-1.1 EXERCISES

-1.1.1 Problems

-1.1. Write the following sets explicitly:

(i) $\{x \in \mathbb{N} \mid 1 < x \leq 7\}$
(ii) $\{x \in \mathbb{Z} \mid x < 5\}$
(iii) $\left\{x \in \mathbb{R} \mid x^2 = -1\right\}$
(iv) $\{x \in \mathbb{N} \wedge x \in \mathbb{Q}\}$
(v) $\left\{x \in \mathbb{R} \mid x^2 - 3x - 4 = 0\right\}$
(vi) $\{x \in \mathbb{R} \mid x < 5 \wedge x \geq 2\}$

-1.2. Determine the relation between the sets:

(i) $A = \{1,\ 2,\ 3\},\ B = \{1,\ 2\}$
(ii) $A = \varnothing,\ B = \{2,\ -5,\ \pi\}$
(iii) $A = \mathbb{Z},\ B = \{\pm x \mid x \in \mathbb{N} \cup \{0\}\}$
(iv) $A = \left\{\pi, \mathrm{e}, \sqrt{2}\right\},\ B = \mathbb{Q}$

-1.3. Write all elements in $S^2 \times W$, where $S = \{\alpha, \beta, \gamma\}$ and $W = \{x, y, z\}$. Find a condition that guarantees $S^2 \times W = W \times S^2$.

-1.4. How many different **bijective** functions exist between a set with 2 elements and another set with 2 elements (e.g. $f : \{1,2\} \to \{\alpha, \beta\}$)? How many exist between two sets, each with 3 elements? Between two sets each with n elements?

-1.5. For each of the real functions below, find a set on which it is surjective (use a graphing calculator if you are not familiar with the shape of a function):

$$x^2,\ x^3 - 5,\ \mathrm{e}^{-x^2/2},\ \sin(x),\ \sin(x) + \cos(x),\ x\mathrm{e}^x.$$

-1.6. Given two sets A, B such that $|A| \neq |B|$, can a bijective function $f : A \to B$ exist? Explain your answer.

-1.7. Find all real roots of the following polynomial function:

$$f(x) = x^3 + x^2 - 6x.$$

-1.8. Given a real $b > 0$ and k, prove that for any real $x > 0$

$$\log_b\left(x^k\right) = k \log_b(x).$$

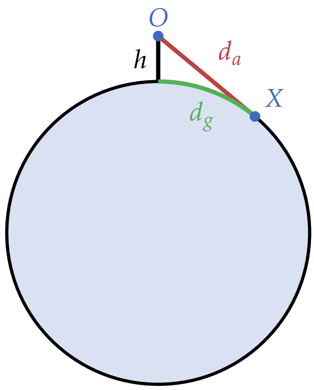
-1.9. Show that for any positive real x, b

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x).$$

-1.10. Solve the following equation for any real $x > 0$: (CHECK SOLUTION! MIGHT BE WRONG)

$$\log_2(x) + \log_4(x-1) = \log_{16}\left(x^3\right).$$

-1.11. The horizon on a spherical planet such as the earth[1] is defined as the distance from an observer to the point where the ground disappears behind the planet's curve. Following is a 2-dimensional depiction, where $\boldsymbol{O}$ is the observer, $\boldsymbol{h}$ its height above the planet surface, $\boldsymbol{X}$ the horizon point and $\boldsymbol{d_a}$ the air-distance from the observer to the horizon and $\boldsymbol{d_g}$ the ground-distance from the observer to the horizon:



(i) Find an expression for the air-distance $\boldsymbol{d_a}$ and ground-distance $\boldsymbol{d_g}$ to the horizon as a function of the radius $\boldsymbol{R}$ and height $\boldsymbol{h}$. (**hint**: find a relevant right triangle containing $\boldsymbol{d_a}$ and the radius of the planet)

(ii) Given that the Earth's radius is about 6371km (6.371×10^6m) and an average person is 1.75m tall - what is the distance to the horizon for a person standing at sea-level (both air- and ground-distances)? What would these distances be at the following heights: 165m (Eiffel tower's observation deck), 9.1km (average cruising altitude of a passenger jet) and 408km (average altitude of the International Space Station)?

(iii) How many degrees does the horizon drops from eye-level as function of $\boldsymbol{h}$? (*eye-level* in this context means the direction tangent to the planet's surface)

-1.12. Calculate the following complex product - first using the algebraic form and then the polar form, showing that the result is the same in both cases:

$$z = z_1^2 z_2 = \left(\sqrt{3} + \mathrm{i}\right)^2 \left(-2 + \sqrt{12}\mathrm{i}\right).$$

-1.13. Prove that the **sum** of all the roots of the complex equation $z^n = 1$ is always zero when $n \geq 2$, i.e. if $w_0, w_1, \ldots, w_{n-1}$ are the roots of the equation, then

$$\sum_{k=0}^{n-1} w_k = 0.$$

[1] yes.

Hint: for $|r| \neq 1$,

$$\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}.$$

-1.14. MORE EXERCISES TO BE WRITTEN...

-1.1.2 Solutions

-1.1. For each of the sets we first write how to read the notation in words, followed by its explicit form:

(i) Any **natural number** such that it is bigger than 1 and smaller or equal to 7. These are of course the numbers

$$\{2, 3, 4, 5, 6, 7\}.$$

(ii) Any **integer** such that it is smaller than 5. These are the numbers

$$\{4, 3, 2, 1, 0, -1, -2, -3, \dots\}.$$

(iii) Any **real number** x such that $x^2 = -1$. Since for any $x \in \mathbb{R}$, $x^2 \geq 0$ - there is no such real number x whose square equals -1. Therefore this definition describes the empty set, i.e. $\emptyset$.

(iv) Any **natural number** that is also a rational number. Since any natural number is also a rational number (e.g. $4 = \frac{4}{1} = \frac{8}{2}$, etc.) the definition actually simply describes the set of natural numbers, $\mathbb{N}$. This fact can also be written as

$$\mathbb{N} \cap \mathbb{Q} = \mathbb{N}.$$

(v) Any **real number** such that it solves the equation $x^2 - 3x - 4 = 0$. The solutions can be found using the quadratic formula:

$$\frac{3 \pm \sqrt{3^2 + 4 \cdot 4}}{2} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = 4, -1.$$

Therefore the set described by the definition is simply

$$\{4, -1\}.$$

(vi) Any **real number** that is smaller than 5 **and** is bigger than or equal to 2. This definition describes the half-open interval

$$[2, 5).$$

-1.2. Relations between sets:

(i) All the elements in the set B are also in the set A $(1, 2)$, but there's an element in A which is not in B (namely 3). Therefore, B is a subset of A:

$$B \subset A.$$

(ii) The empty set is a subset of any set (and a proper subset of any set except itself), therefore

$$A \subset B.$$

(iii) The set B is defined as all the natural numbers, their negatives and zero. This is exactly the definition of the integers $\mathbb{Z}$, which set A in this case. Therefore

$$A = B.$$

(iv) All of the elements in A are irrational numbers. The set B is the set of **rational numbers**, and therefore the sets are disjoined:

$$A \cap B = \emptyset.$$

-1.3. S^2 is a Cartesian product of S with itself:

$$S^2 = \{(\alpha,\alpha),(\alpha,\beta),(\alpha,\gamma),(\beta,\alpha),(\beta,\beta),(\beta,\gamma),(\gamma,\alpha),(\gamma,\beta),(\gamma,\gamma)\}.$$

Therefore, to form the Cartesian product $S^2 \times W$ we simply take each of the elements is S^2 and add to it an element from W:

$$\begin{aligned} S^2 \times W = \{&(\alpha,\alpha,x),(\alpha,\beta,x),(\alpha,\gamma,x),(\beta,\alpha,x),(\beta,\beta,x),(\beta,\gamma,x),(\gamma,\alpha,x),(\gamma,\beta,x),(\gamma,\gamma,x)\\ &(\alpha,\alpha,y),(\alpha,\beta,y),(\alpha,\gamma,y),(\beta,\alpha,y),(\beta,\beta,y),(\beta,\gamma,y),(\gamma,\alpha,y),(\gamma,\beta,y),(\gamma,\gamma,y)\\ &(\alpha,\alpha,z),(\alpha,\beta,z),(\alpha,\gamma,z),(\beta,\alpha,z),(\beta,\beta,z),(\beta,\gamma,z),(\gamma,\alpha,z),(\gamma,\beta,z),(\gamma,\gamma,z)\}. \end{aligned}$$

Note that the number of elements in S is 3, and so the number of elements in S^2 is $3 \times 3 = 9$. The number of elements in W is also 3, and so the number of elements in $S^2 \times W$ is $9 \times 3 = 27$.

The Cartesian product $W \times S^2$ has the same structure as $S^2 \times W$, except that the elements from W are now on the left (remember that the order of elements is important in tuples, unlike with sets):

$$\begin{aligned} S^2 \times W = \{&(x,\alpha,\alpha),(x,\alpha,\beta),(x,\alpha,\gamma),(x,\beta,\alpha),(x,\beta,\beta),(x,\beta,\gamma),(x,\gamma,\alpha),(x,\gamma,\beta),(x,\gamma,\gamma)\\ &(y,\alpha,\alpha),(y,\alpha,\beta),(y,\alpha,\gamma),(y,\beta,\alpha),(y,\beta,\beta),(y,\beta,\gamma),(y,\gamma,\alpha),(y,\gamma,\beta),(y,\gamma,\gamma)\\ &(z,\alpha,\alpha),(z,\alpha,\beta),(z,\alpha,\gamma),(z,\beta,\alpha),(z,\beta,\beta),(z,\beta,\gamma),(z,\gamma,\alpha),(z,\gamma,\beta),(z,\gamma,\gamma)\}. \end{aligned}$$

One way of ensuring that $S^2 \times W = W \times S^2$ is by making all tuples equal, i.e. if

$$\alpha = \beta = \gamma = x = y = z,$$

then

$$S^2 \times W = \{(\alpha,\alpha,\alpha)\} = W \times S^2.$$

-1.4. We start by counting the number of possible bijective functions $f_2 : \{1,2\} \to \{\alpha,\beta\}$. For each element in the domain of f_2 there are two options to connect to an element in the function's image: either 1 or 2. So we can have

$$1 \mapsto \alpha, \text{ or}$$
$$1 \mapsto \beta.$$

(recall that the symbol $x \mapsto y$ means that the element x is mapped by the function to the element y)

For each of the above options, there is only a single option left for the element 2:

$$2 \mapsto \beta \text{ if } 1 \mapsto \alpha,$$
$$2 \mapsto \alpha \text{ if } 2 \mapsto \beta.$$

Therefore, we have two possible choices for mapping 1, each of which dictates the choice for mapping 2. Altogether there are two such bijective functions:

We can use the same logic for the case of 3 elements in each set. Let's define the function as

$$f_3 : \{1, 2, 3\} \to \{\alpha, \beta, \gamma\}.$$

We start with all the possibilities for connecting 1:

$$1 \mapsto \alpha, \text{ or}$$
$$1 \mapsto \beta, \text{ or}$$
$$1 \mapsto \gamma.$$

For each of the above choices, we have two remaining choices for connecting 2 (since one element in the image of f_3 is already decided by our choice for 1). Then, for the last element 3 we remain with a single option only, since two elements in the image of f_3 are already connected to by 1 and 2. Altogether we therefore have

$$3 \times 2 \times 1 = 6$$

total bijective functions f_3.

You probably already noticed the pattern: for a function

$$f_n : \{n \text{ elements}\} \to \{n \text{ elements}\},$$

we have n choices for connecting the first element, then $n-1$ options for connecting the second element, then $n-2$ options for connecting the the third element... and by the last element we have only a single option. Therefore, the number of bijective functions f_n is

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 = n!$$

Note that indeed $2! = 2$ and $3! = 6$, which agrees with the results we got for f_2 and f_3, respectively.

-1.5. solution...

-1.6. A function is bijective if and only if it is both a injective and surjective. There are two cases for $|A| \neq |B|$:

-1.6.1. $|A| > |B|$, in which case there is at least one element in A which is not connected to any element in B: otherwise, there are at least two elements in A that connect to the same element in B. In the first case the relation is not a function, and in the second it is not injective and therefore not bijective.

-1.6.2. $|A| < |B|$, in which case there must be at least one element in B that is not connected to by any element from A (by the defintion of a function, there cannot be any element in A that is connected to more than a single element in B). Therefore such a function is not surjective and thus not bijective.

-1.7. The polynomial $f(x)$ can be re-written as

$$f(x) = x\left(x^2 + x - 6\right).$$

Therefore one of its roots are when $x = 0$, and the other when $x^2 + x - 6 = 0$. Using the quadratic formula we get that $x^2 + x - 6 = 0$ when

$$x_{1,2} = \frac{-1 \pm \sqrt{1 - 4(-6)}}{2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3.$$

Thus, altogether the roots of f are $\{-3, 0, 2\}$.

-1.8. As with most logarithm-related proofs, we transform the problem into the realm of exponents: let us set $m = \log_b(x)$. We then get that $x = b^m$. If we raise both by to the k-th power, we get

$$\begin{aligned} x^k &= (b^m)^k \\ &= b^{mk}. \end{aligned}$$

Taking the logarithm in base b of both sides of the above relation gives

$$\begin{aligned} \log_b\left(x^k\right) &= \log_b\left(b^{mk}\right) \\ &= mk \\ &= k \log_b(x). \end{aligned}$$

The last step results from our original defintion that $m = \log_b(x)$.

-1.9. Using the relation proved in the previous question and setting $k = -1$ we get

$$\log_b\left(\frac{1}{x}\right) = \log_b\left(x^{-1}\right) = -1 \cdot \log_b(x) = -\log_b(x).$$

-1.10. Using the logarithm base-change rule (**??**), we set all logarithms to same base ($b = 16$):

$$\log_2(x) = \log_{16}(x) \cdot \log_2(16) = 4 \log_{16}(x).$$
$$\log_4(x-1) = \log_{16}(x-1) \cdot \log_4(16) = 2 \log_{16}(x-1).$$

Therefore, the expression is equivalent to

$$4\log_{16}(x) = 2\log_{16}(x-1) + \log_{16}(x^3).$$

We can now bring the coefficients 4 and 2 inside the logarithm expression:

$$\log_{16}(x^4) = \log_{16}([x-1]^2) + \log_{16}\left(x^3\right).$$

Using the logarithmic addition law, we get:

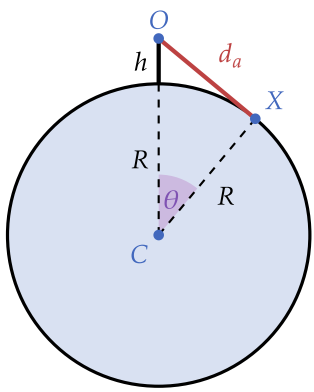
$$\log_{16}(x^4) = \log_{16}(x^3[x-1]^3).$$

We can now discard $\log_{16}$ on both sides, and we're left with a simple equation to solve:

$$x^4 = x^3(x-1)^2,$$

the solutions of which are $x_1 = 0$ and $x_{2,3} = \frac{3\pm\sqrt{5}}{2}$, of which only $x_2 = \frac{3+\sqrt{5}}{2}$ is valid: x_1 isn't valid since $x > 0$, and x_3 isn't valid since $x_3 - 1 < 0$, and thus $\log_b(x_3 - 1)$ isn't defined over the real numbers.

-1.11. (i) We start with drawing two radial lines from the center of the planet $\boldsymbol{C}$: one to the point on the surface where the observer meets the ground, and one to the horizon. The triangle $\triangle COX$ is a right triangle: the angle $\angle CXO = 90°$:



Using the Pythagorean theorem (with $R+h$ as the hypotenuse) we can calculate d_a:

$$d_a^2 + R^2 = (R+h)^2.$$

By expanding the right-hand side, cancelling R^2 and rearranging we get

$$d_a = \sqrt{2Rh + h^2}.$$

To get d_g we need to find the angle θ between the lines CX and CO. For that purpose we can use the law of sines (**??**):

$$\frac{d_a}{\sin(\theta)} = \frac{R+h}{\sin(90°)} = R+h.$$

(since $\sin(90°) = 1$)

Isolating $\sin(\theta)$ and subtituting the value of d_a as function of R and h yields:

$$\sin(\theta) = \frac{d_a}{R+h} = \frac{\sqrt{2Rh+h^2}}{R+h}.$$

Therefore

$$\theta = \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

When θ is given in radians, the length d_g then simply becomes

$$d_g = R\theta = R \cdot \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

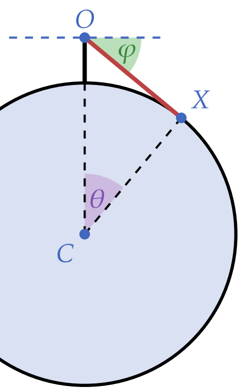
(ii) For an average person on Earth ($h = 1.75\text{m},\ R = 6.371 \times 10^6\text{m}$), standing at sea level, the air-distance to the horizon is therefore

$$d_a = \sqrt{2Rh+h^2} = \sqrt{2 \cdot 6.371 \times 10^6 \cdot 1.75 + 1.75^2} \approx 4722\text{m} = 4.722\text{km}.$$

The ground-distance, on the other hand, is

$$\begin{aligned} d_g &= R \cdot \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right) \\ &= 6.371 \times 10^6\text{m} \cdot \arcsin\left(\frac{4722\text{m}}{6.371 \times 10^6\text{m} + 1.75\text{m}}\right) \\ &\approx 4722\text{m}. \end{aligned}$$

(iii) Let us call the angle representing the drop of the horizon from eye-level φ:



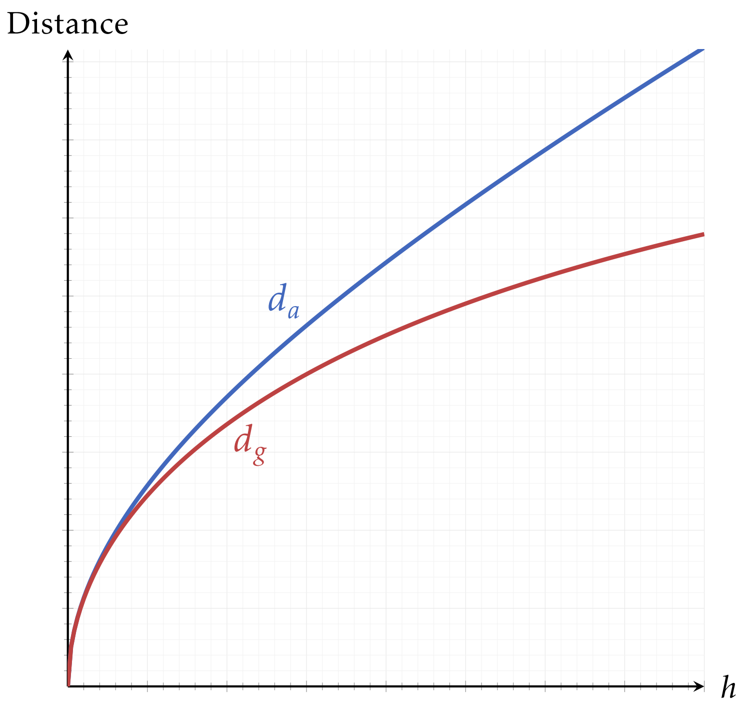
Since $\triangle COX$ is a right triangle ($\angle CXO$ being the right angle), we know θ from previously and all angles in a triangle sum up to deg 180, the angle $\angle COX$ is equal to $90° - \theta$. This in turn means that φ is equal to

$$90° - (90° - \theta) = \theta = \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

The following table sums up all the (approximate) air- and ground-distances to the horizon and the drop of the horizon from eye-level for each of the heights mentioned in the exercise:

Position	Height [m]	d_a[km]	d_g[km]	$\theta[°]$
Person at sea level	1.75	4.7	4.7	0.04
Eiffel Tower observation	165	45.8	45.8	0.41
Average cruising altitude	9100	340.6	340.3	3.06
Internation Space Station	408000	2316	2221	19.98

Note that as the height h grows, the difference between d_a and d_g grows too. We can see this clearly when plotting $d_a(h)$ and $d_g(h)$ in the same graph (disregarding the units and values for now, since we are only interested in the qualitative behaviour of both distances):



For small values of h the two functions are very close to eachother, and as h grows they grow apart, with $d_a > d_g$.

-1.12. • **Algebraic form**: we simply expand all parantheses and multiply everything:

$$
\begin{aligned}
\left(\sqrt{3}+\mathrm{i}\right)^2\left(-2+\sqrt{12}\mathrm{i}\right) &= \left(\sqrt{3}+\mathrm{i}\right)\left(\sqrt{3}+\mathrm{i}\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\
&= \left(3+\sqrt{3}\mathrm{i}+\sqrt{3}\mathrm{i}-1\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\
&= \left(2+2\sqrt{3}\mathrm{i}\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\
&= 2\left(1+\sqrt{3}\mathrm{i}\right)\left(-2+\sqrt{4\cdot 3}\mathrm{i}\right)\\
&= 2\left(1+\sqrt{3}\mathrm{i}\right)\left(-2+2\sqrt{3}\mathrm{i}\right)\\
&= 4\left(1+\sqrt{3}\mathrm{i}\right)\left(-1+\sqrt{3}\mathrm{i}\right)\\
&= 4\left(-1+\sqrt{3}\mathrm{i}-\sqrt{3}\mathrm{i}-3\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 4(-4) \\
&= -16.
\end{aligned}
$$

- **Polar form**: first we use **??** to find the polar form of the two complex numbers:

$$
z_1 = \sqrt{3} + \mathrm{i} \Rightarrow \begin{cases} r_1 = \sqrt{3+1} = 2, \\ \theta_1 = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}. \end{cases}
$$

$$
z_2 = -2 + \sqrt{12} \Rightarrow \begin{cases} r_2 = \sqrt{4+12} = 4, \\ \theta_2 = \arctan\left(-\frac{\sqrt{12}}{2}\right) = \frac{2\pi}{3}. \end{cases}
$$

Therefore, $z_1^2 z_2$ in polar form is

$$
\begin{aligned}
z_1^2 z_2 &= \left(r_1 e^{\theta_1 \mathrm{i}}\right)^2 r_2 e^{\theta_2 \mathrm{i}} \\
&= r_1^2 r_2 e^{(2\theta_1 + \theta_2)\mathrm{i}} \\
&= 2^2 \cdot 4 e^{\left(\frac{2\pi}{6} + \frac{2\pi}{3}\right)\mathrm{i}} \\
&= 16 e^{\pi \mathrm{i}}.
\end{aligned}
$$

Since $e^{\pi \mathrm{i}} = -1$ (**??**), we get that indeed

$$
z_1^2 z_2 = -16,
$$

just as we got in the algebraic form.

-1.13. It is easy to see that for even values of n the statement holds: for each w_k theres an oposing w_m ($m \neq k$) such that $w_k + w_m = 0$. See for example $n = 4$ and $n = 6$ in **??**.

For a more general proof which includes the odd values of n we must work a bit harder. Recall that the k-th root of the equation $z^n = 1$ has the following form (**??**):

$$
w_k = e^{\frac{2\pi}{n} \mathrm{i} k}.
$$

We can re-write the sum of the roots as

$$
\sum_{k=0}^{n-1} \left(e^{\frac{2\pi \mathrm{i}}{n}}\right)^k,
$$

(since $x^{ab} = (x^a)^b$)

Using the hint we note that in this sum $r = e^{\frac{2\pi \mathrm{i}}{n}}$, and thus

$$
\sum_{k=0}^{n-1} \left(e^{\frac{2\pi \mathrm{i}}{n}}\right)^k = \frac{1 - \left(e^{\frac{2\pi \mathrm{i}}{n}}\right)^n}{1 - e^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - e^{\frac{2\pi \mathrm{i}}{n} n}}{1 - e^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - e^{2\pi \mathrm{i}}}{1 - e^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - 1}{1 - e^{\frac{2\pi \mathrm{i}}{n}}} = 0.
$$