Assignment 1

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1) If a line intersects two concentric circles (circles with the same centre) with centre **O** at **A**, **B**, **C** and **D**, prove that $AB = CD$.
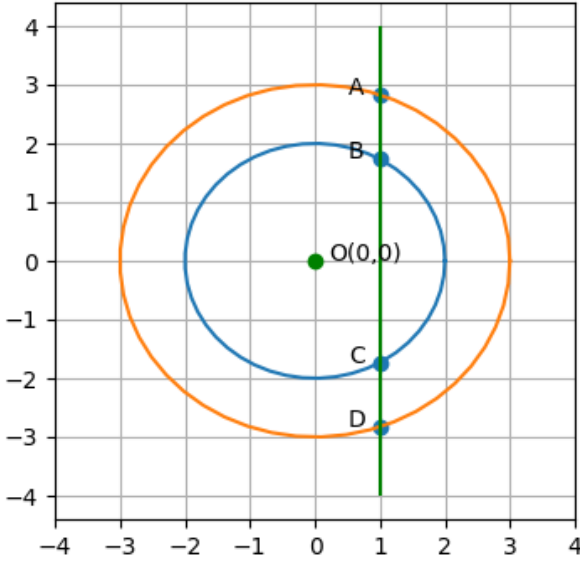

Fig. 1: Graph

Solution: Let the equations of two concentric circles be,

$$\|\mathbf{x}\|^2 = 4 \quad (0.0.1)$$
$$\|\mathbf{x}\|^2 = 9 \quad (0.0.2)$$

The parameter μ of the points of intersection of line (0.0.3) with the conic section (0.0.4)

$$\mathbf{x} = \mathbf{h} + \mu\mathbf{m} \quad (0.0.3)$$
$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} + f = 0 \quad (0.0.4)$$

is given by the equation

$$\mu^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2\mu\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \quad (0.0.5)$$

For the line to intersect the conic at 2 points, the discriminant of the quadratic equation

(0.0.9) should be greater than 0.

$$\Delta > 0 \quad (0.0.6)$$
$$(\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}))^2 - g(\mathbf{h})(\mathbf{m}^\top\mathbf{V}\mathbf{m}) > 0 \quad (0.0.7)$$

Because the circles are concentric, if the line intersects the smaller circle then, we can say that the line satisfies the given conditions.

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = -4 \quad (0.0.8)$$

Consider

$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.0.9)$$
$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = 1 \quad (0.0.10)$$
$$\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \quad (0.0.11)$$
$$g(\mathbf{h}) = -3 \quad (0.0.12)$$
$$\Delta = 0^2 - 4(1)(-3) \quad (0.0.13)$$
$$\Delta > 0 \quad (0.0.14)$$

The points of intersection of circle (0.0.1) and the line (0.0.3) with parameters given by (0.0.9) **B**, **C** are given by,

$$\mu^2 - 3 = 0 \quad (0.0.15)$$
$$\mu = \pm\sqrt{3} \quad (0.0.16)$$
$$\mathbf{B} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (0.0.17)$$

The points of intersection of circle (0.0.2) and the line (0.0.3) with parameters given by (0.0.9) **A**, **D** are given by,

$$\mathbf{V} = \mathbf{I}, \mathbf{u} = \mathbf{O}, f = -9 \quad (0.0.18)$$
$$\mathbf{h} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0.0.19)$$
$$\mathbf{m}^\top\mathbf{V}\mathbf{m} = 1 \quad (0.0.20)$$
$$\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) = 0 \quad (0.0.21)$$
$$g(\mathbf{h}) = -8\mu^2 - 8 \quad = 0 \quad (0.0.22)$$
$$\mu = \pm 2\sqrt{2} \quad (0.0.23)$$

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2\sqrt{2} \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 \\ -2\sqrt{2} \end{pmatrix} \tag{0.0.24}$$

$$\|\mathbf{A} - \mathbf{B}\| = \left\| \begin{pmatrix} 0 \\ 2\sqrt{2} - \sqrt{3} \end{pmatrix} \right\| \tag{0.0.25}$$

$$= 2\sqrt{2} - \sqrt{3} \tag{0.0.26}$$

$$\|\mathbf{C} - \mathbf{D}\| = \left\| \begin{pmatrix} 0 \\ 2\sqrt{2} - \sqrt{3} \end{pmatrix} \right\| \tag{0.0.27}$$

$$= 2\sqrt{2} - \sqrt{3} \tag{0.0.28}$$

Hence $AB = CD$. The parameters used in the construction are shown in the below table 1

Parameter	Description	Value
$\mathbf{O}$	center of both circles	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
r_1	radius of smaller circle	2
r_2	radius of larger circle	3
$\mathbf{h}$	point on the line	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\mathbf{m}$	direction vector of the line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

TABLE 1