Assignment 1

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1) A coin is tossed three times, where. Determine $\Pr(E \mid F)$ where
   a) E : head on third toss, F : heads on first two tosses
   b) E : at least two heads, F : at most two heads
   c) E : at most two tails, F : at least one tail

   Solution: Consider the random variables X_1, X_2, X_3, X, which denotes the first, second, third toss and number of heads in the 3 tosses respectively as described in table 1.

RV	Values	Description
X	$\{0, 1, 2, 3\}$	Number of heads in 3 tosses
X_1	$\{0, 1\}$	0: Heads , 1: Tails
X_2	$\{0, 1\}$	0: Heads , 1: Tails
X_3	$\{0, 1\}$	0: Heads , 1: Tails

TABLE 1: Random variables X_1, X_2, X_3, X

The random variable X follows binomial distribution

$$X = X_1 + X_2 + X_3 \qquad (0.0.1)$$

The PMF of the random variable X is given by,

$$P_X(n) = {}^N C_n \, p^n \, (1-p)^{N-n} \qquad (0.0.2)$$

Here we have

$$N = 3, \; p = \frac{1}{2} \qquad (0.0.3)$$

The CDF of the random variable X is given by,

$$F_X(n) = \Pr(X \le n) = \sum_{i=0}^{n} {}^N C_i \, p^i \, (1-p)^{N-i} \qquad (0.0.4)$$

a) The events E, F can be described by the RV as

$$E : X_3 = 0 \qquad (0.0.5)$$
$$F : X_1 + X_2 = 0 \qquad (0.0.6)$$

Y is another random variable which repre-sents the number of heads in first two tosses.

$$Y = X_1 + X_2 \qquad (0.0.7)$$

The PMF of the random variable Y is given by,

$$P_Y(n) = {}^N C_n \, p^n \, (1-p)^{N-n} \qquad (0.0.8)$$

Here we have

$$N = 2, \; p = \frac{1}{2} \qquad (0.0.9)$$

The event EF can be expressed as,

$$X_3 = 0 \cap X_1 + X_2 = 0 \qquad (0.0.10)$$
$$\triangleq X_1 + X_2 + X_3 = 0 \qquad (0.0.11)$$
$$\implies X = 0 \qquad (0.0.12)$$

The required probability is given by,

$$\Pr(X_3 = 0 \mid Y = 0) \qquad (0.0.13)$$

$$= \frac{\Pr(X = 0)}{\Pr(Y = 0)} \qquad (0.0.14)$$

$$= \frac{1}{2} \qquad (0.0.15)$$

b) The events E, F, F' can be described by the RV as

$$E : X \le 1 \qquad (0.0.16)$$
$$F : X \ge 1 \qquad (0.0.17)$$
$$F' : X = 0 \qquad (0.0.18)$$

The required probability is given by,

$$= \frac{\Pr(EF)}{1 - \Pr(F')} \qquad (0.0.19)$$

The event EF can be expressed as,

$$X \le 1 \cap X \ge 1 \qquad (0.0.20)$$
$$\implies X = 1 \qquad (0.0.21)$$

Hence, the required probability is given by,

$$= \frac{\Pr(X = 1)}{1 - \Pr(X = 0)} \quad (0.0.22)$$

$$= \frac{\frac{3}{8}}{1 - \frac{1}{8}} \quad (0.0.23)$$

$$= \frac{3}{7} \quad (0.0.24)$$

c) For the events E, F, their complements are E' : all 3 tails, F' : zero tails. The events E', F' can be described by the RV as

$$E' : X = 3 \quad (0.0.25)$$

$$F' : X = 0 \quad (0.0.26)$$

By using property of conditional probability we have,

$$\Pr(E \mid F) = \frac{\Pr(EF)}{\Pr(F)} \quad (0.0.27)$$

$$= \frac{1 - \Pr(E' + F')}{\Pr(F)} \quad (0.0.28)$$

The required probability is given by,

$$= \frac{1 - \Pr(X = 0 + X = 3)}{1 - \Pr(X = 0)} \quad (0.0.29)$$

$$= \frac{1 - (\Pr(X = 0) + \Pr(X = 3) - \Pr(\phi))}{1 - \Pr(X = 0)} \quad (0.0.30)$$

$$= \frac{1 - \left(\frac{1}{8} + \frac{1}{8} - 0\right)}{1 - \frac{1}{8}} \quad (0.0.31)$$

$$= \frac{6}{7} \quad (0.0.32)$$