AM II

LEAmb, LEB, LEC, LEQ, LET e LQ

4ª ficha de exercícios - soluções

1/ Como u e v são contínuas em $\mathbb{R}$, pelo Teo. fund. do Calc. Integral:

$$\underbrace{\int_a^x u(\tau)\,d\tau = \int_b^x v(\tau)\,d\tau}_{(*)} \Rightarrow \underbrace{\left(\int_a^x u(\tau)\,d\tau\right)'}_{=u(x)} = \underbrace{\left(\int_b^x v(\tau)\,d\tau\right)'}_{=v(x)}$$

Logo $u(x) = v(x)\ \forall x \in \mathbb{R}$, isto é, $u = v$.

Além disso, $\int_a^b u(x)\,dx = \underbrace{\int_b^b v(\tau)\,d\tau}_{=0}$, tendo em conta $(*)$.

Logo $\int_a^b u(x)\,dx = 0$

2/

$$\lim_{x\to+\infty}\left(1+\int_0^{1/x} f(\tau)\,d\tau\right)^x \underset{(1^\infty)}{=} \lim_{x\to+\infty} e^{\log\left(1+\int_0^{1/x} f(\tau)\,d\tau\right)^x} =$$

$$= \lim_{x\to+\infty} e^{x\log\left(1+\int_0^{1/x} f(\tau)\,d\tau\right)}.$$ Vamos calcular à parte:

$$\lim_{x\to+\infty} x\log\left(1+\int_0^{1/x} f(\tau)\right)d\tau \underset{(\infty\cdot 0)}{=} \lim_{x\to+\infty}\frac{\log\left(1+\int_0^{1/x} f(\tau)\,d\tau\right)}{\frac{1}{x}}$$

$$= \lim_{x\to+\infty}\frac{\frac{f(\frac{1}{x})\cancel{\left(-\frac{1}{x^2}\right)}}{1+\int_0^{1/x} f(\tau)\,d\tau}}{\cancel{-\frac{1}{x^2}}} = \lim_{x\to+\infty}\frac{f(\frac{1}{x})}{1+\int_0^{1/x} f(\tau)\,d\tau} = f(0)$$

Regra de Cauchy & Teo. fund. do Calc. Integral:

$$\left(\int_0^{1/x} f(\tau)\,d\tau\right)' = f\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)$$

(f é contínua)

f é contínua no ponto 0

Logo, $\lim_{x\to+\infty}\left(1+\int_0^{1/x} f(\tau)\,d\tau\right)^x =$

3) a) $\int_0^x t\,dt = \left[\frac{t^2}{2}\right]_0^x = \frac{x^2}{2}$ b) $\int_x^{x+1} (3 \operatorname{sen} t + 2t^5)\,dt =$

$= \left[-3\cos t + \frac{t^6}{3}\right]_x^{x+1} = -3\cos(x+1) + \frac{(x+1)^6}{3} + 3\cos x - \frac{x^6}{3}$

4) $\int_0^1 x^m (1-x)^n\,dx = \int_1^0 (1-t)^m t^n (-1)\,dt = -\int_0^1 (1-t)^m t^n (-$

$= \int_0^1 (1-x)^m x^n\,dx$

$1 - x = t \iff x = 1 - t \iff \varphi(t) = 1 - t \ ; \ \varphi'(t) =$

$x = 0 \Rightarrow t = 1$

$x = 1 \Rightarrow t = 0$

$m, n \in \mathbb{N}$

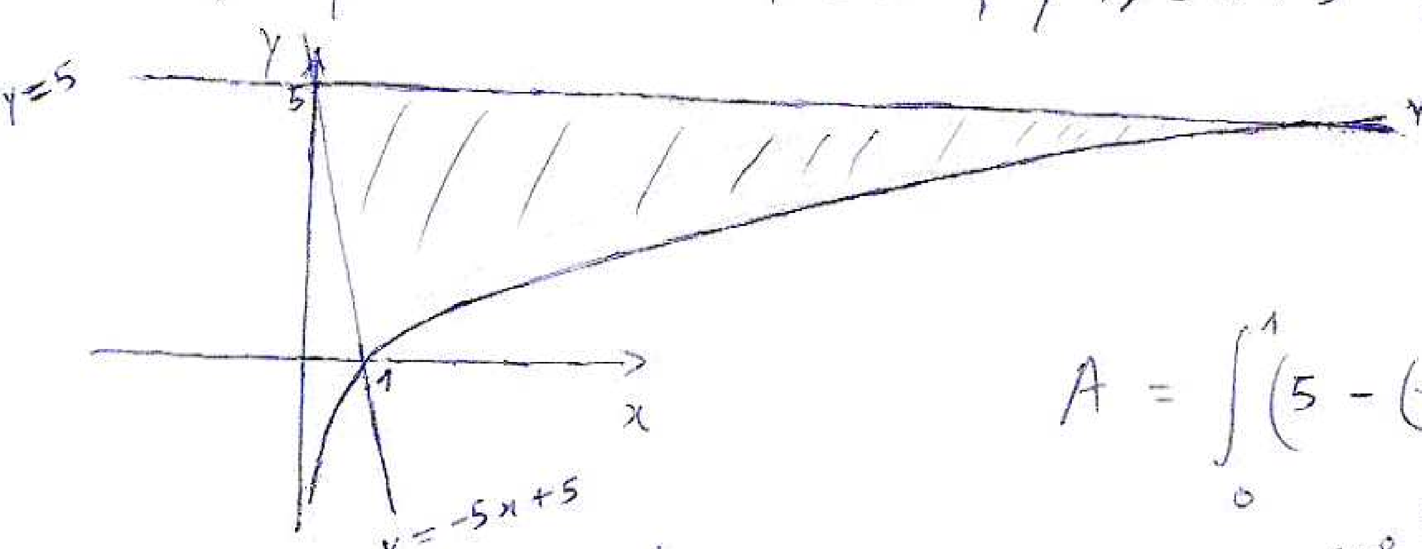
5) $\int_1^x \frac{e^t}{t}\,dt = \int_e^{e^x} \frac{s}{\log s} \cdot \frac{1}{s}\,ds = \int_e^{e^x} \frac{1}{\log s}\,ds$

$e^t = s \iff t = \log s \iff \varphi(s) = \log s \ ; \ \varphi'(s) = \frac{1}{s}$

$t = 1 \Rightarrow s = e$

$t = x \Rightarrow s = e^x$

6) a) $\{(x,y) \in \mathbb{R}^2 : y \leq 5 \ , \ y \geq -5x + 5 \ , \ y \geq \log x\}$



$A = \int_0^1 \left(5 - (-5x + 5)\right)dx +$

$\log x = 5$

$x = e^5$

$+ \int_1^{e^5} (5 - \log x)\,dx =$

$= \left[\frac{5x^2}{2}\right]_0^1 + [5x]_1^{e^5} - \int_1^{e^5} \log x\,dx = \frac{5}{2} + 5e^5 - 5 - \left(\left[x \log x\right]_1^{e^5} - \int_1^{}\right.$

$= -\frac{5}{2} + 5\cancel{e^5} - 5\cancel{e^5} + [x]_1^{e^5} = e^5 - \frac{7}{2}$

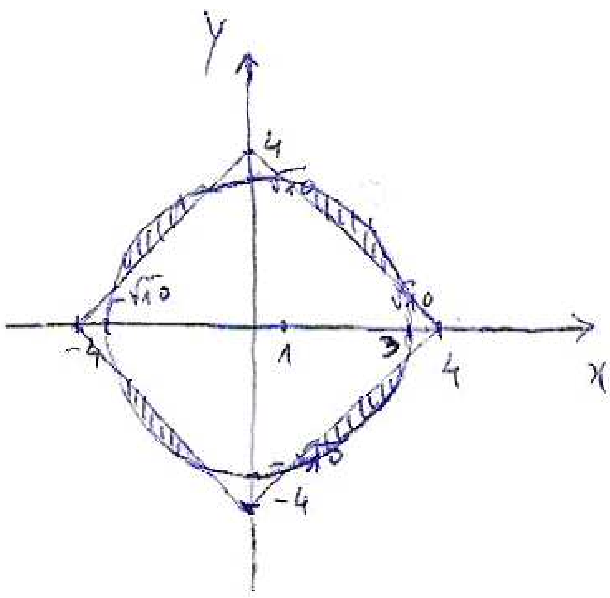
b/ $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 10 \ , \ |x|+|y| \geq 4\}$

$x, y \geq 0 \Rightarrow x + y \geq 4$; $x, y < 0 \Rightarrow -x-y \leq$

$y \geq 4 - x$; $y \geq -4$

$x \geq 0 \wedge y < 0 \Rightarrow x - y \geq 4$; $x < 0 \wedge y \geq 0 \Rightarrow -x+y$

$y \leq x - 4$; $y \geq x$



$$A = 4\int_1^3 \left(\sqrt{10-x^2} - (4-x)\right) dx =$$

$$= 4\int_1^3 \sqrt{10-x^2}\, dx -$$

$y = \sqrt{10-x^2} = 4 - x$

$x = 1 \Rightarrow y = 3$

$-\left[16x - 2x^2\right.$

$x = \sqrt{10}\ \text{sen}\ t = \varphi(t)$

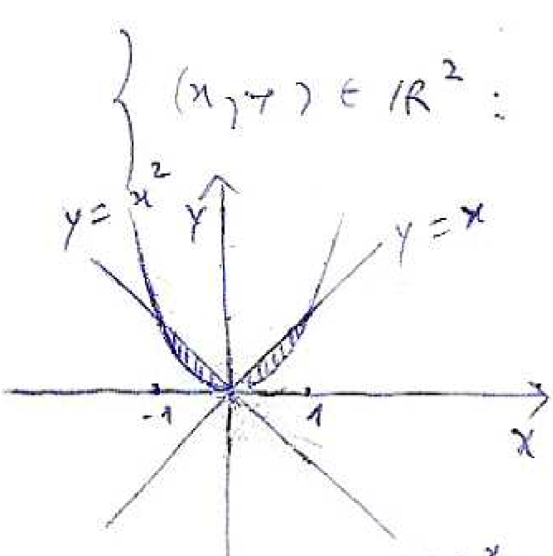
$\varphi'(t) = \sqrt{10} \cos t$ $\qquad t = \text{arcsen} \frac{x}{\sqrt{10}}$

$$= 4\int_{\text{arcsen}\frac{1}{\sqrt{10}}}^{\text{arcsen}\frac{3}{\sqrt{10}}} \underbrace{\sqrt{10}\cos t}_{f(\varphi(t))}\ \underbrace{\sqrt{10}\cos t}_{\varphi'(t)}\ dt - 16\sqrt{10} + 16 + 20 - 2 =$$

$$= 40\left[\frac{t}{2} + \frac{\text{sen}\,2t}{4}\right]_{\text{arcsen}\frac{1}{\sqrt{10}}}^{\text{arcsen}\frac{3}{\sqrt{10}}} - 16\sqrt{10} + 34 =$$

$$= 20\left(\text{arcsen}\frac{3}{\sqrt{10}} - \text{arcsen}\frac{1}{\sqrt{10}}\right) + 20\left(\frac{3}{\sqrt{10}}\sqrt{1-\frac{9}{10}} - \frac{1}{\sqrt{10}}\sqrt{1-\frac{1}{10}}\right) - 16\sqrt{10} + 34$$

c/ $\{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq |x|\}$



$x \geq 0$:

$y \leq x$

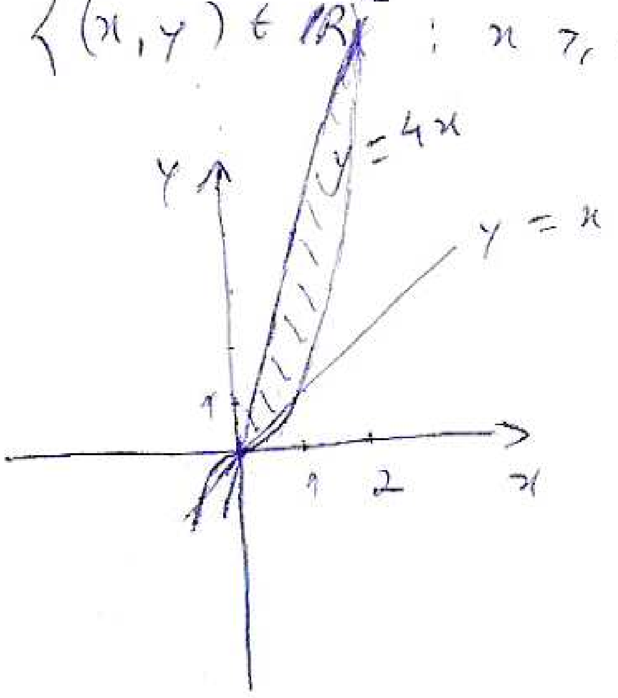
$x < 0$:

$y \leq -x$

$$A = 2\int_0^1 (x - x^2)\, dx =$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

d/ $\left\{(x,y)\in\mathbb{R}^2 : x\geq 0 \ , \ y \geq x \ , \ y\geq x^3, \ \ y\leq 4x\right\}$



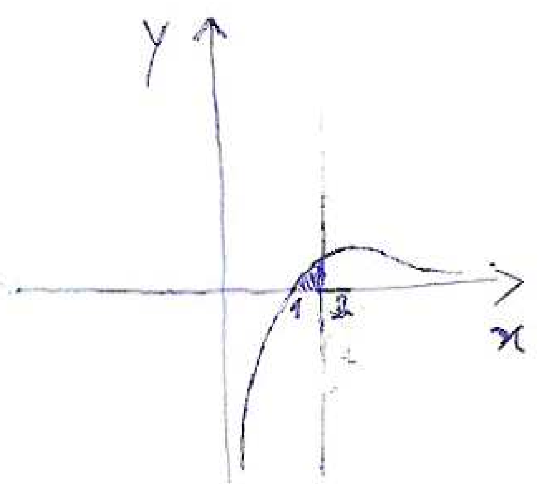
$$4x = x^3$$

$$x = 2$$

$$A = \int_0^1 (4x - x)\,dx + \int_1^2 (4x - x^3)\,dx =$$

$$= \left[2x^2 - \frac{x^2}{2}\right]_0^1 + \left[2x^2 - \frac{x^4}{4}\right]_1^2 = \frac{3}{2} + 4 - \frac{7}{4} = \frac{15}{4}$$

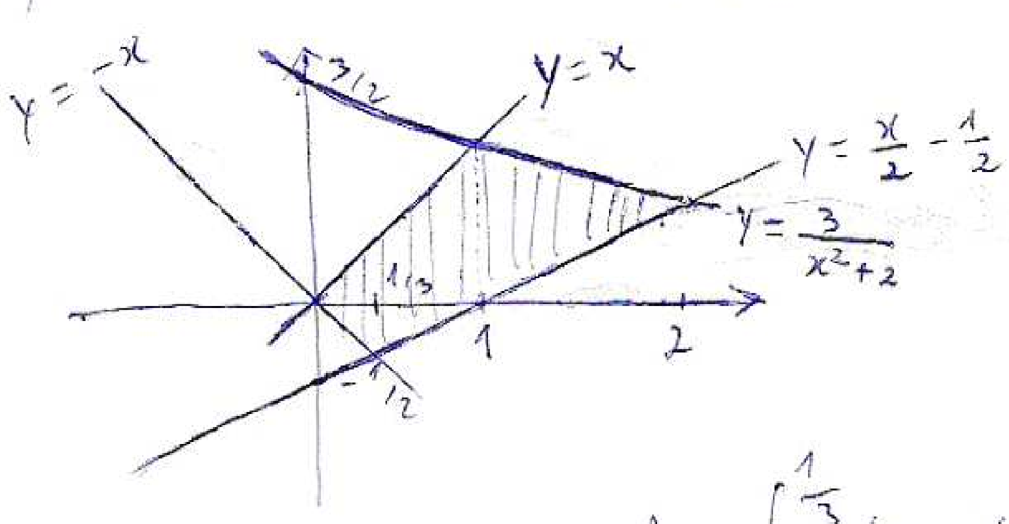
7/ a/ $y = 0 \ , \ y = \frac{\log x}{\sqrt{x}} \ ; \ x = e$



$$A = \int_1^e \frac{\log x}{\sqrt{x}}\,dx = \left[2\sqrt{x}\log x\right]_1^e - \int_1^e 2\sqrt{x}\,\frac{1}{x}$$

$$= 2\sqrt{e} - \left[4\sqrt{x}\right]_1^e = -2\sqrt{e} + 4$$

b/ $|x| = y \ , \ y = \frac{3}{x^2+2} \ , \ y = \frac{x}{2} - \frac{1}{2}$



$$\frac{3}{x^2+2} = \frac{x}{2} - \frac{1}{2}$$

$$x = 2$$

$$\frac{x}{2} - \frac{1}{2} = -x$$

$$A = \int_0^{1/3} (x - (-x))\,dx + \int_{1/3}^1 \left(x - \left(\frac{x}{2} - \frac{1}{2}\right)\right)dx +$$

$$+ \int_1^2 \left(\frac{3}{x^2+2} - \left(\frac{x}{2} - \frac{1}{2}\right)\right)dx = \left[x^2\right]_0^{1/3} + \left[\frac{x^2}{4} + \frac{x}{2}\right]_{1/3}^1 + \left[\frac{3\sqrt{2}}{2}\operatorname{arctg} x - \frac{x^2}{4}\right.$$

$$= \frac{1}{9} + \frac{3}{4} - \frac{1}{36} - \frac{1}{6} \cancel{-1} \cancel{+1} + \frac{3\sqrt{2}}{2}\operatorname{arctg} 2 - \frac{3\sqrt{2}}{2}\frac{\pi}{4} + \frac{1}{4}$$

$$= \frac{5}{12} + \frac{3\sqrt{2}}{2}\left(\operatorname{arctg} 2 - \frac{\pi}{4}\right)$$

10/ a/ $f_m(x) = e^{-mx}$; Em $[0,1]$: $f_m(x) \to f(x) = \begin{cases} 0 & \text{se } x \in]0 \\ 1 & \text{se } x = 0 \end{cases}$

Logo $f_m \to f$ pontualmente em $[0,1]$ mas não uniformement
pois f_m é contínua $\forall m$ mas f não é contínua em $[0,1]$

Em $[1,2]$: $f_m(x) \to 0$ $\forall x \in [1,2]$. Logo $f_m \to 0$ pontual/ em

$\sup_{x \in [1,2]} |f_m(x) - f(x)| = \sup_{x \in [1,2]} e^{-mx} = e^{-m} \to 0$. Logo $f_m \to 0$ uniforme/ em [

b/ $f_m(x) = \frac{mx}{1+m^2x^2}$; Em $[0,1]$: $f_m \to 0$ pontual/ em [

No entanto $\sup_{x \in [0,1]} |f_m(x) - f(x)| \not\to 0$ pois para $x_m = \frac{1}{m} \in [0,1$

$f_m(x_m) = \frac{1}{1+1} = \frac{1}{2} \not\to 0$. Logo (f_m) não converge uniforme/ para
em [

Em $[1,2]$: $f_m \to 0$ pontual/ em $[1,2]$.

$\sup_{x \in [1,2]} |f_m(x) - f(x)| = \sup_{x \in [1,2]} \frac{mx}{1+m^2x^2} \leq \frac{2m}{1+m^2} \to 0$. Logo $f_m \to 0$
forme/ em [

c/ $f_m(x) = mx\,e^{-mx}$ em IR, $f_m \to 0$ pontual/ em IR.

No entanto para $x_m = \frac{1}{m}$, $f_m(x_m) = \frac{1}{e} \not\to 0$. Logo (f_m) não con
uniforme/ para 0 em IR.

d/ $f_m(x) = \frac{mx}{1+mx} = 1 - \frac{1}{1+mx} \to f(x) = \begin{cases} 0 & , x = 0 \\ 1 & , x \neq 0 \end{cases}$

Como f_m é contínua $\forall m$ e f não é contínua em IR, $f_m \to$
pontual/ em IR mas (f_m) não converge uniforme/ para f em IR.

e/ $f_m(x) = x + \frac{x}{m} \to x$ em IR (pontual/). No entanto, para
$x_m = m \in IR$, $f_m(x_m) - f(x_m) = 1 \not\to 0$, logo $\sup_{x \in IR} |f_m(x) - f(x)| \not\to$
pelo que (f_m) não converge uniforme/ para $f(x) = x$ em IR.

f/ $f_m(x) = \begin{cases} 1 - mx & \text{se } 0 \leq x \leq \frac{1}{m} \\ 0 & \text{se } \frac{1}{m} \leq x \leq 1 \end{cases}$

Em $[0,1]$:

$f_m(x) \to f(x) = \begin{cases} 1 & x = \\ 0 & x \end{cases}$

Como f_m é contínua $\forall m$ e f não é contínua em $[0,1]$,

$f_m \to f$ pontual/ em $[0,1]$ mas (f_m) não converge uniforme/ para
em [

Em $]0,1]$: $f_m \to 0$ pontual/, no entanto, para $x_m = \frac{1}{m^2} \in]0,$

$f_m(x_m) = 1 - \frac{1}{m} \to 1 \neq 0$ Logo (f_m) não converge uniforme/ para 0 em
$]0,$

Em $[a,1]$ com $a > 0$: $f_m \to 0$ pontual/ e $f_m \to 0$ uniforme/ p

$\sup_{x \in [a,1]} |f_m(x) - f(x)| = \sup 0 = 0$ (a partir de uma certa ordem)

g/ $f_m(x) = \frac{x}{1+mx^3} \to 0$ pontual/ em $\mathbb{R}$ e uniforme/ pois:

e $\sup_{x \in \mathbb{R}} |f_m(x)| = \frac{\frac{1}{\sqrt[3]{2m}}}{1 + \frac{1}{2}} \to 0$ (Estuda-se o sinal de:

$f'_m(x) = \frac{1 - 2mx^3}{(1+mx^3)^2}$ e o

seguir a monotonia de f_m

h/ $f_m(x) = \int_0^x \frac{T}{1+mT^3}\, dT$. Por g), $\frac{T}{1+mT^3} \to 0$ pontual e uni
me/ em [

Logo $f_m \to \int_0^x 0 = 0$ pontual/ em $[0,a]$, $(a>0)$.

$\sup_{x \in [0,a]} |f_m(x) - f(x)| = \int_0^a \frac{T}{1+mT^3}\, dT \xrightarrow[\text{por } g)]{} \int_0^a 0 = 0$. Logo $f_m \to 0$ uni
me/ em $[0,a$

i/ $f_m(x) = e^{-mx^2} \to f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$ pontual/ em $\mathbb{R}$. No entanto, co

f_m é contínua $\forall m$ e f não é contínua em $\mathbb{R}$, (f_m) não converge uni
me/ para f em $\mathbb{R}$.

j) $f_m(x) = x e^{-mx^2} \longrightarrow 0$ pontual/ em $\mathbb{R}$.

$\sup_{x \in \mathbb{R}} |f_m(x) - f(x)| = \frac{1}{\sqrt{2m}} e^{-\frac{1}{2}} \longrightarrow 0$. Logo $f_m \to 0$ uniforme/ e

(Estuda-se o sinal de $f_m'(x) = e^{-mx^2}(1-2mx^2)$ e a seguir a monotonia de f_m)

k) $f_m(x) = \frac{m e^x}{m e^x + 1}$; Em $\mathbb{R}$: $f_m \longrightarrow 1$ pontual/ em $\mathbb{R}$.

entanto para $x_m = -m \in \mathbb{R}$, $f(x_m) = \frac{m e^{-m}}{m e^{-m} + 1} \longrightarrow 0$.

Logo $\sup_{x \in \mathbb{R}} |f_m(x) - \underbrace{f(x)}_{1}| \not\longrightarrow 0$, pelo que (f_m) não converge uniform para 1 em

Em $]-\infty; a]$ $(a>0)$ a situação é idêntica à anterior.

Em $[a, +\infty[$, $\sup_{x \in [a,+\infty[} \left| \frac{m e^x}{m e^x + 1} - 1 \right| = \sup_{x \in [a,+\infty[} \frac{1}{1 + m e^x} = \frac{1}{1 + m e^a}$

Logo (f_m) converge uniforme/ para 1 em $[a, +\infty[$ $(a>0)$ (e também pontual/.)

l) $f_m(x) = \frac{x^m}{1 + x^{2m}}$; Em $[0, +\infty[$: $f_m(x) \longrightarrow f(x) = \begin{cases} 0 & , x \ldots \\ & 0 \leq x \ldots \\ 1 & , x = 1 \end{cases}$

$f_m \to f$ pontual/ em $[0, +\infty[$ mas não uniforme/

pois f_m é contínua $\forall m$ e f não é contínua em $[0; +\infty[$.

Em: $[0, a]$ $(0 \leq a < 1)$: $f_m \longrightarrow 0$ pontual/ e $\sup_{x \in [0,a]} |f_m(x) - \underbrace{f(x)}_{0}| \leq a \ldots$

Logo $f_m \longrightarrow 0$ uniforme/ em $[0, a]$.

Em $[b, +\infty[$ $(b>1)$: $f_m \longrightarrow 0$ pontual/.

$\sup_{x \in [b,+\infty[} |f_m(x) - f(x)| \leq \sup_{x \in [b,+\infty[} \left| \frac{x^m}{x^{2m}} \right| = \sup_{x \in [b,+\infty[} \left| \frac{1}{x} \right|^m = \frac{1}{b^m} \longrightarrow 0$

Logo $f_m \longrightarrow 0$ uniforme/ em $[b; +\infty[$

Em: $\left[\frac{1}{2} ; \frac{3}{2} \right]$ a situação é idêntica à verificada em $[0; + \ldots$

18/ a/ $\sum_{n=0}^{+\infty} \frac{x^{\frac{n}{2}}}{\frac{n}{2}+1}$, $\sum_{n=0}^{+\infty} x^{\frac{n}{2}} = \sum_{n=0} (\sqrt{x})^n = \frac{1}{1-\sqrt{x}}$

$\forall x \in [0,1[$

Logo, $\sum_{n=0}^{+\infty} \frac{x^{\frac{n}{2}+1}}{\frac{n}{2}+1} + c = P\, \frac{1}{1-\sqrt{x}}$

$\sqrt{x} = t \Leftrightarrow x = t^2 = \varphi(t)$; $\varphi'(t) = 2t$

$P(f\circ\varphi)(t)\,\varphi'(t) = P\,\frac{1}{1-t}\cdot 2t = 2P\,\frac{t-1+1}{1-t} = 2P\left(-1+\frac{1}{1-t}\right)$

$= -2t - \log|1-t|$

Logo, $P\,\frac{1}{1-\sqrt{x}} = -2\sqrt{x} - \log|1-\sqrt{x}|$ e assim:

$\sum_{n=0}^{+\infty} \frac{x^{\frac{n}{2}+1}}{\frac{n}{2}+1} + c = -2\sqrt{x} - \log|1-\sqrt{x}|$

$x=0 \Rightarrow c=0$. Logo, $\sum_{n=0}^{+\infty} \frac{x^{\frac{n}{2}}}{\frac{n}{2}+1} = \begin{cases} \frac{1}{x}\left(-2\sqrt{x} - \log|1-\sqrt{x}|\right) & 0<x< \\ 0 & , x=0 \end{cases}$

Logo a série converge pontual/ mas não uniforme / em $[0,1[$ pois a sua soma não é contínua.

b/ $\sum_{n=0}^{+\infty} \frac{1}{n+1} e^{-nx}$; $\sum_{n=0}^{+\infty} e^{-nx} = \sum_{n=0}^{+\infty} (e^{-x})^n = \frac{1}{1-e^{-x}}$, $\forall x \in$

Logo, $\sum_{n=0}^{+\infty} \frac{e^{-(n+1)x}}{n+1} = \sum_{n=0}^{+\infty} P\left(-e^{-(n+1)x}\right)$ $(e^{-x}<1)$

$= P\left(-\sum_{n=0}^{+\infty} (e^{-x})^{n+1}\right) = P\, -\frac{e^{-x}}{1-e^{-x}} = -\log|1-e^{-x}| + c$

Logo, $\sum_{n=0}^{+\infty} \frac{e^{-nx}}{n+1} = e^{x}\left(-\log|1-e^{-x}| + c\right)$, $\forall x \in]0;+\infty[$

c/ $\sum_{n=0}^{+\infty} \frac{n+2}{2^{n+1}} x^n$; $\sum_{n=0}^{+\infty} \left(\frac{x}{2}\right)^{n+2} = \frac{\left(\frac{x}{2}\right)^2}{1-\frac{x}{2}}$, $\forall x \in]-2;2[$

$\left|\frac{x}{2}\right| < 1$

$$\sum_{n=0}^{+\infty} (n+2)\frac{1}{2}\left(\frac{x}{2}\right)^{n+1} = \sum_{n=0}^{+\infty} \left(\left(\frac{x}{2}\right)^{n+2}\right)' = \left(\sum_{n=0}^{+\infty}\left(\frac{x}{2}\right)^{n+2}\right)' = \left(\frac{x^2}{4-2x}\right)' =$$

$$= \frac{2x(4-2x)-x^2(-2)}{(4-2x)^2} = \frac{8x-4x^2+2x^2}{(4-2x)^2} = \frac{8x-2x^2}{(4-2x)^2}$$

Logo, $\sum_{n=0}^{+\infty} \frac{n+2}{2^{n+1}} x^n = \begin{cases} \frac{2}{\cancel{x}} \frac{8\cancel{x}-2x^{\cancel{2}}}{(4-2x)^2} & , x \in]-2;2[\setminus \{0\} \\ 1 & , x = 0 \end{cases} =$

$$= \frac{16-4x}{(4-2x)^2} \quad , \forall x \in]-2;2[$$

d/ $\sum_{n=0}^{+\infty} (-1)^{n+1}(n+1)x^n$; $\sum_{n=0}^{+\infty} (-x)^{n+1} = \frac{-x}{1-(-x)} = \frac{-x}{1+x}$, $\forall x \in$

$|-x| < 1$

$$\sum_{n=0}^{+\infty} (-1)^{n+1}(n+1)x^n = \sum_{n=0}^{+\infty}\left((-x)^{n+1}\right)' =$$

$$= \left(\sum_{n=0}^{+\infty}(-x)^{n+1}\right)' = \left(\frac{-x}{1-(-x)}\right)' = \frac{-(1+x)-(-x)}{(1+x)^2} = -\frac{1}{(1+x)^2} \quad , \forall x \in]-$$

e/ $\sum_{n=1}^{+\infty} (n+1)x^{n-1}$; $\sum_{n=1}^{+\infty} x^{n+1} = \frac{x^2}{1-x}$, $\forall x \in]-1;1[$

$$\sum_{n=1}^{+\infty} (n+1)x^n = \sum_{n=1}^{+\infty}\left(x^{n+1}\right)' = \left(\sum_{n=1}^{+\infty} x^{n+1}\right)' = \left(\frac{x^2}{1-x}\right)' = \frac{2x(1-x)-}{(1-x)}$$

$$= \frac{2x-x^2}{(1-x)^2} \quad , \forall x \in]-1,1[$$

Logo $\sum_{n=1}^{+\infty}(n+1)x^{n-1} = \begin{cases} \frac{2-x}{(1-x)^2} & , x \in]-1;1[\setminus \{0\} \\ 2 & x = 0 \end{cases} = \frac{2-x}{(1-x)^2}$

$\forall x \in]-1$

f/ $\sum_{n=1}^{+\infty} n e^{-\frac{nx}{2}}$; $\sum_{n=1}^{+\infty}\left(e^{-\frac{x}{2}}\right)^n = \frac{e^{-\frac{x}{2}}}{1-e^{-\frac{x}{2}}}$, $\forall x \in]0;+\infty[$,

$e^{-\frac{x}{2}} < 1$

$$\sum_{n=1}^{+\infty} \frac{-n}{2} e^{-\frac{nx}{2}} = \sum_{n=1}^{+\infty}\left(e^{-\frac{nx}{2}}\right)' =$$

$$= \left(\sum_{n=1}^{+\infty} e^{-\frac{nx}{2}}\right)' = \left(\frac{e^{-\frac{x}{2}}}{1-e^{-\frac{x}{2}}}\right)' = \frac{-\frac{1}{2}e^{-\frac{x}{2}}\left(1-e^{-\frac{x}{2}}\right) - e^{-\frac{x}{2}}\,\frac{1}{2}e^{-\frac{x}{2}}}{\left(1-e^{-\frac{x}{2}}\right)^2} =$$

$$= \frac{-\frac{1}{2}e^{-\frac{x}{2}}}{\left(1-e^{-x/2}\right)^2}$$

Logo, $\sum_{n=1}^{+\infty} n e^{-\frac{nx}{2}} = \frac{e^{-\frac{x}{2}}}{\left(1-e^{-\frac{x}{2}}\right)^2}$, $\forall x \in]0;+\infty[$

g/ $\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1}$; $\sum_{n=0}^{+\infty} (x^2)^n = \frac{1}{1-x^2}$, $\forall x \in]-1;1[$

$$\sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{+\infty} P\, x^{2n} = P \sum_{n=0}^{+\infty} x^{2n} = P\,\frac{1}{1-x^2} = P\,\frac{1}{(1-x)(1+x)}$$

$$= \frac{1}{2}\log\left|\frac{1-x}{1+x}\right| + C$$; $x = 0 \Rightarrow C = 0$

h/ $\sum_{n=1}^{+\infty} n x^{2n}$; $\sum_{n=1}^{+\infty} x^{2n} = \sum_{n=1}^{+\infty}(x^2)^n = \frac{x^2}{1-x^2}$, $\forall x \in]-1,$

$$\sum_{n=1}^{+\infty} 2n x^{2n-1} = \sum_{n=1}^{+\infty}\left(x^{2n}\right)' = \left(\sum_{n=1}^{+\infty} x^{2n}\right)' = \left(\frac{x^2}{1-x^2}\right)' = \frac{2x}{(1-x^2)^2}$$

Logo, $\sum_{n=1}^{+\infty} n x^{2n} = \frac{1}{2}x\sum_{n=1}^{+\infty} 2n x^{2n-1} = \frac{1}{\cancel{2}}x\,\frac{\cancel{2}x}{(1-x^2)^2} = \frac{x^2}{(1-x^2)^2}$

$\forall x \in]-1$

19/ a) $(\log(1+x))' = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{+\infty}(-x)^n = \sum_{n=0}^{+\infty}(-1)^n x^n$

$x_0 = 0$

Logo, $\log(1+x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1} x^{n+1} + c$ $\quad |-x|<1 \; ; \quad \forall x \in]-1,$

$x = 0 \Rightarrow c = 0 \quad ; \quad \log(1+x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1} x^{n+1}, \; \forall x \in]-1$

b/ $\frac{1}{x-2} \quad ; \quad x_0 = 0$

$\frac{1}{x-2} = -\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} = -\frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{+\infty}(-1)\frac{1}{2^{n+1}} x^n, \; \forall x \in]-2;2$

$\left|\frac{x}{2}\right| < 1$

c/ $\frac{1}{x^2-3x+2} \quad ; \quad x_0 = 0 \quad ; \quad \frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)} = \frac{1}{x-2} - \frac{}{x}$

$= \underbrace{\sum_{n=0}^{+\infty}(-1)\frac{1}{2^{n+1}} x^n}_{\forall x \in]-2;2[} + \underbrace{\sum_{n=0}^{+\infty} x^n}_{\forall x \in]-1;1[} = \sum_{n=0}^{+\infty}\left(1 - \frac{1}{2^{n+1}}\right) x^n, \; \forall x \in]-1;$

b/

d/ $\int_0^x e^{-\tau^2} d\tau, \; x_0 = 0 \quad ; \quad \left(\int_0^x e^{-\tau^2} d\tau\right)' = e^{-x^2} = \sum_{n=0}^{+\infty}\frac{(-x^2)}{n!}$

$\forall x \in$

$= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!} x^{2n}$. Logo, $\int_0^x e^{-\tau^2} d\tau = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1} + c$

$x = 0 \Rightarrow c = 0 \quad ; \quad \int_0^x e^{-\tau^2} d\tau = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!(2n+1)} x^{2n+1}, \; \forall x \in \mathbb{R}$

e/ $x^3 + \sqrt{x}, \; x_0 = 1; \quad x^3 + \sqrt{x} = (x-1+1)^3 + (1+x-1)^{1/2} =$

$= (x-1)^3 + 1 + 3(x-1)^2 + 3(x-1) + \sum_{n=0}^{+\infty}\binom{1/2}{n}(x-1)^n =$

$= (x-1)^3 + 1 + 3(x-1)^2 + 3(x-1) + \sum_{n=2}^{+\infty}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\cdots(\frac{1}{2}-n+1)}{n!}(x-1)^n + $

$-(n-1)$

$+ \frac{1}{2}(x-1$

$n=1$

$|x-1|<1 \qquad \forall x \in]0,2$

obs: $\binom{\alpha}{n} = \frac{\alpha!}{(\alpha-n)!\,n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$

$n=0 \Rightarrow \binom{\alpha}{n} = 1 \; ; \; n=1 \Rightarrow \binom{\alpha}{n} = \alpha$

$$= 2 + \frac{7}{2}(x-1) + 3(x-1)^2 + (x-1)^3 + \sum_{m=2}^{+\infty} \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(\frac{-2m+3}{2})}{m!}(x-$$

$$= 2 + \frac{7}{2}(x-1) + 3(x-1)^2 + (x-1)^3 + \sum_{m=2}^{+\infty} \frac{(-1)^{m+1}\, 1.3.5\ldots(2m-3)}{2^m m!}(x-1)^m =$$

$$= 2 + \frac{7}{2}(x-1) + 3(x-1)^2 + (x-1)^3 \underbrace{- \frac{1}{8}(x-1)^2}_{(m=2 \text{ da série anterior})} + \underbrace{\frac{1}{16}}_{m=3}(x-1)^3 + \sum_{m=4}^{+\infty} \frac{(-1)^{m+1}\, 1.3.5\ldots(2m-3)}{2^m m!}(x$$

$$= \underbrace{2}_{a_0} + \underbrace{\frac{7}{2}}_{a_1}(x-1) + \underbrace{\frac{23}{8}}_{a_2}(x-1)^2 + \underbrace{\frac{17}{16}}_{a_3}(x-1)^3 + \sum_{m=4}^{+\infty} \frac{(-1)^{m+1}\, 1.3.5\ldots(2m-3)}{2^m m!}(x-1)^m$$

$\forall x \in]0,2[$

$$a_m = \frac{f^{(m)}(1)}{m!} \quad ; \quad m = 0, 1, 2, \ldots$$

f/ $\frac{\cos x - 1}{x^2}$; $x_0 = 0$; $\frac{\cos x - 1}{x^2} = \frac{\sum_{m=0}^{+\infty} \frac{(-1)^m}{(2m)!} x^{2m} - 1}{x^2} = \sum_{m=1}^{+\infty} \frac{(-1)^m}{(2m)!} x^{2m-}$

$\forall x \in \mathbb{R}$

g/ $(x+1)\,\text{arctg}(x^2+2x+1)$, $x_0 = -1$;

$$(x+1)\,\text{arctg}(x^2+2x+1) = (x+1)\,\text{arctg}(x+1)^2 =$$

$$(\text{arctg}(x+1)^2)' = \frac{2(x+1)}{1+(x+1)^4} = 2(x+1)\sum_{m=0}^{+\infty}\left(-(x+1)^4\right)^m = \sum_{m=0}^{+\infty}(-1)^m 2(x+1)^{4m}$$

Logo $\text{arctg}(x+1)^2 = \sum_{m=0}^{+\infty} \frac{(-1)^m 2}{4m+2}(x+1)^{4m+2} + c$, $\forall x \in]-2,0[$

$$x = -1 \Rightarrow c = 0$$

Logo $(x+1)\,\text{arctg}(x^2+2x+1) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{2m+1}(x+1)^{4m+3}$, $\forall x \in]-2;0[$

h/ $\frac{4}{3x} = \frac{4}{3(x-2)+6} = \frac{2}{3}\,\frac{1}{1-\left[-\frac{(x-2)}{2}\right]} = \frac{2}{3}\sum_{m=0}^{+\infty}\left(-\frac{(x-2)}{2}\right)^m = \sum_{m=0}^{+\infty}\frac{(-1)^m}{2^{m-1}}\cdot\frac{1}{3}(x$

$x_0 = 2$

$\left|-\frac{(x-2)}{2}\right| < 1$ $\quad \forall x \in]0;4$