CHAPTER

# 0

# INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explainations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is adviced to first get an intuitive understanding of the ideas before diving into their more formal construction.

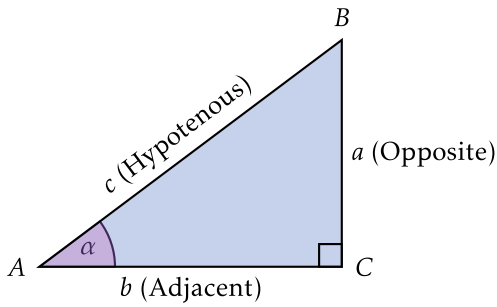
**Note 0.1 In case you are already familiar with the topics**

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concept presented here. !

## 0.1 TRIGONOMETRIC FUNCTIONS

### 0.1.1 Basic Definitions

Consider a **right triangle** $\triangle ABC$ with sides $a, b$, and Hypotenous $c$, where the angle $\angle ACB$ is 90°, and the angle $\angle BAC$ is denoted as $\alpha$:




We use the ratios between the three sides of the triangle to define three functions of $\alpha$:

- The **sine** of the angle $\alpha$ is $\sin(\alpha) = \frac{a}{c}$,
- the **cosine** of the angle $\alpha$ is $\cos(\alpha) = \frac{b}{c}$, and
- the **tangent** of the angle $\alpha$ is $\tan(\alpha) = \frac{a}{b}$, which in turn is equal to $\frac{\sin(\alpha)}{\cos(\alpha)}$.

We can rearrange the above definitions:

$$\begin{aligned} a &= c\sin(\alpha), \\ b &= c\cos(\alpha). \end{aligned} \tag{0.1.1}$$

Normaly, the Hypotenous is the longest side of a right triangle. We will consider here the two edge cases where one of the sides $a$ or $b$ is equal to the Hypotenous (and the other side is thus 0):

- if $a = c$ then $\alpha = 90°$,
- if $b = c$ then $\alpha = 0°$.

The posssible length of $a$ is therefore in the range $0 \leq a \leq c$, which means that $0 \leq \frac{a}{c} \leq 1$. Since $\sin(\alpha) = \frac{a}{c}$ this means that the image of $\sin(\alpha)$ is $[0, 1]$. The same idea is also true for $b$, and therefore $[0, 1]$ is the image of $\cos(\alpha)$ as well.

As a reminder, the **Pythagorean theorem**[1] states that for a right triangle with sides $a, b$ and $c$,

$$a^2 + b^2 = c^2. \tag{0.1.2}$$

By substituting **Equation** 0.1.1 into the Pythagorean theorem we get

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= [c\sin(\alpha)]^2 + [c\cos(\alpha)]^2 \\ &= c^2\sin^2(\alpha) + c^2\cos^2(\alpha) \\ &= c^2\left[\sin^2(\alpha) + \cos^2(\alpha)\right], \end{aligned}$$

and therefore

$$\sin^2(x) + \cos^2(x) = 1. \tag{0.1.3}$$

### 0.1.2 The Unit Circle

We defined $\sin(\alpha)$ and $\cos(\alpha)$ so far in way such that their domains are both $[0°, 90°]$, and their images are both $[0, 1]$. However, there is a simple way to extend these functions such that both their domains are $\mathbb{R}$, and both their images are $[-1, 1]$: by using a **unit circle**.

**Figure 0.1** depicts a unit circle: it is simply a circle of radius $R = 1$, which is placed such that its center lies at the origin of a 2-dimensional axis system (i.e. at the point $\boldsymbol{O} = (0, 0)$. We then draw a line from $\boldsymbol{O}$ to a point $\boldsymbol{P} = (x, y)$ on the circumference of the unit circle. We call the angle between the line $\boldsymbol{OP}$ and the $x$-axis $\theta$. We then draw another line, this time from the point $\boldsymbol{P}$ to a point $\boldsymbol{D}$ on the $x$-axis, such that $\boldsymbol{PD}$ is perpendicular to the $x$-axis.

The triangle $\triangle OPD$ is a right triangle. Therefore, we can use the trigonometic functions to calculate the coordinates of the point $\boldsymbol{P} = (x, y)$:

$$\begin{aligned} x &= R\cos(\theta) = \cos(\theta), \\ y &= R\sin(\theta) = \sin(\theta). \end{aligned} \tag{0.1.4}$$

We then define $\cos(\theta)$ and $\sin(\theta)$ as a function of $\theta$:

$$\begin{aligned} \sin(\theta) &= y, \\ \cos(\theta) &= x. \end{aligned} \tag{0.1.5}$$

Using this definition, the angle $\theta$ can take any value between $0°$ and $360°$. In fact, the values of $\theta$ can be extended to any real number in degrees: if it is more than $360°$... (MUST FINISH THIS PART + QUADRENTS)

[1] It's worth mentioning that no three positive integers $a, b$, and $c$ satisfy the equation $a^n + b^n = c^n$ for any integer value of $n > 2$. This can be proven, however the proof is too large to fit in the footnotes.

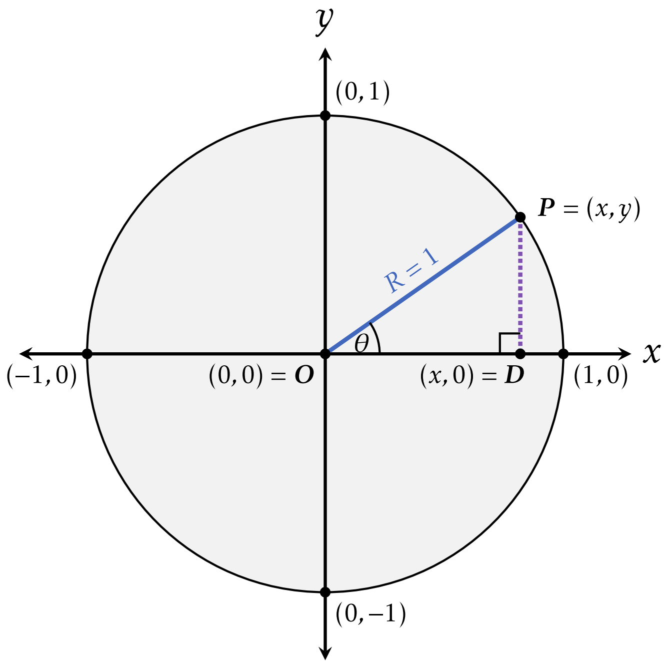



**Figure 0.1** A unit circle with (...)

### 0.1.3 Radians

Using degrees to measure angles in a sphere creates an inconvinience: the domain and image of the trigonometric functions have different units. In order to measure both these magnitudes using the same unit we switch to measuring angles on a circle using **radians** instead of degrees. One radian equals the length of a single radius $R$ of the circle (in the case of the unit circle this is always $R = 1$). We define an inner angle $\theta$ to equal one radian if the arc length it represents is equal to $R$ (see **Figure 0.2** ).

How much is a radian in degrees? The full circumference of any circle with radius $R$ equals $2\pi R$, which means that a single radian $R$ is equivalent to $\frac{180°}{\pi} \approx 57.3°$. **Table** 0.1 shows some common angles and their equivalent value in radians.

**Table 0.1** Common angles in radians.

| Degrees | Radians |
|---|---|
| 0° | 0 |
| 45° | $\frac{\pi}{4}$ |
| 90° | $\frac{\pi}{2}$ |
| 180° | $\pi$ |
| 270° | $\frac{3\pi}{2}$ |
| 360° | $2\pi$ |

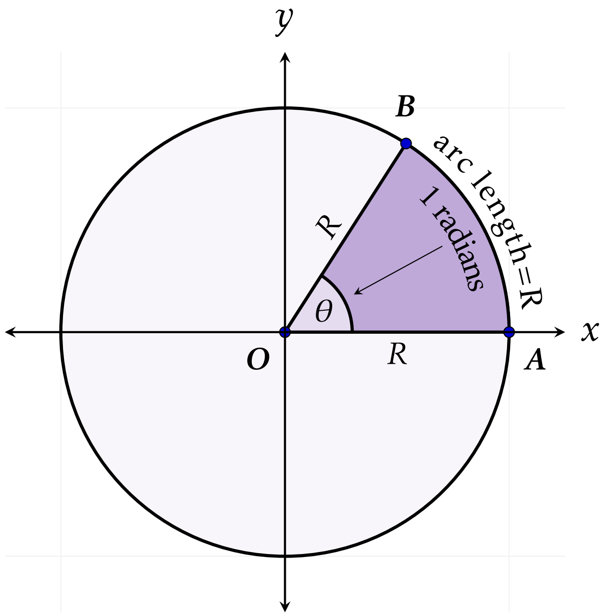



**Figure 0.2** In this figure the arc ***AB*** has the same length of the radii ***OA*** and ***OB*** (all are equal to $R$), and therefore $\theta = 1$ radians.

MORE WILL BE WRITTEN HERE!

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**Table 0.2** Text

| Quadrant | $\cos(\theta) = x$ | $\sin(\theta) = y$ |
|---|---|---|
| 1 | $[0,1]$ | $[0,1]$ |
| 2 | $[-1,0]$ | $[0,1]$ |
| 3 | $[-1,0]$ | $[-1,0]$ |
| 4 | $[0,1]$ | $[-1,0]$ |

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## 0.2 EXERCISES

0.1. First question.

0.2. Second question.

0.3. More questions...