Assignment 1

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1) Two congruent circles intersect each other at points **A** and **B**. Through **A** any line segment **PAQ** is drawn so that **P**, **Q** lie on the two circles. Prove that **BP** = **BQ**.

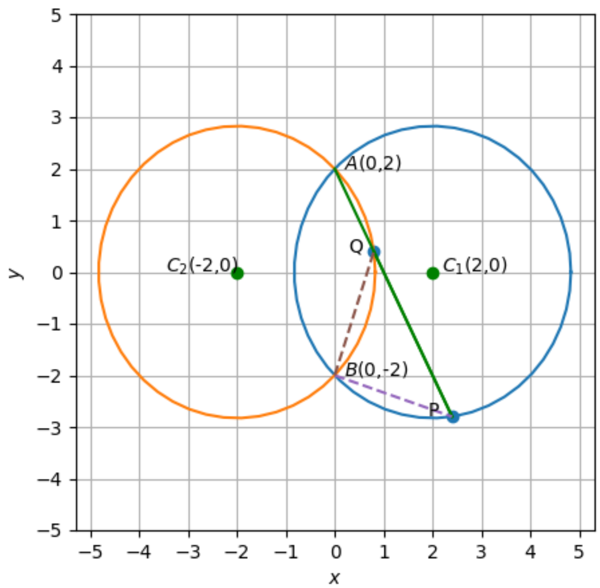

Fig. 1: Graph

Solution: Consider two congruent circles with radius $2\sqrt{2}$ with centres at $(2,0)$, $(-2,0)$. The equations of these circles is given by,

$$\|\mathbf{x}\|^2 + 2\mathbf{u_1}^\top\mathbf{x} + f_1 = 0 \tag{0.0.1}$$

$$\|\mathbf{x}\|^2 + 2\mathbf{u_2}^\top\mathbf{x} + f_2 = 0 \tag{0.0.2}$$

$$\mathbf{u_1} = -\begin{pmatrix}2\\0\end{pmatrix}, \mathbf{u_2} = -\begin{pmatrix}-2\\0\end{pmatrix} \tag{0.0.3}$$

$$f_1 = -4,\ f_2 = -4 \tag{0.0.4}$$

To get points of intersection **A**, **B**

$$\|\mathbf{x}\|^2 + 2\mathbf{u_1}^\top\mathbf{x} + f_1 = \|\mathbf{x}\|^2 + 2\mathbf{u_2}^\top\mathbf{x} + f_2 \tag{0.0.5}$$

$$2\left(\mathbf{u_1} - \mathbf{u_2}\right)^\top \mathbf{x} = f_2 - f_1 \tag{0.0.6}$$

$$2\begin{pmatrix}-4 & 0\end{pmatrix}\mathbf{x} = 0 \tag{0.0.7}$$

$$\mathbf{x} = \begin{pmatrix}\alpha\\\beta\end{pmatrix} \tag{0.0.8}$$

$$\alpha = 0 \tag{0.0.9}$$

$$\mathbf{x} = \begin{pmatrix}0\\\beta\end{pmatrix} \tag{0.0.10}$$

Substituting (0.0.10) in (0.0.1) we get,

$$\beta^2 - 2\begin{pmatrix}2 & 0\end{pmatrix}\begin{pmatrix}0\\\beta\end{pmatrix} - 4 = 0 \tag{0.0.11}$$

$$\beta^2 = 4 \tag{0.0.12}$$

$$\beta = \pm 2 \tag{0.0.13}$$

$$\mathbf{A} = \begin{pmatrix}0\\2\end{pmatrix}, \mathbf{B} = \begin{pmatrix}0\\-2\end{pmatrix} \tag{0.0.14}$$

Equation of the line passing through **A** is given by

$$\mathbf{n}^\top\mathbf{x} = \mathbf{n}^\top\mathbf{A} \tag{0.0.15}$$

Let the noRmal vector **n** given by,

$$\mathbf{n} = \begin{pmatrix}2\\1\end{pmatrix} \tag{0.0.16}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = \begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} \tag{0.0.17}$$

$$\begin{pmatrix}2 & 1\end{pmatrix}\mathbf{x} = 2 \tag{0.0.18}$$

Let **P** be the intersection of the line (0.0.18) and the circle (0.0.1), **Q** be the intersection of the line (0.0.18) and the circle (0.0.2).

$$\mathbf{P} = \begin{pmatrix}x_1\\y_1\end{pmatrix}, \mathbf{Q} = \begin{pmatrix}x_2\\y_2\end{pmatrix} \tag{0.0.19}$$

$$2x_1 + y_1 = 2 \tag{0.0.20}$$

$$2x_2 + y_2 = 2 \tag{0.0.21}$$

To find the point $\mathbf{P}$,

$$y_1 = 2 - 2x_1 \quad (0.0.22)$$

$$x_1^2 + y_1^2 - 2\begin{pmatrix} 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - 4 = 0 \quad (0.0.23)$$

$$x_1^2 + y_1^2 - 4x_1 - 4 = 0 \quad (0.0.24)$$

$$x_1^2 + (2 - 2x_1)^2 - 4x_1 - 4 = 0 \quad (0.0.25)$$

$$5x_1^2 - 12x_1 = 0 \quad (0.0.26)$$

$$x_1 = 0, \frac{12}{5} \quad (0.0.27)$$

$$\mathbf{P} = \begin{pmatrix} \frac{12}{5} \\ -\frac{14}{5} \end{pmatrix} \quad (0.0.28)$$

To find the point $\mathbf{Q}$,

$$y_2 = 2 - 2x_2 \quad (0.0.29)$$

$$x_2^2 + y_2^2 - 2\begin{pmatrix} -2 & 0 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - 4 = 0 \quad (0.0.30)$$

$$x_2^2 + y_2^2 + 4x_2 - 4 = 0 \quad (0.0.31)$$

$$x_2^2 + (2 - 2x_2)^2 + 4x_2 - 4 = 0 \quad (0.0.32)$$

$$5x_2^2 - 4x_2 = 0 \quad (0.0.33)$$

$$x_2 = 0, \frac{4}{5} \quad (0.0.34)$$

$$\mathbf{Q} = \begin{pmatrix} \frac{4}{5} \\ \frac{2}{5} \end{pmatrix} \quad (0.0.35)$$

$$\|\mathbf{BP}\| = \left\| \begin{pmatrix} 0 \\ -2 \end{pmatrix} - \begin{pmatrix} \frac{12}{5} \\ -\frac{14}{5} \end{pmatrix} \right\| \quad (0.0.36)$$

$$= \left\| \begin{pmatrix} -\frac{12}{5} \\ \frac{4}{5} \end{pmatrix} \right\| \quad (0.0.37)$$

$$= \frac{4\sqrt{10}}{5} \quad (0.0.38)$$

$$\|\mathbf{BQ}\| = \left\| \begin{pmatrix} 0 \\ -2 \end{pmatrix} - \begin{pmatrix} \frac{4}{5} \\ \frac{2}{5} \end{pmatrix} \right\| \quad (0.0.39)$$

$$= \left\| \begin{pmatrix} -\frac{4}{5} \\ -\frac{12}{5} \end{pmatrix} \right\| \quad (0.0.40)$$

$$= \frac{4\sqrt{10}}{5} \quad (0.0.41)$$

Hence, $\mathbf{BP} = \mathbf{BQ}$.