Assignment 1

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1) Find the equation of the ellipse that satisfies the conditions - Centre at $(0,0)$, major axis on the y-axis and passes through the points $(3,2)$ and $(1,6)$.
Solution: The equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top\mathbf{x} = c$ and eccentricity e is given by

$$\mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{0.0.1}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2\mathbf{n}\mathbf{n}^\top \tag{0.0.2}$$
$$\mathbf{u} \triangleq ce^2\mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{0.0.3}$$
$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2e^2 \tag{0.0.4}$$

Given that the conic is an ellipse with major axis along the y-axis, we get

$$\mathbf{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{0.0.5}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1-e^2 \end{pmatrix} \tag{0.0.6}$$
$$\mathbf{u} = ce^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \mathbf{F} \tag{0.0.7}$$
$$f = \|\mathbf{F}\|^2 - c^2e^2 \tag{0.0.8}$$

The centre of the conic is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{0.0.9}$$

Since $\mathbf{c} = \mathbf{0}$ and $\mathbf{V}^{-1} \neq \mathbf{0}$, it follows from (0.0.9) that

$$\mathbf{u} = \mathbf{0} \tag{0.0.10}$$

Thus, from (0.0.7),

$$\mathbf{F} = \begin{pmatrix} 0 \\ ce^2 \end{pmatrix} \tag{0.0.11}$$

and so,

$$f = c^2e^2\left(e^2 - 1\right) \tag{0.0.12}$$

Given that the conic passes throught point,

$$\mathbf{P} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{0.0.13}$$

Putting $\mathbf{x} = \mathbf{P}$ in (0.0.1) we get,

$$\begin{pmatrix} 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1-e^2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + f = 0 \tag{0.0.14}$$
$$\implies 4e^2 - f = 13 \tag{0.0.15}$$

Given that the conic passes throught point,

$$\mathbf{Q} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \tag{0.0.16}$$

Putting $\mathbf{x} = \mathbf{Q}$ in (0.0.1), we get

$$\begin{pmatrix} 1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1-e^2 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} + f = 0 \tag{0.0.17}$$
$$\implies 36e^2 - f = 37 \tag{0.0.18}$$

The equations (0.0.15) and (0.0.18) can be formulated as the following matrix equation

$$\begin{pmatrix} 4 & -1 \\ 36 & -1 \end{pmatrix}\begin{pmatrix} e^2 \\ f \end{pmatrix} = \begin{pmatrix} 13 \\ 37 \end{pmatrix} \tag{0.0.19}$$

The augmented matrix is given by,

$$\left(\begin{array}{cc|c} 4 & -1 & 13 \\ 36 & -1 & 37 \end{array}\right) \tag{0.0.20}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{cc|c} -32 & 0 & -24 \\ 36 & -1 & 37 \end{array}\right) \tag{0.0.21}$$
$$\xleftrightarrow{R_1 \leftarrow -\frac{R_1}{8}} \left(\begin{array}{cc|c} 4 & 0 & 3 \\ 36 & -1 & 37 \end{array}\right) \tag{0.0.22}$$
$$\xleftrightarrow{R_2 \leftarrow R_2 - 9R_1} \left(\begin{array}{cc|c} 4 & 0 & 3 \\ 0 & -1 & 10 \end{array}\right) \tag{0.0.23}$$
$$\xleftrightarrow[R_2 \leftarrow -R_2]{R_1 \leftarrow \frac{R_1}{4}} \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -10 \end{array}\right) \tag{0.0.24}$$

Thus,

$$e^2 = \frac{3}{4}, \ f = -10 \tag{0.0.25}$$

The equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \mathbf{x} - 10 = 0 \qquad (0.0.26)$$

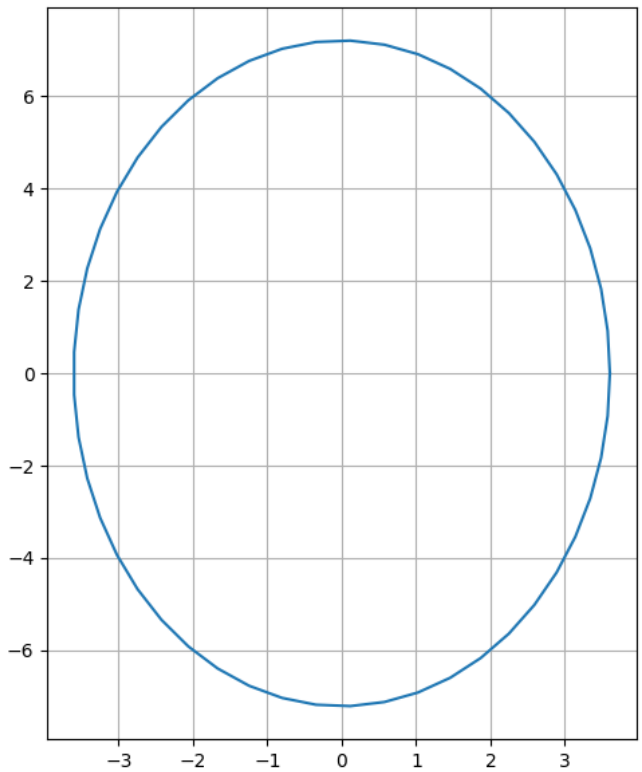

Fig. 1: Graph

Parameter	Description	Value
$\mathbf{C}$	center of the ellipse	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\mathbf{m}$	Direction vector of major axis	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\mathbf{P}$	Point on the ellipse	$\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
$\mathbf{Q}$	Point on the ellipse	$\begin{pmatrix} 1 \\ 6 \end{pmatrix}$

TABLE 1