Assignment 1

Jaswanth Chowdary Madala

1) Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$
Solution: The lines l_1 and l_2 in vector form can be written as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{0.0.1}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{0.0.2}$$

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{0.0.3}$$

The distance between the lines is given by,

$$d = \|(\mathbf{x_2} + \lambda_2 \mathbf{m_2}) - (\mathbf{x_1} + \lambda_1 \mathbf{m_1})\| \tag{0.0.4}$$

$$\implies d = \|\mathbf{x_2} - \mathbf{x_1} - \lambda_1 \mathbf{m_1} + \lambda_2 \mathbf{m_2}\| \tag{0.0.5}$$

Consider the following definitions

$$\mathbf{A} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{0.0.6}$$

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{0.0.7}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{0.0.8}$$

From the above definitions (0.0.6), (0.0.7), (0.0.8) we get,

$$d = \|\mathbf{A} - \mathbf{M}\boldsymbol{\lambda}\| \tag{0.0.9}$$

Here we have the values of $\mathbf{A}, \mathbf{M}$ as

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{0.0.10}$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{0.0.11}$$

Consider the function

$$f(\boldsymbol{\lambda}) = \|\mathbf{A} - \mathbf{M}\boldsymbol{\lambda}\|^2 \tag{0.0.12}$$

The given problem can be formulated as

$$\min_{\boldsymbol{\lambda}} f(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^\top \mathbf{M}^\top \mathbf{M} \boldsymbol{\lambda} - 2\mathbf{A}^\top \mathbf{M} \boldsymbol{\lambda} + \|\mathbf{A}\|^2 \tag{0.0.13}$$

Here we have,

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{0.0.14}$$

$$\mathbf{A}^\top \mathbf{M} = \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{0.0.15}$$

$$\|\mathbf{A}\|^2 = 2 \tag{0.0.16}$$

As there are no constraints on $\boldsymbol{\lambda}$, From Lagrange Multipliers the solution for (0.0.13) is obtained from the following equation

$$\nabla f(\boldsymbol{\lambda}) = 0 \tag{0.0.17}$$

Here we have the gradient as

$$\nabla f(\boldsymbol{\lambda}) = 2\mathbf{M}^\top \mathbf{M} \boldsymbol{\lambda} - 2\left(\mathbf{A}^\top \mathbf{M}\right)^\top = 0 \tag{0.0.18}$$

$$\implies \mathbf{M}^\top \mathbf{M} \boldsymbol{\lambda} = \mathbf{M}^\top \mathbf{A} \tag{0.0.19}$$

Solving (0.0.19) we get

$$\boldsymbol{\lambda} = \begin{pmatrix} 0.4237 \\ -0.1186 \end{pmatrix} \tag{0.0.20}$$

The shortest distance is

$$\min_{\boldsymbol{\lambda}} d = 1.3019 \tag{0.0.21}$$

The shortest distance between the given lines is 1.3019 units, and this answer matches with the other methods.

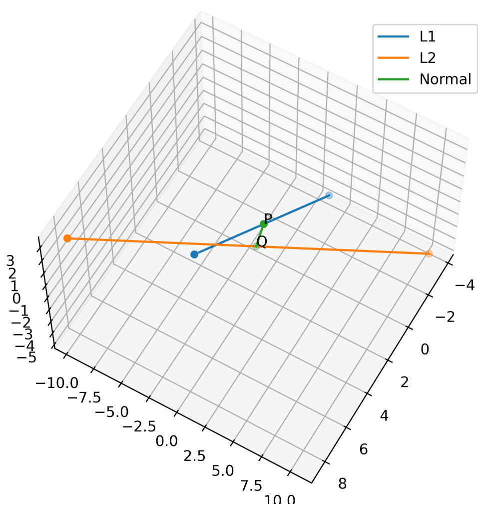

Fig. 1: PQ is the required shortest distance.