Assignment 1

Jaswanth Chowdary Madala

1) Find the equation of the hyperbola that satisfies the conditions - Foci $(\pm 4, 0)$, the latus rectum is of length 12.
Solution: The equation of the conic with focus $\mathbf{F}$, directrix $\mathbf{n}^\top \mathbf{x} = c$ and eccentricity e is given by

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{0.0.1}$$

where

$$\mathbf{V} \triangleq \|\mathbf{n}\|^2 \mathbf{I} - e^2 \mathbf{n}\mathbf{n}^\top \tag{0.0.2}$$
$$\mathbf{u} \triangleq ce^2 \mathbf{n} - \|\mathbf{n}\|^2 \mathbf{F} \tag{0.0.3}$$
$$f \triangleq \|\mathbf{n}\|^2 \|\mathbf{F}\|^2 - c^2 e^2 \tag{0.0.4}$$

a) Major Axis: Given that the conic has foci as

$$\mathbf{F_1} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{0.0.5}$$
$$\mathbf{F_2} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{0.0.6}$$

The direction vector of F_1F_2 is given by

$$\mathbf{m} = \mathbf{F_1} - \mathbf{F_2} \tag{0.0.7}$$
$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.0.8}$$

Hence the normal to the directrix is given by,

$$\mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.0.9}$$

Thus,

$$\mathbf{V} = \begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.0.10}$$
$$\mathbf{u} = ce^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \mathbf{F} \tag{0.0.11}$$
$$f = \|\mathbf{F}\|^2 - c^2 e^2 \tag{0.0.12}$$

b) Centre: The centre of the conic is given by

$$\mathbf{c} = \frac{\mathbf{F_1} + \mathbf{F_2}}{2} \tag{0.0.13}$$
$$\mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.0.14}$$
$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{0.0.15}$$

Since $\mathbf{c} = \mathbf{0}$ and $\mathbf{V}^{-1} \neq \mathbf{0}$, it follows from (0.0.15) that

$$\mathbf{u} = \mathbf{0} \tag{0.0.16}$$

Thus, from (0.0.11),

$$\mathbf{F} = \begin{pmatrix} ce^2 \\ 0 \end{pmatrix} \tag{0.0.17}$$

and so,

$$ce^2 = 4 \tag{0.0.18}$$
$$f = c^2 e^2 \left(e^2 - 1\right) \tag{0.0.19}$$

c) eccentricity: Given that the conic has the latus rectum length 12,

$$l = 2\frac{\sqrt{|f_0 \lambda_2|}}{\lambda_1} \tag{0.0.20}$$
$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \tag{0.0.21}$$
$$= -f \tag{0.0.22}$$
$$= -c^2 e^2 \left(e^2 - 1\right) \tag{0.0.23}$$

From equation (0.0.10) the eigen values λ_1, λ_2 are given by,

$$\lambda_1 = 1 - e^2 \tag{0.0.24}$$
$$\lambda_2 = 1 \tag{0.0.25}$$

Substituting the expressions of $\lambda_1, \lambda_2, f_0$ from the equations (0.0.23), (0.0.24), (0.0.25) in (0.0.20) gives

$$l = \frac{2ce}{\sqrt{e^2 - 1}} = 12 \tag{0.0.26}$$
$$\frac{ce}{\sqrt{e^2 - 1}} = 6 \tag{0.0.27}$$

Substitute the expression of c from (0.0.18) gives,

$$\frac{4}{e\sqrt{e^2-1}} = 6 \tag{0.0.28}$$

$$\tag{0.0.29}$$

Squaring on both sides gives,

$$9e^2\left(e^2-1\right) = 4 \tag{0.0.30}$$

$$9e^4 - 9e^2 - 4 = 0 \tag{0.0.31}$$

The equation (0.0.31) is a quadratic equation in e^2

$$e^2 = \frac{-(-9) \pm \sqrt{(-9)^2 - 4\times 9\times(-4)}}{2\times 9} \tag{0.0.32}$$

$$e^2 = \frac{4}{3} \tag{0.0.33}$$

From equation (0.0.18), (0.0.19), we get

$$f = 4 \tag{0.0.34}$$

The equation of the conic is given by

$$\mathbf{x}^\top \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 4 = 0 \tag{0.0.35}$$

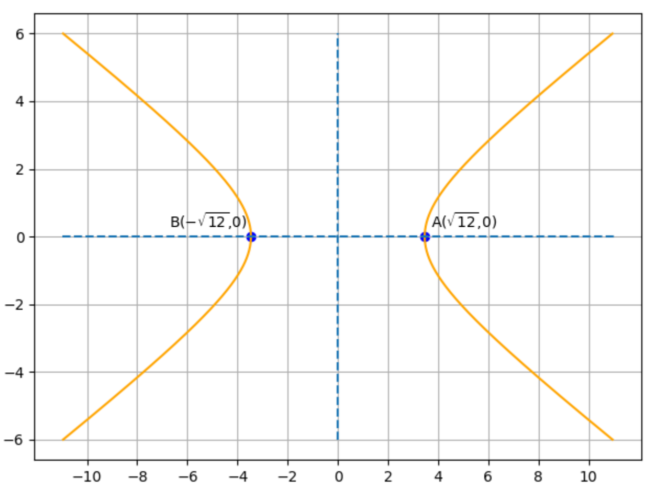

Fig. 1: Graph

Parameter	**Description**	**Value**
$\mathbf{F_1}$	Focus 1 of hyperbola	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$
$\mathbf{F_2}$	Focus 2 of hyperbola	$\begin{pmatrix} -4 \\ 0 \end{pmatrix}$
l	Length of latus rectum	12

TABLE 1