Assignment 1

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1) Two congruent circles intersect each other at points **A** and **B**. Through **A** any line segment **PAQ** is drawn so that **P**, **Q** lie on the two circles. Prove that $BP = BQ$.

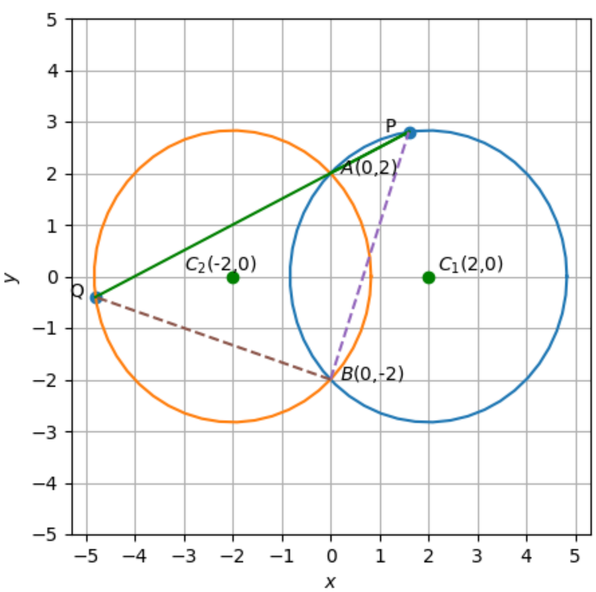

Fig. 1: Graph

Solution: Consider two congruent circles with radius $2\sqrt{2}$ with centres at $\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}-2\\0\end{pmatrix}$. The equations of these circles is given by,

$$\|\mathbf{x}\|^2 + 2\mathbf{u_1}^\top\mathbf{x} + f_1 = 0 \tag{0.0.1}$$

$$\|\mathbf{x}\|^2 + 2\mathbf{u_2}^\top\mathbf{x} + f_2 = 0 \tag{0.0.2}$$

$$\mathbf{u_1} = -\begin{pmatrix}2\\0\end{pmatrix}, \mathbf{u_2} = -\begin{pmatrix}-2\\0\end{pmatrix} \tag{0.0.3}$$

$$f_1 = -4,\ f_2 = -4 \tag{0.0.4}$$

To get points of intersection **A**, **B**, the common chord of the circles is given by,

$$2\mathbf{u_1}^\top\mathbf{x} - 2\mathbf{u_2}^\top\mathbf{x} + f_1 - f_2 = 0 \tag{0.0.5}$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\mathbf{x} = 0 \tag{0.0.6}$$

The equation (0.0.6) can be written in parametric form as,

$$\mathbf{h} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{0.0.7}$$

$$\mathbf{x} = \begin{pmatrix}0\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\end{pmatrix} \tag{0.0.8}$$

The parameter μ of the points of intersection of line (0.0.9) with the conic section (0.0.10)

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \tag{0.0.9}$$

$$g(\mathbf{x}) = \mathbf{x}^\top\mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \tag{0.0.10}$$

is given by the equation

$$\mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{0.0.11}$$

For the line to intersect the conic at 2 points, the discriminant of the quadratic equation (0.0.10) should be greater than 0.

$$\Delta > 0 \tag{0.0.12}$$

$$\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - g(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right) > 0 \tag{0.0.13}$$

To find the points of intersection **A**, **B**, we find the intersection of the circle (0.0.1), common chord (0.0.8)

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix}-2\\0\end{pmatrix}, f = -4 \tag{0.0.14}$$

$$\mathbf{h} = \begin{pmatrix}0\\0\end{pmatrix}, \mathbf{m} = \begin{pmatrix}0\\1\end{pmatrix} \tag{0.0.15}$$

$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = 1 \tag{0.0.16}$$

$$\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \tag{0.0.17}$$

$$g(\mathbf{h}) = -4 \tag{0.0.18}$$

Checking whether two circles intersect each other gives,

$$\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - g(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right) = 0^2 - (-4) \times 1 \tag{0.0.19}$$

$$= 4 \tag{0.0.20}$$

Hence, the two circles taken intersect each other.

$$\mu^2 - 4 = 0 \tag{0.0.21}$$
$$\mu = \pm 2 \tag{0.0.22}$$

From the equation (0.0.8) the points $\mathbf{A}, \mathbf{B}$ are given by,

$$\mathbf{A} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{0.0.23}$$
$$\mathbf{B} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{0.0.24}$$

Let us consider the direction vector, $\mathbf{m}$ of the line passing through the point $\mathbf{A}$ to be,

$$\mathbf{m} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{0.0.25}$$

The equation of the line passing through the point $\mathbf{A}$ with the direction vector (0.0.25) in parametric form is given by,

$$\mathbf{x} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{0.0.26}$$

Let $\mathbf{P}$ be the intersection of the line (0.0.26) and the circle (0.0.1), $\mathbf{Q}$ be the intersection of the line (0.0.26) and the circle (0.0.2).
To find the point $\mathbf{P}$,

$$\mathbf{V} = \mathbf{I},\ \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix},\ f = -4 \tag{0.0.27}$$
$$\mathbf{h} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\ \mathbf{m} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{0.0.28}$$
$$\mathbf{m}^\top \mathbf{V} \mathbf{m} = 5 \tag{0.0.29}$$
$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = -2 \tag{0.0.30}$$
$$g(\mathbf{h}) = 0 \tag{0.0.31}$$

Checking whether the line (0.0.26) and circle (0.0.1) intersect each other gives,

$$\left(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - g(\mathbf{h})\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) = (-2)^2 - (0) \times 5 \tag{0.0.32}$$
$$= 4 \tag{0.0.33}$$

Hence, the line (0.0.26) and circle (0.0.1) intersect each other.

$$5\mu^2 - 4\mu = 0 \tag{0.0.34}$$
$$\mu = 0, \frac{4}{5} \tag{0.0.35}$$
$$\mathbf{P} = \begin{pmatrix} \frac{8}{5} \\ \frac{14}{5} \end{pmatrix} \tag{0.0.36}$$

To find the point $\mathbf{Q}$,

$$\mathbf{V} = \mathbf{I},\ \mathbf{u} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},\ f = -4 \tag{0.0.37}$$
$$\mathbf{h} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},\ \mathbf{m} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{0.0.38}$$
$$\mathbf{m}^\top \mathbf{V} \mathbf{m} = 5 \tag{0.0.39}$$
$$\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u}) = 6 \tag{0.0.40}$$
$$g(\mathbf{h}) = 0 \tag{0.0.41}$$

Checking whether the line (0.0.26) and circle (0.0.2) intersect each other gives,

$$\left(\mathbf{m}^\top (\mathbf{V}\mathbf{h} + \mathbf{u})\right)^2 - g(\mathbf{h})\left(\mathbf{m}^\top \mathbf{V}\mathbf{m}\right) = 6^2 - (0) \times 5 \tag{0.0.42}$$
$$= 36 \tag{0.0.43}$$

Hence, the line (0.0.26) and circle (0.0.1) intersect each other.

$$5\mu^2 + 12\mu = 0 \tag{0.0.44}$$
$$\mu = 0, -\frac{12}{5} \tag{0.0.45}$$
$$\mathbf{Q} = \begin{pmatrix} -\frac{24}{5} \\ -\frac{2}{5} \end{pmatrix} \tag{0.0.46}$$
$$\|\mathbf{B} - \mathbf{P}\| = \left\| \begin{pmatrix} 0 \\ -2 \end{pmatrix} - \begin{pmatrix} \frac{8}{5} \\ \frac{14}{5} \end{pmatrix} \right\| \tag{0.0.47}$$
$$= \left\| \begin{pmatrix} -\frac{8}{5} \\ -\frac{24}{5} \end{pmatrix} \right\| \tag{0.0.48}$$
$$= \frac{8\sqrt{10}}{5} \tag{0.0.49}$$
$$\|\mathbf{B} - \mathbf{Q}\| = \left\| \begin{pmatrix} 0 \\ -2 \end{pmatrix} - \begin{pmatrix} -\frac{24}{5} \\ -\frac{2}{5} \end{pmatrix} \right\| \tag{0.0.50}$$
$$= \left\| \begin{pmatrix} \frac{24}{5} \\ -\frac{8}{5} \end{pmatrix} \right\| \tag{0.0.51}$$
$$= \frac{8\sqrt{10}}{5} \tag{0.0.52}$$

Hence, $BP = BQ$.