Assignment 1

Jaswanth Chowdary Madala

1) Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$
Solution: The lines l_1 and l_2 in vector form can be written as

$$\mathbf{x} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda_1\begin{pmatrix}2\\-1\\1\end{pmatrix} \tag{0.0.1}$$

$$\mathbf{x} = \begin{pmatrix}2\\1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{0.0.2}$$

$$\mathbf{x_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{x_2} = \begin{pmatrix}2\\1\\-1\end{pmatrix}, \mathbf{m_1} = \begin{pmatrix}2\\-1\\1\end{pmatrix}, \mathbf{m_2} = \begin{pmatrix}3\\-5\\2\end{pmatrix} \tag{0.0.3}$$

The distance between the lines is given by,

$$d = \|(\mathbf{x_2} + \lambda_2\mathbf{m_2}) - (\mathbf{x_1} + \lambda_1\mathbf{m_1})\| \tag{0.0.4}$$

$$\implies d = \|\mathbf{x_2} - \mathbf{x_1} - \lambda_1\mathbf{m_1} + \lambda_2\mathbf{m_2}\| \tag{0.0.5}$$

Consider the following definitions

$$\mathbf{A} \triangleq \mathbf{x_2} - \mathbf{x_1} \tag{0.0.6}$$

$$\mathbf{M} \triangleq \begin{pmatrix}\mathbf{m_1} & \mathbf{m_2}\end{pmatrix} \tag{0.0.7}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix}\lambda_1\\-\lambda_2\end{pmatrix} \tag{0.0.8}$$

From the above definitions (0.0.6), (0.0.7), (0.0.8) we get,

$$d = \|\mathbf{A} - \mathbf{M}\boldsymbol{\lambda}\| \tag{0.0.9}$$

Here we have the values of $\mathbf{A}, \mathbf{M}$ as

$$\mathbf{M} = \begin{pmatrix}2 & 3\\-1 & -5\\1 & 2\end{pmatrix} \tag{0.0.10}$$

$$\mathbf{A} = \begin{pmatrix}1\\0\\-1\end{pmatrix} \tag{0.0.11}$$

Consider the function

$$f(\boldsymbol{\lambda}) = \|\mathbf{A} - \mathbf{M}\boldsymbol{\lambda}\|^2 \tag{0.0.12}$$

The given problem can be formulated as

$$\min_{\boldsymbol{\lambda}} f(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^\top\mathbf{M}^\top\mathbf{M}\boldsymbol{\lambda} - 2\mathbf{A}^\top\mathbf{M}\boldsymbol{\lambda} + \|\mathbf{A}\|^2 \tag{0.0.13}$$

Here we have,

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix}6 & 13\\13 & 38\end{pmatrix} \tag{0.0.14}$$

$$\mathbf{A}^\top\mathbf{M} = \begin{pmatrix}1 & 1\end{pmatrix} \tag{0.0.15}$$

$$\|\mathbf{A}\|^2 = 2 \tag{0.0.16}$$

A numerical solution for (0.0.13) is obtained as

$$\boldsymbol{\lambda}_{n+1} = \boldsymbol{\lambda}_n - \alpha\nabla f(\boldsymbol{\lambda}_n) \tag{0.0.17}$$

where λ_0 is an inital guess and α is a variable parameter. These parameters decide how fast the algorithm converges. Here the gradient is given by,

$$\nabla f(\boldsymbol{\lambda}) = 2\mathbf{M}^\top\mathbf{M}\boldsymbol{\lambda} - 2\left(\mathbf{A}^\top\mathbf{M}\right)^\top \tag{0.0.18}$$

From (0.0.17) we get

$$\boldsymbol{\lambda}_{n+1} = \boldsymbol{\lambda}_n - \alpha\left(2\mathbf{M}^\top\mathbf{M}\boldsymbol{\lambda}_n - 2\mathbf{M}^\top\mathbf{A}\right) \tag{0.0.19}$$

By taking the parameters as listed in the below table

Parameter	Description	Value
λ_0	Initial guess	$\begin{pmatrix}1\\2\end{pmatrix}$
α	Variable parameter	0.01
N	Number of iterations	10000
ϵ	Tolerance in λ	10^{-6}

TABLE 1

λ obtained is

$$\boldsymbol{\lambda} = \begin{pmatrix}0.4237\\-0.1186\end{pmatrix} \tag{0.0.20}$$

The shortest distance is

$$\min_{\lambda} d = 1.3019 \tag{0.0.21}$$

The shortest distance between the given lines is 1.3019 units, and this answer matches with the other methods.

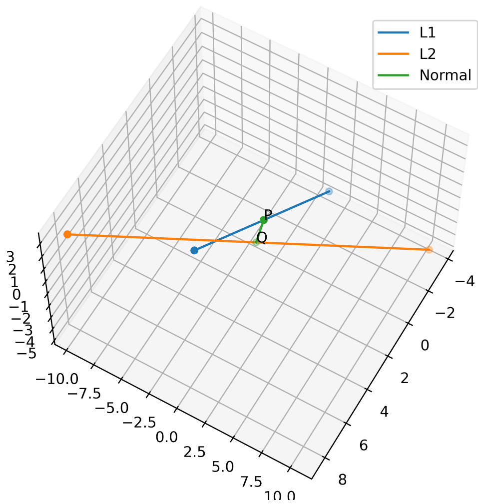

Fig. 1: PQ is the required shortest distance.