MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

Written, illustrated and typeset (mostly) by

PELEG BAR SAPIR

with contributions from others

$$a^b = e^{b \log(a)}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$T\left(\alpha \vec{u} + \beta \vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$$

$$A = Q \Lambda Q^{-1}$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t$$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

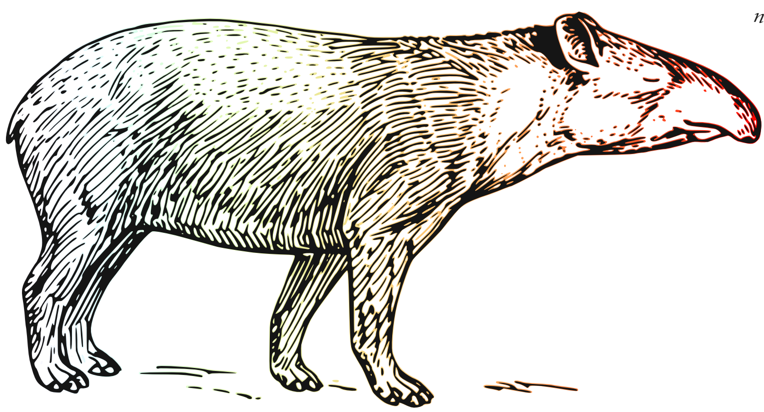
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$e^{\pi i} + 1 = 0$$

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$



PUBLISHED IN THE WILD

! **To be written/to do**: Rights, lefts, etc. will be written here in the future !

HERE BE TABLE

CHAPTER 0

INTRODUCTION

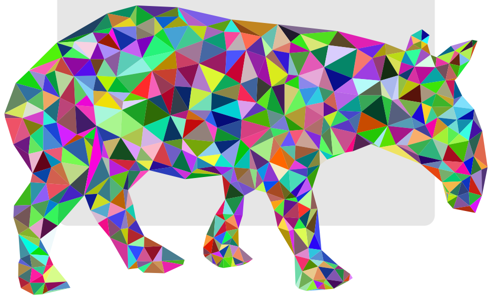
In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

!

CHAPTER 1



LINEAR ALGEBRA

(INTUITIVE APPROACH)

Linear algebra is one of the most important and often used fields, both in theoretical and applied mathematics. It brings together the analysis of systems of linear equations and the analysis of linear functions (in this context usually called linear transformations), and is employed extensively in almost any modern mathematical field, e.g. approximation theory, vector analysis, signal analysis, error correction, 3-dimensional computer graphics and many, many more.

In this book, we divide our discussion of linear algebra into to chapters: the first (this chapter) deals with a wider, birds-eye view of the topic: it aims to give an intuitive understanding of the major ideas of the topic. For this reason, in this chapter we limit ourselves almost exclusively to discussing linear algebra using 2- and 3-dimensional analysis (and higher dimensions when relevant) using real numbers only. This allows us to first create an intuitive picture of what is linear algebra all about, and how to use correctly the tools it provides us with.

The next chapter takes the opposite approach: it builds all concepts from the ground-up, defining precisely (almost) all basic concepts and proving them rigorously, and only then using them to build the next steps. This approach has two major advantages: it guarantees that what we build has firm foundations and does not fall apart at any future point, and it also allows us to generalize the ideas constructed during the process to such extent that they can be used as foundation to build ever newer tools we can apply in a wide range of cases.

1.1 VECTORS

1.1.1 Basics

Vectors are the fundamental objects of linear algebra: the entire field revolves around manipulation of vectors. In this chapter we deal with the so-called **real vectors**, which can be be defined in a geometric way:

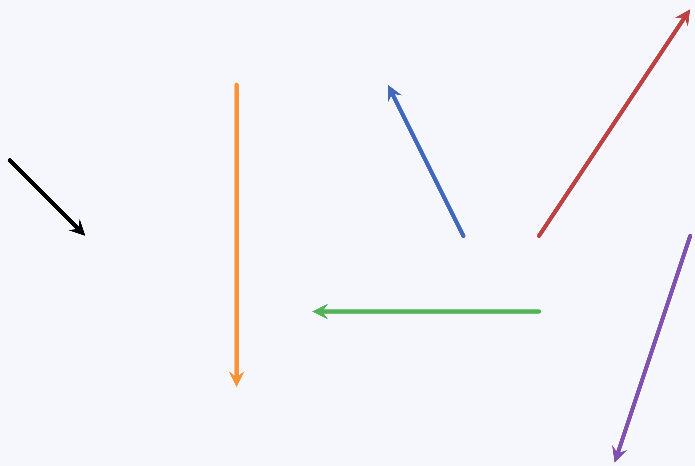
Definition 1.1 Real vectors

A *real vector* is an object with a **magnitude** (also called **norm**) and a **direction**.

In this chapter we refer to real vectors simply as *vectors*.

Example 1.1 Real vectors

The following are all vectors in 2-dimensional space depicted as arrows:



Vectors are usually denoted in one of the following ways:

- **Arrow above letter**: $\vec{u}$, $\vec{v}$, $\vec{x}$, $\vec{a}$, …

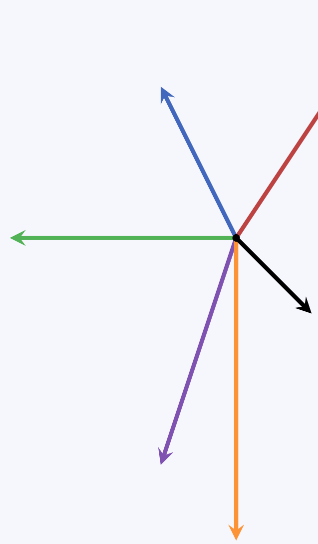
- **Bold letter**: $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{a}, \ldots$
- **Bar below letter**: $\underline{u}, \underline{v}, \underline{x}, \underline{a}, \ldots$

In this book we use the first notation style, i.e. an arrow above the letter. In addition vectors will almost always be denoted using lowercase Lating script.

When discussing vectors in a single context, we always consider them starting at the same point, called the **origin**, and **translating** (moving) vectors around in space does not change their properties: only their norms and directions matter.

Example 1.2 Real vectors

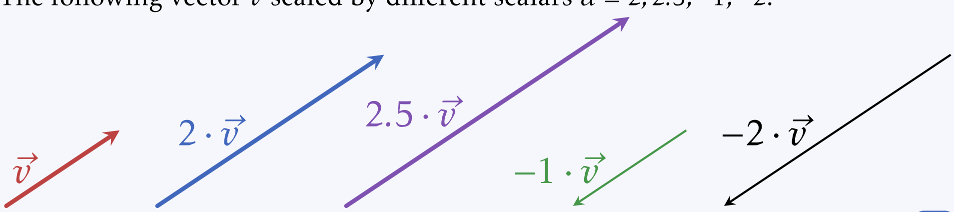
The vectors from the previous example, translated (moved) such that their origins all lie on the same point:



A vector can be scaled by a real number α: when this happens, its norm is multiplied by α while its direction stays the same. We call α a **scalar**.

Example 1.3 Scaling vectors

The following vector $\vec{v}$ scaled by different scalars $\alpha = 2, 2.5, -1, -2$:



Note 1.1 Negative scale

As can be seen in the example above, when scaling a vector by a negative amount its direction reverses. However, we consider two opposing direction (i.e. directions that are 180° apart) as being the same direction.

!

In this chapter we use the following notation for the norm of a vector $\vec{v}$: $\left\|\vec{v}\right\|$.

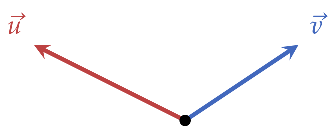
A vector $\vec{v}$ with norm $\left\|\vec{v}\right\| = 1$ is called a **unit vector**, and is usually denoted by replacing the arrow symbol by a hat symbol: $\hat{v}$. Any vector (except $\vec{0}$) can be scaled into a unit vector by scaling the vector by 1 over its own norm, i.e.

$$\hat{v} = \frac{1}{\left\|\vec{v}\right\|}\vec{v}. \tag{1.1.1}$$

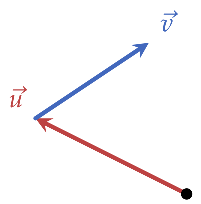
The result of normalization is a vector of unit norm which points in the same direction of the original vector.

Two vectors can be added together to yield a third vector: $\vec{u} + \vec{v} = \vec{w}$. To find $\vec{w}$ we use the following procedure (depicted in Figure 1.1):

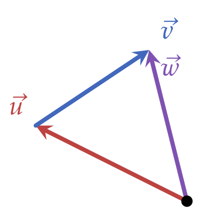
1. Move (translate) $\vec{v}$ such that its origin lies on the head of $\vec{u}$.
2. The vector $\vec{w}$ is the vector drawn from the origin of $\vec{u}$ to the head of $\vec{v}$.



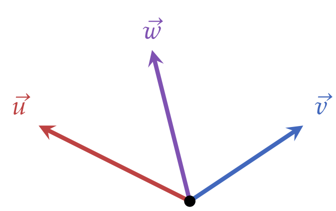
1 The vectors $\vec{u}$ and $\vec{v}$.



2 Translating $\vec{v}$ such that its origin lies at the head of $\vec{u}$.



3 Drawing the vector $\vec{w}$ from the origin to the head of $\vec{v}$.



4 Showing all three vectors.

Figure 1.1 Vector addition.

The addition of vectors as depicted here is commutative, i.e. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. This can be seen by using the **parallogram law of vector addition** as depicted in Figure 1.2: drawing the two vectors $\vec{u}, \vec{v}$ and their translated copies (each such that its origin lies on the other vector's head) results in a parallelogram.

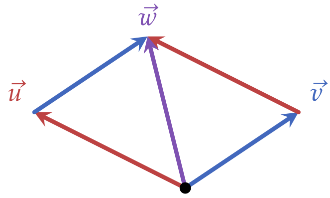


Figure 1.2 The parallogram law of vector addition.

An important vector is the **zero-vector**, denoted as $\vec{0}$. The zero-vector has a unique property: it is neutral in respect to vector addition, i.e. for any vector $\vec{v}$,

$$\vec{v} + \vec{0} = \vec{v}. \tag{1.1.2}$$

(we also say that $\vec{0}$ is the **additive identity** in respect to vectors.)

Any vector $\vec{v}$ always has an **opposite** vector, denoted $-\vec{v}$. The addition of a vector and its opposite always result in the zero-vector, i.e.

$$\vec{v} + \left(-\vec{v}\right) = \vec{0}. \tag{1.1.3}$$

1.1.2 Components

In order to be able to use vectors for actual calculations we must somehow quantify their properties. When quantifying geometric shapes we often first define some unit of measurement, for example a centimeter (cm). We then use this unit to measure the length of different objects (Figure 1.3).

While this simple approach works well for describing lengths, in the case of vectors we also want to quantify directions - which becomes a bit complicated in higher dimensions if we use angles. Instead, we use more than one unit of measurement; in fact, we use a unit of measurement for each dimension, which we call a **unit vector**. For example, we can choose the following two vectors $\vec{e}_1$ and $\vec{e}_2$ to be our unit vectors:

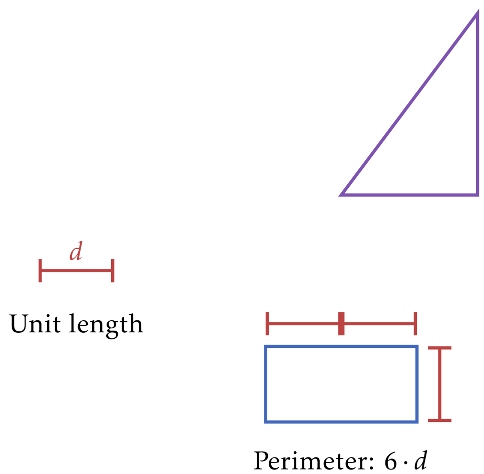


Figure 1.3 Using a defined unit measurement to measure the lengths of different geometric objects.