Assignment 1

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1) Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$

Solution: The lines l_1 and l_2 in vector form can be written as

$$\mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{0.0.1}$$

$$\mathbf{x} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{0.0.2}$$

$$\mathbf{x_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{x_2} = \begin{pmatrix}2\\1\\-1\end{pmatrix}, \mathbf{m_1} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{m_2} = \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{0.0.3}$$

We first check whether the given lines are skew. The lines

$$\mathbf{x} = \mathbf{x_1} + \lambda_1\mathbf{m_1}, \mathbf{x} = \mathbf{x_2} + \lambda_2\mathbf{m_2} \tag{0.0.4}$$

intersect if

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x_2} - \mathbf{x_1} \tag{0.0.5}$$

$$\mathbf{M} \triangleq \begin{pmatrix}\mathbf{m_1} & \mathbf{m_2}\end{pmatrix} \tag{0.0.6}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} \tag{0.0.7}$$

$$\tag{0.0.8}$$

Here we have,

$$\mathbf{M} = \begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix} \tag{0.0.9}$$

$$\mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{0.0.10}$$

We check whether the equation (0.0.11) has a solution

$$\begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\boldsymbol{\lambda} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{0.0.11}$$

the augmented matrix is given by,

$$\left(\begin{array}{cc|c}2 & 3 & 1\\-1 & -5 & 0\\1 & 2 & -1\end{array}\right) \tag{0.0.12}$$

$$\xleftrightarrow[R_3\leftarrow R_3-\frac{1}{2}R_1]{R_2\leftarrow R_2+\frac{1}{2}R_1} \tag{0.0.13}$$

$$\left(\begin{array}{cc|c}2 & 3 & 1\\0 & -\frac{7}{2} & \frac{1}{2}\\0 & \frac{1}{2} & -\frac{3}{2}\end{array}\right) \tag{0.0.14}$$

$$\xleftrightarrow{R_3\leftarrow R_3+7R_2} \tag{0.0.15}$$

$$\left(\begin{array}{cc|c}2 & 3 & 1\\0 & -\frac{7}{2} & \frac{1}{2}\\0 & 0 & -10\end{array}\right) \tag{0.0.16}$$

The rank of the matrix is 3. So the given lines are skew. The closest points on two skew lines defined by (0.0.4) are given by

$$\mathbf{M}^\top\mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top\left(\mathbf{x_2} - \mathbf{x_1}\right) \tag{0.0.17}$$

$$\begin{pmatrix}2 & -1 & 1\\3 & -5 & 2\end{pmatrix}\begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix}\boldsymbol{\lambda} = \begin{pmatrix}2 & -1 & 1\\3 & -5 & 2\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{0.0.18}$$

$$\begin{pmatrix}6 & 13\\13 & 38\end{pmatrix}\boldsymbol{\lambda} = \begin{pmatrix}1\\1\end{pmatrix} \tag{0.0.19}$$

The augmented matrix of the above equation

(0.0.19) is given by,

$$\left(\begin{array}{cc|c} 6 & 13 & 1 \\ 13 & 38 & 1 \end{array}\right) \tag{0.0.20}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{13}{6} R_1} \tag{0.0.21}$$

$$\left(\begin{array}{cc|c} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{array}\right) \tag{0.0.22}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{78}{59} R_2} \tag{0.0.23}$$

$$\left(\begin{array}{cc|c} 6 & 0 & \frac{150}{59} \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{array}\right) \tag{0.0.24}$$

So, we get

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{0.0.25}$$

The closest points $\mathbf{A}$ on line l_1 and $\mathbf{B}$ on line l_2 are given by,

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{0.0.26}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{25}{59} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{0.0.27}$$

$$= \frac{1}{59} \begin{pmatrix} 109 \\ 34 \\ 25 \end{pmatrix} \tag{0.0.28}$$

$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{0.0.29}$$

$$= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \frac{7}{59} \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{0.0.30}$$

$$= \frac{1}{59} \begin{pmatrix} 139 \\ 24 \\ -45 \end{pmatrix} \tag{0.0.31}$$

The minimum distance between the lines is given by,

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \frac{1}{59} \begin{pmatrix} 30 \\ -10 \\ -70 \end{pmatrix} \right\| \tag{0.0.32}$$

$$= \frac{\sqrt{30^2 + 10^2 + 70^2}}{59} \tag{0.0.33}$$

$$= \frac{10}{\sqrt{59}} \tag{0.0.34}$$

The shortest distance between the given lines is $\frac{10}{\sqrt{59}}$ units.

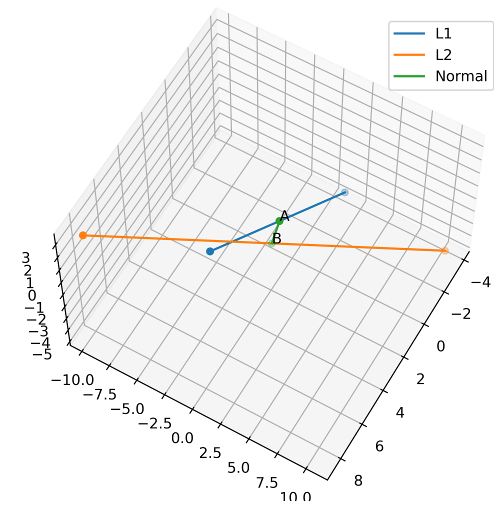

Fig. 1: AB is the required shortest distance.