Assignment 1

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1) If a line intersects two concentric circles (circles with the same centre) with centre $\mathbf{O}$ at $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, prove that $AB = CD$.

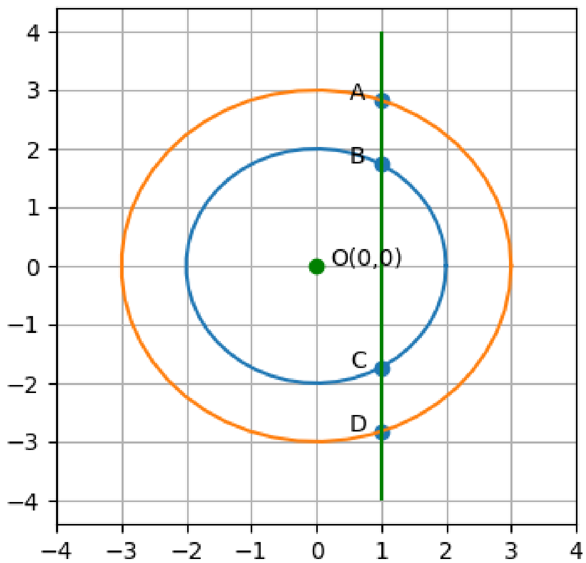

Fig. 1: Graph

Solution: Let the equations of two concentric circles be,

$$\|\mathbf{x}\|^2 = 4 \tag{0.0.1}$$

$$\|\mathbf{x}\|^2 = 9 \tag{0.0.2}$$

The line should intersect both the circles i.e., distance between the line and the center of circle should be less than the radius of smaller circle. Let the equation of the line be,

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{0.0.3}$$

The normal vector of this line is given by,

$$\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{m} \tag{0.0.4}$$

The equation of the line is given by,

$$\mathbf{n}^\top\mathbf{x} - \mathbf{n}^\top\mathbf{h} = 0 \tag{0.0.5}$$

The distance between origin and line is given by,

$$d = \frac{\left|\mathbf{n}^\top\mathbf{h}\right|}{\|\mathbf{n}\|} \tag{0.0.6}$$

The condition for intersection is

$$d < 2 \tag{0.0.7}$$

Consider

$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.8}$$

then we get the distance as,

$$\mathbf{n} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{0.0.9}$$

$$d = 1 \tag{0.0.10}$$

Hence, the taken parameters $\mathbf{h}, \mathbf{m}$ satisfies the reqiuired conditions.
The parameter μ of the points of intersection of line (0.0.11) with the conic section (0.0.12)

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{0.0.11}$$

$$g(\mathbf{x}) = \mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{0.0.12}$$

is given by the equation

$$\mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{0.0.13}$$

The points of intersection of circle (0.0.1) and the line (0.0.11) $\mathbf{B}, \mathbf{C}$ are given by,

$$\mathbf{V} = \mathbf{I}, \ \mathbf{u} = \mathbf{O}, \ f = -4 \tag{0.0.14}$$

$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.15}$$

Substituting the above expressions (0.0.14), (0.0.15) in (0.0.13), we get

$$\mathbf{m}^\top \mathbf{V}\mathbf{m} = 1 \tag{0.0.16}$$
$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{0.0.17}$$
$$g(\mathbf{h}) = -3 \tag{0.0.18}$$
$$\mu^2 - 3 = 0 \tag{0.0.19}$$
$$\mu = \pm\sqrt{3} \tag{0.0.20}$$
$$\mathbf{B} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{0.0.21}$$

The points of intersection of circle (0.0.2) and the line (0.0.11) $\mathbf{A}, \mathbf{D}$ are given by,

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = -9 \tag{0.0.22}$$
$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.23}$$

Substituting the above expressions (0.0.22), (0.0.23) in (0.0.13), we get

$$\mathbf{m}^\top \mathbf{V}\mathbf{m} = 1 \tag{0.0.24}$$
$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{0.0.25}$$
$$g(\mathbf{h}) = -8 \tag{0.0.26}$$
$$\mu^2 - 8 = 0 \tag{0.0.27}$$
$$\mu = \pm 2\sqrt{2} \tag{0.0.28}$$
$$\mathbf{A} = \begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 \\ -2\sqrt{2} \end{pmatrix} \tag{0.0.29}$$

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \begin{pmatrix} 0 \\ 2\sqrt{2} - \sqrt{3} \end{pmatrix} \right\| \tag{0.0.30}$$
$$= 2\sqrt{2} - \sqrt{3} \tag{0.0.31}$$
$$\|\mathbf{C} - \mathbf{D}\| = \left\| \begin{pmatrix} 0 \\ 2\sqrt{2} - \sqrt{3} \end{pmatrix} \right\| \tag{0.0.32}$$
$$= 2\sqrt{2} - \sqrt{3} \tag{0.0.33}$$

Hence $AB = CD$. The parameters used in the construction are shown in the below table 1

Parameter	**Description**	**Value**
$\mathbf{O}$	center of both circles	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
r_1	radius of smaller circle	2
r_2	radius of larger circle	3
$\mathbf{h}$	point on the line	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\mathbf{m}$	direction vector of the line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

TABLE 1