

Assignment 1

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- 1) If a line intersects two concentric circles (circles with the same centre) with centre **O** at **A**, **B**, **C** and **D**, prove that $AB = CD$.

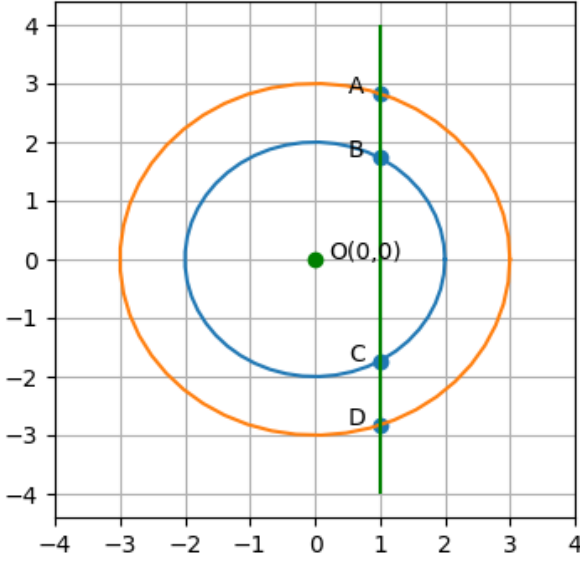


Fig. 1: Graph

Solution: Let the equations of two concentric circles be,

$$\|\mathbf{x}\|^2 = 4 \quad (0.0.1)$$

$$\|\mathbf{x}\|^2 = 9 \quad (0.0.2)$$

Let the equation of the line be,

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \quad (0.0.3)$$

The parameter μ of the points of intersection of line (0.0.4) with the conic section (0.0.5)

$$\mathbf{x} = \mathbf{h} + \mu \mathbf{m} \quad (0.0.4)$$

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (0.0.5)$$

is given by the equation

$$\mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + 2\mu \mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \quad (0.0.6)$$

For the line to intersect the conic at 2 points, the discriminant of the quadratic equation (0.0.6) should be greater than 0.

$$\Delta > 0 \quad (0.0.7)$$

$$\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u})^2 - g(\mathbf{h}) (\mathbf{m}^T \mathbf{V} \mathbf{m}) > 0 \quad (0.0.8)$$

Because the circles are concentric, if the line intersects the smaller circle then, we can say that the line satisfies the given conditions.

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = -4 \quad (0.0.9)$$

$$\mu^2 \mathbf{m}^T \mathbf{m} + 2\mu \mathbf{m}^T \mathbf{h} + g(\mathbf{h}) = 0 \quad (0.0.10)$$

$$\mu^2 \mathbf{m}^T \mathbf{m} + 2\mu \mathbf{m}^T \mathbf{h} + \mathbf{h}^T \mathbf{h} - 4 = 0 \quad (0.0.11)$$

$$(0.0.12)$$

Consider

$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.0.13)$$

$$\mathbf{m}^T \mathbf{m} = 1 \quad (0.0.14)$$

$$2\mathbf{m}^T \mathbf{h} = 0 \quad (0.0.15)$$

$$\mathbf{h}^T \mathbf{h} - 4 = -3 \quad (0.0.16)$$

$$\Delta = 0^2 - 4(1)(-3) \quad (0.0.17)$$

$$\Delta > 0 \quad (0.0.18)$$

The points of intersection of circle (0.0.1) and the line (0.0.4) **B**, **C** are given by, Substituting the above expressions (0.0.9), (0.0.12) in (0.0.6), we get

$$\mathbf{m}^T \mathbf{V} \mathbf{m} = 1 \quad (0.0.19)$$

$$\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) = 0 \quad (0.0.20)$$

$$g(\mathbf{h}) = -3 \quad (0.0.21)$$

$$\mu^2 - 3 = 0 \quad (0.0.22)$$

$$\mu = \pm \sqrt{3} \quad (0.0.23)$$

$$\mathbf{B} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (0.0.24)$$

The points of intersection of circle (0.0.2) and

the line (0.0.4) \mathbf{A}, \mathbf{D} are given by,

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = -9 \quad (0.0.25)$$

$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.0.26)$$

Substituting the above expressions (0.0.25), (0.0.26) in (0.0.6), we get

$$\mathbf{m}^T \mathbf{V} \mathbf{m} = 1 \quad (0.0.27)$$

$$\mathbf{m}^T (\mathbf{V} \mathbf{h} + \mathbf{u}) = 0 \quad (0.0.28)$$

$$g(\mathbf{h}) = -8 \quad (0.0.29)$$

$$\mu^2 - 8 = 0 \quad (0.0.30)$$

$$\mu = \pm 2\sqrt{2} \quad (0.0.31)$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 \\ -2\sqrt{2} \end{pmatrix} \quad (0.0.32)$$

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \begin{pmatrix} 0 \\ 2\sqrt{2} - \sqrt{3} \end{pmatrix} \right\| \quad (0.0.33)$$

$$= 2\sqrt{2} - \sqrt{3} \quad (0.0.34)$$

$$\|\mathbf{C} - \mathbf{D}\| = \left\| \begin{pmatrix} 0 \\ 2\sqrt{2} - \sqrt{3} \end{pmatrix} \right\| \quad (0.0.35)$$

$$= 2\sqrt{2} - \sqrt{3} \quad (0.0.36)$$

Hence $AB = CD$. The parameters used in the construction are shown in the below table 1

Parameter	Description	Value
\mathbf{O}	center of both circles	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
r_1	radius of smaller circle	2
r_2	radius of larger circle	3
\mathbf{h}	point on the line	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
\mathbf{m}	direction vector of the line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

TABLE 1