} \frac{1}{x}\right) = \frac{1}{n+1} x^{n+1} \log x - \frac{1}{(n+1)^2} x^{n+1}$,

h) $P\left(\frac{x^7}{(1-x^4)^2}\right) = P\left(x^4 \frac{x^3}{(1-x^4)^2}\right) = x^4 \frac{1}{4(1-x^4)} - P\left(4x^3 \frac{1}{4(1-x^4)}\right) =$
$\frac{x^4}{4(1-x^4)} + \frac{1}{4} \log(1-x^4)$.

13.

a) $e^x(e^x + x - 1) - e^{2x}/2$,

b) $e^x(\operatorname{sen} x - \cos x)/2$,

c) $-e^{-x^2}(x^2+1)/2$,

d) $x \operatorname{arctg} x - \frac{1}{2} \log(1+x^2)$,

e) $\frac{2}{3} x^{\frac{3}{2}} \left(\log x - \frac{2}{3}\right)$

f) $\frac{1}{4}(1+x^2)^2 \operatorname{arctg} x - x/4 - x^3/12$,

g) $\frac{2}{3} x^3 \sqrt{1+x^3} - \frac{4}{9}(1+x^3)^{3/2}$,

h) $x \log|1/x + 1| + \log|x+1|$,

i) $\frac{x^3}{3} \log^2 x - \frac{2}{9} x^2 \log x + \frac{2}{27} x^3$,

j) $x \log^2 x - 2x \log x + 2x$,

k) $-\frac{1}{x} \operatorname{sen} \frac{1}{x} - \cos \frac{1}{x}$,

l) $\frac{1}{2} \operatorname{sen}(2x) \log(\operatorname{tg} x) - x$,

m) $-(1-x^2)^{3/2} \operatorname{arcsen} x + x - x^3/3$,

n) $-\frac{\log x}{1+x} + \log\left|\frac{x}{1+x}\right|$,

o) $\frac{1}{2}(\operatorname{sh} x \cos x + \operatorname{ch} x \operatorname{sen} x)$,

p) $\frac{1}{1+\log^2 3} 3^x(\operatorname{sen} x + \log 3 \cos x)$,

q) $\frac{x}{2}(\cos(\log x) + \operatorname{sen}(\log x))$,

r) $-\frac{1}{2} \frac{x}{1+x^2} + \frac{1}{2} \operatorname{arctg} x$.

14. c) $P\left(\frac{1}{(1+x^2)^2}\right) = \frac{x}{2(1+x^2)} + \frac{1}{2} \operatorname{arctg} x$.
$P\left(\frac{1}{(1+x^2)^3}\right) = \frac{x}{4(1+x^2)^2} + \frac{3x}{8(1+x^2)} + \frac{3}{8} \operatorname{arctg} x$.

15.

a) $\frac{1}{2}e^{2x} - \frac{1}{2}\log\left(e^{2x}+1\right)$, b) $\frac{3}{2}\operatorname{arctg}\sqrt[3]{x^2}$, c) $2\sqrt{x-1} - 2\operatorname{arctg}\sqrt{x-1}$,

d) $\frac{6}{7}x\sqrt[6]{x} - \frac{6}{5}\sqrt[6]{x^5} - \frac{3}{2}\sqrt[3]{x^2} + 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} - 3\log\left|1+\sqrt[3]{x}\right| + 6\operatorname{arctg}\sqrt[6]{x}$,

e) $\frac{1}{4}\log\left|\frac{e^x-1}{e^x+1}\right| - \frac{1}{2(1+e^x)}$, f) $-2\operatorname{arctg}\sqrt{1-x}$,

g) $\log|\cos x| + \log|\operatorname{tg}x+1|$, h) $\log|\log x - 1| - \frac{1}{\log x - 1}$,

i) $3\log(\sqrt[3]{x}+1)$,

16. a) Fazendo a substituição $\sqrt{x} = t \Leftrightarrow x = t^2$, com $x > 0$, $x \neq 16$, e $t > 0$, $t \neq 4$, temos

$$P\left(\frac{1+\sqrt{x}}{x(4-\sqrt{x})}\right) = P\left(\frac{1+t}{t^2(4-t)}\,2t\right) = 2P\left(\frac{1+t}{t(4-t)}\right).$$

Usando a decomposição em fracções simples:

$$\frac{2+2t}{t(4-t)} = \frac{A}{t} + \frac{B}{4-t}$$

temos $A = \frac{1}{2}$, $B = \frac{5}{2}$, logo

$$2P\left(\frac{1+t}{t(4-t)}\right) = \frac{1}{2}P\left(\frac{1}{t} + \frac{5}{4-t}\right) = \frac{1}{2}\log\left|\frac{t}{(4-t)^5}\right|$$

e assim,

$$P\left(\frac{1+\sqrt{x}}{x(4-\sqrt{x})}\right) = \frac{1}{2}\log\left|\frac{\sqrt{x}}{(4-\sqrt{x})^5}\right|.$$

b) Fazendo a substituição $\sqrt[4]{1+x} = t \Leftrightarrow x = t^4 - 1$, com $x > -1$ e $t > 0$, temos

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = P\left(\frac{1}{(t^4-1)t}\,4t^3\right) = P\left(\frac{4t^2}{t^4-1}\right).$$

Usando a decomposição em fracções simples:

$$\frac{4t^2}{t^4-1} = \frac{4t^2}{(t-1)(t+1)(t^2+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1},$$

temos $A = 1$, $B = -1$, $C = 0$, $D = 2$. Logo,

$$P\left(\frac{4t^2}{t^4-1}\right) = P\left(\frac{1}{t-1} - \frac{1}{t+1} + \frac{2}{t^2+1}\right) = \log\left|\frac{t-1}{t+1}\right| + 2\operatorname{arctg} t$$

e assim,

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = \log\left|\frac{\sqrt[4]{1+x}-1}{\sqrt[4]{1+x}+1}\right| + 2\operatorname{arctg}\sqrt[4]{1+x}.$$

c) Fazendo a substituição $e^{2x} = t \Leftrightarrow x = \frac{1}{2}\log t$, com $x \in \mathbb{R}$ e $t > 0$, temos

$$P\left(\frac{1}{1+e^{2x}}\right) = P\left(\frac{1}{1+t}\cdot\frac{1}{2t}\right).$$

Usando a decomposição em fracções simples:

$$\frac{1}{(1+t)2t} = \frac{A}{1+t} + \frac{B}{t}$$

temos $A = -\frac{1}{2}$, $B = \frac{1}{2}$, logo

$$P\left(\frac{1}{1+t}\cdot\frac{1}{2t}\right) = P\left(-\frac{1}{2(1+t)} + \frac{1}{2t}\right) = \frac{1}{2}\log\left|\frac{t}{1+t}\right|$$

e assim,

$$P\left(\frac{1}{1+e^{2x}}\right) = \frac{1}{2}\log\left|\frac{e^{2x}}{1+e^{2x}}\right|.$$

d) Fazendo a substituição $e^x = t \Leftrightarrow x = \log t$, com $x \in \mathbb{R}\setminus\{0\}$ e $t > 0$, $t \neq 1$, temos

$$P\left(\frac{e^{3x}}{(1+e^{2x})(e^x-1)^2}\right) = P\left(\frac{t^3}{(1+t^2)(t-1)^2}\,\frac{1}{t}\right) = P\left(\frac{t^2}{(1+t^2)(t-1)^2}\right).$$

Usando a decomposição em fracções simples:

$$\frac{t^2}{(1+t^2)(t-1)^2} = \frac{At+B}{1+t^2} + \frac{C}{t-1} + \frac{D}{(t-1)^2}$$

temos $A = -\frac{1}{2}$, $B = 0$, $C = D = \frac{1}{2}$, logo

$$\begin{aligned} P\left(\frac{t^2}{(1+t^2)(t-1)^2}\right) &= \frac{1}{2}P\left(-\frac{t}{1+t^2} + \frac{1}{t-1} + \frac{1}{(t-1)^2}\right) \\ &= -\frac{1}{4}\log(1+t^2) + \frac{1}{2}\log|t-1| - \frac{1}{2}\frac{1}{t-1} \end{aligned}$$

e assim

$$P\left(\frac{e^{3x}}{(1+e^{2x})(e^x-1)^2}\right) = -\frac{1}{4}\log(1+e^{2x}) + \frac{1}{2}\log|e^x-1| - \frac{1}{2}\frac{1}{e^x-1}.$$

e) Fazendo a substituição $\log x = t \Leftrightarrow x = e^t$, com $x \in \mathbb{R}^+\setminus\{1,e\}$ e $t \in \mathbb{R}\setminus\{0,1\}$, temos

$$P\left(\frac{2\log x - 1}{x\log x(\log x - 1)^2}\right) = P\left(\frac{2t-1}{e^t t(t-1)^2}\,e^t\right) = P\left(\frac{2t-1}{t(t-1)^2}\right).$$

Usando a decomposição em fracções simples:

$$\frac{2t-1}{t(t-1)^2} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2}$$

temos $A = -1$, $B = C = 1$, logo

$$P\left(\frac{2t-1}{t(t-1)^2}\right) = P\left(-\frac{1}{t} + \frac{1}{t-1} + \frac{1}{(t-1)^2}\right) = \log\left|\frac{t-1}{t}\right| - \frac{1}{t-1}$$

e assim

$$P\left(\frac{2\log x - 1}{x\log x(\log x - 1)^2}\right) = \log\left|\frac{\log x - 1}{\log x}\right| - \frac{1}{\log x - 1}.$$

f) Fazendo a substituição $\operatorname{sen} x = t \Leftrightarrow x = \operatorname{arcsen} t$, obtem-se (verifique)

$$P\left(\frac{1}{\operatorname{sen}^2 x\cos x}\right) = -\frac{1}{\operatorname{sen} x} + \frac{1}{2}\log\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right|.$$

17.

a) $\frac{1}{2}\log\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right|$, b) $\sqrt{1-\frac{1}{x^2}}$, c) $\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\operatorname{arcsen} x$,

d) $\log\left|1+\operatorname{tg}\frac{x}{2}\right|$, e) $-\frac{1}{3}\left(\frac{1}{x^2}-1\right)^{3/2}$, f) $-2\operatorname{arcsen}\sqrt{1-e^x}$,

g) $-x+\operatorname{tg} x+\sec x$, h) $2\operatorname{arcsen}\sqrt{x}$, i) $\log|\frac{1+2\operatorname{sen} x}{1-\operatorname{sen} x}|$,

j) $\frac{1}{4}\log\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right| + \frac{1}{4(1-\operatorname{sen} x)} - \frac{1}{4(1+\operatorname{sen} x)} = \frac{1}{2}\log\left|\frac{1+\operatorname{sen} x}{\cos x}\right| + \frac{\operatorname{sen} x}{2\cos^2 x}$

$= \frac{1}{2}\log\left|\sec x+\operatorname{tg} x\right| + \frac{1}{2}\sec x\operatorname{tg} x$, k) $\log|x+\sqrt{x^2+1}|$,

l) $\log\left|\frac{\operatorname{sen} x}{1+\operatorname{sen} x}\right|$, m) $\log\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right|$, n) $\log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|$,

o) $2\log\left|\sqrt{1+\left(\frac{x}{2}\right)^2}+\frac{x}{2}\right| + x\sqrt{1+\left(\frac{x}{2}\right)^2}$,

p) $\frac{\sqrt{x^2-1}}{2}(x-2) + \frac{1}{2}\log\left|x+\sqrt{x^2+1}\right|$.

18. a) $f(x) = \frac{1}{2}\operatorname{arctg}^2 x + c$, com $c \in \mathbb{R}$; $\lim_{x\to+\infty} f(x) = \frac{\pi^2}{8} + c$, logo $c = -\frac{\pi^2}{8}$.

b) $g(x) = \frac{1}{2}\log|\frac{\sqrt{x}}{(4-\sqrt{x})^5}| + c$, para $x > 16$ (Ex. 16.a)); $\lim_{x\to+\infty} g(x) = +\infty$, logo não existe g nas condições do enunciado.

19. (ver Ex. 16.c))

20. a) $\frac{1}{2}x|x|$,

b) $\frac{x^2}{2} \operatorname{arcsen} \frac{1}{x} + \frac{1}{2}x\sqrt{1 - \frac{1}{x^2}}$, (por partes, por ex.)

c) $\frac{x}{2}\operatorname{sen}(\log x + 1) - \frac{x}{2}\cos(\log x + 1)$, (por partes, por ex.)

d) $\frac{x}{8} - \frac{1}{32}\operatorname{sen} 4x$,

e) $\frac{2}{3}x^{3/2} \operatorname{arctg} \sqrt{x} - \frac{1}{3}x + \frac{1}{3}\log(1 + x)$, (por partes, por ex.)

f) $-\log x + 2\log|1 + \log x| + \frac{\log^2 x}{2}$, (substituição $t = \log x$, por ex.)

g) $\frac{x}{2} - \frac{1}{2}e^{-x} - \frac{1}{4}\log(e^{2x} - 2e^x + 2)$, (substituição $t = e^x$, por ex.)

h) $\frac{2}{3}\sqrt{x^3} - x + 4\sqrt{x} - 4\log(\sqrt{x} + 1)$, (substituição $t = \sqrt{x}$, por ex.)

i) $\operatorname{sen} x - \frac{1}{3}\operatorname{sen}^3 x$,

j) $\frac{3}{8}x + \frac{1}{4}\operatorname{sen} 2x + \frac{1}{8}\operatorname{sen} 4x$,

k) $\frac{1}{2}(x^2 - 1)\log\left|\frac{1-x}{1+x}\right| - x$,

l) $\frac{1}{2}\log\left|\frac{(x-1)(x+3)}{(x+2)^2}\right|$,

m) $\frac{1}{2}\log^2(\log x)$,

n) $x\log(x + \sqrt{x}) - x + \sqrt{x} - \log(1 + \sqrt{x})$, (substituição $t = \sqrt{x}$ e por partes, por ex.)

o) $-\left(\frac{1}{x} + 1\right)e^{\frac{1}{x}}$, (por partes, por ex.)

p) $\operatorname{sen} x \log(1 + \operatorname{sen}^2 x) - 2\operatorname{sen} x + 2\operatorname{arctg}(\operatorname{sen} x)$,

q) $\log x \log(\log x) - \log x$,

r) $\frac{x^2+1}{2}\operatorname{arctg}^2 x - x\operatorname{arctg} x + \frac{1}{2}\log(1 + x^2)$,

s) $2\sqrt{1 + x}(\log(1 + x) - 2)$,

t) $\log\left|\frac{\operatorname{sen} x}{\cos x+1}\right|$,

u) $-\frac{x}{\operatorname{sen} x} + \log\left|\frac{\operatorname{sen} x}{\cos x+1}\right|$,

v) $-\frac{\sqrt{3}}{3}\operatorname{arctg}\left(\sqrt{3}\cos x\right)$,

w) $-\frac{1}{2}\log^2(\cos x)$,

x) $\log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right|$ (substituição $t = \sqrt{x + 2}$, por ex.),

y) $x(\operatorname{arcsen} x)^2 + 2\sqrt{1 - x^2}\operatorname{arcsen} x - 2x$ (por partes, por ex.),

z) $\frac{1}{4}\log\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right| + \frac{1}{2(1-\operatorname{sen} x)}$ (substituição $t = \operatorname{sen} x$, por ex.).

21. $\log(1 + e^{-x}) + \frac{\pi}{2}$.