Assignment 1

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1) If a line intersects two concentric circles (circles with the same centre) with centre $\mathbf{O}$ at $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, prove that $AB = CD$.

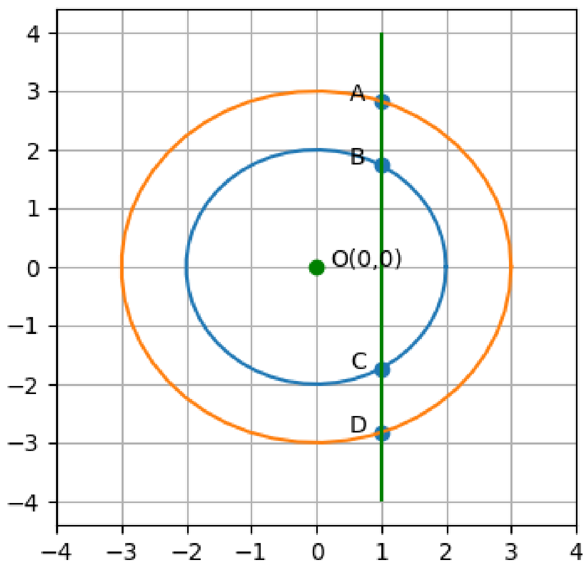

Fig. 1: Graph

Solution: Let the equations of two concentric circles be,

$$\|\mathbf{x}\|^2 = 4 \tag{0.0.1}$$

$$\|\mathbf{x}\|^2 = 9 \tag{0.0.2}$$

Let the equation of the line be,

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{0.0.3}$$

The parameter μ of the points of intersection of line (0.0.4) with the conic section (0.0.5)

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{0.0.4}$$

$$g(\mathbf{x}) = \mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{0.0.5}$$

is given by the equation

$$\mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{0.0.6}$$

For the line to intersect the conic at 2 points, the discriminant of the quadratic equation (0.0.6) should be greater than 0.

$$\Delta > 0 \tag{0.0.7}$$

Because the circles are concentric, if the line intersects the smaller circle then, we can say that the line satisfies the given conditions.

$$\mathbf{V} = \mathbf{I},\ \mathbf{u} = \mathbf{O},\ f = -4 \tag{0.0.8}$$

$$\mu^2\mathbf{m}^\top\mathbf{m} + 2\mu\mathbf{m}^\top\mathbf{h} + g(\mathbf{h}) = 0 \tag{0.0.9}$$

$$\mu^2\mathbf{m}^\top\mathbf{m} + 2\mu\mathbf{m}^\top\mathbf{h} + \mathbf{h}^\top\mathbf{h} - 4 = 0 \tag{0.0.10}$$

$$\tag{0.0.11}$$

Consider

$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.12}$$

$$\mathbf{m}^\top\mathbf{m} = 1 \tag{0.0.13}$$

$$2\mathbf{m}^\top\mathbf{h} = 0 \tag{0.0.14}$$

$$\mathbf{h}^\top\mathbf{h} - 4 = -3 \tag{0.0.15}$$

$$\Delta = 0^2 - 4(1)(-3) \tag{0.0.16}$$

$$\Delta > 0 \tag{0.0.17}$$

The points of intersection of circle (0.0.1) and the line (0.0.4) $\mathbf{B}, \mathbf{C}$ are given by, Substituting the above expressions (0.0.8), (0.0.11) in (0.0.6), we get

$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = 1 \tag{0.0.18}$$

$$\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{0.0.19}$$

$$g(\mathbf{h}) = -3 \tag{0.0.20}$$

$$\mu^2 - 3 = 0 \tag{0.0.21}$$

$$\mu = \pm\sqrt{3} \tag{0.0.22}$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},\ \mathbf{C} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{0.0.23}$$

The points of intersection of circle (0.0.2) and the line (0.0.4) $\mathbf{A}, \mathbf{D}$ are given by,

$$\mathbf{V} = \mathbf{I},\ \mathbf{u} = \mathbf{O},\ f = -9 \tag{0.0.24}$$

$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.25}$$

Substituting the above expressions (0.0.24), (0.0.25) in (0.0.6), we get

$$\mathbf{m}^\top \mathbf{V} \mathbf{m} = 1 \tag{0.0.26}$$

$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{0.0.27}$$

$$g(\mathbf{h}) = -8 \tag{0.0.28}$$

$$\mu^2 - 8 = 0 \tag{0.0.29}$$

$$\mu = \pm 2\sqrt{2} \tag{0.0.30}$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix},\ \mathbf{D} = \begin{pmatrix} 1 \\ -2\sqrt{2} \end{pmatrix} \tag{0.0.31}$$

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \begin{pmatrix} 0 \\ 2\sqrt{2} - \sqrt{3} \end{pmatrix} \right\| \tag{0.0.32}$$

$$= 2\sqrt{2} - \sqrt{3} \tag{0.0.33}$$

$$\|\mathbf{C} - \mathbf{D}\| = \left\| \begin{pmatrix} 0 \\ 2\sqrt{2} - \sqrt{3} \end{pmatrix} \right\| \tag{0.0.34}$$

$$= 2\sqrt{2} - \sqrt{3} \tag{0.0.35}$$

Hence $AB = CD$. The parameters used in the construction are shown in the below table 1

Parameter	**Description**	**Value**
$\mathbf{O}$	center of both circles	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
r_1	radius of smaller circle	2
r_2	radius of larger circle	3
$\mathbf{h}$	point on the line	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\mathbf{m}$	direction vector of the line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

TABLE 1