Optimization and Algorithms
November 25, 2021

Write your name:	______________________
Write your student number:	______________________

Exam

1. *Nonconvex function.* (3 points) One of the following functions $f\colon \mathbf{R} \to \mathbf{R}$ is **not** convex:

(A) $f(x) = (x^2 - x)_+ - x$
(B) $f(x) = -\left((x_+)\right)^2 + x^2 + x$
(C) $f(x) = (x - x_+)^2 - x$
(D) $f(x) = \left((x_+)\right)^2 - x^2 + x$
(E) $f(x) = x_+ + x^2 - x$
(F) $f(x) = (x + x_+)^2 - (x_+)^2$

Which one?

Write your answer (A, B, C, D, E, or F) here: __________

2. *Least-squares.* (2 points) Consider the following six optimization problems:

(A)
$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|A(c+x) - b\|_2^2 + \rho\,\|x\|_2^2$$

(B)
$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax - (Bx + b)\|_2^2 + \rho\,\|x - c\|_2^2$$

(C)
$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|Ax\|_2^2 + \rho\,\|(B(x-c) + b)\|_2^2$$

(D)
$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \|(Ax - b) + \rho(Bx)\|_2^2 + \|x - c\|_2^2$$

(E)
$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad \left(\|Ax - b\|_2 + \rho\,\|Bx\|_2\right)^2 + \|x - c\|_2^2$$

(F)
$$\underset{x\in\mathbf{R}^n}{\text{minimize}} \quad (Ax + Bx - b)^T\,(Ax + Bx - b) + \rho x^T x$$

In each of the six problems above, the variable to optimize is $x \in \mathbf{R}^n$. The matrices A and B, and the vector c are given. The scalar ρ is also given and is positive: $\rho > 0$.

One of the optimization problems above is **not** a least-squares problem.

Which one?

Write your answer (A, B, C, D, E, or F) here: __________

3. *Optimal value of a constrained problem.* (3 points) Consider the constrained problem

$$\begin{array}{ll} \underset{x_1,\ldots,x_N}{\text{minimize}} & \underbrace{\frac{1}{2}\sum_{n=1}^{N} x_n^T R_n x_n}_{f(x_1,\ldots,x_N)} \\ \text{subject to} & x_1 + \cdots + x_N = s, \end{array}$$

where the variable to optimize is $(x_1, \ldots, x_N)$, with $x_n \in \mathbf{R}^d$ for $1 \leq n \leq N$. The matrices $R_n \in \mathbf{R}^{d \times d}$ are given for $1 \leq n \leq N$. Assume that each R_n is a symmetric, positive-definite matrix. The vector $s \in \mathbf{R}^d$ is also given.

One of the following expressions is the minimum value that f attains over the feasible set, that is, one of the following expressions is the number $\min\{f(x_1, \ldots, x_N) : x_1 + \cdots + x_N = s\}$:

(A) $\frac{1}{2}s^T \left(R_1 + \cdots + R_N\right) s$

(B) $\frac{1}{2}s^T \left(R_1^{-1} + \cdots + R_N^{-1}\right)^{-1} s$

(C) $\frac{1}{2}s^T \left(R_1^{-2} + \cdots + R_N^{-2}\right)^{-1} s$

(D) $\frac{1}{2}s^T \left(R_1^{-1} + \cdots + R_N^{-1}\right) s$

(E) $\frac{1}{2}s^T \left(R_1 + \cdots + R_N\right)^{-1} s$

(F) $\frac{1}{2}s^T \left(R_1^2 + \cdots + R_N^2\right) s$

Which one?

Write your answer (A, B, C, D, E, or F) here: __________

4. *Sparse linear regression with asymmetric loss.* (4 points) Consider the optimization problem

$$\underset{s\in\mathbf{R}^n, r\in\mathbf{R}}{\text{minimize}} \quad \underbrace{\sum_{k=1}^{K} \alpha\left((s^T x_k + r - y_k)_-\right)^2 + \beta\left((s^T x_k + r - y_k)_+\right)^2 + \rho\,\|s\|_1}_{f(s,r)},$$

where the variable to optimize is $(s,r) \in \mathbf{R}^n \times \mathbf{R}$. The vectors $x_k \in \mathbf{R}^n$ and the scalars $y_k \in \mathbf{R}$ are given for $1 \leq k \leq K$. The scalars α, β, and ρ are given and denote positive constants: $\alpha > 0$, $\beta > 0$, and $\rho > 0$. The functions $(\cdot)_-$ and $(\cdot)_+$ are defined as $(z)_- = \max\{-z, 0\}$ and $(z)_+ = \max\{z, 0\}$ for $z \in \mathbf{R}$.

Show that the function f is convex.

5. *A simple control problem.* (4 points) Consider the optimization problem

$$\begin{array}{ll} \underset{\{x_t\colon 1\leq t\leq T\},\{u_t\colon 1\leq t\leq T-1\}}{\text{minimize}} & \frac{1}{2}\|x_T\|_2^2 + \frac{\rho}{2}\sum_{t=1}^{T-1}\|u_t\|_2^2 \\ \text{subject to} & x_1 = x_{\text{initial}} \\ & x_{t+1} = x_t + D_t u_t \quad \text{for } 1 \leq t \leq T-1, \end{array}$$

where the variables to optimize are $x_t \in \mathbf{R}^n$ for $1 \leq t \leq T$ and $u_t \in \mathbf{R}^p$ for $1 \leq t \leq T-1$. The vector $x_{\text{initial}} \in \mathbf{R}^n$ and the matrices $D_t \in \mathbf{R}^{n\times p}$ are given for $1 \leq t \leq T-1$. The scalar ρ is also given and denotes a positive constant: $\rho > 0$.

Give a closed-form solution for the optimal $\{u_t\colon 1 \leq t \leq T-1\}$.

6. *Moureau envelope.* (4 points) Let $f\colon \mathbf{R} \to \mathbf{R}$ be a convex function. For $\lambda > 0$, we define a function $e_\lambda[f]\colon \mathbf{R} \to \mathbf{R}$ as follows: for $x \in \mathbf{R}$, the image of x under the function $e_\lambda[f]$ is the number $\min\{f(u) + \frac{1}{2\lambda}(u-x)^2\colon u \in \mathbf{R}\}$.

That is, the function $e_\lambda[f]$ maps each number x to the number $e_\lambda[f](x)$, where $e_\lambda[f](x)$ is the minimum value attained by $f(u) + \frac{1}{2\lambda}(u-x)^2$ as u varies in $\mathbf{R}$.

Let $\lambda_1 > 0$ and $\lambda_2 > 0$. Show that

$$e_{\lambda_1}[e_{\lambda_2}[f]](x) = e_{\lambda_1+\lambda_2}[f](x),$$

for each $x \in \mathbf{R}$.