Assignment 1

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1) If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.

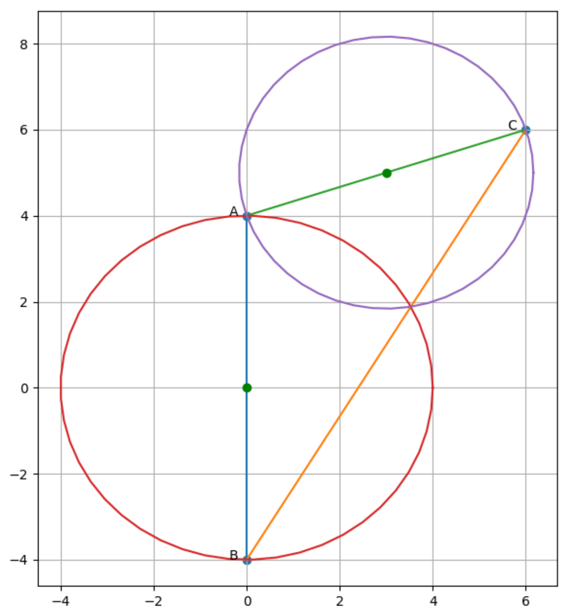

Fig. 1: Graph

Solution: Let the traingle be ABC, Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are given by,

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \tag{0.0.1}$$

The equation of circle taking $\mathbf{AB}$ as diameter is given by,

$$\|\mathbf{x}\|^2 + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0 \tag{0.0.2}$$

$$\mathbf{u_1} = -\left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) \tag{0.0.3}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.0.4}$$

$$r_1 = \frac{\|\mathbf{AB}\|}{2} \tag{0.0.5}$$

$$= 4 \tag{0.0.6}$$

$$f_1 = \|\mathbf{u_1}\|^2 - r_1^2 \tag{0.0.7}$$

$$= -4 \tag{0.0.8}$$

$$\|\mathbf{x}\|^2 - 16 = 0 \tag{0.0.9}$$

The equation of circle taking $\mathbf{AC}$ as diameter is given by,

$$\|\mathbf{x}\|^2 + 2\mathbf{u_2}^\top \mathbf{x} + f_2 = 0 \tag{0.0.10}$$

$$\mathbf{u_2} = -\left(\frac{\mathbf{A}+\mathbf{C}}{2}\right) \tag{0.0.11}$$

$$= -\begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{0.0.12}$$

$$r_2 = \frac{\|\mathbf{AC}\|}{2} \tag{0.0.13}$$

$$= \sqrt{10} \tag{0.0.14}$$

$$f_2 = \|\mathbf{u_2}\|^2 - r_2^2 \tag{0.0.15}$$

$$= 24 \tag{0.0.16}$$

$$\|\mathbf{x}\|^2 - 2\begin{pmatrix} 3 & 5 \end{pmatrix}\mathbf{x} + 24 = 0 \tag{0.0.17}$$

Let the other point of intersection of circles (0.0.9) and (0.0.17) be point $\mathbf{P}$. The equation of the common chord of intersection of two circles, AP is given by,

$$\mathbf{u_1}^\top \mathbf{x} - \mathbf{u_2}^\top \mathbf{x} + f_1 - f_2 = 0 \tag{0.0.18}$$

$$2\begin{pmatrix} 3 & 5 \end{pmatrix}\mathbf{x} - 16 - 24 = 0 \tag{0.0.19}$$

$$\begin{pmatrix} 3 & 5 \end{pmatrix}\mathbf{x} = 20 \tag{0.0.20}$$

The equation (0.0.20) can be written in parametric form as,

$$\mathbf{h} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \tag{0.0.21}$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 3 \end{pmatrix} \tag{0.0.22}$$

The parameter μ of the points of intersection of line (0.0.23) with the conic section (0.0.24)

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{0.0.23}$$

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.0.24}$$

is given by the equation

$$\mu^2 \mathbf{m}^\top \mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{0.0.25}$$

The points of intersection of the circle (0.0.9) and line (0.0.20) are the points $\mathbf{A}, \mathbf{P}$. Here we have,

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = 16 \tag{0.0.26}$$

$$\mathbf{h} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \tag{0.0.27}$$

$$\mathbf{m}^\top \mathbf{V}\mathbf{m} = 34 \tag{0.0.28}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 12 \tag{0.0.29}$$

$$g(\mathbf{h}) = 0 \tag{0.0.30}$$

$$34\mu^2 + 24\mu = 0 \tag{0.0.31}$$

$$\mu = 0, -\frac{12}{17} \tag{0.0.32}$$

$\mu = 0$ corresponds to point $\mathbf{A}$.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \frac{12}{17} \begin{pmatrix} -5 \\ 3 \end{pmatrix} \tag{0.0.33}$$

$$\mathbf{P} = \begin{pmatrix} \frac{60}{17} \\ \frac{32}{17} \end{pmatrix} \tag{0.0.34}$$

The equation of the the line $\mathbf{BC}$ is given by,

$$\mathbf{m} = \mathbf{C} - \mathbf{B} \tag{0.0.35}$$

$$= \begin{pmatrix} 6 \\ 10 \end{pmatrix} \tag{0.0.36}$$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} \tag{0.0.37}$$

$$\mathbf{n} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \tag{0.0.38}$$

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{B} \tag{0.0.39}$$

$$\begin{pmatrix} -5 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \end{pmatrix} \tag{0.0.40}$$

$$\begin{pmatrix} -5 & 3 \end{pmatrix} \mathbf{x} = -12 \tag{0.0.41}$$

$$\begin{pmatrix} -5 & 3 \end{pmatrix} \begin{pmatrix} \frac{60}{17} \\ \frac{32}{17} \end{pmatrix} = -12 \tag{0.0.42}$$

It is clear that the point $\mathbf{P}$ satisfies the equation of line $\mathbf{BC}$ (0.0.41). Hence, the point of intersection of the circles drawn by taking two sides of a triangle as diameters lies on the third side.