Optimization and Algorithms
November 25, 2021

Write your name:	____________________
Write your student number:	____________________

Mock exam

1. *Non strongly convex function.* (3 points) One of the following functions $f\colon \mathbf{R}^2 \to \mathbf{R}$ is **<u>not</u>** strongly convex:

(A) $f(x_1, x_2) = |x_1 + x_2| + x_1^2 + (x_1 - x_2)^2$
(B) $f(x_1, x_2) = 4x_1^2 + e^{x_1 + x_2} + 4x_1 x_2 + x_2^2$
(C) $f(x_1, x_2) = (x_1 + x_2)^2 + |x_1| + (x_1 - x_2)^2$
(D) $f(x_1, x_2) = e^{x_1 - x_2} + 4x_1^2 + 3x_1 - 2x_2 - 2x_1 x_2 + x_2^2$
(E) $f(x_1, x_2) = -3x_1 x_2 + (x_1 + 2x_2)^2 + (x_1 - x_2)_+$
(F) $f(x_1, x_2) = x_1 + x_1^2 - x_2 + x_2^2 + \log(1 + e^{x_1 + x_2})$

Which one?

Write your answer (A, B, C, D, E, or F) here: __________

2. *True statement about convexity.* (2 points) One of the following statements is true:

(A) if $f\colon \mathbf{R}^n \to \mathbf{R}$ is convex, then f has at least one global minimizer
(B) if $f_1\colon \mathbf{R}^n \to \mathbf{R}$ and $f_2\colon \mathbf{R} \to \mathbf{R}$ are both convex functions, then $f_2 \circ f_1$ is convex
(C) if $f\colon \mathbf{R}^n \to \mathbf{R}$ is strictly convex, then f has exacly one global minimizer
(D) if $f_1\colon \mathbf{R} \to \mathbf{R}$ is strongly convex , $f_2\colon \mathbf{R}^n \to \mathbf{R}$ is convex, and $f_2(x) \geq f_1(x)$ for each $x \in \mathbf{R}^n$, then f_2 is strongly convex
(E) if $f\colon \mathbf{R}^n \to \mathbf{R}$ is strictly convex, then f has at most one global minimizer
(F) if $f\colon \mathbf{R}^n \to \mathbf{R}$ is convex, then f^2 is strongly convex

Which one?

Write your answer (A, B, C, D, E, or F) here: __________

3. *Augmented Lagrangian method.* (3 points) Consider the constrained problem

$$\begin{array}{ll} \underset{x \in \mathbf{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & h(x) = 0, \end{array} \tag{1}$$

where $f\colon \mathbf{R}^n \to \mathbf{R}$ and $h\colon \mathbf{R}^n \to \mathbf{R}$ are differentiable functions.

The augmented Lagrangian method applied to (1) solves, at each iteration, an optimization problem of one of the following forms:

(A)

$$\begin{array}{ll} \underset{x \in \mathbf{R}^n}{\text{minimize}} & f(x) + \lambda h(x) + c h(x)^2 \ , \\ \text{subject to} & h(x) = 0 \end{array}$$

where $\lambda \in \mathbf{R}$ and $c > 0$

(B)

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad f(x) + \lambda h(x) + c h(x)^2,$$

where $\lambda \in \mathbf{R}$ and $c > 0$

(C)

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad f(x) + \lambda |h(x)| + c h(x)^2,$$

where $\lambda \in \mathbf{R}$ and $c > 0$

(D)

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad f(x) + \lambda h(x)^2,$$

where $\lambda \in \mathbf{R}$

(E)

$$\underset{x \in \mathbf{R}^n}{\text{minimize}} \quad f(x) + c|h(x)|,$$

where $c > 0$

(F)

$$\begin{array}{ll} \underset{x \in \mathbf{R}^n}{\text{minimize}} & f(x) + \lambda |h(x)| + c h(x)^2 \ , \\ \text{subject to} & h(x) = 0 \end{array}$$

where $\lambda \in \mathbf{R}$ and $c > 0$

Which one?

Write your answer (A, B, C, D, E, or F) here: __________

4. *Existence of global minimizers.* (4 points) Consider the optimization problem

$$\begin{array}{ll} \underset{c,R}{\text{minimize}} & \sum_{m=1}^{M} \omega_m \left((\|c - x_m\|_2 - R)_+ \right)^2 + \rho R^2 \\ \text{subject to} & R \geq 0, \end{array} \tag{2}$$

where the variables to optimize are $c \in \mathbf{R}^n$ and $R \in \mathbf{R}$. The vectors x_m and the scalars ω_m are given for $1 \leq m \leq M$, with $\omega_m > 0$ for all m. The scalar ρ is also given and denotes a positive constant: $\rho > 0$.

Show that (2) has at least one global minimizer.

5. *Smooth control of an uncertain system.* (4 points) Consider the optimization problem

$$\begin{array}{ll} \underset{u_1,\ldots,u_K}{\text{minimize}} & \underbrace{\sum_{k=1}^{K} \max\{|a_k^T u_k - b_k|, |c_k^T u_k - d_k|\}}_{f(u_1,\ldots,u_K)} \\ \text{subject to} & \|u_{k+1} - u_k\|_2 \leq U, \quad k = 1, \ldots, K-1, \end{array} \tag{3}$$

where the variable to optimize is $u_1, \ldots, u_K$, with $u_k \in \mathbf{R}^d$ for $1 \leq k \leq K$. The vectors $a_k \in \mathbf{R}^d$ and $c_k \in \mathbf{R}^d$ and the scalars $b_k \in \mathbf{R}$ and $d_k \in \mathbf{R}$ are given for $1 \leq k \leq K$. Also, the scalar U is given and denotes a positive constant: $U > 0$.

Show that (3) is a convex optimization problem.

6. *Penalty method.* (4 points) Consider the optimization problem

$$\begin{array}{ll} \underset{x \in \mathbf{R}^2}{\text{minimize}} & f(x) \\ \text{subject to} & s^T x = r, \end{array} \tag{4}$$

where the vector $s \in \mathbf{R}^2$ ($s \neq 0$) and the scalar r are given. Assume that the function f is differentiable and strongly convex. Let $x^\star \in \mathbf{R}^2$ be the global minimizer of (4).

Consider now the penalized problem

$$\underset{x \in \mathbf{R}^2}{\text{minimize}} \quad f(x) + \frac{c_k}{2}(s^T x - r)^2, \tag{5}$$

where $c_k > 0$. Let $x_k^\star \in \mathbf{R}^2$ be the global minimizer of (5).

Assume that $(c_k)_{k \geq 1}$ is an increasing sequence converging to $+\infty$; that is, $0 < c_1 < c_2 < c_3 < \cdots$ and $\lim_{k \to +\infty} c_k = +\infty$. Also, assume that the sequence $(x_k^\star)_{k \geq 1}$ converges to some vector $\overline{x}$, that is, $\lim_{k \to +\infty} x_k^\star = \overline{x}$.

Show that $\overline{x} = x^\star$.

(You cannot invoke theorems about penalty methods; you must prove the equality above by yourself.)