Assignment 1

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1) Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$

Solution: The shortest distance between the lines whose vector equations are

$$L_1 : \mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{0.0.1}$$

$$L_2 : \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{0.0.2}$$

is given by,

$$d = \left\| \left(\mathbf{U} \left(\Sigma \Sigma^{-1} \right) \mathbf{U}^\top - \mathbf{I} \right) \mathbf{x} \right\| \tag{0.0.3}$$

with the parameter λ given by

$$\lambda = \mathbf{V} \Sigma^{-1} \mathbf{U}^\top \mathbf{x} \tag{0.0.4}$$

where

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{0.0.5}$$

$$\lambda \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{0.0.6}$$

$$\mathbf{x} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{0.0.7}$$

We use singular value decomposition of the matrix $\mathbf{M}$

$$\mathbf{M} = \mathbf{U} \Sigma \mathbf{V}^\top \tag{0.0.8}$$

where $\mathbf{U}, \mathbf{V}$ are orthogonal and Σ is diagonal with nonnegative diagonal entries.

a) In this problem we have the lines l_1 and l_2 as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \tag{0.0.9}$$

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{0.0.10}$$

We first need to check whether the given lines are skew. The lines (0.0.1), (0.0.2)

intersect if

$$\mathbf{M}\lambda = \mathbf{x_2} - \mathbf{x_1} \tag{0.0.11}$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{0.0.12}$$

$$\mathbf{x} = \mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{0.0.13}$$

We check whether the equation (0.0.14) has a solution

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \lambda = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{0.0.14}$$

the augmented matrix is given by,

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{array} \right) \tag{0.0.15}$$

$$\xrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{array} \right) \tag{0.0.16}$$

$$\xrightarrow{R_3 \leftarrow R_3 + 7R_2} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -10 \end{array} \right) \tag{0.0.17}$$

The rank of the matrix is 3. So the given lines are skew.

b) From (0.0.12) we have

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \tag{0.0.18}$$

$$= \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \tag{0.0.19}$$

$$\mathbf{M}\mathbf{M}^{\top} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \quad (0.0.20)$$

$$= \begin{pmatrix} 13 & -17 & 8 \\ -17 & 26 & -11 \\ 8 & -11 & 5 \end{pmatrix} \quad (0.0.21)$$

We perform the eigen decompositions for the matrics (0.0.21), (0.0.19) and write them in the form

$$\mathbf{M}\mathbf{M}^{\top} = \mathbf{P_1}\mathbf{D_1}\mathbf{P_1}^{\top} \quad (0.0.22)$$

$$\mathbf{M}^{\top}\mathbf{M} = \mathbf{P_2}\mathbf{D_2}\mathbf{P_2}^{\top} \quad (0.0.23)$$

The characteristic polynomial of the matrix $\mathbf{M}\mathbf{M}^{\top}$ is given by,

$$\text{char}\,(\mathbf{M}\mathbf{M}^{\top}) = \begin{vmatrix} 13-x & -17 & 8 \\ -17 & 26-x & -11 \\ 8 & -11 & 5-x \end{vmatrix}$$
$$(0.0.24)$$

$$= -x^3 + 44x^2 - 59x \quad (0.0.25)$$

Thus, the eigenvalues are given by

$$\lambda_1 = 22 + 5\sqrt{17}, \ \lambda_2 = 22 - 5\sqrt{17}, \ \lambda_3 = 0$$
$$(0.0.26)$$

From the augmented matrix formed from the eigen value - eigen vector equation we get, the normalized eigen vectors as

$$\mathbf{p_1} = \frac{\sqrt{5}}{\sqrt{68 - 6\sqrt{17}}} \begin{pmatrix} \frac{12-\sqrt{17}}{5} \\ \frac{1-3\sqrt{17}}{5} \\ 1 \end{pmatrix} \quad (0.0.27)$$

$$\mathbf{p_2} = \frac{\sqrt{5}}{\sqrt{68 + 6\sqrt{17}}} \begin{pmatrix} \frac{12+\sqrt{17}}{5} \\ \frac{1+3\sqrt{17}}{5} \\ 1 \end{pmatrix} \quad (0.0.28)$$

$$\mathbf{p_3} = \frac{1}{\sqrt{59}} \begin{pmatrix} -3 \\ 1 \\ 7 \end{pmatrix} \quad (0.0.29)$$

where $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ corresponds to the eigen values $\lambda_1, \lambda_2, \lambda_3$ respectively. Using (0.0.22), we get

$$\mathbf{P_1} = \begin{pmatrix} \frac{12-\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{12+\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & -\frac{3}{\sqrt{59}} \\ \frac{1-3\sqrt{17}}{\sqrt{5}\sqrt{68-6\sqrt{17}}} & \frac{1+3\sqrt{17}}{\sqrt{5}\sqrt{68+6\sqrt{17}}} & \frac{1}{\sqrt{59}} \\ \frac{\sqrt{5}}{\sqrt{68-6\sqrt{17}}} & \frac{\sqrt{5}}{\sqrt{68+6\sqrt{17}}} & \frac{7}{\sqrt{59}} \end{pmatrix}$$
$$(0.0.30)$$

$$\mathbf{D_1} = \begin{pmatrix} 22 + 5\sqrt{17} & 0 & 0 \\ 0 & 22 - 5\sqrt{17} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$(0.0.31)$$

For $\mathbf{M}^{\top}\mathbf{M}$, the characteristic polynomial is

$$\text{char}\,(\mathbf{M}^{\top}\mathbf{M}) = \begin{vmatrix} 6-x & 13 \\ 13 & 38-x \end{vmatrix} \quad (0.0.32)$$

$$= x^2 - 44x + 59 \quad (0.0.33)$$

Thus, the eigenvalues are given by

$$\lambda_1 = 22 + 5\sqrt{17}, \ \lambda_2 = 22 - 5\sqrt{17}$$
$$(0.0.34)$$

From the augmented matrix formed from the eigen value - eigen vector equation we get, the normalized eigen vectors as

$$\mathbf{p_1} = \frac{13}{\sqrt{850 - 160\sqrt{17}}} \begin{pmatrix} \frac{-16+5\sqrt{17}}{13} \\ 1 \end{pmatrix} \quad (0.0.35)$$

$$\mathbf{p_2} = \frac{13}{\sqrt{850 + 160\sqrt{17}}} \begin{pmatrix} \frac{-16-5\sqrt{17}}{13} \\ 1 \end{pmatrix} \quad (0.0.36)$$

where $\mathbf{p_1}, \mathbf{p_2}$ corresponds to the eigen values λ_1, λ_2 respectively. Using (0.0.23), we get

$$\mathbf{P_2} = \begin{pmatrix} \frac{-16-5\sqrt{17}}{\sqrt{850+160\sqrt{17}}} & \frac{13}{\sqrt{850-160\sqrt{17}}} \\ \frac{13}{\sqrt{850+160\sqrt{17}}} & \frac{-16+5\sqrt{17}}{\sqrt{850-160\sqrt{17}}} \end{pmatrix} \quad (0.0.37)$$

$$\mathbf{D_2} = \begin{pmatrix} 22 - 5\sqrt{17} & 0 \\ 0 & 22 + 5\sqrt{17} \end{pmatrix} \quad (0.0.38)$$

Therefore, from (0.0.8) we have

$$\mathbf{U} = \mathbf{P_1} \quad (0.0.39)$$

$$\mathbf{V} = \mathbf{P_2} \quad (0.0.40)$$

$$\mathbf{\Sigma} = \begin{pmatrix} \sqrt{22 + 5\sqrt{17}} & 0 \\ 0 & \sqrt{22 - 5\sqrt{17}} \\ 0 & 0 \end{pmatrix}$$
$$(0.0.41)$$

and substituting into (0.0.4), we get

$$\lambda = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \quad (0.0.42)$$

The minimum distance between the lines is

given by,

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \frac{1}{59} \begin{pmatrix} 30 \\ -10 \\ -70 \end{pmatrix} \right\| \quad (0.0.43)$$

$$= \frac{\sqrt{30^2 + 10^2 + 70^2}}{59} \quad (0.0.44)$$

$$= \frac{10}{\sqrt{59}} \quad (0.0.45)$$

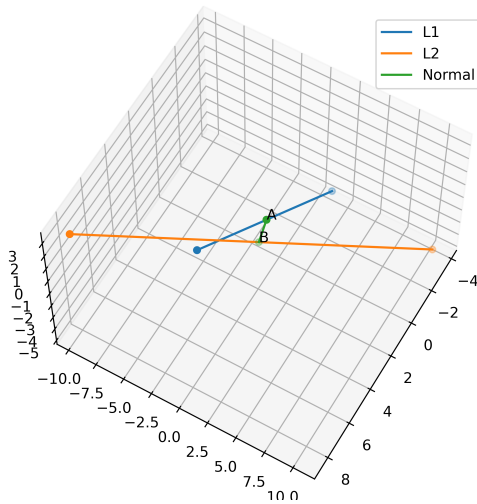The shortest distance between the given lines is $\frac{10}{\sqrt{59}}$ units.

Fig. 1: *AB* is the required shortest distance.