Assignment 1

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1) A coin is tossed three times, where. Determine $\Pr(E \mid F)$ where
   a) E : head on third toss, F : heads on first two tosses
   b) E : at least two heads, F : at most two heads
   c) E : at most two tails, F : at least one tail

   Solution: Consider the random variables X_1, X_2, X_3, X, which denotes the first, second, third toss and number of heads in the 3 tosses respectively as described in table 1.

RV	Values	Description
X	$\{0,1,2,3\}$	Number of heads in 3 tosses
X_1	$\{0,1\}$	0: Heads , 1: Tails
X_2	$\{0,1\}$	0: Heads , 1: Tails
X_3	$\{0,1\}$	0: Heads , 1: Tails

TABLE 1: Random variables X_1, X_2, X_3, X

The random variable X follows binomial distribution

$$X = X_1 + X_2 + X_3 \tag{0.0.1}$$

The PMF of the random variable X is given by,

$$P_X(n) = {}^N C_n \, p^n (1-p)^{N-n} \tag{0.0.2}$$

Here we have

$$N = 3,\ p = \frac{1}{2} \tag{0.0.3}$$

The CDF of the random variable X is given by,

$$F_X(n) = \Pr(X \le n) = \sum_{i=0}^{n} {}^N C_i \, p^i (1-p)^{N-i} \tag{0.0.4}$$

a) The events E, F can be described by the RV as

$$E : X_3 = 0 \tag{0.0.5}$$
$$F : X_1 + X_2 = 0 \tag{0.0.6}$$

Y is another random variable which represents the number of heads in first two tosses.

$$Y = X_1 + X_2 \tag{0.0.7}$$

The PMF of the random variable Y is given by,

$$P_Y(n) = {}^N C_n \, p^n (1-p)^{N-n} \tag{0.0.8}$$

Here we have

$$N = 2,\ p = \frac{1}{2} \tag{0.0.9}$$

The event EF can be expressed as,

$$X_3 = 0 \cap X_1 + X_2 = 0 \tag{0.0.10}$$
$$\triangleq X_1 + X_2 + X_3 = 0 \tag{0.0.11}$$
$$\implies X = 0 \tag{0.0.12}$$

The required probability is given by,

$$\Pr(X_3 = 0 \mid Y = 0) \tag{0.0.13}$$
$$= \frac{\Pr(X = 0)}{\Pr(Y = 0)} \tag{0.0.14}$$
$$= \frac{1}{2} \tag{0.0.15}$$

b) The events E, F, F' can be described by the RV as

$$E : X \le 1 \tag{0.0.16}$$
$$F : X \ge 1 \tag{0.0.17}$$
$$F' : X = 0 \tag{0.0.18}$$

The required probability is given by,

$$= \frac{\Pr(EF)}{1 - \Pr(F')} \tag{0.0.19}$$

The event EF can be expressed as,

$$X \le 1 \cap X \ge 1 \tag{0.0.20}$$
$$\implies X = 1 \tag{0.0.21}$$

Hence, the required probability is given by,

$$= \frac{\Pr(X = 1)}{1 - \Pr(X = 0)} \tag{0.0.22}$$

$$= \frac{\frac{3}{8}}{1 - \frac{1}{8}} \tag{0.0.23}$$

$$= \frac{3}{7} \tag{0.0.24}$$

c) For the events E, F, their complements are E' : all 3 tails, F' : zero tails. The events E', F' can be described by the RV as

$$E' : X = 3 \tag{0.0.25}$$

$$F' : X = 0 \tag{0.0.26}$$

By using property of conditional probability we have,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} \tag{0.0.27}$$

$$= \frac{1 - \Pr(E' + F')}{\Pr(F)} \tag{0.0.28}$$

The required probability is given by,

$$= \frac{1 - \Pr(X = 0 + X = 3)}{1 - \Pr(X = 0)} \tag{0.0.29}$$

$$= \frac{1 - (\Pr(X = 0) + \Pr(X = 3))}{1 - \Pr(X = 0)} \tag{0.0.30}$$

$$= \frac{1 - \left(\frac{1}{8} + \frac{1}{8}\right)}{1 - \frac{1}{8}} \tag{0.0.31}$$

$$= \frac{6}{7} \tag{0.0.32}$$