

Assignment 1

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- 1) If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.

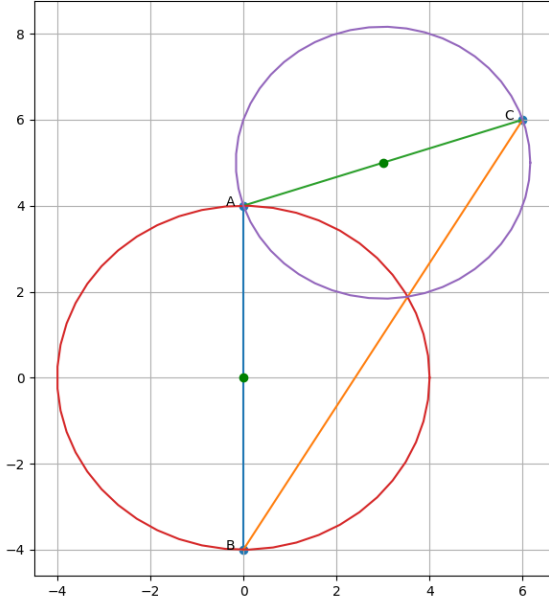


Fig. 1: Graph

Solution: Let the triangle be ABC , Points A, B, C are given by,

$$\mathbf{A} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \quad (0.0.1)$$

The equation of circle taking AB as diameter

is given by,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_1^\top \mathbf{x} + f_1 = 0 \quad (0.0.2)$$

$$\mathbf{u}_1 = -\left(\frac{\mathbf{A} + \mathbf{B}}{2}\right) \quad (0.0.3)$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (0.0.4)$$

$$r_1 = \frac{\|\mathbf{A} - \mathbf{B}\|}{2} \quad (0.0.5)$$

$$= 4 \quad (0.0.6)$$

$$f_1 = \|\mathbf{u}_1\|^2 - r_1^2 \quad (0.0.7)$$

$$= -4 \quad (0.0.8)$$

$$\|\mathbf{x}\|^2 - 16 = 0 \quad (0.0.9)$$

The equation of circle taking AC as diameter is given by,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}_2^\top \mathbf{x} + f_2 = 0 \quad (0.0.10)$$

$$\mathbf{u}_2 = -\left(\frac{\mathbf{A} + \mathbf{C}}{2}\right) \quad (0.0.11)$$

$$= -\begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (0.0.12)$$

$$r_2 = \frac{\|\mathbf{A} - \mathbf{C}\|}{2} \quad (0.0.13)$$

$$= \sqrt{10} \quad (0.0.14)$$

$$f_2 = \|\mathbf{u}_2\|^2 - r_2^2 \quad (0.0.15)$$

$$= 24 \quad (0.0.16)$$

$$\|\mathbf{x}\|^2 - 2\begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} + 24 = 0 \quad (0.0.17)$$

Let the other point of intersection of circles (0.0.9) and (0.0.17) be point \mathbf{P} . The equation of the common chord of intersection of two circles, AP is given by,

$$2\mathbf{u}_1^\top \mathbf{x} - 2\mathbf{u}_2^\top \mathbf{x} + f_1 - f_2 = 0 \quad (0.0.18)$$

$$2\begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} - 16 - 24 = 0 \quad (0.0.19)$$

$$\begin{pmatrix} 3 & 5 \end{pmatrix} \mathbf{x} = 20 \quad (0.0.20)$$

The equation (0.0.20) can be written in para-

metric form as,

$$\mathbf{h} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \quad (0.0.21)$$

$$\mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 3 \end{pmatrix} \quad (0.0.22)$$

The parameter μ of the points of intersection of line (0.0.23) with the conic section (0.0.24)

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \quad (0.0.23)$$

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (0.0.24)$$

is given by the equation

$$\mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \quad (0.0.25)$$

The points of intersection of the circle (0.0.9) and line (0.0.20) are the points **A**, **P**. Here we have,

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f = 16 \quad (0.0.26)$$

$$\mathbf{h} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} \quad (0.0.27)$$

$$\mathbf{m}^T \mathbf{V} \mathbf{m} = 34 \quad (0.0.28)$$

$$\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) = 12 \quad (0.0.29)$$

$$g(\mathbf{h}) = 0 \quad (0.0.30)$$

$$34\mu^2 + 24\mu = 0 \quad (0.0.31)$$

$$\mu = 0, -\frac{12}{17} \quad (0.0.32)$$

$\mu = 0$ corresponds to point **A**.

$$\mathbf{P} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \frac{12}{17} \begin{pmatrix} -5 \\ 3 \end{pmatrix} \quad (0.0.33)$$

$$\mathbf{P} = \begin{pmatrix} \frac{60}{17} \\ \frac{32}{17} \end{pmatrix} \quad (0.0.34)$$

The equation of the the line **BC** is given by,

$$\mathbf{m} = \mathbf{C} - \mathbf{B} \quad (0.0.35)$$

$$= \begin{pmatrix} 6 \\ 10 \end{pmatrix} \quad (0.0.36)$$

$$= \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad (0.0.37)$$

$$\mathbf{n} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \quad (0.0.38)$$

$$\mathbf{n}^T \mathbf{x} = \mathbf{n}^T \mathbf{B} \quad (0.0.39)$$

$$\begin{pmatrix} -5 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \end{pmatrix} \quad (0.0.40)$$

$$\begin{pmatrix} -5 & 3 \end{pmatrix} \mathbf{x} = -12 \quad (0.0.41)$$

$$\begin{pmatrix} -5 & 3 \end{pmatrix} \begin{pmatrix} \frac{60}{17} \\ \frac{32}{17} \end{pmatrix} = -12 \quad (0.0.42)$$

It is clear that the point **P** satisfies the equation of line **BC** (0.0.41). Hence, the point of intersection of the circles drawn by taking two sides of a triangle as diameters lies on the third side.

The parameters used in the construction are shown in the below table 1

Parameter	Description	Value
A	vertex of the triangle	$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$
B	vertex of the triangle	$\begin{pmatrix} 0 \\ -4 \end{pmatrix}$
C	vertex of the triangle	$(6, 6)$

TABLE 1