MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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$a^b = e^{b\log(a)}$ $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $T\left(\alpha\vec{u}+\beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$

$A = Q\Lambda Q^{-1}$ $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$

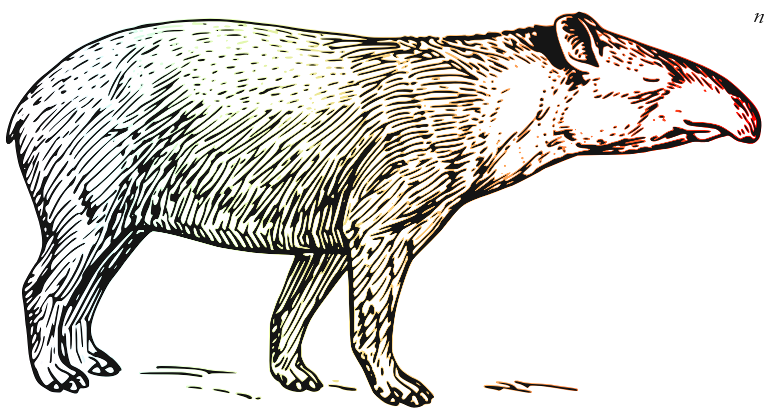
$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ $\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$ $\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$

$e^{\pi i} + 1 = 0$

$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$



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HERE BE TABLE

CHAPTER 0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

0.1 TAYLOR SERIES

Unlike most real functions, polynomials are relatively easy to calculate for any real value x. For example, to correctly calculate the value of the polynomial $P(x) = x^2 - 2x + 5$ we simply need to know how to perform the following operations: addition, subtraction, muliplication and raising by an integer power, all operations that are easy for both humans and computers to perform.

Ideally, we would like to use polynomials for all calculations, e.g. given the function $f(x) = \cos(x)$ it would be great if we could find some polynomial $P(x)$ of a finite order n for which $P(x) = \cos(x)$. Unfortunately such a polynomials does not exist, nor does such polynomials exist for $\sin(x), \sqrt(x), \exp(x), a^x \;\; (a > 0)$ or any of the so-called fundamental functions and their compositions (except, of course, polynomials). The reason for this fact is rather complicated, but in short it lies in the fact that these functions are *non-algebric*[1].

However, we do know at least one value of each of these functions with infinite precision in at least one $x \in \mathbb{R}$, sometimes at several values of x or even at infinitely many values of x. For example, we know that $\cos(0) = 1$ and $\cos\left(\frac{1}{2}\pi\right) = 0$. From symmetry we thus know that $\cos(\pi) = -1$ and $\cos\left(\frac{3}{4}\pi\right) = 0$. Since $\cos(x)$ is periodic, i.e. $\cos(x + 2\pi k) = \cos(x)$ for any $k \in \mathbb{Z}$, we actually know infinitely many values of the function. For $\exp(x)$ we know that $\exp(0) = 1$ and $\exp(1) = \mathrm{e}$. One can argue that since e is not algebraic, we don't *really* know its value (i.e. we can't calculate it an infinite precision), but even it that case we still definitely know the value of $\exp(0)$ with inifnite precision.

We can use this knowledge to constract a polynomial which approximates the function's value in any $x \in \mathbb{R}$ to whatever precision we wish, given that a specific condition is met. We will describe this condition later, but first let's use an example function to construct such a polynomial - $\exp(x)$.

[1]technically $\sqrt{x}$ is algebraic, and there are indeed many methods to calculate its values - but non of these methods are as simple as calculating a polynomial. . . except the one we discuss in this section.