MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

Written, illustrated and typeset (mostly) by

PELEG BAR SAPIR

with contributions from others

$a^b = e^{b\log(a)}$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$T\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$

$A = Q\Lambda Q^{-1}$

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$

$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

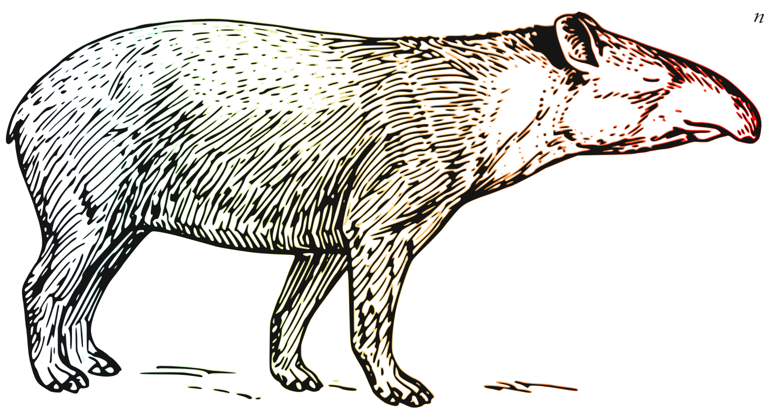
$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$

$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$

$e^{\pi i} + 1 = 0$

$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$



PUBLISHED IN THE WILD

! **To be written**: Rights, lefts, etc. will be written here in the future !

HERE BE TABLE

CHAPTER 0

INTRODUCTION

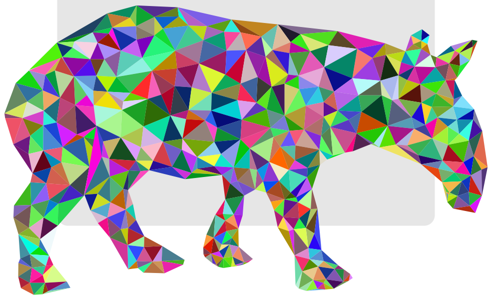
In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

!

CHAPTER 1



REAL CALCULUS IN 1D

1.1 SEQUENCES AND SERIES

1.1.1 Basics

A **sequence** is an indexed collection of **elements**. By *indexed* we mean that the order of the elements in a sequence matters (unlike with sets): changing the order of any element changes the sequence as a whole. The following are some examples of sequences composed of real numbers:

- $1, -3, 0, -7, 2, 1.5, 4, 0, 1, -0.35, \sqrt{2}$.
- $0, 1, 2, 1, 1, -1, 0$.
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$

- $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots$

The examples above present two more properties of sequences:

- Elements may repeat (unlike in the case of sets), and
- sequences can be either **finite** (as in the first two examples), or **infinite** (as in the latter two examples).

The number of elements in a sequence is called its **length**. In the case of infinite sequences we say that their length equals ∞ (inifinity). The elements of a sequence a are usually indexed using a subscript, such that a_1 is the first element in the sequence, a_2 is the second element in the sequence, etc. - and generally a_i is the i-th element in the sequence, where $i \in \mathbb{N}$.

We can therefore define a sequence somewhat more formally as a function from a subset of the natural numbers to the real numbers:

$$a : N \to \mathbb{R}, \tag{1.1.1}$$

where $N \subseteq \mathbb{N}$.

Example 1.1 Sequences as functions

The following 9-element sequence a

$$\begin{array}{ccccccccc} 3, & 4, & \frac{1}{2}, & 0, & 2, & 6, & -\frac{2}{3}, & 0, & -1. \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a(1) & a(2) & a(3) & a(4) & a(5) & a(6) & a(7) & a(8) & a(9) \end{array}$$

can be viewed as a function

$$a : \{1, 2, \dots, 9\} \to \mathbb{R},$$

or more precisely as a function

$$a : \{1, 2, \dots, 9\} \to \left\{-1, -\frac{2}{3}, 0, \frac{1}{2}, 2, 3, 4, 6\right\}.$$

The follow infinite sequence b

$$\begin{array}{cccccccc} 1, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \frac{1}{6}, & \frac{1}{7}, & \dots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ b(1) & b(2) & b(3) & b(4) & b(5) & b(6) & b(7) & \end{array}$$

can be viewed as a function

$$b : \mathbb{N} \to (0, 1].$$

Since sequences can be viewed as functions, they can be defined using formulas: for example, the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

can be defined using the simple formula

$$a_n = \frac{1}{n}.$$

Example 1.2 Some sequences defined using formulas

$$(-1)^n \quad \Rightarrow \quad -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

$$3n + 4 \quad \Rightarrow \quad 7, 10, 13, 16, 19, 22, \ldots$$

$$(n+1)^2 \quad \Rightarrow \quad 4, 9, 16, 25, 36, 49, \ldots$$

$$\begin{cases} \text{2n+1} & \text{if } n \text{ is odd,} \\ \text{n-1} & \text{if } n \text{ is even.} \end{cases} \quad \Rightarrow \quad 3, 1, 7, 3, 11, 5, 15, 7, \ldots$$

Sequences can also be defined using **recursion**, where the value of an element is defined using previous values and a **starting value**. For example:

$$a_n = a_{n-1}^2 - 2,$$

with the starting value $a_1 = 3$. We the get that

$$a_2 = a_1^2 - 2 = 3^2 - 2 = 7,$$

and thus

$$a_3 = a_2^2 - 2 = 7^2 - 2 = 47,$$

etc.

Example 1.3 The Fibonacci sequnce

The **Fibonacci sequences** is a well-known sequence defined using the following recursive rule:

$$F_n = F_{n-1} + F_{n-2},$$

with $F_1 = F_2 = 1$. The first few elements of the sequence are therefore

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

See Figure 1.1 for a graphical representation of the Fibonacci sequence.

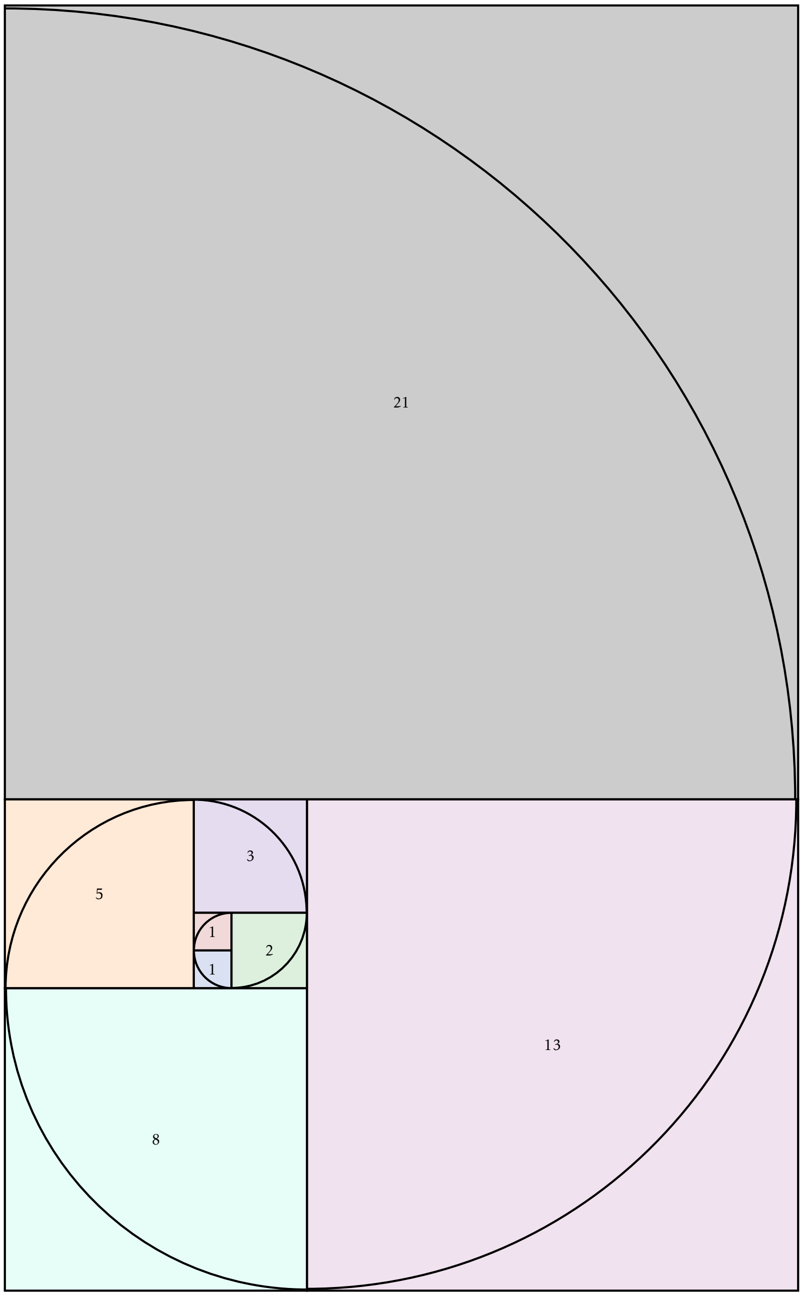


Figure 1.1 A graphical representation of the Fibonacci sequence: two squares of side 1 are placed adjacent to each other on the plane. In each subsequent step a new square is place such that its side is equal to the combined sides of the previous two squares. This way, the side of each square in the sequence follows the Fibonacci sequence. In each square we draw a quarter circle centered on one of the vertices, such that we get the famous **golden ratio** helix.

Note 1.1 Focus of section

From now on in the section we will focus on infinite sequences only.

!

1.1.2 Types of sequences

Consider the sequence $a_n = n^2$. Since $n \in \mathbb{N}$, for any n, $a_{n+1} > a_n$, since $(n+1)^2 > n^2$ (see Figure 1.2). We say that such a sequence is **increasing**. In fact, for a sequence to be increasing some sequential elements can be equal: for example, the sequence $c_n = 1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,\dots$ is also a increasing sequence. Thus, the definition of a increasing sequence is the following:

Definition 1.1 increasing sequence

sequence a_n is said to be *increasing* if for any $n \in \mathbb{N}$, $a_{n+1} \geq a_n$.

π

If we change the condition to $a_{n+1} > a_n$ we say that such a sequence is **strictly increasing**. In the above examples a_n is a strictly increasing sequence, while c_n is just increasing (since for some indeces n, $c_{n+1} = c_n$).

Similarly, a **decreasing** sequence is a sequence b_n for which for any $n \in \mathbb{N}$, $b_{n+1} \leq b_n$. An example of such sequence is $b_n = \frac{1}{n}$ (see Figure 1.3). ANd of course, if we change the condition to $b_{n+1} < b_n$ then the sequence is **strictly decreasing**.

Generally, a sequence that is either increasin or decreasing is said to be **monotone**. If a sequence is monotone starting only from a certain n, we say that the sequence is **eventually monotone** (i.e. *eventually increasing* or *eventually decreasing*). An example of such sequence is $d_n = (n-5)^2$ (Figure 1.4): for $N \in 1,2,3,4,5$ it is decreasing, but starting from $n = 5$ it is increasing for any n.

As an example of a sequence which isn't monote, consider the sequence $e_n = \sin(n)$: for some values of n, $e_{n+1} > e_n$ and for some other values $e_{n+1} < e_n$ (see Figure 1.5).

The following are two ways to determine whether a sequence a_n is monotone:

- **Difference test**: if $a_{n+1} - a_n \geq 0$ for all $n \in \mathbb{N}$, then the sequence is increasing. If $a_{n+1} - a_n \leq 0$ for all $n \in \mathbb{N}$ then the sequence is decreasing.

- **Ratio test**: if $\frac{a_n+1}{a_n} \geq 1$ for all $n \in \mathbb{N}$ then the sequence is increasing, and if $\frac{a_{n+1}}{a_n} < 1$ for all $n \in \mathbb{N}$ then the sequence is decreasing.

Example 1.4 Difference test

Given the sequence $a_n = \frac{n}{n+1}$, we look at the difference $a_{n+1} - a_n$:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)(n+1) - (n+2)n}{(n+1)(n+2)}$$

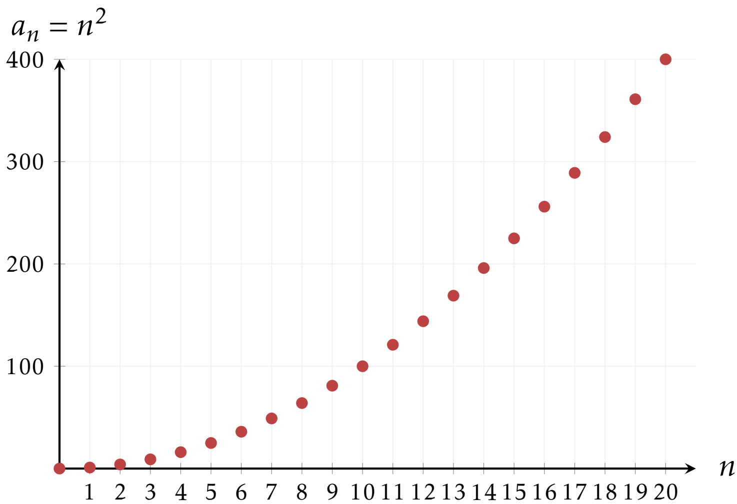


Figure 1.2 The sequence $a_n = n^2$ is increasing, and is in fact *strictly* increasing.

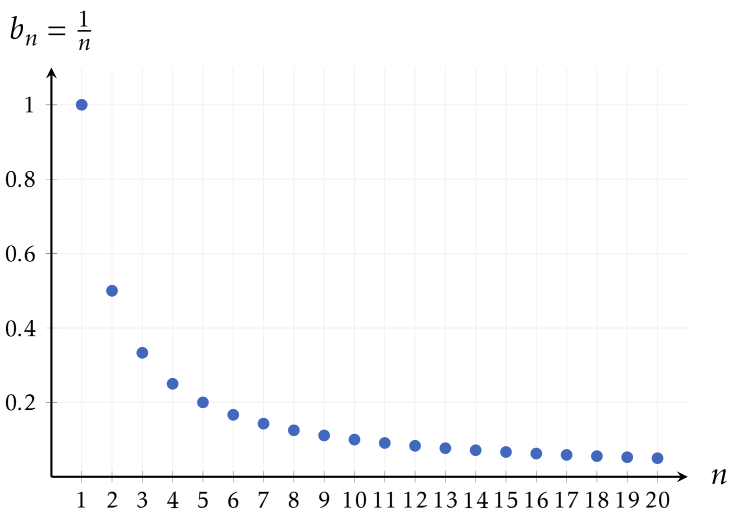


Figure 1.3 The sequence $b_n = \frac{1}{n}$ is decreasing, and is in fact *strictly* decreasing.

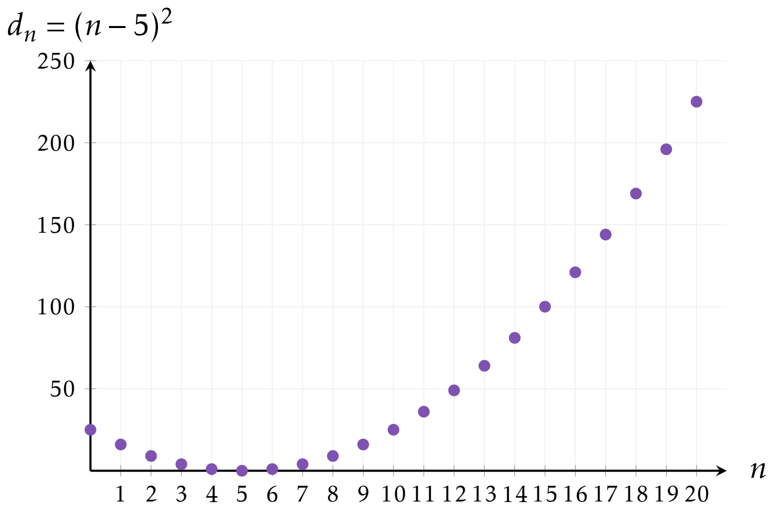


Figure 1.4 The sequence $d_n = (n-5)^2$ starts as a decreasing sequence, but starting from $n = 5$ it is increasing, making it an *eventually increasing sequence*.

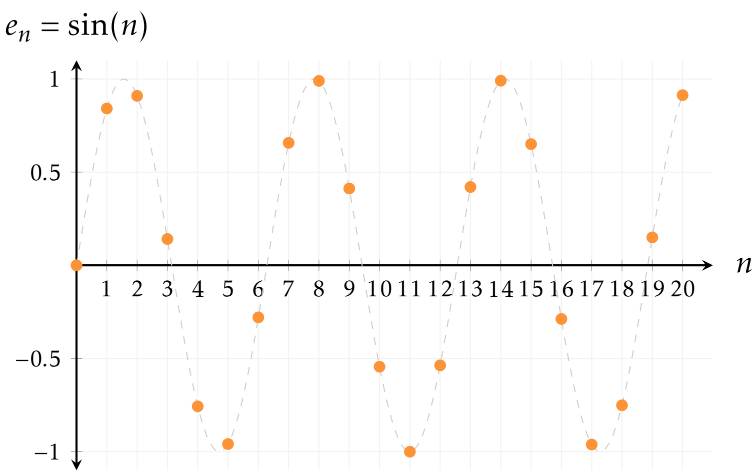


Figure 1.5 The sequence $e_n = \sin(n)$ is neither increasing nor decreasing. For reference, the function $\sin(x)$ is plotted as a dashed line behind e_n.

$$= \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < 1 \quad \forall n \in \mathbb{N}.$$

The last (in)equality stems from the fact that no matter what n we subtitute into $(n+1)(n+2)$, the result will be greater than 1, and thus $\frac{1}{(n+1)(n+2)}$ is always smaller than 1. Therefore, a_n is a decreasing sequence.

Example 1.5 Ratio test

Given the sequence $b_n = \frac{2^n}{n^2}$, the ratio of a_{n+1} to a_n is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = 2\frac{n^2}{(n+1)^2}.$$

Let's look at the first few approximated values of the ratio $\frac{n^2}{(n+1)^2}$:

n	$\frac{n^2}{(n+1)^2}$
0	0
1	0.25
2	0.44444...
3	0.5625
4	0.64
5	0.69444...
6	0.7346938775510204
7	0.765625
8	0.7901234567901234
9	0.81
10	0.8264462809917356
11	0.840277777...
12	0.8520710059171598
13	0.8622448979591837

We see that for any $n \geq 3$, $\frac{n^2}{(n+1)^2} > \frac{1}{2}$, and therefore $2\frac{n^2}{(n+1)^2} > 1$. Thus, the sequence is eventually increasing.

Some sequences are **bounded** from below: this means that their elements never get smaller than some constant $\underline{M} \in \mathbb{R}$. For example, consider the simple sequence $a_n = n$, where $n = \{1, 2, 3, 4, \ldots\}$: there is no element in the sequence that is smaller than 1. Therefore, a_n is bounded from below by 1. Of course, one may argue that b_n is also bounded from below by 0, or -6, or in fact any negative number. This is true, however we are usually interested in the *maximal* number $\underline{M}$ that bounds the sequence from below, which in this case is $\underline{M} = 1$. We call that number the **infimum** of the sequence, and denote it as $\inf a_n$.

Similarly, a sequence a_n can be bounded from above by some number $\overline{M} \in \mathbb{R}$, i.e. there exist no n for which $a_n > \overline{M}$. We call the *minimal* such number the **supremum** of the sequence a_n, denoted $\sup a_n$. For example, the sequence $b_n = \frac{1}{n}$ is bounded from above by any real number $x \geq 1$, and therefore $\sup b_n = 1$. In fact, b_n is also bounded from below by $\underline{M} = 0$, and therefore we say that it is **bounded**. Another example of a sequence that is bounded is $e_n = \sin(n)$, which is bounded from below by $\underline{M} = -1$ and from above by $\overline{M} = 1$.

Example 1.6 Bounded and unbounded sequences

The following table shows some examples of sequences that are bounded from below, from above, or neither:

a_n	**First 5 elements**	$\inf a_n$	$\sup a_n$
$n^2 - n$	$0, 2, 6, 12, 20, \ldots$	0	-
$\frac{n}{n+1}$	$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$	$\frac{1}{2}$	1
e^{-n}	$e^{-1}, e^{-2}, e^{-3}, e^{-4}, e^{-5}, \ldots$	0	e^{-1}
$\log(n)$	$0, \log(2), \log(3), \log(4), \log(5), \ldots$	0	-
$(-1)^n$	$-1, 1, -1, 1, -1, \ldots$	-1	1
$(-1)^n n$	$-1, 2, -3, 4, -5, \ldots$	-	-
$(-2)^n$	$-2, 4, -8, 16, -32, \ldots$	-	-

1.1.3 Subsequences

Given any sequence a_n, we can remove from it any number of its elements (including 0 elements) and get a new sequence b_n which is a **subsequence** of a_n. For example, let $a_n = n^2 - 5n$. We can remove each 2nd element from a_n (i.e. those with indices $2, 4, 6, 8, \ldots$) and get the following sequence b_n:

$$a_n = -4, -6, -6, -4, 0, 6, 14, 24, 36, 50, 66, 84, 104, 126, \ldots$$

$$b_n = -4, -6, 0, 14, 36, 66, 104, \ldots$$