Open Vibrations

Austin Downey
Department of Mechanical Engineering
Department of Civil and Enviormental Engineering
University of South Carolina, Colubmia SC, USA

Laura Micheli
Department of Civil and Enviormental Engineering
Catholic University of America, Wahington DC, USA

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1 Basic Concepts in Vibrations

The study of vibrations, within the broader field of classical mechanics, is the investigation of oscillations that occur about an equilibrium point. Vibrations, both desired and undesired, are present in all mechanical systems and can be helpful (e.g. a soil sieve, rotary sander) or destructive (e.g. an aircraft frame in resonance). The oscillations that form a vibrating system may be periodic (e.g., pendulum) or random (e.g. turbulence in an airplane), or a combination of the two.

Vibrations impact our daily lives in a variety of ways, from the sound made by banjo strings that vibrates between 140 and 400 Hz to the 4-6 Hz vibration felt by a passenger in a car seat. The consideration of the vibrations and their associated mathematical modeling are an important factor in the design of mechanical systems. In the material that follows, the fundamental theories of vibration are presented and modeled using fundamental physical principles such and Newton's three laws of motion. These models and analyzed using the mathematical tools of calculus and differential equations.

Review 1.1 Newton's three laws of motion:

1. In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.
2. In an inertial reference frame, the vector sum of the forces F on an object is equal to the mass m of that object multiplied by the acceleration of the object: $F = ma$. (It is assumed here that the mass m is constant)
3. When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

1.1 Single Degree-of-Freedom Systems

In its simplest form, the phenomenon of vibration is the exchange of energy between potential and kinetic energy. Therefore, a vibrating system must have a component that stores potential energy. This component must also be capable of releasing the energy as kinetic energy. This kinetic energy is stored in the movement of a mass where the measure of this movement is the velocity of the system and the continuous interchange between potential and kinetic energy is the vibration of the system. The simplest vibrating systems can be modeled as a single-degree-of-freedom (1-DOF) system. In a 1-DOF system, one variable can describe the motion of a system. Potential examples of 1-DOF systems include:

1. yo-yo
2. pogo stick
3. door swinging on axis
4. throttle (gas pedal)

Variables often used for describing 1-DOF systems are $x(t)$, $y(t)$, $z(t)$, and $\theta(t)$. Examples of 1-DOF systems are presented in figure 1.1 where the assumption of small displacements is made. Note: we will often drop the "(t)" for simplicity in this material.

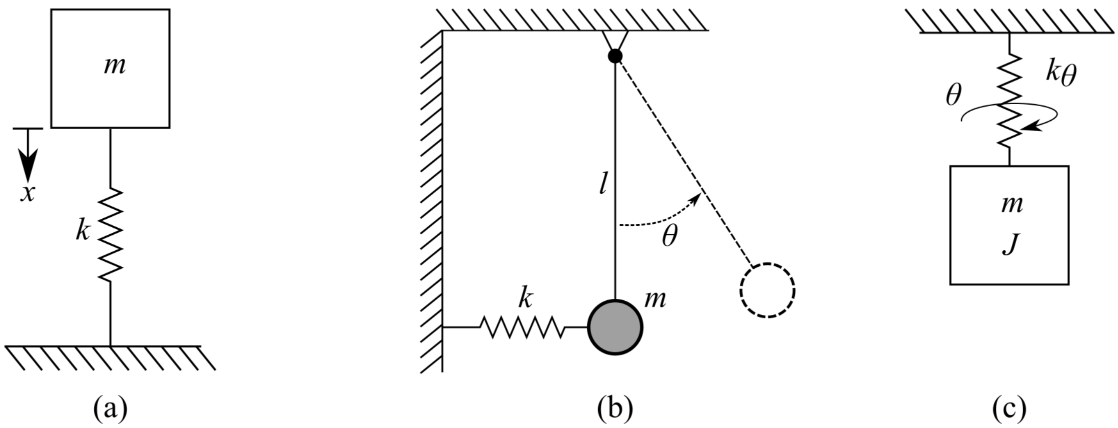

Figure 1.1: Examples of single degree of freedom (DOF) systems showing: (a) a vertical spring-mass system; (b) a simple pendulum; and (c) a rotational spring-mass system.

1.1.1 Spring-Mass Model

> "All models are wrong, but some are useful"
> George E.P. Box (1919 - 2013)

Newtonian physics describes the motion of particles in terms of displacement x, velocity $\dot{x}$, and acceleration $\ddot{x}$ vectors. Moreover, from Newton's second law of motion says that the change in the velocity of mass in motion is a product of the force acting on the mass. A simple way to express this phenomenon is though a spring-mass model as presented in figure 1.2. These spring-mass models neglect the mass of the spring and concentrate all the mass of the system into a single point. Note that in this case the force vector and mass-acceleration vectors lie on the same axis and as such are collinear. Therefore, these vectors can be easily treated as scalers simplifying the math used in the modeling of the system.

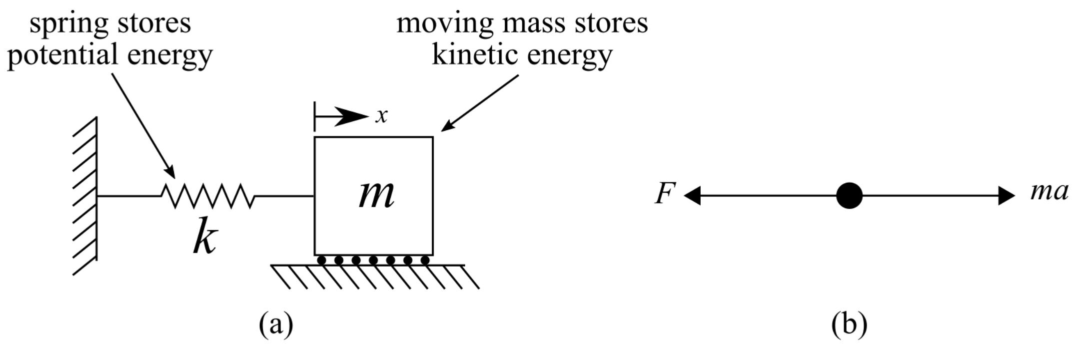

Figure 1.2: A single-degree-of-freedom (1-DOF) spring mass model showing: (a) annotated schematic of a mass-spring system; and (b) the equivalent free-body diagram represented as a point-mass system.

1.1.2 Linear Springs

Springs are mechanical devices that store energy, moreover, ideal spring is a theoretical representation of this mechanical device that is massless and responds with a linear increase in force for a unit increase in displacement (i.e. $F = kx$). For simplicity, the spring in the spring-mass model considered here is assumed aways ideal linear springs. A graphical representation of the idealized linear spring is presented in figure 1.3 where a unit force F applied to the free end of the spring results in a unite displacement x of the spring. The resulting mathematical relationships, $F = kx$, is known as Hooke's Law. Nonlinear springs add considerable complexity to the modeling of spring-mass systems, therefore, these are not considered in this introductory work.

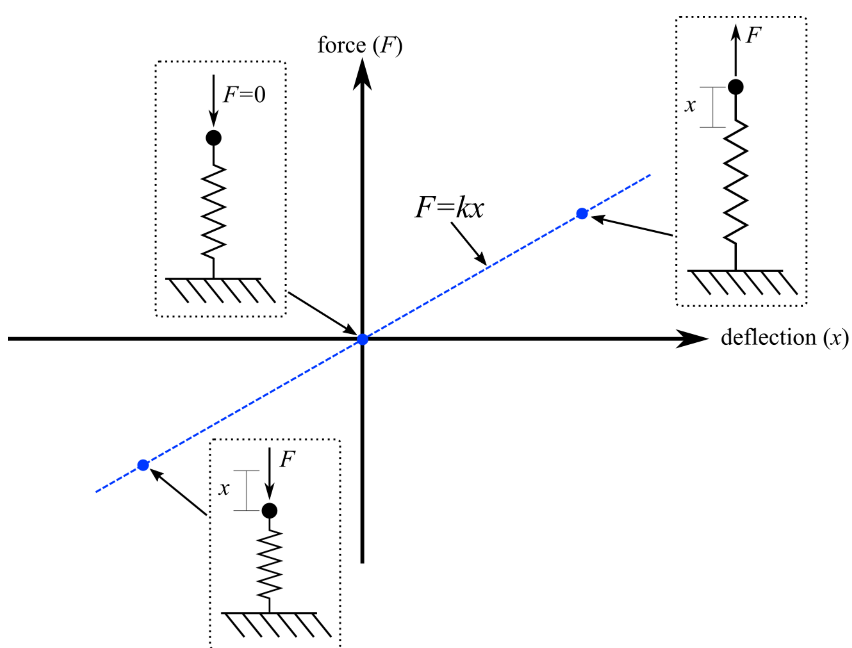

Figure 1.3: Force-displacement plot for a linear spring.

1.2 Equivalent Stiffness

The generalized concept of stiffness can be directly related to mechanical systems and structural components through Hooke's law.

Review 1.2 Hooke's Law states that the force (F) needed to extend or compress a spring by some distance x scales linearly with respect to that distance. This law can be extend to tensional stress of an uniform and elastic bar where the length, area, and Young's modulus of the bar are represented by l, A, and E, respectively. Knowing the tensile stress in the bar:

$$\sigma = \frac{F}{A} \tag{1.1}$$

and the definition of strain:

$$\varepsilon = \frac{\Delta l}{l} \tag{1.2}$$

Hooke's law can be expanded to represent an uniform and elastic bar:

$$\sigma = E\varepsilon \tag{1.3}$$

It follows that the change in length Δl can be expressed as:

$$\Delta l = \varepsilon l = \frac{FL}{AE} \tag{1.4}$$

Note: Hooke's law is often expressed using the convention that F is the restoring force exerted by the spring on the applied force at the free end. Defining the stiffness and displacement as $k = \frac{AE}{L}$ and $\Delta l = x$, respectively. The equation for Hooke's Law becomes:

$$F = -kx \tag{1.5}$$

since the direction of the restoring force is opposite the spring displacement.

1.2.1 Equivalent Stiffness of Structural Systems

For a rod with a uniform cross-section, a direct representation of the system can be developed as expressed in figure 1.4 where the vibration along the axis of the rod is to be considered. The stiffness of the rod, k, is a measure of the resistance offered by an elastic body to deformation.

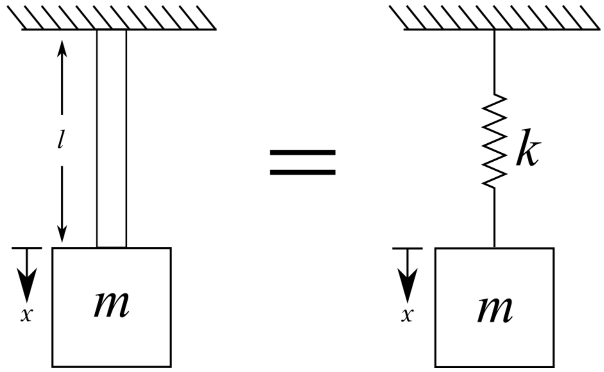

Figure 1.4: Equivalency between a vertical bar with mass attached to the bottom and a spring-mass model of the system

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For this 1-DOF system, the equation of a spring can be rearranged such that the stiffness can be defined as:

$$k = \frac{F}{x} \tag{1.6}$$

The stiffness of the spring can be more closely related to materials properties of the bar A, E, and l considering that Hooke's Law for the uniform tension on a bar can be expressed as:

$$\sigma = E\varepsilon \tag{1.7}$$

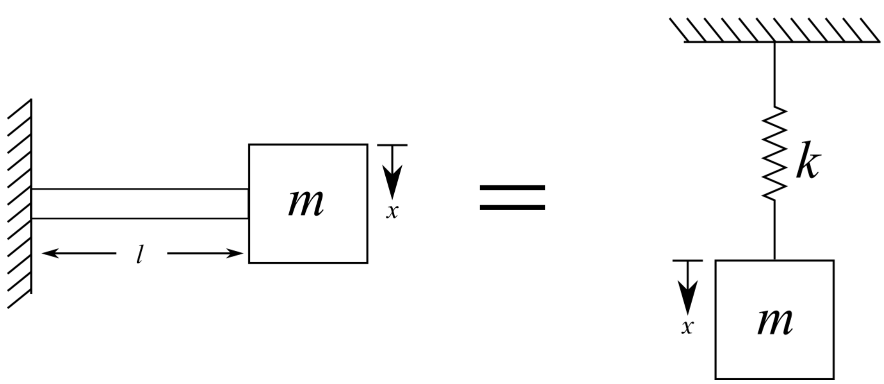
This expression can be expanded into the form:

$$\frac{F}{A} = E\left(\frac{x}{l}\right) \tag{1.8}$$

rearranging the terms and recalling the expression $k = \frac{F}{x}$ leads to:

$$k = \frac{EA}{l} \tag{1.9}$$

In a similar fashion, we can also solve the equivalent system for a mass at the end of a cantilever beam.



From engineering mechanics we can compute the deflection at the point of a beam δ with a point load P. This expression is typically expressed as:

$$\delta = \frac{Pl^3}{3EI} \tag{1.10}$$

If we transform this matrix into our variable system by exchanging P for F and δ for x. Thereafter, the point load is replaced with the equivalent force F generated by the mass and the pull of gravity(mg). As before, knowing that the stiffness of the system can be expressed as $k = F/x$ we can show that:

$$k = \frac{3EI}{l^3} \tag{1.11}$$

Example 1.1 Considering the rod diagrammed below; calculate an equivalent spring constant for the rod using the length of the rod l, its area A, and Young's modulus E for a compressive force F that compresses the rod a distance x. Additionally, is a linear spring a useful model for a rod under compression? What if the rod is under tension?

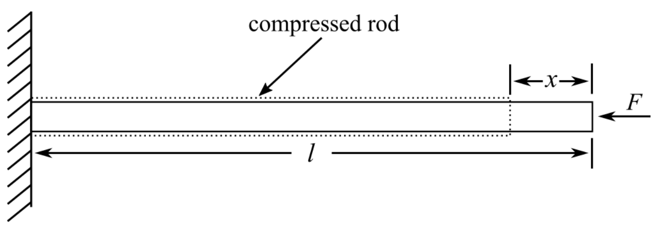


Figure 1.5: Compressed cantilever rod.

Solution: The rod shortens by a distance x under the axial force F, this can be related to the equation of a linear spring $F = kx$ by recalling from solid mechanics that the elongation (or shortening) of a rod is expressed as

$$x = \frac{x}{l}l = \varepsilon l = \frac{\sigma}{E}l = \frac{Fl}{AE} \tag{1.12}$$

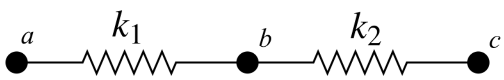
where $\varepsilon = \frac{x}{l}$ is the strain value and $\sigma = F/A$ is the stress induced in the rod. Combining this expression with the equation of a linear spring yields:

$$k = \frac{F}{x} = \frac{AE}{l} \tag{1.13}$$

As per the usefulness of the linear spring to represent an axial rod under compression or tension, this would be application-specific but could generally be considered an excellent first-order approximation.

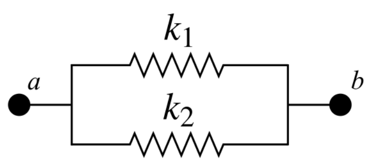
1.2.2 Springs in Series and Parallel

In many cases, it becomes necessary to model a mechanical system as a set of springs (e.g., a composite material, a table with multiple legs). For these systems, or for systems with more than one spring acting on a body, equivalent stiffness can be calculated as:



$$k_{ac} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

(a)



$$k_{ab} = k_1 + k_2$$

(b)

Figure 1.6: Equations for calculating the equivalent stiffness of two springs (k_1 and k_2); (a) in series; and (b) in parallel.

These are derived considering the displacement δ of the systems. For two springs in series:

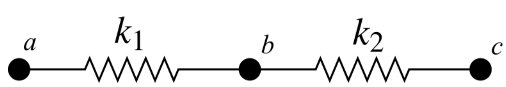


Figure 1.7: Two springs k_1 and k_2 combined in series.

where the total displacement is

$$\delta_{ac} = \delta_{ab} + \delta_{bc} \tag{1.14}$$

Using the equation for stiffness $k = F/\delta$, this converts to:

$$\frac{F}{k_{ac}} = \frac{F}{k_1} + \frac{F}{k_2} \tag{1.15}$$

As F is the same throughout the system, we can cancel out F. Solving for the equivalent stiffness yields:

$$k_{ac} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} \tag{1.16}$$

Similarly for a system of springs in parallel:

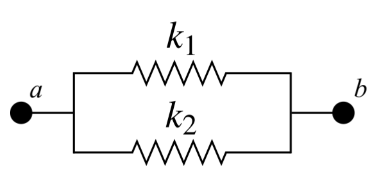


Figure 1.8: Two springs k_1 and k_2 combined in parallel.

The displacement in both springs is the same, so the total displacement is

$$\delta_{ab} = \delta_1 = \delta_2 = \delta \tag{1.17}$$

The forces in the direction of spring elongation sum to zero, therefore:

$$F_{ab} = F_1 + F_2 \tag{1.18}$$

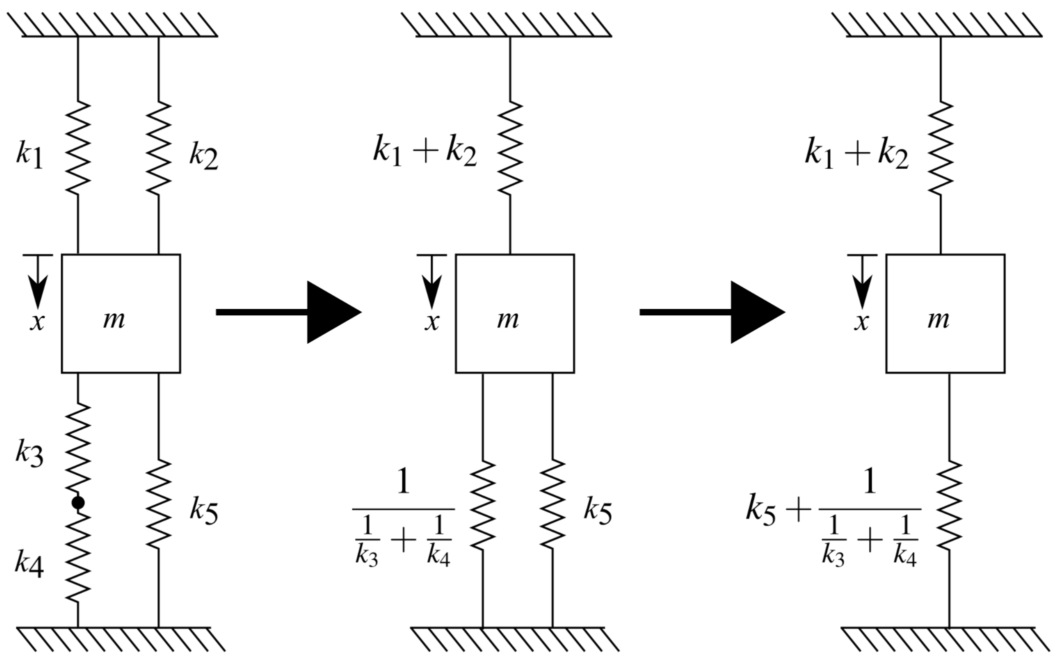
Substituting the displacement and stiffness into the force equation yields:

$$\delta k_{ab} = \delta k_1 + \delta k_2 \tag{1.19}$$

this simplifies to:

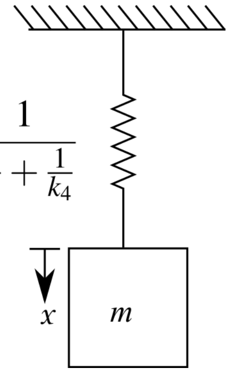
$$k_{ab} = k_1 + k_2 \tag{1.20}$$

Example 1.2 Calculate the equivalent stiffness of the following system:



The springs are combined as shown, using the equations defined before. Now, considering that the displacement (δ) of the top spring, and the bottom spring are the same we can state the total stiffness k, is the summation of the two. Therefore,

$$k_1 + k_2 \;+\; k_5 + \frac{1}{\frac{1}{k_3}+\frac{1}{k_4}}$$



where the final addition, $(k_1+k_2)+(k_5+\frac{1}{\frac{1}{k_3}+\frac{1}{k_4}})$ is applied at two springs in parallel as each spring is connected between the mass and the fixity. Rearranging this new expression to get a common denominator:

$$k = \frac{(k_1+k_2+k_5)(k_3+k_4)+k_3k_4}{k_3+k_4} \tag{1.21}$$

1.3 Equation of Motion for an Oscillating System

An Equation of Motion (EOM) is an equation that provides a basis for modeling a vibrating system about its equilibrium point and relates the transfer of the potential energy from the spring to the kinetic energy mass. In developing the EOM we assume that any surfaces are frictionless and as such, no energy is extracted from the vibrating system. Referencing the 1-DOF system in figure 1.9(a), and assuming the mass only moves in the x direction, the only force acting on the mass in the x direction is the force that results from the elongation of the spring as annotated in figure 1.9(b). Therefore, the sum of forces in along the x axis must equal the mass (m) times the acceleration of the mass ($a\dot{x}$).

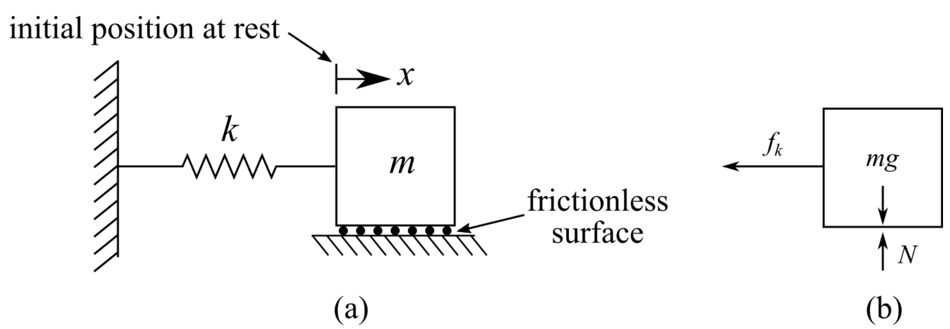


Figure 1.9: A spring mass model of a 1-DOF system showing: (a) a schematic of the system; (b) free-body diagram of the system at its initial position.

Considering that positive displacements are to the right, the standard form of the equation of motion for an undamped system without any excitation is expressed as:

$$s_1\ddot{x} + s_2 x = 0 \tag{1.22}$$

where s_1 and s_2 are constants to be determined for the specific system. A systematic approach to obtaining free-body diagram (FBD) of a system under vibration can be expressed in three steps:

1. Draw a free-body diagram (FBD) at the system's equilibrium and displaced position (without a displacing force).
2. Apply Newton's second law to both FBDs (equilibrium and displaced).
3. Combine the equations to write the EOM in standard form with the forcing component on the right-hand side. For free vibration, the forcing component is 0.

Solving these three steps for 1-DOF system presented in figure 1.9 results in the EOM:

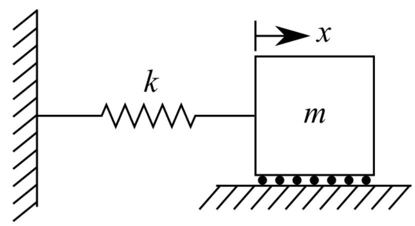
$$m\ddot{x} + kx = 0 \tag{1.23}$$

Review 1.3 A second-order linear homogeneous differential equation has the form:

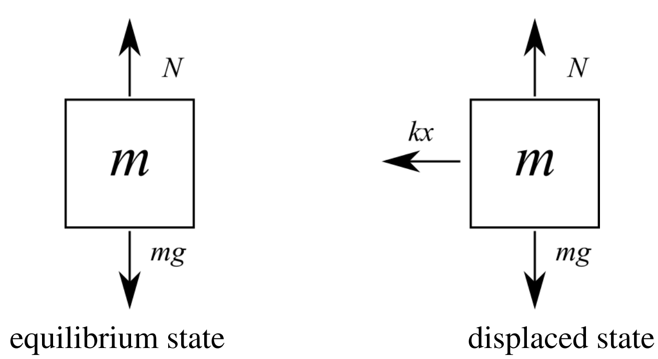
$$a\ddot{x} + b\dot{x} + cx = 0 \tag{1.24}$$

The EOM for a 1-DOF system under a free vibration is a a second-order differential equation due to acceleration ($\ddot{x}$) being the second derivative of displacement (x) and homogeneous as the forcing function (right-hand side of the equations) is zero. In EOM's current form, $m = k$, $b = 0$, and $c = k$. In future work, b will account for damping in the vibrating system.

Example 1.3 Considering the system:



Step-1 Define the direction of displacement, and draw the FBD for the equilibrium and displaced state.



The equation for the equilibrium state is:

$$\overset{+}{\rightarrow} \sum F_x = 0 \tag{1.25}$$

and in the displaced state:

$$\overset{+}{\rightarrow} \sum F_x = -kx \tag{1.26}$$

This equation does not equal zero as the FBD does not account for the restoring force.
Step-2 Apply Newton's second law (we want to store energy in the kinetic state) of motion to the sum of forces for the displaced position we get:

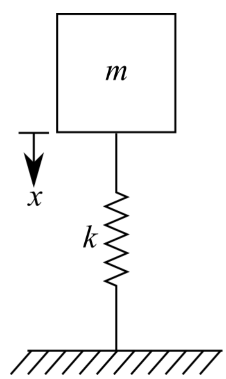
$$ma = m\ddot{x} = \overset{+}{\rightarrow} \sum F_x = -kx \tag{1.27}$$

$$m\ddot{x} = -kx \tag{1.28}$$

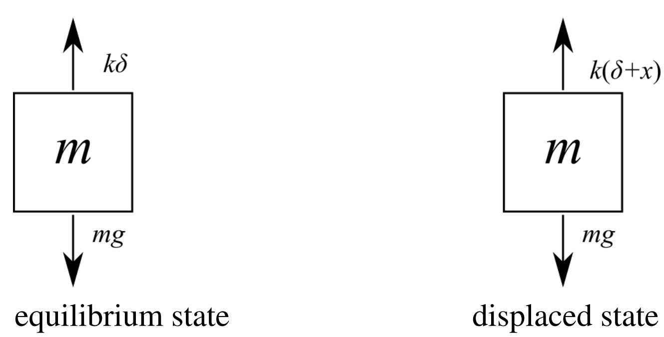
Step-3 Rearrange in the Equation to construct an EOM:

$$m\ddot{x} + kx = 0 \tag{1.29}$$

Example 1.4 Some systems will have an initial displacement, as the system will oscillate around this position we need to define the EOM about this position. Considering the system:



Step-1 Define the direction of displacement, and draw the FBD for the equilibrium and displaced state.



The equation for the equilibrium state is:

$$+\downarrow \sum F_x = mg - k\delta = 0 \tag{1.30}$$

and in the displaced state:

$$+\downarrow \sum F_x = mg - k(\delta + x) \tag{1.31}$$

This equation does not equal zero as the FBD does not account for the restoring force.
Step-2 Apply Newton's second law (we want to store energy in the kinetic state) of motion to the sum of forces for the displaced position we get:

$$m\ddot{x} = +\downarrow \sum F_x = mg - k\delta - kx \tag{1.32}$$

We can than use the information from the equilibrium state to cancel out some terms, this becomes:

$$m\ddot{x} = -kx \tag{1.33}$$

Step-3 Rearrange in the Equation to construct an EOM:

$$m\ddot{x} + kx = 0 \tag{1.34}$$

2 Free Vibration of Single-Degree-of-Freedom Systems

Vibrations (i.e. the exchange of potential and kinetic energy) requires oscillatory motion that may repeat itself regularly or irregularly. A motion that is repeated on time intervals is called periodic motion. If this motion has a single frequency and amplitude it is called simple harmonic motion are represents the most basic form of oscillatory motion as depicted in figure 2.2. For a 1-DOF system simple harmonic motion is defined as a periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

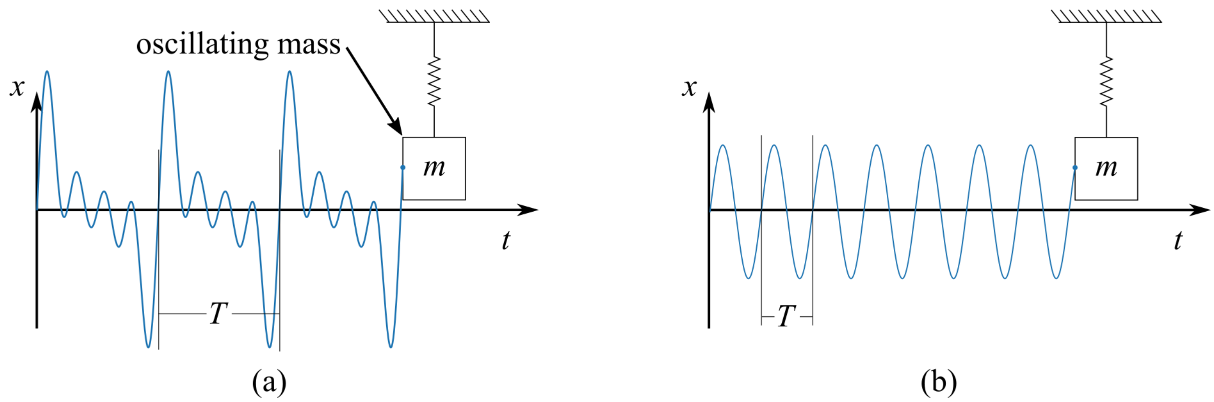


Figure 2.2: Oscillatory motion for a single degree of freedom system showing (a) periodic motion; and (b) simple harmonic motion.

Given the nature of simple harmonic motion, constant amplitude and frequency, the wave starting at the origin O can be modeled at a point on the end of a vector with length A rotating rotating at a constant angular velocity ω_{n} where the angle from the origin of the vector is ϕ, defined as $\phi = \omega t$. Where ω is the lowercase Greek letter Omega and ϕ is the lowercase Coptic letter phi. This is similar to a Greek phi (φ) and either can be used in this context. The subscript n on ω denotes that this frequency relates to the natural frequency of the system, the only frequency in simple harmonic motion. A visualization of the harmonic motion obtained from projecting the point on the edge of a vector onto the $\omega_{\mathrm{n}}t$ space is presented in figure 2.3.

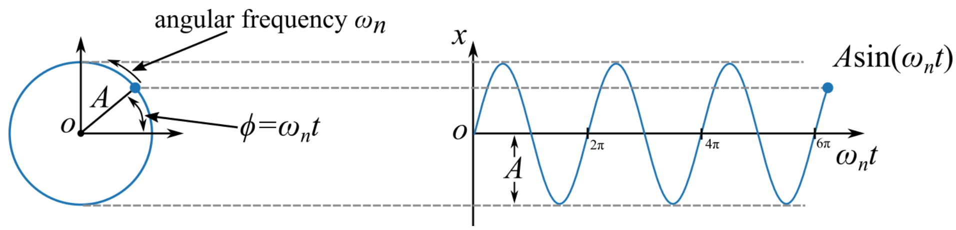


Figure 2.3: Harmonic motion represented at the projection of a point on the end of a vector moving on a circle. Note the axis $\omega_{\mathrm{n}}t$.

2.1 Mathematical Modeling of Free Vibration

The Development of a mathematical model for a system under free vibration would enable the practitioner to predict, or model, the vibrating system of interest. Therefore, considering that the following system,

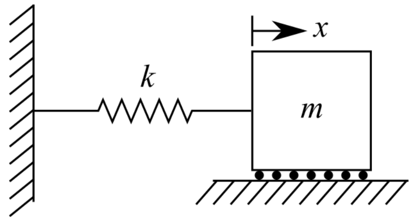


Figure 2.4: 1-DOF spring-mass system.

can be modeled expressed with the following EOM

$$m\ddot{x}(t) + kx(t) = 0 \tag{2.2}$$

it becomes prudent to solve this homogeneous ordinary differential equation (ODE) to obtain a model of the vibrating system. The simplest method for solving an ODE is to propose a solution based on observations of a vibrating physical system. Figure 2.5 reports and annotates the key components from an observation of a vibrating system.

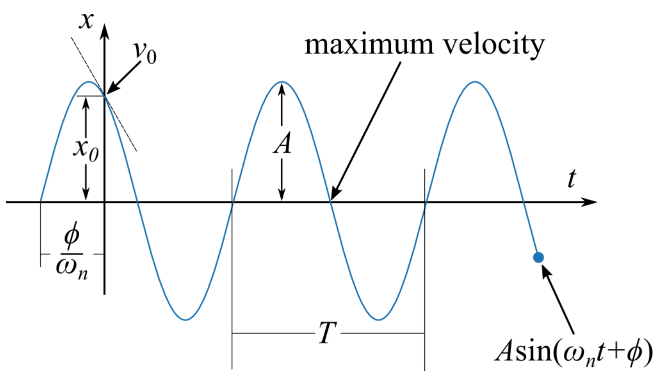


Figure 2.5: Summary of the temporal response for a 1-DOF system.

where x_0 and v_0 are the is the displacement and velocity at t=0 (i.e. the initial displacement).

A mathematical expression can now be formulated to represent the observed simple harmonic motion. This expression can be based on the projection of a point on a vector (transposed into the time domain) or assembled from constituent parts as done in what follows. Solving for a location x, at a time t; $x(t)$, the various characteristics of the expression can be identified:

2.1.1 Solve for the Natural Frequency (ω_n) of the System

- System oscillates $\rightarrow$ a sin function models this
- System oscillates at different speed $\rightarrow$ use a parameter to adjust ω_n in rad/s.
- Systems have different amplitudes $\rightarrow$ use a parameter to adjust A in meters.
- System has different starting points $\rightarrow$ use a parameter to adjust ϕ in rad.

Using these four constituent components, an equation can be proposed:

$$x(t) = A\sin(\omega_{\mathrm{n}} t + \phi) \tag{2.3}$$

Take the derivative to get velocity:

$$\dot{x}(t) = A\omega_{\mathrm{n}}\cos(\omega_{\mathrm{n}} t + \phi) \tag{2.4}$$

Take the derivative again to get acceleration:

$$\ddot{x}(t) = -A\omega_{\mathrm{n}}^2\sin(\omega_{\mathrm{n}} t + \phi) \tag{2.5}$$

Substituting x and $\ddot{x}$ into the EOM for the considered 1-DOF system ($m\ddot{x}(t) + kx(t) = 0$) yields:

$$m\big(-A\omega_{\mathrm{n}}^2\sin(\omega_{\mathrm{n}} t + \phi)\big) + k\big(A\sin(\omega_{\mathrm{n}} t + \phi)\big) = 0 \tag{2.6}$$

Thereafter, dividing both sides by $A\sin(\omega_{\mathrm{n}} t + \phi)$ results in the expression:

$$-m\omega_{\mathrm{n}}^2 + k = 0 \tag{2.7}$$

This expression can be rearranged into the more useful standard form:

$$\omega_{\mathrm{n}} = \sqrt{\frac{k}{m}} \tag{2.8}$$

Equation 2.8 represents a solution to the EOM presented in equation 2.2. This solution is not in the form of an ODE so, therefore, we can experientially prove that this is the correct solution. For example, we could build a systems with known mass and stiffness and measure the natural frequency of the system. Equation 2.8 equation leads to:

$$T = \frac{2\pi}{\omega_{\mathrm{n}}} \tag{2.9}$$

where T is the period of oscillations and

$$f_n = \frac{\omega_{\mathrm{n}}}{2\pi} \tag{2.10}$$

where f_n is the frequency of the oscillations.

2.1.2 Solve for Initial Phase (ϕ) of the System

The EOM is a second-order ODE so there needs to exist two initial conditions (constants) to solve it. For the systems under considerations, the displacement (x) and velocity ($\dot{x}$ or v) at $t = 0$ are the initial conditions. For simplicity, these are written as

$$x(0) = x_0 \tag{2.11}$$

$$\dot{x}(0) = v(0) = v_0 \tag{2.12}$$

Setting the equation to its initial state $t = 0$, the equations for displacement and velocity can be simplified to:

$$x(0) = x_0 = A\sin(\omega_n 0 + \phi) = A\sin(\phi) \tag{2.13}$$

$$\dot{x}(0) = v_0 = A\omega_n\cos(\omega_n 0 + \phi) = A\omega_n\cos(\phi) \tag{2.14}$$

Thereafter, mathematical meanings for ϕ and A can be derived. To do this, ϕ can be solved for by rearranging equations 2.13 and 2.14 for A:

$$A = \frac{x_0}{\sin(\phi)} \tag{2.15}$$

and:

$$A = \frac{v_0}{\omega_n\cos(\phi)} \tag{2.16}$$

Setting these two equations equal to each other cancels out A and creates:

$$\frac{x_0\omega_n}{\sin(\phi)} = \frac{v_0}{\cos(\phi)} \tag{2.17}$$

therefore:

$$\frac{x_0\omega_n}{v_0} = \frac{\sin(\phi)}{\cos(\phi)} \tag{2.18}$$

finally:

$$\phi = \tan^{-1}\left(\frac{x_0\omega_n}{v_0}\right) \tag{2.19}$$

2.1.3 Solve for Amplitude (A) of the System

The amplitude of the vibrating system (A) is solved for in a similar manner to ϕ where the expressions for x and $\dot{x}$ are solved for at $t = 0$ and rearranged as to isolate ϕ. This operations results in the the equations:

$$\sin(\phi) = \frac{x_0}{A} \tag{2.20}$$

and:

$$\cos(\phi) = \frac{v_0}{\omega_n A} \tag{2.21}$$

From these equations a value for ϕ can be obtained knowing that $\sin(\phi)^2 + \cos(\phi)^2 = 1$. Therefore:

$$\left(\frac{x_0}{A}\right)^2 + \left(\frac{v_0}{\omega_n A}\right)^2 = 1 \tag{2.22}$$

multiplying each expression by 1 (also expressed as $\frac{\omega_n}{\omega_n}$), gives the equation:

$$\left(\frac{\omega_n}{\omega_n}\right)^2\left(\frac{x_0}{A}\right)^2 + 1\left(\frac{v_0}{\omega_n A}\right)^2 = 1 \times 1 \tag{2.23}$$

which becomes:

$$\left(\frac{\omega_n x_0}{\omega_n A}\right)^2 + \left(\frac{v_0}{\omega_n A}\right)^2 = 1 \tag{2.24}$$

Further simplification is obtained by multiplying each side by $(\omega_\mathrm{n} A)^2$ to obtain:

$$\omega_\mathrm{n}^2 x_0^2 + v_0^2 = A^2 \omega_\mathrm{n}^2 \tag{2.25}$$

Solving for A, this expression rearranges to:

$$A = \frac{\sqrt{\omega_\mathrm{n}^2 x_0^2 + v_0^2}}{\omega_\mathrm{n}} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_\mathrm{n}}\right)^2} \tag{2.26}$$

2.1.4 Response for Simple Harmonic Motion

The time-varying displacement of a 1-DOF vibrating system under free response is expressed by the equation $x(t) = A\sin(\omega_\mathrm{n} t + \phi)$. Substituting in the expressions for A and ϕ results in:

$$x(t) = \frac{\sqrt{\omega_\mathrm{n}^2 x_0^2 + v_0^2}}{\omega_\mathrm{n}} \sin\left(\omega_\mathrm{n} t + \left(\tan^{-1}\left(\frac{x_0 \omega_\mathrm{n}}{v_0}\right)\right)\right) \tag{2.27}$$

This equation provides a mathematical solutions that relates displacement of the mass to the initial conditions x_0 and v_0. The solution is considered a free response because no input is applied after t=0. The relationship between the initial conditions (x_0 and v_0) and the amplitude and phase of the response can be expressed using the Pythagorean theorem, $a^2 + b^2 = c^2$, as annotated in figure 2.6.

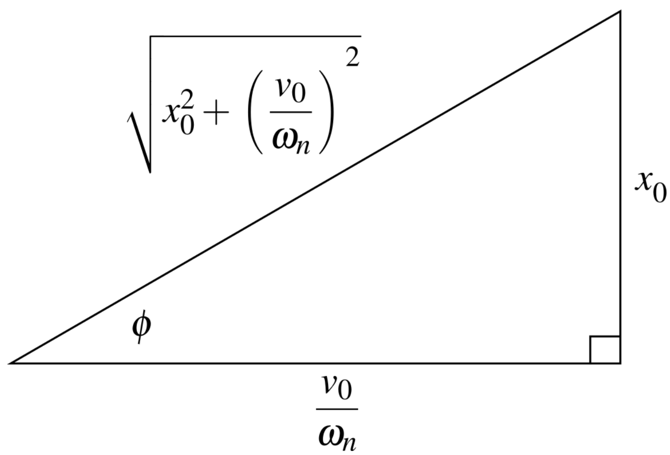


Figure 2.6: Trigonometric relationship between the initial conditions (x_0 and v_0), amplitude A, and phase ϕ for free vibration of a 1-DOF system.

2.1.5 Special Considerations for No Initial Velocity ($v_0 = 0$)

Upon close inspection of the temporal solution in equation 2.27, it becomes evident that any system without initial velocity (i.e. $v_0 = 0$) results in an undefined number for $(x_0 \omega_\mathrm{n})/v_0$. A solution to this challenge lies in the fact that limit of $\tan^{-1}(x)$ approaches $-\pi/2$ at $-\infty$ and $\pi/2$ at ∞, as depicted in figure 2.7. Therefore, the solution at $-\infty$ and ∞ is undefined, resulting in the expression:

$$\left(\frac{x_0 \omega_\mathrm{n}}{v_0}\right) = \pm\frac{\pi}{2}, \text{ when } v_0 = 0 \tag{2.28}$$

This step is applied in IEEE floating point arithmetic (IEEE 754) and results in either $\pi/2$ or $\pm\pi/2$ depending on the rounding format used. From the practitioner's side, it becomes important to recognize the situation $v_0 = 0$ and correct this value as needed.

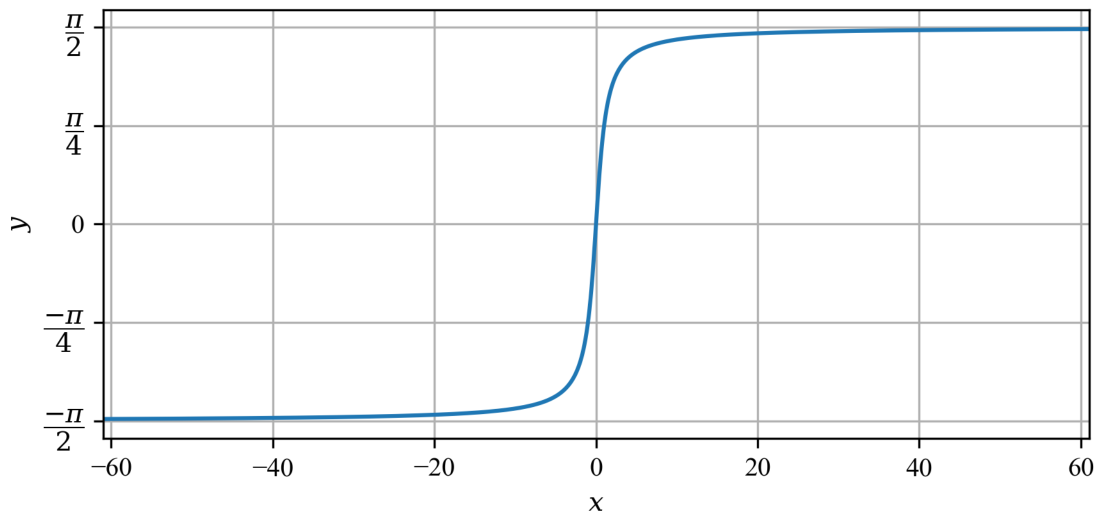


Figure 2.7: Response of $\tan^{-1}$ (or arctan) for x=-60 to 60 showing that the $\tan^{-1}$ is undefined as x approaches $-\infty$ and ∞.

Example 2.1 A vehicle wheel, tire, and suspension can be modeled as a SDOF spring and mass as depicted below: The mass of the wheel and tire is measured to be 300 kg and its frequency of oscillation is observed to be 10 rad/sec. What is the stiffness of the wheel assembly?

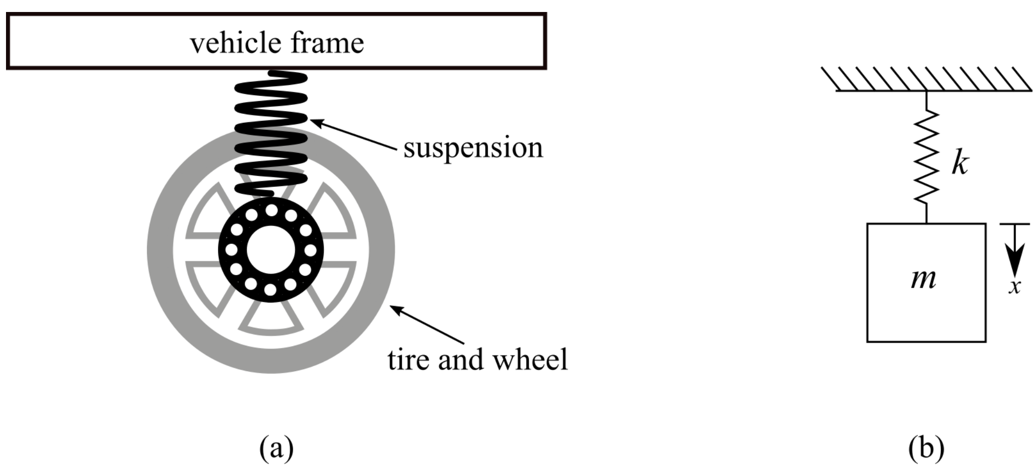


Figure 2.8: Modeling of a vehicle wheel, tire, and suspension showing: (a) Graphical representation; and (b) a spring-mass model.

Solution: Considering:

$$\omega_{\text{n}} = \sqrt{\frac{k}{m}} \tag{2.29}$$

therefore, $k = m\omega_{\text{n}}^2 = (300 \text{ kg})(10 \text{ rad/s})^2 = 30$ KN/m. Note: radians are a dimensionless quantity and as such the units of $m\omega_{\text{n}}^2$ become $\frac{\text{kg}}{\text{s}^2} \cdot \frac{\text{m}}{\text{m}}$ where the unit value $\frac{\text{m}}{\text{m}}$ is added such that the stiffness of the spring can be expressed as $\frac{\text{kg}\cdot\text{m}}{\text{s}^2} \cdot \frac{1}{\text{m}} = \frac{\text{N}}{\text{m}}$.

Example 2.2 Consider the following 1-DOF system, where $k = 857.8$ N/m and $m = 49.2 \times 10^{-3}$ kg, calculate the natural frequency in rad/s and Hz. Also find the period of oscillations and the maximum displacement if the spring is initially displaced 10 mm with no initial velocity.

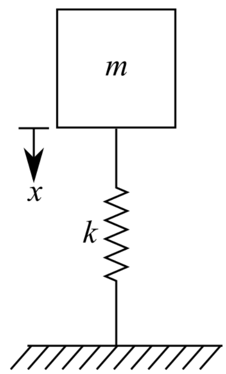


Figure 2.9: 1-DOF spring-mass system.

Solution:

$$\omega_{\text{n}} = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132 \text{ rad/sec} \tag{2.30}$$

In Hz, this is:

$$f_n = \frac{\omega_{\text{n}}}{2\pi} = 21 \text{ Hz} \tag{2.31}$$

The period is:

$$T = \frac{2\pi}{\omega_{\text{n}}} = 0.0476 \text{ s} \tag{2.32}$$

The maximum displacement will happen when $\sin(\omega_{\text{n}} t + \phi) = 0$, therefore, the value of A is the maximum displacement. For an undamped system, $A = \frac{\sqrt{\omega_{\text{n}}^2 x_0^2 + v_0^2}}{\omega_{\text{n}}}$,

$$A = \frac{\sqrt{\omega_{\text{n}}^2 x_0^2 + v_0^2}}{\omega_{\text{n}}} = \frac{\sqrt{132^2 0.01^2 + 0^2}}{132} = 0.01 \text{ m} \tag{2.33}$$

2.2 General Solution for Vibrating Systems

The EOM for a vibrating system has many solutions and can be be expressed in various forms including a general solution. These forms offer different mathematical approaches to solve the same 1-DOF spring-mass system and relate to each other through Euler's equations.

Review 2.1 Vibration analysis uses complex numbers to solve the EOM's differential equation. In this text the imaginary number is termed j (sometimes referred to as i): such that:

$$j = \sqrt{-1} \tag{2.34}$$

and:

$$j^2 = -1 \tag{2.35}$$

A general complex number, x, can be expressed as:

$$x = a + bj \tag{2.36}$$

here, a is referred to as the real number and b is the imaginary part of the number x. Such complex numbers can be represented in the complex plane, also called a Argand plot. The absolute value or modules is defined as $|x|$ presented on the complex plot.

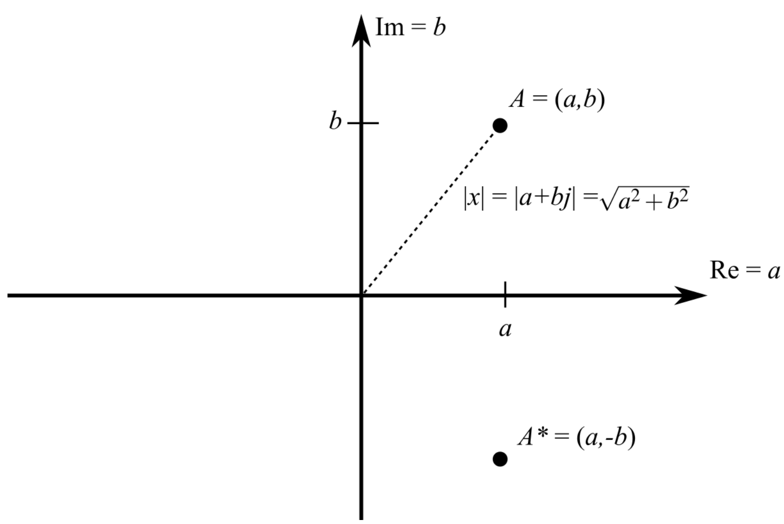


Figure 2.10: A conjugate pair of numbers (A and $A*$) represented on the complex plane.

A and A^* prime are complex conjugate pairs. In mathematics, the complex conjugate of a complex number is the number with an equal real part and an imaginary part equal in magnitude but opposite in sign. In other words, a conjugate pair is $a + bj$ and $a - bj$.

Definition - con·ju·gate (adjective): Coupled, connected, or related.

Review 2.2 Euler's (pronounced oy-ler) formula, named after Swiss engineer and mathematician Leonhard Euler (1707-1783), is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number x,

$$e^{j\psi} = \cos(\psi) + j\sin(\psi) \tag{2.37}$$

where $j = \sqrt{-1}$. This equation can also be expressed as:

$$e^{-j\psi} = \cos(\psi) - j\sin(\psi) \tag{2.38}$$

This can be expressed in terms of polar coordinates as:

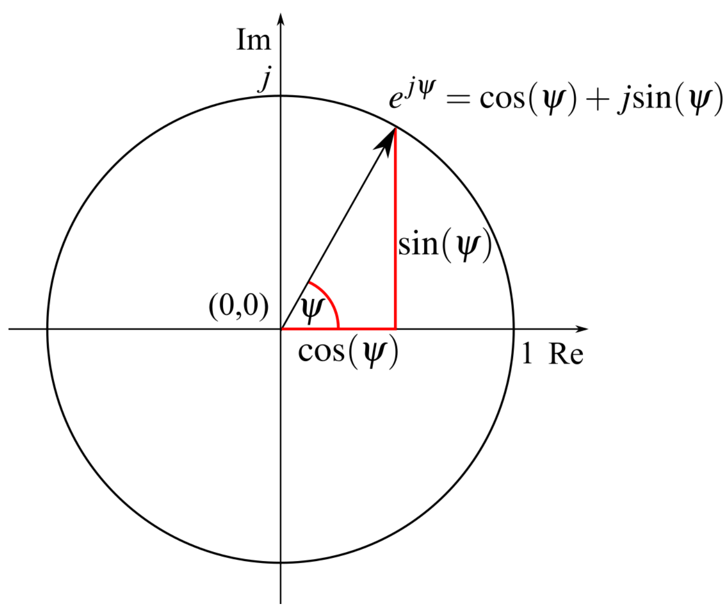


Figure 2.11: Euler's formula illustrated on the unit circle in the complex plane.

2.2.1 Formulating the General Solution for a 1-DOF Spring-Mass System

We can also solve the following EOM as an elementary differential equation:

$$m\ddot{x} + kx = 0 \tag{2.39}$$

in a more analytical manner using the theory of elementary differential equations. To do this the form:

$$x(t) = ae^{\lambda t} \tag{2.40}$$

is assumed where, a and t are nonzero constants that need to be determined. Using successive differentiation,the assumed solution becomes:

$$\dot{x}(t) = \lambda ae^{\lambda t} \tag{2.41}$$

and

$$\ddot{x}(t) = \lambda^2 a e^{\lambda t} \tag{2.42}$$

therefore, $m\ddot{x}(t) + kx(t) = 0$ becomes:

$$m\lambda^2 a e^{\lambda t} + k a e^{\lambda t} = 0 \tag{2.43}$$

Next, the above expressions is divide by $ae^{\lambda t}$ to obtain the characteristic equation:

$$m\lambda^2 + k = 0 \tag{2.44}$$

This can be done because $ae^{\lambda t}$ is never zero, therefore, the expressions is never divide by zero. The quadratic formula gives us:

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_{\mathrm{n}} j \tag{2.45}$$

remember that $\omega_{\mathrm{n}} = \sqrt{\frac{k}{m}}$. Notice that the $\pm$ tells us there are two solutions to this problem. So, putting λ back into the assumed solution results in two solutions (one positive, one negative):

$$x(t) = a_1 e^{+\omega_{\mathrm{n}} jt} \tag{2.46}$$

and

$$x(t) = a_2 e^{-\omega_{\mathrm{n}} jt} \tag{2.47}$$

As these solutions only consider, and are only valid for, linear systems, the sum of the solutions is also a solution. This simplification results in:

$$x(t) = a_1 e^{+\omega_{\mathrm{n}} jt} + a_2 e^{-\omega_{\mathrm{n}} jt} \tag{2.48}$$

where a_1 and a_2 are complex valued constants of integration.

Example 2.3 Show that $x(t) = a_1 e^{+\omega_{\mathrm{n}} jt} + a_2 e^{-\omega_{\mathrm{n}} jt}$ is equal to $A\sin(\omega_{\mathrm{n}} t + \phi)$.

Solution: This equation was derived using Euler's formula and it can be shown that this equation is equivalent to the $A\sin(\omega_{\mathrm{n}} + \phi)$. To recover the previously assumed solution, the knowledge that a_1 and a_2 are complex congregate pairs and as such the magnitude can be expressed as $a_1 = a_2$ is leveraged. Using Euler's polar notation, a_1 and a_2 can be expressed as

$$a_1 = a_2 = ae^{j\psi} \tag{2.49}$$

where a and ψ are real numbers, the equation becomes:

$$x(t) = ae^{j(\omega_{\mathrm{n}} t + \psi)} + ae^{-j(\omega_{\mathrm{n}} t + \psi)} \tag{2.50}$$

this becomes:

$$x(t) = a\big(e^{j(\omega_{\mathrm{n}} t + \psi)} + e^{-j(\omega_{\mathrm{n}} t + \psi)}\big) \tag{2.51}$$

Remembering Euler's equations from before, this becomes:

$$x(t) = a\big(\cos(\omega_{\mathrm{n}} t + \psi) + j\sin(\omega_{\mathrm{n}} t + \psi) + \cos(\omega_{\mathrm{n}} t + \psi) - j\sin(\omega_{\mathrm{n}} t + \psi)\big) \tag{2.52}$$

combining the "cos" terms and canceling out the "sin" terms this becomes:

$$x(t) = 2a \cdot \cos(\omega_n t + \psi) \tag{2.53}$$

This is equivalent to $x(t) = A\sin(\omega_n t + \phi)$ considering that $A = 2a$ and knowing $\sin(\phi) = \cos(\phi + \psi)$. To expand, this is because the sin and cos are only differentiated by a phase shift.

Next, a general solution for the EOM is obtained. Using the previous solution:

$$x(t) = a_1 e^{+\omega_n jt} + a_2 e^{-\omega_n jt} \tag{2.54}$$

we can expand this into the form:

$$x(t) = a_1\big(\cos(\omega_n t) + j\sin(\omega_n t)\big) + a_2\big(\cos(\omega_n t) - j\sin(\omega_n t)\big) \tag{2.55}$$

using trigonometric functions. This equates to:

$$x(t) = (a_1 + a_2) \cdot \cos(\omega_n t) + (a_1 - a_2)j \cdot \sin(\omega_n t) \tag{2.56}$$

As $x(t)$ is always real, A_1 and A_2 can be defined as:

$$A_1 = (a_1 + a_2) \tag{2.57}$$

and

$$A_2 = (a_1 - a_2)j \tag{2.58}$$

Lastly, as the general solution is written as:

$$x(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) \tag{2.59}$$

This is the general solution for the EOM ($m\ddot{x} + kx = 0$) of the considered oscillating system where A_1 and A_2 are defined as:

$$A = \sqrt{A_1^2 + A_2^2} \tag{2.60}$$

and

$$\phi = \tan^{-1}\left(\frac{A_1}{A_2}\right) \tag{2.61}$$

These are obtained from a trigonometric relationship, similar to that used before:

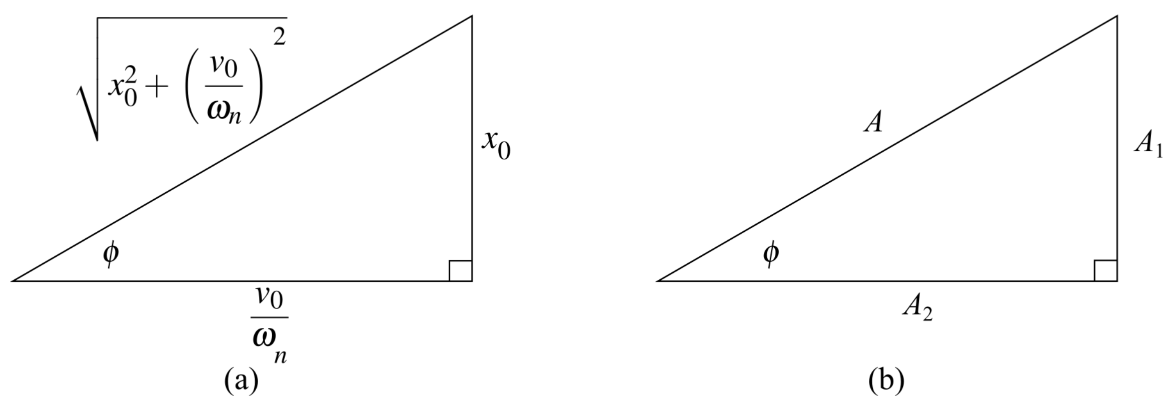


Figure 2.12: Trigonometric relationship between the initial conditions, amplitude, and phase, for free vibration of a 1-DOF system expressed with: (a) variables for initial conditions; and (b) generic variables A_1 and A_2.

again, A and ϕ are:

$$A = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2} \tag{2.62}$$

$$\phi = \tan^{-1}\left(\frac{x_0 \omega_n}{v_0}\right) \tag{2.63}$$

Example 2.4 Using the general solution:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{2.64}$$

Calculate the values of A_1 and A_2 in terms of their initial conditions x_0 and v_0.
Solution: Knowing the following for x and $\dot{x}$:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{2.65}$$

$$\dot{x}(t) = -A_1 \omega_n \sin(\omega_n t) + A_2 \omega_n \cos(\omega_n t) \tag{2.66}$$

Now apply the initial conditions, $x(0) = 0$ and $v(0) = 0$, this yields:

$$x(0) = x_0 = A_1 \tag{2.67}$$

$$\dot{x}(0) = v_0 = A_2 \omega_n \tag{2.68}$$

Solving for A_1 and A_2 shows us:

$$A_1 = x_0, \text{ and } A_2 = \frac{v_0}{\omega_n} \tag{2.69}$$

thus:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{2.70}$$

2.2.2 Solution of 1-DOF System in Three Forms

Form one, for $m\ddot{x} + kx = 0$ subject to nonzero initial conditions can be written as:

$$x(t) = a_1 e^{+\omega_n jt} + a_2 e^{-\omega_n jt} \tag{2.71}$$

where a_1 and a_2 are complex terms. Form two is:

$$x(t) = A \sin(\omega_n t + \phi) \tag{2.72}$$

while form three is:

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{2.73}$$

where A, ϕ, A_1, and A_2, are all real-valued constants. Each set of constants can be related to each other by:

$$A = \sqrt{A_1^2 + A_2^2} \qquad \phi = \tan^{-1}\left(\frac{A_1}{A_2}\right) \tag{2.74}$$

$$A_1 = (a_1 + a_2) \qquad A_2 = (a_1 - a_2)j \tag{2.75}$$

$$a_1 = \frac{A_1 - A_2 j}{2} \qquad a_2 = \frac{A_1 + A_2 j}{2} \tag{2.76}$$

Which follow from trigonometric identities and the Euler's formulas.

2.3 Damping

The response of a spring-mass system predicts that a system will oscillate indefinitely. However, we know that this is not true from observing real-world solutions. So based on real-world observations and mathematical conveniences, we need to add a term that will remove "energy" from the system with time. To do this the idea of the ideal dashpot is introduced. A linear dashpot is diagrammed in figure 2.13 and is a mechanical device that resits motion via viscous friction and therefore converts the mechanical energy of the system into thermal energy that is dissipated.

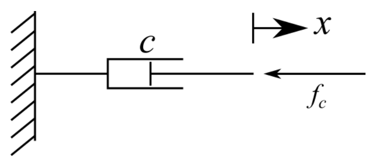


Figure 2.13: Schematic of a liner dashpot showing the damping force (f_c) acting in the opposite direction of the displacement (x).

Just as spring forms a physical model of the cause vibration, through its storage and release of energy, a dashpot (sometimes called a damper) forms a physical model for dissipating energy. Dashpots create a resisting or damping force that acts opposite to the direction of travel (as annotated in figure 2.13) and is proportional to the velocity. Therefore, the damping forces f_c can be computed as:

$$f_c = c\dot{x} \tag{2.77}$$

the constant c, called the damping coefficient, has the units of kg/s. Dashpots are a mathematical representation of viscous dampers installed in automobiles, aircraft, structures, and other mechanical devices. However, all systems have inherent damping not just systems with physical dampers. The spring-mass system can be used as a representations of real-world systems with inherent damping as demonstrated by the rubber engine mount depicted in figure 2.14.

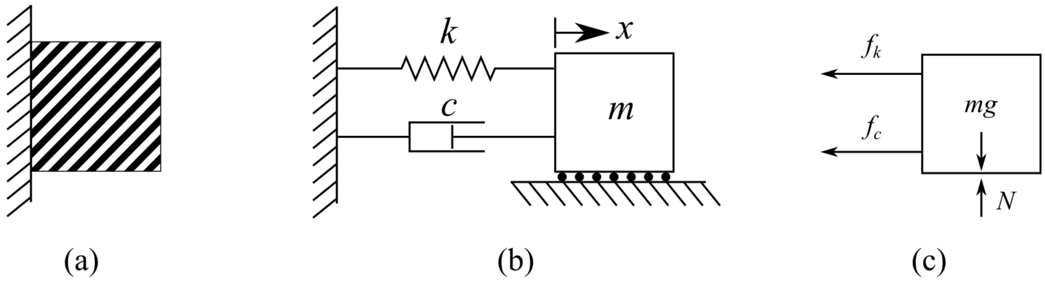


Figure 2.14: Modeling of a rubber engine mount as an spring-dashpot-mass model showing (a) the rubber engine mount; (b) idealized model of the rubber month; and (c) the FBD of the idealized model

Depending on the amount of damping present in a system, the temporal response of the system will represent itself in various way, as represented in figure 2.15. To reiterate, an undamped case will oscillates around the equilibrium and does not decay. If a limited amount of damping is present in a system it will oscillates around the equilibrium and slowly decay with time to the equilibrium position, this is termed underdamped. If an excessive amount of damping is present, the system will not oscillate but decay directly to the equilibrium position, this is termed the overdamped case. Lastly, there exists a special case that results in the system converging as quickly as possible to the equilibrium position without oscillations; this cases is termed the critically damped case. Furthermore, the amount of damping required to obtain a critically damped system is the damping value that the underdamped and overdamped cases for a specific system. To recap, the key types of damping are:

- **Undamped** - Oscillates around the equilibrium and does not decay.
- **Underdamped** - Oscillates around the equilibrium and slowly decays and is the most common case.
- **Overdamped** - Does not pass the equilibrium position and is a simple a decay with no oscillation.
- **Critically damped** - provides the quickest approach to zero amplitude for a damped oscillator.

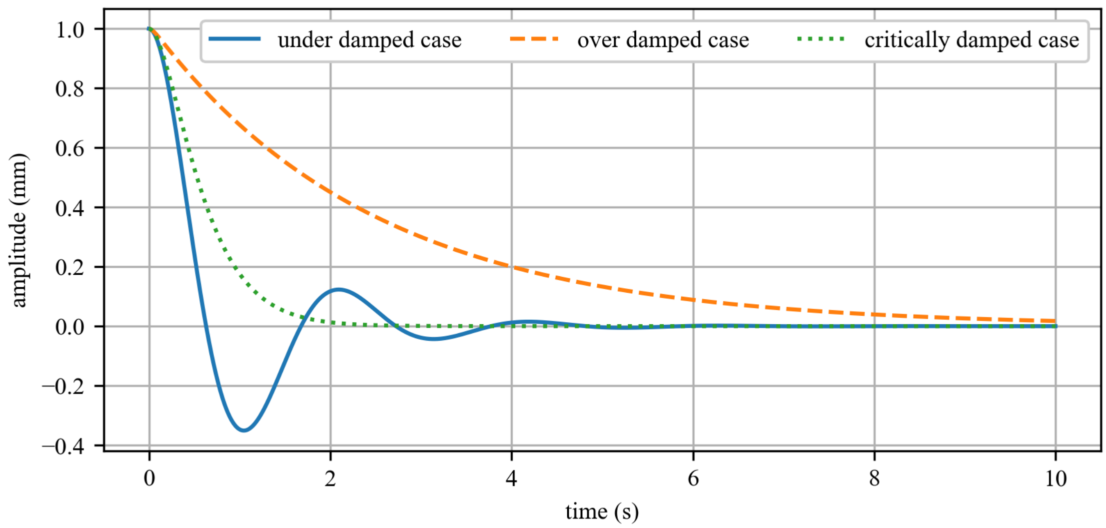


Figure 2.15: Temporal responses for the three types of damping: underdamped, over damped, and critically damped.

2.3.1 Modeling Vibrating Systems with Damping

The spring-mass system of chapter 1 can be expanded to a spring-dashpot-mass system that considers the damping component of the system. A mathematical model of the spring-dashpot-mass system can be developed for the case present in figure 2.16. Using the FBD for the system, it can conclude that the EOM for this system:

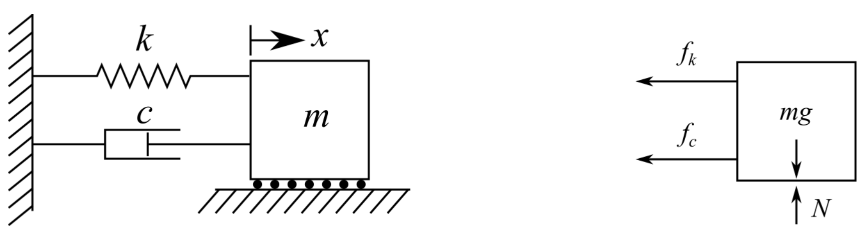


Figure 2.16: Spring-dashpot-mass model showing: (a) a schematic of the system; and (b) the FBD of the system.

is:

$$m\ddot{x}(t) = -f_c - f_k \tag{2.78}$$

Rearranging into standard form and concerting forces into parameters c and k results in:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{2.79}$$

This system is subject to the same initial conditions as before, $x(0) = x_0$ and $\dot{x}(0) = v_0$. Again, choosing to model it this way for convinces, so let's solve it in a similar manner to the EOM without damping. Again, assume the solution:

$$x(t) = ae^{\lambda t} \tag{2.80}$$

here, a and t are nonzeros constants that need to be determined. Using successive differentiation, we get:

$$\dot{x}(t) = \lambda a e^{\lambda t} \tag{2.81}$$

and

$$\ddot{x}(t) = \lambda^2 a e^{\lambda t} \tag{2.82}$$

therefore, $m\ddot{x} + c\dot{x} + kx = 0$ becomes:

$$m\lambda^2 a e^{\lambda t} + c\lambda a e^{\lambda t} + k a e^{\lambda t} = 0 \tag{2.83}$$

Now we divide by $ae^{\lambda t}$ to obtain the **characteristic equation**:

$$m\lambda^2 + c\lambda + k = 0 \tag{2.84}$$

We can do this because $ae^{\lambda t}$ is never zero, therefore, we never divide by zero. The quadratic formula gives us:

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{-c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \tag{2.85}$$

Some key points from this equation:

- The $\pm$ tells us there are two solutions to this problem
- if $c^2 - 4km < 0$, system is Underdamped, solutions are complex conjugate pairs
- if $c^2 - 4km = 0$, system is critically damped, solutions are equal negative real numbers
- if $c^2 - 4km > 0$, system is Overdamped, solutions are distinct negative real numbers

From this, we can see that $c^2 - 4km = 0$ is a special value, let us define a value for c that will give us this critical damping number. We will call it the **critical damping coefficient** (c_{cr}). So setting the equation as:

$$c_{\mathrm{cr}}^2 - 4km = 0 \tag{2.86}$$

giving us:

$$c_{\mathrm{cr}}^2 = 4km \tag{2.87}$$

next we can derive the function:

$$c_{\mathrm{cr}} = 2\sqrt{km} = 2\left(\frac{\sqrt{m}}{\sqrt{m}}\right)\sqrt{km} = 2m\omega_{\mathrm{n}} \tag{2.88}$$

remember that $\omega_{\mathrm{n}} = \sqrt{\frac{k}{m}}$ for an undamped system. Next, we generate a non-dimensional number (ζ), pounced 'zeta' that will allow us to distinguish between different types of damping. ζ is called the **critical damping ratio**.

$$\zeta = \frac{c}{c_{\mathrm{cr}}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_{\mathrm{n}}} \tag{2.89}$$

Now if we put the ζ back into the characteristic equation and resolve using the quadratic equation we get:

$$\lambda_{1,2} = -\zeta\omega_{\mathrm{n}} \pm \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1} \tag{2.90}$$

From this equation it become clear that ζ determines whether the roots are complex or real, this in turn determines the nature of the response of the structure. Listing our possible responses we get: For each damping case, we will have a different solution to the problem.

damping case	critical damping ratio	radicand	solutions
under damped	$0 < \zeta < 1$	$c^2 - 4km < 0$	complex conjugate pairs
critically damped	$\zeta = 1$	$c^2 - 4km = 0$	equal negative real numbers
over damped	$1 < \zeta$	$c^2 - 4km > 0$	distinct negative real numbers

2.3.2 Modeling Underdamped Motion

In the case that $0 < \zeta < 1$, a complex conjugate pair of roots are the solutions to the characteristic equation after pulling out a $\sqrt{-1}$:

$$\lambda_1 = -\zeta\omega_\mathrm{n} + \omega_\mathrm{n}\sqrt{1-\zeta^2}\,j \tag{2.91}$$

and:

$$\lambda_2 = -\zeta\omega_\mathrm{n} - \omega_\mathrm{n}\sqrt{1-\zeta^2}\,j \tag{2.92}$$

Where the j is pulled out because:

$$\sqrt{1-\zeta^2}\,j = \sqrt{(1-\zeta^2)(-1)} = \sqrt{\zeta^2 - 1} \tag{2.93}$$

Next, let us "arbitrarily" define:

$$\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2} \tag{2.94}$$

where ω_d is the **damped natural frequency**. Therefore, the equations become:

$$\lambda_1 = -\zeta\omega_\mathrm{n} + \omega_\mathrm{d}\,j \tag{2.95}$$

and:

$$\lambda_2 = -\zeta\omega_\mathrm{n} - \omega_\mathrm{d}\,j \tag{2.96}$$

Again, we have two solutions to a linear problem, so we can combine these into one solution and insert λ into the assumed solution $ae^{\lambda t}$ to obtain:

$$x(t) = a_1 e^{-\zeta\omega_\mathrm{n}t + \omega_\mathrm{d}t\,j} + a_2 e^{-\zeta\omega_\mathrm{n}t - \omega_\mathrm{d}t\,j} \tag{2.97}$$

where a_1 and a_2 are complex valued constants. This can now be simplified into:

$$x(t) = e^{-\zeta\omega_\mathrm{n}t}\left(a_1 e^{\omega_\mathrm{d}t\,j} + a_2 e^{-\omega_\mathrm{d}t\,j}\right) \tag{2.98}$$

Using Euler's equations, (same as before) and choosing:

$$A_1 = (a_1 - a_2)\,j \tag{2.99}$$

and

$$A_2 = (a_1 + a_2) \tag{2.100}$$

The **general form** of this solution is then:

$$x(t) = e^{-\zeta\omega_\mathrm{n}t}\big(A_1\sin(\omega_\mathrm{d}t) + A_2\cos(\omega_\mathrm{d}t)\big) \tag{2.101}$$

Recall that for undamped 1-DOF systems we showed

$$x(t) = A\sin(\omega_n t + \phi) = A_1\sin(\omega_n t) + A_2\cos(\omega_n t) \tag{2.102}$$

As $e^{-\zeta\omega_n t}$ accounts for the damping, our current solution becomes:

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \tag{2.103}$$

Now that we have x and $\dot{x}$, we can solve for the boundary conditions x_0 and v_0 by setting $t = 0$, we get:

$$x(0) = x_0 = A\sin(\phi) \tag{2.104}$$

and taking the directive of $x(t)$ using the product rule (fg)'= f'g+fg', we get:

$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + Ae^{-\zeta\omega_n t}\omega_d\cos(\omega_d t + \phi) \tag{2.105}$$

$$\dot{x}(0) = v_0 = -\zeta\omega_n A\sin(\phi) + A\omega_d\cos(\phi) \tag{2.106}$$

a simplification can be made to the prior equation by letting $A = x_0/\sin(\phi)$. This gives us the equation:

$$\dot{x}(0) = v_0 = -\zeta\omega_n\left(\frac{x_0}{\sin(\phi)}\right)\sin(\phi) + \left(\frac{x_0}{\sin(\phi)}\right)\omega_d\cos(\phi) \tag{2.107}$$

that can be simplified to:

$$\dot{x}(0) = v_0 = -\zeta\omega_n x_0 + x_0\omega_d\cot(\phi) \tag{2.108}$$

The above equation related v_0 to ϕ using terms that are known for a giving system (ζ, ω_n, x_0, and ω_d). Therefore, this expression can be used to solve for ϕ:

$$\cot(\phi) = \frac{v_0 + \zeta\omega_n x_0}{x_0\omega_d} \tag{2.109}$$

and as $\tan(\phi) = 1/\cot(\phi)$:

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right) \tag{2.110}$$

Thereafter, we can solve for A considering the fact that we sent $A = x_0/\sin(\phi)$. Using the trigonometric relationship between expressed in equation 2.109 and visualized in figure 2.17:

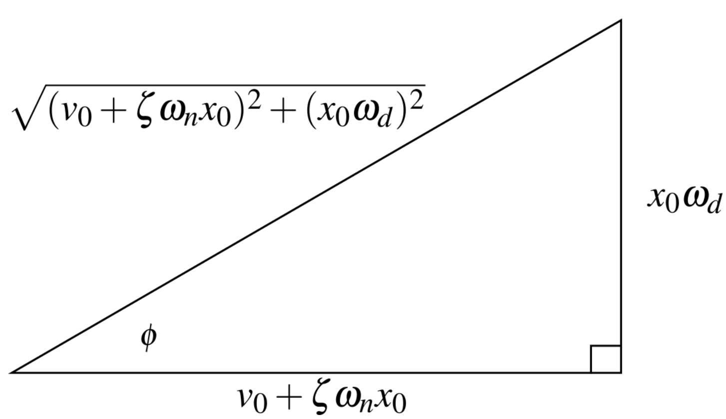


Figure 2.17: Trigonometric relationship between the initial conditions (x_0 and v_0), amplitude A, and phase ϕ for underdamped motion of a 1-DOF system.

we show that $\sin(\phi)$ can be expressed as:

$$\sin(\phi) = \frac{x_0 \omega_\mathrm{d}}{\sqrt{(v_0 + \zeta \omega_\mathrm{n} x_0)^2 + (x_0 \omega_\mathrm{d})^2}} \tag{2.111}$$

and applying $A = x_0 / \sin(\phi)$ we get:

$$A = \frac{\sqrt{(v_0 + \zeta \omega_\mathrm{n} x_0)^2 + (x_0 \omega_\mathrm{d})^2}}{\omega_\mathrm{d}} = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta \omega_\mathrm{n} x_0}{\omega_\mathrm{d}} \right)^2} \tag{2.112}$$

Finally, collecting all of our important equations:

- Critical damping coefficient: $c_\mathrm{cr} = 2\sqrt{km} = 2m\omega_\mathrm{n}$
- Damping ratio: $\zeta = \frac{c}{c_\mathrm{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_\mathrm{n}}$
- Damped natural frequency: $\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$
- Solution for underdamped system: $x(t) = Ae^{-\zeta \omega_\mathrm{n} t} \sin(\omega_\mathrm{d} t + \phi)$, where:

$$A = \frac{\sqrt{(v_0 + \zeta \omega_\mathrm{n} x_0)^2 + (x_0 \omega_\mathrm{d})^2}}{\omega_\mathrm{d}} \qquad \phi = \tan^{-1}\left(\frac{x_0 \omega_\mathrm{d}}{v_0 + \zeta \omega_\mathrm{n} x_0} \right)$$

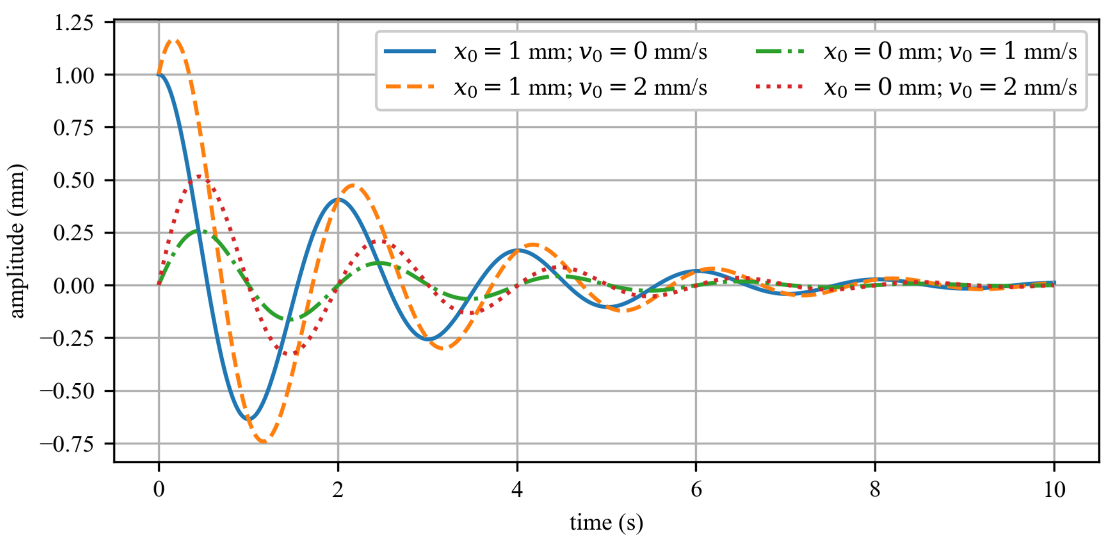


Figure 2.18: Four example responses for an under damped 1-DOF system ($\zeta = 0.142$) with various initial conditions.

Example 2.5 Consider the following 1-DOF system, where $k = 857.8$ N/m, $c = 7.8$ kg/s, and $m = 49.2 \times 10^{-3}$ kg, calculate the damped frequency in rad/s and Hz. What damping case is this system?

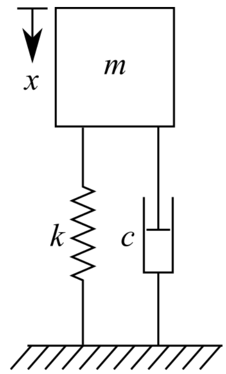


Figure 2.19: 1-DOF spring-dashpot-mass system.

Solution:

Calculate the undamped frequency:

$$\omega_\mathrm{n} = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132 \ \text{ rad/s} \tag{2.113}$$

The systems critical damping value:

$$c_\mathrm{cr} = 2\sqrt{km} = 2\sqrt{k = 857.8 \cdot 49.2 \times 10^{-3}} = 12.993 \ \text{ kg/s} \tag{2.114}$$

And the critical damping ratio:

$$\zeta = \frac{c}{c_\mathrm{cr}} = \frac{7.8}{12.993} = 0.600 \tag{2.115}$$

This can also be expressed as 60% damped, this is a underdamped system, and the system will oscillate. Now we can calculate the damped frequency:

$$\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1 - \zeta^2} = \omega_\mathrm{n}\sqrt{1 - 0.600^2} = 105.6 \ \text{ rad/s} \tag{2.116}$$

Therefore, the system oscillates at 105.6 rad/sec or 16.81 Hz

Example 2.6 For a damped one DOF system where m, c, and k are known to be m = 1 kg, c = 2 kg/s, and k = 10 N/m. Calculate the value of ζ and ω_n. Is the system overdamped, underdamped, or critically damped?

Solution:

The natural frequency is calculated as

$$\omega_{\mathrm{n}} = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{1}} = 3.16 \ \text{rad/s} \tag{2.117}$$

The damping can be calculated as:

$$\zeta = \frac{c}{2\omega_{\mathrm{n}} m} = \frac{2}{2\left(\sqrt{\frac{10}{1}}\right)(1)} = \frac{1}{\sqrt{10}} = 0.316 \tag{2.118}$$

So the damped natural frequency is equal to:

$$\omega_{\mathrm{d}} = \omega_{\mathrm{n}} \sqrt{1 - \zeta^2} = \sqrt{10} \sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2} = 0.3 \ \text{rad/s} \tag{2.119}$$

As $0 < \zeta < 1$ the system is underdamped.

Example 2.7 For the following industrial device consisting of a mass isolated from its fixtures by two rubber dampers and a offset spring provide an estimate of the system's damped natural frequency in the vertical direction. Assume the the rubber dampers add damping and only negligible stiffness to the system and that the spring is long enough such that the angles remain constant.

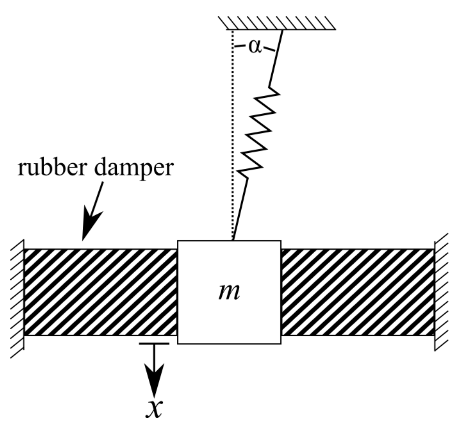


Figure 2.20: Industrial device (mass) connected to a fixed point with a rubber damper and spring at an angle.

Solution:

First and foremost, we need to develop a mass-spring-dashpot representation of the system. The is presented in what follows:

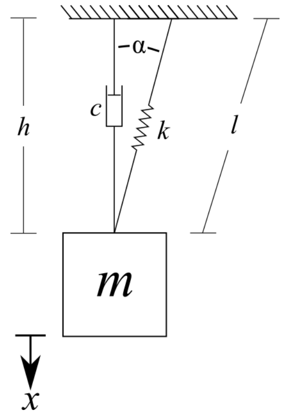
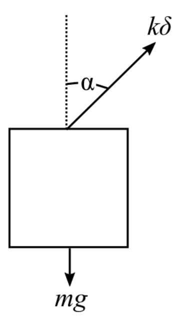


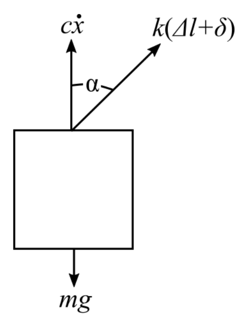
Figure 2.21: Mass-spring-dashpot representation of the industrial system represented if figure 2.20.

where the damping in the vertical direction provided by the rubber damper is modeled a dashpot in the vertical direction. As we only want an estimate of the frequency, the assumption that the is small and as such α of the displaced state is equal α of the equilibrium state. This leads to the FBD for the equilibrium and displaced states:

equilibrium position



displaced position "x"



The equation for the equilibrium state is:

$$+\downarrow \sum F_x = mg - k\delta\cos(\alpha) = 0$$

and in the displaced state:

$$+\downarrow \sum F_x = mg - c\dot{x} - k\cos(\alpha)(\Delta l + \delta)$$

Applying Newton's second law and combining these equations yields:

$$m\ddot{x} + c\dot{x} + k\Delta l\cos(\alpha) = 0$$

Looking at the triangles formed by the dashpot and spring it can be shown that:

$$\cos(\alpha) = h/l = x/\Delta l$$

As we assumed the displacement is small and α remains unchanged. Therefore the prior equation becomes:

$$m\ddot{x} + c\dot{x} + k\Delta l \frac{x}{\Delta l} = 0$$

This simplifies to the "normal" EOM for a 1-DOF system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

Therefore, once the values for the system are measured the system's damped natural frequency in the vertical direction can be estimated as

$$\boxed{\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{1-\zeta^2}}$$

2.3.3 Modeling Overdamped Motion

In the case of overdamped systems, $1 < \zeta$, the solutions for λ are distinct real roots that are written as:

$$\lambda_1 = -\zeta\omega_{\mathrm{n}} - \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1} \tag{2.120}$$

and:

$$\lambda_2 = -\zeta\omega_{\mathrm{n}} + \omega_{\mathrm{n}}\sqrt{\zeta^2 - 1} \tag{2.121}$$

The solution for the EOM using the assumed solution then becomes:

$$x(t) = e^{-\zeta\omega_{\mathrm{n}}t}\left(a_1 e^{-\omega_{\mathrm{n}}t\sqrt{\zeta^2-1}} + a_2 e^{+\omega_{\mathrm{n}}t\sqrt{\zeta^2-1}}\right) \tag{2.122}$$

This equation represents a non-oscillating response of the system. Again, a_1 and a_2 are solved for using known boundary conditions x_0 and v_0 such that:

$$a_1 = \frac{-v_0 + \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_{\mathrm{n}}x_0}{2\omega_{\mathrm{n}}\sqrt{\zeta^2 - 1}} \tag{2.123}$$

$$a_2 = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_{\mathrm{n}}x_0}{2\omega_{\mathrm{n}}\sqrt{\zeta^2 - 1}} \tag{2.124}$$

Typical responses for a overdamped system with various initial conditions are shown below:

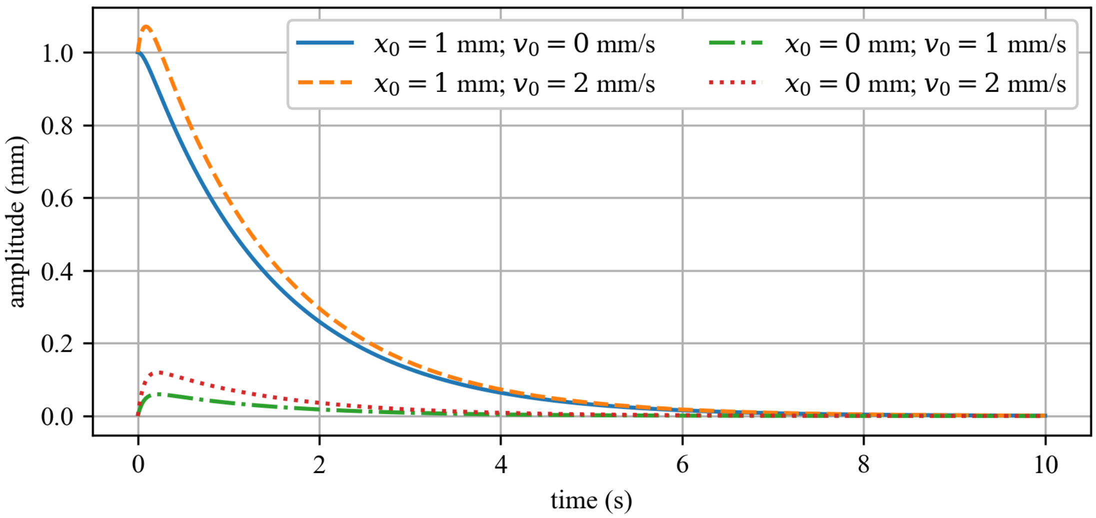


Figure 2.22: Four example responses for an over damped 1-DOF system ($\zeta = 2.371$) with various initial conditions.

2.3.4 Modeling critically damped motion

In the case of critically damped systems, $\zeta = 1$, the solutions for λ will be equal negative real numbers, therefore from before:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{2.125}$$

We get:

$$\lambda_1 = \lambda_2 = -\omega_n \tag{2.126}$$

Because both solutions (a_1 and a_2) are the same, we multiply the second solution by t so the solution for a critically damped system is in the same form as before. The solution for the EOM using the assumed solution then becomes:

$$x(t) = a_1 e^{-\omega_n t} + a_2 t e^{-\omega_n t} \tag{2.127}$$

This simplifies into:

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t} \tag{2.128}$$

This equation represents a non-oscillating response of the system. Again, a_1 and a_2 are solved for using known boundary conditions x_0 and v_0 such that:

$$a_1 = x_0 \tag{2.129}$$

$$a_2 = v_0 + \omega_n x_0 \tag{2.130}$$

2.3.5 Standard Form of the EOM

The EOM for a damped 1-DOF system is written in a "standard form" in which the effect of the damping ration and natural frequencies are more obvious. To get to the standard form, the normal form of the EOM:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.131}$$

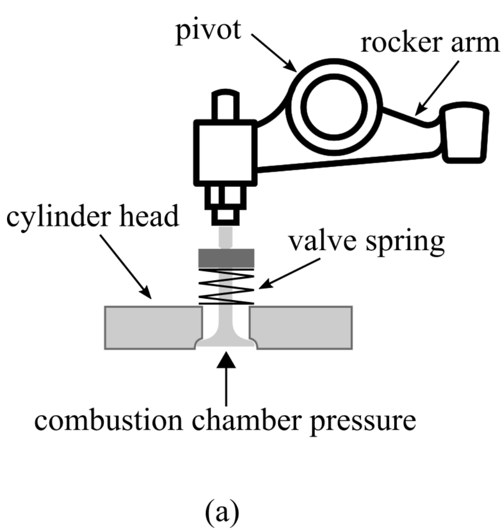
is divided by what the constant terms associated with the acceleration term. In this example, this is m. Dividing every term by m yields:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{2.132}$$

Numerical manipulations can be undertaken to get the coefficients of the velocity and displacement terms into coefficients that more clearly express the characteristics of the vibrating system:

$$\ddot{x} + 2\zeta\omega_{\mathrm{n}}\dot{x} + \omega_{\mathrm{n}}^2 x = 0 \tag{2.133}$$

Example 2.8 A engine valve assembly depicted is figure 2.23 where J is the inertia caused by the right-hand side of the rocker arm. Derive an analytical solution for the natural frequency of the rocker arm. Use the assumptions $\sin(\theta) = \theta$ and $\cos(\theta) = 1$.



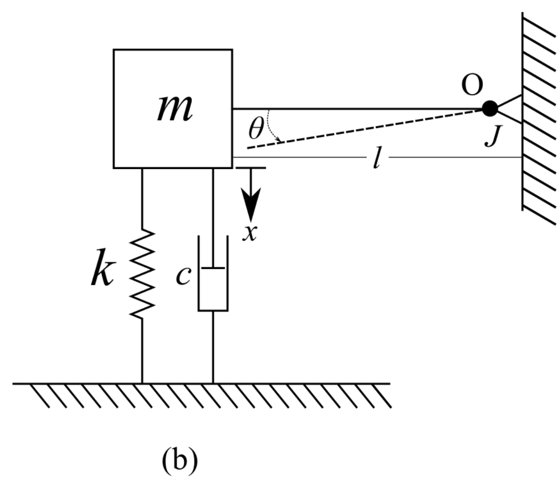
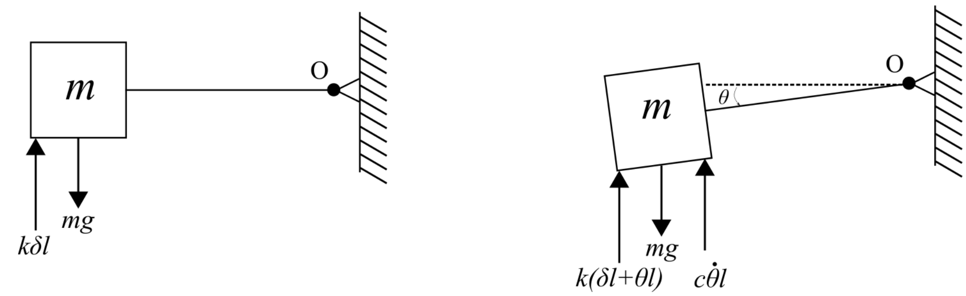


Figure 2.23: Rocker arm assembly of an internal combustion engine showing: (a) a diagram of the system and; (b) the FBD of the system.

Solution:

Taking the sum of the moments about O and considering the inertia caused by the right-hand side of the rocker arm, J, the FBDs can be written as:

equilibrium position displaced position "x"



The equation for the equilibrium state is:

$$\circlearrowleft+\sum M_o = mgl - kl^2\delta = 0$$

and in the displaced state:

$$\circlearrowleft+\sum M_o = mgl - kl^2\delta - kl^2\theta - cl^2\dot{\theta} = 0$$

Applying Newton's second law and combining these equations yields:

$$(J+ml^2)\ddot{\theta} + cl^2\dot{\theta} + kl^2\theta = 0 \tag{2.134}$$

Therefore, the standard form of the EOM is:

$$\ddot{\theta} + \frac{cl^2}{J+ml^2}\dot{\theta} + \frac{kl^2}{J+ml^2}\theta = 0 \tag{2.135}$$

Results in the following analytical solution for the natural frequency:

$$\omega_{\mathrm{n}} = \sqrt{\frac{kl^2}{J+ml^2}}\ \text{rad/s} \tag{2.136}$$

2.4 Logarithmic decrement

For a vibrating system, the mass (m) and stiffness (k) can be measured using scales and static deflection tests. However, the damping coefficient (c) is a more difficult quantity to determine. From k and m we can compute the natural frequency (ω_{n}) and the critical damping coefficient (c_{cr}). Therefore, knowing that the critical damping ratio (ζ) is defined as:

$$\zeta = \frac{c}{c_{\mathrm{cr}}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_{\mathrm{n}}} \tag{2.137}$$

if we calculate ζ, we can obtain c for the system of interest. This is made possible because c_{cr} can be calculated from k and m. Observing the temporal response for the underdamped system,

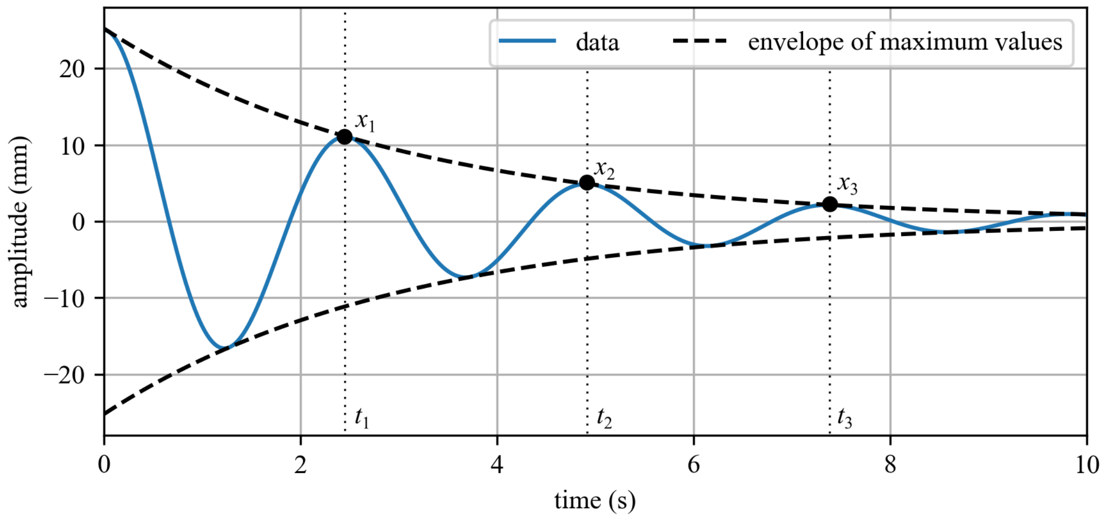


Figure 2.24: Measuring the peak displacements points in an experimental system with decay caused by damping.

we mark three points of maximum amplitude, x_1, x_2, and x_3 that happen at t_1, t_2, and t_3, respectively. Considering displacement values for the first two points x_1 and x_2. separated by a complete period (T). Knowing that one cycle is 2π, the time period for this complete cycle is given by:

$$t_2 - t_1 = \frac{2\pi}{\omega_\mathrm{d}} = \frac{2\pi}{\omega_\mathrm{n}\sqrt{1-\zeta^2}} \tag{2.138}$$

where ω_d is the damped natural frequency. This is the time period (T) of damped oscillations. If we derive an equation for the values of the peaks, also called the envelope of maximum values, we get:

$$x_\mathrm{peaks} = Ae^{-\zeta\omega_\mathrm{n}t} \tag{2.139}$$

Knowing that the system is underdamped, A can be solved for using the initial conditions x_0 and v_0, therefore:

$$A = \frac{\sqrt{(v_0 + \zeta\omega_\mathrm{n}x_0)^2 + (x_0\omega_\mathrm{d})^2}}{\omega_\mathrm{d}} \tag{2.140}$$

In terms of t_1 and t_2, we can express the displacement at these times as:

$$x_1 = Ae^{-\zeta\omega_\mathrm{n}t_1} \tag{2.141}$$

and

$$x_2 = Ae^{-\zeta\omega_\mathrm{n}t_2} \tag{2.142}$$

therefore:

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_\mathrm{n}t_1}}{e^{-\zeta\omega_\mathrm{n}t_2}} = e^{\zeta\omega_\mathrm{n}(t_2-t_1)} \tag{2.143}$$

However, from before we know that $t_2 - t_1 = \frac{2\pi}{\omega_\mathrm{d}} = \frac{2\pi}{\omega_\mathrm{n}\sqrt{1-\zeta^2}}$. Therefore, we can express this last equation as:

$$\frac{x_1}{x_2} = e^{\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \tag{2.144}$$

Next, we take the natural log of both sides to get the logarithmic decrement, denoted by δ:

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln\left(\frac{x(t_1)}{x(t_1+T)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{2.145}$$

This shows us that the ratio of any two successive amplitudes for an underdamped system, vibrating freely, is constant and is a function of the damping only. Sometimes, in experiments, it is more convenient/accurate to measure the amplitudes after say "n" peaks rather than two successive peaks (because if the damping is very small, the difference between the successive peaks may not be significant). The logarithmic decrement can then be given by the equation

$$\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{\mathrm{n+1}}}\right) = \frac{1}{n}\ln\left(\frac{x(t_1)}{x(t_1+nT)}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{2.146}$$

Once we use the experimental data to obtain δ, and knowing that:

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{2.147}$$

we can calculate the value of ζ:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} \tag{2.148}$$

Therefore, having ζ we can solve for the coefficient of damping, c, as:

$$c = \zeta 2\sqrt{km} \tag{2.149}$$

Example 2.9 Calculate the damping coefficient for the system with the measured amplitude as expressed below given that m = 3 kg and k = 43 N/m. Use t_1 = 1 sec, and $t_{n+1} = t_4 = 6$ sec.

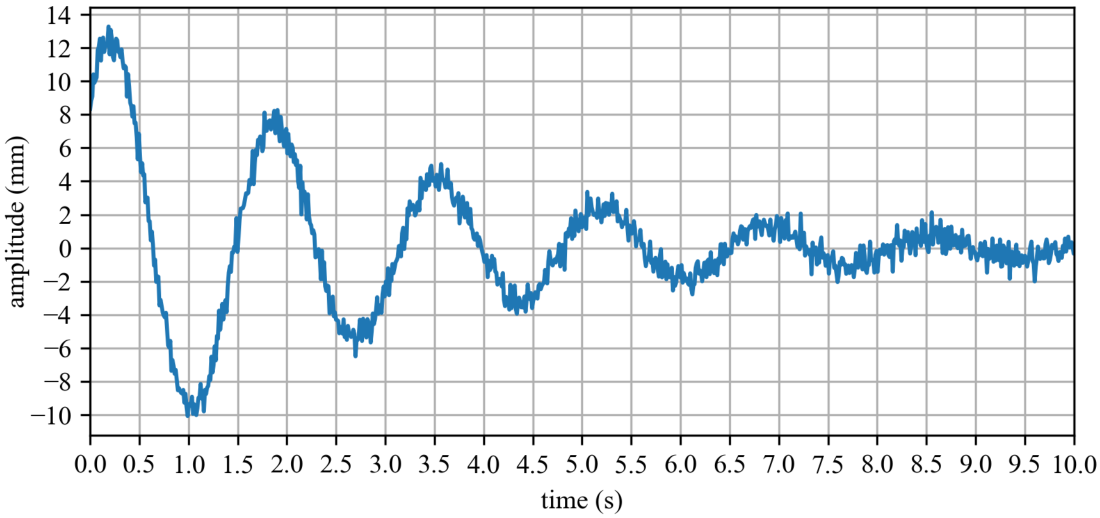


Figure 2.25: Response from an experimental system with noise.

Solution:

First, from the plot we can determine that $x_1 = -9.5$ mm and $x_3 = -1.8$ mm where $n = 3$. Thereafter, we can solve for δ:

$$\delta = \frac{1}{3}\ln\left(\frac{x_1}{x_4}\right) = \frac{1}{3}\ln\left(\frac{-9.5}{-1.8}\right) = 0.554 \tag{2.150}$$

Next, we can calculate ζ, as:

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.554}{\sqrt{4\pi^2 + 0.554^2}} = 0.0879 \tag{2.151}$$

And lastly:

$$c = \zeta 2\sqrt{km} = 0.0879 \cdot 2\sqrt{43 \cdot 3} = 2.0 \text{ kg/s} \tag{2.152}$$

Example 2.10 The free response of a 1000-kg automobile with a stiffness of $k = 400{,}000$ N/m is observed to be underdamped. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, as annotated in figure 2.8, determine the damping coefficient if the displacement at t_1 is measured to be 2 cm and 0.22 cm at t_2.

Solution:

Knowing $x_1 = 2$ cm and $x_2 = 0.22$ cm and $t_2 = T + t_1$, therefore:

$$\delta = \ln\frac{x_1}{x_2} = \ln\frac{2}{0.22} = 2.207 \tag{2.153}$$

and:

$$\zeta = \left(\frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \right) = \left(\frac{2.207}{\sqrt{4\pi^2 + 2.207^2}} \right) = 0.331 \tag{2.154}$$

therefore, we can obtain the damping coefficient as

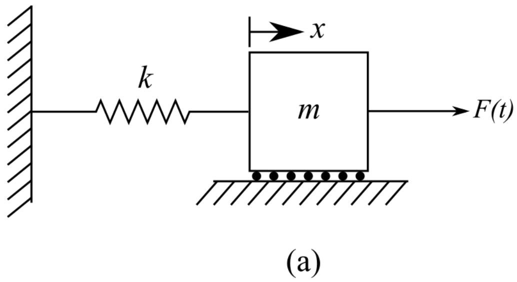
$$c = 2\zeta\sqrt{km} = 2(0.331)\sqrt{400{,}000 \cdot 1{,}000} = 13{,}256 \ \text{kg/s} \tag{2.155}$$

3 Forced Vibrations

Mechanical systems are ofter subjected to an external loading. For example, a piston in a engine if forced up and down my a crankshaft or a seat in an airplane my vibrate due to movement of the jet engines transmitted through the aircraft structure.

3.1 Harmonic Excitations of Undamped Systems

Investigating a single-degree of freedom system for a harmonic input is a useful as it can be solved mathematically with straightforward techniques. Consider the system:



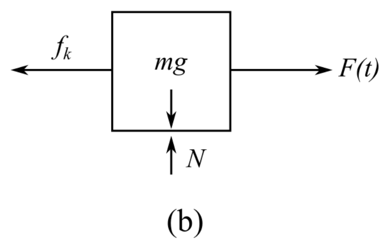


Figure 3.3: 1-DOF system with an external force ($F(t)$) applied, showing: (a) the system configuration; and (b) the free body diagram

where $F(t)$ is the external force applied to the mass. For simplicity, let us consider a harmonic excitation for $F(t)$ such that:

$$F(t) = F_0 \cos(\omega t) \tag{3.3}$$

note that here, ω has no subscript and is the frequency in rad/sec of the driving force. F_0 represents the magnitude of the applied force. This is often called the input frequency, driving frequency, or forcing frequency. Building the EOM for system in figure 3.3 yields:

$$m\ddot{x}(t) + kx(t) = F_0 \cos(\omega t) \tag{3.4}$$

For convinces we drop the "(t)" to make the writing easier can convert this to the standard form by dividing the equation through by m:

$$\ddot{x} + \omega_n^2 x = f_0 \cos(\omega t) \tag{3.5}$$

where:

$$f_0 = \frac{F_0}{m} \tag{3.6}$$

The EOM in this form is a second order, linear nonhomogeneous differential equation. It is nonhomogeneous because there are no terms related to x on the right-hand side of the equation. One way to solve such such an ODE is to recall that the solution for a nonhomogeneous equation is the sum of the homogeneous and particular solutions.

$$x = x_h + x_p \tag{3.7}$$

again, noting that this is a temporal solution where "(t)" is implied. First, knowing that the solution is the sum of two parts: 1) oscillations caused by the spring/mass system; and 2) vibrations caused by the forcing function. The oscillations caused by the spring/mass system will form the homogeneous while the vibrations caused by the forcing function will form the particular solution. As we know the solution for oscillations caused by the spring/mass system form our prior investigation of unforced system we set the equation for the homogeneous solution to be:

$$x_h = A\sin(\omega_n t + \phi) \tag{3.8}$$

Next, we will denote the particular solution as x_p. x_p can be determined by assuming that it is in form of the forcing function, therefore:

$$f_0\cos(\omega t) \tag{3.9}$$

becomes:

$$x_p = X\cos(\omega t) \tag{3.10}$$

where, x_p is the particular solution and X is the amplitude of the forced response. Our total solution for the harmonic excitations of undamped systems now becomes:

$$x(t) = A\sin(\omega_n t + \phi) + X\cos(\omega t) \tag{3.11}$$

This approach, of assuming that $x_p = X\cos(\omega t)$, in order to determine the particular solution is called the **method of undetermined coefficients**. To calculate X, first we take the equations for x_p and $\ddot{x}_p$:

$$x_p = X\cos(\omega t) \tag{3.12}$$

$$\ddot{x}_p = -\omega^2 X\cos(\omega t) \tag{3.13}$$

and substituting these into the equation of motion in standard from yields:

$$-\omega^2 X\cos(\omega t) + \omega_n^2 X\cos(\omega t) = f_0\cos(\omega t) \tag{3.14}$$

As long as $\cos(\omega t) \neq 0$, solving for X yields:

$$X = \frac{f_0}{\omega_n^2 - \omega^2} \tag{3.15}$$

Therefore, as long as $\omega_n \neq \omega$, the particular solution can take the form:

$$x_p = \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t) \tag{3.16}$$

This then expands to the total form:

$$x(t) = A\sin(\omega_n t + \phi) + \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t) \tag{3.17}$$

Expanding this to the general form for the homogeneous solution obtains the equation:

$$x(t) = A_1\sin(\omega_n t) + A_2\cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t) \tag{3.18}$$

As before, we need to determine the values for the coefficients A_1 and A_2 by enforcing the initial conditions x_0 and v_0. Setting the time to zero ($t = 0$) and solving the initial displacement leads to:

$$x(0) = x_0 = A_2 + \frac{f_0}{\omega_n^2 - \omega^2} \tag{3.19}$$

or:

$$A_2 = x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \tag{3.20}$$

again, solving the equation in terms of velocity:

$$\dot{x}(t) = A_1\omega_n \cos(\omega_n t) - A_2\omega_n \sin(\omega_n t) - \omega\frac{f_0}{\omega_n^2 - \omega^2}\sin(\omega t) \tag{3.21}$$

and solving for the initial velocity at $t = 0$:

$$\dot{x}(0) = v_0 = A_1\omega_n \tag{3.22}$$

or:

$$A_1 = \frac{v_0}{\omega_n} \tag{3.23}$$

Therefore, combining the equations we get:

$$x(t) = \left(\frac{v_0}{\omega_n}\right)\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right)\cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t) \tag{3.24}$$

As before, we can relate A_1 and A_2 to each other through the basic trigonometric identities. This yields,

$$x(t) = A\sin(\omega_n t + \phi) + X\cos(\omega t) \tag{3.25}$$

$$A = \sqrt{\left(\frac{v_0}{\omega_n}\right)^2 + (x_0 - X)^2} \tag{3.26}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n(x_0 - X)}{v_0}\right) \tag{3.27}$$

$$X = \frac{f_0}{\omega_n^2 - \omega^2} \tag{3.28}$$

Example 3.1 For a the 1-DOF system:

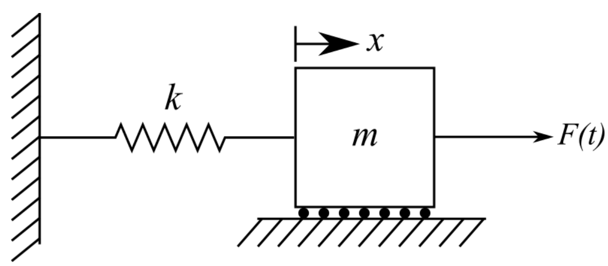


Figure 3.4: 1-DOF spring-mass system subjected to an external force $F(t)$.

with with k = 10 N/m, m = 2.5 kg, ω = 4 rad/sec, F_0 = 0.1 kN, x_0 = 1 mm, and v_0 = 0

mm plot the temporal responses of system considering the free-vibration case and the excited case. Plot these on a single plot to compare the responses.
Solution: The free-vibration response can be plotted using the expression:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{3.29}$$

while the force vibration is expressed using:

$$x(t) = \left(\frac{v_0}{\omega_n}\right) \sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right) \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t) \tag{3.30}$$

These temporal responses are plotted as (Note that the forcing function uses the axis on the right):

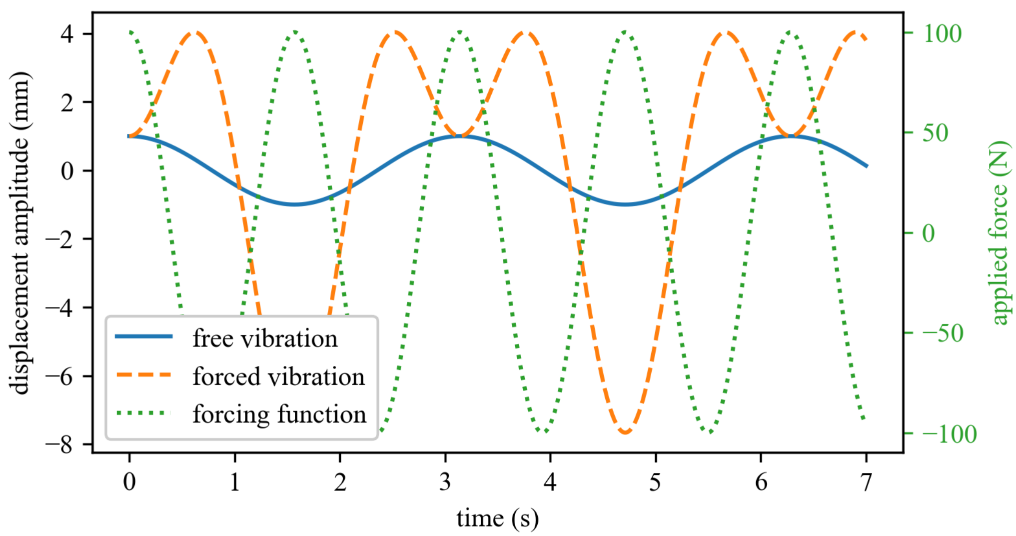


Figure 3.5: Comparison of the temporal response for a 1-DOF system; expressing how the forcing function changes the vibrational temporal response of the system.

3.2 Harmonic Resonance

Recall that our solution from before assumed that $\omega_n \neq \omega$, however, if $\omega_n = \omega$ then the system will develop the phenomenon of resonance. Mathematically, this means the amplitude of the vibrations becomes unbounded. The prior choice of $X\cos(\omega t)$ for the particular solution fails as it is also a solution for a homogeneous equation. Therefore, a new particular solution is needed for the case where $\omega_n = \omega$. This new particular solution can be written as:

$$x_p(t) = tX\sin(\omega t) \tag{3.31}$$

Substituting this into the EOM of the system in standard form equation (from Boyce and DiPrima (1997)) and solving for X yields:

$$x_p(t) = \frac{f_0}{2\omega} t \sin(\omega t) \tag{3.32}$$

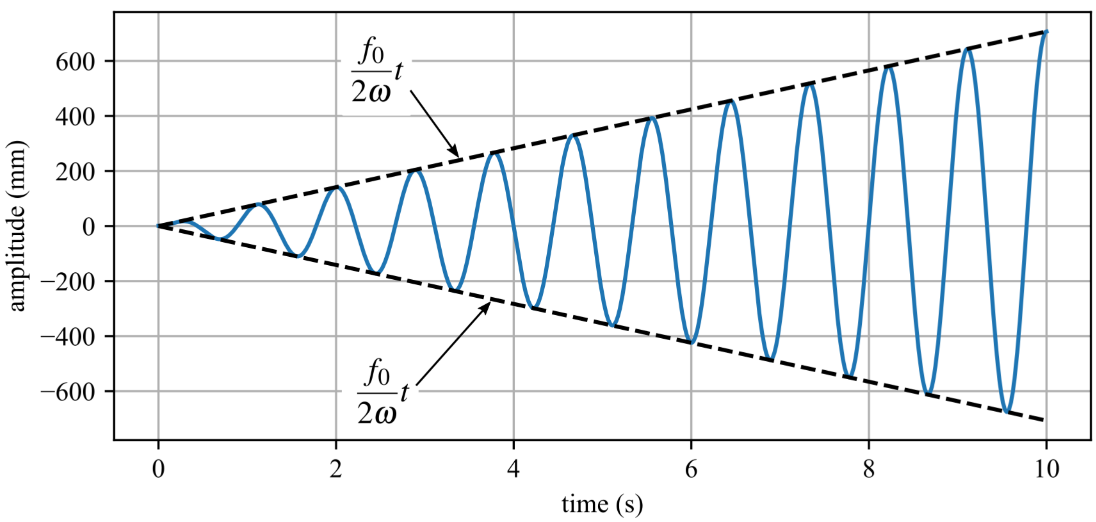
thus, the total solution can now be written as:

$$x(t) = A_1 \sin(\omega t) + A_2 \cos(\omega t) + \frac{f_0}{2\omega} t \sin(\omega t) \tag{3.33}$$

Note that $\omega_n = \omega$, therefore, the frequencies are all in terms of the driving frequency ω. Again, evaluating the solution at $t = 0$ for the initial conditions x_0 and v_0 yields:

$$x(t) = \left(\frac{v_0}{\omega}\right) \sin(\omega t) + x_0 \cos(\omega t) + \frac{f_0}{2\omega} t \sin(\omega t) \tag{3.34}$$

Where the first two terms account for the oscillations while the third terms accounts for the continued increase of the maximum amplitude. The following plot shows the forced response of a spring-mass system driven harmonically at its natural frequency.



Example 3.2 Compute solutions for the homogeneous and particular solution separately, than compute the total response of a spring-mass system with the following values: k = 1000 N/m, m = 10 kg, subject to a harmonic force of magnitude F_0 = 100 N and frequency of 8.162 rad/s, and initial conditions given by x_0 = 0 m and v_0 = 0 m/s. Plot the response.

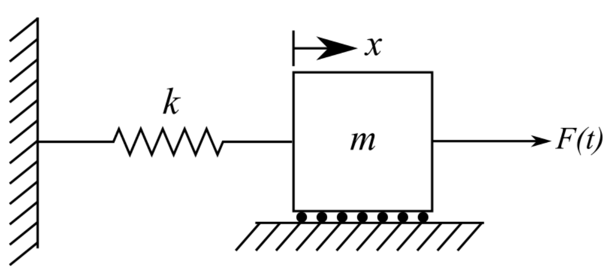


Figure 3.6: 1-DOF spring-mass system subjected to an external force $F(t)$.

Solution: First, make sure that the system is not in resonance. Calculating that $\omega_n = \sqrt{1000/10} =$

10 shows us that $\omega_n \neq \omega$. Next knowing that $f_0 = F_o/m = 10$ we can find the homogeneous and particular solutions as:

$$x_h(t) = A\sin(\omega_n t + \phi) \tag{3.35}$$

$$x_p(t) = X\cos(\omega t) \tag{3.36}$$

also:

$$x(t) = x_h(t) + x_p(t) \tag{3.37}$$

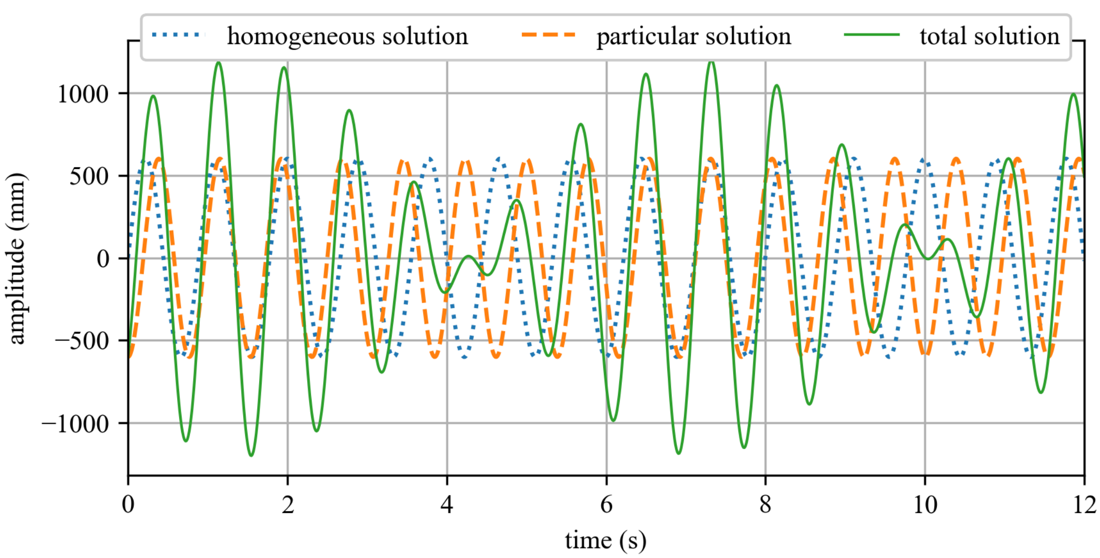
where:

$$A = \sqrt{\left(\frac{v_0}{\omega_n}\right)^2 + (x_0 - X)^2} = \tag{3.38}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n(x_0 - X)}{v_0}\right) \tag{3.39}$$

$$X = \frac{f_0}{\omega_n^2 - \omega^2} \tag{3.40}$$

This leads to the following results.



Example 3.3 Considering the following system, write the equation of motion and calculate the response assuming a) that the system is initially at rest, and b) that the system has an initial displacement of 0.005 m. Use k = 2000 N/m, m = 100 kg, $F(t)$ = 10sin(10t) N.

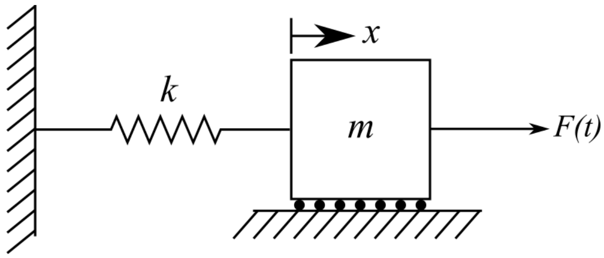


Figure 3.7: 1-DOF spring-mass system subjected to an external force $F(t)$.

Solution: The equation of motion is

$$m\ddot{x} + kx = 10\sin(10t) \tag{3.41}$$

or in standard form:

$$\ddot{x} + \omega_n^2 x = f_0 \sin(\omega t) \tag{3.42}$$

Note that the forcing function is in terms of sin, not cos as before, so we will have to resolve for the constants A_1 and A_2. Again, setting the particular solution to $x_p = X\sin(\omega t)$ and solving for X as before yields:

$$x(t) = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2}\sin(\omega t) \tag{3.43}$$

Now we can solve for A_1 and A_2 by setting the initial conditions x_0 and v_0 to $t = 0$. First, setting $t = 0$ in the equation for $x(t)$ yields:

$$A_2 = x_0 \tag{3.44}$$

Then, a function for the velocity of the system is obtained:

$$\dot{x}(t) = v_0 = A_1 \omega_n \cos(\omega_n t) - A_2 \omega_n \sin(\omega_n t) + \omega \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t) \tag{3.45}$$

This allows us to obtains:

$$A_1 = \frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2} \tag{3.46}$$

at $t = 0$. These lead to the full equation for the general solution:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2}\right)\sin(\omega_n t) + x_0 \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2}\sin(\omega t) \tag{3.47}$$

Also, knowing:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{20} \text{ rad/sec} = 4.472 \text{ rad/sec} \tag{3.48}$$

and

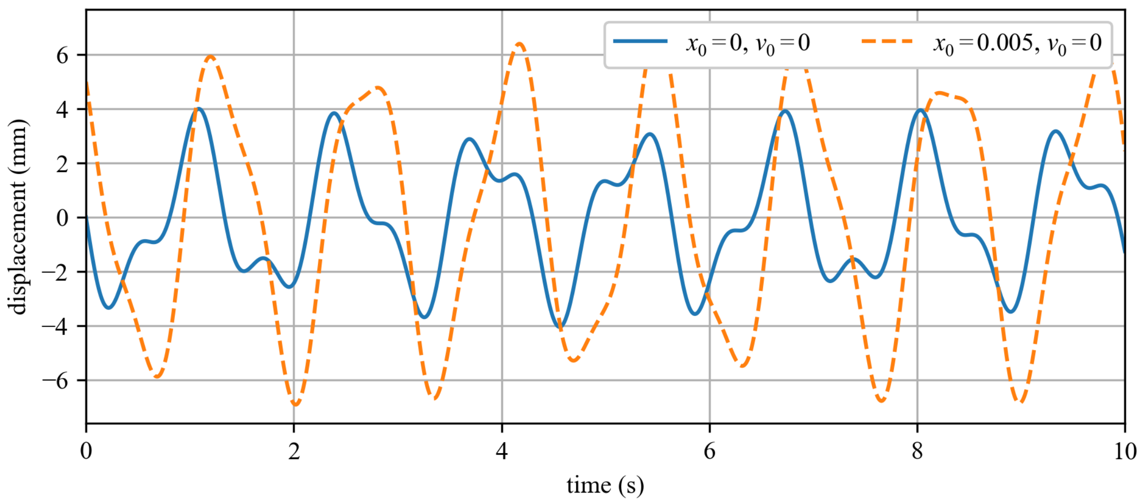
$$f_o = \frac{F_0}{m} = \frac{F_0}{m} = 0.1 \text{ N/kg} \tag{3.49}$$

a) using the initial conditions x_0 = 0 m and v_0 = 0 m/s and the general expression obtained above:

$$x(t) = \left(0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{20-10^2}\right)\sin(\sqrt{20}t) + 0 + \frac{0.1}{20-10^2}\sin(10t) \tag{3.50}$$

b) using the initial conditions x_0 = 0.005 m and v_0 = 0 m/s and the general expression obtained above:

$$x(t) = \left(0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{20-10^2}\right)\sin(\sqrt{20}t) + 0.05\cos(\sqrt{20}t) + \frac{0.1}{20-10^2}\sin(10t) \tag{3.51}$$



3.3 Harmonic Excitations of Underdamped Systems

Consider the system:

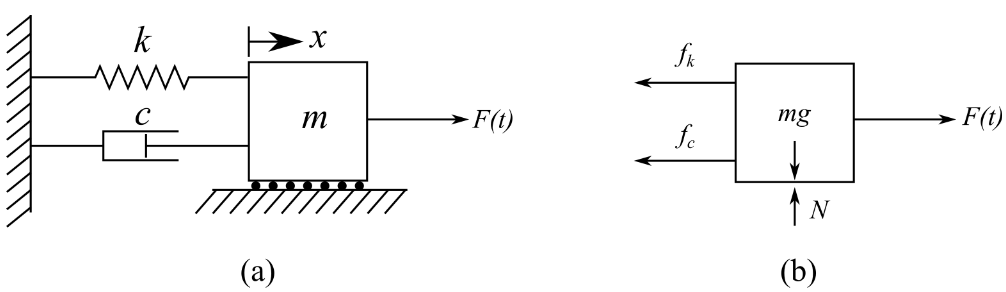


Figure 3.8: Damped 1-DOF system with an external force ($F(t)$) applied, showing: (a) the system configuration; and (b) the free body diagram

Again, for simplicity let us consider a harmonic excitation for $F(t)$ such that:

$$F(t) = F_0 \cos(\omega t) \tag{3.52}$$

Building the EOM for the above system results in:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos(\omega t) \tag{3.53}$$

For convinces we can convert this to the standard form:

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t) \tag{3.54}$$

again, where:

$$f_0 = \frac{F_0}{m} \tag{3.55}$$

Recall that one way to solve such an equation is to obtain the sum of the homogeneous and particular solutions.

$$x(t) = x_h(t) + x_p(t) \tag{3.56}$$

However, now that we have damping force to consider, our particular solution will have to consider this damping. Therefore:

$$x_p(t) = X\cos(\omega t - \phi_p) \tag{3.57}$$

where ϕ_p represents the phase shift. Note: ϕ_p is represented in other texts as θ, θ_p, or even just ϕ but we will use ϕ_p throughout the remainder of this text. Again, the phase shift is expected because of the effect of the damping force. Now, our total equation is:

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + X\cos(\omega t - \phi_p) \tag{3.58}$$

We can use the method of undetermined coefficients to obtain X and ϕ_p for the particular solution. First, considering that we write the particular solution in equivalent form:

$$x_p(t) = X\cos(\omega t - \phi_p) = A_s \cos(\omega t) + B_s \sin(\omega t) \tag{3.59}$$

Taking the derivative of the assumed forms of the particular solution yields:

$$x_p(t) = A_s \cos(\omega t) + B_s \sin(\omega t) \tag{3.60}$$

$$\dot{x}_p(t) = -\omega A_s \sin(\omega t) + \omega B_s \cos(\omega t) \tag{3.61}$$

$$\ddot{x}_p(t) = -\omega^2 A_s \cos(\omega t) - \omega^2 B_s \sin(\omega t) \tag{3.62}$$

Recall that the homogeneous and particular solutions are each solutions on their own, therefor, the EOM can be used to describe just the particular solution. Substituting x_p. $\dot{x}_p$, and $\ddot{x}_p$ for x. $\dot{x}$, and $\ddot{x}$ in the EOM in standard form:

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t) \tag{3.63}$$

yields:

$$\big(-\omega^2 A_s \cos(\omega t) - \omega^2 B_s \sin(\omega t)\big)(t) + 2\zeta\omega_n\big(-\omega A_s \sin(\omega t) + \omega B_s \cos(\omega t)\big)(t) + \tag{3.64}$$

$$\omega_n^2\big(A_s\cos(\omega t)+B_s\sin(\omega t)\big)(t)=f_0\cos(\omega t)$$

and rearranging in terms of $\sin(\omega t)$ and $\cos(\omega t)$ yields:

$$(-\omega^2 A_s+2\zeta\omega_n\omega B_s+\omega_n^2 A_s-f_0)\cos(\omega t)+ \tag{3.65}$$
$$(-\omega^2 B_s-2\zeta\omega_n\omega A_s+\omega_n^2 B_s)\sin(\omega t)=0$$

From this expression it is clear that there are two special moments in time where $\cos(\omega t)$ and $\sin(\omega t)$ equal zero. First, considering that $t=\pi/(2\omega)$ results in $\cos(\omega t)$=0, $\sin(\omega t)$=1 and the equation simplifies to:

$$(-2\zeta\omega_n\omega)A_s+(\omega_n^2-\omega^2)B_s=0 \tag{3.66}$$

Additionally, at $t=0$, $\sin(\omega t)$=0 and $\cos(\omega t)$=1. Therefore, the equation yields

$$(\omega_n^2-\omega^2)A_s+(2\zeta\omega_n\omega)B_s=f_0 \tag{3.67}$$

We can solve two equations for two unknowns. Writing the two linear equations as the singular matrix equation yields:

$$\begin{bmatrix}\omega_n^2-\omega^2 & 2\zeta\omega_n\omega\\ -2\zeta\omega_n\omega & \omega_n^2-\omega^2\end{bmatrix}\begin{bmatrix}A_s\\ B_s\end{bmatrix}=\begin{bmatrix}f_0\\ 0\end{bmatrix} \tag{3.68}$$

This can be solved by computeing the complex inversing, to give us:

$$A_s=\frac{(\omega_n^2-\omega^2)f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2} \tag{3.69}$$

$$B_s=\frac{2\zeta\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2} \tag{3.70}$$

From trigonometric relationships we can see that,

$$X=\sqrt{A_s^2+B_s^2} \tag{3.71}$$

$$\phi_p=\tan^{-1}\left(\frac{B_s}{A_s}\right) \tag{3.72}$$

We can now derive values for our particular solution x_p:

$$X=\frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}} \tag{3.73}$$

$$\phi_p=\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right) \tag{3.74}$$

Now we can build a solution for the particular equation (x_p), therefore, the total solution becomes:

$$x(t)=x_h(t)+x_p(t) \tag{3.75}$$

$$x(t)=Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)+X\cos(\omega t-\phi_p) \tag{3.76}$$

Note for larger values of t, the homogeneous solution approaches zero resulting in the particular solution becoming the total solution. Therefore, the particular solution is sometimes called the steady state response and the homogeneous solution is called the transient response. Solving for the constants A and ϕ using boundary conditions ($x_0 = 0$ and $v_0 = 0$) results a total solution expressed as:

$$A = \frac{x_0 - X\cos(\phi_p)}{\sin(\phi)} \tag{3.77}$$

$$\phi = \tan^{-1}\left(\frac{\omega_d\left(x_0 - X\cos(\phi_p)\right)}{v_0 + \left(x_0 - X\cos(\phi_p)\right)\zeta\omega_n - \omega X\sin(\phi_p)}\right) \tag{3.78}$$

Finally, assembling all the terms:

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\cos(\omega t - \phi_p) \tag{3.79}$$

$$A = \frac{x_0 - X\cos(\phi_p)}{\sin(\phi)} \tag{3.80}$$

$$\phi = \tan^{-1}\left(\frac{\omega_d(x_0 - X\cos(\phi_p))}{v_0 + (x_0 - X\cos(\phi_p))\zeta\omega_n - \omega X\sin(\phi_p)}\right) \tag{3.81}$$

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \tag{3.82}$$

$$\phi_p = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right) \tag{3.83}$$

Example 3.4 Consider the damped 1-DOF system below, plot the total, steady state, and transient responses for the following system configurations with no initial conditions. For each configuration, comment on the temporal response and how it differs from the response of the previous configuration.

a) $k = 100$ N/m, $m = 10$ kg, $c = 10$ kg/s, $F_0 = 1$ N, and $\omega = 8.162$.

b) $k = 100$ N/m, $m = 10$ kg, $c = 10$ kg/s, $F_0 = 3$ N, and $\omega = 8.162$.

c) $k = 100$ N/m, $m = 10$ kg, $c = 10$ kg/s, $F_0 = 3$ N, and $\omega = 3.162$.

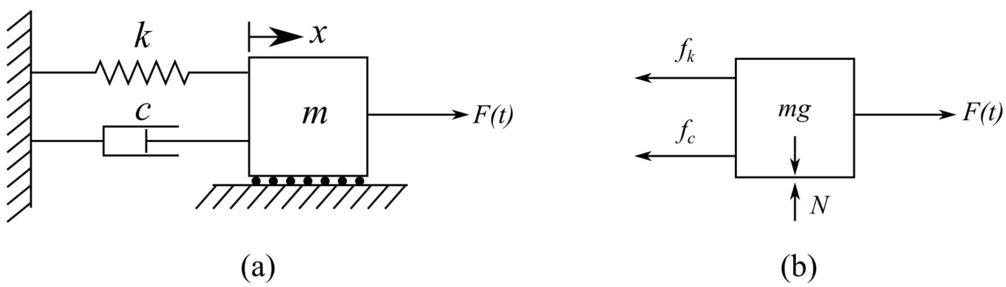


Figure 3.9: Damped 1-DOF system with an external force ($F(t)$) applied, showing: (a) the system configuration; and (b) the free body diagram

Solution: The total response for the damped 1-DOF system subjected to an external force is modeled using equations 3.79 through 3.83 while the transient response consists fo the first half of equation 3.79 and the steady state response consists of the second half of equation 3.79.

Solution a): Therefore, plotting the temporal responses for configuration a yields:

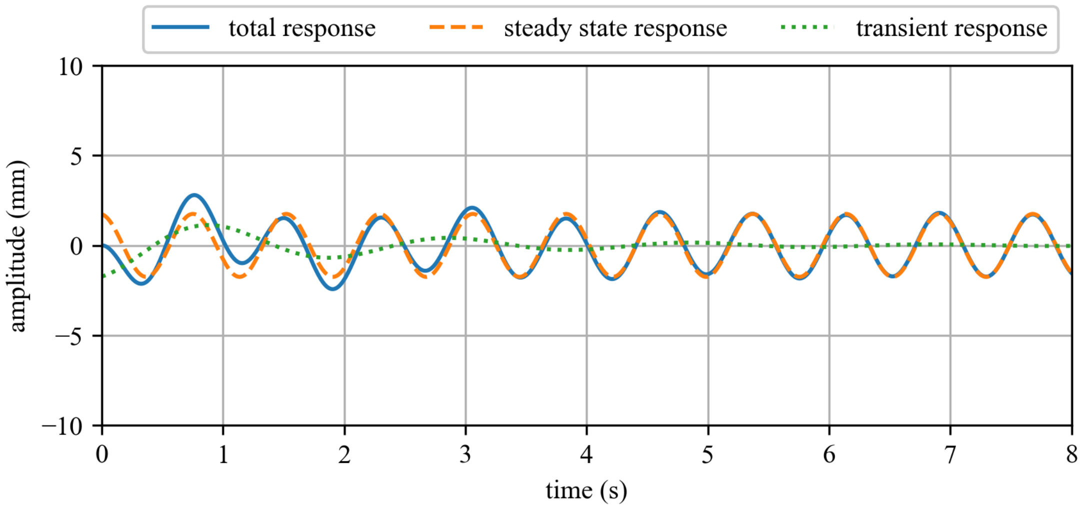


Figure 3.10: Temporal responses for a underdamped system with $k = 100$ N/m, $m = 10$ kg, $c = 10$ kg/s, $F_0 = 1$ N, and $\omega = 8.162$.

Solution b): Configuration b increases the forcing function F_0 to 3 N. This results a similar response to configuration a but with a lineally scaled amplitude:

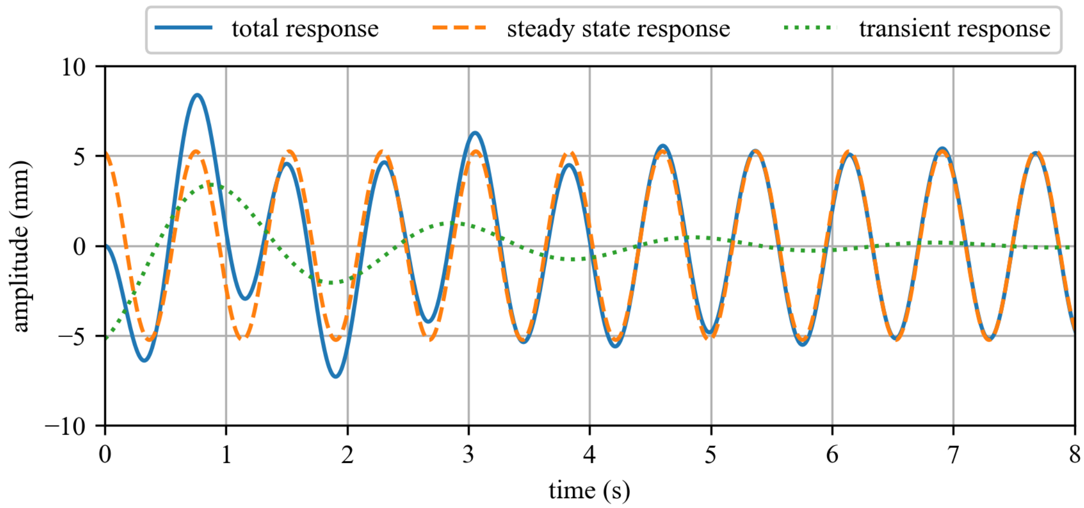


Figure 3.11: Temporal responses for a underdamped system with $k = 100$ N/m, $m = 10$ kg, $c = 10$ kg/s, $F_0 = 3$ N, and $\omega = 8.162$.

Solution c): Now, using $\omega = 3.162$ rad/sec we put the system into resonance as $\omega = \omega_n$. However, unlike the undamped system the amplitude of the displacement is not unbounded as the damper absorbs energy from the system. Therefore, after about 7 seconds the systems enters an equilibrium state where any additional increase in amplitude caused by the system entering into resonance is canceled out by the damping in the system as demonstrated in the plot below:

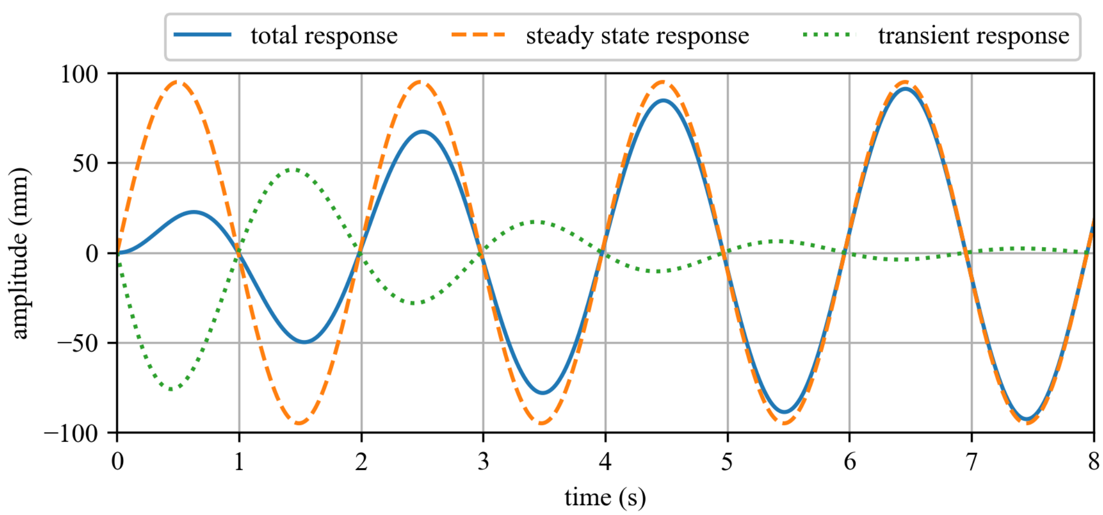


Figure 3.12: Temporal responses for a underdamped system with $k = 100$ N/m, $m = 10$ kg, $c = 10$ kg/s, $F_0 = 3$ N, and $\omega = 3.162$.

3.4 Frequency Response of Underdamped Systems

From equations 3.79 through 3.83 and the figures in example 3.4 we can see that for larger values of t the transient response dies out while only the steady state response controls the displacement of the total response. This is always true if the system has any significant damping. Therefore, it is often prudent to ignore the transient part and focus only on the steady-state response. Considering the equation for the particular solution:

$$x_p(t) = X\cos(\omega t - \phi_p) \tag{3.84}$$

and knowing the values for X and ϕ_p:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \tag{3.85}$$

$$\phi_p = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right) \tag{3.86}$$

We want to find a way to plot the responses of the system only in terms of the system's natural and driving frequencies, and its damping. First, we define a frequency ratio as the dimensionless quantity

$$\beta = \frac{\omega}{\omega_n} \tag{3.87}$$

Another common way to express β is r. Next, Recall that:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{\frac{F_0}{m}}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \tag{3.88}$$

If we factor out ω_n^2 from the denominator and substitute in $\omega_n^2 = k/m$ and $r = \omega/\omega_n$, we get:

$$X = \frac{\frac{F_0}{m}}{\omega_n^2\sqrt{\left(1-(\frac{\omega}{\omega_n})^2\right)^2 + (2\zeta\frac{\omega}{\omega_n})^2}} = \frac{\frac{F_0}{k}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3.89}$$

this becomes:

$$\frac{Xk}{F_0} = \frac{X\omega_n^2}{f_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \tag{3.90}$$

in a similar fashion, if we manipulate the equation for ϕ_p we can get ϕ_p in term of β:

$$\phi_p = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \tag{3.91}$$

If we solve for a few key values of β we can get the following data points. On the board, we can solve for a few different frequency responses for a few different damping coefficients.

	frequency ratio (β)								
	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2.0
$\zeta = 0.1$	1.00	1.07	1.32	2.16	5.00	1.62	0.78	0.48	0.33
$\zeta = 0.25$	1.00	1.06	1.27	1.74	2.00	1.19	0.69	0.45	0.32
$\zeta = 0.5$	1.00	1.03	1.11	1.15	1.00	0.73	0.51	0.37	0.28
$\zeta = 0.7$	1.00	1.00	0.97	0.88	0.71	0.54	0.41	0.31	0.24

If we plot the values of the normalize amplitude vs β we obtain:

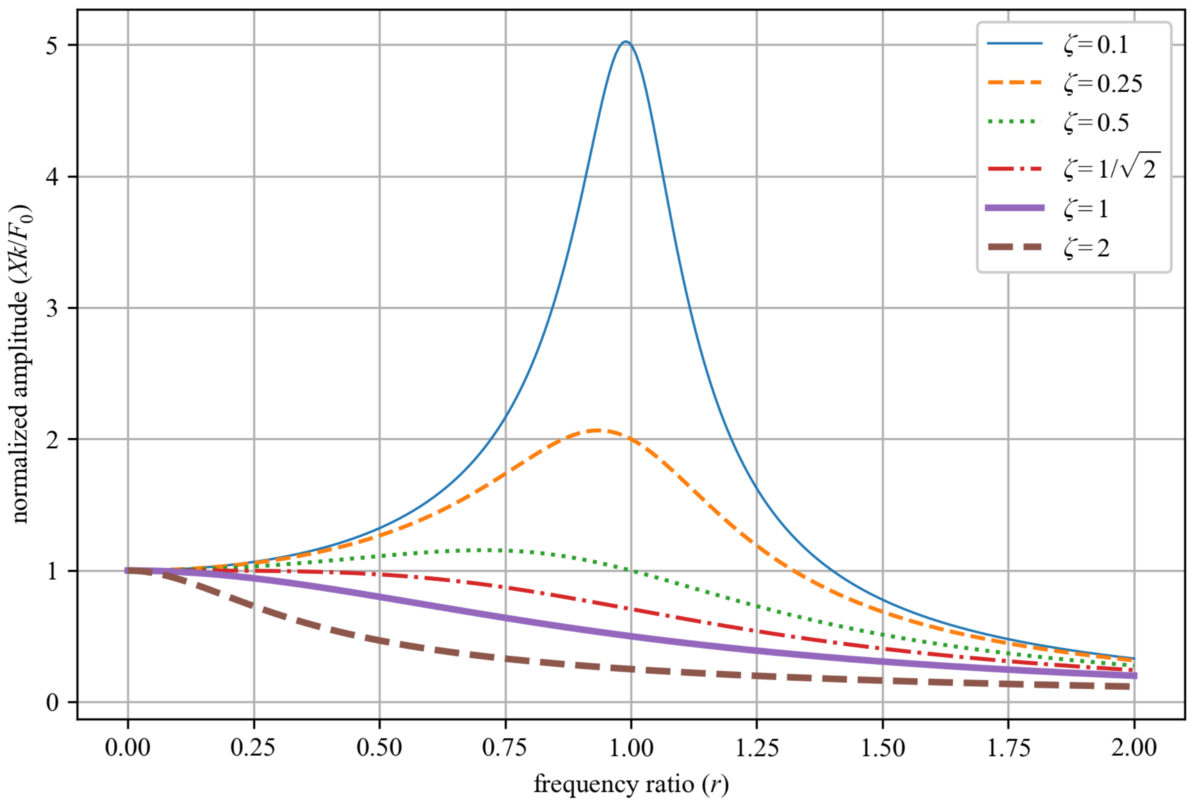


Figure 3.13: Normalized amplitude response for frequency ratio (β) from 0 to 2 for a variety of critical damping ratios.

And again, if we plot the values of the phase vs β (ω/ω_n) we obtain:

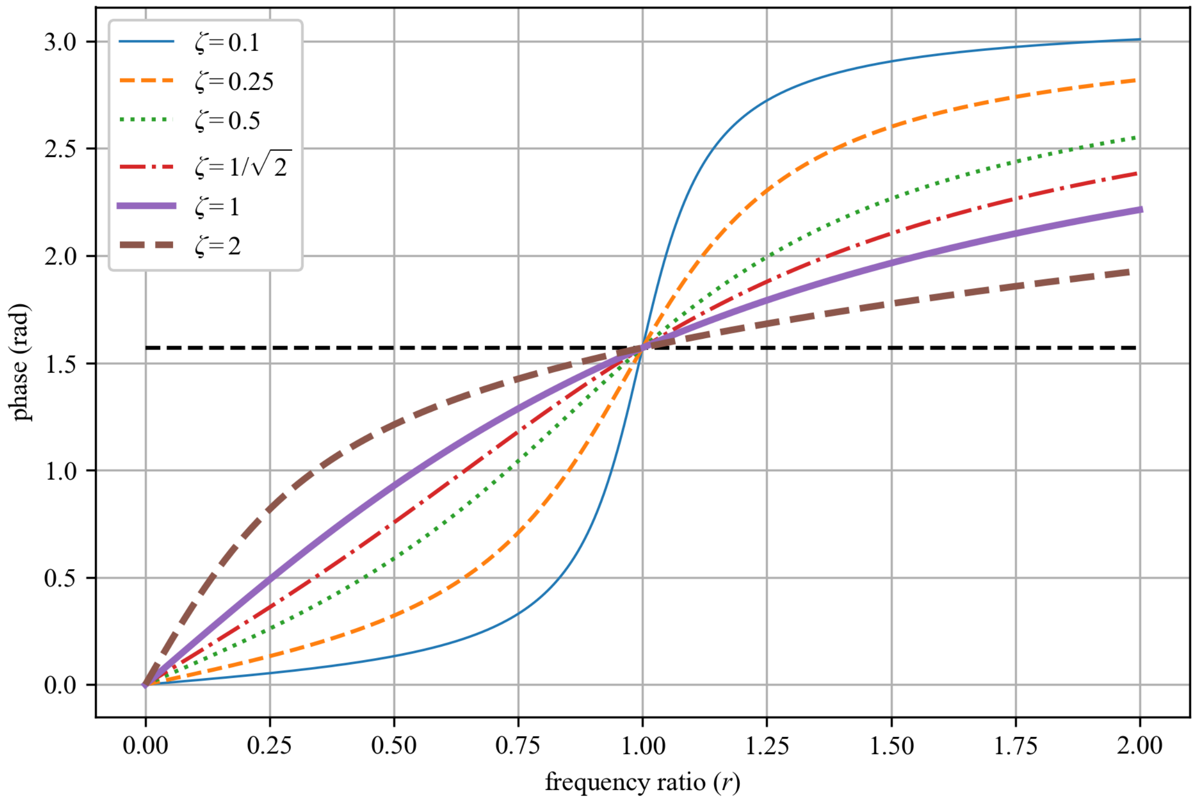


Figure 3.14: Phase response for frequency ratio (β) from 0 to 2 for a variety of critical damping ratios.

note that the dashed black line is there because the phase values after $\pi/2$ need to be adjusted to obtain a continuous plot. An astute observer would notice that the maximum amplitude is not at $\omega = \omega_n$. While resonance is defined as $\omega = \omega_n$, this does not define the point of maximum displacement of the steady state response. Let us solve for the frequency ratio with the maximum displacement. This will happen when

$$\frac{d}{dr}\left(\frac{Xk}{F_0}\right) = 0 \tag{3.92}$$

We can show that:

$$\left(\frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}\right)\frac{d}{dr} = 0 \tag{3.93}$$

when

$$\beta_{\text{peak}} = \sqrt{1-2\zeta^2} = \frac{\omega_p}{\omega_n}, \qquad \zeta < 1/\sqrt{2} \tag{3.94}$$

however, this is only true for under damped system in which $\zeta < 1/\sqrt{2}$. If $\zeta > 1/\sqrt{2}$ than the value in imaginary and the peak value is at $r = 0$. In these cases, the maximum displacement is a function of only ω_n. ω_p represents the driving frequency that correspond to the maximum amplitude ($\frac{Xk}{F_0}$) and is called the **peak frequency**, and can be calculated as:

$$\omega_p = \omega_n \beta_{\text{peak}} = \omega_n \sqrt{1 - 2\zeta^2}, \qquad \zeta < 1/\sqrt{2} \tag{3.95}$$

Example 3.5 Consider the simple spring mass system,

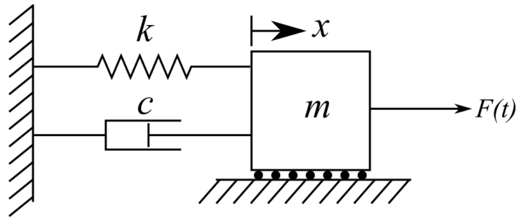


Figure 3.15: Damped 1-DOF spring-mass system subjected to an external force $F(t)$.

where ω_n = 132 rad/sec and ζ = 0.0085. Calculate the displacements of the steady-state response for ω=132 and 125 rad/sec. In both cases, use f_0 = 10 N/kg.

Solution: From before, we know the solution for the displacement of the particular solution for ω=132 rad/sec is:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{10}{2(0.0085)(132)^2} = 0.034 \text{ m} \tag{3.96}$$

while for ω=125 rad/sec X is:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{10}{\sqrt{(1799)^2 + (280.5)^2}} = 0.005 \text{ m} \tag{3.97}$$

Therefore, a slight change in the driving frequency (about 5%) results in a 85% change in the amplitude of the steady-state response.

Example 3.6 The steady state response for an engineered system must not surpass 1 cm, if the system can be modeled as the spring and mass system below, what value of c must be used?

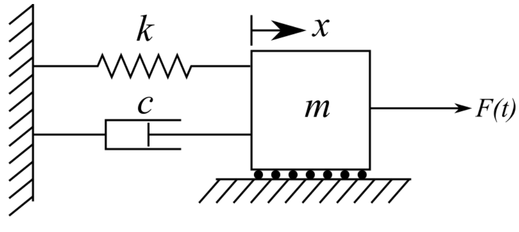


Figure 3.16: Damped 1-DOF spring-mass system subjected to an external force $F(t)$.

Use k = 2000 N/m, m = 100 kg, $F(t)$ = 20 cos(6.3t) N.
Solution: The steady state solution is:

$$x_p(t) = X\cos(\omega t - \phi_p) \tag{3.98}$$

knowing the amplitude is controlled by X:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \tag{3.99}$$

and recalling from the EOM in standard form that $2\zeta\omega_n = c/m$ we can obtain:

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (\frac{c}{m}\omega)^2}} \tag{3.100}$$

rearranging for c gives:

$$c = m\sqrt{\frac{f_0^2}{\omega^2 X^2} - \frac{\left(\omega_n^2 - \omega^2\right)^2}{\omega^2}} = \sqrt{\frac{F_0^2}{\omega^2 X^2} - m^2\frac{\left(\omega_n^2 - \omega^2\right)^2}{\omega^2}} \tag{3.101}$$

Therefore, if we set $X = 0.01$ m we can solve the above equation to yield c = 55.7 kg/s.

3.5 Base Excitation

Often, loading is not applied directly to the mass, but rather the mass of the system is excited when the base of mount that it is attached to is excited. This is called base excitation, or sometimes support motion. Examples of base excitation, or where base excitation is considered, include:

- machines on rubber mounts
- automobiles excited by the road
- building under earthquake loading
- hospital equipment

Consider the following system base excited system

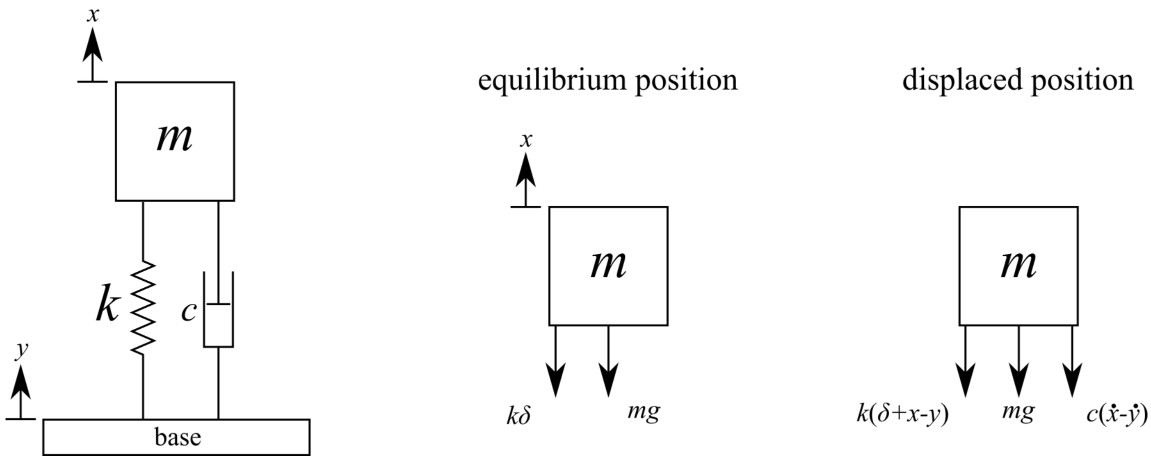


Figure 3.17: Damped 1-DOF spring-mass system subjected to a displacement controlled base excitation showing the FBDs for the equilibrium and displaced positions.

where x is the displacement of the mass and y is the displacement of the base. Note that we consider positive upward here. The EOM can be constructed the same as before, but now considering that the displacement of the springs and damper is $x-y$. In the equilibrium state, where a positive x is up and the base displaces down:

$$\sum F_x = -k\delta - mg = 0 \tag{3.102}$$

Note that these are both negative because the base displacing down "pulls" the mass down with a force $k\delta$ (i.e. if you hold the mass and let the base "fall"). Conversely, the equation for the displaced state is:

$$\sum F_x = -k(\delta + x - y) - mg - c(\dot{x} - \dot{y}) \tag{3.103}$$

Apply Newton's second law about the mass ($m\ddot{x}$) of motion to the sum of forces for the displaced position we get:

$$\sum F_x = m\ddot{x} = -k\delta - kx + ky - mg - c\dot{x} + c\dot{y} \tag{3.104}$$

applying the equation $-k\delta - mg = 0$, and rearrange into the EOM yields:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{3.105}$$

Now as before we assume an input for the base excitation. For simplicity we assume:

$$y(t) = Y\sin(\omega_b t) \tag{3.106}$$

Taking the derivative of the assumed input yields:

$$\dot{y}(t) = Y\omega_b\cos(\omega_b t) \tag{3.107}$$

where Y is the amplitude and ω_b is the frequency of the base excitation. Adding these terms into our EOM yields:

$$m\ddot{x} + c\dot{x} + kx = cY\omega_b\cos(\omega_b t) + kY\sin(\omega_b t) \tag{3.108}$$

We can get this in standard form if we divide by m and apply the equations for the critical damping ratio and natural frequency:

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x = 2\zeta\omega_n\omega_bY\cos(\omega_bt)+\omega_n^2Y\sin(\omega_bt) \tag{3.109}$$

This equation can be related to a spring-mass-damper system with two harmonic inputs, one cos and one sin as shown below:

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x = C\cos(\omega_bt)+D\sin(\omega_bt) \tag{3.110}$$

where C and D are arbitrary coefficients.

3.5.1 Displacement Transmissibility Solution for Base Excitation

The steady-state solution is often of more important than the transient solution when designing systems for continuous use. The particular solution for the base excited system annotated in figure 3.17 with the EOM presented in equation 3.110 can be expressed as $x_p(t)$. To solve for this expression we will use the linearity of the system and solve for a solution that is the sum of two particular solutions. Resulting in:

$$x_p(t) = x_p^{(1)}(t)+x_p^{(2)}(t) \tag{3.111}$$

Recall that the steady state solution for a harmonically excited spring-mass-damper can be expressed as $x_p(t) = X\cos(\omega t-\phi_p)$, as denoted in equation 3.57. For the base excitation problem, we will convert this expression to $x_p(t) = X\cos(\omega_bt-\phi_1)$. Therefore, for a base excited problem, the forcing function can be expressed as the sum of particular solutions:

$$C\cos(\omega_bt)+D\sin(\omega_bt) = x_p = x_p^{(1)}+x_p^{(2)} \tag{3.112}$$

where we dropped the (t) term from the expression for simplicity in writing and:

$$x_p^{(1)} = X^{(1)}\cos(\omega_bt-\phi_1) \tag{3.113}$$

$$x_p^{(2)} = X^{(2)}\sin(\omega_bt-\phi_1) \tag{3.114}$$

Note that $x_p^{(1)}$ uses a cos term while $x_p^{(1)}$ uses a sin term. Both solutions use ϕ_1 as the damping term as the phase angle is independent of the excitation amplitude and the sin and cos terms account for the difference in phase.

For $x_p^{(1)}$, we use the method of undetermined coefficients to obtain a solution for $x_p^{(1)} = X^{(1)}\cos(\omega_bt-\phi_1)$. This can be as simple as setting $2\zeta\omega_n\omega_bY$ equal to f_0 from equation 3.73 that defines X for underdamped systems. Again, $2\zeta\omega_n\omega_bY$ comes from the EOM in standard form as presented in equation 3.109. We can do this because both terms can be considered a “driving force”. This results in the equation:

$$x_p^{(1)} = \frac{2\zeta\omega_n\omega_bY}{\sqrt{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}}\cos(\omega_bt-\phi_1) \tag{3.115}$$

where:

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right) \tag{3.116}$$

Next, the particular solution associated with $x_p^{(2)} = X^{(2)}\sin(\omega_b t - \phi_1)$ can be obtained using the same method of undetermined coefficients an setting f_0 from equation 3.73 to the driving force for $x_p^{(2)}$ in equation 3.109, ω_n^2. This results in:

$$x_p^{(2)} = \frac{\omega_n^2 Y}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}\sin(\omega_b t - \phi_1) \tag{3.117}$$

As both equation 3.115 and 3.117 have the same argument $(\omega_b t - \phi_1)$, these can be added as:

$$x_p = x_p^{(1)} + x_p^{(2)} \tag{3.118}$$

to obtain:

$$x_p = \omega_n Y\sqrt{\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}\cos(\omega_b t - \phi_1 - \phi_2) \tag{3.119}$$

and:

$$\phi_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta\omega_b}\right) \tag{3.120}$$

where ϕ_2 is added to account for the cos and sin terms being combined. Again, the (t) has been dropped for simplicity.

As before, if we want to investigate how a frequency input will affect the response (frequency response) we can substitute substitute

$$\beta = \frac{\omega_b}{\omega_n} \tag{3.121}$$

into the temporal response to obtain:

$$X = Y\sqrt{\frac{1 + (2\zeta\beta)^2}{(1-\beta^2)^2 + (2\zeta\beta)^2}} \tag{3.122}$$

Next, if we divide by Y we can obtain a normalized expression for the displacement:

$$\frac{X}{Y} = \sqrt{\frac{1 + (2\zeta\beta)^2}{(1-\beta^2)^2 + (2\zeta\beta)^2}} \tag{3.123}$$

Plotting this for several critical damping ratios:

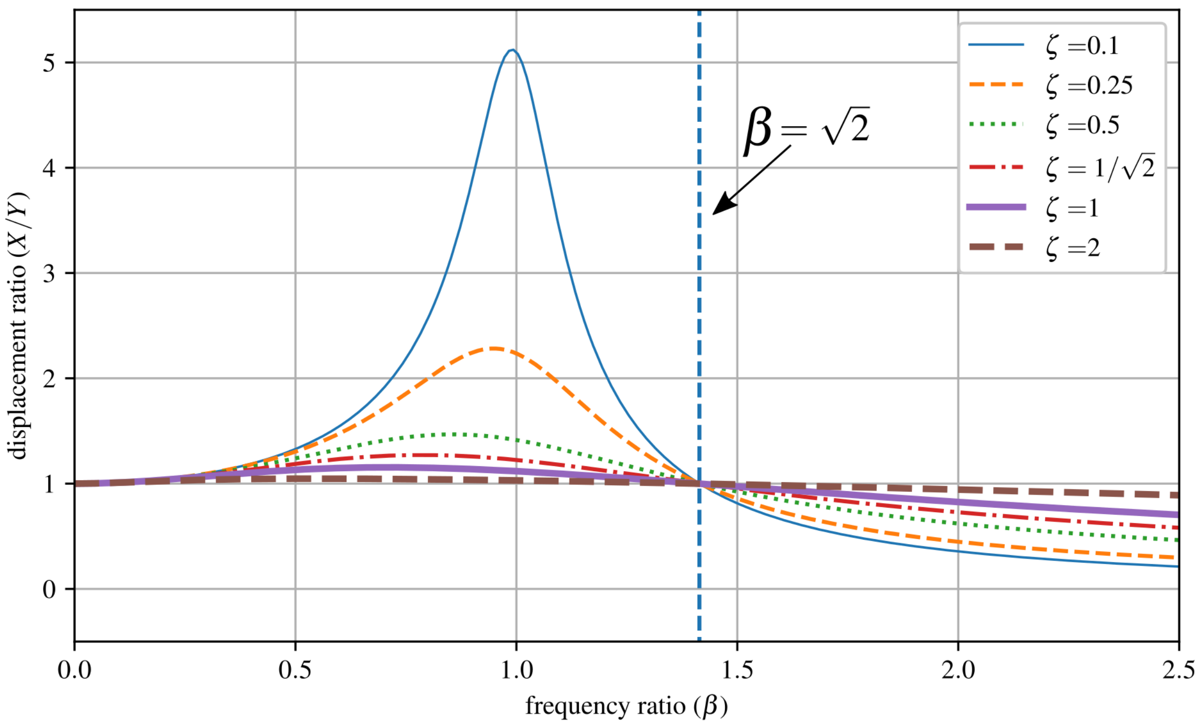
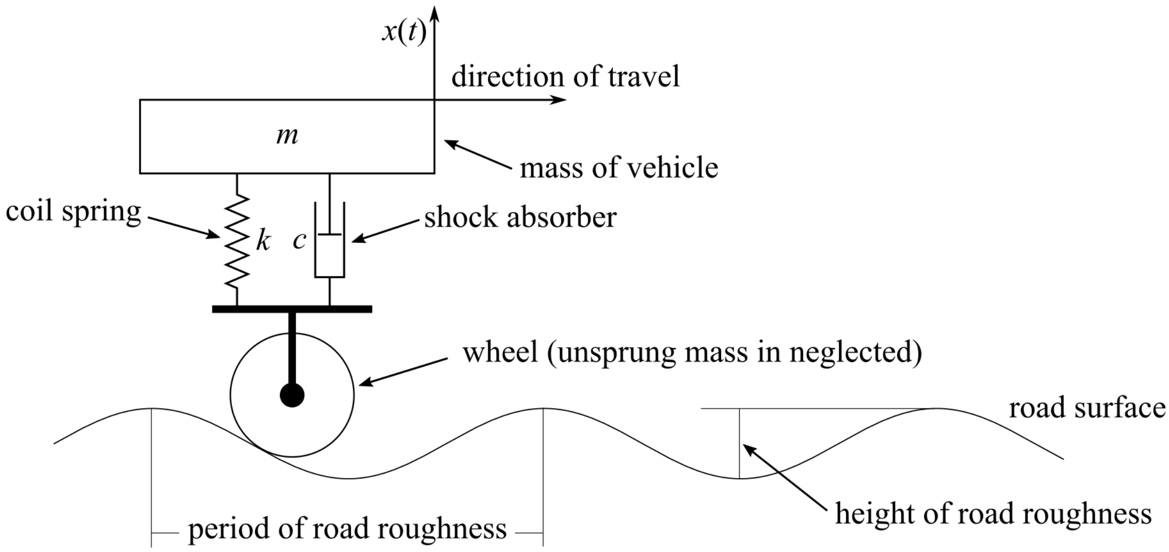


Figure 3.18: Displacement transmissibility for an underdamped 1-DOF system.

Around resonance, the maximum amount of displacement is transmitted to the mass. Additionally, the above plot shows that at $\beta = \sqrt{2}$ the displacement transmissibility X/Y is 1. Note the "flip" where overdamped systems have a greater response to excitations after $\beta = \sqrt{2}$ than do underdamped systems.

Example 3.7 A very common example of base motion is the SDOF model of a vehicle wheel driving over a "rough" road as shown below.



where k = 400,000 N, m = 1007 kg, c = 20,000 kg/s, the period of road roughness = 6 m, and the height of road roughness = 0.02 m. What is the deflection experience by the car at v = 20 km/h?

Solution: The road is applying a base excitation that can be approximated as

$$Y = 0.01 \text{ m} \tag{3.124}$$

$$v \text{ m/s} = 20 \text{ km/hr}\left(\frac{1000 \text{ m}}{1\text{km}}\right)\left(\frac{1 \text{ hours}}{3600 \text{ s}}\right) = \frac{50}{9} \text{ m/s} = 5.555 \text{ m/sec} \tag{3.125}$$

$$\omega_b = \left(\frac{5.55 \text{ m}}{s}\right)\left(\frac{1 \text{ cycle}}{6 \text{ m}}\right)\left(\frac{2\pi \text{ rad}}{\text{cycle}}\right) = \frac{11.11\pi}{6} \text{ rad/s} = 5.817 \text{ rad/s} \tag{3.126}$$

Therefore, the sinusoidal for the base excitation is then:

$$y(t) = (0.01)\sin(5.817t) \tag{3.127}$$

Next, we can calculate the natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{400,000}{1007}} = 19.93 \text{ rad/s} \tag{3.128}$$

Therefore:

$$r = \frac{\omega_b}{\omega_n} = \frac{5.817}{19.93} = 0.292 \tag{3.129}$$

and:

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{20,000}{2\sqrt{1007 \cdot 400,000}} = 0.498 \tag{3.130}$$

Then it can be found that the maximum deflection of the car is: into the temporal response to obtain:

$$X = Y\sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} = Y\sqrt{\frac{1+(2\cdot 0.498\cdot 0.292)^2}{(1-0.292^2)^2+(2\cdot 0.498\cdot 0.292)^2}} \\ = 0.0108 \text{ m} \tag{3.131}$$

3.5.2 Force Transmissibility Solution for Base Excitation

For some systems, such as those with weak connections, the force transmitted to the mass is more important than the displacement of the mass. The force transmitted to the mass are the sums of the forces applied by the spring and damper. From the FBD above,

$$F(t) = k(x-y) + c(\dot{x}-\dot{y}) \tag{3.132}$$

where this force is counteracted by the inertial force of the mass:

$$F(t) = -m\ddot{x}(t) \tag{3.133}$$

Only considering the steady state we found that

$$x_p(t) = \omega_n Y\sqrt{\frac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}}\cos(\omega_b t-\phi_1-\phi_2) \tag{3.134}$$

if we differentiate this twice, to obtain $\ddot{x}(t)$ and combine this with $F(t) = -m\ddot{x}(t)$ we get:

$$F(t) = m\omega_b^2\omega_n Y\sqrt{\frac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}}\cos(\omega_b t-\phi_1-\phi_2) \tag{3.135}$$

where the negative sign $F(t) = -m\ddot{x}(t)$ as the force transmitted to the mass is both positive and negative and we are solving for the amplitude of the transmitted force. Again applying:

$$\beta = \frac{\omega_b}{\omega_n} \tag{3.136}$$

this becomes:

$$F(t) = F_{\mathrm{T}}\cos(\omega_b t-\phi_1-\phi_2) \tag{3.137}$$

where F_T is the magnitude of the transmitted force and is

$$F_{\mathrm{T}} = kYr^2\sqrt{\frac{1+(2\zeta\beta)^2}{(1-\beta^2)^2+(2\zeta\beta)^2}} \tag{3.138}$$

Again, this can be converted to a force transmissibility to provide a normalized response such that:

$$\frac{F_{\mathrm{T}}}{kY} = \beta^2\sqrt{\frac{1+(2\zeta\beta)^2}{(1-r^2)^2+(2\zeta\beta)^2}} \tag{3.139}$$

Plotting this for several critical damping ratios:

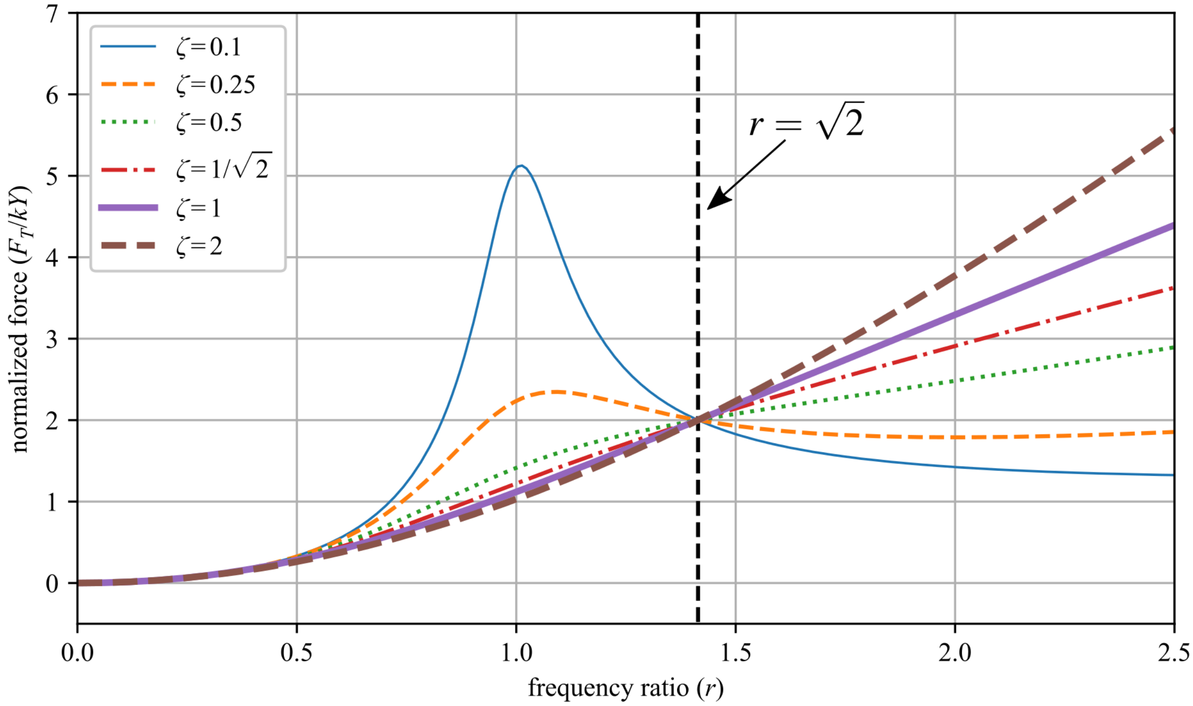


Figure 3.19: Force transmissibility for an underdamped 1-DOF system.

Again, note the key location $\beta = \sqrt{2}$. At $\beta = \sqrt{2}$ the force transmitted to the system is $2\ \frac{F_T}{kY}$. However, also note that the normalized force does not necessarily fall off for β values greater than $\beta = \sqrt{2}$.

Example 3.8 For the system given below and excited at the base, should the system be excited above or below the natural frequency if the transmitted force is the design limitation? Consider the under damped with $\zeta = 0.1$, and the over damped with $\zeta = 2$ conditions.

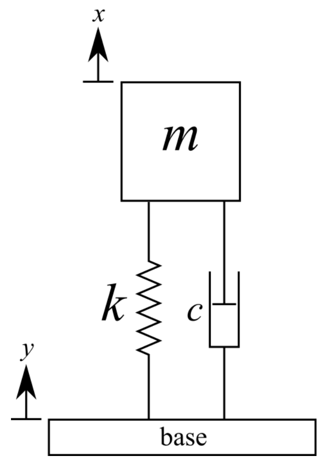


Figure 3.20: Force transmissibility for an underdamped 1-DOF system.

Solution: We can plot the transmissibility of both the force and displacement onto one plot. For $\zeta = 0.1$

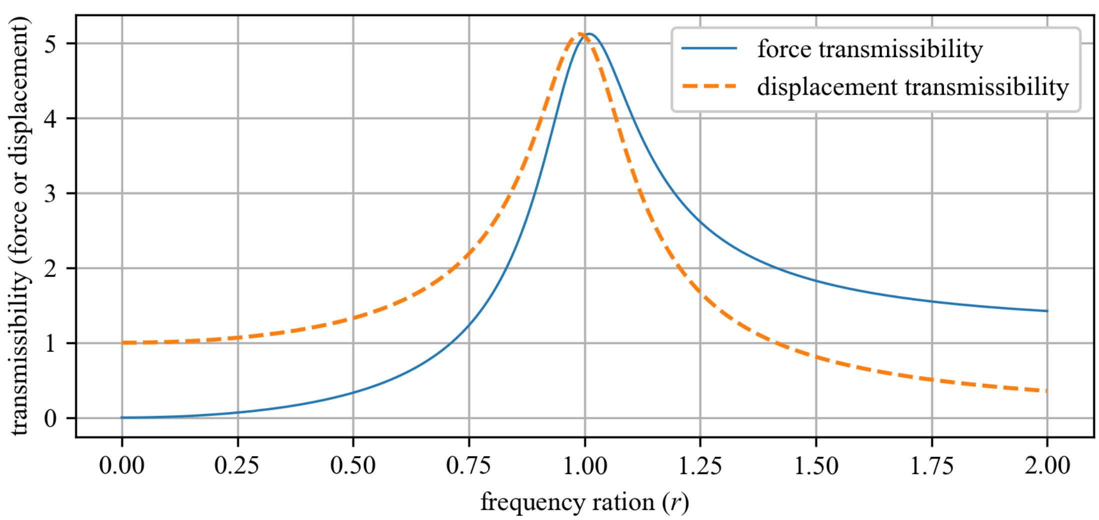


Figure 3.21: Force and displacement transmissibility for the considered base excited system with $\zeta = 0.1$.

it is clear that to minimize the force, the system should be driven with a frequency below the natural frequency. Next for $\zeta = 2$:

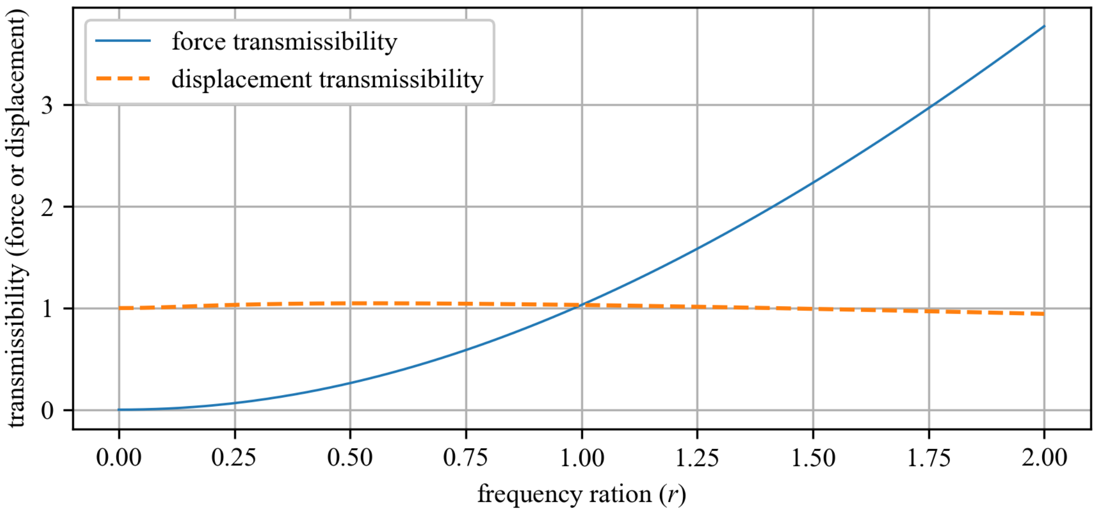
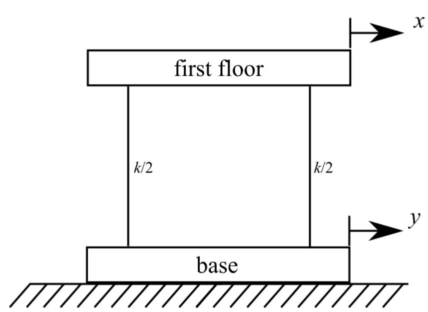


Figure 3.22: Force and displacement transmissibility for the considered base excited system with $\zeta = 2$.

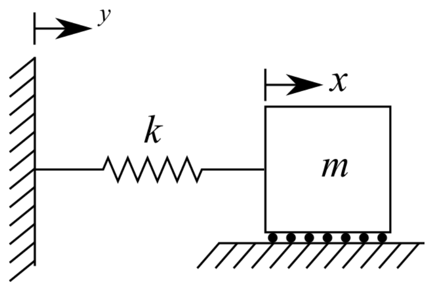
it can be seen that the same rationale applies. Therefore, for both $\zeta = 0.1$ and $\zeta = 2$ the system should be excited below the natural frequency.

Example 3.9 A single story building is subjected to a harmonic ground motion, $\ddot{y}(t) = A\cos(\omega_b t)$. a) Find the steady-state solution for the structure. b) If a damper was added between the base and the floor, and $r = 2$, what would be the ideal critical damping coefficient to insure the safety of the building. (Think of safety as limiting displacement and transmitted force.)



Solution (a):

For simplicity, we can rearrange the system as what follows:



solving for the EOM yields:

$$m\ddot{x} + kx = ky \tag{3.140}$$

Notice that this is the same as the EOM for a damped 1-DOF system if $c = 0$.

$$m\ddot{x} + c\dot{x} + kx = +c\dot{y} + ky \rightarrow m\ddot{x} + kx = ky \tag{3.141}$$

Therefore, we can use the solution:

$$x_p(t) = \omega_n Y \sqrt{\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}} \cos(\omega_b t - \phi_1 - \phi_2) \tag{3.142}$$

where:

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right) \tag{3.143}$$

$$\phi_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta\omega_b}\right) \tag{3.144}$$

Now we have, or can easily get, values for ω_n, ω_b, and ζ. However, we do not have an expression for Y. We can extract the displacement (and therefore the Y) from the acceleration as:

$$\ddot{y}(t) = A\cos(\omega t) \tag{3.145}$$

$$\dot{y}(t) = \frac{A}{\omega}\sin(\omega t) + C_1 \tag{3.146}$$

$$y(t) = -\frac{A}{\omega^2}\cos(\omega t) + C_1 t + C_2 \tag{3.147}$$

Resulting in

$$Y = -\frac{A}{\omega^2} \tag{3.148}$$

Solution (b): From the plots we solved for before, we can see that we want a critical damping coefficient that is as low as possible. This means any damping added to the system will decrease its safety. This may seem counter-intuitive, but this is because we are attempting to drive the structure at a frequency higher than its natural frequency, something that does not commonalty happen. Typically excitations for a structure are well below its natural frequency.

4 Transfer Function for Vibrating Systems

Thusfar, this text has only considered forced vibrations for 1-DOF systems excited with forcing functions that can be easily expressed using either sin or cos examples. Therefore, the previously developed solutions are only acceptable for systems with known and simple excitations. This chapter will introduce the concept of transfer functions for solving vibration related problems. The transfer function, in particular the Laplace transfer function, is an important tool in the study of vibrations as it allows the practitioner to solve for the temporal response of a system for a variety of inputs using a single approach. Examples of force excitation that can be calculated include using this method include:

- sinusoidal
- base excitation
- impulse
- arbitrary input
- arbitrary periodic input

4.1 Transfer Function Method (Generic)

Consider the following system

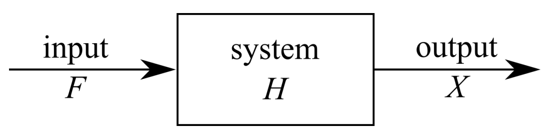


Figure 4.4: Generic system H subjected to an input F and its corresponding output X.

where F is the input, H is the system, and X is the output from the system. This formulation is called the transfer-function approach and is commonly used for the formulation and solution of dynamic problems in the control literature. It can also be used for solving the various forced-vibration problems including those from complex or stochastic inputs.

Review 4.1 Laplace transforms, or more boradly integral transform, are a procedure for integrating the time (t) dependence of a function into a function of of position or space (s). By transforming the whole whole differential equation from the time domain into a lower order function of space the problem becomes easier to solve as the function can often be manipulated algebraically.

Figure 4.5: Portrait of Pierre-Simon Laplace by Johann Ernst Heinsius (1775). Johann Ernst Heinsius, CC BY-SA 4.0 <https://creativecommons.org/licenses/by-sa/4.0>, via Wikimedia Commons

The Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace (23 March 1749 - 5 March 1827). Pierre-Simon Laplace was one of the greatest scientists of all time and is often considered the French Newton. He taught Napoleon at the École Militaire in 1784, became a count of the empire in 1806, and a marquis in 1817 after the restoration of the monarchy. He is credited with advancements in engineering, mathematics, statistics, physics, astronomy, and philosophy; however, maybe his greatest achievement is not only surviving but benefiting from the change from the Ancien Régime $\rightarrow$ Bonaparte $\rightarrow$ Bourbon Restoration.

Of interest to this class is the Laplace transform ($\mathscr{L}[\ ]$) of the function $f(t)$, expressed as $\mathscr{L}[f(t)]$. Here, a Laplace transform is used as a method of solving the differential equations of motion by reducing the computation needed to that of integration and algebraic manipulation.

The definition of the Laplace transform of the function $f(t)$ is:

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{4.4}$$

where s represents a variable in the complex plane (also called the s-plane) and $f(t) = 0$ for all values of $t < 0$. Here, the s is a complex value. Lastly, the term $F(s)$ is a generic term that represents the input to a system. As this class needs the derivatives of the base function, we will calculate these next:

$$\mathscr{L}\left[\dot{f}(t)\right] = \int_0^\infty \dot{f}(t)e^{-st}dt = \int_0^\infty e^{-st}\frac{d[f(t)]}{dt}dt \tag{4.5}$$

integration by parts yields:

$$\mathscr{L}\left[\dot{f}(t)\right] = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt \tag{4.6}$$

Astutely, it can be noticed that the second term $s\int_0^\infty e^{-st}f(t)dt$ is the input to the system $F(s)$. Therefor, with a little rearranging this becomes:

$$\mathscr{L}\left[\dot{f}(t)\right] = sF(s) - f(0) \tag{4.7}$$

Taking the derivative of again yields:

$$\mathscr{L}\left[\ddot{f}(t)\right] = s^2F(s) - sf(0) - \dot{f}(0) \tag{4.8}$$

A few key points of the Laplace transforms are:

- The domain of the problem changes from the real number line (t) to the complex plane (s-plane).
- The integration of the Laplace transform changes differentiation into multiplication.
- The transform procedure is linear. Therefore, the transform of the linear combination of two transforms is the same as the linear transformation of these functions.
- To move from the time domain to the complex number plane we typically use tables of pre-solved integral.
- The function $x(t)$ can be obtained by taking the inverse Laplace transform defined as $x(t) = \mathscr{L}[X(s)]^{-1}$

The Laplace transform can be calculated in symbolic form. In particular interest to this text is the Laplace form of the system input $F(s)$ and output $X(s)$. To expand the symbolic form of the Laplace transform for the system inputs are and for system outputs:

$$\mathscr{L}[f(t)] = F(s) \tag{4.9}$$

$$\mathscr{L}\left[\dot{f}(t)\right] = sF(s) - f(0) \tag{4.10}$$

$$\mathscr{L}\left[\ddot{f}(t)\right] = s^2F(s) - sf(0) - \dot{f}(0) \tag{4.11}$$

here, $f(0)$ and $\dot{f}(0)$ are the initial values of the function $f(t)$. Furthermore, the for system outputs are:

$$\mathscr{L}[x(t)] = X(s) \tag{4.12}$$

$$\mathscr{L}[\dot{x}(t)] = sX(s) - x(0) \tag{4.13}$$

$$\mathscr{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0) \tag{4.14}$$

here, $x(0)$ and $\dot{x}(0)$ are the initial values of the function $x(t)$.

4.2 Transfer Function Method for Solving Vibrating Systems

As mentioned in the introduction to this chapter, a variety of systems can be solved for using the transfer function method. The procedure for using the Laplace transform to solve equations of motion expressed as an inhomogeneous ordinary differential equation is:

1. Take the Laplace transform of both sides of the EOM while treating the time derivatives symbolically.
2. Solve for $X(s)$ in the obtained equation.
3. Apply the inverse transform $x(t) = \mathscr{L}[X(s)]^{-1}$

4.2.1 Free Vibration for Undamped Systems

Consider the undamped single-DOF system:

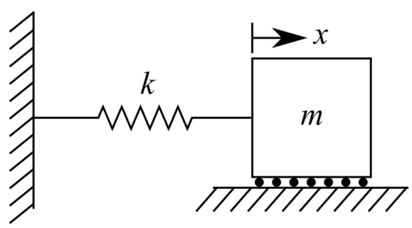


Figure 4.6: A spring mass model of a 1-DOF system.

The EOM for this system is a homogeneous differential equation becasue the right-hand side is equal to zero:

$$m\ddot{x}(t) + kx(t) = 0 \tag{4.15}$$

Here we will leave the "(t)" for clarity to differentiate the time domain solution from Laplace solution "(s)" in the s-plane, as discussed in review 4.1. The EOM can be rewritten in standard form as:

$$\ddot{x}(t) + \omega_n^2 x(t) = 0 \tag{4.16}$$

where the initial conditions at $t = 0$ are $x(0) = x_0$ and $\dot{x}(0) = v_0$. Taking the Laplace transforms, in symbplic form using equations 4.12 - 4.14, of both sides of the EOM yields:

$$\left[s^2X(s) - sx_0 - v_0\right] + \left[\omega_n^2 X(s)\right] = 0 \tag{4.17}$$

using equations 4.12 and 4.14 from section 4.1. Solving for the output of the system $X(s)$ yields:

$$X(s) = \frac{sx_0 + v_0}{s^2 + \omega_n^2} \tag{4.18}$$

We can expand this form of $X(s)$ to obtain equations listed in our Laplace Transform table:

$$X(s) = \frac{sx_0}{s^2 + \omega_n^2} + \frac{v_0}{s^2 + \omega_n^2} \cdot \frac{\omega_n}{\omega_n} \tag{4.19}$$

This becomes:

$$X(s) = x_0 \frac{s}{s^2 + \omega_n^2} + \left(\frac{v_0}{\omega_n}\right) \cdot \frac{\omega_n}{s^2 + \omega_n^2} \tag{4.20}$$

Next, using the inverse Laplace transform $x(t) = \mathscr{L}[X(s)]^{-1}]$ and the two following Laplace transforms (#5 and #6):

$$f(t) \text{ is } \cos(\omega t) \text{ when } F(s) \text{ is } \frac{s}{s^2 + \omega^2} \tag{4.21}$$

$$f(t) \text{ is } \sin(\omega t) \text{ when } F(s) \text{ is } \frac{\omega}{s^2 + \omega^2} \tag{4.22}$$

Therefore, we can obtain the solution for the system output $X(s)$ as:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{4.23}$$

The same procedure can be used to calculate the under damped and forced responses. However, when calculating these responses the algebraic solution for $X(s)$, s often contains quotients of polynomials. These Polynomial ratios may not be found in simple Laplace tables and must be solved using the method of partial fractions. An example of this procedure can be found in Appendix B of Inman.

4.2.2 Impulse Response Function

Shock loads on mechanical systems represent a very common source of vibration. These short-duration forces are also called called an impulse. An impulse excitation is defined as a force that is applied for a very short, or infinitesimal, length of time. An impulse is a nonperiodic force that is represented by the symbol δ. The response of a system to an impulse load is the same as the system's free response provided that the correct initial conditions are applied. This is illustrated in the following where the applied force $F(t)$ is impulsive in nature (i.e., large magnitude over a very short time).

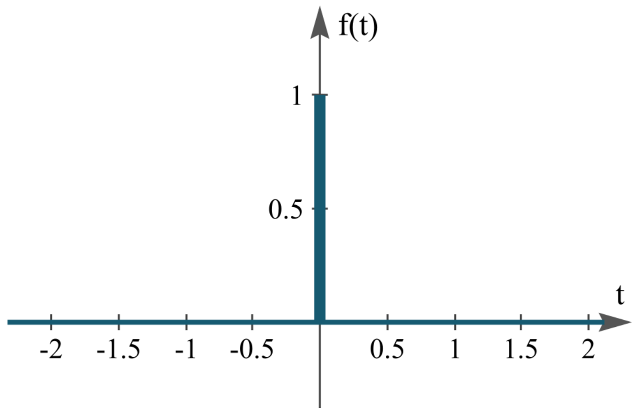


Figure 4.7: An impulse function with the impulse at $t = 0$.

The impulse response function can be solved for analytically, however, we will solve it using the transfer function approach. Here we will consider the under-damped spring-mass system. First, assume that the system is at rest (no initial conditions). Next, we write the EOM as:

$$m\ddot{x} + c\dot{x} + kx = \delta(t) \tag{4.24}$$

Taking the Laplace transform of both sides of the equation yields

$$m\big(s^2X(s) - sx(0) - \dot{x}(0)\big) + c\big(sX(s) - x(0)\big) + kX(s) = 1 \tag{4.25}$$

note that the $\mathscr{L}[\delta] = 1$ per #1 in the transform table. However, if we assume zero initial conditions (a system at rest when the impulse happens), the equation simplifies to.

$$ms^2X(s) + csX(s) + kX(s) = 1 \tag{4.26}$$

or

$$(ms^2 + cs + k)X(s) = 1 \tag{4.27}$$

Solving this equation for $X(s)$:

$$X(s) = \frac{1}{m} \cdot \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.28}$$

Again, the mass is extracted to develop a formulation that can be found in the Laplace tables. Setting the constraint that $\zeta < 1$ and consulting #10 in the table for Laplace transforms results in:

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{4.29}$$

where this is the general solution for a damped system subjected to an impulse loading function. For the undamped case a solution can be obtained by setting $\zeta = 0$. This Results in the following form for the undamped case:

$$x(t) = \frac{1}{m\omega_n} \sin(\omega_n t) \tag{4.30}$$

Below is a typical response for both a undamped and underdamped 1-DOF system subject to an impulse response at $t = 0$ seconds.

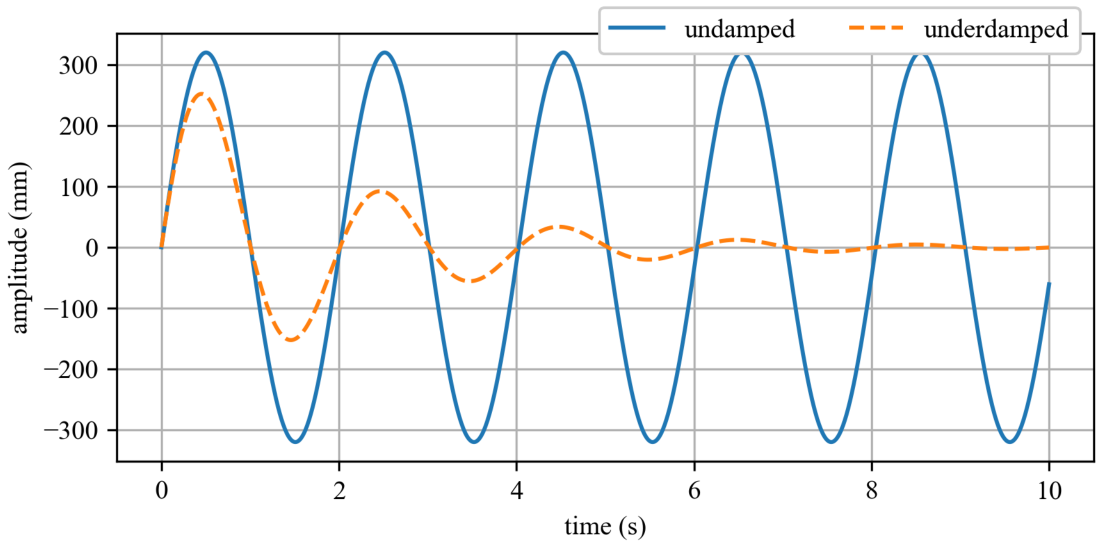


Figure 4.8: Temporal responses from a underdamped and undamped 1-DOF systems to a impulse response function.

4.2.3 Unit step function

Now consider a unit step function, denoted with a capital Greek Phi Φ:

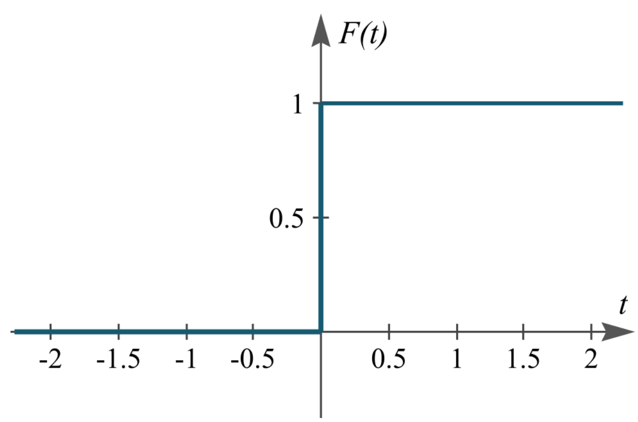


Figure 4.9: A Step function.

A step function is a common loading situation and can represent the dropping of a load into a truck, a car going over a curve, or a motor starting up.

The Laplace transform of the function, for a unit step function Φ, is:

$$\mathscr{L}[\Phi(t)] = \int_0^{\infty} e^{-st} dt = -\frac{e^{-\infty}}{s} + \frac{e^{-0}}{s} = \frac{1}{s}$$

This also lines up with Laplace Transform #3 from the Laplace table. This would be expected as Φ is used to represent the unit step function (i.e. a step function with a displacement of 1). As we consider linear systems in this class, we can scale the magnitude of the response by the magnitude of the impulse after the transform is performed.

4.2.4 Undamped spring-mass system

For a spring-mass system subjected to a unit step, assuming both initial conditions are zero, the solution can be obtained using the transform method. First, the EOM is

$$m\ddot{x}(t) + kx(t) = \Phi(t) \tag{4.31}$$

Taking the Laplace transform of both sides and assuming zero initial conditions yields:

$$ms^2X(s) + kX(s) = \frac{1}{s} \tag{4.32}$$

Next, this equation is solved for $X(s)$ as:

$$X(s) = \frac{1}{s(ms^2+k)} \tag{4.33}$$

This can be rearranged as:

$$X(s) = \frac{1}{m} \cdot \frac{1}{s(s^2+\omega_n^2)} \tag{4.34}$$

where $\frac{1}{m}$ will pass through the Laplace function. Therefore, taking the inverse Laplace transform using #9 of the provided Laplace transforms yields:

$$x(t) = \frac{1}{m} \cdot \frac{1}{\omega_n^2}\big(1-\cos(\omega_n t)\big) = \frac{1}{k}\big(1-\cos(\omega_n t)\big) \tag{4.35}$$

4.2.5 Under damped spring-mass system

For a spring-mass-damper system subjected to a unit step, assuming both initial conditions are zero, the solution can be obtained using the transform method. First, the EOM is:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \Phi(t) \tag{4.36}$$

Converting to the standard form results in:

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \frac{1}{m} \cdot \Phi(t) \tag{4.37}$$

taking the Laplace transform of both sides and assuming zero initial conditions yields:

$$s^2X(s) + 2\zeta\omega_n sX(s) + \omega_n^2 X(s) = \frac{1}{m} \cdot \frac{1}{s} \tag{4.38}$$

Next, this equation is solved for $X(s)$ as:

$$X(s) = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2} \cdot \frac{1}{m} \cdot \frac{1}{s} \tag{4.39}$$

multiplying the right-hand-side of this equation by $\frac{\omega_n^2}{\omega_n^2}$ results in:

$$X(s) = \frac{1}{m\omega_n^2} \cdot \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{4.40}$$

Again, the $\frac{1}{m\omega_n^2}$ will pass through the Laplace function. Therefore, taking the inverse Laplace transform using #11 on the Laplace transform sheet yields:

$$x(t) = \frac{1}{m\omega_n^2} \cdot \left(1 - \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)\right), \text{ where } \phi = \cos^{-1}(\zeta), \text{ where } \zeta < 1 \tag{4.41}$$

After obtaining equations for the undamped and under damped cases, the responses for the unit step, solved with the transform method, can be plotted as:

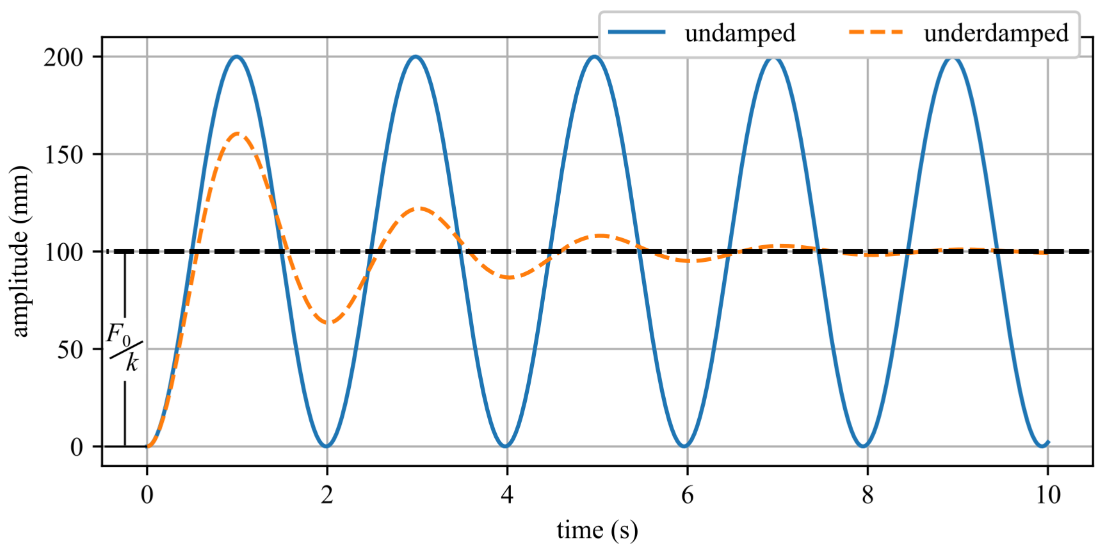


Figure 4.10: Temporal responses from a underdamped and undamped 1-DOF systems to a impulse response function.

Note that the system will settle out around F_0/k where $F_0\Phi$ is a scaling factor for the step loading.

Example 4.1 A load of roosters m_r is dumped into the back of a dump truck. Assuming the roosters do not move, the bed of the truck can be modeled as a spring-mass-damper system (of values k, m, and c, respectively). Next, the load of roosters is modeled as a force $F(t) = m_r g$ that is applied to the spring-mass-damper system, as illustrated in the following figure. The simplification of the system allows for the analysis of the truck bed's vibrations as a simple spring-mass-damper system. First assume that the trucks damper is broken, how does the maximum dynamic displacement compare to the static displacement. What would happen to the maximum displacement if the damper was repaired on the truck?

(a) (b)

Figure 4.11: Dump truck being loaded with roosters showing (a) roosters going into the truck bed; and (b) the single-degree-of-freedom vibration model.

Solution: First, setting the load applied to the truck as 1 unit, it can be seen that this is a unit step loading condition and a broken damper represents an undamped case. To obtain a rough idea about the nature of static and dynamic displacement, the undamped displacement is

$$x(t) = \frac{1}{k}\big(1 - \cos(\omega_n t)\big) = \frac{m_r g}{k}\big(1 - \cos(\omega_n t)\big) \tag{4.42}$$

This equation has a maximum amplitude when the $\cos(\omega_t) = -1$, resulting in:

$$x(t) = \frac{m_r g}{k}\big(1 - (-1)\big) \tag{4.43}$$

This can be rearranged for the maximum displacement value $x_{\max}$ as:

$$x_{\max} = 2\frac{m_r g}{k} \tag{4.44}$$

Note that the dynamic displacement of the truck bed is twice that of the static displacement. Therefore, if the truck manufacturer designed the truck to only take the static load of rosters (i.e., if the roosters were placed gently into the truck bed), the frame of the truck would be damaged when the roosters were loaded into the truck dynamically. From this, it can be understood that it is important to consider dynamic responses of a system during the design phase.

4.2.6 Arbitrary Inputs to a System

The time domain response of a system to an arbitrary input force in time can be calculated using a series of impulses as shown in figure 4.12. This method allows the practitioner to easily calculate the response of an arbitrary input to a system using a single expression executed in a "for loop". This type of analysis is often more efficient in terms of programing than more direct methods such as the transfer functions sown in this text.

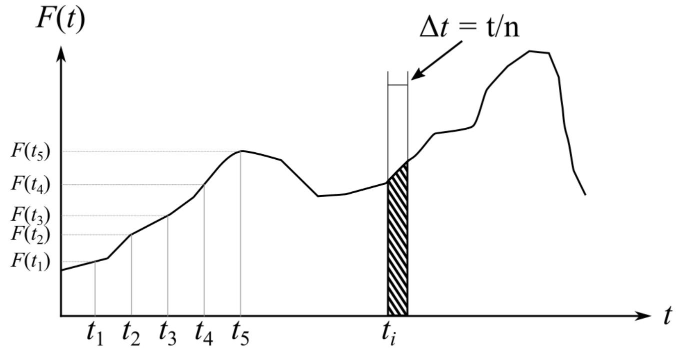


Figure 4.12: Generalized response showing that any signal can be represented as a series of impulse signals.

Example 4.2 In testing, an hammer is used to excite a 1-DOF system with an impact (i.e. impulse), however, the hammer ascendantly impacts the system twice. The first impact has a force of 0.2 N, while the second has a force of 0.1 N and happens 0.1 seconds after the first impact. Plot the response for the double impact. The system has the parameters m = 1 kg, c = 0.5 kg/s, k = 4 N/m.
Solution: First, we can define the forcing function as:

$$F(t) = 0.2\delta(t) + 0.1\delta(t-\tau) \tag{4.45}$$

where τ is the offset between the first and second impacts. Next, considering that the unit impulse has a magnitude of 1 we can obtain solutions for the first impact by first writing it's EOM:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0.2\delta(t) \tag{4.46}$$

Taking the Laplace transform of both sides of the equation yields

$$m\big(s^2X(s) - sx(0) - \dot{x}(0)\big) + c\big(sX(s) - x(0)\big) + kX(s) = 0.2 \tag{4.47}$$

However, assuming zero initial conditions, the equation simplifies to.

$$(ms^2 + cs + k)X(s) = 0.2 \tag{4.48}$$

Solving this equation for $X(s)$:

$$X(s) = \frac{0.2}{m} \cdot \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.49}$$

Again, consulting #10 in the table for Laplace transforms results in:

$$x_1(t) = \frac{0.2}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{4.50}$$

where this is the general solution for a damped system subjected to an impulse loading function. The second impact can now be solved for using the same method. However, now the time (t) must be offset by (τ) to allow the impact to still be located at $t = 0$ in terms of the second impact. This results in:

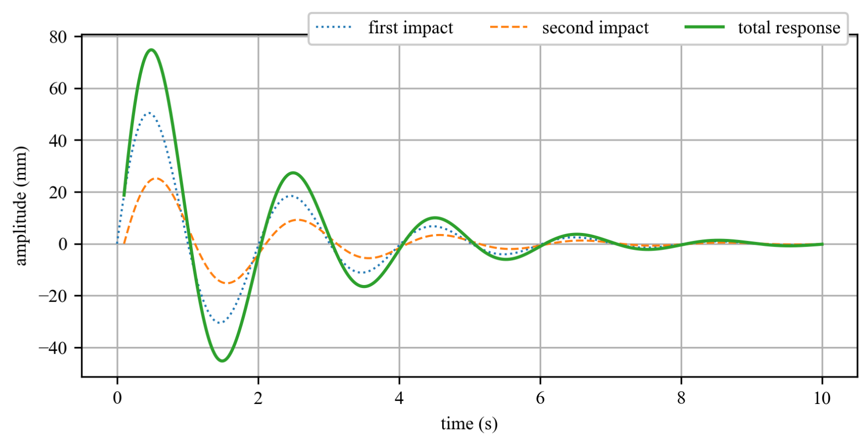
$$x_1(t) = \frac{0.2}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) \tag{4.51}$$

$$x_2(t) = \frac{0.1}{m\omega_d} e^{-\zeta\omega_n (t-\tau)} \sin\big(\omega_d (t-\tau)\big) \tag{4.52}$$

Next, using the knowledge that the systems are linear and that the Laplace transform of a linear combination of two transforms is the same as the linear transformation of these functions we can build the piecewise function:

$$x(t) = \begin{cases} \frac{0.2}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) & \text{if } t < \tau \\ \frac{0.2}{m\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t) + \frac{0.1}{m\omega_d} e^{-\zeta\omega_n (t-\tau)} \sin\big(\omega_d (t-\tau)\big) & \text{if } \tau \le t \end{cases}$$

For the mass, damping, and stiffness values given above this can be plotted as:



4.2.7 Base excitation

add nodes on base excitation,

4.3 Transfer Function for Response to Random Inputs

Consider the following system

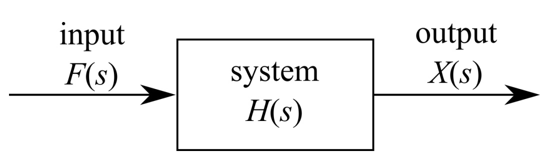


Figure 4.13: Generic block diagram of a system $H(s)$ subjected to an input $F(s)$ and its corresponding output $X(s)$ where the (s) denotes that the considered system is in the s-plane.

where $F(s)$ is the input, $H(s)$ is the system, and $X(s)$ is the output from the system. This formulation is called the transfer-function approach and is commonly used for the formulation and solution of dynamic problems in the control literature. It can also be used for solving the various forced-vibration problems including those from complex or stochastic inputs.

4.3.1 Defining the transfer function $\mathbf{H}(\mathbf{s})$

Again, consider the generic system represented in figure 4.13. For this system representation, $F(s)$ is the Laplace of the transform of the driving force and $H(s)$ is the Laplace transform of the response of the system $h(t)$.

We need to define transfer function $H(s)$ for a generic system. To do this let us show the reasoning behind the transfer function. Here we will show that the output of any system ($x(t)$) can be related to the input of the system ($f(t)$) through a series of polynomial coefficients (a and b). Consider the general n^{th}-order linear, time-invariant differential equation that governs the behavior of the dynamic system.

$$a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + ... + a_0 x(t) = b_m \frac{d^m f(t)}{dt^m} + b_{m-1} \frac{d^{m-1} f(t)}{dt^{m-1}} + ... + b_0 f(t) \tag{4.53}$$

where $x(t)$ is the output and $f(t)$ is the input. Note that this is similar to the formulation we have had before for the EOM. Taking the Laplace transformation of both side of the above equation yields

$$\begin{aligned} a_n s^n X(s) + a_{n-1} s^{n-1} X(s) + ... + a_0 X(s) + \text{initial condition for } x(t) = \\ b_m s^m F(s) + b_{m-1} s^{m-1} F(s) + ... + b_0 F(s) + \text{initial condition for } f(t) \end{aligned} \tag{4.54}$$

It can be seen that this equation is a purely algebraic expression. If we assume the initial conditions to be zero, the equation reduces to the following:

$$(a_n s^n + a_{n-1} s^{n-1} + ... + a_0) X(s) = (b_m s^m + b_{m-1} s^{m-1} + ... + b_0) F(s) \tag{4.55}$$

if we rearrange equation 4.55 to solve for the relationship between the Laplace variables $\big(X(s)$ and $F(s)\big)$ and the algebraic expressions we get:

$$\frac{X(s)}{F(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_0} \tag{4.56}$$

this shows that the ratio of the input algebraic expressions over the output algebraic expressions is equal to the ratio of the output Laplace variable over the input Laplace variable. This shows that we can relate the Laplace variables to the algebraic expressions. Therefore, we can define the transfer function $H(s)$ as:

$$H(s) = \frac{X(s)}{F(s)} \tag{4.57}$$

In a more formal term, the transfer function that is defined as: “The ratio of the Laplace transforms of the output or response function to the laplace transform of the input or forcing function assuming zero initial conditions”.

Equation 4.57 can be rearranged to show that the output of the system $X(s)$, can be obtained if we know the input $F(s)$ and the transfer function $H(s)$:

$$X(s) = H(s)F(s) \tag{4.58}$$

4.3.2 Transfer Function method (Steady-State solution)

Considering the forced system:

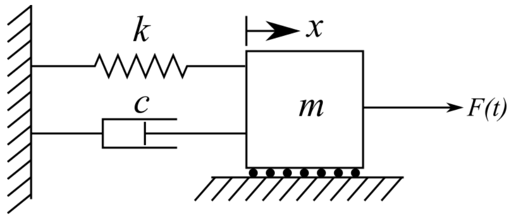


Figure 4.14: A spring-dashpot-mass model of a 1-DOF system with external excitation.

that can be expressed as the equation of motion

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos(\omega t) \tag{4.59}$$

Here $F_0 \cos(\omega t)$, is used at the input but any input will develope the same transfer function as the transfer function is bounded to the system and not the input. From the #6 in the table for Laplace Transforms, we know that

$$\mathscr{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2} \tag{4.60}$$

Therefore,

$$F(s) = \frac{F_0 s}{s^2 + \omega^2} \tag{4.61}$$

Ignoring the initial conditions, and therefore considering only the particular solution, and taking the Laplace transform of the EOM equation yields:

$$(ms^2 + cs + k)X(s) = \frac{F_0 s}{s^2 + \omega^2} \tag{4.62}$$

where $X(s)$ denotes the Laplace transform of the unknown function $x(t)$ and s is the complex transform variable. Rearranging the above equation for $X(s)$ yields:

$$X(s) = \frac{F_0 s}{(ms^2 + cs + k)(s^2 + \omega^2)} \tag{4.63}$$

Now that we have $F(s)$ and $X(s)$ we can obtain $H(s)$ as

$$H(s) = \frac{X(s)}{F(s)} = \frac{F_0 s}{(ms^2 + cs + k)(s^2 + \omega^2)} \cdot \frac{s^2 + \omega^2}{F_0 s} = \frac{1}{ms^2 + cs + k} \tag{4.64}$$

or

$$H(s) = \frac{1}{ms^2 + cs + k} \tag{4.65}$$

This ratio is termed the transfer function of a system and is an important tool in vibration analysis.

Sometimes, how the system responds to a inputs with certain frequency components is important in understanding they system in general, therefore, we want to solve for the frequency response function of the system. The frequency response function is denoted as $H(j\omega)$ where the complex number s is replaced by the frequency component of the system while considering the imaginary portion in the complex plane (i.e., $s = j\omega$). Therefor, the frequency response function of the system becomes:

$$H(j\omega) = \frac{1}{m(j\omega)^2 + cj\omega + k} = \frac{1}{-m\omega^2 + cj\omega + k} \tag{4.66}$$

rearranging into a standard form yields:

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{4.67}$$

recall that $j^2 = -1$. This is the frequency response function of the system. Therefore, it can be seen that the frequency response function of the system is the transfer function of the system evaluated along the imaginary axis $s = j\omega$. However, this expression contains imaginary values (that help to account for the phase in the system) and therefore can be challenging to work with. As the amplitude $|H(j\omega)|$ of the response (the real portion of the equation) is useful to the practitioner, it is prudent to consider the special case of amplitude response while neglecting the phase response. Consider that:

$$H(j\omega) = \mathrm{R} + \mathrm{I}j \tag{4.68}$$

so

$$|H(j\omega)| = \sqrt{\mathrm{R}^2 + \mathrm{I}^2} \tag{4.69}$$

multiplying $H(j\omega)$ by 1 that is represented by the its unit complex conjugate yields:

$$\begin{aligned} H(j\omega) &= \left(\frac{1}{k - m\omega^2 + c\omega j}\right)\left(\frac{k - m\omega^2 - c\omega j}{k - m\omega^2 - c\omega j}\right) && (4.70) \\ &= \left(\frac{k - m\omega^2}{(k - m\omega^2)^2(c\omega)^2}\right)\left(\frac{-c\omega}{(k - m\omega^2)^2(c\omega)^2}j\right) && (4.71) \end{aligned}$$

therefore, $\mathrm{R} = \frac{k - m\omega^2}{(k - m\omega^2)^2(c\omega)^2}$ and $\mathrm{I} = \frac{-c\omega}{(k - m\omega^2)^2(c\omega)^2}$. Now, calculating the amplitude of $H(j\omega)$ we get:

$$\begin{aligned} H(\omega) &= |H(j\omega)| && (4.72) \\ &= \sqrt{\mathrm{R}^2 + \mathrm{I}^2} && (4.73) \\ &= \sqrt{\frac{(k - m\omega)^2 + (-c\omega)^2}{\left((k - m\omega^2)^2 + (c\omega)^2\right)^2}} && (4.74) \\ &= \sqrt{\frac{1}{(k - m\omega^2)^2 + c^2\omega^2}} && (4.75) \\ &= \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} && (4.76) \end{aligned}$$

where $H(\omega)$ represents only the amplitude of the frequency response function and therefore drops the j term from the expression.

To recap, for a single DOF damped spring-mass system the transfer function is:

$$H(s) = \frac{1}{ms^2 + cs + k} \tag{4.77}$$

And the frequency response function is:

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{4.78}$$

While the amplitude of the frequency response is:

$$H(\omega) = |H(j\omega)| = \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \tag{4.79}$$

Example 4.3 Considering the forced system:

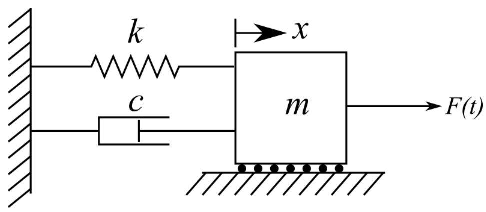


Figure 4.15: A spring-dashpot-mass model of a 1-DOF system with external excitation.

Set the forcing function to be $F_0 \sin(\omega t)$ and calculate the transfer function.

Solution: The equation of motion for the system is:

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t) \tag{4.80}$$

From the #6 in the table for Laplace Transforms, we know that:

$$\mathscr{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2} \tag{4.81}$$

Therefore,

$$F(s) = \frac{F_0\omega}{s^2 + \omega^2} \tag{4.82}$$

Ignoring the initial conditions and taking the Laplace transform of the EOM equation yields:

$$(ms^2 + cs + k)X(s) = \frac{F_0\omega}{s^2 + \omega^2} \tag{4.83}$$

Solving algebraically for the $X(s)$ yields:

$$X(s) = \frac{F_0\omega}{(ms^2 + cs + k)(s^2 + \omega^2)} \tag{4.84}$$

Now that we have $F(s)$ and $X(s)$ we can obtain $H(s)$ as

$$H(s) = \frac{X(s)}{F(s)} = \frac{F_0\omega}{(ms^2+cs+k)(s^2+\omega^2)} \cdot \frac{s^2+\omega^2}{F_0\omega} = \frac{1}{ms^2+cs+k} \tag{4.85}$$

or

$$H(s) = \frac{1}{ms^2+cs+k} \tag{4.86}$$

This is identical to the solution obtained using $F_0\cos(\omega t)$ as would be expected because the transfer function is related to the system and not to the input.

4.3.3 Response to Random Inputs

The transfer and frequency response functions can be very useful for determining the system's response to random inputs. Up to this point we have solved for deterministic input.

- **Deterministic**-For a known time t, the value of the input force $F(t)$ is precisely known.
- **Random** For a known time t, the value of the input force $F(t)$ is known only statistically.

To expand, a random signal is a signal with no obvious pattern. For these types of it is not possible to focus on the details of the input signal, as is done with a deterministic signal, rather the signal is classified and manipulated in terms of its statistical properties.

Randomness in vibration analysis can be thought of as the result of a series of results obtained from testing a system repeatability for various inputs under varying conditions. In these cases, one record or time history is not enough to describe the system. Rather, an ensemble of various tests are used to describe how the system will respond to the various inputs.

First, let us consider two inputs, a deterministic input (typical sin wave), and a random input (white noise). These inputs are shown in figure 4.16.

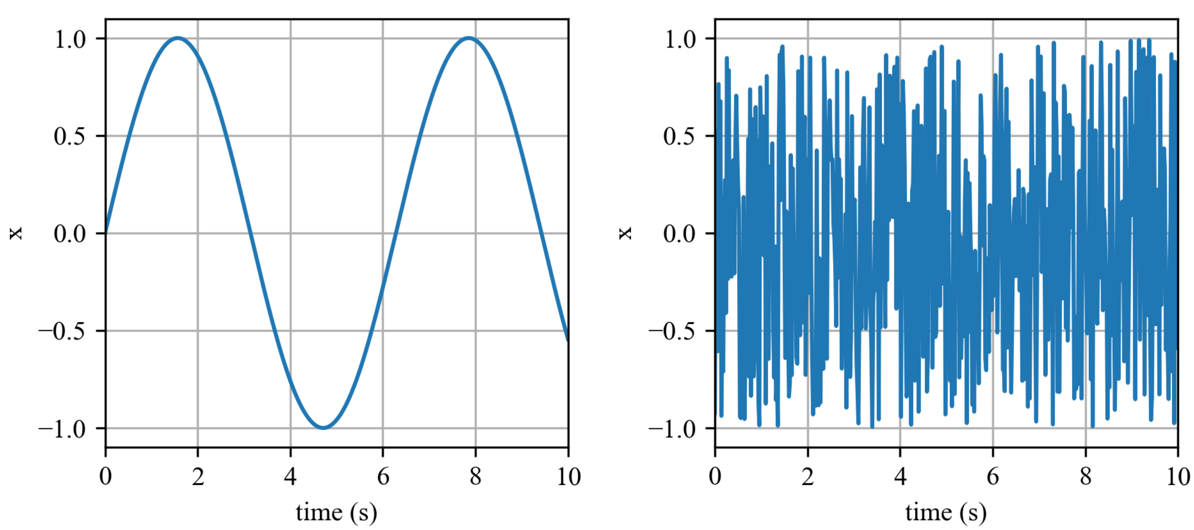


Figure 4.16: Two arbitrary inputs: (a) sinusoidal; and (b) uniform random noise.

One of the first factors to consider is the mean of the random signal $x(t)$, defined as:

$$E[x] = \bar{x} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)dt \tag{4.87}$$

where T is the length in time of the data collected. However, for random signals we often want to consider signals with an average mean of zero (i.e. $\bar{x}(t) = 0$). Therefore, for signals not centered around zero we can obtain a zero centered signal if the signal is stationary and we subtract the mean value from $\bar{x}$ from the signal $x(t)$. This can be written as:

$$x'(t) = x(t) - \bar{x} \tag{4.88}$$

where the $x'(t)$ is now centered around zero. As mentioned before, it is important to consider whether or not the input signals are stationary. A signal is stationary if its statistical properties (usually expressed by its mean) do not change with time. Here, it can be seen that for our inputs considered the signals are stationary if a long enough time period is considered.

Another important variable is variance (or mean-square value) of the random variable $x(t)$ defined as:

$$E[(x-\bar{x})^2] = \lim_{T\to\infty} \frac{1}{T} \int_0^T (x(t)-\bar{x})^2 dt \tag{4.89}$$

and provides a measure of the magnitude of the fluctuations in the signal $x(t)$. If the signal has an expected value of zero, or $E[x] = 0$, this simplifies to.

$$E[x^2] = \overline{x^2} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t)dt \tag{4.90}$$

This expression leads to the calculation of the root-mean-square (RMS) of the signal:

$$x_{\text{rms}} = \sqrt{\overline{x^2}} \tag{4.91}$$

Considering a nonstationary signal, a important measure of interests is how fast the value of the variables change. This is important to understand as it provides context for how long a signal must be sampled to before a meaningful representation of the signal can be calculated in a statistical sense. One way to quantify how fast the values of signal change is the autocorrelation function:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt \tag{4.92}$$

The subscript xx denotes that this is a measure of the response for the variable xx, τ is the time difference between the values at which the signal $x(t)$ is sampled. The auto collation for the two inputs considered above are expressed in figure 4.17.

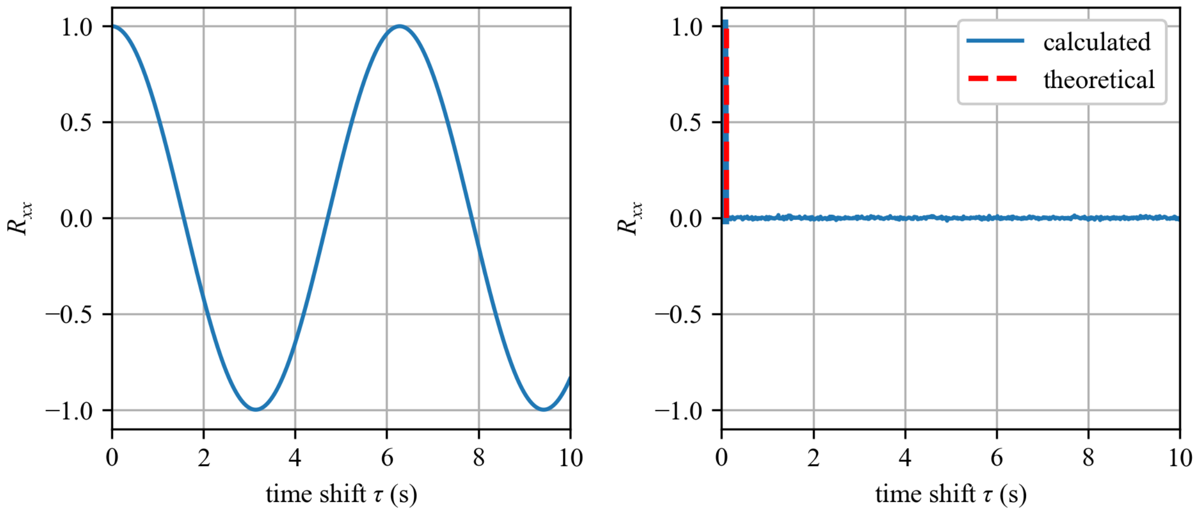


Figure 4.17: Responses from the autocorrelation function for the inputs shown in figure 4.16 showing: (a) a sinusoidal; and (b) uniform random noise.

Note that the value of τ selected in the auto correlation function greatly affects its response for the sinusoidal input. This is because the values for the sinusoidal are highly correlated. To expand, the value at any time t is greatly effected by the values immediately before and after it. This is not the case for the random input where the signal is not correlated and therefore there is little difference in changing the value of τ on the response of the autocorrelation function.

Next, if we take the Fourier transform of the autocorrelation function we obtain the power spectral density (PSD) defined as:

$$S_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau \tag{4.93}$$

where the integral of $R_{xx}(\tau)$ changes the real number τ into the frequency-domain value ω. The frequency spectrum is denoted with S and the subscript of the considered variable (e.g., $S_{xx}(\omega)$). The frequency spectrum for the two input cases considered are plotted in figure 4.18.

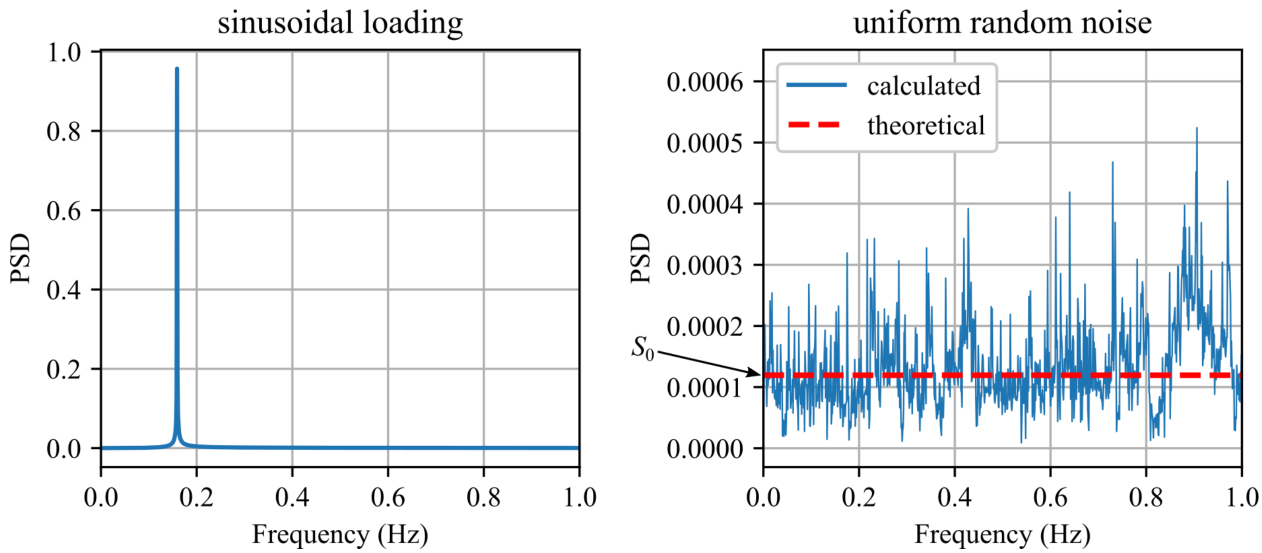


Figure 4.18: Power spectral density plots for the inputs shown in figure 4.16 showing: (a) a sinusoidal; and (b) uniform random noise.

where the flat frequency response for the random input denotes that the random input is white noise input. This flat frequency response in the frequency domain can be denoted S_0, such that $S_{ff}(\omega) = S_0$ or $S_{xx}(\omega) = S_0$, depending on whether the frequency spectrum of the input (ff) or output (xx) is being considered. While a true white noise input would be perfectly flat, white noise is really just a theoretical concept as all real-world data will have some variation in the frequency domain as diagrammed in figure 4.18(b).

Recall that S_{xx} is the spectrum of the response of the system. For the one-DOF system considered here, we can express the arbitrary input as a series of impulse inputs as discussed in section 4.2.6. This knowledge, along with the frequency response function can be used to relate the spectrum of the input $S_{ff}(\omega)$ to the output through the transfer function as:

$$S_{xx}(\omega) = |H(j\omega)|^2 \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ff}(\tau) e^{-j\omega\tau} d\tau \right] \tag{4.94}$$

This can also be expressed in symbolic form as:

$$S_{xx}(\omega) = |H(j\omega)|^2 S_{ff}(\omega) \tag{4.95}$$

where R_{ff} denotes the autocorrelation function of $F(t)$ and S_{ff} denotes the PSD of the forcing function $F(t)$. The notation $|H(j\omega)|^2$ is the square of the magnitude of the complex frequency response function. A more detailed derivation can be found in [Engineering Vibrations, Inman (2001)],[Random Vibrations, Spectral & Wavelet Analysis, Newland (1993)], but here it is more important to study the results rather than the derivations.

Example 4.4 Consider the following system

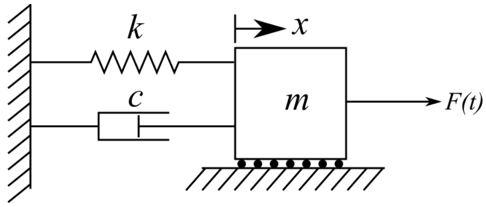


Figure 4.19: A spring-dashpot-mass model of a 1-DOF system with external excitation.

Calculate the PSD of the response $x(t)$ given that the PSD of the applied force $S_{ff}(\omega)$ is white noise.

Solution: From the system we know that the EOM is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{4.96}$$

The frequency response function for this system is

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{4.97}$$

while the amplitude of the response is:

$$H(\omega) = |H(j\omega)| = \frac{1}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \tag{4.98}$$

Applying the equation that relates $S_{ff}(\omega)$ to $S_{xx}(\omega)$ we get:

$$S_{xx}(\omega) = |H(j\omega)|^2 S_{ff}(\omega) = \left|\frac{1}{k - m\omega^2 + c\omega j}\right|^2 S_{ff}(\omega) \tag{4.99}$$

White noise means the forcing function $S_{ff}(\omega)$ is constant across the frequency spectrum, therefore, $S_{ff}(\omega) = S_0$. Additionally as:

$$|H(j\omega)|^2 = \left|\frac{1}{k - m\omega^2 + c\omega j}\right|^2 = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \tag{4.100}$$

where the absolute value is the amplitude of the system. Therefore, we obtain:

$$S_{xx}(\omega) = |H(j\omega)|^2 S_0 = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} S_0 = \frac{S_0}{(k - m\omega^2)^2 + c^2\omega^2} \tag{4.101}$$

Using various values for the elements in the system, the PSD for the system considered looks like:

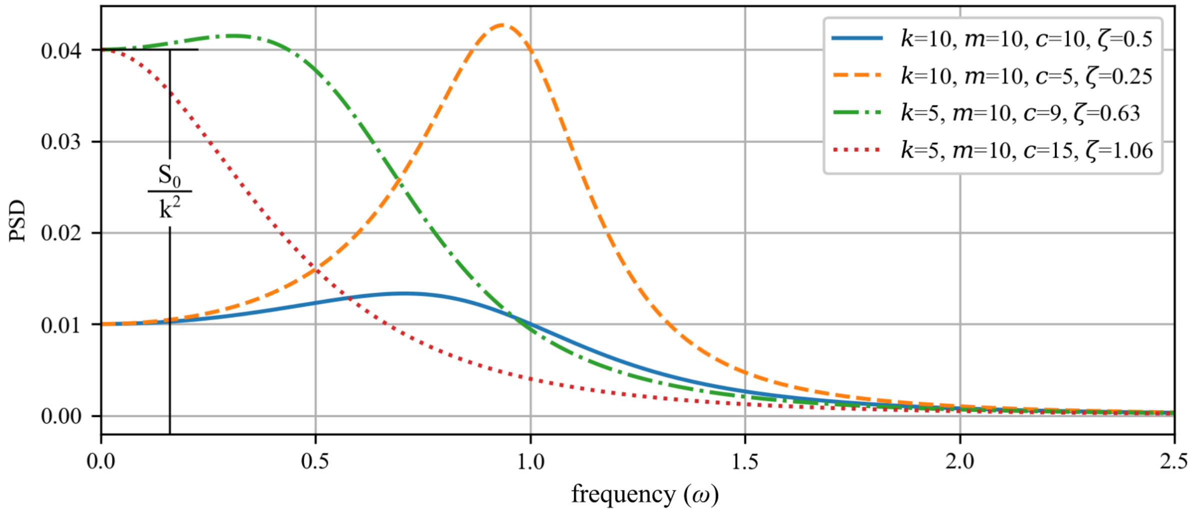


Figure 4.20: Response for considered 1-DOF systems subjected to a white noise input.

Another useful quantity to consider is the expected output, in terms of its mean and variance, for a given input. Working within the constraint that the system will oscillate about zero, $E[x] = 0$, the mean-square value can be directly related to the PSD function as:

$$E[x^2] = \overline{x^2} = \int_{-\infty}^{\infty} |H(j\omega)|^2 S_{ff}(\omega) d\omega \tag{4.102}$$

For a constant input S_0, as diagrammed in figure 4.18(b), the mean-square value can be expressed as:

$$E[x^2] = \overline{x^2} = S_0 \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \tag{4.103}$$

After inspecting the above equation, it becomes clear that to obtain the square of the expected value, a solution for $\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega$ must be obtained. For cases where $S_{ff}(\omega) = S_0$ and as such $S_{ff}(\omega)$ can be pulled out of the integral, these integrals have been solved [Random Vibrations, Spectral & Wavelet Analysis, Newland (1993)]. For example, given $\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega$:

$$\int_{-\infty}^{\infty} \left| \frac{B_0}{A_0 + j\omega A_1} \right|^2 d\omega = \frac{\pi B_0^2}{A_0 A_1} \tag{4.104}$$

and

$$\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi (A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{4.105}$$

When combined with equation 4.103, these integrals allow for the easy calculation of the expected values.

Example 4.5 For system below, calculate the mean-square response of the system given that the the spectrum of the input force $F(t)$ is a perfect theoretical white noise.

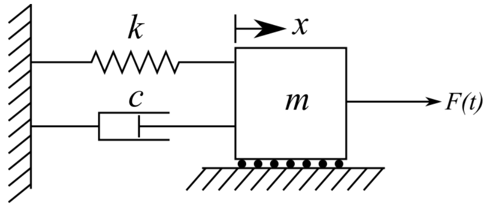


Figure 4.21: A spring-dashpot-mass model of a 1-DOF system with external excitation.

Solution: Again, as the forcing function $S_{ff}(\omega)$ is constant across the frequency spectrum $S_{ff}(\omega) = S_0$ the mean-square response can be calculated as:

$$E[x^2] = \overline{x^2} = S_0 \int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega \tag{4.106}$$

Using the already tabulated response:

$$\int_{-\infty}^{\infty} \left| \frac{B_0 + j\omega B_1}{A_0 + j\omega A_1 - \omega^2 A_2} \right|^2 d\omega = \frac{\pi(A_0 B_1^2 + A_2 B_0^2)}{A_0 A_1 A_2} \tag{4.107}$$

and the frequency response function for the system as derived in equation 4.67:

$$H(j\omega) = \frac{1}{k - m\omega^2 + c\omega j} \tag{4.108}$$

when $B_0 = 1$, $B_1 = 0$, $A_0 = k$, $A_1 = c$, and $A_2 = m$. Therefore, using the tabulated expression we can show that:

$$E[x^2] = S_0 \frac{\pi m}{kcm} = \frac{S_0 \pi}{kc} \tag{4.109}$$

Table of Laplace Transforms for Vibrations

This is a partial lists of important Laplace transforms for vibrations that assumes zero initial conditions, $0 < t$, and $\zeta < 1$.

$f(t)$	$\mathscr{L}[f(t)] = F(s)$	
$\delta(t)$	1	(1)
$\delta(t - t_0)$	e^{-st_0}	(2)
1	$\frac{1}{s}$	(3)
e^{at}	$\frac{1}{s-a}$	(4)
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	(5)
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	(6)
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	(7)
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	(8)
$\frac{1}{\omega^2}\big(1-\cos(\omega t)\big)$	$\frac{1}{s(s^2+\omega^2)}$	(9)
$\frac{1}{\omega_d}e^{-\zeta\omega t}\sin(\omega_d t)$	$\frac{1}{s^2+2\zeta\omega s+\omega^2}$	(10)
$1-\frac{\omega}{\omega_d}e^{-\zeta\omega t}\sin(\omega_d t+\phi),\ \phi=\cos^{-1}(\zeta)\ldots$	$\frac{\omega^2}{s(s^2+2\zeta\omega s+\omega^2)}$	(11)
$\frac{t^{n-1}}{(n-1)!},\ n=1,2,\ldots$	$\frac{1}{s^n}$	(12)
$t^n,\ n=1,2,\ldots$	$\frac{n!}{s^{n+1}}$	(13)
$t^n e^{\omega t},\ n=1,2,\ldots$	$\frac{n!}{(s-\omega)^{n+1}}$	(14)
$\frac{1}{\omega}(1-e^{-\omega t})$	$\frac{1}{s(s+\omega)}$	(15)
$\frac{1}{\omega^2}(e^{-\omega t}+\omega t-1)$	$\frac{1}{s^2(s+\omega)}$	(16)

$f(t)$	$\mathscr{L}[f(t)] = F(s)$	
$\frac{1}{\omega^3}\big(\omega t-\sin(\omega t)\big)$	$\frac{1}{s^2(s^2+\omega^2)}$	(17)
$\frac{1}{2\omega^3}\big(\sin(\omega t)-\omega t\cos(\omega t)\big)\ldots$	$\frac{1}{(s^2+\omega^2)^2}$	(18)
$\frac{t}{2\omega}\sin(\omega t)$	$\frac{s}{(s^2+\omega^2)^2}$	(19)
$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	(20)
$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	(21)
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$	(22)
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$	(23)
$e^{at}\sinh(\omega t)$	$\frac{\omega}{(s-a)^2-\omega^2}$	(24)
$e^{at}\cosh(\omega t)$	$\frac{s-a}{(s-a)^2-\omega^2}$	(25)
$\frac{1}{\omega_2}\sin(\omega_2 t)-\frac{1}{\omega_1}\sin(\omega_1 t)\ldots$	$\frac{\omega_1^2-\omega_2^2}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$	(26)
$\cos(\omega_2 t)-\cos(\omega_1 t)$	$\frac{s(\omega_1^2-\omega_2^2)}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$	(27)
$e^{at}f(t)$	$F(s-a)$	(28)
$f(t-a)\Phi(t-a)$	$e^{-as}F(s)$	(29)
$\Phi(t-a)$	$\frac{e^{-as}}{s}$	(30)
$f'(t)$	$sF(s)-f(0)$	(31)

5 Multiple degree-of-freedom systems

Until now we have only considered and modeled systems that can require one coordinate system to describe their motion. In this chapter we will develop the mathematical tools required to model multiple degree-of-freedom system that require multiple independent coordinates to describe their motion. As before, the equations that describe the motion of rigid bodies in space are developed from Newton's second law of motion. However, unlike before, there exists an independent equation for each body in motion. These equations are therefore coupled by the system and are often expressed in matrix notation such that the mass, damping, and stiffness matrices are easily defined.

Review 5.1 Linear Algebra

Linear algebra allows for the efficient solving of these coupled equations.

The dot product allows us to multiply matrices and is defined as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \end{bmatrix} \tag{5.5}$$

Another arrangement of the same principle, in a format more related to vibrations, is:

$$\begin{bmatrix} a_1+a_2 & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} (a_1+a_2)e+bf \\ ce+df \end{bmatrix} \tag{5.6}$$

The transpose of a matrix is an operator which flips a matrix over its diagonal. For a matrix A, the transpose A^{T} can be written as:

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \rightarrow A^{\mathrm{T}} = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \tag{5.7}$$

A matrix is symmetric if $A = A^{\mathrm{T}}$. Therefore, symmetric matrix must be square and can be written as:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = A^{\mathrm{T}} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}, \text{ where } b=d,\ c=g,\ f=h \tag{5.8}$$

The determent of a matrix is a scalar value that is a function of the entries of a square matrix. The determent characterizes the matrix and its linear map. The determent is often writted as det(A), det A, or $|A|$. For a 2 $\times$ 2 matrix this is defined as:

$$\det(A) = ad - bc, \text{ when } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{5.9}$$

The inverse of a square matrix is such that $AA^{-1} = A^{-1}A = I$ where I is the identity matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.10}$$

and the inverse of a 2×2 matrix is defined as:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ when } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{5.11}$$

Matrices that do not have an inverse are called a singular matrix. add notes on notation of vectors

5.1 General Discussion on Mode Shapes

Studying and characterizing the natural frequencies of a system allows for the detailed investigation of the system response. Modern vibration analysis relies heavily on the concepts of mode shapes for various engineering tasks.

Mode shapes are not the displacement of a system, rather the describe the configurations into which a structure will naturally displace at a given frequency. For example, consider the 4-DOF system shown in figure 5.5 that represents a pole (i.e. cantilever beam). Assuming that the system experience a linear response and using the mode-superposition method we can see that the displaced shape $\vec{x}$ is a function of all of the mode shapes u_i and their corresponding participation factors q_i. Note that the mode shapes associated with the lower frequencies tend to provide the greatest contribution to structural response. As the frequencies that excite the modes increase, the mode shapes contribute less, are predicted less reliably, and are harder to measure. Therefore, the analysis of the system is often truncated after the first few modes and rarely exceeds the 10$^{\text{th}}$ mode.

Figure 5.5 shows a structure with N degrees of freedom that therefore had N corresponding mode shapes. Each mode shape is independent and normalized such that the maximum displacements are the same. The summation of the mode shapes multiplied by their corresponding participation factors (q_i) yields the deflection of the structure.

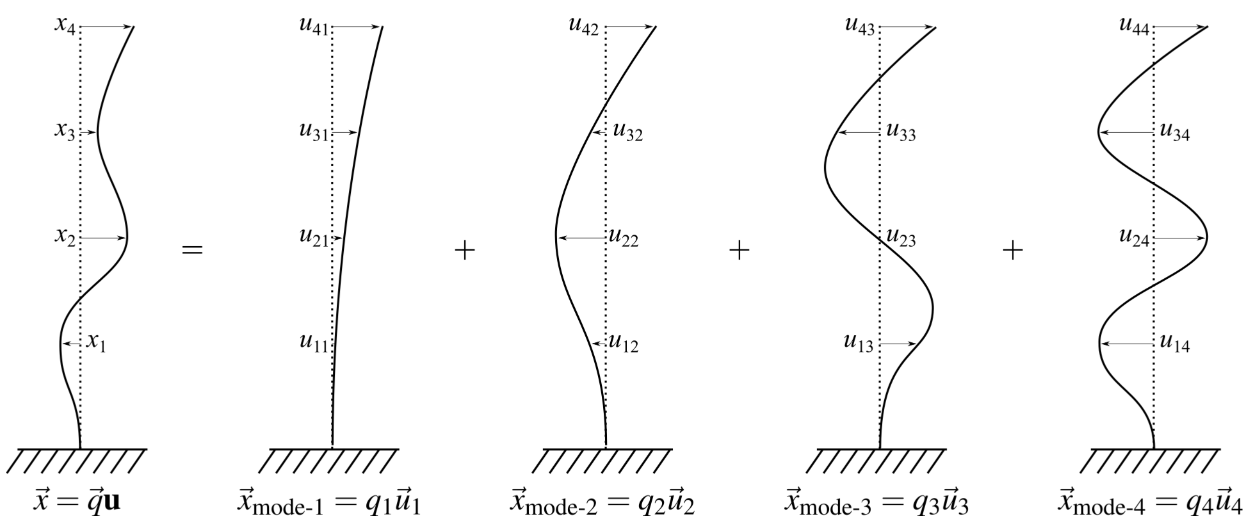
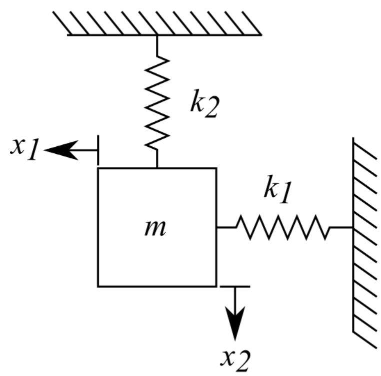


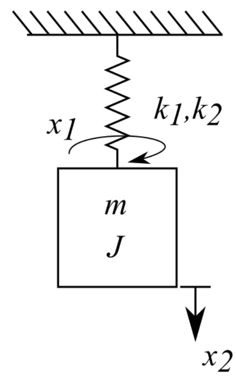
Figure 5.5: Deflection of a vertical cantilever, $\vec{x}$, is a function of the considered mode shapes u_i and their corresponding participation factors q_i.

5.2 Modeling Undamped Two Degree of Freedom Systems

Consider the undamped 2-DOF systems presented in figure 5.6. These system with a single mass capable of moving in two directions. To expand, figure 5.6(a) reports a mass that can move horizontally or vertically in space. However, this mass does not rotate during its movements. Moreover, figure 5.6(b) presents a system that rotates about the spring and displaces vertically. These are examples of 2 DOF systems because each system has two independent coordinate systems that express the movement of the mass.



(a)



(b)

Figure 5.6: Examples of single mass 2-DOF systems that: (a) displace in the vertical and horizontal directions; and (b) rotates about the spring and displaces in the vertical direction.

Another example of a 2-DOF system with two masses, each with their own independent coordinate system, is presented in figure 5.7. The two coordinates that describe the systems movements are x_1 and x_2.

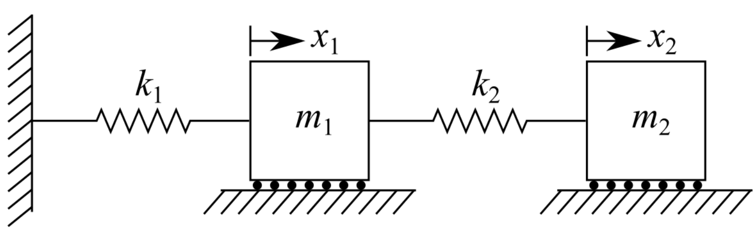


Figure 5.7: 2-DOF system with two masses and two independent confidante systems x_1 and x_2.

5.2.1 Solution for the Two-Degree-of-Freedom System

Before we derive a model for undamped 2-DOF systems, let us first consider the solution to the system shown in figure 5.7. The solution consists of two equations, one for each mass. This solution will be derived in section 5.2.2 and is expressed by the coupled equations:

$$x_1(t) = A_1 \sin(\omega_1 t + \phi_1)u_{11} + A_2 \sin(\omega_2 t + \phi_2)u_{12} \tag{5.12}$$

$$x_2(t) = A_1 \sin(\omega_1 t + \phi_1)u_{21} + A_2 \sin(\omega_2 t + \phi_2)u_{22}, \qquad \omega_1 \text{ or } \omega_2 \neq 0$$

These two equations ban be written as a single equation in matrix form as:

$$\vec{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\vec{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\vec{u}_2, \qquad \omega_1 \text{ or } \omega_2 \neq 0 \tag{5.13}$$

Where the bold text denotes vectors. Therefore, the vectors $\vec{u}_1$ and $\vec{u}_2$ are the mathematical expressions that "couple" or tie the equations together. Expanding these vectors shows:

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}, \tag{5.14}$$

The four key components of the solution expressed in equation 5.13 are:

1. ω_1 and ω_2 are the natural frequencies of the system. They are not the frequencies of the masses. The solution states that each of the masses oscillates at the two frequencies ω_1 and ω_2. Moreover, consider the special case where the initial conditions are selected to force $A_2 = 0$, in this case, each mas would only oscillates at only one frequency, ω_1.

2. A_1 and A_2 are the constants of integration and determine the amplitude of the system.

3. ϕ_1 and ϕ_2 represent the phase shift of the system

4. $\vec{u}_1$ and $\vec{u}_2$ are the first and second mode shapes of the system and couple the system together.

5.2.2 Deriving the Solution for the Two-Degree-of-Freedom System

To derive this solution for the system under consideration a FBD for figure 5.7 can be constructed for the forces acting on each mass. First we have to make the assumption that $x_1 < x_2$, this allows us to say that m_2 pulls on m_1 and results in:

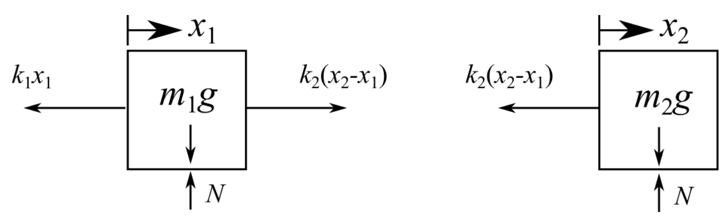


Figure 5.8: Free body diagram for the 2-DOM system presented in figure 5.7.

Applying Newton's second law and summing the forces on each mass in the horizontal direction yields:

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1) \end{aligned} \tag{5.15}$$

These equations can be rearranged in terms of x_1 and x_2 as:

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \\ m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 \end{aligned} \tag{5.16}$$

where these are two coupled second-order differential equations that each require two initial conditions to solve. These initial conditions can be obtained form the displacement and velocity terms as:

$$\begin{aligned} x_1(0) &= x_{10} \\ \dot{x}_1(0) &= \dot{x}_{10} = v_{10} \\ x_2(0) &= x_{20} \\ \dot{x}_2(0) &= \dot{x}_{20} = v_{20} \end{aligned} \tag{5.17}$$

As before, these initial conditions will be the constants of integration used to solve the two second-order differential equations. This solution will provide the free response of each mass in the system. There is a multitude of ways to solve these two coupled second-order differential equations, however, here we will just consider a matrix notation solution. This matrix notation solution is used as this formulation is readably solved using computers and is expandable to more than 2 DOF.

To initiate the solution, let us first develop the matrix formulation of the two coupled ODEs:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x_1} \\ \ddot{x_2} \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.18}$$

This equation can also be expressed as the vector equation:

$$M\ddot{\vec{x}} + K\vec{x} = 0 \tag{5.19}$$

and is known as the EOM in vector form. In this formulation the mass matrix (M) is defined as:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{5.20}$$

while the stiffness matrix (K) is:

$$K = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{5.21}$$

along with the displacement, velocity, and acceleration matrices:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}, \quad \ddot{\vec{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} \tag{5.22}$$

Beyond these equations we can write the initial conditions as:

$$\vec{x}_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \qquad \dot{\vec{x}}_0 = \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} \tag{5.23}$$

This simple connection between vibration analysis and matrix analysis allows computers to be used to solve large and complicated vibration problems quickly.

Recall that the 1-DOF version of the equation of motion was solved by calculating the values of the constants in an assumed harmonic solution. The same approach is applied here in order to solve for the displacement of the two-DOF system. This time, the the solution is assumed in the form:

$$\vec{x}(t) = \vec{u}e^{j\omega t} \tag{5.24}$$

where $\vec{u}$ is a vector of constants to be demerited and can be written as:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5.25}$$

From before, ω is also a constant to be determined. Again, $j = \sqrt{-1}$. In the same manner as before, $e^{j\omega t}$ represents harmonic motion as $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$. Taking the derivatives of $\vec{x}(t) = \vec{u}e^{j\omega t}$ yields:

$$\dot{\vec{x}}(t) = j\omega\vec{u}e^{j\omega t} \tag{5.26}$$

$$\ddot{\vec{x}}(t) = -\omega^2\vec{u}e^{j\omega t} \tag{5.27}$$

Substituting this into the EOM in vector form ($M\ddot{\vec{x}} + K\vec{x} = 0$) yields:

$$-\omega^2 M\vec{u}e^{j\omega t} + K\vec{u}e^{j\omega t} = 0 \tag{5.28}$$

or

$$(-\omega^2 M + K)\vec{u}e^{j\omega t} = 0 \tag{5.29}$$

As $e^{j\omega t} \neq 0$ for any value of t and not allowing $\vec{u}$ to be zero it can be demerited that $(-\omega^2 M + K)$ must satisfy the vector equation. Therefore,

$$(-\omega^2 M + K)\vec{u} = 0, \qquad \vec{u} \neq 0 \tag{5.30}$$

This forms a homogeneous set of algebraic equations. To be useful, these equations have a nonzero solution for the system must exist. For this to be true, the the inverse of the coefficient matrix $(-\omega^2 M + K)$ must not exist. To expand, assume that the the inverse of $(-\omega^2 M + K)$ does exist, by multiplying both sides of the equation by $(-\omega^2 M + K)^{-1}$ yields $\vec{u} = 0$. This is trivial solution (its not useful) as it no motion in the system is implied. Therefore, the logical connection can be drawn between the solution of equation and the inverse of the coefficient matrix $(-\omega^2 M + K)$.

Applying the singularity condition to the coefficient matrix of equation $(-\omega^2 M + K)\vec{u} = 0,\ \vec{u} \neq 0$ results a nonzero solution of $\vec{u}$. However, for this to exist the following must be true:

$$\det(-\omega^2 M + K) = 0 \tag{5.31}$$

Solving this expression results in one algebraic equation with one unknown (ω). Expanding the above equation to consider the values for the matrices M and K results in:

$$\det\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} = 0 \tag{5.32}$$

Using the definition of the determinant yields that the unknown quantity ω^2 must satisfy:

$$m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2)\omega^2 + k_1 k_2 = 0 \tag{5.33}$$

This expression is called the characteristic equation for the system and is used to determine the constants $\omega_{1,2}$, in the assumed form of the solution given by the assumed solution $\vec{x}(t) = \vec{u}e^{j\omega t}$, once the values of the physical parameters m_1, m_2, k_1, and k_2 are known. Note that $\omega_{1,2}$ are not in the characteristic equation, therefore, solving for $\omega_{1,2}$ will be done by factoring the equation above to obtain two solutions ω_1 and ω_2. The characteristic equation is in the form of the quadratic formula if you set $x = \omega^2$, as:

$$ax^2 + bx + c = 0 \tag{5.34}$$

After finding the value of $\omega_{1,2}$ using the characteristic equation, the values in $\vec{u}$ can be found using equation $(-\omega^2 M + K)\vec{u} = 0,\ \ \vec{u} \neq 0$ for each value of ω^2. That is, for both ω_1 and ω_2 there is a vector $\vec{u}$ that satisfies the equation. These solutions can be written as:

$$(-\omega_1^2 M + K)\vec{u}_1 = 0 \tag{5.35}$$

and

$$(-\omega_2^2 M + K)\vec{u}_2 = 0 \tag{5.36}$$

The direction of the vectors $\vec{u}_1$ and $\vec{u}_2$ can be obtained by solving the above expressions, however, the information regarding the magnitude of is not contained in this expression. To verify this, assume that $\vec{u}_1$ satisfies the equation, therefore, the vector $a\vec{u}_1$ also satisfies the equation where a is any nonzero number. Hence the vectors satisfying the above are of arbitrary magnitude.

The values obtained for $\vec{u}_1$ and $\vec{u}_2$ can now be combined with the assumed solution:

$$\vec{x}(t) = \vec{u}e^{j\omega t} \tag{5.37}$$

to form a set of solution:

$$\vec{x}(t) = \vec{u}_1 e^{-j\omega_1 t}, \quad \vec{u}_1 e^{j\omega_1 t}, \quad \vec{u}_2 e^{-j\omega_2 t}, \quad \vec{u}_2 e^{j\omega_2 t} \tag{5.38}$$

Since the equation to be solved is linear, the solution is the sum of these solutions. This results in:

$$\vec{x}(t) = (ae^{j\omega_1 t} + be^{-j\omega_1 t})\vec{u}_1 + (ce^{j\omega_2 t} + de^{-j\omega_2 t})\vec{u}_2 \tag{5.39}$$

where a, b, c, and d are the arbitrary constants of integration to be determined by the initial conditions. Applying Euler's formulas for the sin functions (where ω_1 or $\omega_2 \neq 0$) reorganizes this equation as:

$$\vec{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\vec{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\vec{u}_2, \qquad \omega_1 \text{ or } \omega_2 \neq 0 \tag{5.40}$$

Another way to write this equation is in the form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}, \qquad \omega_1 \text{ or } \omega_2 \neq 0 \tag{5.41}$$

Where the values for A_1 and A_2 can be obtained by setting applying the boundary conditions and taking the derivatives of the equations as done in the 1-DOF problems.

The final form of the equation provides a physical insight into the solution of the system. It states that the each mass in the system oscillates at both of the natural frequencies of the system (ω_1 and ω_2). Furthermore, the importance of the initial conditions can be understood. Assume that initial conditions are chosen that result in $A_2 = 0$, this cancels out the second natural frequency such that each mass oscillates at only one frequency, ω_1. Moreover, the positions of the masses can determined by the values of the vector $\vec{u}_1$ at any given time. For this reason, $\vec{u}_1$ is termed the first mode shape of the system. Likewise, if the opposite initial conditions are chosen such that $A_1 = 0$, then both system coordinates (e.g., masses in the systems we have studied) will oscillate at ω_2 and again, the positions can be obtained from the vector $\vec{u}_2$. Where $\vec{u}_2$ is termed the second mode shape. The interactions between mode shapes and natural frequencies are very important and form the basis of several area in the field of vibrations.

Example 5.1 Considering the following system:

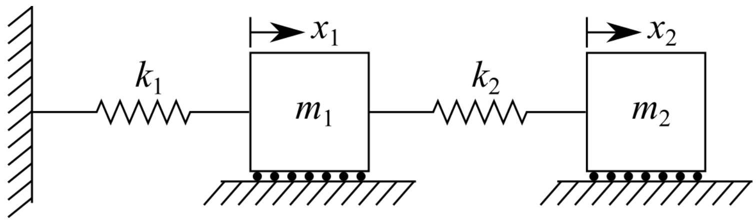


Figure 5.9: 2-DOF system with two masses and two independent confidante systems x_1 and x_2.

Calculate response for the system if m_1=9 kg, m_2=1 kg, k_1 = 24 N/m, and k_2 = 3 N/m with the initial conditions $x_{10} = 1$ mm, $v_{10} = 0$ mm/s, $x_{20} = 0$ mm, and $v_{20} = 0$ mm/s.

Solution: We have already obtained a characteristic equation for this system. This is shown in Equation 5.33 and is given as:

$$m_1 m_2 \omega^4 - (m_1 k_2 + m_2 k_1 + m_2 k_2)\omega^2 + k_1 k_2 = 0 \tag{5.42}$$

Substituting our values into this obtains:

$$9 \cdot 1\omega^4 - (9 \cdot 3 + 1 \cdot 24 + 1 \cdot 3)\omega^2 + 24 \cdot 3 = 0 \tag{5.43}$$

or

$$\omega^4 - 6\omega^2 + 8 = 0 \tag{5.44}$$

This can then be factored into:

$$(\omega^2 - 2)(\omega^2 - 4) = 0 \tag{5.45}$$

This results in solutions of $\omega_1^2 = 2$ and $\omega_2^2 = 4$. Leading to:

$$\omega_1 = \pm\sqrt{2} \text{ rad/sec}, \quad \omega_2 = \pm 2 \text{ rad/sec} \tag{5.46}$$

Next, we need to obtain solutions for $\vec{u}_1$ and $\vec{u}_2$. Having solved for ω_1 and ω_2 we can obtained. First, knowing $\vec{u}_1 = [u_{11} u_{21}]^{\mathrm{T}}$ and using $\omega_1 = \sqrt{2}$ and the follwoing equation:

$$(-\omega_1^2 M + K)\vec{u}_1 = 0 \tag{5.47}$$

yields simplified to

$$\left(-2\begin{bmatrix}9 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}24+3 & -3\\-3 & 3\end{bmatrix}\right)\begin{bmatrix}u_{11}\\u_{21}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{5.48}$$

simplified to

$$\begin{bmatrix}27-9\cdot 2 & -3\\-3 & 3-2\end{bmatrix}\begin{bmatrix}u_{11}\\u_{21}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{5.49}$$

or

$$\begin{bmatrix}9 & -3\\-3 & 1\end{bmatrix}\begin{bmatrix}u_{11}\\u_{21}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{5.50}$$

Taking the dot product of the matrix equation yields:

$$9u_{11}-3u_{21}=0, \text{ and } -3u_{11}+u_{21}=0 \tag{5.51}$$

Both of these equations yield the same equation, that is:

$$\frac{u_{11}}{u_{21}}=\frac{1}{3} \tag{5.52}$$

As mentioned before, only the ratio of the elements is determined here. To show this is true it is easily seen that:

$$u_{11}=u_{21}\frac{1}{3}\rightarrow au_{11}=au_{21}\frac{1}{3} \tag{5.53}$$

To obtain a numerical value, we arbitrarily assign a value to one of the elements. Here, let $u_{21}=1$ so let $u_{11}=1/3$. Therefore,

$$\vec{u}_1=\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix} \tag{5.54}$$

The same processes can be used for obtaining $\vec{u}_1$ using $\omega_2=2$, this results in:

$$\begin{bmatrix}-9 & -3\\-3 & -1\end{bmatrix}\begin{bmatrix}u_{12}\\u_{22}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{5.55}$$

Taking the dot product of the matrix equation yields:

$$-9u_{12}-3u_{22}=0, \text{ and } -3u_{12}-u_{22}=0 \tag{5.56}$$

Both of these equations yield the same equation, that is:

$$\frac{u_{12}}{u_{22}}=-\frac{1}{3} \tag{5.57}$$

Again, assuming $u_{22}=1$ this can be rearranged into $\vec{u}_2$ as:

$$\vec{u}_2=\begin{bmatrix}-\frac{1}{3}\\1\end{bmatrix} \tag{5.58}$$

Where $\vec{u}_1$ and $\vec{u}_2$ represent only the directions and shape of the mode shapes and not the magnitude of the mode shapes. Now that we have the mode shapes, we can solve for the initial conditions A_1 and A_2. To do this, let us use the following formulation of the solution:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}, \qquad \omega_1 \text{ or } \omega_2 \neq 0 \tag{5.59}$$

Adding our values for the problem at $t = 0$ this becomes:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} A_1 \sin(\phi_1) \\ A_2 \sin(\phi_2) \end{bmatrix} \tag{5.60}$$

and after applying the dot product:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} A_1 \sin(\phi_1) - \frac{1}{3} A_2 \sin(\phi_2) \\ A_1 \sin(\phi_1) + A_2 \sin(\phi_2) \end{bmatrix} \tag{5.61}$$

Next we can differentiate the equation for $x(t)$ to obtain the velocity solution. Adding our values for the problem at $t = 0$ obtains:

$$\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{3} A_1 \cos(\phi_1) - \frac{2}{3} A_2 \cos(\phi_2) \\ \sqrt{2} A_1 \cos(\phi_1) + 2 A_2 \cos(\phi_2) \end{bmatrix} \tag{5.62}$$

Now that we have 4 equations for 4 unknowns we can use these equations to solve for A_1, A_2, ϕ_1, and ϕ_2. The 4 equations are:

$$3 = A_1 \sin(\phi_1) - A_2 \sin(\phi_2) \tag{5.63}$$

$$0 = A_1 \sin(\phi_1) + A_2 \sin(\phi_2) \tag{5.64}$$

$$0 = \sqrt{2} A_1 \cos(\phi_1) - 2 A_2 \cos(\phi_2) \tag{5.65}$$

$$0 = \sqrt{2} A_1 \cos(\phi_1) + 2 A_2 \cos(\phi_2) \tag{5.66}$$

Setting these last two equations equal to each other yields:

$$0 = \sqrt{2} A_1 \cos(\phi_1) + 2 A_2 \cos(\phi_2) = \sqrt{2} A_1 \cos(\phi_1) - 2 A_2 \cos(\phi_2) \tag{5.67}$$

or:

$$0 = -4 A_2 \cos(\phi_2) \tag{5.68}$$

For this equation to be true, $\phi_2 = \frac{\pi}{2}$. Therefore, applying this to $0 = \sqrt{2} A_1 \cos(\phi_1) + 2 A_2 \cos(\phi_2)$ results in:

$$0 = \sqrt{2} A_1 \cos(\phi_1) \tag{5.69}$$

where again, for this equation to be true, $\phi_1 = \frac{\pi}{2}$. Now the first two equations become:

$$3 = A_1 - A_2 \tag{5.70}$$

$$0 = A_1 + A_2 \tag{5.71}$$

Where this shows us that $A_1 = \frac{3}{2}$ and $A_2 = -\frac{3}{2}$. Therefore, now that we have the initial conditions we can find a solution for the temporal response of each mass. Using the equations from before:

$$x_1(t) = A_1 \sin(\omega_1 t + \phi_1)u_{11} + A_2 \sin(\omega_2 t + \phi_2)u_{12} \tag{5.72}$$

$$x_2(t) = A_1 \sin(\omega_1 t + \phi_1)u_{21} + A_2 \sin(\omega_2 t + \phi_2)u_{22} \tag{5.73}$$

And applying our obtained values

$$x_1(t) = \frac{3}{2}\sin(\sqrt{2}t + \frac{\pi}{2})\frac{1}{3} + \left(-\frac{3}{2}\right)\sin(2t + \frac{\pi}{2})\left(-\frac{1}{3}\right) \tag{5.74}$$

$$x_2(t) = \frac{3}{2}\sin(\sqrt{2}t + \frac{\pi}{2}) + \left(-\frac{3}{2}\right)\sin(2t + \frac{\pi}{2}) \tag{5.75}$$

results in:

$$x_1(t) = \frac{1}{2}\left(\sin(\sqrt{2}t + \frac{\pi}{2}) + \sin(2t + \frac{\pi}{2})\right) \tag{5.76}$$

$$x_2(t) = \frac{3}{2}\left(\sin(\sqrt{2}t + \frac{\pi}{2}) - \sin(2t + \frac{\pi}{2})\right) \tag{5.77}$$

These results can be plotted as:

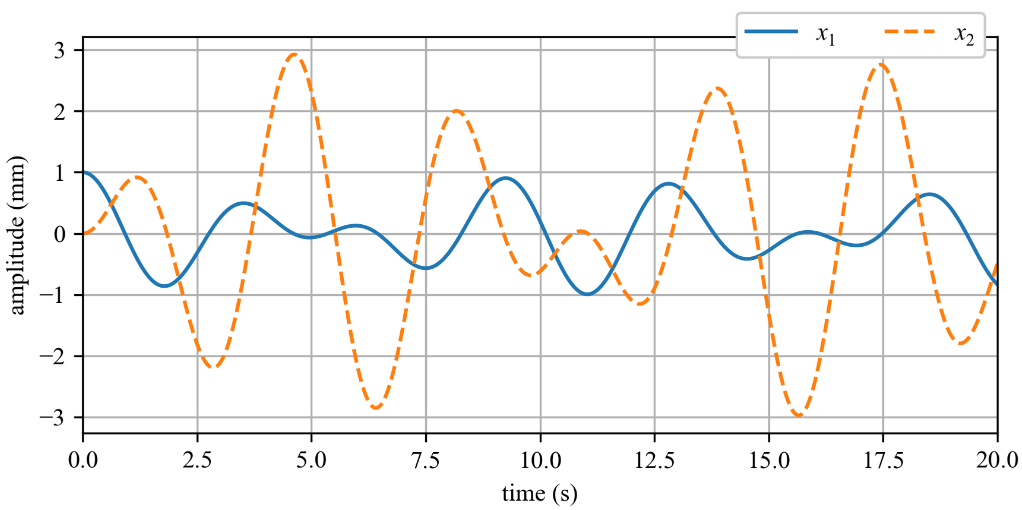


Figure 5.10: Temporal response for each of the rigid bodies in the 2-DOF system.

Example 5.2 Mode shapes can be better understood through a graphical representation. To do this, consider the 2-DOF system presented in figure 5.11(a). Assuming that $x_1 < x_2$ the FBD for the system is expressed in figure figure 5.11(b).

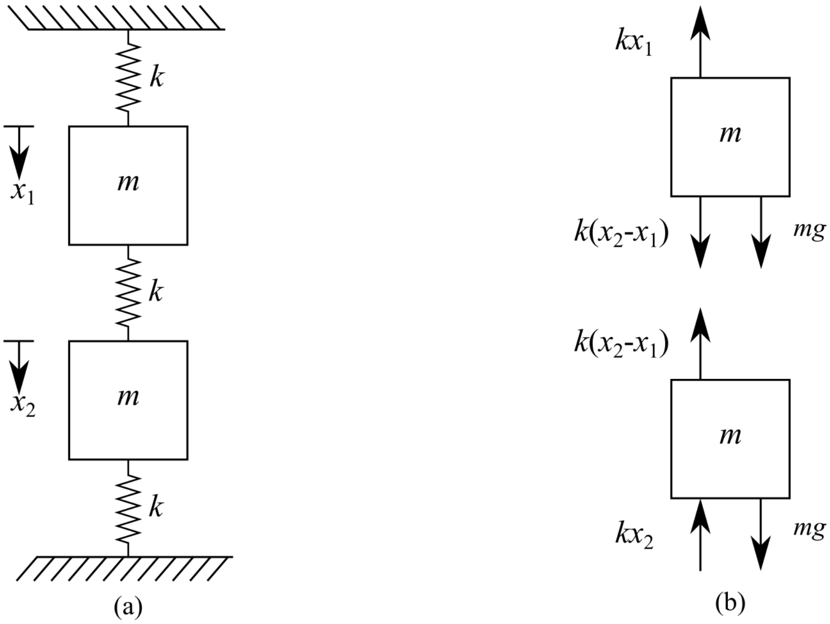


Figure 5.11: (a) 2-DOF system with two masses arranged in a vertical configuration; and (b) FBD of system.

For simplicity, all masses and spring stiffness are considered equal and that $m = 1$ and $k = 1$. From the previous investigations in this text we know that the forces caused by gravity will cancel out. Therefore, the EOM for the system can be written as:

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\ m\ddot{x}_2 &= -k(x_2 - x_1) - kx_2 \end{aligned} \tag{5.78}$$

These equations can be written in matrix notation as:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.79}$$

Substituting the values of the matrices M and K into this expression $\det(-\omega^2 M + K) = 0$ yields:

$$\det \begin{bmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 m + 2k \end{bmatrix} = 0 \tag{5.80}$$

The determinant yields that the unknown quantity, ω^2, must satisfy:

$$m^2\omega^4 - 4km\omega^2 + 3k = 0 \tag{5.81}$$

Therefore,

$$\omega_1 \pm \sqrt{\frac{k}{m}} = 1 \text{ rad/sec}, \quad \omega_2 \pm \sqrt{\frac{3k}{m}} = \sqrt{3} \text{ rad/sec} \tag{5.82}$$

Now, we need to obtain solutions for $\vec{u}_1$ and $\vec{u}_2$. Knowing $(-\omega_1^2 M + K)\vec{u}_1 = 0$ yields:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.83}$$

Taking the dot product of the matrix equation yields:

$$u_{11} - u_{21} = 0, \text{ and } -u_{11} + u_{21} = 0 \tag{5.84}$$

Setting $u_{11} = 1$ results in $u_{21} = 1$. The same processes can be performed for $\vec{u}_2$ to show that if we set $u_{12} = 1$, $u_{22} = -1$. Therefore, the mode shapes can be expresses as:

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{5.85}$$

The displacement of the masses as a function of time and the general mode shape plots are graphically represented in figure 5.12. In the 2-DOF system considered here, the second mode shape has a spot at the center of the middle spring spring that does not move (i.e. has zero displacement). This point is called a node. Nodes correspond to points in the mode shape where the displacement is always zero. Furthermore, the displacement of the node points remain zero at all times, as diagrammed in the top-right of figure 5.12.

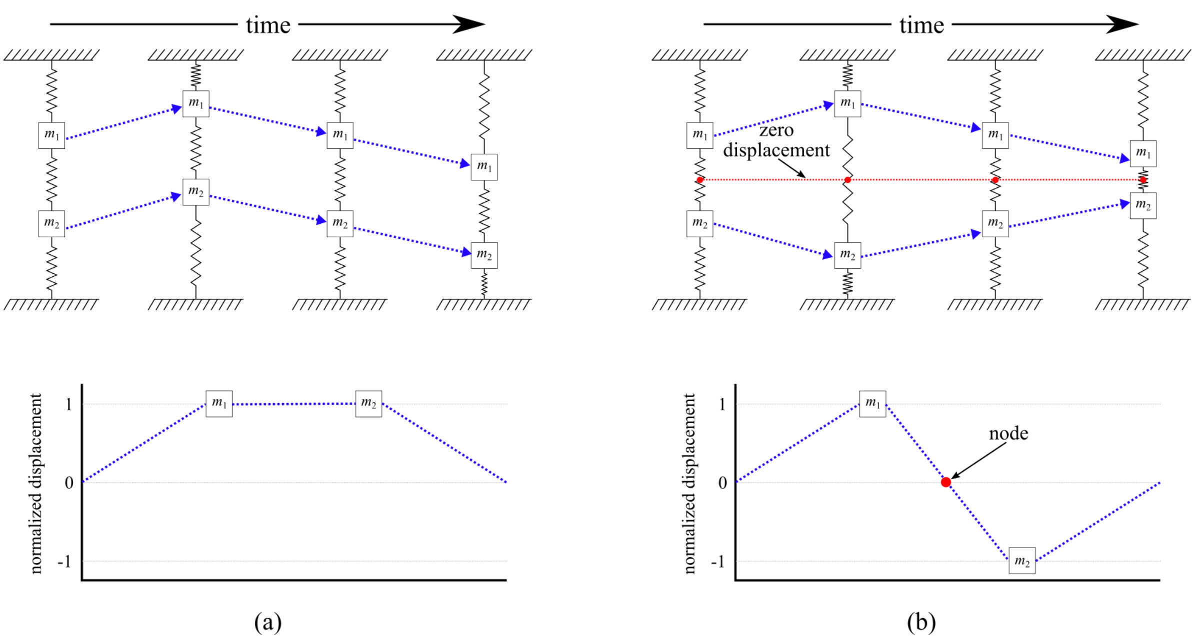


Figure 5.12: Modes of vibration for the system shown in figure 5.11 showing the: (a) first mode; and (b) second mode.

5.3 Eigenvalue-based Solution for Natural Frequencies and Mode Shapes

The process of calculating the mode shapes presented in section 5.2 is long and tedious. Therefore, methods that can be easily deployed on computers are of great interest to the practitioner. An eigenvalue-based solution that takes advantage of the symmetry in the M and K matrices and can be easily implemented on a computer is discussed in this section.

Review 5.2 Eigenvalues and Eigenvectors

In linear algebra, eigenvalues and eigenvectors are concepts that appear prominently in the analysis of linear transformations. By definition, If $\mathbf{v}$ is a vector (in vector space V over a field F) and T is a linear transformation into itself, then $\mathbf{v}$ is an eigenvector of T if $T(\mathbf{v})$ is a scalar multiple of $\mathbf{v}$:

$$T(\mathbf{v}) = \lambda \mathbf{v} \tag{5.86}$$

where λ is a scalar in the field F, known as the eigenvalue associated with the eigenvector v. If the linear transformation is expressed in the form of an $n \times n$ matrix A, then the eigenvalue equation for a linear transformation above can be rewritten as the matrix multiplication

$$A\mathbf{v} = \lambda \mathbf{v} \tag{5.87}$$

where $\mathbf{v}$ is a $n \times 1$ matrix of the eigenvectors. For the matrix A, eigenvalues and eigenvectors can be used to decompose the matrix.

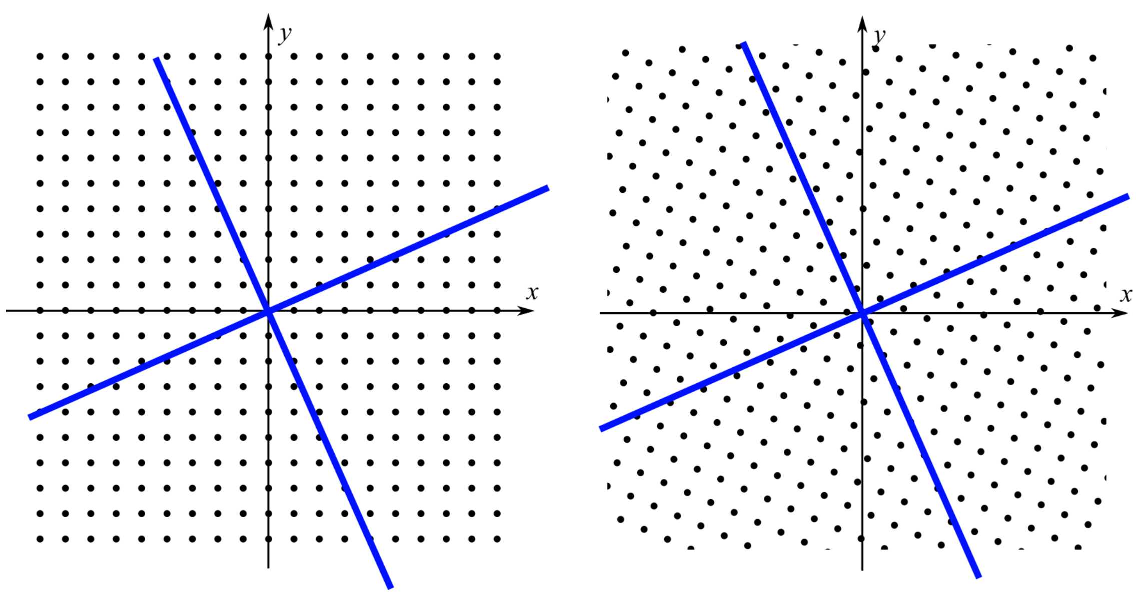


Figure 5.13: The stretching and shearing of the plane on which a matrix lies where the eigenvectors of the matrix are denoted by the blue lines and are the special directions such that point on the lines will just slide on them.

Need to add a discussion on generalized eigenvalues, normal, orthogonal, and orthonormal matrices.

A matrix is positive definite if the scalar $\mathbf{x}^{\mathrm{T}}A\mathbf{x}$ is positive for any non-zero vector x comprised of real numbers:

$$\mathbf{x}^{\mathrm{T}}A\mathbf{x} > 0 \tag{5.88}$$

The Cholesky decomposition of a real positive-definite matrix A is a decomposition of the form:

$$A = LL^{\mathrm{T}} \tag{5.89}$$

where L is the lower triangular matrix of A.

Review 5.3 Cholesky Decomposition

The vast majority of mass and stiffness matrices are symmetric and positive definite due to the physical meaning of these matrices. Therefore, M can be factored into two terms using the Cholesky decomposition:

$$M = LL^{\mathrm{T}} \tag{5.90}$$

For diagonal mass matrices (all the mass values lie along the diagonal of the matrix) the Cholesky decomposition (L) is defined as:

$$L = M^{1/2} = \begin{bmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{bmatrix} \tag{5.91}$$

This expression factors into:

$$M = M^{1/2}M^{1/2} \tag{5.92}$$

Moreover, the inverse of the diagonal matrix ($M^{1/2}$) is denoted as $M^{-1/2}$ and defined as:

$$L^{-1} = M^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{m_1}} & 0 \\ 0 & \frac{1}{\sqrt{m_2}} \end{bmatrix} \tag{5.93}$$

5.3.1 Deriving the Eigenvalue-based Solution

To derive an eigenvalue-based solution for calculating the natural frequencies and mode shapes let us consider the previously derived EOM for a undamped 2-DOF system:

$$M\ddot{\vec{x}} + K\vec{x} = 0 \tag{5.94}$$

This expression can be transformed into a symmetric eigenvalue problem, allowing us to leverage the strengths of symmetric eigenvalue mathematics and computer solvers. To perform this transform, we set $\vec{x} = M^{-1/2}\vec{q}$ and multiply the equation by $M^{-1/2}$ such that the EOM becomes:

$$M^{-1/2}MM^{-1/2}\ddot{\vec{q}} + M^{-1/2}KM^{-1/2}\vec{q} = 0 \tag{5.95}$$

As $M^{-1/2}MM^{-1/2}$ is equal to the identity matrix I and defining $M^{-1/2}KM^{-1/2}$ as the mass normalized stiffness $\widetilde{K}$ yields the simplified expression:

$$I\ddot{\vec{q}} + \widetilde{K}\vec{q} = 0 \tag{5.96}$$

where $\widetilde{K} = M^{-1/2}KM^{-1/2}$ is analogous to $\sqrt{k/m}$ from the 1-DOF system.

As before, a solution is found by assuming a solution, taking the derivatives of the solution and substituting it into the EOM. Following these steps and assuming a solution of:

$$\vec{q} = \mathbf{v}e^{j\omega t} \tag{5.97}$$

results in an EOM in the from:

$$-\mathbf{v}\omega^2 e^{j\omega t} + \widetilde{K}\mathbf{v}e^{j\omega t} = 0 \tag{5.98}$$

driving out the nonzero scaler $e^{j\omega t}$ and rearranging the above expression results in:

$$\widetilde{K}\mathbf{v} = \omega^2\mathbf{v} \tag{5.99}$$

Knowing that $\mathbf{v} \neq 0$, as a vector with zeros would mean no motion is present in the system, this equation can be expressed in a typical eigenvalue formulation:

$$\widetilde{K}\mathbf{v} = \lambda\mathbf{v} \tag{5.100}$$

where $\lambda = \omega^2$. Or more importantly, $\omega = \sqrt{\lambda}$. As $\widetilde{K}$ is symmetric, this is a symmetric eigenvalue problem. Moreover, the vector λ represent the eigenvalues of the system. Given that we set $\vec{x} = M^{-1/2}\vec{q}$, the eigenvectors are not a direct representation of the mode shapes. To develop a link between the eigenvectors and mode shapes we first need to normalize the lengths of the eigenvectors obtained by solving equation 5.100 to that of a unit vector. The norm of a unit vector is defined as:

$$1 = ||\mathbf{v}|| = \sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}} = \sqrt{\sum_{i=1}^{n}(v_i^2)} \tag{5.101}$$

where a scalar is used in conjunction with vector such that $\alpha\mathbf{u} = 1$ computed the normalized unit vector directions as outlined in example 5.3. In general, a nonzero vector of any length can be normalized by calculating:

$$\frac{1}{\sqrt{\mathbf{v}^{\mathrm{T}}\mathbf{v}}}\mathbf{v} \tag{5.102}$$

We can relate the eigenvectors to the modes shapes by a factor of the mass matrix:

$$\vec{u}_1 = M^{-1/2}\vec{v}_1 \tag{5.103}$$

The important thing to remember is that the natural frequencies are the square root of the eigenvalues and the mode shapes are related to the eigenvectors through the mass matrix.

Example 5.3 Normalize the vector $\vec{v}_1 = [1/3 \;\; 1]^T$

Solution: To normalize the vector $\vec{v}_1$, a scalar (α) is calculated to make $\alpha\mathbf{v} = 1$. Therefore, following the definition of an orthogonal vector:

$$(\alpha\vec{v}_1)^T(\alpha\vec{v}_1) = 1 \tag{5.104}$$

or:

$$\alpha[1/3 \;\; 1]\alpha\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix} = \alpha^2(1/9+1) = 1 \tag{5.105}$$

Therefore, $\alpha = 3/\sqrt{10}$. Resulting in a normalized unit vector of $\alpha\vec{v}_1 = [1/\sqrt{10} \;\; 3/\sqrt{10}]^T$

Example 5.4 Consider the system presented in example 5.1 and repeated below where m_1=9 kg, m_2=1 kg, k_1 = 24 N/m, and k_2 = 3 N/m with the initial conditions $x_{10} = 1$ mm, $v_{10} = 0$ mm/s, $x_{20} = 0$ mm, and $v_{20} = 0$ mm/s. Calculate the natural frequencies and the mode shapes using the eigenvalue solution.

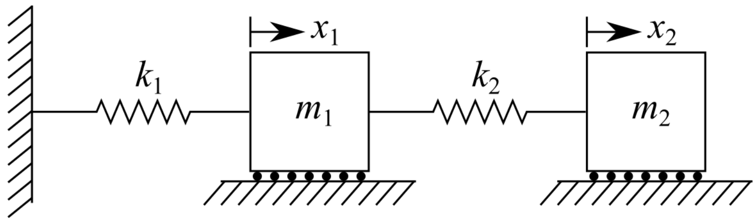


Figure 5.14: 2-DOF system with two masses and two independent confidante systems x_1 and x_2.

Solution: Writing the mass and stiffness matrix of the system as:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.106}$$

and

$$K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \tag{5.107}$$

we can compute $\widetilde{K}$ using the following expression:

$$\widetilde{K} = M^{-1/2} K M^{-1/2} \tag{5.108}$$

where $KM^{-1/2}$ is computed first to maintain symmetry. This results in:

$$KM^{-1/2} = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -1 & 3 \end{bmatrix} \tag{5.109}$$

and:

$$\widetilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \tag{5.110}$$

Now a solution must be obtained for the eigenvalue problem:

$$\widetilde{K}\mathbf{v} = \lambda \mathbf{v} \tag{5.111}$$

While this can be obtained using computers for such a simple case it is more appropriate to solve this expression by had. Therefore, the above expression can be rewritten as:

$$(\widetilde{K} - \lambda I)\mathbf{v} = 0 \tag{5.112}$$

as $\mathbf{v} \neq 0$ the matrix must be singular, resulting in the determinant of the matrix equaling zero. Or:

$$\det \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = 0 \tag{5.113}$$

This can be expanded ot the characteristic equation:

$$\lambda^2 - 6\lambda + 8 = 0 \tag{5.114}$$

with the roots (eigenvalues):

$$\lambda_1 = 2 \text{ and } \lambda_2 = 4 \tag{5.115}$$

Therefore, $\omega_1 = \sqrt{2}$ and $\omega_2 = 2$. These are the same values computed in example 5.1. The eigenvectors for λ_1 are computed as:

$$(\widetilde{K} - \lambda_1 I)\mathbf{v} = 0 \tag{5.116}$$

or:

$$\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.117}$$

This results in two dependent scalar equations:

$$v_{11} - v_{21} = 0 \text{ and } -v_{11} + v_{21} = 0 \tag{5.118}$$

That show us that $v_{11} = v_{21}$ or $\vec{v}_1 = [1\ \ 1]^T$. Therefore, using $(\alpha\vec{v}_1)^T(\alpha\vec{v}_1) = 1$ we obtain:

$$\alpha[1\ \ 1]\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha^2(2) = 1 \tag{5.119}$$

or $\alpha = 1/\sqrt{2}$. This allows us to normalize the vector knowing $\alpha\vec{v}_1 = 1$, resulting in a normalized vector of:

$$\alpha\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5.120}$$

A similar processes is followed for $\lambda_2 = 4$ that leads to the normalized vector

$$\alpha\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{5.121}$$

Lastly, the normalized eigenvectors can be converted to mode shapes using $\mathbf{u} = M^{-1/2}\mathbf{v}$. Resulting in:

$$\vec{u}_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \tag{5.122}$$

and:

$$\vec{u}_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix} \tag{5.123}$$

Therefore, the first mode shape is $[1 \;\; 1]$ while the second mode shape is $[\frac{1}{3} \;\; -\frac{1}{3}]$. Again, this is using our prior definition of mode shapes to set $[1 \;\; 1]$ as the first mode shape. Note that these are the same mode shape vectors as computed in example 5.1.

Expanding on equation 5.103, the one can go from the eigenvector to the mode shapes through:

$$\vec{u}_1 = M^{-1/2}\vec{v}_1 \text{ and } \vec{v}_1 = M^{1/2}\vec{u}_1 \tag{5.124}$$

therefore, it can be seen that the eigenvectors and mode shapes are related through the term $\vec{v}_1$.

5.4 Transfer-function method

As in 1-DOF systems, transfer functions can be used to solve for the temporal response of 2-DOF systems under a variety of inputs. Again, the transfer function of a differential equation is defined as the ratio of the Laplace transform of the output (system response) to the Laplace transform of the input (forcing function). Moreover, the procedure for using the Laplace transform to solve the equations of motion is the same and follows three steps:

1. Take the Laplace transform of both sides of the EOM while treating the time derivatives symbolically.

2. Solve for $X(s)$ in the obtained equation.

3. Apply the inverse transform $x(t) = \mathscr{L}[X(s)]^{-1}$

Example 5.5 2-DOF System Subjected to Impulse

Two masses are connected thorough a spring, as shown in figure 5.15.

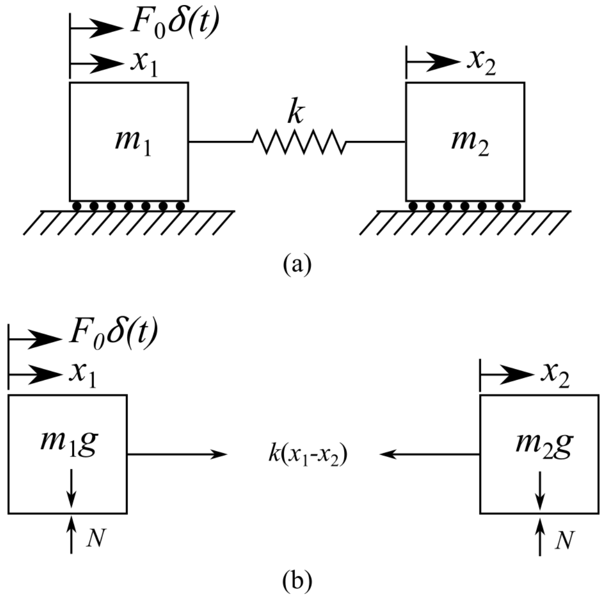


Figure 5.15: 2-DOF system subjected to an impulse showing: (a) system, and (b) FBD.

Assuming that x_1 displaces more than x_2, the equations of motion are:

$$
\begin{aligned}
m_1\ddot{x}_1 + k(x_1 - x_2) &= F_0\delta(t) \\
m_1\ddot{x}_1 + k(x_2 - x_1) &= 0
\end{aligned}
\tag{5.125}
$$

Taking the Laplace of both equations (step 1) yields:

$$
\begin{aligned}
(m_1s^2 + k)X_1(s) - kX_2(s) &= F_0 \\
-kX_1(s) + (m_2^s + k)X_2(s) &= 0
\end{aligned}
\tag{5.126}
$$

solving these two equations for X_1 and X_2 (step 2) results in:

$$
\begin{aligned}
X_1(s) &= \frac{F_0(m_2s^2 + k)}{s^2[m_1m_2s^2 + k(m_1 + m_2)]} \\
X_2(s) &= \frac{F_0k}{s^2[m_1m_2s^2 + k(m_1 + m_2)]}
\end{aligned}
\tag{5.127}
$$

Using partial fractions, or a symbolic toolbox in MATLAB or Python, these expressions can be rewritten as:

$$
\begin{aligned}
X_1(s) &= \frac{F_0}{m_1 + m_2}\left(\frac{1}{s^2} + \frac{m_2}{\omega m_1}\frac{\omega}{s^2 + \omega^2}\right) \\
X_2(s) &= \frac{F_0}{m_1 + m_2}\left(\frac{1}{s^2} + \frac{1}{\omega}\frac{\omega}{s^2 + \omega^2}\right)
\end{aligned}
\tag{5.128}
$$

where:

$$\omega^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \tag{5.129}$$

Taking the inverse transform of the expressions for $X_1(s)$ and $X_2(s)$ (step 3) results in expressions in the time domain and yields:

$$\begin{aligned} x_1(t) &= \frac{F_0}{m_1 + m_2}\left(t + \frac{m_2}{\omega m_1}\sin(\omega t)\right) \\ x_2(t) &= \frac{F_0}{m_1 + m_2}\left(t + \frac{1}{\omega}\sin(\omega t)\right) \end{aligned} \tag{5.130}$$

Considering a system where $F_0 = 10$ N, $m_1 = 1000$ kg, $m_2 = 1000$ kg, and $k = 1500$ N/m the temporal response is annotated in figure 5.16.

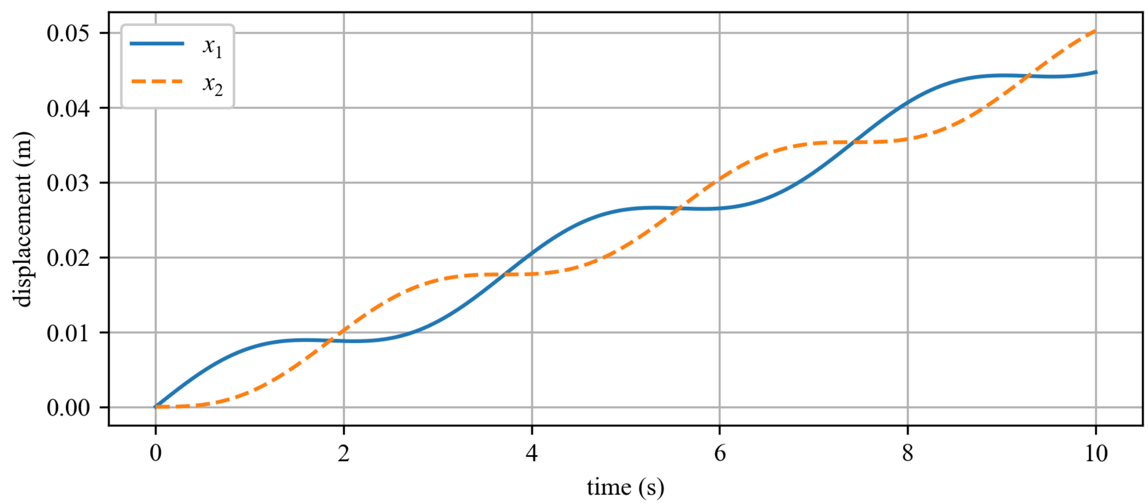


Figure 5.16: Temporal response for the considered 2-DOF system subjected to a impact load.

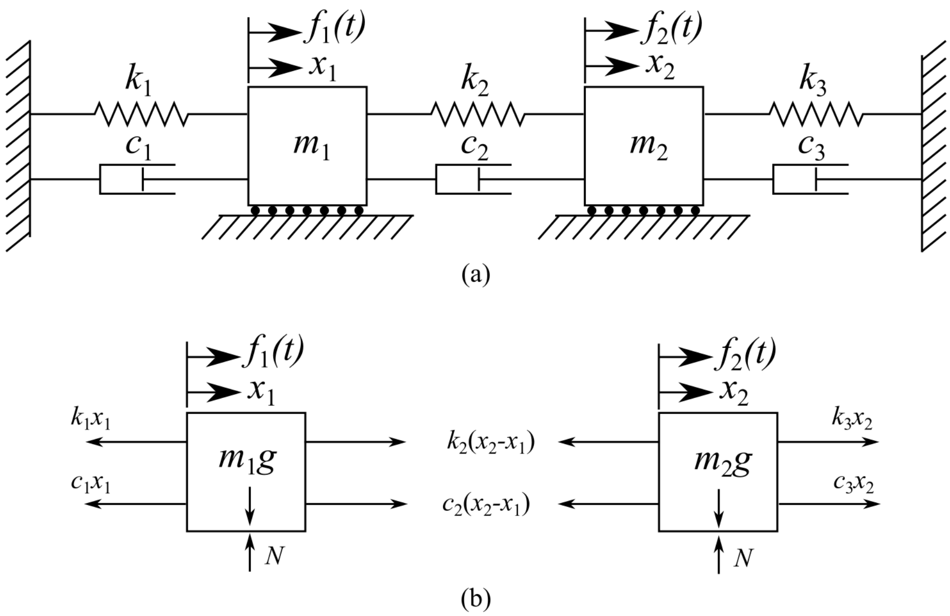


Figure 5.17: Forced 2-DOF damped system showing: (a) system, and (b) FBD.

5.5 Cramer's Rule for Solving 2-DOF Systems

Cramer's rule is an explicit formula for the solution of a system of linear equations with as many equations as unknown. Cramer's rule is valid whenever the system has a unique solution and can be used as a more generalized approach to solve for the temporal solution to a 2-DOF. Consider the 2-DOF systems shown in figure 5.17, where the two coupled equations of motion are expressed as:

$$\begin{aligned} m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2x_2 = 0 \\ m_2\ddot{x}_2 + (c_2+c_3)\dot{x}_2 - c_2\dot{x}_1 + (k_2+k_3)x_2 - k_2x_1 = 0 \end{aligned} \tag{5.131}$$

As before, taking the Laplace of the EOM (while ignoring the initial conditions) changes the equation from the temporal domain to the complex s-plane. This yields:

$$\begin{aligned} m_1s^2X_1(s) + (c_1+c_2)sX_1(s) - c_2sX_2(s) + (k_1+k_2)X_1(s) - k_2X_2(s) = F_1(s) \\ m_2s^2X_2(s) + (c_2+c_3)sX_2(s) - c_2sX_1(s) + (k_2+k_3)X_2(s) - k_2X_1(s) = F_2(s) \end{aligned} \tag{5.132}$$

these equations can be rearranged in terms of X_1 and X_2 as follows:

$$\begin{aligned} [m_1s^2 + (c_1+c_2)s + (k_1+k_2)]X_1(s) - [c_2s+k_2]X_2(s) = F_1(s) \\ [m_2s^2 + (c_2+c_3)s + (k_2+k_3)]X_2(s) - [c_2s+k_2]X_1(s) = F_2(s) \end{aligned} \tag{5.133}$$

These equations show two linear equations in terms of X_1 and X_2 that can be solved for using Cramer's rule, resulting in the expression:

$$\begin{aligned} X_1(s) = \frac{D_1(s)}{D(s)} \\ X_2(s) = \frac{D_2(s)}{D(s)} \end{aligned} \tag{5.134}$$

where:

$$D_1 = \begin{vmatrix} F_1(s) & -(c_2s+k_2) \\ F_2(s) & m_2s^2+(c_2+c_3)s+(k_2+k_3) \end{vmatrix} \tag{5.135}$$
$$= [m_2s^2X_2(s)+(c_2+c_3)s+(k_2+k_3)]F_1(s)+(c_2s+k_2)F_2(s)$$

$$D_2 = \begin{vmatrix} m_1s^2+(c_1+c_2)s+(k_1+k_2) & F_1(s) \\ -(c_2s+k_2) & F_2(s) \end{vmatrix} \tag{5.136}$$
$$= [m_1s^2+(c_1+c_2)s+(k_1+k_2)]F_2(s)+(c_2s+k_2)F_1(s)$$

$$D = \begin{vmatrix} m_1s^2+(c_1+c_2)s+(k_1+k_2) & m_2s^2+(c_2+c_3)s+(k_2+k_3) \\ -(c_2s+k_2) & -(c_2s+k_2) \end{vmatrix} \tag{5.137}$$
$$\begin{aligned} &= m_1m_2s^4+[m_2(c_1+c_3)+m_1(c_2+c_3)]s^3 \\ &+[m_2(k_1+k_2)+m_1(k_2+k_3)+c_1c_2+c_2c_3+c_3c_1]s^2 \\ &+[(k_1+k_2)(c_2+c_3)+c_1k_2+c_1k_3-c_2k_2+c_2k_3]s \\ &+(k_1k_2+k_2k_3+k_3k_1) \end{aligned}$$

The denominator, $D(s)$ is a 4^{th} polynomial in s and is the characteristic polynomial of the system. The system is considered a 4^{th} order system because the characteristic polynomial of the system is of order 4.

Example 5.6 content... content... content...
example 5.11 in Rao ed 6 has a good example of this nature.

5.6 Frequency Response Transfer Functions

5.7 Computational Methods for 2-DOF Systems

5.7.1 Python

5.8 Multiple Degrees of Freedom

This chapter introduces methodologies for the solving of systems with more than 2-DOF. As shown in Chapter 5, 2-DOF systems can be solved analytically using 2 EOM coupled through their mode shapes. However, these methods become tedious when extended to systems with damping or even beyond beyond 2-DOF system.

Example 5.7 Multiple Mode Shapes

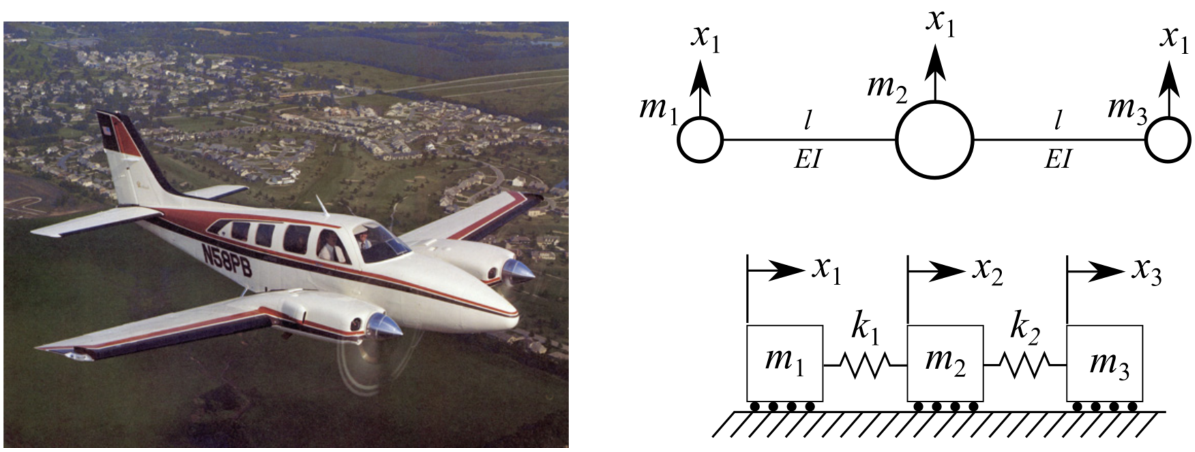


Figure 5.18: A Beechcraft Baron in flight[1] along with the Free-Free 3-DOF model simplified as a mass-spring model. This footnote is messed up.

Modeling the vibrations of a dual engine airplane as a three-degree-of-freedom system can be done as shown in figure 5.18 where the fuselage is a center mass, and the engines are point masses suspended by cantilevers from the center mass. The stiffness of the wing corresponds to the modulus of the wing E and its moment of inertia I. assuming that $m_1 = m_3 = 1m$, $m_2 = 4m$, and $k = \frac{3EI}{l}$, the EOM can be written as:

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \frac{3EI}{l} \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.138}$$

calculate the natural frequencies and mode shapes of the system and plot the mode-shapes in relation to the considered Beechcraft Baron.

Solution using the mass normalized stiffness matrix $\tilde{K}$.

Solving for the modes shapes using the mass normalized stiffness matrix $\tilde{K}$ requires solving for $M^{-1/2}$ and $\tilde{K}$ such that:

$$M^{-1/2} = \frac{1}{\sqrt{m}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.139}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \frac{3EI}{l} \begin{bmatrix} 3 & -1.5 & 0 \\ -1.5 & 1.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix} \tag{5.140}$$

Than, the eigenvalue problem, formulated as:

$$\tilde{K}\mathbf{v} = \lambda \mathbf{v} \tag{5.141}$$

is solved for the eigenvalues and eigenvectors, resulting in:

$$\lambda_1 = 0, \;\; \lambda_2 = 3, \;\; \lambda_3 = 4.5 \tag{5.142}$$

$$\vec{v}_1 = \begin{bmatrix} 0.408248 \\ 0.816497 \\ 0.408248 \end{bmatrix}, \; \vec{v}_2 = \begin{bmatrix} -0.707107 \\ 0.0 \\ -0.707107 \end{bmatrix}, \; \vec{v}_3 = \begin{bmatrix} 0.57735 \\ -0.57735 \\ 0.57735 \end{bmatrix} \tag{5.143}$$

When the eigenvalue problem is solved using the mass normalized stiffness matrix $\tilde{K}$ the natural frequencies are $\omega_i = \sqrt{\lambda_i}$ while the mode shapes are derived from the eigenvectors as $\vec{u} = M^{-1/2}\vec{v}$. This results in:

$$\omega_1 = 0 \text{ rad/sec}, \;\; \omega_2 = 1.732 \text{ rad/sec}, \;\; \omega_3 = 2.121 \text{ rad/sec} \tag{5.144}$$

$$\vec{u}_1 = \begin{bmatrix} 0.408248 \\ 0.408248 \\ 0.408248 \end{bmatrix}, \; \vec{u}_2 = \begin{bmatrix} -0.707107 \\ 0.0 \\ -0.707107 \end{bmatrix}, \; \vec{u}_3 = \begin{bmatrix} 0.57735 \\ -0.288675 \\ 0.57735 \end{bmatrix} \tag{5.145}$$

Next, normalizing the mode shapes by the norm of the vector as $\vec{u}_i = \frac{\vec{u}_i}{|\vec{u}_i|}$ results in:

$$\vec{u}_1 = \begin{bmatrix} 0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix}, \; \vec{u}_2 = \begin{bmatrix} -0.707107 \\ 0.0 \\ 0.707107 \end{bmatrix}, \; \vec{u}_3 = \begin{bmatrix} 0.666667 \\ -0.333333 \\ 0.666667 \end{bmatrix} \tag{5.146}$$

these mode-shapes can than be plotted as:

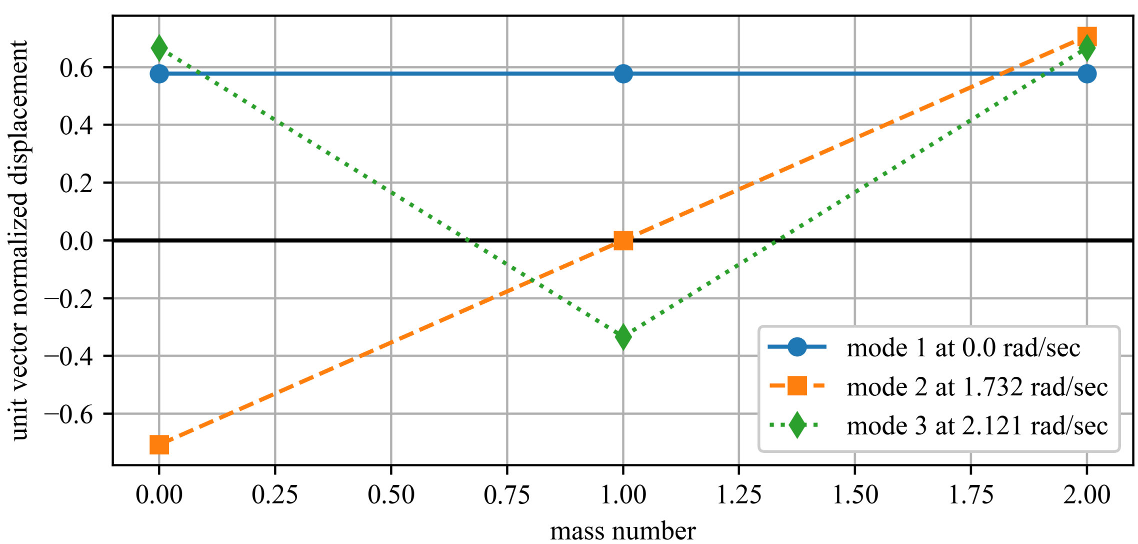


Figure 5.19: The unit vector normalized displacement of the mode shapes solved for using the mass normalized stiffness matrix $\tilde{K}$.

Solution using the generalized eigenvalue approach.

The mode shapes can also be solved for using the generalized eigenvalue approach where the eigenvalue problem is written as:

$$\lambda M\mathbf{v} = K\mathbf{v} \tag{5.147}$$

solving for the eigenvalues and eigenvectors yields:

$$\lambda_1 = 0, \ \ \lambda_2 = 3, \ \ \lambda_3 = 4.5 \tag{5.148}$$

$$v_1 = \begin{bmatrix} -0.57735 \\ -0.57735 \\ -0.57735 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0.707107 \\ 0.0 \\ -0.707107 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0.666667 \\ -0.333333 \\ 0.666667 \end{bmatrix} \tag{5.149}$$

Note that the eigenvalues are the same as those solved for using the normalized stiffness matrix approach while the eigenvectors appear to be different. This is because for generalized eigenvalue approach $u_i = v_i\alpha$ where α is a scaling value. Note that here $\alpha = 1$ as the mode shapes are already normalized. Note, that some software tools may not return normalized modes and the nodes should be normalized by the norm of the vector ($u_i = \frac{u_i}{|u_i|}$) to achieve the same mode shapes as before. Moreover, recall that the information stored in the eigenvectors are just the direction of the transform. Therefore, the mode shapes are correct but some amplitudes are reversed but as the mode shapes only report the shape these are corrected. Plotting the mode shapes results in:

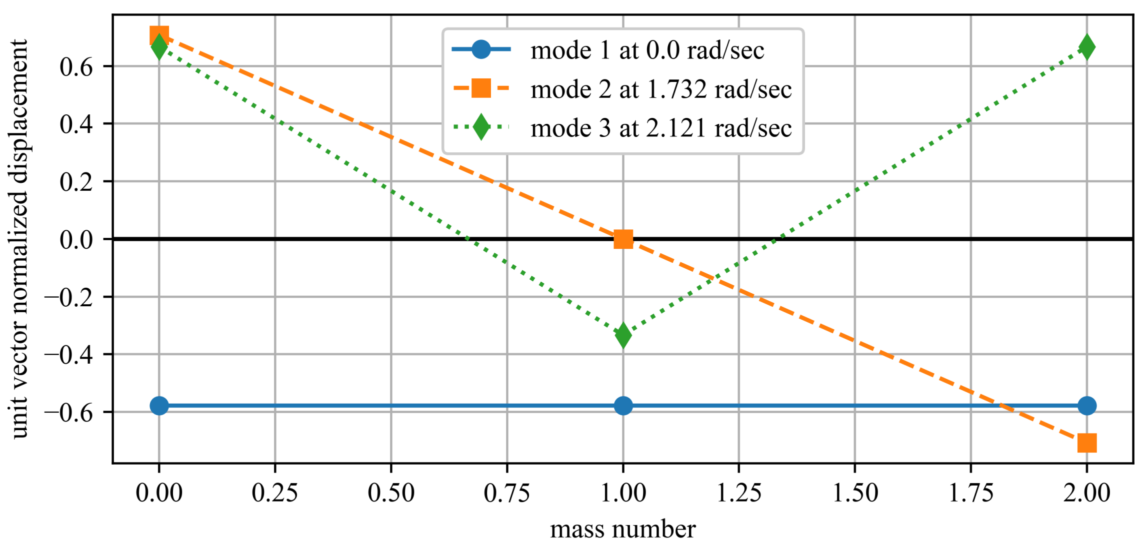


Figure 5.20: The unit vector normalized displacement of the mode shapes solved for using the generalized eigenvalue approach.

[a]"A Beechcraft Baron 58 in flight" by San Diego Air & Space Museum Archives, Public Domain.

5.9 Modal Analysis

Modal analysis is the study of a system's dynamic properties and is done in the frequency domain. Consider a system with n degrees of motion, modal analysis allows for the uncoupling of the EOM into n single-degree-of-freedom system (represented as 2^{nd}-order DOF systems) where the displacements of the masses are expressed as the linear summations of the normal modes of the system. If every mode shape is considered, the solution is equivalent to the solution obtained from the original n^{th}-degree-of-freedom system.

Consider the generic multidegree-of-freedom system under external forces, expressed as:

$$M\ddot{\vec{x}} + K\vec{x} = \vec{F} \tag{5.150}$$

where damping is not considered and the vector $\vec{F}$ is a set of deterministic inputs. To expand this equation by modal analysis, the eigenvalue problem must first be solved. The generalized eigenvalue problem is written at:

$$\lambda M\mathbf{v} = K\mathbf{v} \tag{5.151}$$

For the n^{th}-degree-of-freedom, the generalized eigenvalue problem can be simplified to:

$$\omega_i^2 M\vec{v}_i = K\vec{v}_i \tag{5.152}$$

Considering that the total displacement of the system, expressed as $\vec{x}(t)$, is the summation of the of the displacement of each of the noncontributing modes; assuming a linear system, the temporal response of the system can be written as:

$$\vec{x}(t) = q_1(t)\vec{v}_1 + q_2(t)\vec{v}_2 + q_3(t)\vec{v}_3 + \cdots + q_n(t)\vec{v}_n \tag{5.153}$$

where the time-dependent generalized scalars $q_1(t), q_1(t), \cdots, q_1(t)$ are the modal participation coefficients (also called principal coordinates). Defining the modal matrix P as:

$$P = [\vec{v}_1 \;\; \vec{v}_2 \;\; \vec{v}_3 \cdots \vec{v}_n] \tag{5.154}$$

Therefore, the linear combination of the normal modes (equation 5.153) can be more concisely written as:

$$\vec{x}(t) = P\vec{q}(t) \tag{5.155}$$

where $\vec{q}(t) = [q_1 \;\; q_2 \;\; q_3 \cdots q_n]^{\mathrm{T}}$. Next, the relationship that relates the physical space to the modal space for the acceleration component is written as:

$$\ddot{\vec{x}}(t) = P\ddot{\vec{q}}(t) \tag{5.156}$$

combining these two terms results in the EOM that can be written as:

$$MP\ddot{\vec{q}}(t) + KP\vec{q}(t) = \vec{F} \tag{5.157}$$

To convert the EOM into the standard form, first the P^{T} is multiplied through the equation as:

$$P^{\mathrm{T}}MP\ddot{\vec{q}}(t) + P^{\mathrm{T}}KP\vec{q}(t) = P^{\mathrm{T}}\vec{F} \tag{5.158}$$

If the modes are normalized, the following is true:

$$P^{\mathrm{T}}MP = I \tag{5.159}$$

where I is the identity matrix and

$$P^{\mathrm{T}}KP = \begin{bmatrix} \nwarrow & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \searrow \end{bmatrix} \tag{5.160}$$

Moreover, defining $\vec{Q}(t) = P^{\mathrm{T}}\vec{F}$ results in a EOM expressed as:

$$\ddot{\vec{q}}(t) + \begin{bmatrix} \nwarrow & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \searrow \end{bmatrix} \vec{q}(t) = \vec{Q}(t) \tag{5.161}$$

For a system with n degrees of freedom, this equation can be broken down into:

$$\ddot{\vec{q}}_i(t) + \omega_i^2 \vec{q}_i(t) = \vec{Q}(t), \;\; i = 1, 2, \cdots, n \tag{5.162}$$

This expression is the same ODE that we have solved multiple times in this text. Therefore, we know the solution to be:

$$q_i(t) = q_{i0}\cos(\omega_i t) + \frac{\dot{q}_{i0}}{\omega_i}\sin(\omega_i t) \tag{5.163}$$

Lastly, to solve for a solution in the modal space, the initial conditions that were given in the physical space must be converted to the modal space. This can be done by generalizing the velocities in terms of the modal matrix:

$$\vec{q}(0) = P^{\mathrm{T}}M\vec{x}(0) \tag{5.164}$$

$$\dot{\vec{q}}(0) = P^{\mathrm{T}}M\vec{v}(0) \tag{5.165}$$

Example 5.8 Free vibration response

Solve for the free vibration response of the 2-DOF presented in figure 5.21 using modal analysis. Show the temporal response for the entire system for its first 20 seconds using the full modal reconstruction and the reconstruction truncated to just include the first mode. Also, plot the variations in the modal participation coefficients though time. Apply the parameters, $f_1 = 0$ kg, $f_2 = 0$ N, $m_1 = 10$ kg, $m_2 = 1$ kg, $k_1 = 30$ N/m, $k_2 = 5$ N/m, $k_3 = 1$ N/m, $x_1(0) = 1$ mm, $x_2(0) = 0$ mm, $v_1(0) = 0$ mm, and $v_2(0) = 0$ mm.

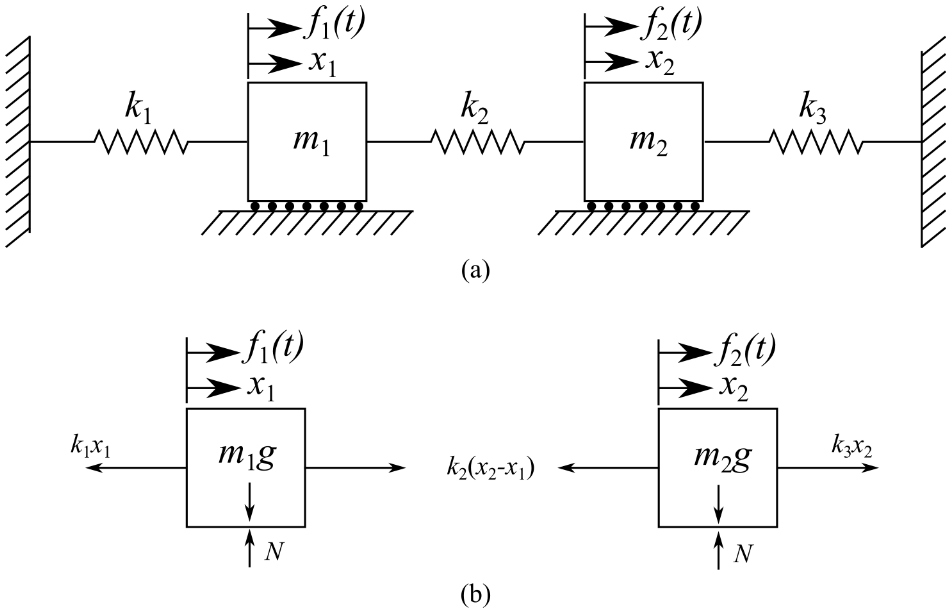


Figure 5.21: Forced 2-DOF damped system showing: (a) system, and (b) FBD.

The equations of motion that couple the system are:

$$\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0 \\ m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0 \end{aligned} \tag{5.166}$$

In matrix form, these become:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.167}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{v}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies and mode shapes can than be obtained by solving the eigenvalue problem. Setting up the generalized eigenvalue problem: these are symmetric M and K matrices so I don't know why $\tilde{K}$ does not work in a normal eigenvalue problem. Maybe norms of vectors? It is not simple scaling. This example is based off or Rao 6.16 6ed I think it has to do with solving the generalized eigenvalue matrix vs the mass normalized one and when you apply the second $\mathrm{M}^{-1/2}$ as you need it all the time.

$$K\mathbf{v} = \lambda M\mathbf{v} \tag{5.168}$$

and solving yields:

$$\lambda_1 = 2.734, \ \vec{v}_1 = \begin{bmatrix} 0.546819 \\ 0.837251 \end{bmatrix} \tag{5.169}$$

$$\lambda_2 = 6.765,\, \vec{v}_2 = \begin{bmatrix} -0.151349 \\ 0.988480 \end{bmatrix}$$

this is than related to the natural frequency and mode shapes as:

$$\omega_1 = \sqrt{\lambda_1} = 1.65 \text{ rad/s},\, \vec{v}_1 = \vec{v}_1\alpha_1 = \begin{bmatrix} 0.546819 \\ 0.837251 \end{bmatrix} \alpha_1 \tag{5.170}$$

$$\omega_2 = \sqrt{\lambda_2} = 2.60 \text{ rad/s},\, \vec{v}_2 = \vec{v}_2\alpha_2 = \begin{bmatrix} -0.151349 \\ 0.988480 \end{bmatrix} \alpha_2 \tag{5.171}$$

recall that the eigenvalues only contain information about the direction of the linear transform, and therefore, there magnitudes are arbitrary.Therefore, they must be scaled proportionally to each other. For this reason, the scalars α_1 and α_2 are added. My orthogonalizing the modal vectors with respect to the mass matrix, the values of α_1 and α_2 are found as:

$$1 = \vec{v}_1^{\mathrm{T}} M \vec{v}_1 \tag{5.172}$$

$$1 = \alpha_1^2 \begin{bmatrix} 0.546819 & 0.837251 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.546819 \\ 0.837251 \end{bmatrix} \tag{5.173}$$

and:

$$1 = \vec{v}_2^{\mathrm{T}} M \vec{v}_2 \tag{5.174}$$

$$1 = \alpha_2^2 \begin{bmatrix} -0.151349 & 0.98848 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -0.151349 \\ 0.98848 \end{bmatrix} \tag{5.175}$$

therefore, $\alpha_1 = 0.520$ and $\alpha_2 = 0.910$. Now, applying the proper scaling values to the modal vector, the modal matrix becomes:

$$P = [\vec{v}_1 \;\; \vec{v}_2] = \begin{bmatrix} 0.284 & -0.137 \\ 0.435 & 0.900 \end{bmatrix} \tag{5.176}$$

Next, check that the normal modes in the modal matrix (P) are normalized, per equation 5.159. This yields,

$$P^{\mathrm{T}} M P = \begin{bmatrix} 1 & -2.775e-16 \\ -2.775e-16 & 1 \end{bmatrix} \approx I \tag{5.177}$$

which is close enough to I. Considering that $\vec{x}(t) = P\vec{q}(t)$, the EOM for the system can be expressed as:

$$\ddot{\vec{q}}(t) + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \vec{q}(t) = \vec{Q} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.178}$$

rewriting this in scalar form for each modal coefficient yields:

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = 0,\;\; i = 1,2 \tag{5.179}$$

where the solution for this ODE is:

$$q_i(t) = q_{i0}\cos(\omega_i t) + \frac{\dot{q}_{i0}}{\omega_i}\sin(\omega_i t) \tag{5.180}$$

where q_{i0} and $\dot{q}_{i0}$ are the initial conditions in modal space. Therefore, the given initial conditions must be transferred into modal space as:

$$\vec{q}(0) = P^{\mathrm{T}} M \vec{x}(0) = \begin{bmatrix} 0.284 & 0.435 \\ -0.137 & 0.900 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.846 \\ -1.378 \end{bmatrix} \tag{5.181}$$

$$\dot{\vec{q}}(0) = P^{\mathrm{T}} M \vec{v}(0) = \begin{bmatrix} 0.284 & 0.435 \\ -0.137 & 0.900 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.182}$$

therefore,

$$\begin{aligned} q_1(t) &= 2.846 \cdot \cos(1.65t) \\ q_2(t) &= -1.378 \cdot \cos(2.6t) \end{aligned} \tag{5.183}$$

converting back into the time domain is done knowing $\vec{x}(t) = P\vec{q}(t)$, therefore,

$$\vec{x}(t) = P\vec{q}(t) = \begin{bmatrix} 0.284 & -0.137 \\ 0.435 & 0.900 \end{bmatrix} \begin{bmatrix} 2.846 \cdot \cos(1.65t) \\ -1.378 \cdot \cos(2.6t) \end{bmatrix} \tag{5.184}$$

This further simplified into:

$$\begin{aligned} x_1(t) &= 0.808 \cdot \cos(1.65t) + 0.189 \cdot \cos(2.6t) \\ x_2(t) &= 1.238 \cdot \cos(1.65t) - 1.240 \cdot \cos(2.6t) \end{aligned} \tag{5.185}$$

These results are plotted in figure 5.22.

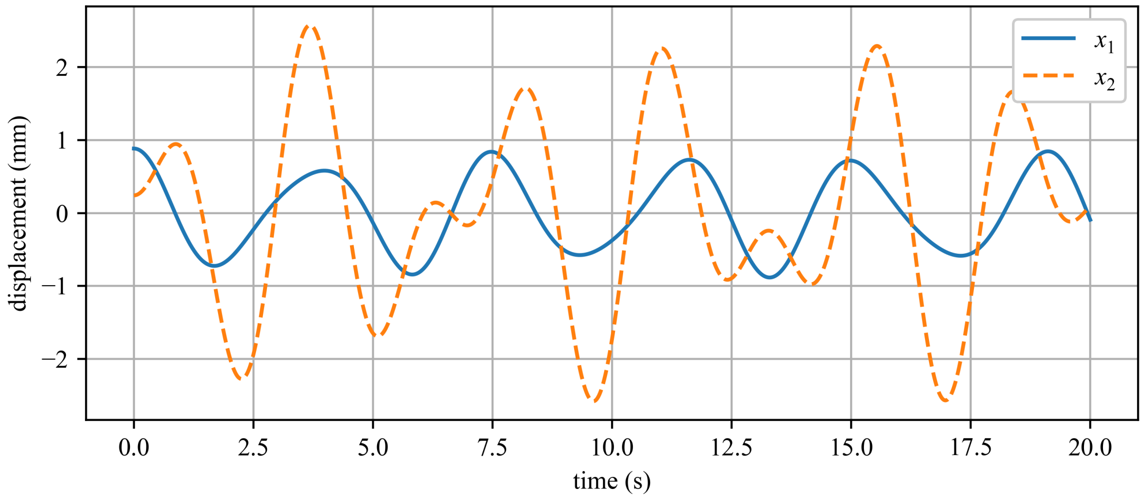


Figure 5.22: Temporal response for the 2-DOF reconstructed using just all the modal coordinates.

Next, the truncated response can be computed by only considering the first mode response for the system (i.e. $\vec{x}(t) = q_1(t)\vec{v}_1$). This is obtain as:

$$\vec{x}(t) = P\vec{q}_{\text{truncated}}(t) = \begin{bmatrix} 0.284 & -0.137 \\ 0.435 & 0.900 \end{bmatrix} \begin{bmatrix} 2.846 \cdot \cos(1.65t) \end{bmatrix} \tag{5.186}$$

This further simplified into:

$$\begin{aligned} x_1(t) &= 0.808 \cdot \cos(1.65t) \\ x_2(t) &= 1.238 \cdot \cos(1.65t) \end{aligned} \tag{5.187}$$

These results are plotted in figure 5.23. Note that this only considers the response of the system that is a function of the first mode. Note that this captures some of the "general" idea of the system while missing out on the finer points that the 2^{nd} mode contributes.

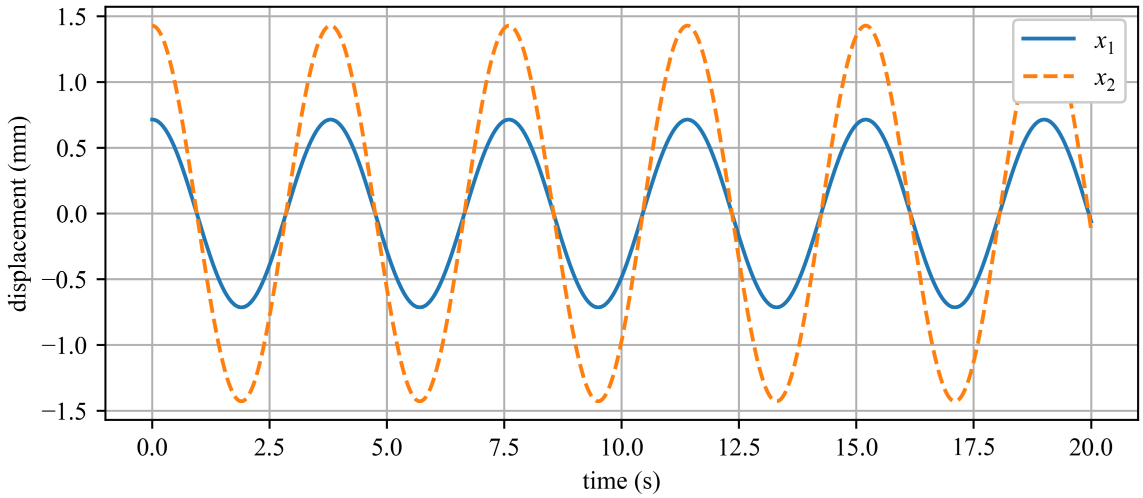


Figure 5.23: Truncated temporal response for the 2-DOF reconstructed using just the first modal coordinates.

Lastly, the participation of the two modes can be plotted from the time series response of equation 5.184.

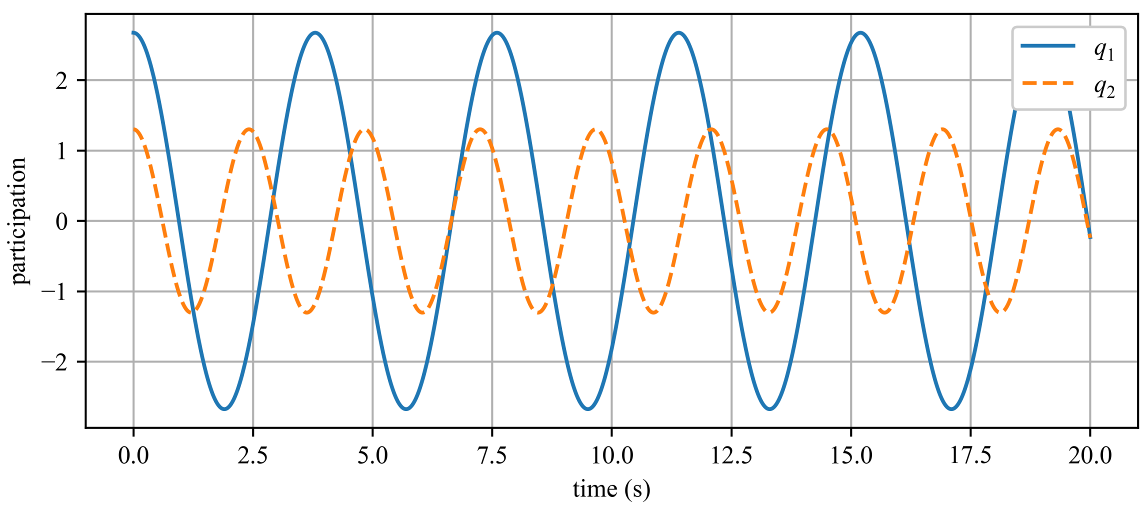


Figure 5.24: Modal participation coefficients.

6 Vibration Control

Throughout this text, we have studied various aspects related to analyzing and modeling vibrating systems. Therefore, it becomes prudent to look at methods for reducing or eliminating unwanted vibrations. However, before vibrations in a system can be effectively reduced they must be better understood in terms of their effects on the system under study. For this reason, this chapter first introduces the vibration Nomograph, this is then followed by vibration isolation, absorption, and active suppression.

6.1 Vibration Nomograph

There exist various methods and standards for measuring and describing acceptable levels of vibrations in systems, these include ISO/AWI 2631 for the evaluation of human exposure to whole-body vibrations and ISO 4866 for the measurement and effects of vibrations on structures. A common way to present the acceptable limit of vibration is in a vibration nomograph. A vibration nomograph is a simplified way to express the acceptable limits on a system while considering the displacement, velocity, acceleration, and frequency of a system. A typical nomograph with various limits is presented in figure 6.6.

A vibration nomograph is a logarithmic plot that allows us to easily express the relationships between displacement, velocity, acceleration, and frequency of a system. The vibration nomograph presented in figure 6.6 considers an undamped 1-DOF system with constant amplitude (A) experiencing harmonic motion that can be modeled as:

$$x(t) = A\sin(\omega t) \tag{6.6}$$

Therefore, the velocity and acceleration terms can be found by taking the derivatives of the displacement expression to yield:

$$\dot{x}(t) = A\omega\cos(\omega t) \tag{6.7}$$

and:

$$\ddot{x}(t) = -A\omega^2\sin(\omega t) \tag{6.8}$$

These equations are converted from a circular frequency in rad/sec to a linear frequency (f) in Hz, such that $\omega = 2\pi f$. Therefore, equations 6.6-6.8 become:

$$x(t) = A\sin(\omega t) \tag{6.9}$$

$$v(t) = \dot{x}(t) = 2\pi f A\cos(\omega t) \tag{6.10}$$

$$a(t) = \ddot{x}(t) = -4\pi^2 f^2 A\sin(\omega t) \tag{6.11}$$

Thereafter, the maximum values for velocity $v_{\max}$ and acceleration $a_{\max}$ are related to amplitude through:

$$v_{\max} = 2\pi f A \tag{6.12}$$

$$a_{\max} = -4\pi^2 f^2 A = -2\pi f v_{\max} \tag{6.13}$$

by taking the natural log of both side of equation 6.12 we obtain:

$$\ln v_{\max} = \ln(2\pi f) + \ln A \tag{6.14}$$

doing the same for equation 6.13 leads to:

$$\ln a_{\max} = -\ln(2\pi f) - \ln v_{\max} \tag{6.15}$$

It can be seen that both of these expressions are linear.

The nomograph sets the $x-$axis as frequency in Hz and the $y-$axis as velocity in mm/s. Equation 6.14 tells us that For a constant amplitude of displacement (A), $\ln v_{\max}$ is linearly proportional to $\ln(2\pi f)$, at a rate of 2π. As the $x-$axis in a nomograph is frequency, measured in Hz and thereby accounting for the 2π, $\ln(2\pi f)$ is a straight line with a positive slope of 1 with respect the frequency axis (i.e. $x-$axis). Therefore, a line on the nomograph that represents a constant displacement is at a 45° angle from the x-axis.

For a constant value of velocity, ($v_{\max}$), equation 6.15 shows that acceleration ($\ln a_{\max}$) is linearly proportional to $-\ln(2\pi f)$, at a rate of 2π. Again, as the $x-$axis in a nomograph is frequency, measured in Hz, acceleration is represented by a straight line that varies with $-\ln(2\pi f)$, therefore, $\ln a_{\max}$ is a straight line with the slope of -1. This is also represented by a line of constant acceleration set at a -45° angle from the x-axis. These equations are expressed in the vibration nomograph plot of figure 6.6 where each point on the plot represents a specific sinusoidal (harmonic) vibration for a 1-DOF system.

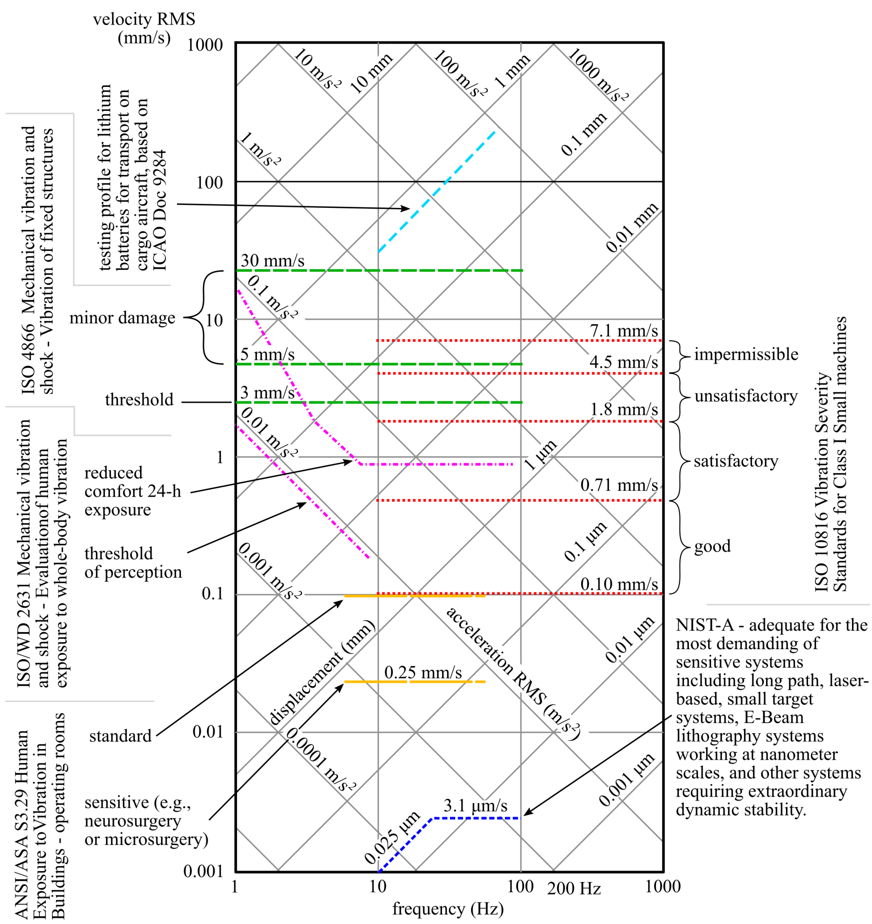


Figure 6.6: Vibration nomograph showing the acceptable limits of vibration for various applications.

6.2 Vibration Isolation

6.3 Vibration Absorption

Vibration absorbers, also termed dynamic vibration absorber, are a class of mechanical devices that seek to reduce unwanted vibrations in a system. In contrast to a traditional dash-pot style damper, these systems seek to "redirect" the vibrations from the system to another mass connected to the system. In this way the main system is protected form the bandwidth of vibrations that the vibration absorbers is tuned for. As the vibration absorbers must be tuned for the system, it is generally limited to devices that operate at a fixed frequency like industrial equipment or cables suspended in the air and subjected to wind loading. Figure 6.7 presents a Stockbridge and a Dogbone damper designed to remove certain bandwidths of excitation form wind-excited cables.

(a) (b)

Figure 6.7: Vibration absorbers deployed on wind excited cables showing: (a) a Stockbridge damper on a high-power transmission line[1], and; (b) a dogbone damper on a suspender cable of a suspension bridge[2].

Vibration absorbers are most often designed to shift the resonance frequency of the first mode of the system away from the expected excitation frequency. This is done by adding an additional degree-of-freedom in the form of a mass (the vibration absorber) connected to system with a spring to alter the natural frequency of the combined system away from the original excitation frequency. Dashpots may also be added in parallel to the spring element if additional energy dissipation in needed beyond that provided by the original system.

[1] "Stockbridge dampers installed on high voltage power lines" by Badics CC BY-SA 3.0

[2] "Dogbone dampers on the road-support cables of the Severn Bridge" by Bassaar CC BY-SA 4.0

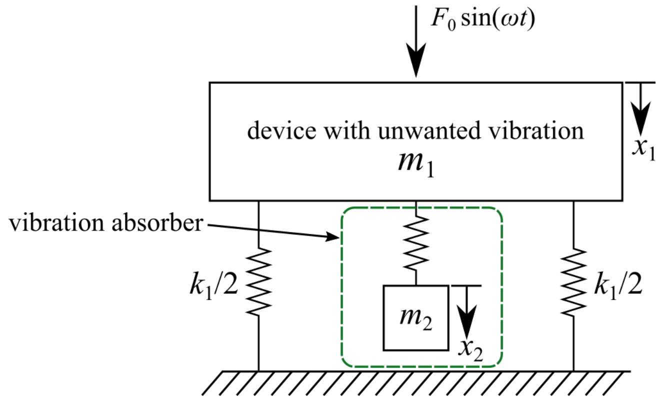


Figure 6.8: A vibration absorber (m_2) for mitigating unwanted dynamics in a device (m_1).

The tuning of a 2-DOF system can be done by setting the displacement of the mass to be controlled to zero and solving for the mass and stiffness of vibration absorber. Consider the system presented in figure 6.8, here m_1 and k_1 are the mass and stiffness of the system while m_2 and k_2 are the mass and stiffness of the vibration absorber. A good assumption to make when designing a vibration absorber is that the mass of the absorber should be between 1% and 5% of the mass of the system to be damped. Therefore, for this case let m_1 = 20 kg, m_2 = 1 kg, and k_1 = 20 kN. Assuming a sinusoidal input where $F_0 = 1$ kN, the equations of motion are:

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = F_0\sin(\omega t) \tag{6.16}$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \tag{6.17}$$

Assuming the temporal solution is of a harmonic form, the following is true:

$$x_i(t) = X_i\sin(\omega t), \;\; i = 1,2 \tag{6.18}$$

using the transfer function approach and assuming no initial conditions, the following steady state solution can be obtained for m_1 and m_2:

$$X_1 = \frac{(k_2 - m_2\omega^2)F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2} \tag{6.19}$$

$$X_2 = \frac{k_2F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2} \tag{6.20}$$

Next, the natural frequency of m_1 (ω_1) can be solved for as $\omega_1 = \sqrt{k_1/m_1}$. In order to eliminate movement for m_1 at a given driving frequency ω, the numerator of equation 6.19 should be set to zero. Note that setting F_0 is a trivial solution and provides no real benefit to the system. Therefore:

$$k_2 = m_2\omega^2 \tag{6.21}$$

note that this will force the frequency of the tuned vibration absorber to match that of the system, therefore $\omega_1 = \omega_2 = \sqrt{k_2/m_2}$. Next, normalizing the input force F_0 by the stiffness of the main system k_1 yields:

$$\delta_{\mathrm{st}} = \frac{F_0}{k_1} \tag{6.22}$$

using this term, equations 6.19 and 6.20 can be rearranged as:

$$\frac{X_1}{\delta_{st}} = \frac{1-\left(\frac{\omega}{\omega_2}\right)^2}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}} \tag{6.23}$$

$$\frac{X_2}{\delta_{st}} = \frac{1}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_2}{k_1}} \tag{6.24}$$

Figure 6.9 reports the normalized displacement of the system over a frequency range for system with and without a vibration absorber. Note that at $\omega = 1$ the original system is in resonance while the system with the vibration absorber has no displacement. However, no system is without compromise. From equation 6.24 it can be seen that at $\omega = \omega_1 = \omega_2$ the second mass needs a displacement equal to:

$$X_2 = -\frac{k_1}{k_2}\delta_{st} = -\frac{F_0}{k_2} \tag{6.25}$$

or 1 m using the given parameters. Therefore, the mass and stiffness values of the vibration absorber should be selected based the the allowable travel of the absorber (i.e. X_2), among other factors. Moreover, from this equation it can be seen the force exerted by the second mass operates in the direction opposite the original force ($-F_0 - k_2X_2$), thereby canceling it. Lastly, note that the addition of the vibration absorber creates two resonate frequencies of the system, termed Ω_1 and Ω_2. These resonate frequencies represent roots of the system and care should be taken to limit the time the system spends at these frequencies (i.e. on startup). The locations of these roots can be solved for analytically by setting the denominators of equation 6.23 to zero.

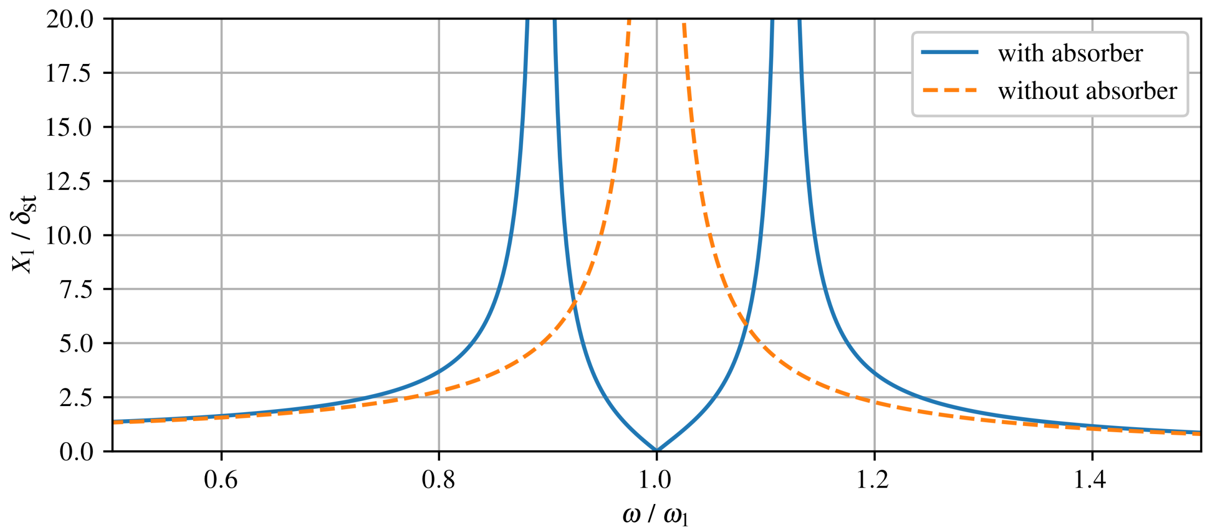


Figure 6.9: Frequency response of the undamped system with and without the vibration absorber.

6.4 Active Vibration Suppression

Active vibration control add energy to the system in order to mitigate the vibrations in the system. As depicted in figure 6.10(a), an active vibration control system requires a sensor to acquire data from the system, control hardware and algorithms to processes this data, and an actuator to exert a physical control on the system. These system together are called a feedback loop, as a movement in the mass results in a control force (f_u) being exerted on the system. This control force is diagrammed in figure 6.10(b).

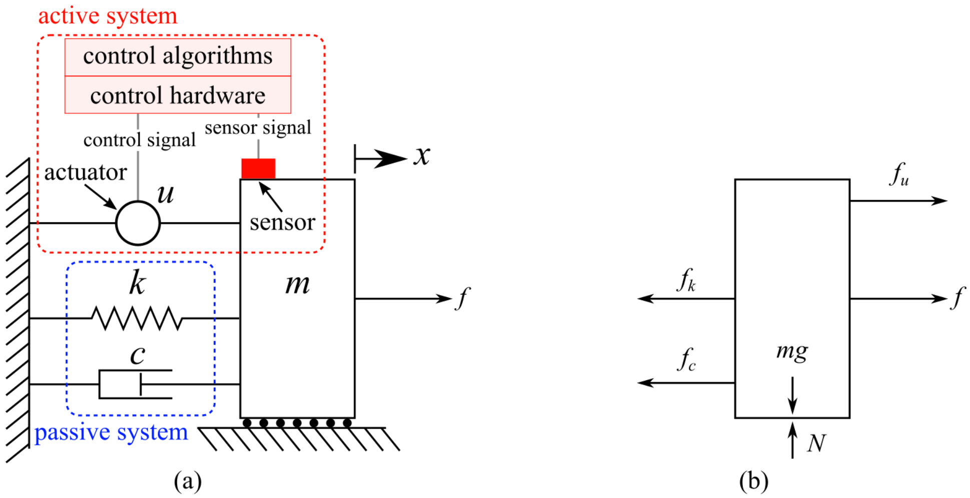


Figure 6.10: Active vibration control system showing: (a) the system with a feed-back loop that takes a signal from the sensor, converts it to a control signal, and drives the actuator; and (b) the free body diagram.

Adding the control force to the EOM for the 1-DOF system presented in figure 6.10 results in:

$$m\ddot{x} + c\dot{x} + kx = F(t) = f + f_u \tag{6.26}$$

A common method for providing control for vibration suppression is called position and derivative control or PD-control. A PD-controller is a state-variable feedback controller as it uses velocity and displacement obtained from the measured acceleration, assuming that the acceleration is properly integrated. PD-control measures the position and velocity of the mass and uses these to compute the control force needed to mitigate the vibration to an acceptable level. A simple way to code a PD-controller is to provide a control force proportional to the displacement velocity (derivative of displacement) of the mass such that:

$$f_u = -g_1 x - g_2 \ddot{x} \tag{6.27}$$

where g_1 and g_2 are the proportional gains of the systems. The control gains can be constants determined by the designer or variables updated through time by an algorithm. Here we will consider the gains to be constant, therefore, the EOM for the closed-loop system in figure 6.10 becomes:

$$m\ddot{x} + (c + g_2)\dot{x} + (k + g_1)x = F(t) = f \tag{6.28}$$

This formulation lets g_1 act as additional stiffness while g_2 acts as additional damping. This closed-loop EOM can be used to solve for the effective natural frequency of the system, given by:

$$\omega_n = \sqrt{\frac{k+g_p}{m}} \tag{6.29}$$

and the effective damping ratio of the system

$$\zeta = \frac{c+g_2}{2\sqrt{m(k+g_1)}} \tag{6.30}$$

7 Structural Dynamics

7.1 Single-story frame

The dynamic response of civil infrastructures, including buildings, bridges, and towers, can be studied by applying fundamental vibrations concepts studied in the previous chapters.

Let's start by considering the single-story frame shown in Fig. 7.7 (a) in free vibration (no external load is applied to the structure). The frame has height H and bay width L. As shown in Fig. 7.7, the frame consists of two columns with modulus of elasticity E and moment of inertia (second moment of the cross sectional area) I. The columns are fixed at the base. The frame in Fig. 7.7 (a) can be modeled as a single-degree of freedom (SDOF) system under the following assumptions:

- Shear building: flexible columns ($EI \neq 0$), beam infinitely rigid ($EI_b = \infty$), axial deformations of beams and columns negligible ($EA = 0$);

- Lumped mass system: floor-mass concentrated at the floor level.

Fig. 7.7 (b) illustrates a SDOF with mass m and stiffness k that can be used to model the dynamic behavior of the single-story frame considering no damping ($\zeta = 0$).

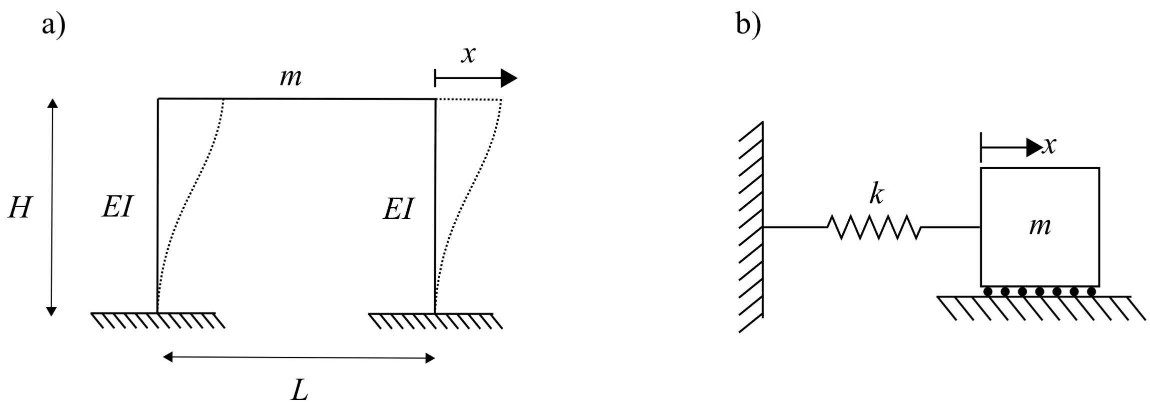


Figure 7.7: (a) Single story frame; (b) undamped single degree of freedom system.

The response of a SDOF system can be written in general notation as:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{7.7}$$

where ω_n is the natural frequency of the frame, x_0 and v_0 are the initial conditions. In order to find ω_n, we need to calculate the stiffness of the system. The mass is usually given.

The stiffness of the system can be found by applying the Hooke's law: $F = kx$. To find k, let's imagine to apply an arbitrary lateral force F to the frame and analyze a single column. At the top, the column will be subjected to a force F and to a moment M_0, as schematically shown in Fig. 7.8 (a). Applying the equilibrium equations to the column, it can be found that $M_0 = \frac{FH}{2}$.

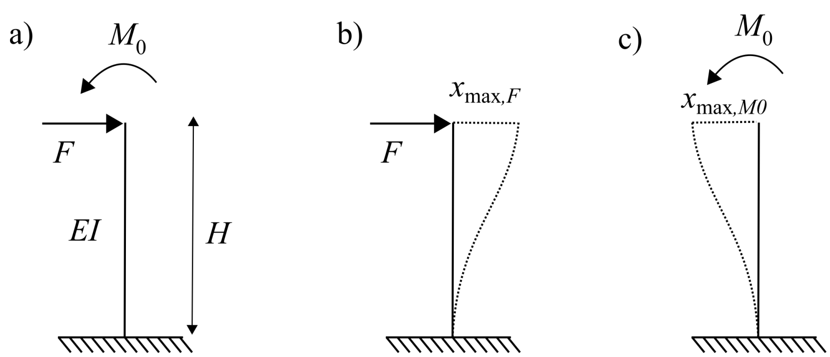


Figure 7.8: Single column subjected to: (a) force and moment; (b) force only; (c) moment only.

Since the system is linear, we can calculate the effects of F and M_0 separately and then summed them together (superposition principle). The maximum deflection due to F occurs at the top of the column, as shown in Fig. 7.8 (b), and it is equal to:

$$x_{\max,F} = \frac{FH^3}{3EI} \tag{7.8}$$

while the maximum deflection caused by M_0 (Fig. 7.8 (c)) is:

$$x_{\max,M0} = \frac{M_0 H^2}{2EI} \tag{7.9}$$

The displacements in Eq. (7.8) and (7.9) were found using engineering tables. The total displacement x at the top of the column is obtained from the sum of the two displacements:

$$x = \frac{FH^3}{3EI} - \frac{M_0 H^2}{2EI} \tag{7.10}$$

where the $x_{\max,M0}$ is negative in sign because the displacement caused by M_0 goes in opposite direction to $x_{\max,F}$. Replacing $M_0 = \frac{FH}{2}$ in Eq. (7.10):

$$x = \frac{FH^3}{3EI} - \frac{FH^3}{4EI} = \frac{FH^3}{12EI} \tag{7.11}$$

Applying Hooke's law:

$$F = k_c x = k_c \frac{FH^3}{12EI} \tag{7.12}$$

where k_c is the stiffness of the column. Therefore:

$$k_c = \frac{12EI}{H^3} \tag{7.13}$$

Since the frame has two columns, the total stiffness of the SDOF system will be:

$$k = \sum_{columns} k_c = \sum_2 \frac{12EI}{H^3} \tag{7.14}$$

where k is also called *lateral stiffness*. Note that the lateral stiffness of the frame is independent on the length of the bay L and it depends only on the properties of the columns (E, I, and H). It is possible at this point to calculate the natural frequency of the frame:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{\sum_2 \frac{12EI}{H^3}}{m}} \tag{7.15}$$

If the columns have same properties, Eq. (7.16) becomes:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{24EI}{H^3 m}} \tag{7.16}$$

Finally, the response of the system to initial conditions x_0 and v_0 can be obtained:

$$x(t) = x_0 \cos(\omega_n t) + \frac{v_0}{\omega_n} \sin(\omega_n t) \tag{7.17}$$

Example 7.1 Let's consider the single-story frame shown in Fig. 7.7 with mass m = 0.15 kip s^2/ft, L = 12 ft, EI= 1800 kip ft^2. a) Determine the EOM and the natural period of the frame; b) assume that the moment of inertia of the right column is $2I$. Will the EOM change?

Solution:

a) The frame can be modeled as a single-degree of freedom in free vibration. Therefore, the EOM is:

$$m\ddot{x} + kx = 0 \tag{7.18}$$

The lateral stiffness of the system is:

$$k = \sum_{columns} k_c = \sum_{2} \frac{12EI}{H^3} = \frac{24EI}{H^3} \tag{7.19}$$

Thus, the natural frequency and period are:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{24EI}{mH^3}} = \sqrt{\frac{1800}{0.15 \cdot 12^3}} = 12.91\frac{rad}{s} \tag{7.20}$$

$$T_n = \frac{2\pi}{\omega_n} = 0.48s \tag{7.21}$$

b) The EOM won't change, but the lateral stiffness of the system will be:

$$k = \sum_{columns} k_c = \frac{12EI}{H^3} + \frac{24EI}{H^3} = \frac{36EI}{H^3} \tag{7.22}$$

The same principle can be applied to a single-story frame with damping ratio $\zeta \neq 0$. In this case, the displacement of the frame will be given by:

$$x(t) = e^{(-\zeta\omega_n t)}\left(\frac{(v_0 + x_0)\omega_n}{\omega_d}\cos(\omega_d t) + x_0\sin(\omega_d t)\right) \tag{7.23}$$

where ω_d is the damped natural frequency of the system:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{7.24}$$

7.2 Duhamel's Integral

In Chapter 4, the frequency response method was used to solve the EOM of a SDOF system subjected to an arbitrary force. Here, an alternative method widely employed in structural dynamics to find the solution of the EOM is presented. This method exploits a specif integral, named Duhamel's integral.

Let's consider an underdamped SDOF system subjected to an arbitrary force $F(t)$. The EOM is:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{7.25}$$

Let's assume that the system is at rest: $x(0) = 0$ and $\dot{x} = 0$. The assumption underlying the Duhamel's integral method is that a generic force $F(t)$ can be expressed as a sequence of impulses of very small duration and the response of the system as the sum of the response to individual unit impulses.

An impulsive force can be defined as a very large force applied in a very short time interval. Fig. 7.9 (a) shows an impulsive force $F(t) = \frac{1}{\varepsilon}$ applied at time $t = \tau$. Assuming to apply an impulsive force to a generic mass m and applying Netwon's second law:

$$m\ddot{x} = F(t) \tag{7.26}$$

and integrating both sides between two generic time instants t_1 and t_2 yields:

$$\int_{t_1}^{t_2} F(t)dt = m(\dot{x}_1 - \dot{x}_2) \tag{7.27}$$

where the left-hand side of the equation represents the magnitude of the force and the right-hand side the change in momentum.

In the limit case in which ε tends to 0, $F(t)$ tends to 1 and the impulsive force is called *unit impulse*. In the case of a unit impulse, $\int_{t_1}^{t_2} F(t)dt = 1$ and t_1 tends to t_2. Therefore, the velocity of the mass can be found as:

$$\dot{x}(\tau) = \frac{1}{m} \tag{7.28}$$

A similar concept applies to a SDOF system. Since the impulse is applied in a very short time interval, the spring and the damper do not have the time to react. When we apply a unit impulse to an underdamped SDOF, the system will start vibrating with velocity $\dot{x}(\tau)$ given by Eq. (7.28) and displacement $x(\tau) = 0$. The response of the system is given by the following equation:

$$x(t) = h(t - \tau) = \frac{1}{m\omega_d} e^{-\zeta\omega_n(t-\tau)} sin(\omega_d(t - \tau)) \tag{7.29}$$

where τ is the time instant at which the impulse is applied. Note that the Dirac delta function $\delta(t - \tau)$ mathematically defines a unit impulse centered at $t = \tau$.

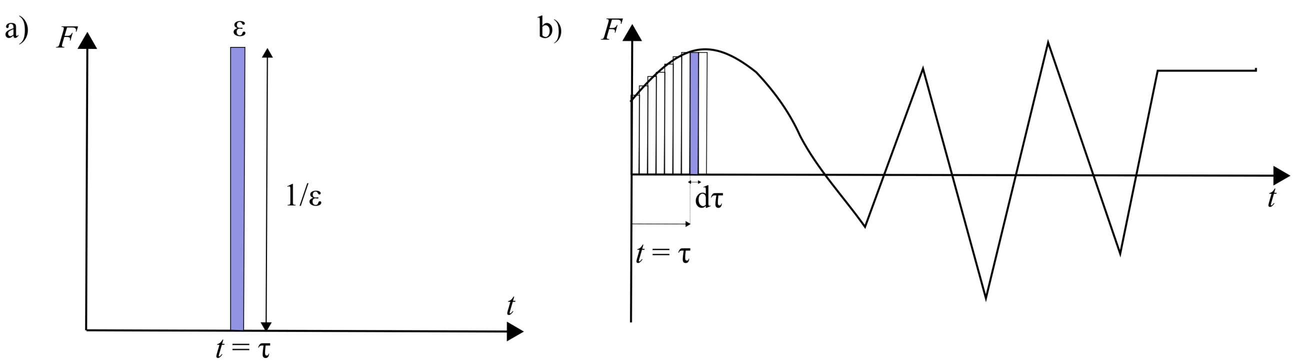


Figure 7.9: (a) Impulsive force; (b) arbitrary force decomposed in a series of impulses.

Let's now consider a force $F(t)$ varying arbitrarily with time. As shown in Fig. 7.9 (b), $F(t)$ can be represented as a sequence of infinitesimaly short impulses. The response of a linear system to $F(t)$ can be therefore expressed as the response to a series impulses, following:

$$x(t) = \int_0^t p(\tau)h(t-\tau)d\tau \tag{7.30}$$

where $h(t-\tau)$ is the response to a unit impulse and $p(\tau)$ is the magnitude of the actual impulse. For the case of an underdamped SDOF system, Eq. (7.30) can be re-written as:

$$x(t) = \frac{1}{m\omega_d}\int_0^t p(\tau)e^{-\zeta\omega_n(t-\tau)}sin(\omega_d(t-\tau))d\tau \tag{7.31}$$

Eq. (7.31) represents the *Duhamel's integral*.

Similarly, the response of an undamped SDOF system to an arbitrary force can be expressed through the Duhamel's integral as:

$$x(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)sin(\omega_n(t-\tau))d\tau \tag{7.32}$$

If $F(t)$ is characterized by a simple function, the Duhamel's integral can be evaluated in closed form. If the equation of $F(t)$ is complicated, the Duhamel's integral can be solved with numerical methods.

Note that Eq. (7.31) and (7.32) apply when the initial conditions are zero (the system is at rest). If the initial conditions are different than zero, we need to add the free vibration response of the system to Eq. (7.31) and (7.32), respectively.

Example 7.2 Let's consider an undamped SDOF system subjected to a step function force with constant amplitude F_0, as schematically represented in Fig. 7.10. Assume that the system is at rest (initial conditions: $x(0) = \dot{x}(0) = 0$) and compute the system response $x(t)$.

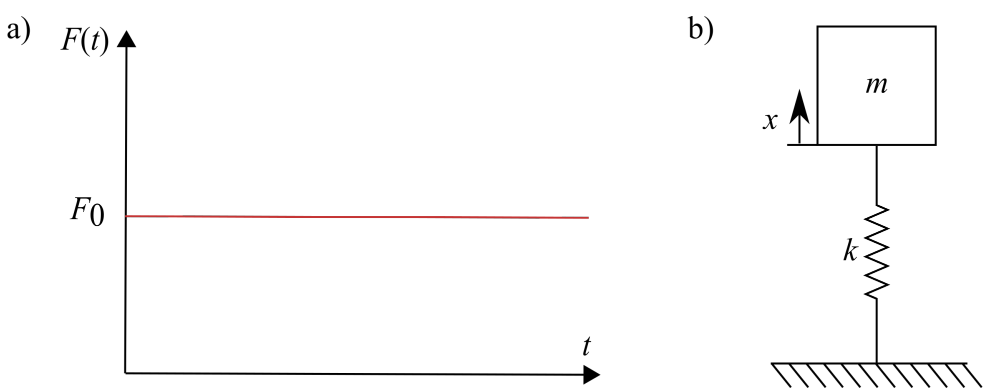


Figure 7.10: (a) Step function force; (b) undamped SDOF system.

Solution:

The system is undamped, therefore we can use the Duhamel's integral in Eq. (7.32) to find

$x(t)$:

$$x(t) = \frac{1}{m\omega_n} \int_0^t F_0 sin(\omega_n(t-\tau))d\tau \tag{7.33}$$

Considering that F_0 is constant:

$$x(t) = \frac{F_0}{m\omega_n} \left[\frac{cos(\omega_n(t-\tau))}{\omega_n} \right]_0^t = \frac{F_0}{m\omega_n^2}[1 - cos(\omega_n t)] \tag{7.34}$$

Reminding that $\omega_n^2 = k/m$, $x(t)$ becomes:

$$x(t) = \frac{F_0}{k}[1 - cos(\omega_n t)] \tag{7.35}$$

where $\frac{F_0}{k}$ is the displacement that the system would undergo if the force F_0 was applied statically.

In the case of underdamped SDOF system, the response becomes:

$$x(t) = \frac{F_0}{k} \left[1 - e^{-\zeta\omega_n t} \left(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t) \right) \right] \tag{7.36}$$

7.3 Two-story frame

The concepts discussed in Sec. 1 can be extended to the 2-story frame represented in Fig. 7.11. In fact, a 2-story frame can be modeled as a 2-DOF system under the following assumptions:

- shear building: flexible columns ($EI \neq 0$), beam infinitely rigid ($EI_b = \infty$), axial deformations of beams and columns negligible ($EA = 0$);
- lumped mass system: floor-mass concentrated at the floor level.

Under such assumptions and free vibrations, we expect that the building moves following the deformed shape reported in Fig. 7.11 (dotted line). Let's call the degrees of freedom of the frame $x_1(t)$ and $x_2(t)$.

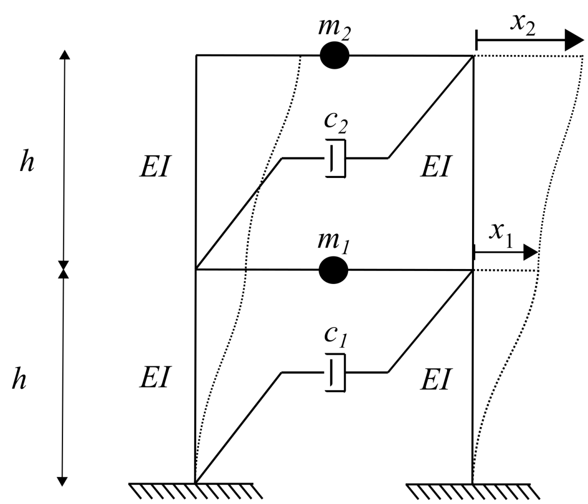


Figure 7.11: 2-story frame with lumped masses.

The forces acting on the 2-DOF system are reported in Fig. 7.12. It follows that the equation of motion of the two masses are:

$$\begin{aligned} m_1\ddot{x}_1 + k_1x_1 + k_2(x_2 - x_1) + c_1\dot{x}_1 \; + c_2(\dot{x}_2 - \dot{x}_1) &= 0 \\ m_2\ddot{x}_2 - k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) &= 0 \end{aligned} \tag{7.37}$$

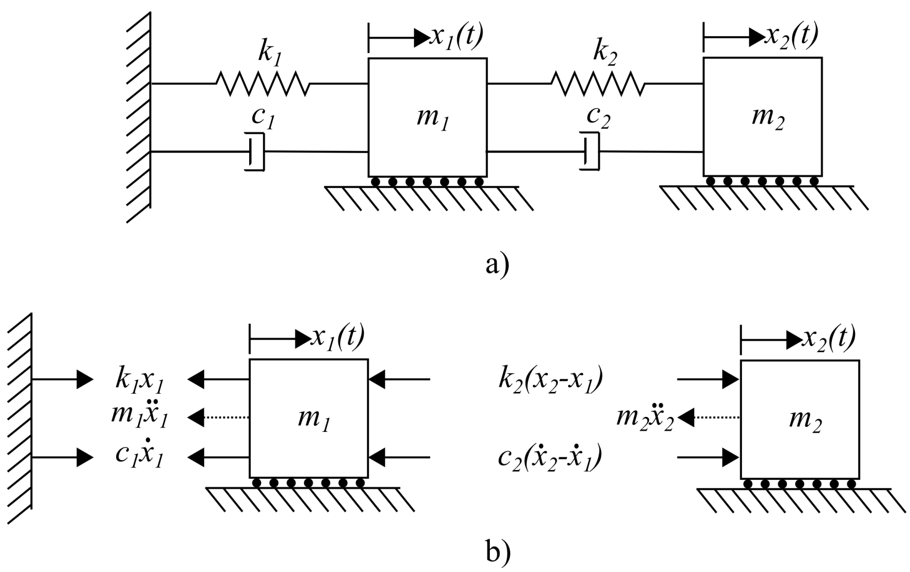


Figure 7.12: (a) 2-DOF system used to model the 2-story frame; (b) free body diagram of the two masses.

In matrix notation, these two equations become:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.38}$$

where we can define the mass matrix M as:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{7.39}$$

the stiffness matrix K as:

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{7.40}$$

and the damping matrix C as:

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \tag{7.41}$$

While mass and damping of a frame are usually given, the stiffness values k_1 and k_2 need to be calculated as a function of the columns properties (EI) and geometry (h). As demonstrated in Sec. 1, the stiffness of a column with clamped ends can be determined as:

$$k_c = \frac{12EI}{h^3} \tag{7.42}$$

The lateral stiffness of each floor can be computed as the sum of the stiffness of the columns at that floor:

$$k = \sum_{columns} k_c = \sum_{2} \frac{12EI}{h^3} \tag{7.43}$$

Therefore, for the frame in Fig. 7.11, the stiffness values are:

$$k_1 = k_2 = \frac{24EI}{h^3} \tag{7.44}$$

The solution of the EOM in Eq.(7.38) was derived in Chapter 5 and can be summarized as:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}, \qquad \omega_1 \text{ or } \omega_2 \neq 0 \tag{7.45}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors (or mode shapes), ω_1 and ω_2 are the natural frequency of vibration, ϕ_1, ϕ_2, A_1, and A_2 are constants that can be found based on the initial conditions (see Chapter 5 for more details).

Example 7.3 Consider the frame in Fig. 7.13. Determine natural frequency of vibration and mode shapes of the system.

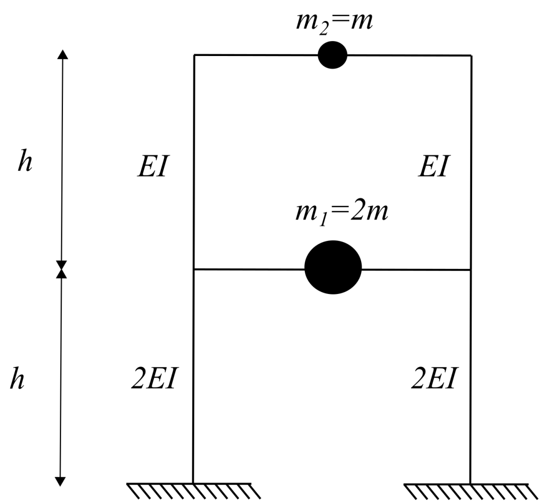


Figure 7.13: Example of a 2-story frame with floors with different dynamic properties.

Solution:

Assumption: the frame can be modeled as a shear building with mass lumped at the floor levels. The lateral stiffness at the first floor is:

$$k_1 = 2\frac{12(2EI)}{h^3} = \frac{48EI}{h^3} \tag{7.46}$$

The lateral stiffness at the second floor is:

$$k_2 = 2\frac{12(EI)}{h^3} = \frac{24EI}{h^3} \tag{7.47}$$

Therefore, the stiffness matrix can be written as:

$$K = \begin{bmatrix} \frac{48EI}{h^3} + \frac{24EI}{h^3} & -\frac{24EI}{h^3} \\ -\frac{24EI}{h^3} & \frac{24EI}{h^3} \end{bmatrix} = \frac{24EI}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \tag{7.48}$$

The EOM of the system is:

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \frac{24EI}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{7.49}$$

In order to determine the natural frequency of vibration and the mode shapes of the system, we need to solve the characteristic equation:

$$\det(-\omega^2 M + K) = 0 \tag{7.50}$$

leading to:

$$2m^2\omega^4 + 5km\omega^2 + 2k = 0 \tag{7.51}$$

This equation has two solutions:

$$\omega_1^2 = \frac{k}{2m} \tag{7.52}$$

$$\omega_2^2 = \frac{2k}{m} \tag{7.53}$$

Therefore, the two natural frequencies of vibration of the system are:

$$\omega_1 = \sqrt{\frac{k}{2m}} \tag{7.54}$$

$$\omega_2 = \sqrt{\frac{2k}{m}} \tag{7.55}$$

where $k = \frac{24EI}{h^3}$. The mode shapes of the frame can be found by solving the following equation:

$$(-\omega_1^2 M + K)\mathbf{u}_1 = 0 \tag{7.56}$$

Replacing mass and stiffness matrix the equation becomes:

$$\left(-\frac{k}{2m}\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} + k\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.57}$$

simplified to

$$\begin{bmatrix} 2k & -k \\ -k & \frac{k}{2} \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.58}$$

leading to two equations:

$$2ku_{11} - ku_{21} = 0, \text{ and } -ku_{11} + \frac{k}{2}u_{21} = 0 \tag{7.59}$$

It follows that:

$$2u_{11} = u_{21}, \text{ and } u_{11} = \frac{1}{2}u_{21} \tag{7.60}$$

To obtain a numerical value, we arbitrarily assign a value to one of the elements. Here, let $u_{21} = 1$ so let $u_{11} = 1/2$. Therefore,

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \tag{7.61}$$

Similarly, $\mathbf{u}_2$ can be obtained by solving the following equation:

$$(-\omega_2^2 M + K)\mathbf{u}_2 = 0 \tag{7.62}$$

leading to:

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{7.63}$$

Fig. 7.14 represents the two mode shapes of the building.

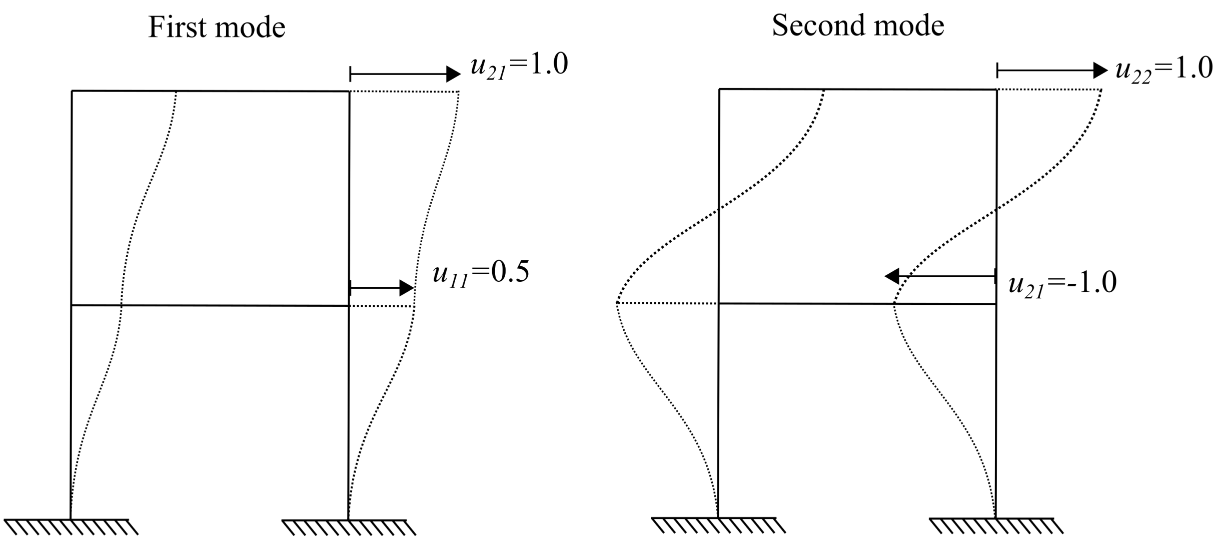


Figure 7.14: Mode shapes of the 2-story frame.

The temporal response of the system is given by:

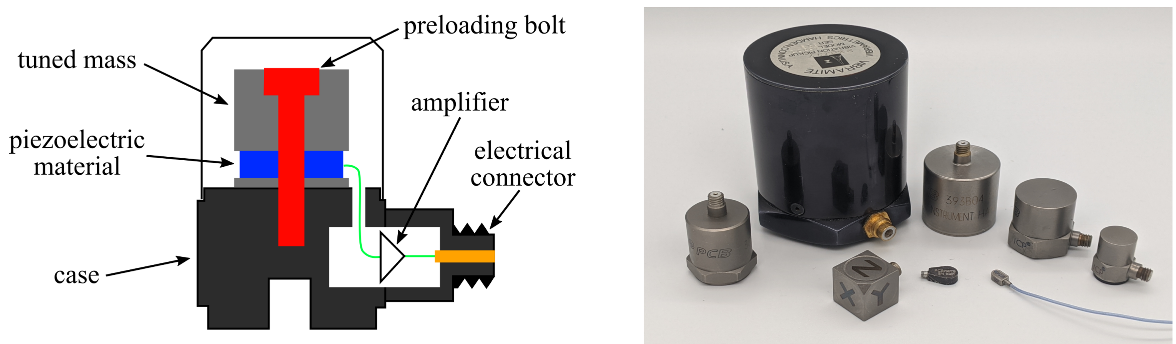
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) \\ A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}, \qquad \omega_1 \text{ or } \omega_2 \neq 0 \tag{7.64}$$

where $\mathbf{u} = [\mathbf{u}_1, \mathbf{u}_2]$ is the time invariant part of the equation.

8 Experimental Vibrations

Experimental vibration testing requires the practitioner to understand the basics of testing hardware and digital signal processing.

8.1 Hardware for vibration testing



(a) (b)

Figure 8.8: Integrated Electronics Piezo-Electric (IEPE) accelerometers, showing: (a) the cross section of a typical IEPE) accelerometer with key components annotated, and; (b) selection of IEPE accelerometers for various applications.

Table 1: Parameters for various IEPE accelerometers.

parameter	accelerometers				
model number	PCB 393B31	PCB 393B04	PCB 352C67	PCB 352A21	PCB 352A92
Sensitivity($\pm$ 10 %)	10.0 V/g	1000 mV/g	100 mV/g	10 mV/g	0.25 mV/g
Measurement Range	$\pm$ 0.5 g pk	$\pm$ 5 g pk	$\pm$ 50 g pk	$\pm$ 500 g pk	$\pm$ 20 kg pk
Frequency Range($\pm$ 5 %)	0.1 to 200 Hz	0.06 to 450 Hz	0.5 to 10 kHz	1.0 to 10 kHz	1.2 to 10 kHz
Resonant Frequency	>700 Hz	>2.5 kHz	>35 kHz	>50 kHz	>100 kHz
Non-Linearity	$\leq$1%	$\leq$1%	$\leq$1%	$\leq$1%	
Transverse Sensitivity	$\leq$5%	$\leq$5%	$\leq$5%	$\leq$5 %	

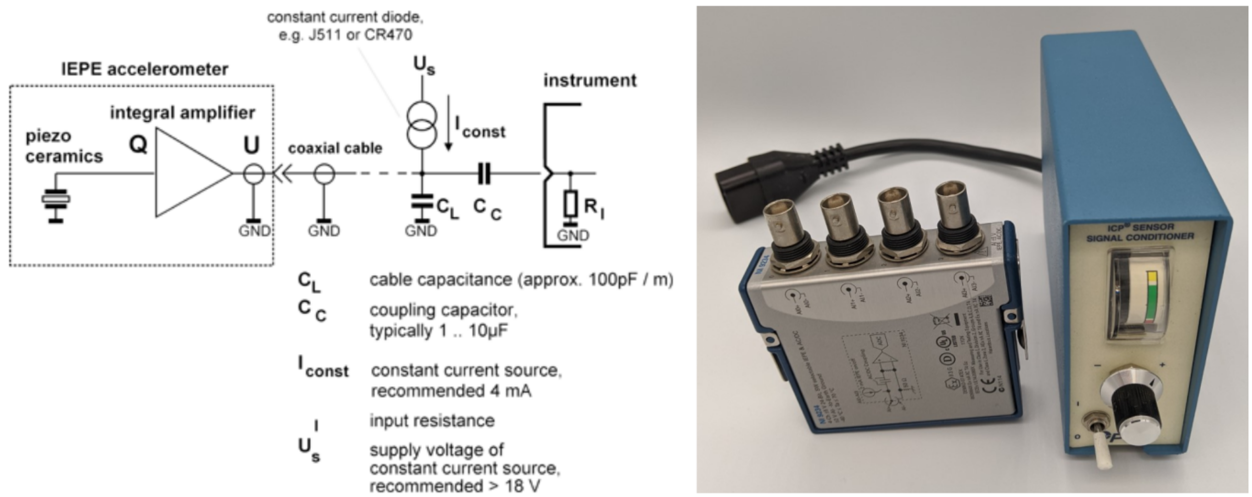


Figure 8.9: Integrated Electronics Piezo-Electric (IEPE)-based measurement system showing the: (a) simplified circuit schematic[1]; and (b) IEPE data acquisition systems in various form factors.

8.2 Digital Signal Processing

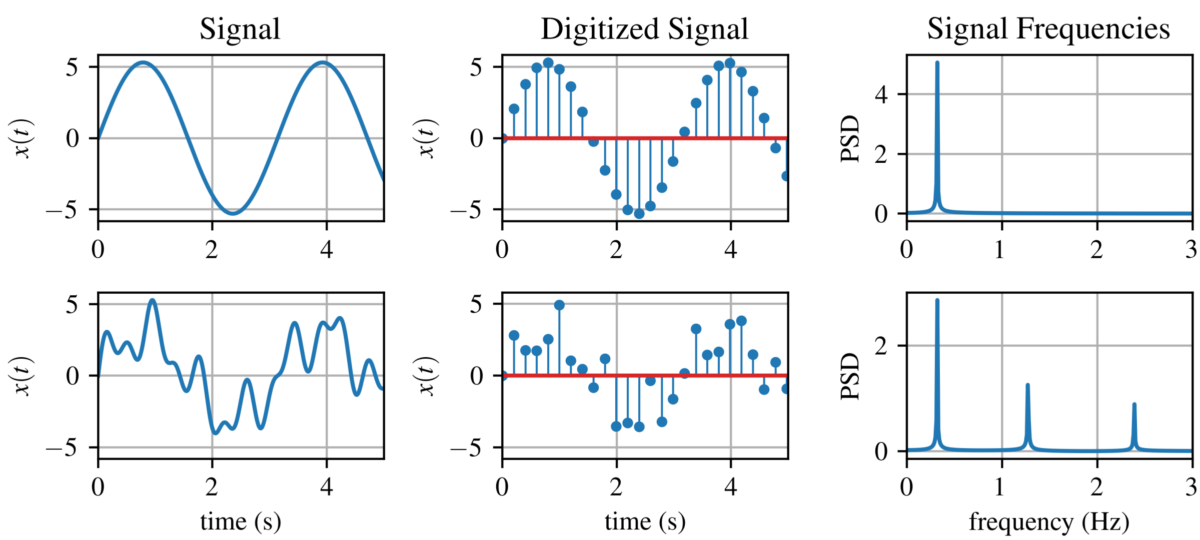


Figure 8.10: Digitization of two continuous time-series signals sampled at 5 S/s.

The Nyquist-Shannon sampling theorem is a theorem in the field of signal processing that defines the sample rate that permits a discrete sequence of samples (i.e. discrete-time) to sample a continuous-time signal of a finite bandwidth.

[1] "IEPE sensor connected to the input of an instrument" by JanBurg CC BY-SA 4.0

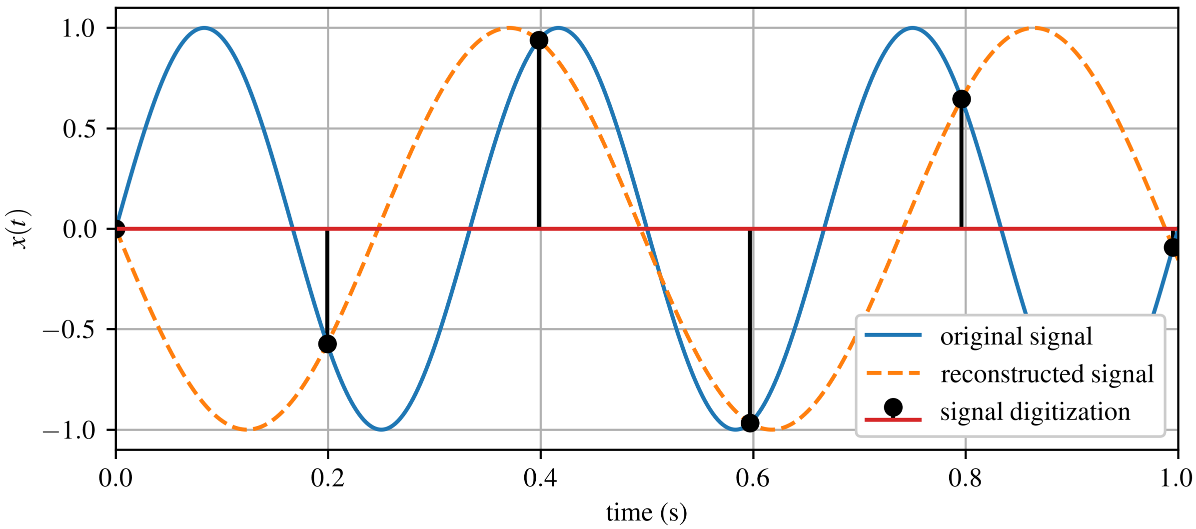


Figure 8.11: Aliasing of a 3 Hz signal that is sampled at 5 S/s.

In signal processing, aliasing is an effect that causes different signals to become indistinguishable from each other. In this way, the signals become an aliases of one another when sampled. Aliasing also accounts for the development of distortion or artifact in a reconstructed signal when compared to the original continuous signal.

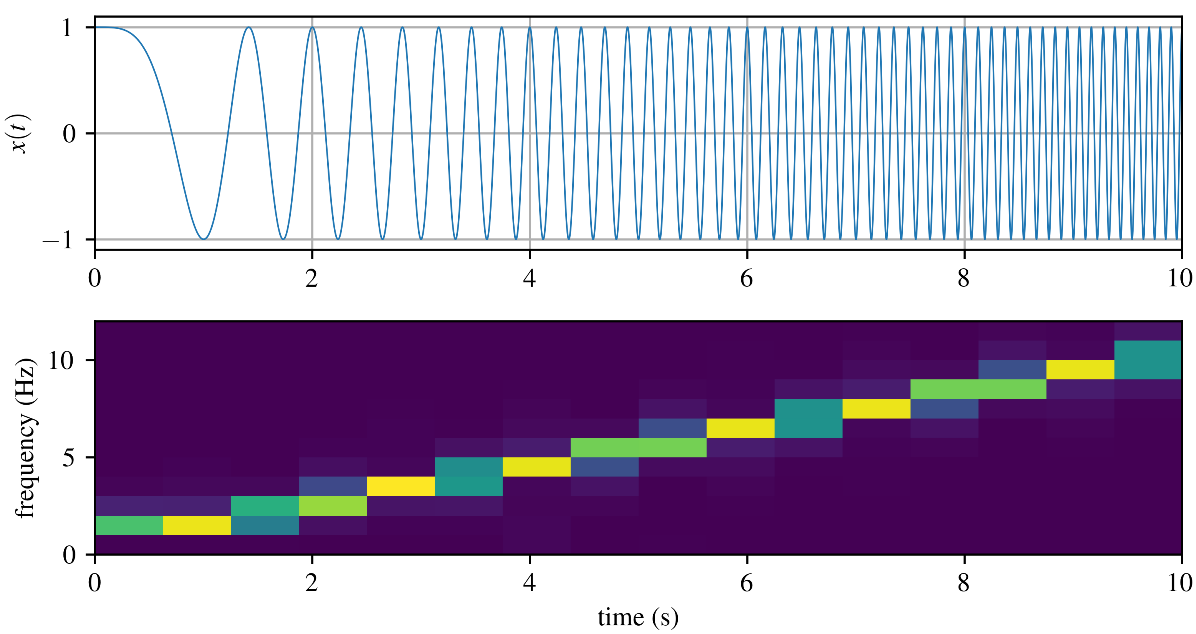


Figure 8.12: Spectrogram of a 0-10 Hz chirp signal.

Some of the key parameters in a spectrogram include:

- window
- segment length
- overlap