MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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$a^b = e^{b\log(a)}$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$T\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$

$A = Q\Lambda Q^{-1}$

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$

$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

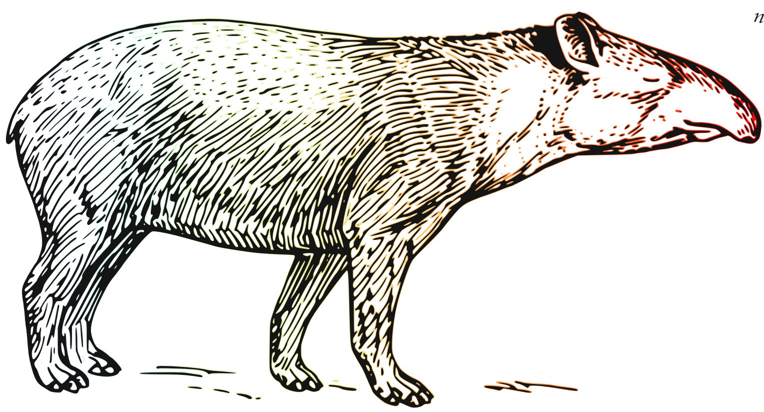
$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$

$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$

$e^{\pi i} + 1 = 0$

$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$



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CHAPTER 0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

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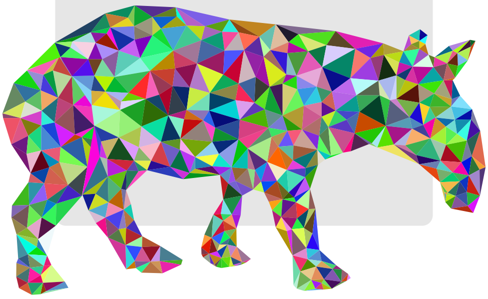
CHAPTER 1



LINEAR ALGEBRA
(INTUITIVE APPROACH)

Linear algebra is one of the most important and often used fields, both in theoretical and applied mathematics. It brings together the analysis of systems of linear equations and the analysis of linear functions (in this context usually called linear transformations), and is employed extensively in almost any modern mathematical field, e.g. approximation theory, vector analysis, signal analysis, error correction, 3-dimensional computer graphics and many, many more.

In this book, we divide our discussion of linear algebra into to chapters: the first (this chapter) deals with a wider, birds-eye view of the topic: it aims to give an intuitive understanding of the major ideas of the topic. For this reason, in this chapter we limit ourselves almost exclusively to discussing linear algebra using 2- and 3-dimensional analysis (and higher dimensions when relevant) using real numbers only. This allows us to first create an intuitive picture of what is linear algebra all about, and how to use correctly the tools it provides us with.

The next chapter takes the opposite approach: it builds all concepts from the ground-up, defining precisely (almost) all basic concepts and proving them rigorously, and only then using them to build the next steps. This approach has two major advantages: it guarantees that what we build has firm foundations and does not fall apart at any future point, and it also allows us to generalize the ideas constructed during the process to such extent that they can be used as foundation to build ever newer tools we can apply in a wide range of cases.

1.1 DUAL SPACES

! **To be written**: the section. !

1.2 THE BRA-KET NOTATION

In this section we introduce a special vector notation widely used in physics: the **Bra-ket notation**, also known as **Dirac's notation**. This notation helps simplify many aspects of linear algebra, and make its use more streamlined.

Note 1.1 Importance of this section

A person can have a pretty good grasp of linear algebra without ever learning about the bra-ket notation. This section, while interesting and providing a useful tool in working with linear algebra - is not obligatory, especially to people who do not intend to ever learn topics such as quantum physics, relativity theory, statistical mechanics, etc. It is however recommended even for those readers.

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1.2.1 Definition

Up until now we presented the theory of linear algebra based on real numbers: all of the vectors we used were real vectors, i.e. of the form $\vec{v} \in \mathbb{R}^n$, where $n \in \mathbb{N}$. All of the matrices used were also made up of real components - and so were of course the scalars themselves, which we defined simply as real numbers.

However, it is aparently useful in many cases to use linear algebra in the context of complex numbers: instead of working with spaces of the form $\mathbb{R}^n$ we can use spaces of the form $\mathbb{C}^n$, e.g. a vector in $\mathbb{C}^3$ can be the following:

$$\vec{v} = \begin{bmatrix} 1-i \\ 2+3i \\ \pi+\sqrt{2}i \end{bmatrix}.$$

When using complex numbers instead of real numbers, a small change must be made to the way we conceptualize row vs. column vectors. Before we said that essentially both forms can be used interchangeably without affecting the outcome. However now we define row vectors a bit differently: given the column vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \tag{1.2.1}$$

we can get its row form by transposing it, i.e. we look at $\vec{v}^\top$. However, when doing this we must change all the components of $\vec{v}$ to their respective complex conjugates, i.e.

$$\vec{v}^\top = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^\top = \left[\overline{v_1},\ \overline{v_2},\ \dots,\ \overline{v_n}\right]. \tag{1.2.2}$$

$$\vec{v}^\top = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^\top = \left[v_1^*,\ v_2^*,\ \dots,\ v_n^*\right]. \tag{1.2.3}$$

To be consistent with the usual notation used in physics (and have more succisent, we introduce the following changes:

- The complex conjugate of the number $z \in \mathbb{C}$ is changed to C^*.
- The star notation is also used for transpose (this is normaly called **conjugate transpose**, and is sometimes denoted by $u^\dagger$).
- The arrow is dropped from the vector notation.

Applying these changes, Equation 1.2.3 has the form

$$v^* = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^* = \left[v_1^*,\ v_2^*,\ \dots,\ v_n^*\right]. \tag{1.2.4}$$

Example 1.1 Row vectors

The row form of the vector

$$v = \begin{bmatrix} 1+2i \\ 3-i \\ \sqrt{2}+5i \\ 4 \\ -3i \end{bmatrix}$$

is

$$v^* = \begin{bmatrix} 1+2i \\ 3-i \\ \sqrt{2}+5i \\ 4 \\ -3i \end{bmatrix}^* = \begin{bmatrix} 1-2i, \ 3+i, \ \sqrt{2}-5i, \ 4, \ 3i \end{bmatrix}.$$

Next, we make sure that the scalar product between any two vectors u, v is such that the left vector is a row vector, and the right vector is a column vector, i.e. given $u, v \in \mathbb{C}^n$ such that

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix},$$

their scalar product is

$$u \cdot v = u^* \cdot v = \begin{bmatrix} u_1^*, \ u_2^*, \ \dots, \ u_n^* \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n. \tag{1.2.5}$$

Recall that a common notation for the scalar product of two vectors uses triangular brakets, i.e.

$$u \cdot v = \langle u | v \rangle.$$

We can use Equation 1.2.5 and "separate" this product into two parts: a **bra** $\langle u|$ and a **ket** $|v\rangle$, define as

$$\begin{aligned} \langle u| &= \begin{bmatrix} u_1^*, \ u_2^*, \ \dots, \ u_n^* \end{bmatrix}, \\ |v\rangle &= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \end{aligned} \tag{1.2.6}$$

1.2.2 Norm and products

The norm of a vector v can be calculated by taking the square root of its scalar product with itself (**??**). Using the bra-ket notation this becomes

$$\left\|\vec{v}\right\| = \sqrt{\langle v|v\rangle}. \tag{1.2.7}$$

Let us write the properties of the scalar product adjusted to the bra-ket notation:

- **Non-negative norm**: for any vector $v \in \mathbb{C}^n$, $\langle v|v\rangle \geq 0$.
- **Uniqeness of zero**: if $\langle v|v\rangle = 0$, then $v = 0$.
- **Conjugate commutativity**: for any two vectors $u, v \in \mathbb{C}^n$, $\langle u|v\rangle = \langle v|u\rangle^*$.
- **Distributivity**: Given three vectors $u, v, w \in \mathbb{C}^n$ and two scalars $\alpha, \beta \in \mathbb{C}$,

$$\langle u|\Big(\alpha\,|v\rangle + \beta\,|w\rangle\Big) = \alpha\,\langle u|v\rangle + \beta\,\langle u|w\rangle.$$

Note 1.2 Hilbert spaces

enerally speaking, any vector space that is "equiped" with a norm complying with these properties is called a **Hilbert space**. We will discuss such spaces in more details later in the book.

!

There is an interesting way one can interpret the scalar product: instead of as an operation acting on two vectors, we can view a bra (i.e. a row vector) as an operator acting on a column vector and returning a scalar. Mathematically this is written as

$$\langle\bigcirc| : \mathbb{C}^n \to \mathbb{C}. \tag{1.2.8}$$

(the empty circle signifies that that the symbol representing the row vector is placed inside the bra symbol)

! **To be written**: this is the dual space of $\mathbb{C}^n$, etc. !

Another product that is easily defined usin the bra-ket notation is the **exterior product** of two vectors (recall that the scalar product is also called the *inner product*). The exterior product arises when we multiply two vectors in the opposite order compared to the scalar product, i.e. instead of $\langle u|v\rangle$ we calculate $|u\rangle\langle v|$:

$$|u\rangle\langle v| = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\begin{bmatrix} v_1^*, & v_2^*, & \dots, & v_n^* \end{bmatrix} = \begin{bmatrix} u_1v_1^* & u_1v_2^* & \dots & u_1v_n^* \\ u_2v_1^* & u_2v_2^* & \dots & u_2v_n^* \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1^* & u_nv_2^* & \dots & u_nv_n^* \end{bmatrix}, \tag{1.2.9}$$

i.e. the result of the outer product is a matrix, in which any element a_{ij} is equal to

$$a_{ij} = u_i v_j^*. \tag{1.2.10}$$

Note 1.3 Bra-ket vs. ket-bra

Using the bra-ket notation, the names for the inner and outer products make sense: in the inner product the vectors stay inside the brackets, while in the outer product they are outside of it. This is just a small demonstration for the "power" of the bra-ket notation in simplifying mathematical expressions.

!

Example 1.2 Inner and outer products

Given the two column vectors

$$|u\rangle = \begin{bmatrix} 1+\mathrm{i} \\ 3 \end{bmatrix}, \; |v\rangle = \begin{bmatrix} -2+3\mathrm{i} \\ 5-\mathrm{i} \end{bmatrix},$$

let us calculate the following:

$$\langle u|v\rangle, \; \langle v|u\rangle, \; |u\rangle\langle v|, \; |v\rangle\langle u|.$$

We start by writing $\langle u|$ and $\langle v|$:

$$\langle u| = \begin{bmatrix} 1-\mathrm{i}, & 3 \end{bmatrix}, \; \langle v| = \begin{bmatrix} -2-3\mathrm{i}, & 5+\mathrm{i} \end{bmatrix}.$$

Then, the 4 requested products are easy to calculate:

$$\langle u|v\rangle = (1-\mathrm{i})(-2+3\mathrm{i}) + 3(5-\mathrm{i}) = -2+3\mathrm{i}+2\mathrm{i}+3+15-3\mathrm{i} = 16+2\mathrm{i}.$$
$$\langle v|u\rangle = (1+\mathrm{i})(-2-3\mathrm{i}) + 3(5+\mathrm{i}) = -2-3\mathrm{i}-2\mathrm{i}+3+15+3\mathrm{i} = 16-2\mathrm{i} = \langle u|v\rangle^*.$$

$$|u\rangle\langle v| = \begin{bmatrix} (1+\mathrm{i})(-2-3\mathrm{i}) & (1+\mathrm{i})(5+\mathrm{i}) \\ 3(-2-3\mathrm{i}) & 3(5+\mathrm{i}) \end{bmatrix} = \begin{bmatrix} 1-5\mathrm{i} & 4+6\mathrm{i} \\ -6-9\mathrm{i} & 15+3\mathrm{i} \end{bmatrix}.$$
$$|v\rangle\langle u| = \begin{bmatrix} (-2+3\mathrm{i})(1-\mathrm{i}) & 3(-2+3\mathrm{i}) \\ (5-\mathrm{i})(1-\mathrm{i}) & 3(5-\mathrm{i}) \end{bmatrix} = \begin{bmatrix} 1+5\mathrm{i} & -6+9\mathrm{i} \\ 4-6\mathrm{i} & 15-3\mathrm{i} \end{bmatrix}.$$

1.2.3 Basis sets

! **To be written**: text text text... !