MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

Written, illustrated and typeset (mostly) by

PELEG BAR SAPIR

with contributions from others

$$a^b = e^{b\log(a)}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$T\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$$

$$A = Q\Lambda Q^{-1}$$

$$\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t}\,\mathrm{d}t$$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

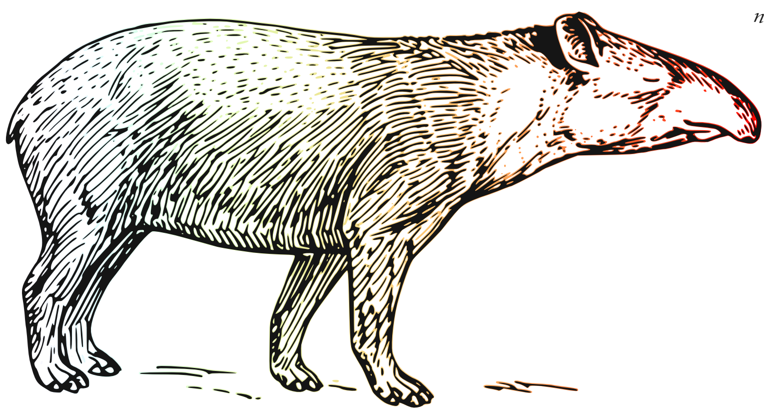
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$e^{\pi i} + 1 = 0$$

$$\int_{a}^{b} f(x)\,\mathrm{d}x = F(b) - F(a)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$



PUBLISHED IN THE WILD

! **To be written**: Rights, lefts, etc. will be written here in the future !

HERE BE TABLE

CHAPTER 0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

!

0.1 MATHEMATICAL SYMBOLS AND SETS

0.1.1 Logical Statements and their Truth Value

We start our discussion with the simplest mathematical concept: a **proposition**. A proposition is simply a statement that might be either **true** or **false**.

Example 0.1 Truth of propositions

- $3 > 1$ (**true**)
- $-2 = 5 - 7$ (**true**)
- $7 < 5$ (**false**)
- The radius of the earth is bigger than that of the moon. (**true**)
- The word 'House' starts with the letter 'G'. (**false**)

We can group together propositions using **logical operators**. Two of the most common logical operators are **AND** and **OR**.

The **AND** operator returns a **true** statement only if **both** the statements it groups are themselves **true**, otherwise it returns **false**.

Example 0.2 The AND operator

- $2+4=6$ is **true**, $4-2=2$ is **true**. $(2+4=6$ **AND** $4-2=2)$ is therefore **true**.
- $2+4=6$ is **true**, $2>6$ is **false**. $(2+4=6$ **AND** $2>6)$ is therefore **false**.
- $\frac{10}{2}=1$ is **false**, $2^4=16$ is **true**. $\left(\frac{10}{2}=1\right.$ **AND** $\left.2^4=16\right)$ is therefore **false**.
- $7<5$ is **false**, $10+2=13$ is **false**. $(7<5$ **AND** $10+2=13)$ is therefore **false**.

The **OR** operator returns **true** if **at least** one of the statements it groups is true.

Example 0.3 The OR operator

- $2+4=6$ is **true**, $4-2=2$ is **true**. $(2+4=6$ **OR** $4-2=2)$ is therefore **true**.
- $2+4=6$ is **true**, $2>6$ is **false**. $(2+4=6$ **OR** $2>6)$ is therefore **true**.
- $\frac{10}{2}=1$ is **false**, $2^4=16$ is **true**. $\left(\frac{10}{2}=1\right.$ **OR** $\left.2^4=16\right)$ is therefore **true**.
- $7<5$ is **false**, $10+2=13$ is **false**. $(7<5$ **OR** $10+2=13)$ is therefore **false**.

Table 0.1 The truth table for the operators **AND** and **OR**.

A	B	A AND B	A OR B
true	true	true	true
true	false	false	true
false	true	false	true
false	false	false	false

Table 0.2 Common mathematical notations used in this Book.

Symbol	In words
$\neg a$	**not** a
$a \wedge b$	a **and** b
$a \vee b$	a **or** b
$a \Rightarrow b$	a **implies** b
$a \Leftrightarrow b$	a **is equivalent to** b
$\forall x$	**For all** x (...)
$\exists x$	**There exists** x **such that** (...)
$a := b$	a **is defined to be** b
$a \equiv b$	a **is equivalent to** b

The behaviour of both operators can be summarized using a **truth table** (see Table 0.1 below).

When writing, it is convenient to use **notations** to represent operators: the **AND** operator is denoted by $\wedge$, while the **OR** operator is denoted by $\vee$.

Example 0.4 Using the notations for AND and OR

$$(\underbrace{2+2=5}_{\text{false}}) \wedge (\underbrace{1-1=0}_{\text{true}}) \Rightarrow \textbf{false}$$

$$(\underbrace{2+2=5}_{\text{false}}) \vee (\underbrace{1-1=0}_{\text{true}}) \Rightarrow \textbf{true}$$

0.1.2 Common mathematical notations

Several more common mathematical notations are given in Table 0.2.

The notation $\Rightarrow$ need a bit of clarification: implication means that we can directly derive a proposition from another proposition. For example, if $x = 3$ then $x > 2$. The opposite implication can be a **false** statement, i.e. for the example above $x > 2$ does not imply $x = 3$ (denoted as $x > 2 \not\Rightarrow x = 3$). Sometimes implication is expressed by using the word *if*: in the above example $x > 2$ if $x = 3$, but the other way around is not **true**.

We say that two propositions are **equivalent** when they imply each other. For example: $x = 2$ implies that $\frac{x}{2} = 1$, while $\frac{x}{2} = 1$ implies that $x = 2$. We can write this as

$$\frac{x}{2} = 1 \Leftrightarrow x = 2.$$

Instead of the word *equivalent*, the phrase *if and only if* (sometimes shortened to **iff**) is commonly used, e.g.

$$x = 2 \text{ iff } \frac{x}{2} = 1.$$

0.1.3 Sets and subsets

The concept of **sets** is perhaps one of the most basic ideas in modern mathematics. Much of the material covered in this book will be built upon sets and their properties. However, as with the rest of the material presented here - our description of sets will not be thorough nor precise.

For our purposes, a set is a collection of **elements**. These elements can be any concept - be it physical (a chair, a bicycle, a tapir) or abstract (a number, an idea). However, we will consider only sets comprised of numbers. Sets can have finite of infinite number of elements in them.

We denote sets by using curly brackets, and if the number of elements in them is not too big - we display the elements, separated by commas, inside the brackets. In other cases we can express the sets as a sentence or a mathematical proposition.

Example 0.5 Simple sets

$$\{1, 2, 3, 4\} \qquad \left\{-4, \frac{3}{7}, 0, \pi, 0.13, -2.5, \frac{e}{3}, 2^{-\pi}\right\} \qquad \{\text{all even numbers}\}$$

Sets have two important properties:

1. Elements in a set do not repeat. i.e each element is unique.
2. The order of elements in a set does not matter.

Example 0.6 Important set properties

Examples demonstrating the two aforementioned important properties of sets:

1. The following is not a set:

$$\{1,1,0,1,0,0,-1,0,0,-1,-1,1\}$$

2. The following sets are all identical:

$$\{1,2,3,4\} \qquad \{1,3,2,4\} \qquad \{3,4,1,2\} \qquad \{1,3,2,4\} \qquad \{4,3,2,1\}$$

Sets can be denoted using **conditions**, with the symbol | representing the phrase ”such that”.

Example 0.7 Defining a set using a condition

he following set contains all the odd whole numbers between 0 and 10, including both:

$$\{0 < x < 10 \mid x \text{ is an odd number}\}.$$

The definition of this set can be read as

all numbers x that are bigger than 0 *and are smaller than* 10*, such that x is odd.*

(note that the requirement of x to be an odd number means that it is necessarily a whole number as well)

This set can be written explicitly as

$$\{1,3,5,7,9\}.$$

Sets are usually denoted with an uppercase Latin letter ($A, B, C, \dots$), while their elements are denoted as lowercase letters ($a, b, \alpha, \phi, \dots$). When we want to denote that an element belongs to a set we use the following symbol: $\in$. Conversely, $\notin$ is used to denote that an element *does not* belong to a set.

Example 0.8 Elements in sets

For the two sets

$$A = \{1,2,5,7\}, \quad B = \{\text{even numbers}\},$$

all the following propositions are **true**:

$$\begin{gathered} 1 \in A, \quad 2 \in A, \quad 5 \in A, \quad 7 \in A, \\ 2 \in B, \quad 1 \notin B, \quad 5 \notin B, \quad 7 \notin B. \end{gathered}$$

The number of elements in a set, also called its **cardinality** is denoted using two vertical bars (similar to the way absolute values are denoted).

Example 0.9 Cardinality

For $S = \left\{-3, 0, -2, 7, 1, \frac{1}{2}, 5\right\}$, $|S| = 7$.

An important special set is the **empty set**, which is the set containing no elements. It is denoted by $\emptyset$, and has the unique property that $|\emptyset| = 0$.

0.1.4 Intersection, union, difference and complement sets

Two sets are equal if they both contain the exact same elements and only these elements, i.e.

$$A = B \iff x \in A \Leftrightarrow x \in B. \tag{0.1.1}$$

This proposition reads 'The sets A and B are equal *if and only if* any element x in A is also in B, and any element x in B is also in A'. When all the elements of a set B are also elements of another set A, we say that B is a **subset** of A, and we denote that as $B \subset A$. In mathematical notation, we write

$$B \subset A \Leftrightarrow \forall x \in B, x \in A. \tag{0.1.2}$$

i.e. B is a subset of A **iff** the following is true: any element in B is also an element in A.

Note 0.2 (not so) Surprising properties of subsets

The definition of a subset (Equation 0.1.2) gives rise to two interesting properties:

- The empty set $\emptyset$ is a subset of any set.
- Any set is a subset of itself.

Note 0.3 The uniqueness of $\emptyset$

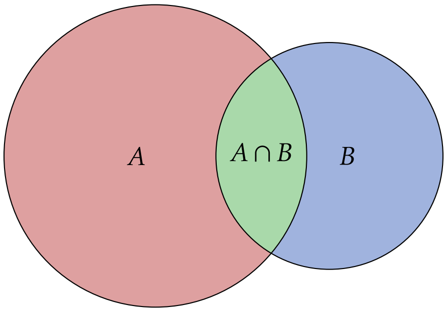
There is only a single empty set, as any set that has no elements is equivalent to any other set with no elements (i.e. they have the same elements). Due to the way subsets are defined, the empty set is a subset of any set (including itself!).

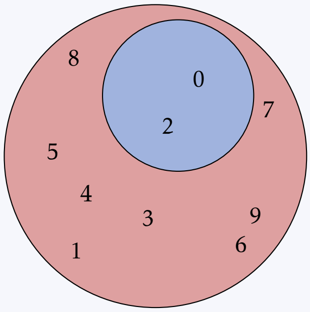
Of course, since we have a definition for a subset, the opposite concept also exists: if B is a subset of A, then we say that A is a **superset** of B.

A very useful way of illustrating the relationship between two or more sets is by using **Venn diagrams**, where sets are represented by circles (or other 2D shapes).

Example 0.10 Subsets and Venn diagrams

A Venn diagram depicting the set $B = \{0, 2\}$ as a subset of $A = \{0, 1, 2, 3, 4, \ldots, 9\}$:





If for two sets A, B both $A \subset B$ and $B \subset A$, then $A = B$. We can write this fact as a mathematical proposition:

$$(A \subset B) \wedge (B \subset A) \Leftrightarrow A = B. \tag{0.1.3}$$

The **intersection** of two sets A and B, denoted $A \cup B$, is the set of all elements x such that $x \in A$ **AND** $x \in B$:

$$A \cup B = \{x \mid x \in A \wedge x \in B\}. \tag{0.1.4}$$

Example 0.11 Intersection of sets

The intersection of the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$ is the set $A \cup B = \{3, 4\}$. The intersection of the sets $C = \{0, 1, 2, 6, 7\}$ and $D = \{3, 9, -4, 5\}$ is the empty set $\emptyset$, since no element is in both sets.

The following Venn diagram depicts the intersection of two sets (the green area):

Note 0.4 Disjoint sets

When the intersection of two sets is the empty set, we say that the set is **disjoint**.

The **union** of two sets (denoted using the symbol $\cup$) is the set composed of all the elements that belong to any of the sets, including elements that are in both sets:

$$A \cup B = \{x \mid x \in A \vee x \in B\}. \tag{0.1.5}$$

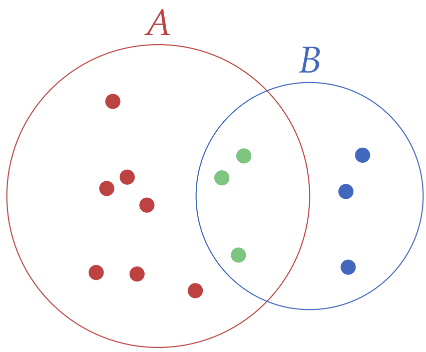


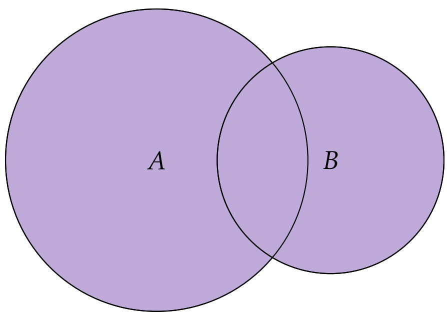
Figure 0.1 Counting the number of elements in the union of two sets: A has 10 elements (red + green dots), while B has 6 elements (blue + green dots). If we count both we get 16 elements, but this counts the joint elements (green dots) twice. Therefore we should subtract the number of joint points, and get that there are only 13 elements in the union.

Example 0.12 Union of sets

The union of the sets $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$ is the set $A \cup B = \{1,2,3,4,5,6\}$.
The union of the sets $C = \{0,1,2,6,7\}$ and $D = \{3,9,-4,5\}$ is the set $C \cup D = \{0,1,2,3,-4,5,6,7,9\}$.

The following Venn diagram depicts the union of two sets (the purple area):

$A \cup B$



Naively, the number of elements of a union $A \cup B$ is simply the sum of the number of elements in A and the number of elements in B. However, this naive approach might count the elements in both sets twice: once for A and once for B (see Figure 0.1) - this is exactly the set $A \cap B$. We therefore subtract the number of elements in $A \cap B$ and get

$$|A \cup B| = |A| + |B| - |A \cap B|. \tag{0.1.6}$$

When two sets A, B are disjoint, then $|A \cap B| = 0$, and so $|A \cup B| = |A| + |B|$.

The definitions of intersections and unions can be easily extended to any whole number of sets.

Example 0.13 Intersection and union of 3 sets

The intersection of 3 sets $A = \{1,2,3,4,5\}$, $B = \{-2,-1,0,1,2\}$ and $C = \{2,3,4,5,6\}$ is the set of all elements that are in A and in B and in C, i.e. the set $A \cap B \cap C = \{2\}$. The union of these sets is the set of all elements that are in either of the sets, i.e. $A \cup B \cup C = \{-2,-1,0,1,2,3,4,5,6\}$.

The most general definition of an intersection of n sets (where n is a whole number), which we will call $A_1, A_2, A_3, \cdots, A_n$ is

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \left\{x \mid (x \in A_1) \land (x \in A_2) \land (x \in A_3) \land \cdots \land (x \in A_n)\right\}. \tag{0.1.7}$$

The left hand side of Equation 0.1.7 can be written as

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i. \tag{0.1.8}$$

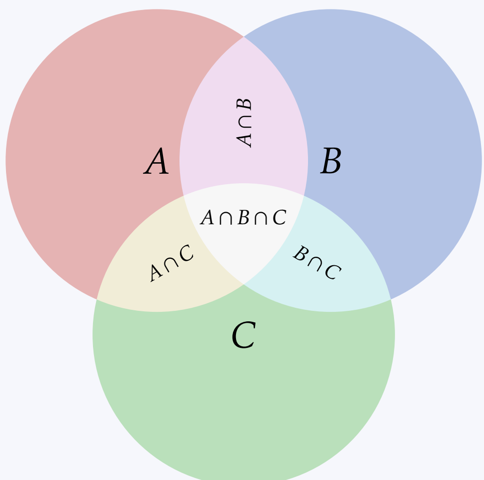
(clarifying the notation? i.e. indexing, etc.)

Similarly, the union of n different sets is defined as

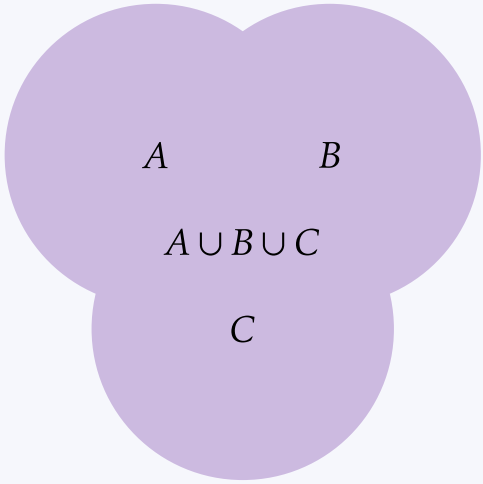
$$\begin{aligned} \bigcup_{i=1}^{n} A_i &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \\ &= \left\{x \mid (x \in A_1) \lor (x \in A_2) \lor (x \in A_3) \lor \cdots \lor (x \in A_n)\right\}. \end{aligned} \tag{0.1.9}$$

Example 0.14 Venn diagrams: intersection and union of 3 sets

The following Venn diagram shows all possible intersections between three sets:



...and the following Venn diagram depicts the union of the same three sets:



The **difference** of two sets A and B (written $A - B$ or $A \setminus B$) is, in a sense, the opposite of their intersection: it is the set of all elements in A that are not in B. Note that $A - B$ doesn't necessarily equal $B - A$, i.e. it is not **commutative**.

Example 0.15 Difference of two sets

Given the two sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5, 6, 7, 8, 9\}$,

$$A - B = \{1, 2\},$$
$$B - A = \{6, 7, 8, 9\}.$$

(note how in this case $A - B \neq B - A$)

Given a set A and a subset of A, $B \subset A$, we can define the **complement** of B in relation to A (notation: $B^{\complement}$) as all the elements in A that are not in B. As the name suggest, the elements of $B^{\complement}$ complete B: $B \cup B^{\complement} = A$.

Example 0.16 Complement of a set

Given the set $A = \mathbb{Z}$ and $B = \{x \in \mathbb{Z} \mid x \text{ is odd}\}$, the complement $B^{\complement}$ in relation to A is the set of all even numbers. The reason is that any integer number can be either odd (in which case it belongs in B) or even (in which case it belongs in $B^{\complement}$).

Given a set A with $|A|$ elements - how many different subsets does it have? We'll start by looking at a practical example: $A = \{1, 2, 3\}$. We can imminently see that any set which contains just one of the elements of A is a subset of A, i.e. $\{1\}, \{2\}, \{3\}$ are all subsets of A. In addition, any set which contains only two elements from A is a subset of A, i.e.

$\{1,2\},\{1,3\},\{2,3\}$. Of course, we must not forget the empty set and A itself - both subsets of A (see Note 0.2). Thus altogether A has 8 subsets:

$$\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}.$$

Generally, any set A with $|A|$ elements has $2^{|A|}$ different subsets. The set of all these subsets is called the **power set** of A, and is denoted as $P(A)$.

Example 0.17 Power set

The power set of $A=\{1,2,3\}$ is

$$P(A)=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}.$$

0.1.5 Important number sets

It is now time to introduce some important number sets. We begin with the simplest of these sets: the **natural numbers**, denoted by $\mathbb{N}$. These are the numbers $1,2,3,4,\ldots$. Adding the opposites to the natural numbers and adding 0 to the set yields the **integers**, denoted by $\mathbb{Z}$. Loosely speaking, we can define the integers as

$$\mathbb{Z}=\{0,\pm1,\pm2,\pm3,\pm4,\ldots\}. \tag{0.1.10}$$

This makes the integers a superset of the natural numbers, i.e.

$$\mathbb{N}\subset\mathbb{Z}. \tag{0.1.11}$$

One can think of the integers as all the number needed for solving an equation of the form $a+x=b$, where a and b are integers themselves, and x is an unknown. No matter which integer values we put in a and b, the unknown x will always be an integer as well (whether it be positive, negative or zero depends on the values of a and b). However, when one wishes to solve an equation of the sort $ax=b$, the integers are not longer sufficient: for example, if $a=2$ and $b=1$, then x is not an integer.

To solve $ax=b$ (where $a,b\in\mathbb{Z}$) we must introduce the **rational numbers**: numbers with values that are ratios of two integers. We denote the set of rational numbers with the symbol $\mathbb{Q}$, and write

$$\mathbb{Q}=\left\{\frac{a}{b}\,\middle|\,a,b\in\mathbb{Z}\wedge b\neq0\right\}. \tag{0.1.12}$$

! **To be written**: discuss briefly why $b\neq0$!

For some combinations of a and b the ratio $\frac{a}{b}$ is an integer. For example: $\frac{3}{1}$, $\frac{8}{4}$, $\frac{-2}{2}$. This makes the integers a subset of the rational numbers, i.e.

$$\mathbb{Z}\subset\mathbb{Q}. \tag{0.1.13}$$

About 2500 years ago it was discovered that some numbers are not rational (and thus also not integers). The most famous example is the number $\sqrt{2}$ - there are not two integers a, b such that $\frac{a}{b} = \sqrt{2}$. We call some of these numbers **algebraic numbers** (denoted by $\mathbb{A}$), and what makes them special is that they are solutions to **polynomial equations**, which we will not define yet (see section xxx). Instead, here is an example for a 2nd order polynomial equation (called a **quadratic equation**):

$$x^2 - 2x - 1 = 0. \tag{0.1.14}$$

Similar to what we saw before, the rational numbers are a subset of the algebraic numbers, i.e.

$$\mathbb{Q} \subset \mathbb{A}. \tag{0.1.15}$$

The algebraic numbers together with other non-rational numbers, such as π and e, form the set of **real numbers**, denoted as $\mathbb{R}$. The definition of real numbers is way beyond the scope of this book, but it is important to understand that the progression we used so far still holds, i.e.

$$\mathbb{A} \subset \mathbb{R}. \tag{0.1.16}$$

The final set of numbers we will touch upon here is the set of **complex numbers**, denoted $\mathbb{C}$, which we can define as

$$\mathbb{C} = \left\{ a + \mathrm{i}b \mid a, b \in \mathbb{R},\ \mathrm{i} = \sqrt{-1} \right\}. \tag{0.1.17}$$

When $b = 0$, Equation 0.1.17 becomes just a single real number - and so

$$\mathbb{R} \subset \mathbb{C}. \tag{0.1.18}$$

(Section 0.7 is dedicated to complex numbers)

Equations 0.1.11-0.1.18 can be merged together to the following single equation:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A} \subset \mathbb{R} \subset \mathbb{C}. \tag{0.1.19}$$

There are more advanced constructions that generalize the complex numbers (i.e. create supersets of the complex number set). These include **quaternions** and **Clifford algebras**. However, as stated before, we will not consider them in this book.

0.1.6 Intervals on the real number line

An important concept that is easily defined over the set $\mathbb{R}$ is an **interval**. A **closed interval** $[a, b]$ is a subset of $\mathbb{R}$ which is defined as

$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}. \tag{0.1.20}$$

An **open interval** (a, b) is a subset of $\mathbb{R}$ which is defined as

$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}. \tag{0.1.21}$$

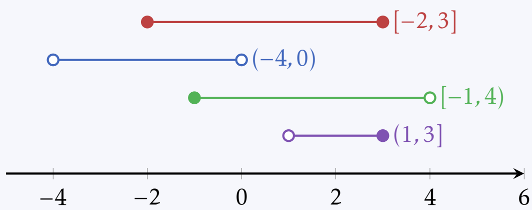
The difference between closed and open intervals is the inclusion and exclusion, respectively, of the edge point: in a closed interval the points a,b are included, while they are not included in an open interval. Of course, we can also create **half open intervals**, i.e.

$$\begin{aligned}[a,b) &= \{x \in \mathbb{R} \mid a \le x < b\},\\ (a,b] &= \{x \in \mathbb{R} \mid a < x \le b\},\end{aligned} \tag{0.1.22}$$

where the first interval includes a but not b, and the second interval includes b but not a.

Example 0.18 Intervals

Intervals can be drawn as colored line segments on top of the real number line:



Note how a full point denotes a closed edge, while an empty point denotes an open edge.

In some cases, it is necessary to use intervals that are infinite in one side, i.e. the left or the right edge are at infinity. In these cases, we use the symbol ∞ to denote infinity, and always keep the interval open at that end:

$$\begin{aligned}(-\infty,b) &= \{x \mid x < b\},\\ (-\infty,b] &= \{x \mid x \le b\},\\ (a,\infty) &= \{x \mid x > a\},\\ [a,\infty) &= \{x \mid x \ge a\}.\end{aligned} \tag{0.1.23}$$

0.1.7 Cartesian Products

The **Cartesian product** of two sets A,B (denoted $A \times B$) is the set of all possible **ordered** pairs, where the first component is an element of A and the second component is an element of B:

$$A \times B = \Big\{(a,b) \mid a \in A,\ b \in B\Big\}. \tag{0.1.24}$$

Example 0.19 Cartesian products

Consider $A = \{1, 2, 3\}$, $B = \left\{x, y\right\}$. Then

$$A \times B = \left\{(1, x),\ \left(1, y\right),\ (2, x),\ \left(2, y\right),\ (3, x),\ \left(3, y\right)\right\}$$

The concept of 'ordered pairs' is paramount: if we reverse the order of the elements in a pair the result might not be in the Cartesian product. We therefore say that the Cartesian product is **not commutative**.

Example 0.20 Non-commutativity of the Cartesian product

The elements $(x, 1)$, $(y, 1)$, $(x, 2)$ and so on **are not** in the Cartesian product $A \times B$ as defined in the previous example, since in each one of the pairs the first element is from B and the second element is from A.

The number of elements in a Cartesian product is the product of the number of elements in each of the sets it is composed of, i.e.

$$|A \times B| = |A| \cdot |B|. \tag{0.1.25}$$

Example 0.21 Number of elements in a Cartesian product

The Cartesian product described in the previous two examples has in total $3 \cdot 2 = 6$ elements, as seen in **??**.

As with intersections and unions, the definition of a Cartesian product can be expanded into any natural number of sets:

$$A_1 \times A_2 \times \cdots \times A_n = \left\{(x_1, x_2, \dots, x_n) \mid x_1 \in A_1,\ x_2 \in A_2,\ \dots, x_n \in A_n\right\}. \tag{0.1.26}$$

Example 0.22 Cartesian product of three sets

The Cartesian product of the sets $A = \{1, 2, 3\}$, $B = \{x, y\}$, $C = \{\alpha, \beta\}$ is

$$\begin{aligned} A \times B \times C = \{&(1, x, \alpha),\ (1, x, \beta),\ (1, y, \alpha),\ (1, y, \beta) \\ &(2, x, \alpha),\ (2, x, \beta),\ (2, y, \alpha),\ (2, y, \beta) \\ &(3, x, \alpha),\ (3, x, \beta),\ (3, y, \alpha),\ (3, y, \beta)\}. \end{aligned}$$

An element in a cartesian product is sometimes called a **tuple**. A tuple can be tought of as a set where the order of elements does matter, and thus elements can repeat. A tuple with n-elements is called an n-tuple.

Example 0.23 Tuples

n	Name	Example
0	empty tuple	$()$
1	monpule	(-5)
2	couple	$(-5,-3)$
3	triple	$(-7,-2,8)$
4	quadruple	$(1,3,3,7)$
5	quintuple	$(-8,8,-9,-5,8)$
6	sextuple	$(-7,7,1,1,0,-5)$
7	septuple	$(5,8,0,-8,-3,7,-7)$
8	octuple	$(-9,8,5,-5,2,-5,-1,2)$
9	nonuple	$(2,4,-3,-1,5,-1,-5,-4,-7)$
10	decuple	$(2,-3,7,2,3,-7,0,-1,-9,8)$
$\vdots$	$\vdots$	$\vdots$

A special case of Cartesian products are those products for which all the sets composing them are the same set. We denote these as the respective integer power, for example the Cartesian product $\mathbb{R}\times\mathbb{R}$ is denoted as $\mathbb{R}^2$, the Cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ is denoted as $\mathbb{R}^3$, etc.

Specifically, the Cartesian product $\mathbb{R}^2$ can be interpreted as the two-dimensional **Euclidean space**, which is the space used to draw graphs in one-dimensional calculus and shapes in two-dimensional analytical geometry. We will explore this idea (and higher dimensional spaces) in more details in upcoming chapters.

0.2 RELATIONS AND FUNCTIONS

0.2.1 Basics

The Cartesian product of two sets can be viewed as describing all possible connections between the elements of the first set to the elements of the second set, and thus any subset of a Cartesian product forms a specific **relation** between the sets.

Example 0.24 Relations as subsets of Cartesian products

Given the following two sets:

$$A=\{1,2,3,4\},\ B=\{\alpha,\beta,\gamma\},$$

then

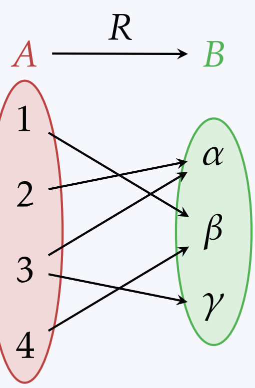
$$A\times B=\Big\{(1,\alpha),\ (1,\beta),\ (1,\gamma),$$

$$
\begin{aligned}
&(2,\alpha),\ (2,\beta),\ (2,\gamma),\\
&(3,\alpha),\ (3,\beta),\ (3,\gamma),\\
&(4,\alpha),\ (4,\beta),\ (4,\gamma)\Big\}.
\end{aligned}
$$

We can choose the following pairs to form a subset of $A \times B$:

$$R = \Big\{(1,\beta),\ (2,\alpha),\ (3,\alpha),\ (3,\beta), (4,\gamma)\Big\}.$$

R is thus a relation between A and B. We can graphically illustrate R as follows:



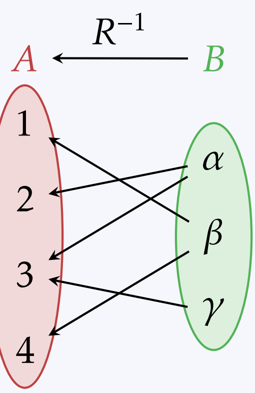
Relations can be inverted by reversing the order of each of its pairs.

Example 0.25 Inverse relation

The inverse relation to the relation in Example 0.24 is

$$R^{-1} = \Big\{(\beta,1),\ (\alpha,2),\ (\alpha,3),\ (\beta,3), (\gamma,4)\Big\}.$$

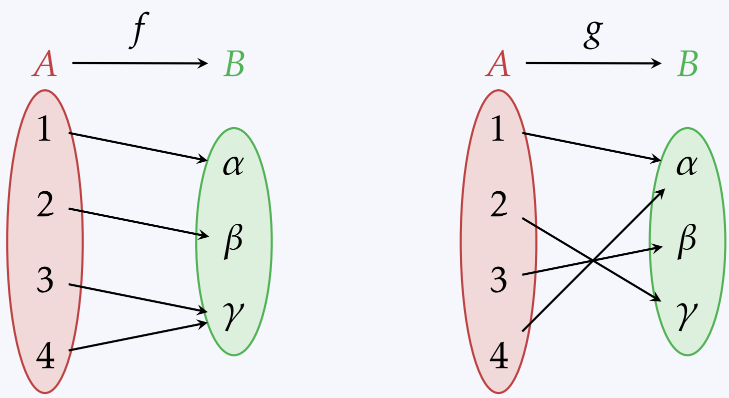
Graphically:



A **function** f from a set A to a set B is a relation for which any element in A is connected to a single element in B.

Example 0.26 Functions

The following are two functions from the set A to the set B defined in Example 0.24:



The pairs making up f are $(1,\alpha)$, $(2,\beta)$, $(3,\gamma)$ and $(4,\gamma)$, and the pairs making up g are $(1,\alpha)$, $(2,\gamma)$, $(3,\beta)$ and $(4,\alpha)$.

Note 0.5 Relations which are not functions

Note that the relation in Example 0.24 is **not** a function, since the element $3 \in A$ is connected to more than one element in B, namely α and γ.

Different names are used in some branches of mathematics to describe functions, such as **maps** and **transformations**. Barring context, they all mean the same thing.

A common way to denote that a function f is connecting elements in A to elements in B is

$$f : A \to B. \tag{0.2.1}$$

A is called the **domain** of f, and B its **image**. In this book and many other sources, the following notation is used: $f(x) = y$, which means that when we apply the function f to an element $x \in A$, the result is the element it is connected to, i.e. $y \in B$. We write this as $x \mapsto y$ (the special symbol $\mapsto$ is called a **mapping notation**).

Example 0.27 Value $\mapsto$ value notation for functions

For the functions f, g as defined in Example 0.26:

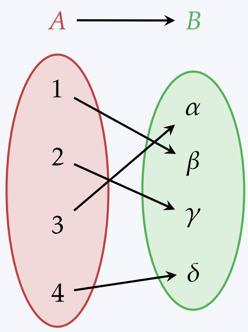
$$f(1) = \alpha,\ f(2) = \beta,\ f(3) = f(4) = \gamma.$$
$$g(1) = g(4) = \alpha,\ g(2) = \gamma,\ g(3) = \beta.$$

0.2.2 Injective, surjective and bijective functions

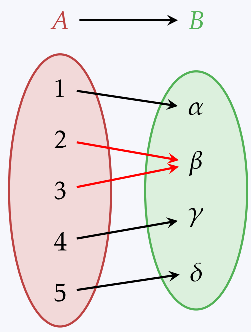
A function is **injective** if each of the elements in its **image** is connected to by at most a single element in its **domain**. An injective function is also known as an **injection**.

Example 0.28 Injective function

Injective



Non-injective



The function on the right in non-injective because the element $\beta \in B$ is connected to by two elements in A (2 and 3, red arrows).

A function is **surjective** if every element in its image is connected to by at least a single element in its domain (see Example 0.29). As with injective functions, a surjective function is also known as a **surjection**. A non surjective function can be made into surjective function by excluding from its image any element that is not connected to by any element from its domain (see Example 0.30).

A function $f : A \to B$ that is both surjective and bijective is called a **bijective function** (also a **bijection**). All elements in the image of a bijection are connected to by exactly a single element in its domain. This means that the direction of the connections can be flipped, yielding the **inverse** of the original function (denoted f^{-1}).

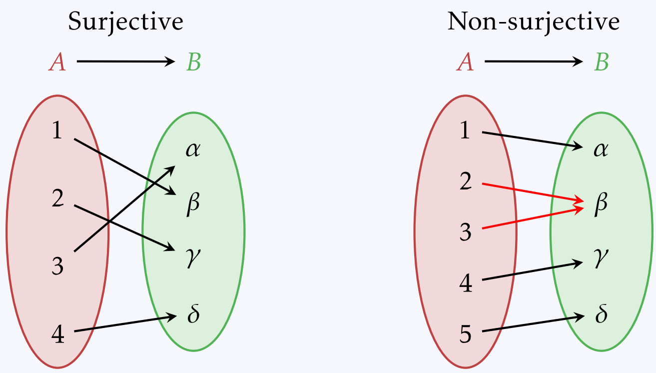
The reason only bijective functions have inverses is as follows: Given a function $f : A \to B$,

- if f is non-injective, then there is at least one element $y_1 \in B$ which is connected to by at least two elements from A. We can name these elements x_1 and x_2. When inverted, $f^{-1} : B \to A$ has an element $y_1 \in B$ (note that for f^{-1}, B is its domain), which is connected to two or more elements in A, the image of f^{-1}. These are of course x_1, x_2. This fact disqualifies f^{-1} from being a function.
- If f is non-surjective, then there exists at least one element $y_2 \in B$ that is not connected to by any element from A. When inverted, y_2 in the domain B of f^{-1} is not connected to any element in its image A. This fact disqualifies f^{-1} from being a function.

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Etiam lobortis facilisis sem.

Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper.

Example 0.29 Surjective function



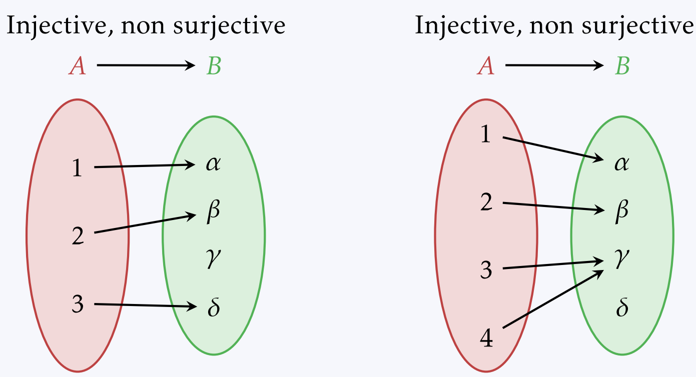
Example 0.30 Making a non-surjective function into a surjection

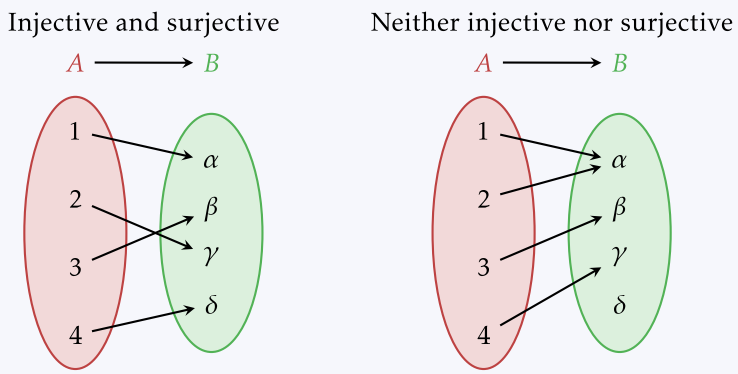
Given the two sets $A = \{1, 2, 3, 4\}$ and $B = \{\alpha, \beta, \gamma, \delta\}$, the following non-surjective function $f : A \to B$ is defined:

$$f = \Big\{(1, \alpha),\ (2, \beta),\ (3, \gamma),\ (4, \gamma)\Big\}.$$

By removing δ from B, the function f becomes surjective (though it remains non-injective).

Example 0.31 Cross examples





Note 0.6 Other names for bijections

Bijections are also called **one-to-one correspondences** and **invertible functions**.

0.2.3 Real functions

In suitable cases, a function is defined via a general mapping rule. This should be very familiar to anyone who learned mathematics in high school, where many times functions are defined this way, e.g.

$$f(x) = x^2 + 3x - 4. \tag{0.2.2}$$

In mapping notation we can write Equation 0.2.2 as $f : x \mapsto x^2 + 3x - 4$. In high school mathematics, both the domain and image of such functions is $\mathbb{R}$, although it is almost never specified explicitly. Such functions are commonly referred to as **real functions**, a convention used in this book as well.

Example 0.32 Functions defined using a mapping rule

The following are real functions:

$$f_1(x) = 2x^2 - 5, \quad f_2(x) = \sin\left(\frac{x}{3}\right), \quad f_3(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\mu)^2}{\sigma^2}}.$$

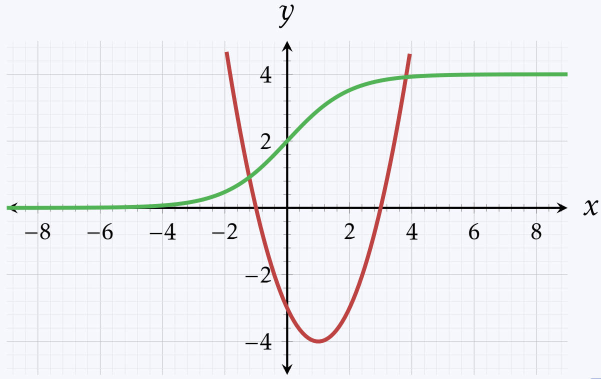
Note that these functions can also be defined using different sets, for example $f_1 : \mathbb{N} \to \mathbb{Z}, \quad f_2 : \mathbb{N} \to [-1, 1], \quad$ etc.

Real functions can be easily plotted in a **Cartesian coordinate system** by drawing all the points $\big(x, f(x)\big)$ (i.e. all the points $\big(x, y\big)$, where $x, y \in \mathbb{R}$ and $x \mapsto y$). We call these points the **graph** of f over $\mathbb{R}$.

Example 0.33 Graphs of real functions

The following two functions are plotted on the domain $[-9, 9]$:

- $f(x) = x^2 - 2x - 3$,
- $g(x) = 4e^x / (e^x + 1)$.

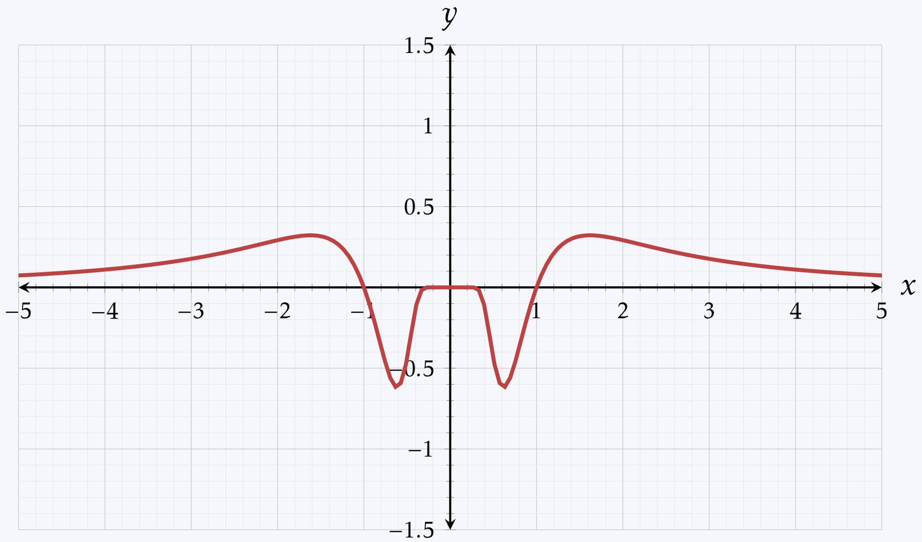


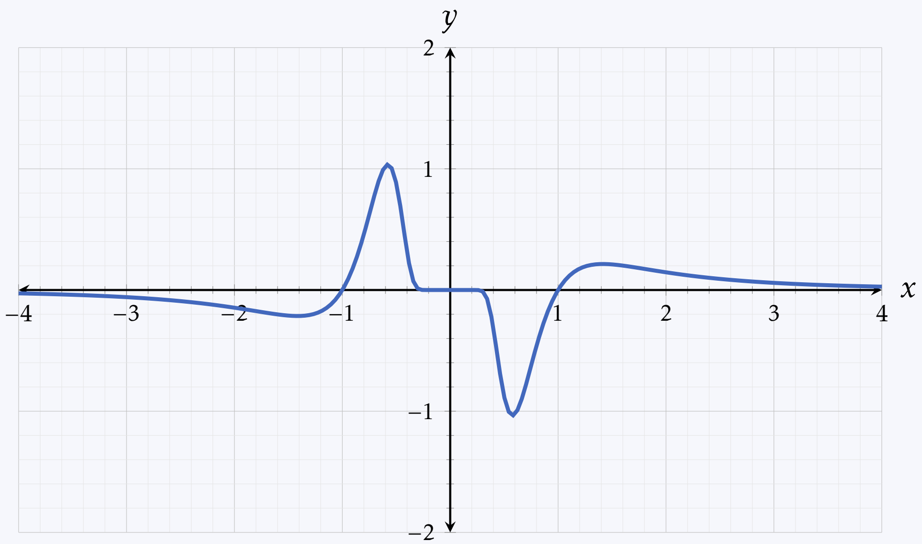
In Example 0.33, the function $g(x)$ always increases in value from left to right. Let's give this notion a more formal tone: a function f is said to be **increasing** on an interval I if for any $x_1, x_2 \in I$, if $x_2 > x_1$ then $f(x_2) > f(x_1)$. We can similarily define the idea of **decreasing** on an interval.

A property of some functions which is visually easy to depict is symmetry. A real function f is said to be **symmetric** if for any $x \in \mathbb{R}$, $f(-x) = f(x)$. This essentially means that the y-axis mirrors the function's plot. If for any $x \in \mathbb{R}$, $f(-x) = -f(x)$, we say that the function is **anti-symmetric**. A function can be neither, but there's only a single function which is both: the zero function, i.e. $f(x) = 0$.

Example 0.34 Symmetric and anti-symmetric functions

In the following graphs, the function on the top is symmetric, while the function on the bottom is anti-symmetric:





(injections/surjections of real functions?)

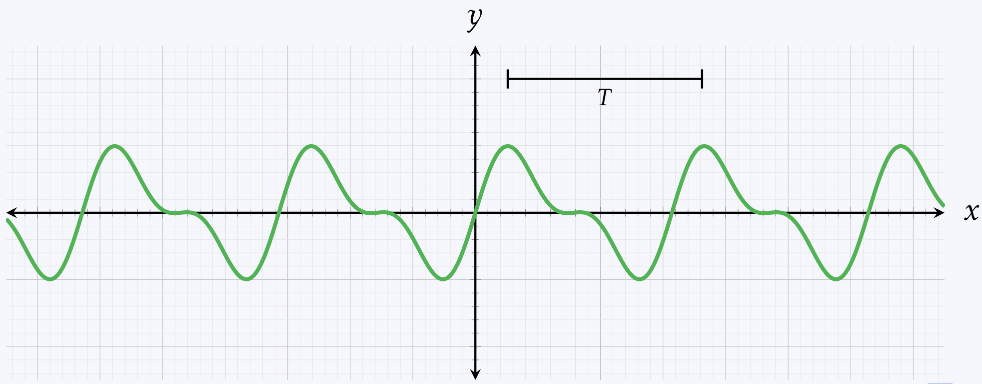
A real function is said to be **periodic** if it repeats its values exactly over and over with increasing x. In more precise terms we define a real function f to be periodic if for any integer value k,

$$f(x+kT) = f(x). \tag{0.2.3}$$

where $T = [a, b]$ is a finite interval of $\mathbb{R}$ which we call the **period** of the function.

Example 0.35 A periodic function

The following graph depicts a periodic function f, with its period T shown. Notice that for any x $f(x+T)=f(x)$, i.e. you can move the period measure left and right along the x-axis and the values of $f(x)$ in both its edges would always be the equal.



Two additional measures that arise from a period T are the **frequency** $f=\frac{1}{T}$, and the **angular frequency** $\omega=2\pi f=\frac{2\pi}{T}$. We will use these measures later in the book.

Note 0.7 Units of period and frequency

In a periodic function such as the one in the above example, the units for the period are the same one used for the horizontal axis, while the units of both frequency and angular frequency are both 1 over the unit used for the horizontal axis/period. For example, if the unit of the horizontal axis is that of seconds, then the frequency units are 1/seconds, i.e. Hertz (SI symbol: Hz).

0.2.4 Composition of functions

Functions can be **composed** together, generating new functions. Given two functions $f:A\to B$ and $g:B\to C$, their composition is denoted as $f\circ g$. For the composition to be well defined, the **image** of f must be the same as the **domain** of g, and the resulting composition would have A as its domain and C as its image, i.e. $f\circ g:A\to C$.

Example 0.36 Composition of functions

Consider the functions

$$f(x)=x^2,\quad g(x)=\sin(x).$$

Using these functions, the two possible compositions are

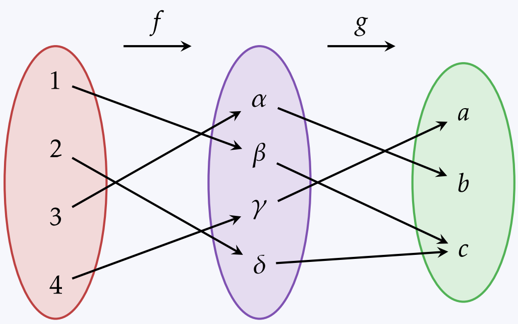
- $f\circ g=f\big(g(x)\big)=[\sin(x)]^2$, and

- $g \circ f = g\left(f(x)\right) = \sin\left(x^2\right)$.

Example 0.37 Graphical representation of function composition

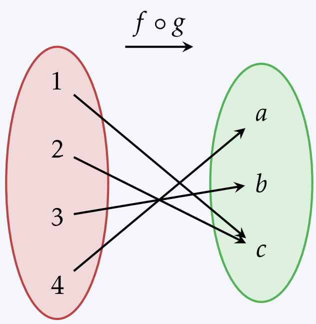
A graphical representation of composing two functions:

$$f : \{1,2,3,4\} \to \{\alpha,\beta,\gamma,\delta\}, \quad g : \{\alpha,\beta,\gamma,\delta\} \to \{a,b,c\}.$$



The composition results in the following function

$$f \circ g : \{1,2,3,4\} \to \{a,b,c,\}.$$



0.3 POLYNOMIAL FUNCTIONS

A very useful family of real functions can be derived using only three fundamental operations: addition, multiplication and exponentiation: the (real) **polynomial functions**. These are functions of the form

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n, \tag{0.3.1}$$

where $a_0, a_1, \ldots, a_n$ are real numbers called the **coefficients** of the polynomial function. Note that $a_n \neq 0$, i.e. the **degree** of the polynomial function is the index of the highest non-zero coefficient (and thus the highest power in the expression). We also call this the **order** of the polynomial function.

Example 0.38 Polynomial

The following is a polynomial function of degree $n = 6$:

$$P(x) = 4 + 2x - 3x^2 + 7x^4 - x^5 + 3x^6.$$

Breaking down this polynomial to its constituent terms:

$$\begin{array}{ccccccc}
P(x) = & 4 & +2x & -3x^2 & +7x^4 & -x^5 & +3x^6 \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & a_0 = 4 & a_1 = 2 & a_2 = -3 & a_4 = 7 & a_5 = -1 & a_6 = 3
\end{array}$$

Note that a_3 is missing from the polynomial function (i.e. there is no x^3 term). This means that $a_3 = 0$.

A shorthand way to write the general form of a polynomial function is by using the **summation notation**:

$$P(x) = \sum_{k=0}^{n} a_k x^k. \tag{0.3.2}$$

This notation, called the **Capital-sigma notation**, essentially represents addition of n elements (in the case shown here), each with its own **index of summation**, in this case i. The most general form of the summation notation is

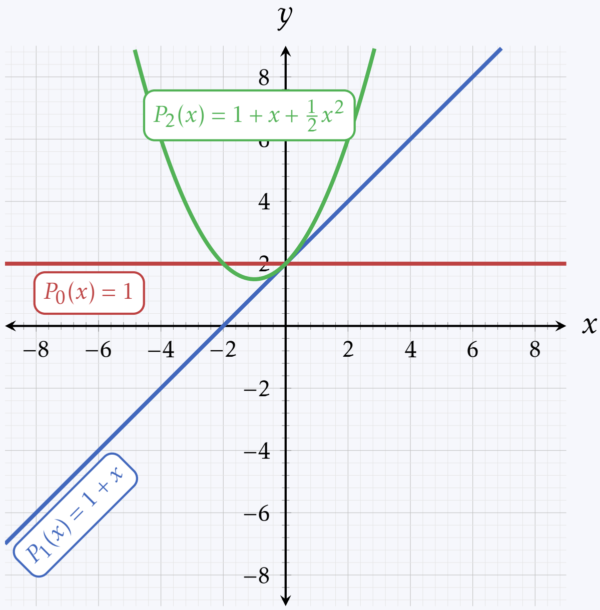
$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + a_{k+2} + \cdots + a_{n-1} + a_n, \tag{0.3.3}$$

i.e. the notation tells us to add those elements a_i for which $k \geq i \geq n$. Note that in the case of Equation 0.3.2, when $k = 0$, $x^k = x^0 = 1$ and the first term of the polynomial function has no x power (i.e. it is simply a_0), and when $k = 1$, $x^k = x^1 = x$ and thus the second term is $a_1 x$. We will encounter the summation notation in more details later in the book.

In the special case $n = 0$, i.e. when $P(x) = a_0$, the function is constant. When $n = 1$ the function $P(x) = a_0 + a_1 x$ is a line, and when $n = 2$, $P(x) = a_0 + a_1 x + a_2 x^2$ is a quadratic function.

Example 0.39 Polynomial functions for $n = 0, 1, 2$

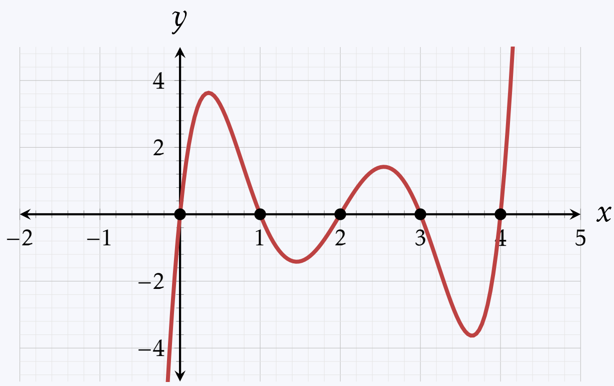
The following graphs represent the polynomial functions of degrees $n = 0, 1, 2$ with coefficients $a_0 = 2$, $a_1 = 1$, $a_2 = \frac{1}{2}$:



The values $x \in \mathbb{R}$ for which $P(x) = 0$ are called the **roots** (also: **zeros**) of the polynomial function.

Example 0.40 Roots of a polynomial function

The polynomial function $P(x) = 24x - 50x^2 + 35x^3 - 10x^4 + x^5$ has the following 5 roots: $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$. In the following graph of $P(x)$ the roots are shown as black dots.



The maximum number of **real** roots of a polynomial function with degree $n \geq 1$ is n, e.g. a polynomial of degree $n = 4$ has at most 4 real roots. This statement is a consequence of a very important theorem called **the fundamental theorem of algebra**, which due to its importance we will mention here without proof:

Theorem 0.1 The fundamental theorem of algebra

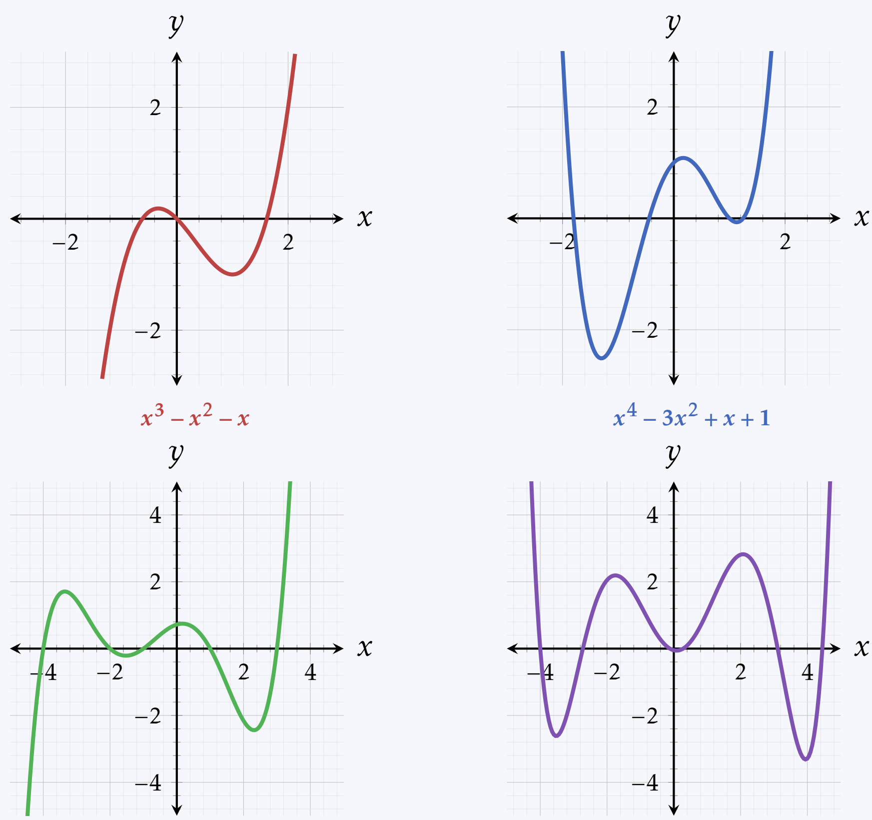
For any $n \geq 1$, the polynomial function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_0, a_1, a_2, \ldots, a_n$ are all **complex numbers** and $a_n \neq 0$, has n complex roots.

Given a polynomial function $P(x)$ with n roots $r_1, r_2, \cdots, r_n$, the function can be written as a product of terms of the form $x - r_i$ (up to a constant), e.g. the polynomial function of degree $n = 3$ with roots $-1, 1, 2$ can be written as

$$P(x) = (x+1)(x-1)(x-2) = x^3 - 2x^2 - x + 2. \tag{0.3.4}$$

Example 0.41 Higher order polynomial functions

The following are the graphs of high-order polynomial functions ($n = 3, 4, 5, 6$):



$x^3 - x^2 - x$

$x^4 - 3x^2 + x + 1$

$\frac{3}{100}\left(x^5 + 3x^4 - 11x^3 - 27x^2 + 10x + 24\right)$

$\frac{1}{100}\left(x^6 - x^5 - 26x^4 + 15x^3 + 150x^2 - 25x - 5\right)$

As can be seen in **??**, the maximal number of 'bends' in a polynomial function of order n is $n - 1$ (i.e. one less than the order of the function).

We will continue to explore polynomial functions in more details in future chapters.

0.4 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In the previous section we dealt with functions composed of integer powers of x. We will now shortly focus on functions where x is in the power itself and their inverse functions.

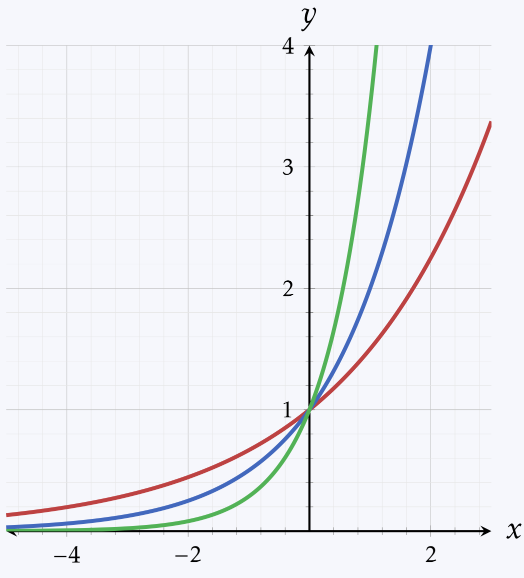
An **exponential function**, or simply an **exponential**, is a real function of the type

$$f(x) = b^x, \tag{0.4.1}$$

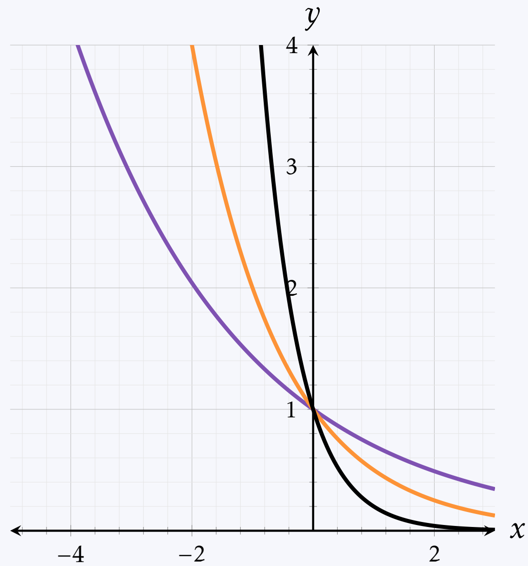
where $b > 0$ is called the **base** of the exponentiation, and x the exponential. All exponents, regardless of base, are always positive. In addition, all exponents pass through the point $(0,1)$ since $b^0 = 1$ for any real positive number, and through the point $(1,b)$ since $b^1 = b$. When $b > 1$ the function is increasing on $\mathbb{R}$, while for $b < 1$ the function is descending on $\mathbb{R}$.

Example 0.42 Exponential functions

The following are graphs of the exponential functions $\mathbf{1.5^x}$, 2^x and $\mathbf{3.5^x}$:



And the following are graphs of the exponential functions $\mathbf{0.7^x}$, $\mathbf{0.5^x}$ and $\mathbf{0.2^x}$:



As a reminder, the following are two well known properties of exponents: given a base $b > 0$,

$$b^{-x} = \frac{1}{b^x}, \tag{0.4.2}$$

$$b^x b^y = b^{x+y}. \tag{0.4.3}$$

A special base for exponential functions is the real, non-algebraic number e. This number has many names, among them is **Euler's number**, but in the constant of exponentials it is known as the **natural base**. Its exact value is not entirely important for the moment: it is about 2.718, and in any case it is not possible to write it as there it has infinitely many digits after the period. It is very common across different fields of mathematics and science to write $\exp(x)$ instead of e^x.

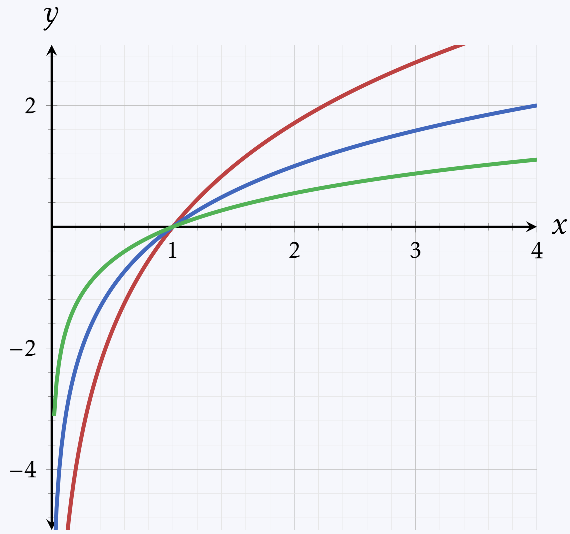
The inverse function to exponentials are the **logarithmic functions** (or simply **logarithms**), i.e. for any real $b > 0,\ b \neq 1$,

$$\log_b\left(b^x\right) = b^{\log_b(x)} = x. \tag{0.4.4}$$

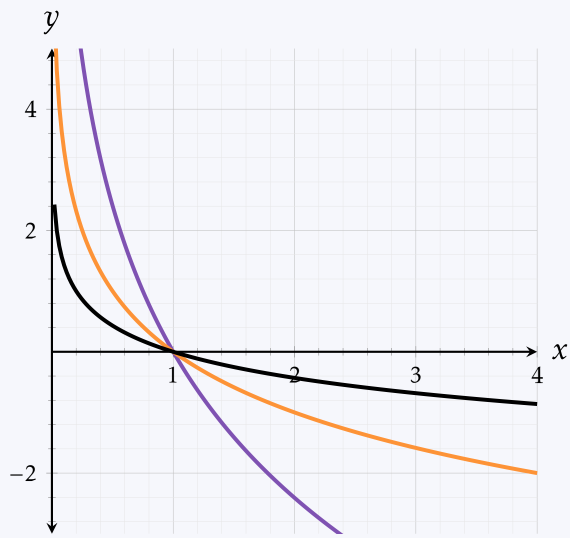
In essence, the logarithm in base b of a number x answers the question *"what is the number a for which $b^a = x$?"*. Being the inverses of exponential functions, all logarithms go through the point $(1,0)$, and each also passes through its own point $(b,1)$.

Example 0.43 Logarithmic functions

The following are graphs of the logarithmic functions $\log_{1.5}(x)$, $\log_2(x)$ and $\log_{3.5}(x)$:



...and the following are graphs of the exponential functions $\log_{0.75}(x)$, $\log_{0.5}(x)$ and $\log_{0.2}(x)$:



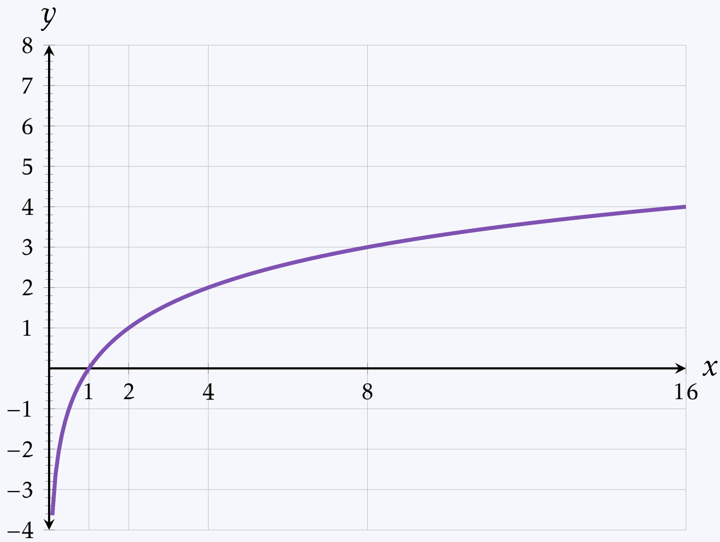
A useful property of logarithms is that they can help reduce ranges spanning several orders of magnitude to numbers humans can deal with. The easiest way to see this is using $b = 10$: $10^1 = 10$, and so $\log_{10}(10) = 1$. $10^2 = 100$, and so $\log_{10}(100) = 2$. $10^3 = 1000$, and so $\log_{10}(1000) = 3$, etc. The value of the logarithm goes by 1 for each raise in order of magnitude of its argument.

Therefore, if we have some measurement x which can hold values spanning several orders of magnitude (say $x \in [3, 1500000000]$), then it can sometimes be useful to use instead the logarithmic value of x (which in our case would span the range $\log_{10}(x) \in$

[0.477,9.176]). This is done in many fields of science, for example some definitions of entropy[1], acid dissociation constants[2], pH[3] and more.

Example 0.44 Logarithms as evaluating orders of magnitude

In the following graph of $\log_2(x)$, each increase by power of two in x (i.e. $x = 1, 2, 4, 8, 16, \ldots$) yields only a single increase in y (i.e. $y = 0, 1, 2, 3, 4, \ldots$). This shows how logarithms shift our perspective from absolute values to orders of magnitude.



Using the definition of the logarithmic function $\log_b(x)$ (Equation 0.4.4) and the product rule for exponentials (Equation 0.4.3), a similar rule can be derived for logarithms. Let $x, y > 0$ and $b > 0,\ b \neq 1$ all be real numbers. We define

$$\log_b(x) = M,\ \log_b(y) = N, \tag{0.4.5}$$

which means

$$b^M = x,\ b^N = y. \tag{0.4.6}$$

From Equation 0.4.3 we know that

$$xy = b^M b^N = b^{M+N}, \tag{0.4.7}$$

and by re-applying the definition of logarithmic functions we get that

$$\log_b(xy) = M + N = \log_b(x) + \log_b(y). \tag{0.4.8}$$

Similarly to Equation 0.4.8, division yields subtraction:

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y). \tag{0.4.9}$$

[1] $S = k_\text{B} \log(\Omega)$
[2] $\text{p}K_a = -\log(K_\text{diss})$
[3] $\text{pH} = -\log\left([\text{H}^+]\right)$

Equations 0.4.8 and 0.4.9 reveal another valuable property of logarithms: they reduce multiplication to addition (and subsequently division to subtraction). While today this property doesn't seem very impressive, in pre-computers days it helped carrying on complicated calculations, using tables of pre-calculated logarithms (called simply **logarithm tables**) - a sight rarely seen today.

Taking one step forward in regards to reduction of operations, logarithms reduce powers to multiplication:

$$\log_b\left(x^k\right) = k\log_b(x). \tag{0.4.10}$$

for any $k \in \mathbb{R}$.

! **To be written**: proving this will be in the chapter questions to the reader !

Any logarithm $\log_b(x)$ can be expressed using another base, i.e. $\log_a(x)$ (where $a > 0$, $a \neq 1$) using the following formula:

$$\log_a(x) = \log_b(x) \cdot \log_a(b). \tag{0.4.11}$$

! **To be written**: proving this too will be a question to the reader !

Example 0.45 Changing logarithm base

Expressing $\log_4(x)$ in terms of $\log_2(x)$:

$$\log_4(x) = \log_2(x) \cdot \underbrace{\log_4(2)}_{=\frac{1}{2}} = \frac{1}{2}\log_2(x).$$

Much like with exponentials, the number e plays an important role when it comes to logarithms, for reasons that are discussed in the calculus chapter (ref). For now, we will just mention that $\log_e(x)$ gets a special notation: $\ln(x)$, which stands for **natural logarithm**. This notation is mainly used in applied mathematics and science, while in pure mathematics the notation is simply $\log(x)$, i.e. without mentioning the base [4].

For reason we will see in the calculus chapter, it is relatively simple to calculate both the exponential and logarithm in base e. Therefore, many operations in modern computations are actually done using these functions, for example calculating logarithms in other bases:

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}. \tag{0.4.12}$$

Another operation commonly using both e^x and $\ln(x)$ is raising a real number a to a real power b: using the properties of both exponential and logarithmic functions, any such power can be expressed as

$$a^b = e^{b\ln(a)}. \tag{0.4.13}$$

[4] Depending on convention and context, this notation can refer to logarithm in any other base, most commonly $\log_{10}(x)$ and $\log_2(x)$.

0.5 SIMPLE COMBINATORICS AND THE BINOMIAL COEFFICIENTS

Suppose we have a set of 3 spheres, each of a different color: red, blue and green - and want to order all of them in a row. How many different ways do we have of organizing the spheres? Figure 0.3 shows all of the options.

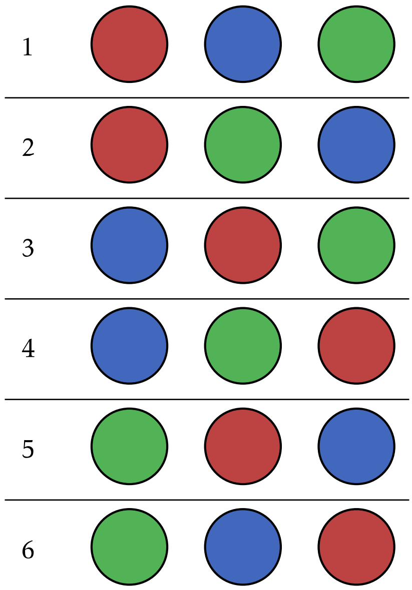


Figure 0.3 All the ways of arranging a set of 3 differently colored spheres.

Going about the options systematically, we can describe them as follows: we have three options for placing the first sphere - red, blue or green. Once we have chosen the first sphere, we're left with only two option for the second sphere: if we chose red, then we're left with choosing between blue and green. The choice of last sphere is then dictated by the previous choices: if for the second sphere we chose green, then we are left with only the blue sphere for the third position (as in option 2 above). We call each of these ways to organize the spheres a **combination**.

Quantitatively the number of ways we have of organizing the spheres is

$$k = 3 \times 2 \times 1 = 6. \tag{0.5.1}$$

We can expand this logic to however many $n \in \mathbb{N}$ different spheres we wish: for n different spheres we have n options for placing the first sphere, then $n-1$ options for placing the second sphere, then $n-2$ options for placing the third sphere... all the way to the last sphere. The number of total combinations is therefore

$$k = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1. \tag{0.5.2}$$

The function used to represent k in the above general form is called the **factorial**, and is denoted using an exclamation mark:

$$k = n! \tag{0.5.3}$$

A somewhat more rigouros (and quite intereseting) way of defining the factorial is as follows:

$$n! := \begin{cases} 1 & \text{if } n = 1, \\ n \times (n-1)! & \text{if } n > 1. \end{cases} \tag{0.5.4}$$

This kind of definition is called a **recursive** definition, since it uses itself in its own definition. For example, for 3! we have $3 > 1$, and thus $3! = 3 \times 2!$. Then $2 > 1$ and thus $2! = 2 \times 1!$, but since $1 = 1$ we get $1! = 1$, and altogether we get $3! = 3 \times 2 \times 1 = 6$.

Going forward we can ask the following question: given that we have a set of 5 spheres, 2 of them red and the rest are blue. How many combinations are there to sort the spheres, assuming there's no way to distinguish spheres of the same color? All the possible combinations are shown in Figure 0.4.

We can again go about solving this by directly counting all possible combinations. We place a red sphere in the 1st spot, and then count all such possible combinations, each time placing the other red sphere in a different place (2nd, 3rd, etc.). This gives us 5 combinations (numbered 1-4 in the table). We then place a red sphere in the 2nd spot, and placing the other red sphere in all possible spots: 3rd, 4th and 5th (combinations 5-7 in the table). Note that we don't count the combination where the red spheres are in the 1st and 2nd spots since this was already counted (combination 1).

We then proceed to puting a red sphere in the 3rd spot, and counting all the possible combinations, i.e. the other red sphere being in the 4th or 5th spots (combinations 8 and 9). Again, all other options were already counted. We're then left with only a single combination: the red spheres being in the 4th and 5th spots. Altogether we have counted 10 distinct combinations.

In the general case, we want to know how many combinations are there of arrainging n spheres in a row, when k of them are red and the other $n - k$ blue. A convinient way of representing this number is using the **binomial coefficients**, which are defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{0.5.5}$$

Note 0.8 Binom pronounciation

Reading the binom notation outloud we say "n choose k", since we are choosing k objects out of a total of n objects.

We expect then that $\binom{5}{2} = 10$, since this is what we got in the example above. Let's verify this:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{120}{2 \cdot 3!} = \frac{120}{12} = 10.$$

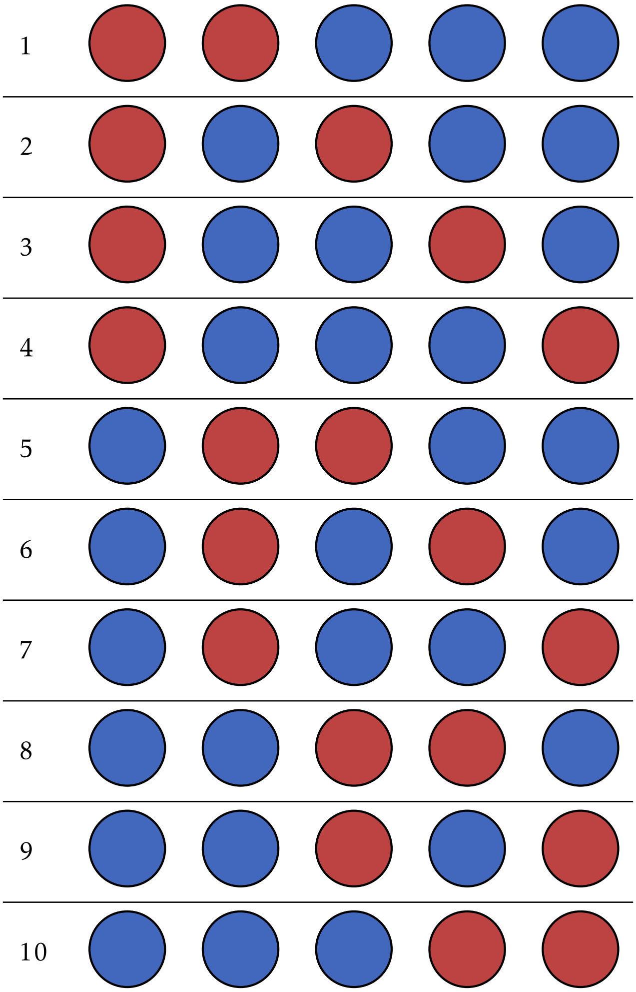


Figure 0.4 All possible combinations of two red spheres and three blue spheres, where spheres of the same color are indistinguishable.

The binomial coefficients are symmetric: note how in the denominator $k!$ is multiplied by $(n-k)!$, i.e. k is always multiplied by the difference between it and n. Thus, if we change our choice of k to be $n-k$ (e.g. for the case of $\binom{5}{2}$ we instead use $\binom{5}{3}$) the result stays the same. This can be visualized in the above example: instead of thinking of two red spheres out of five total spheres, think of three blue spheres out of a total of five spheres. The result doesn't change because we are talking about the same exact problem.

Thus, the following always holds:

$$\binom{n}{k} = \binom{n}{n-k}. \tag{0.5.6}$$

Note 0.9

Some important values of the binomial coefficianets: Given n objecst,

1. there is a single combination of choosing 0 objects, and also a single combination of choosing n objects. Therefore
$$\binom{n}{0} = \binom{n}{n} = 1.$$
2. there are n combinations for choosing a single object, and also n combination of choosing $n-1$ objects. Therefore
$$\binom{n}{1} = \binom{n}{n-1} = n.$$

!

We can visualize the binomial coefficients using **Pascal's triangle**, in which the value of each cell is the result of adding the values of the two cells above it (Figure 0.5). For example: to find the value of $\binom{4}{2}$ we go to the 4-th row (purple color in the figure) and then start from left starting with 0. The value of the resulting cell is 6, and indeed $\binom{4}{2} = 6$.

Why are the binomial coefficients called that way? The name comes from the **binomial expansion**. You probably had to memorize the following expansion in school:

$$(a+b)^2 = a^2 + 2ab + b^2. \tag{0.5.7}$$

We can use the binomial coefficients to easily expand any expression of the form $(a+b)^n$, where $n \in \mathbb{N}$. As always, we start with a basic example - this time the above expansion when $n = 2$: when we expand this expression we first get (color coded for clarity)

$$(a+b)^2 = (a+b)(a+b).$$

Then, to calculate all possible products we multiply each possible pair of numbers systematically:

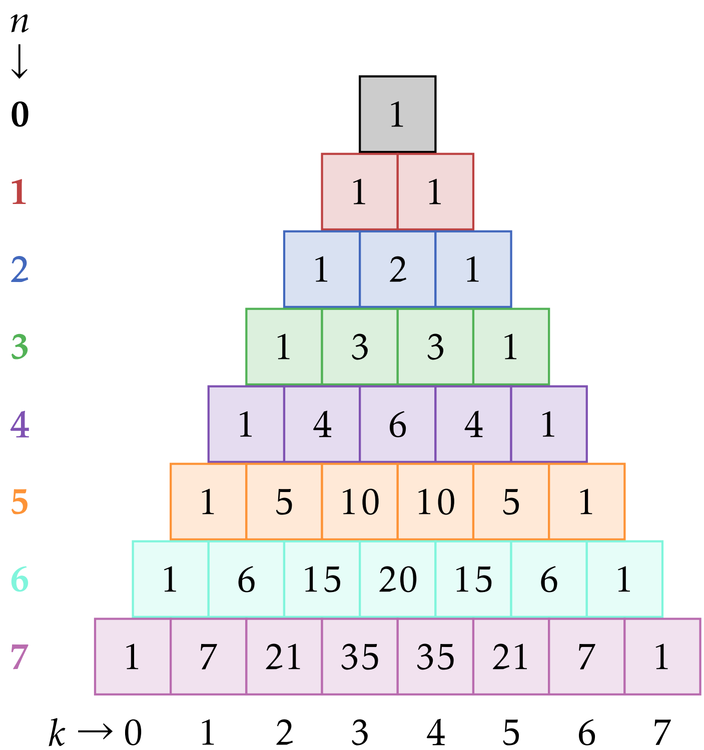


Figure 0.5 Pascal's triangle: the value in each cell is the sum of the values in the two cells above it (empty cells are considered to be with value 0). Each row is enumerated starting from 0: see for example the indeces below the last row, which go from 0 to 7.

$a \cdot a$: $$(a+b)(a+b)$$

$a \cdot b$: $$(a+b)(a+b)$$

$b \cdot a$: $$(a+b)(a+b)$$

$b \cdot b$: $$(a+b)(a+b)$$

Altogether we therefore get

$$(a+b)^2 = (a+b)(a+b) = a \cdot a + ab + ba + b \cdot b = a^2 + 2ab + b^2.$$

(since for real numbers $ab = ba$)

Notice the pattern here: the coefficients of the different terms in $a^2 + 2ab + b^2$ correspond to the binomial coefficients when $n = 2$ (i.e. $1, 2, 1$). This is because there's only a single combination of numbers yielding a^2 (namely $a \cdot a$), two combinations that yield ab (namely $a \cdot b$, and $b \cdot a$), and again a single combination that yields b^2 ($b \cdot b$).

The powers of a and b also follow a pattern: the power of a decreases in each term - starting with 2 (i.e. a^2), then 1 ($2ab$) and finally 0 (b^2). The power of b in each term increases in the same way from 0 to 2. If we call the power of b in each term k, then the power of a is always $2-k$.

The pattern appears in $(a+b)^3$ as well: (we again color code the numbers)

$$(a+b)^3 = (a+b)(a+b)(a+b).$$

The expansion is then

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= a \cdot a \cdot a + a \cdot a \cdot b + a \cdot b \cdot a + a \cdot b \cdot b + b \cdot a \cdot a + b \cdot a \cdot b + b \cdot b \cdot a + b \cdot b \cdot b\\ &= a^3 + 3a^2 b + 3ab^2 + b^3.\end{aligned}$$

Again, we see that the coefficients in the expansion are exactly the binomial coefficients for $n = 3$ (namely $1, 3, 3, 1$), and that the powers of a and b decrease and increase, respectively, just as for $n = 2$: a goes from 3 to 0, and b from 0 to 3. This time the power of a is always $3-k$ (where k is the power of b).

Therefore, in the most general case (i.e. $(a+b)^n$) we expect each term to be of the following structure:

$$a^k b^{n-k},$$

and the coefficients being the binomial coefficients $\binom{n}{k}$. Putting this into a formula:

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.\end{aligned} \tag{0.5.8}$$

Example 0.46 Exapnding $(a+b)^4$ using binomial coefficients

Let's use the above patterns to expand $(a+b)^4$. Looking at Pascal's triangle, the coefficients in row $n=4$ are $1,4,6,4,1$. Starting with the powers a^4 and $b^0=1$ and then decreasing and increasing the powers of a and b, respectively, we write all terms:

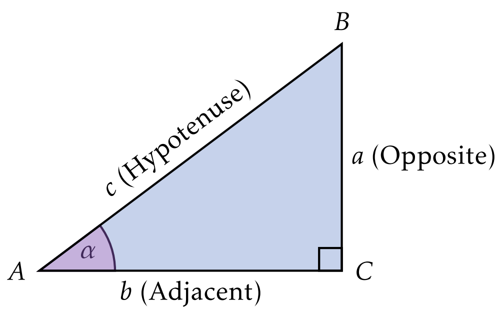
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

You can check this expansion manually and see that it is indeed correct.

0.6 TRIGONOMETRIC FUNCTIONS

0.6.1 Basic Definitions

Consider a **right triangle** $\triangle ABC$ with sides a, b, and Hypotenuse c, where the angle $\angle ACB$ is $90°$, and the angle $\angle BAC$ is denoted as α:



We use the ratios between the three sides of the triangle to define three functions of α:

- The **sine** of the angle α is $\sin(\alpha) = \frac{a}{c}$,

- the **cosine** of the angle α is $\cos(\alpha) = \frac{b}{c}$, and
- the **tangent** of the angle α is $\tan(\alpha) = \frac{a}{b}$, which in turn is equal to $\frac{\sin(\alpha)}{\cos(\alpha)}$.

We can rearrange the above definitions:

$$\begin{aligned} a &= c\sin(\alpha), \\ b &= c\cos(\alpha). \end{aligned} \tag{0.6.1}$$

Normally, the Hypotenuse is the longest side of a right triangle. We will consider here the two edge cases where one of the sides a or b is equal to the Hypotenuse (and the other side is thus 0):

- if $a = c$ then $\alpha = 90°$,
- if $b = c$ then $\alpha = 0°$.

The possible length of a is therefore in the range $0 \leq a \leq c$, which means that $0 \leq \frac{a}{c} \leq 1$. Since $\sin(\alpha) = \frac{a}{c}$ this means that the image of $\sin(\alpha)$ is $[0,1]$. The same idea is also true for b, and therefore $[0,1]$ is the image of $\cos(\alpha)$ as well.

As a reminder, the **Pythagorean theorem**[5] states that for a right triangle with sides a, b and c,

$$a^2 + b^2 = c^2. \tag{0.6.2}$$

By substituting **Equation** 0.6.1 into the Pythagorean theorem we get

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= [c\sin(\alpha)]^2 + [c\cos(\alpha)]^2 \\ &= c^2\sin^2(\alpha) + c^2\cos^2(\alpha) \\ &= c^2\left[\sin^2(\alpha) + \cos^2(\alpha)\right], \end{aligned}$$

and therefore

$$\sin^2(x) + \cos^2(x) = 1. \tag{0.6.3}$$

0.6.2 The Unit Circle

We defined $\sin(\alpha)$ and $\cos(\alpha)$ so far in way such that their domains are both $[0°, 90°]$, and their images are both $[0,1]$. However, there is a simple way to extend these functions such that both their domains are $\mathbb{R}$, and both their images are $[-1,1]$: by using a **unit circle**.

Figure 0.7 depicts a unit circle: it is simply a circle of radius $R = 1$, which is placed such that its center lies at the origin of a 2-dimensional axis system (i.e. at the point

[5] It's worth mentioning that no three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer value of $n > 2$. This can be proven, however the proof is too large to fit in the footnotes.

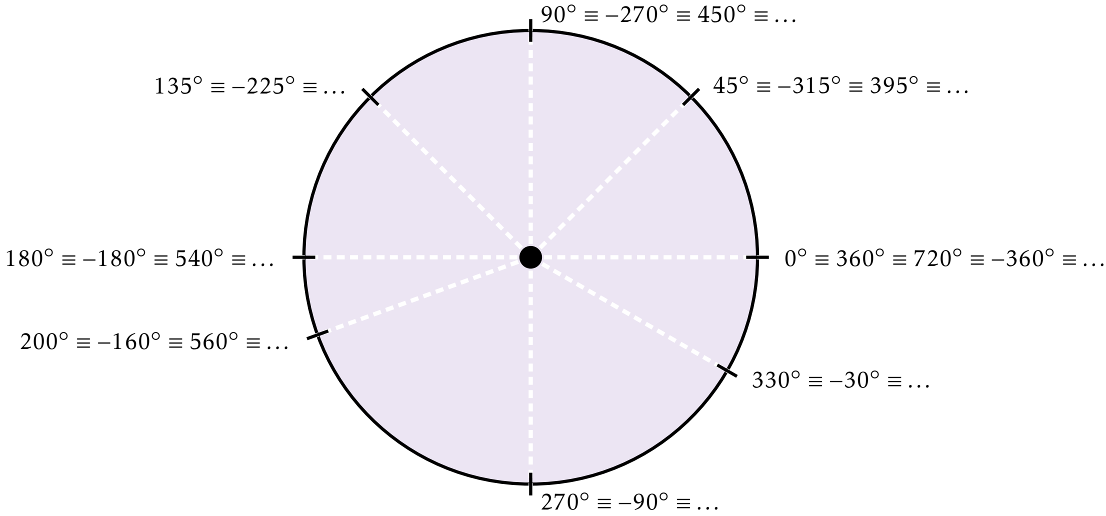


Figure 0.6 Angles equivalency on a circle.

$\boldsymbol{O} = (0,0)$. We then draw a line from $\boldsymbol{O}$ to a point $\boldsymbol{P} = (x, y)$ on the circumference of the unit circle. We call the angle between the line $\boldsymbol{OP}$ and the x-axis θ. We then draw another line, this time from the point $\boldsymbol{P}$ to a point $\boldsymbol{D}$ on the x-axis, such that $\boldsymbol{PD}$ is perpendicular to the x-axis.

The triangle $\triangle OPD$ is a right triangle. Therefore, we can use the trigonometric functions to calculate the coordinates of the point $\boldsymbol{P} = (x, y)$:

$$\begin{aligned} x &= R\cos(\theta) = \cos(\theta), \\ y &= R\sin(\theta) = \sin(\theta). \end{aligned} \tag{0.6.4}$$

We then define $\cos(\theta)$ and $\sin(\theta)$ as a function of θ:

$$\begin{aligned} \sin(\theta) &= y, \\ \cos(\theta) &= x. \end{aligned} \tag{0.6.5}$$

Using this definition, the angle θ can take any value between $0°$ and $360°$. In fact, the values of θ can be extended to any real number in degrees: any real value of degrees is equivalent to some value in the range $[0°, 360°]$, the first and most obvious example is that $360°$ is equivalent to $0°$. Similarly, $-30° \equiv 330°$, $-180° \equiv 180°$, $-90° \equiv 270°$, etc (see Figure 0.6). In fact, this property makes the trigonometric functions periodic, with a period $T = 360°$.

0.6.3 Radians

Using degrees to measure angles in a sphere creates an inconvenience: the domain and image of the trigonometric functions have different units. In order to measure both

Table 0.3 Common angles in radians, and their respective images for the three main trigonometric functions.

$\theta[°]$	θ[rad]	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	**0**	**0**	**1**	**0**
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	undefined
180	π	0	-1	0
270	$\frac{3\pi}{2}$	-1	0	undefined
360	2π	0	1	0

these magnitudes using the same unit we switch to measuring angles on a circle using **radians** instead of degrees. One radian equals the length of a single radius R of the circle (in the case of the unit circle this is always $R = 1$). We define an inner angle θ to equal one radian if the arc length it represents is equal to R (see Figure 0.8).

How much is a radian in degrees? The full circumference of any circle with radius R equals $2\pi R$, which means that a single radian R is equivalent to $\frac{180°}{\pi} \approx 57.3°$. Table 0.3 shows some common angles and their equivalent value in radians.

0.6.4 Graphs

As seen previously, the functions $\sin(x)$ and $\cos(x)$ are periodic, having both the period $T = 2\pi$. Their graphs are depicted in Figure 0.10. The value of $\sin(x)$ is always equal to that of $\cos\left(x - \frac{\pi}{2}\right)$: we say that the two functions have a **phase difference** of $\pi/2$. The graph of $\tan(x)$ is depicted in Figure 0.9.

0.6.5 Identities

The following are some useful facts and connections between trigonometric functions:

- Pythagorean identity:
$$\sin^2(\theta) + \cos^2(\theta) = 1 \tag{0.6.6}$$
- Symmetry/Antisymmetry:
$$\sin(-\theta) = -\sin(\theta). \tag{0.6.7}$$

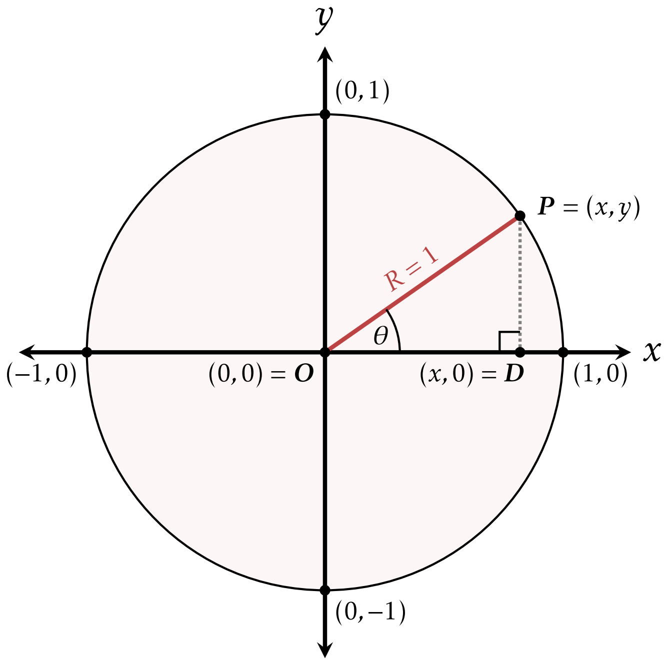


Figure 0.7 A unit circle with a point $\boldsymbol{P} = (x, y)$ on its circumference. The triangle $\triangle \boldsymbol{OPD}$ is a right triangle with sides $\boldsymbol{OD} = x$, $\boldsymbol{OP} = y$ and an angle θ opposing the side $\boldsymbol{DP}$.

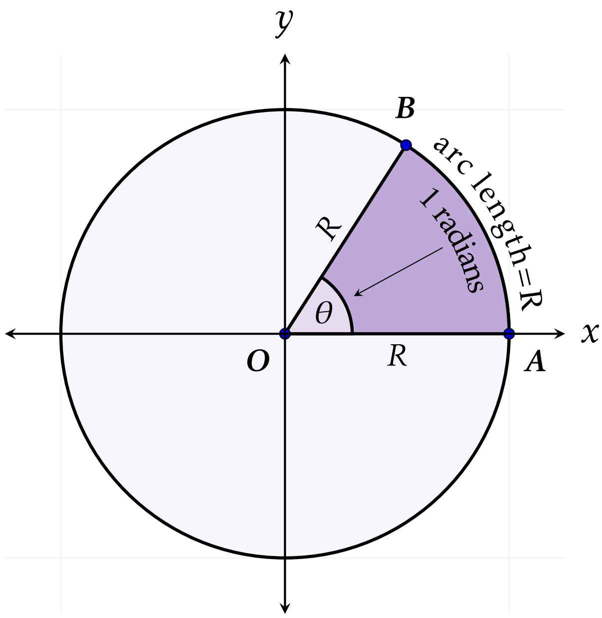


Figure 0.8 In this figure the arc $\boldsymbol{AB}$ has the same length of the radii $\boldsymbol{OA}$ and $\boldsymbol{OB}$ (all are equal to R), and therefore $\theta = 1$ radians.

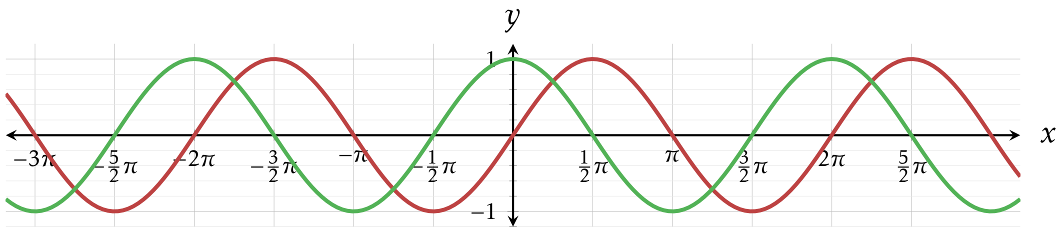


Figure 0.9 The graphs of sin(x) and cos(x) for $x \in [-10, 10]$. Note how the graph of cos(x) is "lagging" behind the graph of sin(x) by $\pi/2$.

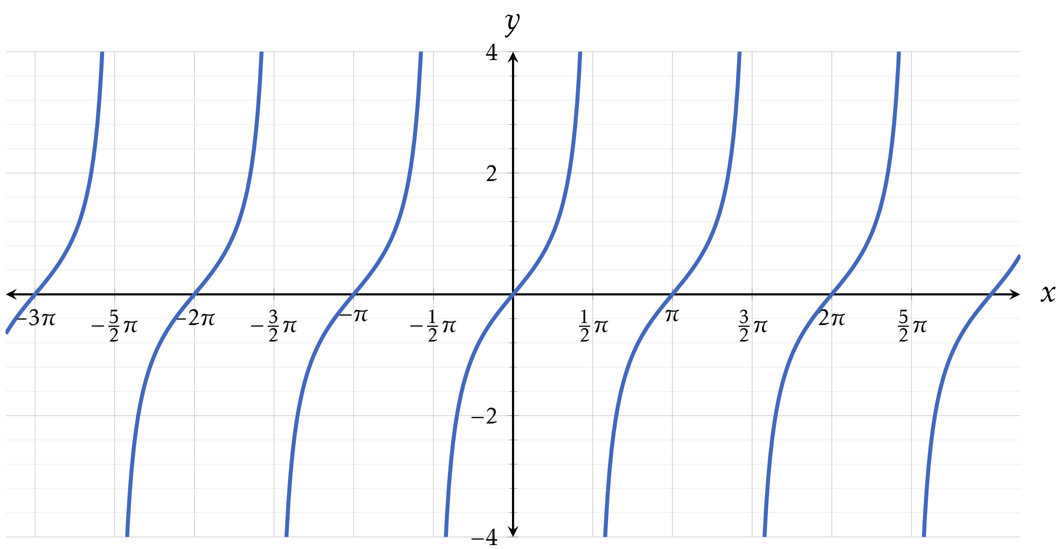


Figure 0.10 The graphs of $\tan(x)$ on the domain $[-3\pi, 3\pi]$.

$$\cos(-\theta) = \cos(\theta). \tag{0.6.8}$$
$$\tan(-\theta) = -\tan(\theta). \tag{0.6.9}$$

- Tangent from sine and cosine:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}. \tag{0.6.10}$$

- Phase between sine and cosine:

$$\sin\left(\theta \pm \frac{\pi}{2}\right) = \pm\cos(\theta). \tag{0.6.11}$$
$$\cos\left(\theta \pm \frac{\pi}{2}\right) = \mp\sin(\theta). \tag{0.6.12}$$

- Half-period shift:

$$\sin(\theta + \pi) = -\sin(\theta). \tag{0.6.13}$$
$$\cos(\theta + \pi) = -\cos(\theta). \tag{0.6.14}$$

- Angle sum:

$$\sin\left(\alpha \pm \beta\right) = \sin(\alpha)\cos\left(\beta\right) \pm \cos(\alpha)\sin\left(\beta\right). \tag{0.6.15}$$
$$\cos\left(\alpha \pm \beta\right) = \cos(\alpha)\cos\left(\beta\right) \mp \sin(\alpha)\sin\left(\beta\right). \tag{0.6.16}$$

- Double angle:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}. \tag{0.6.17}$$
$$\cos(2\theta) = 1 - 2\sin^2(\theta) = \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)}. \tag{0.6.18}$$

- Half angle:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}. \tag{0.6.19}$$
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}. \tag{0.6.20}$$
$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}. \tag{0.6.21}$$

- Product to sum:

$$\sin(\theta)\sin\left(\varphi\right) = \frac{1}{2}\left[\cos\left(\theta-\varphi\right) - \cos\left(\theta+\varphi\right)\right]. \tag{0.6.22}$$
$$\cos(\theta)\cos\left(\varphi\right) = \frac{1}{2}\left[\cos\left(\theta-\varphi\right) + \cos\left(\theta+\varphi\right)\right]. \tag{0.6.23}$$
$$\sin(\theta)\cos\left(\varphi\right) = \frac{1}{2}\left[\sin\left(\theta+\varphi\right) + \sin\left(\theta-\varphi\right)\right]. \tag{0.6.24}$$
$$\tan(\theta)\tan\left(\varphi\right) = \frac{\cos\left(\theta-\varphi\right) - \cos\left(\theta+\varphi\right)}{\cos\left(\theta-\varphi\right) + \cos\left(\theta+\varphi\right)}. \tag{0.6.25}$$

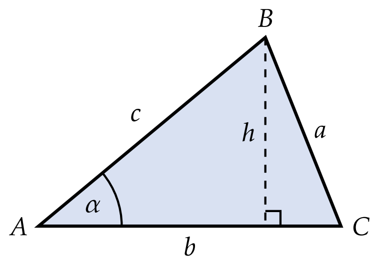


Figure 0.11 The area of a triangle using the side b as a base, and its corresponding height to the point B. The angle opposing the side A is marked as α.

- Sum to product:

$$\sin(\theta) \pm \sin(\varphi) = 2\sin\left(\frac{\theta \pm \varphi}{2}\right)\cos\left(\frac{\theta \mp \varphi}{2}\right). \tag{0.6.26}$$

$$\cos(\theta) + \cos(\varphi) = 2\cos\left(\frac{\theta + \varphi}{2}\right)\cos\left(\frac{\theta - \varphi}{2}\right). \tag{0.6.27}$$

$$\cos(\theta) - \cos(\varphi) = -2\cos\left(\frac{\theta + \varphi}{2}\right)\cos\left(\frac{\theta - \varphi}{2}\right). \tag{0.6.28}$$

$$\tan(\theta) \pm \tan(\varphi) = \frac{\sin(\theta \pm \varphi)}{\cos(\theta)\cos(\varphi)}. \tag{0.6.29}$$

0.6.6 Useful theorems

The area S of a triangle $\triangle ABC$ can be calculated using the length L any side of the triangle (in this context called a **base**) and the height h to its opposing vertex (see Figure 0.11):

$$S = \frac{1}{2}Lh. \tag{0.6.30}$$

The triangle with sides cbh is a right triangle, c being its hypotenuse. We can therefore infer the size of h using α:

$$h = c\sin(\alpha). \tag{0.6.31}$$

Substituting this back to Equation 0.6.30 yields that the area of the triangle is

$$S = \frac{1}{2}bc\sin(\alpha). \tag{0.6.32}$$

There is nothing special about choosing the side b as a base: we can also use a or c for the calculation. This will yield, respectively,

$$S = \frac{1}{2}ac\sin(\gamma), \tag{0.6.33}$$

$$S = \frac{1}{2}ac\sin(\beta), \tag{0.6.34}$$

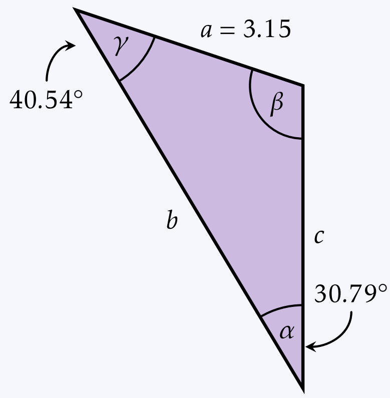
where β is the angle opposing b and γ is the angle opposing c. Since S is the same in all cases, we simply multiply each of the area equations by 2 and divide by abc, which yields

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}, \tag{0.6.35}$$

i.e. in a triangle, the ratio between any side and the sine of its opposing angle is always the same no matter which side we choose. This theorem is called the **law of sines**.

Example 0.47 Law of sines

Given the triangle $\triangle ABC$ below, what are β and b?



Since all angles in a triangle must add up to 180°,

$$\beta = 180° - 30.79° - 40.54° = 108.67°.$$

Using the law of sines,

$$b = \frac{a}{\sin(\alpha)} \cdot \sin(\beta) = \frac{3.15}{\sin(30.79°)} \cdot \sin(108.67°) \approx 5.83,$$

and

$$c = \frac{a}{\sin(\alpha)} \cdot \sin(\gamma) = \frac{3.15}{\sin(30.79°)} \cdot \sin(40.54°) \approx 4.$$

Note 0.10 Ambiguity of solutions

The above example reveals an issue that might arise due to the symmetrical nature of $\sin(x)$ around $x = \pi$ (180°): say we wanted to calculate β using the law of sines instead of by using $\beta = 180° - \alpha - \gamma$. In this case we would solve the equation

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b},$$

which would result in $\beta = \arcsin\left(\frac{b\sin(\alpha)}{a}\right) = \arcsin(0.95)$. However, two angles can fit this requirement: the sines of $71.34°$ and $108.67°$ are both equal to 0.95! Therefore, we must be careful when using the law of sine and make sure we always choose values that make sense (e.g. such that all angles add up to $180°$).

!

Of course, the sine function is not unique in having its own named “Law”: another useful theorem is the so-called **law of cosines** (also **al-Kashi’s theorem**). This theorem states that given a triangle with sides a, b, c and an angle γ opposing c,

$$c^2 = a^2 + b^2 - 2ab\cos\left(\gamma\right). \tag{0.6.36}$$

Much like the law of sines, the choice of angle does not matter, as long as we plug the correct sides to the equation: for α and β being the angles opposing to a and b respectively,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(\alpha), \\ b^2 &= a^2 + c^2 - 2ac\cos\left(\beta\right). \end{aligned} \tag{0.6.37}$$

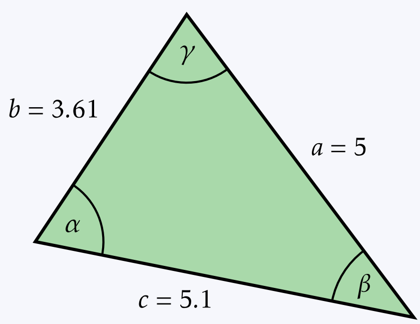
If the triangle in question is a right triangle then one of the angles is equal to $90°$. Without loss of generality, let us assume that this is γ. Since $\cos(90°) = 0$ we get that in the case of a right triangle

$$c^2 = a^2 + b^2, \tag{0.6.38}$$

i.e. we retrieve back the Pythagorean theorem.

Example 0.48 Law of cosines

Calculate all angles in the following triangle:



Using the law of cosines:

$$\cos\left(\gamma\right) = \frac{c^2 - b^2 - a^2}{-2ab} = \frac{5.1^2 - 3.61^2 - 5^2}{-2\cdot 5\cdot 3.61} \approx 0.33302 \Rightarrow \gamma = 70.54°,$$

$$\cos\left(\beta\right) = \frac{b^2 - a^2 - c^2}{-2ac} = \frac{3.61^2 - 5^2 - 5.1^2}{-2 \cdot 5 \cdot 5.1} \approx 0.74466 \Rightarrow \beta = 41.87°,$$
$$\cos(\alpha) = \frac{a^2 - b^2 - c^2}{-2cb} = \frac{5^2 - 3.61^2 - 5.1^2}{-2 \cdot 5.1 \cdot 3.61} \approx 0.38135 \Rightarrow \alpha = 67.58°.$$

0.7 COMPLEX NUMBERS

0.7.1 Algebraic approach

Real numbers, while being extremely useful, are not complete - they can't solve all equations involving numbers. For example, the equation

$$x^2 + 1 = 0 \tag{0.7.1}$$

has no real solutions, since there can be no real number x such that $x^2 = -1$. However, we can choose to define a new number, $\mathrm{i} = \sqrt{-1}$ and using it to build a new number system. This system is of course the set of complex numbers, $\mathbb{C}$. It is defined as the set of all z such that

$$z = a + \mathrm{i}b, \tag{0.7.2}$$

where $a, b \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$. We call a the **real component** of z or $\mathrm{Re}(z)$, and b its **imaginary component** or $\mathrm{Im}(z)$[6]. These numbers appear a lot all throughout the exact sciences (but especially in physics and engineering), so we must at the very least learn their basic properties.

It is not so obvious that we can add two different kinds of numbers together, but it works (the linear algebra chapter sheds more light on this idea). What is important is that we always keep these two parts separated. We see this when we add together two complex numbers z_1, z_2:

$$z = z_1 + z_2 = (a_1 + b_1\mathrm{i}) + (a_2 + b_2\mathrm{i}) = (a_1 + a_2) + (b_1 + b_2)\mathrm{i}. \tag{0.7.3}$$

The real part of z is therefore $a_1 + b_1$, and its imaginary part is $b_1 + b_2$.

What happens when we multiply two complex numbers? Let's check:

$$\begin{aligned} z = z_1 z_2 &= (a_1 + b_1\mathrm{i})(a_2 + b_2\mathrm{i}) \\ &= a_1a_2 + \mathrm{i}a_1b_2 + \mathrm{i}a_2b_1 + \mathrm{i}^2b_1b_2 \\ &= a_1a_2 + \mathrm{i}a_1b_2 + \mathrm{i}a_2b_1 - b_1b_2 \\ &= (a_1a_2 - b_1b_2) + \mathrm{i}(a_1b_2 + a_2b_1). \end{aligned} \tag{0.7.4}$$

We see that we can still separate the real part and imaginary part of the result. What happens in the case of two real numbers? For real numbers $b = 0$, and thus

[6] There is nothing more "real" about real numbers than imaginary numbers, but unfortunately that's the terminology we're stuck with ¯_(ツ)_/¯

Equation 0.7.4 devolves to $z = a_1 a_2 \in \mathbb{R}$, which is exactly what we expect: multiplying two real numbers yields their product, which is a real number. Notice that this doesn't happen with purely imaginary numbers: multiplying together two imaginary numbers (i.e. numbers for which $a = 0$) results in a real number. Will get to understand why this happens very soon.

When discussing real numbers sometimes we like to refer to their *magnitude*, i.e. their absolute value. With complex numbers this is defined as

$$|z| = \sqrt{a^2 + b^2}, \tag{0.7.5}$$

i.e. in a sense, to get the magnitude of a complex number we imagine its two components as being perpendicular and calculate the length of the resulting hypotenuse (cf. the Pythagorean theorem). In fact, this is one very useful interpretation of complex numbers, which we will explore in depth in the next subsection.

A very important operation that can be applied to complex numbers is **conjugation**. The conjugate of a complex number $z = a + \mathrm{i}b$ is defined as

$$\overline{z} = a - \mathrm{i}b, \tag{0.7.6}$$

i.e. conjugating a number is simply negating its imaginary part. When we multiply a complex number by its own complex conjugate we get

$$z\overline{z} = (a + \mathrm{i}b)(a - \mathrm{i}b) = a^2 + \cancel{ab\mathrm{i}} - \cancel{ab\mathrm{i}} - b^2\mathrm{i}^2 = a^2 + b^2, \tag{0.7.7}$$

i.e. $z\overline{z} = |z|^2$. The inverse of a complex number can be expressed as

$$z^{-1} = \frac{\overline{z}}{|z|^2}. \tag{0.7.8}$$

0.7.2 Geometric approach

As alluded to in the previous subsection, we can interpret a complex number $z = a + \mathrm{i}b$ as two components in a 2-dimensional space (called the **complex plane**), in which the horizontal axis represents real components, and the vertical access represents imaginary components:

Drawing a line from z to a (on the real axis) creates a right triangle. We can then define θ to be the angle near the origin and r the length of the hypotenuse: We call r the **magnitude** of z, and θ its **argument**. The ranges for r and θ are, respectively, $[0, \infty)$ and $[0, 2\pi)$.

Using Equation 0.6.4 the real and imaginary components of z are

$$\begin{aligned} a &= r\cos(\theta), \\ b &= r\sin(\theta), \end{aligned} \tag{0.7.9}$$

and z can be re-written as

$$z = r\left(\cos(\theta) + \mathrm{i}\sin(\theta)\right). \tag{0.7.10}$$

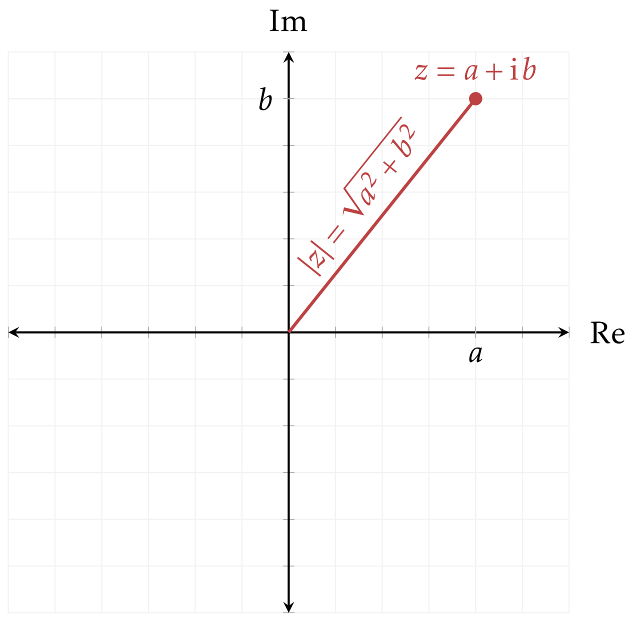


Figure 0.12 A complex number $z = a + ib$ shown on the complex plane: the horizontal and vertical axes represent the real and imaginary components, respectively.

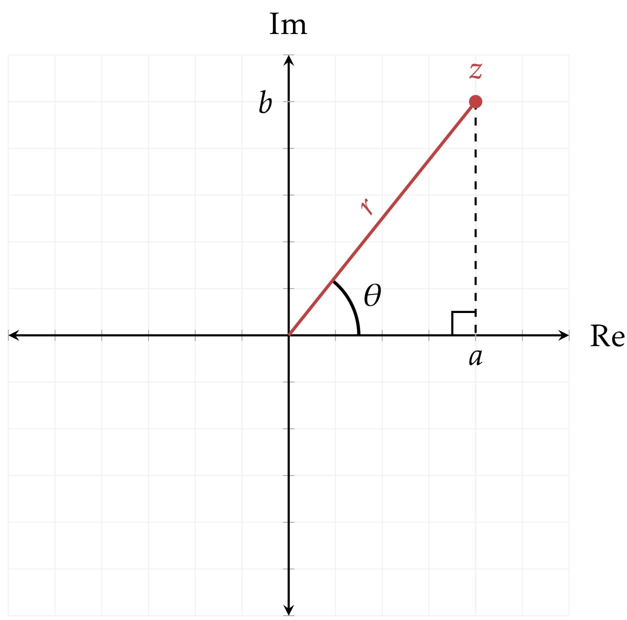


Figure 0.13 The same complex number z from Figure 0.12 shown with its polar components $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arctan\left(\frac{b}{a}\right)$.

Which we call the **polar form** of z (contrasted with $z = a + \mathrm{i}b$ being the **Cartesian form** of z).

Inverting the relations in Equation 0.7.9 yields the relations

$$\begin{aligned} r &= a^2 + b^2, \\ \theta &= \arctan\left(\frac{b}{a}\right). \end{aligned} \tag{0.7.11}$$

Let's examine the same properties of complex numbers shown in Equations 0.7.3, 0.7.4 and 0.7.6, and verify that they work in the polar form of complex numbers. We start with addition (Equation 0.7.3):

$$\begin{aligned} z_1 + z_2 &= r_1\left[\cos(\theta_1) + \mathrm{i}\sin(\theta_1)\right] + r_2\left[\cos(\theta_2) + \mathrm{i}\sin(\theta_2)\right] \\ &= \underbrace{r_1\cos(\theta_1)}_{a_1} + \underbrace{r_2\cos(\theta_2)}_{a_2} + \mathrm{i}\underbrace{r_1\sin(\theta_1)}_{b_1} + \mathrm{i}\underbrace{r_2\sin(\theta_2)}_{b_2} \\ &= (a_1 + a_2) + \mathrm{i}(b_1 + b_2). \end{aligned} \tag{0.7.12}$$

We see that indeed, the polar form of complex numbers adheres to the addition rule in Equation 0.7.3. Next is the product rule:

$$\begin{aligned} z_1 z_2 &= r_1\left[\cos(\theta_1) + \mathrm{i}\sin(\theta_1)\right] \cdot r_2\left[\cos(\theta_2) + \mathrm{i}\sin(\theta_2)\right] \\ &= r_1 r_2\left[\cos(\theta_1)\cos(\theta_2) + \mathrm{i}\cos(\theta_1)\sin(\theta_2) + \mathrm{i}\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right] \\ &= r_1\cos(\theta_1) r_2\cos(\theta_2) - r_1\sin(\theta_1) r_2\sin(\theta_2) + \mathrm{i}\left[r_1\cos(\theta_1) r_2\sin(\theta_2) + r_1\sin(\theta_1) r_2\cos(\theta_2)\right] \\ &= (a_1 a_2 - b_1 b_2) + \mathrm{i}(a_1 b_2 + a_2 b_1), \end{aligned} \tag{0.7.13}$$

which is indeed the result seen in Equation 0.7.4. We can also develop further the second row of Equation 0.7.13 using some trigonometry (specifically the trigonometric identities in Equation 0.6.25):

$$\begin{aligned} z_1 z_2 &= r_1 r_2\left[\cos(\theta_1)\cos(\theta_2) + \mathrm{i}\cos(\theta_1)\sin(\theta_2) + \mathrm{i}\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right] \\ &= r_1 r_2\Big[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\left[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\right]\Big] \\ &= r_1 r_2\left[\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)\right]. \end{aligned} \tag{0.7.14}$$

This is a very important result: it shows that multiplying a complex number z_1 by another complex number z_2 gives a complex number with magnitude $r_1 r_2$, i.e. the product of the magnitudes of the two complex numbers, and argument $\theta_1 + \theta_2$, i.e. the argument of z_1 rotated by the argument of z_2 (or vice-versa). We will consider this result in more detail soon.

In the polar form the complex conjugate of a number $z = r\left[\cos(\theta) + \mathrm{i}\sin(\theta)\right]$ can be brought about by substituting $-\theta$ into the arguments of the trigonometric functions:

$$\begin{aligned} \bar{z} &= r\left[\cos(-\theta) + \mathrm{i}\sin(-\theta)\right] \\ &= r\left[\cos(\theta) - \mathrm{i}\sin(\theta)\right] \\ &= r\cos(\theta) - \mathrm{i}r\sin(\theta) \\ &= a - \mathrm{i}b. \end{aligned} \tag{0.7.15}$$

Table 0.4 Values of e^{ix} for some useful values of x (cf. Table 0.3 for the values of $\sin(\theta)$ and $\cos(\theta)$).

x	$\cos(x)$	$\sin(x)$	z
$\frac{\pi}{2}$	0	1	i
π	-1	0	-1
$\frac{3\pi}{2}$	0	-1	$-\mathrm{i}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}\left(1+\mathrm{i}\sqrt{3}\right)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}(1+\mathrm{i})$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{2}\left(\sqrt{3}+\mathrm{i}\right)$

Lastly, let's show that Equation 0.7.7 can be derived in the polar form:

$$\begin{aligned} z\bar{z} &= r\left[\cos(\theta)+\mathrm{i}\sin(\theta)\right]\cdot r\left[\cos(\theta)-\mathrm{i}\sin(\theta)\right] \\ &= r^2\left[\cos^2(\theta)-\cancel{\mathrm{i}\cos(\theta)\sin(\theta)}+\cancel{\mathrm{i}\sin(\theta)\cos(\theta)}+\sin^2(\theta)\right] \\ &= r^2\left[\sin^2(\theta)+\cos^2(\theta)\right] \\ &= r^2 = a^2+b^2. \end{aligned} \tag{0.7.16}$$

In 1748 Leonhard Euler published his famous work *Introduction to analysis of the infinite*[7]. In it he introduced the following relation, called **Euler's formula**:

$$e^{\mathrm{i}x} = \sin(x)+\mathrm{i}\cos(x). \tag{0.7.17}$$

Using Euler's formula a complex number z can be written as

$$z = re^{\mathrm{i}\theta}. \tag{0.7.18}$$

In Table 0.4 we can see some useful complex exponentials $e^{\mathrm{i}x}$. Specifically, setting $x=\pi$ yields the famous **Eurler's identity**, considered by many to be one of the most beautiful equations in mathematics, as it binds together five important numbers, namely $0,1,\pi,e$ and i:

$$e^{\mathrm{i}\pi}+1=0. \tag{0.7.19}$$

Table 0.4 also shows us the integer behaviours of i:

$$\mathrm{i}^2=-1,\ \mathrm{i}^3=-\mathrm{i}, \mathrm{i}^4=1,\ \mathrm{i}^5=\mathrm{i},\ \mathrm{i}^6=-1,\ \mathrm{i}^7=-\mathrm{i},\ \ldots \tag{0.7.20}$$

[7] Latin for **Introduction to the Analysis of the Infinite**.

0.7.3 Roots of complex numbers

What is the n-th order roots of a complex number z, i.e. $\sqrt[n]{z}$? An illuminating way to approach this problem is by looking at the polar form of z. As an example, we start with the number $z = 1$ and find its 3rd order roots, i.e. all number w such that $w^3 = 1$ (spoiler alert: there are three such numbers).

Equation 0.7.14 taught us that complex numbers not only scale other numbers, but also rotate them: with real numbers the product $x \cdot y$ is equivalent to a scaling of x by y. With complex numbers the product has two components: its magnitude is the scale of x by y, and its argument is the argument of x rotated by the argument of y[8].

Example 0.49 Rotation using complex numbers

Let $z = 2 + 2\mathrm{i}$ and $w = 3\mathrm{i}$. Their polar forms are $z = 2\sqrt{2}\left[\cos(\pi/4) + \mathrm{i}\sin(\pi/4)\right]$ and $w = 3\left[\cos(\pi/2) + \mathrm{i}\sin(\pi/2)\right]$. Their product is

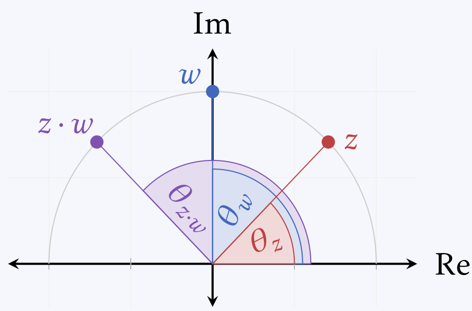
$$z \cdot w = (2 + 2\mathrm{i}) \cdot 3\mathrm{i} = 6\mathrm{i} + 6\mathrm{i}^2 = -6 + 6\mathrm{i},$$

which in polar form is $6\sqrt{2}\left[\cos(3\pi/4) + \mathrm{i}\sin(3\pi/4)\right]$. Note that

$$\begin{aligned} 2\sqrt{2} \cdot 3 &= 6\sqrt{2}, \text{ and} \\ \frac{\pi}{4} + \frac{\pi}{2} &= \frac{3\pi}{4}, \end{aligned}$$

i.e. $z \cdot w$ has magnitude which is the **product** of the magnitudes of z and w, and an argument which is the **sum** of the arguments of z and w.

The figure below depict the arguments of the three numbers $z, w, z \cdot w$. Note that $\theta_{z \cdot w} = \theta_z + \theta_w$.

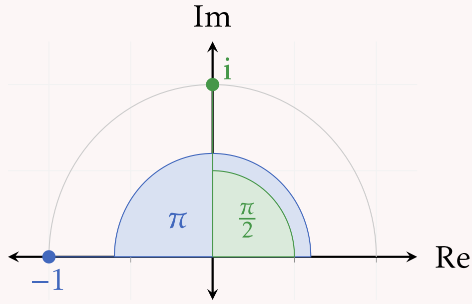


Note 0.11 $\mathrm{i}^2 = -1$ from a geometric (polar) viewpoint

In polar coordinates i has magnitude 1 and argument $\frac{\pi}{2}$, i.e. multiplying by i is equivalent to rotation by $\frac{\pi}{2}$ (90°) counter clockwise. Therefore, multiplying i by

[8] Due to the commutativity of the complex product we can switch the order of z and w and get the same result.

itself, i.e. i^2, rotates i itself by $\frac{\pi}{2}$ counter clockwise, bringing it to -1.



In polar form $1 = \cos(0) + \mathrm{i}\sin(0)$. Finding the arguments of the cube roots of 1 is therefore done by answering the following question: what angles θ will equal 0 (or its equivalent angles $2\pi, 4\pi, 6\pi, \dots$) when multiplied by 3? The answer is very simple: the only possible solutions are

$$\begin{aligned}\theta_1 &= 0,\\ \theta_2 &= \frac{2\pi}{3},\\ \theta_3 &= \frac{4\pi}{3}.\end{aligned} \tag{0.7.21}$$

(any other number in the range $[0, 2\pi)$ will give the same angles)

Therefore, the three cube roots of 1 are [9] (Figure 0.14)

$$\begin{aligned}z_1 &= \cos(\theta_1) + \mathrm{i}\sin(\theta_1) = \cos(0) + \mathrm{i}\sin(0) = 1,\\ z_2 &= \cos(\theta_2) + \mathrm{i}\sin(\theta_2) = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) = -0.5 + \frac{\sqrt{3}}{2}\mathrm{i},\\ z_3 &= \cos(\theta_3) + \mathrm{i}\sin(\theta_1) = \cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right) = -0.5 - \frac{\sqrt{3}}{2}\mathrm{i}.\end{aligned} \tag{0.7.22}$$

The n-th degree roots of 1 will follow the same pattern for $n \in \mathbb{N}$ (see Figure 0.15): their magnitude is always 1, and the argument of the k-th root is

$$\theta_k = \frac{2\pi}{n}k. \tag{0.7.23}$$

Finding the n-th degree roots of a general complex number $z = r\left[\cos(\theta) + \mathrm{i}\sin(\theta)\right]$ can be done in a similar fashion: all roots will have the magnitude $\sqrt[n]{r}$, and their argument θ_k will be such that multiplying it by n gives $\theta + 2\pi m$ for some integer value m, i.e.

$$\theta_k = \frac{\theta + 2\pi m}{n}. \tag{0.7.24}$$

[9] recall that $\sqrt[3]{1} = 1$, and therefore all roots have magnitude 1.

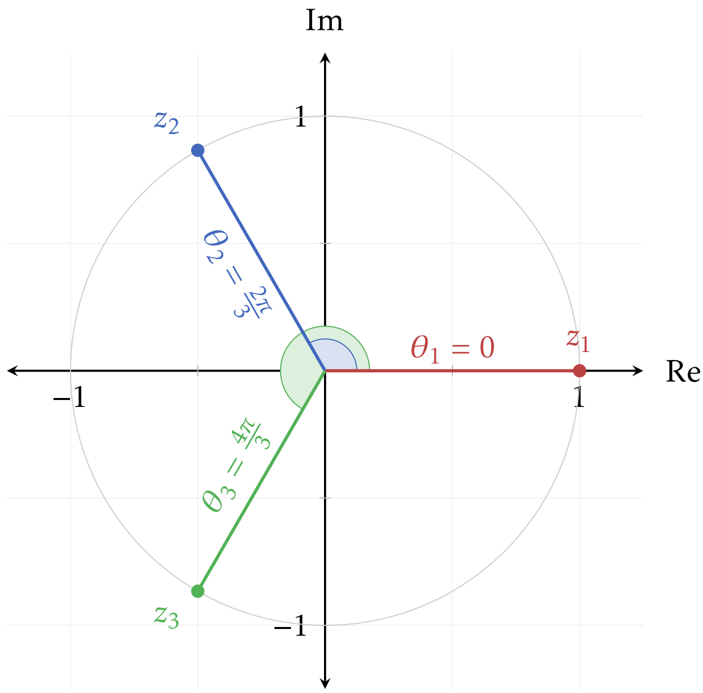


Figure 0.14 The three cube roots of $z = 1$.

0.8 EXERCISES

0.8.1 Problems

1. Write the following sets explicitly:

   (i) $\{x \in \mathbb{N} \mid 1 < x \leq 7\}$

   (ii) $\{x \in \mathbb{Z} \mid x < 5\}$

   (iii) $\left\{x \in \mathbb{R} \mid x^2 = -1\right\}$

   (iv) $\{x \in \mathbb{N} \wedge x \in \mathbb{Q}\}$

   (v) $\left\{x \in \mathbb{R} \mid x^2 - 3x - 4 = 0\right\}$

   (vi) $\{x \in \mathbb{R} \mid x < 5 \wedge x \geq 2\}$

2. Determine the relation between the sets:

   (i) $A = \{1,\ 2,\ 3\},\ B = \{1,\ 2\}$

   (ii) $A = \varnothing,\ B = \{2,\ -5,\ \pi\}$

   (iii) $A = \mathbb{Z},\ B = \{\pm x \mid x \in \mathbb{N} \cup \{0\}\}$

   (iv) $A = \left\{\pi, \mathrm{e}, \sqrt{2}\right\},\ B = \mathbb{Q}$

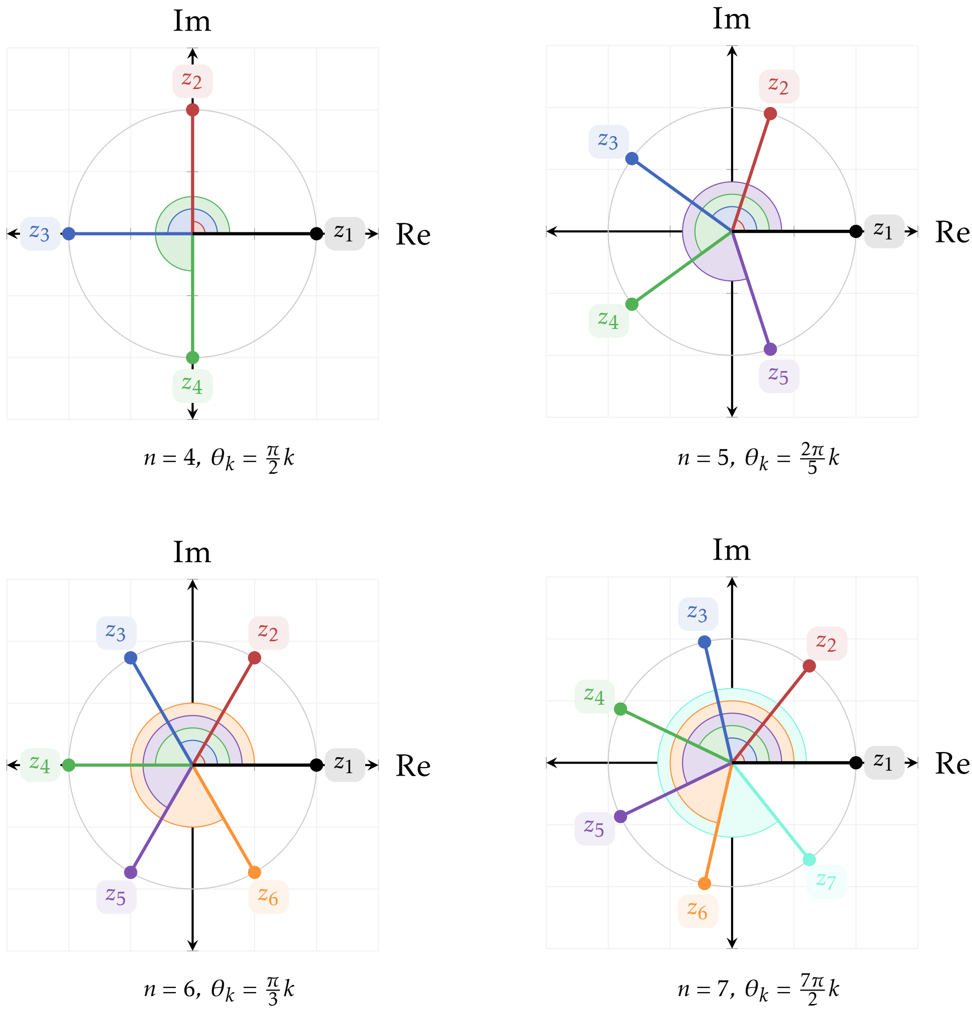


Figure 0.15 Complex n-th roots of $z = 1$ for $n = 4, 5, 6, 7$. Note that for all circles $r = 1$.

3. Write all elements in $S^2 \times W$, where $S = \{\alpha, \beta, \gamma\}$ and $W = \{x, y, z\}$. Find a condition that guarantees $S^2 \times W = W \times S^2$.

4. How many different **bijective** functions exist between a set with 2 elements and another set with 2 elements (e.g. $f: \{1,2\} \to \{\alpha, \beta\}$)? How many exist between two sets, each with 3 elements? Between two sets each with n elements?

5. For each of the real functions below, find a set on which it is surjective (use a graphing calculator if you are not familiar with the shape of a function):
$$x^2,\ x^3 - 5,\ \mathrm{e}^{-x^2/2},\ \sin(x),\ \sin(x) + \cos(x),\ x\mathrm{e}^x.$$

6. Given two sets A, B such that $|A| \neq |B|$, can a bijective function $f: A \to B$ exist? Explain your answer.

7. Find all real roots of the following polynomial function:
$$f(x) = x^3 + x^2 - 6x.$$

8. Given a real $b > 0$ and k, prove that for any real $x > 0$
$$\log_b\left(x^k\right) = k \log_b(x).$$

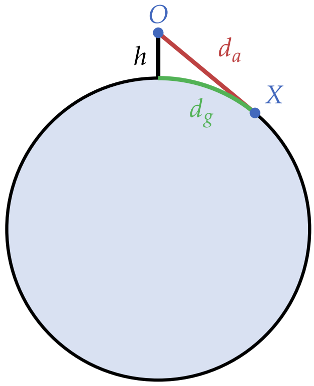
9. Show that for any positive real x, b
$$\log_b\left(\frac{1}{x}\right) = -\log_b(x).$$

10. Solve the following equation for any real $x > 0$: (CHECK SOLUTION! MIGHT BE WRONG)
$$\log_2(x) + \log_4(x-1) = \log_{16}\left(x^3\right).$$

11. During the second age of Middle-earth, 20 rings of power were forged. The following poem describes their distribution amongs the different peoples of the land:

12. The horizon on a spherical planet such as the earth[10] is defined as the distance from an observer to the point where the ground disappears behind the planet's curve. Following is a 2-dimensional depiction, where $\boldsymbol{O}$ is the observer, $\boldsymbol{h}$ its height above the planet surface, $\boldsymbol{X}$ the horizon point and $\boldsymbol{d_a}$ the air-distance from the observer to the horizon and $\boldsymbol{d_g}$ the ground-distance from the observer to the horizon:

[10] yes.



(i) Find an expression for the air-distance d_a and ground-distance d_g to the horizon as a function of the radius R and height h. (**hint**: find a relevant right triangle containing d_a and the radius of the planet)

(ii) Given that the Earth's radius is about 6371km (6.371×10^6m) and an average person is 1.75m tall - what is the distance to the horizon for a person standing at sea-level (both air- and ground-distances)? What would these distances be at the following heights: 165m (Eiffel tower's observation deck), 9.1km (average cruising altitude of a passenger jet) and 408km (average altitude of the International Space Station)?

(iii) How many degrees does the horizon drops from eye-level as function of h? (*eye-level* in this context means the direction tangent to the planet's surface)

13. Calculate the following complex product - first using the algebraic form and then the polar form, showing that the result is the same in both cases:

$$z = z_1^2 z_2 = \left(\sqrt{3} + \mathrm{i}\right)^2 \left(-2 + \sqrt{12}\mathrm{i}\right).$$

14. Prove that the **sum** of all the roots of the complex equation $z^n = 1$ is always zero when $n \geq 2$, i.e. if $w_0, w_1, \dots, w_{n-1}$ are the roots of the equation, then

$$\sum_{k=0}^{n-1} w_k = 0.$$

Hint: for $|r| \neq 1$,

$$\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}.$$

15. MORE EXERCISES TO BE WRITTEN...

0.8.2 Solutions

1. For each of the sets we first write how to read the notation in words, followed by its explicit form:

(i) Any **natural number** such that it is bigger than 1 and smaller or equal to 7. These are of course the numbers

$$\{2,3,4,5,6,7\}.$$

(ii) Any **integer** such that it is smaller than 5. These are the numbers

$$\{4,3,2,1,0,-1,-2,-3,\dots\}.$$

(iii) Any **real number** x such that $x^2 = -1$. Since for any $x \in \mathbb{R}$, $x^2 \geq 0$ - there is no such real number x whose square equals -1. Therefore this definition describes the empty set, i.e. $\emptyset$.

(iv) Any **natural number** that is also a rational number. Since any natural number is also a rational number (e.g. $4 = \frac{4}{1} = \frac{8}{2}$, etc.) the definition actually simply describes the set of natural numbers, $\mathbb{N}$. This fact can also be written as

$$\mathbb{N} \cap \mathbb{Q} = \mathbb{N}.$$

(v) Any **real number** such that it solves the equation $x^2 - 3x - 4 = 0$. The solutions can be found using the quadratic formula:

$$\frac{3 \pm \sqrt{3^2 + 4 \cdot 4}}{2} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = 4, -1.$$

Therefore the set described by the definition is simply

$$\{4,-1\}.$$

(vi) Any **real number** that is smaller than 5 **and** is bigger than or equal to 2. This definition describes the half-open interval

$$[2,5).$$

2. Relations between sets:

(i) All the elements in the set B are also in the set A $(1,2)$, but there's an element in A which is not in B (namely 3). Therefore, B is a subset of A:

$$B \subset A.$$

(ii) The empty set is a subset of any set (and a proper subset of any set except itself), therefore

$$A \subset B.$$

(iii) The set B is defined as all the natural numbers, their negatives and zero. This is exactly the definition of the integers $\mathbb{Z}$, which set A in this case. Therefore

$$A = B.$$

(iv) All of the elements in A are irrational numbers. The set B is the set of **rational numbers**, and therefore the sets are disjoined:

$$A \cap B = \emptyset.$$

3. S^2 is a Cartesian product of S with itself:

$$S^2 = \Big\{(\alpha,\alpha),(\alpha,\beta),(\alpha,\gamma),(\beta,\alpha),(\beta,\beta),(\beta,\gamma),(\gamma,\alpha),(\gamma,\beta),(\gamma,\gamma)\Big\}.$$

Therefore, to form the Cartesian product $S^2 \times W$ we simply take each of the elements is S^2 and add to it an element from W:

$$\begin{aligned} S^2 \times W = \Big\{&(\alpha,\alpha,x),(\alpha,\beta,x),(\alpha,\gamma,x),(\beta,\alpha,x),(\beta,\beta,x),(\beta,\gamma,x),(\gamma,\alpha,x),(\gamma,\beta,x),(\gamma,\gamma,x)\\ &(\alpha,\alpha,y),(\alpha,\beta,y),(\alpha,\gamma,y),(\beta,\alpha,y),(\beta,\beta,y),(\beta,\gamma,y),(\gamma,\alpha,y),(\gamma,\beta,y),(\gamma,\gamma,y)\\ &(\alpha,\alpha,z),(\alpha,\beta,z),(\alpha,\gamma,z),(\beta,\alpha,z),(\beta,\beta,z),(\beta,\gamma,z),(\gamma,\alpha,z),(\gamma,\beta,z),(\gamma,\gamma,z)\Big\}. \end{aligned}$$

Note that the number of elements in S is 3, and so the number of elements in S^2 is $3 \times 3 = 9$. The number of elements in W is also 3, and so the number of elements in $S^2 \times W$ is $9 \times 3 = 27$.

The Cartesian product $W \times S^2$ has the same structure as $S^2 \times W$, except that the elements from W are now on the left (remember that the order of elements is important in tuples, unlike with sets):

$$\begin{aligned} S^2 \times W = \Big\{&(x,\alpha,\alpha),(x,\alpha,\beta),(x,\alpha,\gamma),(x,\beta,\alpha),(x,\beta,\beta),(x,\beta,\gamma),(x,\gamma,\alpha),(x,\gamma,\beta),(x,\gamma,\gamma)\\ &(y,\alpha,\alpha),(y,\alpha,\beta),(y,\alpha,\gamma),(y,\beta,\alpha),(y,\beta,\beta),(y,\beta,\gamma),(y,\gamma,\alpha),(y,\gamma,\beta),(y,\gamma,\gamma)\\ &(z,\alpha,\alpha),(z,\alpha,\beta),(z,\alpha,\gamma),(z,\beta,\alpha),(z,\beta,\beta),(z,\beta,\gamma),(z,\gamma,\alpha),(z,\gamma,\beta),(z,\gamma,\gamma)\Big\}. \end{aligned}$$

One way of ensuring that $S^2 \times W = W \times S^2$ is by making all tuples equal, i.e. if

$$\alpha = \beta = \gamma = x = y = z,$$

then

$$S^2 \times W = \Big\{(\alpha,\alpha,\alpha)\Big\} = W \times S^2.$$

4. We start by counting the number of possible bijective functions $f_2 : \{1,2\} \to \{\alpha,\beta\}$. For each element in the domain of f_2 there are two options to connect to an element in the function's image: either 1 or 2. So we can have

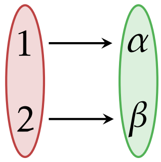
$$\begin{aligned} 1 &\mapsto \alpha, \text{ or}\\ 1 &\mapsto \beta. \end{aligned}$$

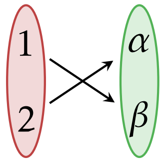
(recall that the symbol $x \mapsto y$ means that the element x is mapped by the function to the element y)

For each of the above options, there is only a single option left for the element 2:

$$\begin{aligned} 2 &\mapsto \beta \text{ if } 1 \mapsto \alpha,\\ 2 &\mapsto \alpha \text{ if } 2 \mapsto \beta. \end{aligned}$$

Therefore, we have two possible choices for mapping 1, each of which dictates the choice for mapping 2. Altogether there are two such bijective functions:





We can use the same logic for the case of 3 elements in each set. Let's define the function as

$$f_3 : \{1, 2, 3\} \to \{\alpha, \beta, \gamma\}.$$

We start with all the possibilities for connecting 1:

$$\begin{aligned} &1 \mapsto \alpha, \text{ or} \\ &1 \mapsto \beta, \text{ or} \\ &1 \mapsto \gamma. \end{aligned}$$

For each of the above choices, we have two remaining choices for connecting 2 (since one element in the image of f_3 is already decided by our choice for 1). Then, for the last element 3 we remain with a single option only, since two elements in the image of f_3 are already connected to by 1 and 2. Altogether we therefore have

$$3 \times 2 \times 1 = 6$$

total bijective functions f_3.

You probably already noticed the pattern: for a function

$$f_n : \{n \text{ elements}\} \to \{n \text{ elements}\},$$

we have n choices for connecting the first element, then $n-1$ options for connecting the second element, then $n-2$ options for connecting the the third element...and by the last element we have only a single option. Therefore, the number of bijective functions f_n is

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 = n!$$

Note that indeed $2! = 2$ and $3! = 6$, which agrees with the results we got for f_2 and f_3, respectively.

5. solution...

6. A function is bijective if and only if it is both a injective and surjective. There are two cases for $|A| \neq |B|$:

   (a) $|A| > |B|$, in which case there is at least one element in A which is not connected to any element in B: otherwise, there are at least two elements in A that connect to the same element in B. In the first case the relation is not a function, and in the second it is not injective and therefore not bijective.

   (b) $|A| < |B|$, in which case there must be at least one element in B that is not connected to by any element from A (by the defintion of a function, there cannot be any element in A that is connected to more than a single element in B). Therefore such a function is not surjective and thus not bijective.

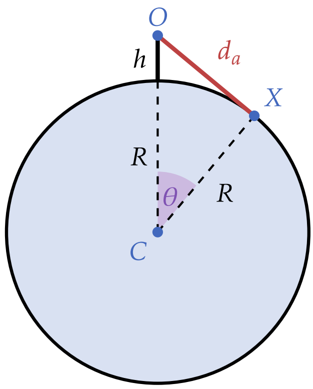
7. The polynomial $f(x)$ can be re-written as
$$f(x) = x\left(x^2 + x - 6\right).$$
Therefore one of its roots are when $x = 0$, and the other when $x^2 + x - 6 = 0$. Using the quadratic formula we get that $x^2 + x - 6 = 0$ when
$$x_{1,2} = \frac{-1 \pm \sqrt{1-4(-6)}}{2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3.$$
Thus, altogether the roots of f are $\{-3, 0, 2\}$.

8. As with most logarithm-related proofs, we transform the problem into the realm of exponents: let us set $m = \log_b(x)$. We then get that $x = b^m$. If we raise both by to the k-th power, we get
$$\begin{aligned} x^k &= \left(b^m\right)^k \\ &= b^{mk}. \end{aligned}$$
Taking the logarithm in base b of both sides of the above relation gives
$$\begin{aligned} \log_b\left(x^k\right) &= \log_b\left(b^{mk}\right) \\ &= mk \\ &= k\log_b(x). \end{aligned}$$
The last step results from our original defintion that $m = \log_b(x)$.

9. Using the relation proved in the previous question and setting $k = -1$ we get
$$\log_b\left(\frac{1}{x}\right) = \log_b\left(x^{-1}\right) = -1 \cdot \log_b(x) = -\log_b(x).$$

10. Using the logarithm base-change rule (Equation 0.4.11), we set all logarithms to same base ($b = 16$):
$$\log_2(x) = \log_{16}(x) \cdot \log_2(16) = 4\log_{16}(x).$$
$$\log_4(x-1) = \log_{16}(x-1) \cdot \log_4(16) = 2\log_{16}(x-1).$$
Therefore, the expression is equivalent to
$$4\log_{16}(x) = 2\log_{16}(x-1) + \log_{16}(x^3).$$
We can now bring the coefficients 4 and 2 inside the logarithm expression:
$$\log_{16}(x^4) = \log_{16}([x-1]^2) + \log_{16}\left(x^3\right).$$
Using the logarithmic addition law, we get:
$$\log_{16}(x^4) = \log_{16}(x^3\,[x-1]^3).$$
We can now discard $\log_{16}$ on both sides, and we're left with a simple equation to solve:
$$x^4 = x^3\,(x-1)^2,$$
the solutions of which are $x_1 = 0$ and $x_{2,3} = \frac{3\pm\sqrt{5}}{2}$, of which only $x_2 = \frac{3+\sqrt{5}}{2}$ is valid: x_1 isn't valid since $x > 0$, and x_3 isn't valid since $x_3 - 1 < 0$, and thus $\log_b(x_3 - 1)$ isn't defined over the real numbers.

11. (i) We start with drawing two radial lines from the center of the planet $\boldsymbol{C}$: one to the point on the surface where the observer meets the ground, and one to the horizon. The triangle $\triangle COX$ is a right triangle: the angle $\angle CXO = 90°$:



Using the Pythagorean theorem (with $R+h$ as the hypotenuse) we can calculate d_a:

$$d_a^2 + R^2 = (R+h)^2.$$

By expanding the right-hand side, cancelling R^2 and rearranging we get

$$d_a = \sqrt{2Rh + h^2}.$$

To get d_g we need to find the angle θ between the lines CX and CO. For that purpose we can use the law of sines (Equation 0.6.35):

$$\frac{d_a}{\sin(\theta)} = \frac{R+h}{\sin(90°)} = R+h.$$

(since $\sin(90°) = 1$)
Isolating $\sin(\theta)$ and subtituting the value of d_a as function of R and h yields:

$$\sin(\theta) = \frac{d_a}{R+h} = \frac{\sqrt{2Rh+h^2}}{R+h}.$$

Therefore

$$\theta = \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

When θ is given in radians, the length d_g then simply becomes

$$d_g = R\theta = R \cdot \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

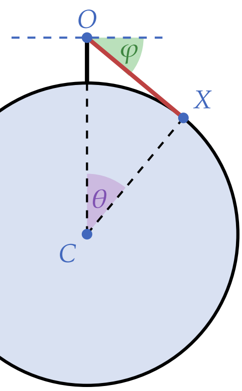
(ii) For an average person on Earth ($h = 1.75\text{m},\ R = 6.371 \times 10^6\text{m}$), standing at sea level, the air-distance to the horizon is therefore

$$d_a = \sqrt{2Rh + h^2} = \sqrt{2 \cdot 6.371 \times 10^6 \cdot 1.75 + 1.75^2} \approx 4722\text{m} = 4.722\text{km}.$$

The ground-distance, on the other hand, is

$$\begin{aligned} d_g &= R \cdot \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right) \\ &= 6.371\times10^6\text{m}\cdot\arcsin\left(\frac{4722\text{m}}{6.371\times10^6\text{m}+1.75\text{m}}\right) \\ &\approx 4722\text{m}. \end{aligned}$$

(iii) Let us call the angle representing the drop of the horizon from eye-level φ:



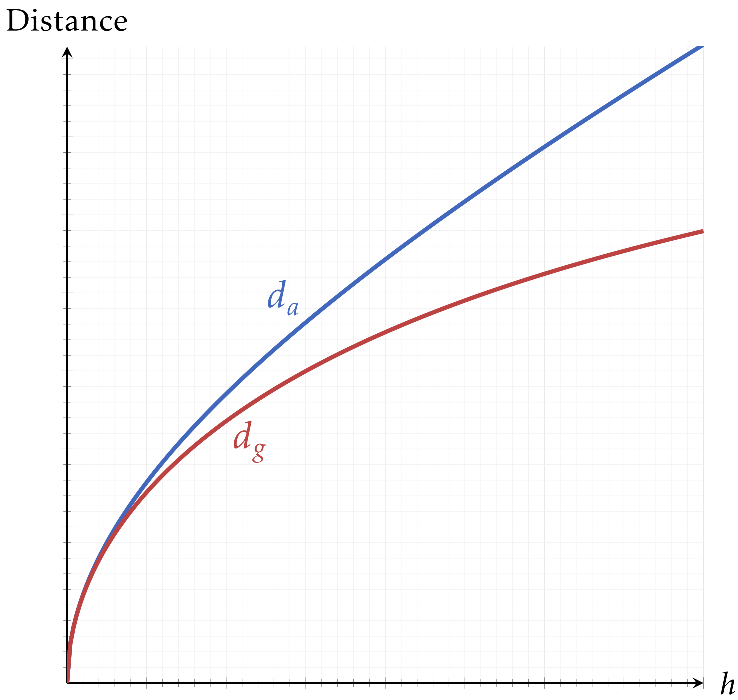
Since $\triangle COX$ is a right triangle ($\angle CXO$ being the right angle), we know θ from previously and all angles in a triangle sum up to deg 180, the angle $\angle COX$ is equal to $90° - \theta$. This in turn means that φ is equal to

$$90° - (90° - \theta) = \theta = \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

The following table sums up all the (approximate) air- and ground-distances to the horizon and the drop of the horizon from eye-level for each of the heights mentioned in the exercise:

Position	Height [m]	d_a[km]	d_g[km]	θ[°]
Person at sea level	1.75	4.7	4.7	0.04
Eiffel Tower observation	165	45.8	45.8	0.41
Average cruising altitude	9100	340.6	340.3	3.06
Internation Space Station	408000	2316	2221	19.98

Note that as the height h grows, the difference between d_a and d_g grows too. We can see this clearly when plotting $d_a(h)$ and $d_g(h)$ in the same graph (disregarding the units and values for now, since we are only interested in the qualitative behaviour of both distances):



For small values of h the two functions are very close to eachother, and as h grows they grow apart, with $d_a > d_g$.

12. • **Algebraic form**: we simply expand all parantheses and multiply everything:

$$\begin{aligned}\left(\sqrt{3}+\mathrm{i}\right)^2\left(-2+\sqrt{12}\mathrm{i}\right) &= \left(\sqrt{3}+\mathrm{i}\right)\left(\sqrt{3}+\mathrm{i}\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\ &= \left(3+\sqrt{3}\mathrm{i}+\sqrt{3}\mathrm{i}-1\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\ &= \left(2+2\sqrt{3}\mathrm{i}\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\ &= 2\left(1+\sqrt{3}\mathrm{i}\right)\left(-2+\sqrt{4\cdot 3}\mathrm{i}\right)\\ &= 2\left(1+\sqrt{3}\mathrm{i}\right)\left(-2+2\sqrt{3}\mathrm{i}\right)\\ &= 4\left(1+\sqrt{3}\mathrm{i}\right)\left(-1+\sqrt{3}\mathrm{i}\right)\\ &= 4\left(-1+\sqrt{3}\mathrm{i}-\sqrt{3}\mathrm{i}-3\right)\\ &= 4(-4)\\ &= -16.\end{aligned}$$

• **Polar form**: first we use Equation 0.7.11 to find the polar form of the two complex numbers:

$$z_1 = \sqrt{3}+\mathrm{i} \Rightarrow \begin{cases} r_1 = \sqrt{3+1} = 2, \\ \theta_1 = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}. \end{cases}$$

$$z_2 = -2 + \sqrt{12} \Rightarrow \begin{cases} r_2 = \sqrt{4+12} = 4, \\ \theta_2 = \arctan\left(-\frac{\sqrt{12}}{2}\right) = \frac{2\pi}{3}. \end{cases}$$

Therefore, $z_1^2 z_2$ in polar form is

$$\begin{aligned} z_1^2 z_2 &= \left(r_1 e^{\theta_1 \mathrm{i}}\right)^2 r_2 e^{\theta_2 \mathrm{i}} \\ &= r_1^2 r_2 e^{(2\theta_1 + \theta_2)\mathrm{i}} \\ &= 2^2 \cdot 4 e^{\left(\frac{2\pi}{6} + \frac{2\pi}{3}\right)\mathrm{i}} \\ &= 16 e^{\pi \mathrm{i}}. \end{aligned}$$

Since $e^{\pi \mathrm{i}} = -1$ (Equation 0.7.19), we get that indeed

$$z_1^2 z_2 = -16,$$

just as we got in the algebraic form.

13. It is easy to see that for even values of n the statement holds: for each w_k theres an oposing w_m ($m \neq k$) such that $w_k + w_m = 0$. See for example $n = 4$ and $n = 6$ in Figure 0.15.

For a more general proof which includes the odd values of n we must work a bit harder. Recall that the k-th root of the equation $z^n = 1$ has the following form (Equation 0.7.23):

$$w_k = \mathrm{e}^{\frac{2\pi}{n}\mathrm{i}k}.$$

We can re-write the sum of the roots as

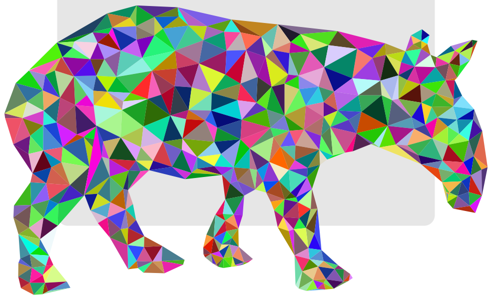
$$\sum_{k=0}^{n-1} \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\right)^k,$$

(since $x^{ab} = (x^a)^b$)

Using the hint we note that in this sum $r = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$, and thus

$$\sum_{k=0}^{n-1} \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\right)^k = \frac{1 - \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\right)^n}{1 - \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - \mathrm{e}^{\frac{2\pi \mathrm{i}}{n} n}}{1 - \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - \mathrm{e}^{2\pi \mathrm{i}}}{1 - \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - 1}{1 - \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}} = 0.$$

CHAPTER 1



REAL CALCULUS IN 1D

1.1 SEQUENCES AND SERIES

1.1.1 Basics

A **sequence** is an indexed collection of **elements**. By *indexed* we mean that the order of the elements in a sequence matters (unlike with sets): changing the order of any element changes the sequence as a whole. The following are some examples of sequences composed of real numbers:

- $1, -3, 0, -7, 2, 1.5, 4, 0, 1, -0.35, \sqrt{2}$.
- $0, 1, 2, 1, 1, -1, 0$.
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$

- $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$

The examples above present two more properties of sequences:

- Elements may repeat (unlike in the case of sets), and
- sequences can be either **finite** (as in the first two examples), or **infinite** (as in the latter two examples).

The number of elements in a sequence is called its **length**. In the case of infinite sequences we say that their length equals ∞ (inifinity). The elements of a sequence a are usually indexed using a subscript, such that a_1 is the first element in the sequence, a_2 is the second element in the sequence, etc. - and generally a_i is the i-th element in the sequence, where $i \in \mathbb{N}$.

We can therefore define a sequence somewhat more formally as a function from a subset of the natural numbers to the real numbers:

$$a : N \to \mathbb{R}, \tag{1.1.1}$$

where $N \subseteq \mathbb{N}$.

Example 1.1 Sequences as functions

The following 9-element sequence a

$$\begin{array}{ccccccccc} 3, & 4, & \frac{1}{2}, & 0, & 2, & 6, & -\frac{2}{3}, & 0, & -1. \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a(1) & a(2) & a(3) & a(4) & a(5) & a(6) & a(7) & a(8) & a(9) \end{array}$$

can be viewed as a function

$$a : \{1, 2, \ldots, 9\} \to \mathbb{R},$$

or more precisely as a function

$$a : \{1, 2, \ldots, 9\} \to \left\{-1, -\frac{2}{3}, 0, \frac{1}{2}, 2, 3, 4, 6\right\}.$$

The follow infinite sequence b

$$\begin{array}{cccccccc} 1, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \frac{1}{6}, & \frac{1}{7}, & \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ b(1) & b(2) & b(3) & b(4) & b(5) & b(6) & b(7) & \end{array}$$

can be viewed as a function

$$b : \mathbb{N} \to (0, 1].$$

Since sequences can be viewed as functions, they can be defined using formulas: for example, the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

can be defined using the simple formula

$$a_n = \frac{1}{n}.$$

Example 1.2 Some sequences defined using formulas

$$(-1)^n \quad \Rightarrow \quad -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

$$3n+4 \quad \Rightarrow \quad 7, 10, 13, 16, 19, 22, \ldots$$

$$(n+1)^2 \quad \Rightarrow \quad 4, 9, 16, 25, 36, 49, \ldots$$

$$\begin{cases} \text{2n+1} & \text{if } n \text{ is odd,} \\ \text{n-1} & \text{if } n \text{ is even.} \end{cases} \quad \Rightarrow \quad 3, 1, 7, 3, 11, 5, 15, 7, \ldots$$

Sequences can also be defined using **recursion**, where the value of an element is defined using previous values and a **starting value**. For example:

$$a_n = a_{n-1}^2 - 2,$$

with the starting value $a_1 = 3$. We the get that

$$a_2 = a_1^2 - 2 = 3^2 - 2 = 7,$$

and thus

$$a_3 = a_2^2 - 2 = 7^2 - 2 = 47,$$

etc.

Example 1.3 The Fibonacci sequence

The **Fibonacci sequences** is a well-known sequence defined using the following recursive rule:

$$F_n = F_{n-1} + F_{n-2},$$

with $F_1 = F_2 = 1$. The first few elements of the sequence are therefore

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

See Figure 1.1 for a graphical representation of the Fibonacci sequence.

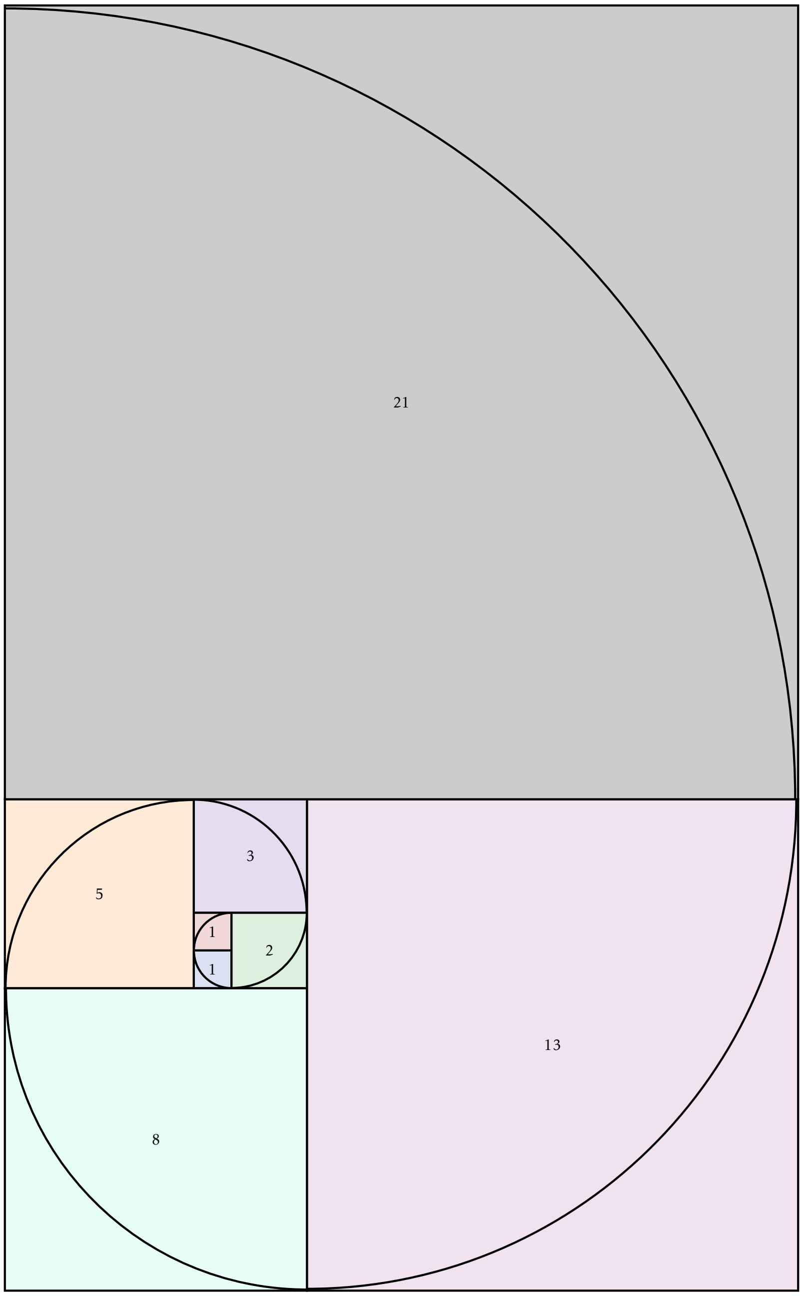


Figure 1.1 A graphical representation of the Fibonacci sequence: two squares of side 1 are placed adjacent to each other on the plane. In each subsequent step a new square is place such that its side is equal to the combined sides of the previous two squares. This way, the side of each square in the sequence follows the Fibonacci sequence. In each square we draw a quarter circle centered on one of the vertices, such that we get the famous **golden ratio** helix.

Note 1.1 Focus of section

From now on in the section we will focus on infinite sequences only.

1.1.2 Types of sequences

Consider the sequence $a_n = n^2$. Since $n \in \mathbb{N}$, for any n, $a_{n+1} > a_n$, since $(n+1)^2 > n^2$ (see Figure 1.2). We say that such a sequence is **increasing**. In fact, for a sequence to be increasing some sequential elements can be equal: for example, the sequence $c_n = 1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,\dots$ is also a increasing sequence. Thus, the definition of a increasing sequence is the following:

Definition 1.1 increasing sequence

sequence a_n is said to be *increasing* if for any $n \in \mathbb{N}$, $a_{n+1} \geq a_n$.

If we change the condition to $a_{n+1} > a_n$ we say that such a sequence is **strictly increasing**. In the above examples a_n is a strictly increasing sequence, while c_n is just increasing (since for some indeces n, $c_{n+1} = c_n$).

Similarly, a **decreasing** sequence is a sequence b_n for which for any $n \in \mathbb{N}$, $b_{n+1} \leq b_n$. An example of such sequence is $b_n = \frac{1}{n}$ (see Figure 1.3). ANd of course, if we change the condition to $b_{n+1} < b_n$ then the sequence is **strictly decreasing**.

Generally, a sequence that is either increasin or decreasing is said to be **monotone**. If a sequence is monotone starting only from a certain n, we say that the sequence is **eventually monotone** (i.e. *eventually increasing* or *eventually decreasing*). An example of such sequence is $d_n = (n-5)^2$ (Figure 1.4): for $N \in 1,2,3,4,5$ it is decreasing, but starting from $n = 5$ it is increasing for any n.

As an example of a sequence which isn't monote, consider the sequence $e_n = \sin(n)$: for some values of n, $e_{n+1} > e_n$ and for some other values $e_{n+1} < e_n$ (see Figure 1.5).

The following are two ways to determine whether a sequence a_n is monotone:

- **Difference test**: if $a_{n+1} - a_n \geq 0$ for all $n \in \mathbb{N}$, then the sequence is increasing. If $a_{n+1} - a_n \leq 0$ for all $n \in \mathbb{N}$ then the sequence is decreasing.

- **Ratio test**: if $\frac{a_n+1}{a_n} \geq 1$ for all $n \in \mathbb{N}$ then the sequence is increasing, and if $\frac{a_{n+1}}{a_n} < 1$ for all $n \in \mathbb{N}$ then the sequence is decreasing.

Example 1.4 Difference test

Given the sequence $a_n = \frac{n}{n+1}$, we look at the difference $a_{n+1} - a_n$:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)(n+1) - (n+2)n}{(n+1)(n+2)}$$

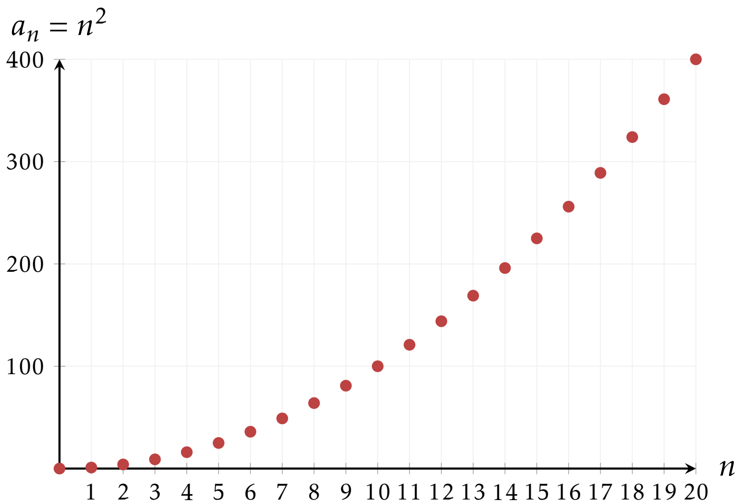


Figure 1.2 The sequence $a_n = n^2$ is increasing, and is in fact *strictly* increasing.

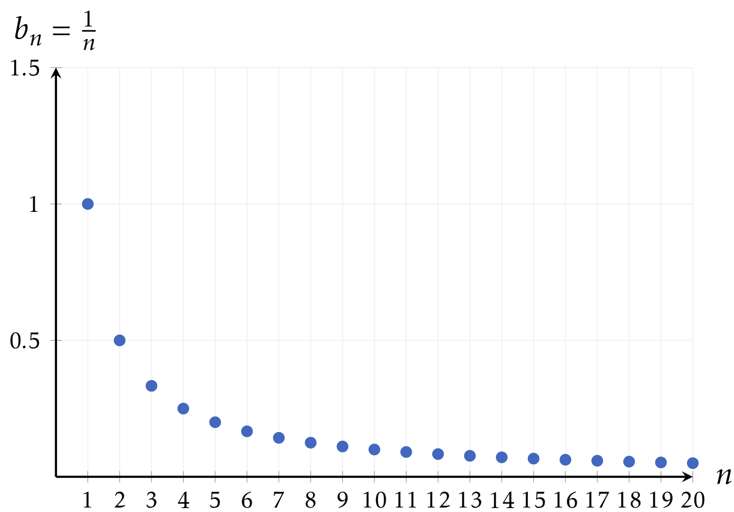


Figure 1.3 The sequence $b_n = \frac{1}{n}$ is decreasing, and is in fact *strictly* decreasing.

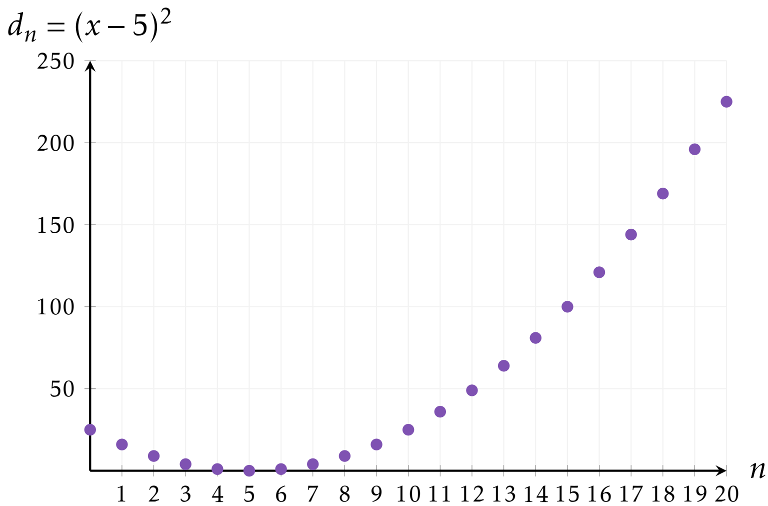


Figure 1.4 The sequence $d_n = (n - 5)^2$ starts as a decreasing sequence, but starting from $n = 5$ it is increasing, making it an *eventually increasing sequence*.

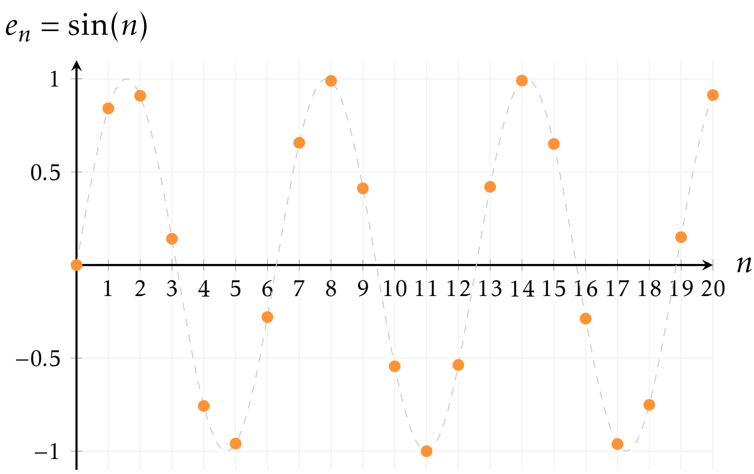


Figure 1.5 The sequence $e_n = \sin(n)$ is neither increasing nor decreasing. For reference, the function $\sin(x)$ is plotted as a dashed line behind e_n.

$$= \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < 1 \quad \forall n \in \mathbb{N}.$$

The last (in)equality stems from the fact that no matter what n we subtitute into $(n+1)(n+2)$, the result will be greater than 1, and thus $\frac{1}{(n+1)(n+2)}$ is always smaller than 1. Therefore, a_n is a decreasing sequence.

Example 1.5 Ratio test

Given the sequence $b_n = \frac{2^n}{n^2}$, the ratio of a_{n+1} to a_n is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = 2\frac{n^2}{(n+1)^2}.$$

Let's look at the first few approximated values of the ratio $\frac{n^2}{(n+1)^2}$:

n	$\frac{n^2}{(n+1)^2}$
0	0
1	0.25
2	0.44444...
3	0.5625
4	0.64
5	0.69444...
6	0.7346938775510204
7	0.765625
8	0.7901234567901234
9	0.81
10	0.8264462809917356
11	0.840277777...
12	0.8520710059171598
13	0.8622448979591837

We see that for any $n \geq 3$, $\frac{n^2}{(n+1)^2} > \frac{1}{2}$, and therefore $2\frac{n^2}{(n+1)^2} > 1$. Thus, the sequence is eventually increasing.

Some sequences are **bounded** from below: this means that their elements never get smaller than some constant $\underline{M} \in \mathbb{R}$. For example, consider the simple sequence $a_n = n$, where $n = \{1, 2, 3, 4, \dots\}$: there is no element in the sequence that is smaller than 1. Therefore, a_n is bounded from below by 1. Of course, one may argue that b_n is also bounded from below by 0, or -6, or in fact any negative number. This is true, however we are usually interested in the *maximal* number $\underline{M}$ that bounds the sequence from below, which in this case is $\underline{M} = 1$. We call that number the **infimum** of the sequence, and denote it as $\inf a_n$.

Similarly, a sequence a_n can be bounded from above by some number $\overline{M} \in \mathbb{R}$, i.e. there exist no n for which $a_n > \overline{M}$. We call the *minimal* such number the **supremum** of the sequence a_n, denoted $\sup a_n$. For example, the sequence $b_n = \frac{1}{n}$ is bounded from above by any real number $x \geq 1$, and therefore $\sup b_n = 1$. In fact, b_n is also bounded from below by $\underline{M} = 0$, and therefore we say that it is **bounded**. Another example of a sequence that is bounded is $e_n = \sin(n)$, which is bounded from below by $\underline{M} = -1$ and from above by $\overline{M} = 1$.

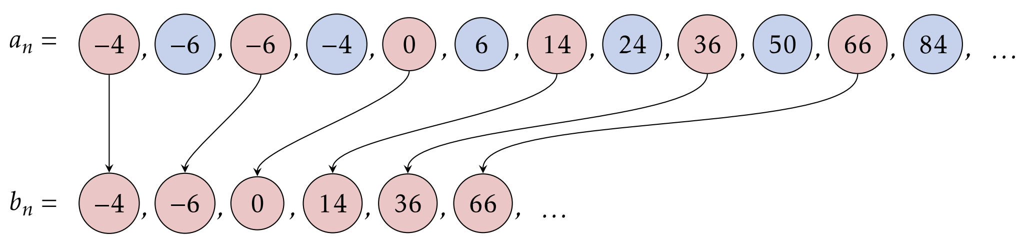
Example 1.6 Bounded and unbounded sequences

The following table shows some examples of sequences that are bounded from below, from above, or neither:

a_n	**First 5 elements**	$\inf a_n$	$\sup a_n$
$n^2 - n$	$0, 2, 6, 12, 20, \dots$	0	-
$\frac{n}{n+1}$	$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$	$\frac{1}{2}$	1
e^{-n}	$e^{-1}, e^{-2}, e^{-3}, e^{-4}, e^{-5}, \dots$	0	e^{-1}
$\log(n)$	$0, \log(2), \log(3), \log(4), \log(5), \dots$	0	-
$(-1)^n$	$-1, 1, -1, 1, -1, \dots$	-1	1
$(-1)^n n$	$-1, 2, -3, 4, -5, \dots$	-	-
$(-2)^n$	$-2, 4, -8, 16, -32, \dots$	-	-

1.1.3 Subsequences

Given any sequence a_n, we can remove from it any number of its elements (including 0 elements) and get a new sequence b_n which is a **subsequence** of a_n. For example, let $a_n = n^2 - 5n$. We can remove each 2nd element from a_n (i.e. those with indices $2, 4, 6, 8, \dots$) and get the following sequence b_n:



Note 1.2 Order of elements in a subsequence

A subsequence must preserve the order of the original sequence, since all we do in practice is removing elements from the original sequence, without changing the order of the remaining elements.

!

Let's look at a more formal defintion of a subsequence, which uses the choice of indices instead of removing elements:

Definition 1.2 Subsequence

A subsequence of a sequence a_n is a sequence a_{n_k}, where n_k is a **strictly increasing** sequence of natural numbers.

π

To create a subsequence using the above defintion, one can first create a sequence of indices n_k, and then subtitute only those indices into n in a_n. For example, given the sequence $a_n = n^2 - 5n$ from before, we can define a sequence of indices $n_k = 1, 3, 5, 7, 9, 11, \dots$ which would then yield the subsequence b_n shown before.

The reason we define n_k to be strictly increasing is to avoid changing the order of the elements from the original sequeunce a_n: for example, if we allowed n_k to be "just" increasing, we might end up with a case where there are two subsequent equal indices, e.g. $n_k = 1, 3, 5, 8, 9, 9, 10, \dots$. That would mean that we repeat an element from a_n **twice or more** in the subsequence (in the example this would be a_9), rendering it invalid as a subsequence, since as mentioned before - a subsequence must perserve the order of the original sequence.

Subsequences share all of the above-mentioned properties of the original sequence: if the original sequence is increasing or decreasing - so do all of its subsequences, and if it is bounded from above or below - so do all of its subsequences. Let's prove two of these properties:

Proof 1.1 Rising sequences and their subsequences

Claim: given an **increasing** sequence a_n, all of its subsequences are also increasing sequences themselves.

Proof: using contradiction. Let a_n be an increasing sequence, and b_n a subsequence of a_n which isn't increasing. From the fact that b_n is not an increasing sequence we know that there exist at least two indices k, m such that $k < m$ but $b_k > b_m$. Since any b_n is an element of a_n without change of order, we can substitute $b_k = a_i$ for some index i and $b_m = a_j$ for some index j, such that $i < j$ (since $k < m$ - this is exactly the idea of preserving the order of a_n). We therefore get that $a_i = b_k > b_m = a_j$, or simply $a_i > a_j$ even though $i < j$ - in contradiction to a_n being an increasing sequence. Therefore there can be no subsequence of a_n that isn't an increasing sequence.

QED

Proof 1.2 Bounded sequences and their subsequences

Claim: given a sequence a_n which is **bounded from below**, all of its subsequences are also bounded from below.

Proof: also using contradiction. Let a_n be a sequence bounded from below by $\inf a_n = \underline{M}$. Let b_n be a subsequence of a_n that isn't bounded from below - i.e. there exist an element b_i such that $b_i < \underline{M}$. Since b_n is a subsequence of a_n, $b_i = a_j$ for some index j. Therefore $a_j = b_i < \underline{M}$ in contradiction to $\underline{M}$ being the infimum of a_n. Therefore, a subsequence of a sequence bounded from below can not be unbounded from below.

QED

Challenge 1.1 Further proofs

1. Prove that all subsequences of a decreasing sequence are themselves decreasing.
2. Prove that all subsequences of a sequence bounded from above are themselves bounded from above.
3. Prove that all subsequences of a **strictly** increasing/decreasing sequence are themselves increasing or decreasing, respectively.
4. Given a bounded sequence a_n with $\inf a_n = \underline{M}$, can it have a subseqeunce b_n with $\inf b_n \neq \underline{M}$? If yes - give an example. If no - prove your claim.

?

1.1.4 Limits

As you probably noticed by now, some infinite sequences seem to approach a certain value as we increase n. That is to say, the bigger n is, the closer such a sequence a_n gets to a certain value $L \in \mathbb{R}$. For example, the sequence $b_n = \frac{1}{n}$ approaches to $L = 0$ as we increase n (see again Figure 1.3). The sequence $a_n = \frac{1}{n^2+1}$ approaches the value $L = 1$ as we increase n (see Figure 1.6). On the contrary, the sequence $d_n = (n-5)^2$ eventually increase in such a way that it "approaches" $L = \infty$, while $e_n = \sin(n)$ doesn't approach any value and instead endlessly "jumps" around in a repeated manner.

The formal term for the behaviour of a_n and b_n is called **convergance**, and it is one of the most important properties of infinite sequences. In this subsection we will define, analyze, and explain it in detail. To begin, we can divide all infinite sequences into two separate categories:

1. Sequences which converge to a finite number $L \in \mathbb{R}$.
2. Sequences which do not converge to any finite number.

Sequences in the second category are said to be **diverging**, and they can be further split into two separate categories:

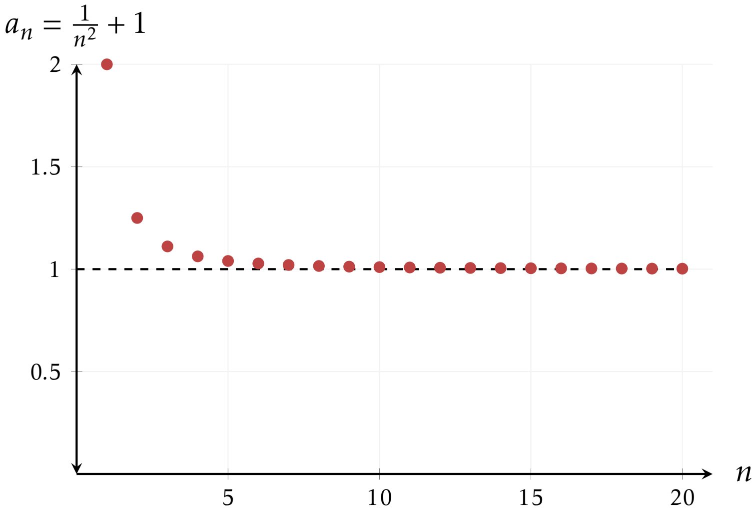


Figure 1.6 The sequence $a_n = \frac{1}{n^2} + 1$ approaches the value $L = 1$ as n increases.

i. Sequences which diverge to either positive or negative infinity.

ii. Sequences which neither converge nor diverge to $\pm\infty$.

Let us start with a more precise analysis of sequences that diverge to $\pm\infty$. In essence, a sequence which diverges to positive infinity is a sequence that is bounded from below but not from above, that is - given any real number R the sequence eventually passes it. In other words, all the elements in the suquence *after a certain value of* n are greater than R, for any R that we choose.

Take for example the sequence $a_n = n^2$. It most certainly has a lower boundry, namely $\inf a_n = 1$. On the other hand, given any $R \in \mathbb{R}$ eventually the values of a_n pass it. For example, given $R = 100$, the elements of a_n pass it after just 10 elements (since $a_{11} = 11^2 = 121 > 100$). The number $R = 1{,}000{,}000$ is passed after 1000 elements, etc. No matter how big R is, **eventually** a_n will pass it. Therefore, we say that a_n *goes to infinity*, and denote it by writing

$$\lim_{n\to\infty} a_n = \infty. \tag{1.1.2}$$

The notation lim is short for **limit**. While it can be argued that a divergent sequence has no limit, sometimes the term is used in the case of divergence to $\pm\infty$.

Note 1.3 Another limit notation

Another common notation used to denote that a sequence a_n is going to infinity as n increases is the following:

$$a_n \xrightarrow{n\to\infty} \infty.$$

A more formal definition of this behaviour is as follows:

Definition 1.3 Sequence going to inifnity

Let a_n be an infinite sequence. If for any $R \in \mathbb{R}$ there exist $n_R \in \mathbb{N}$ such that for any $n > n_R$, $a_n > R$, the sequence is said to be going to positive infinity. We denote this fact by writing

$$\lim_{n\to\infty} a_n = \infty.$$

π

Note 1.4 Chain of inequality

Note the following: say we have 5 real numbers $k_1, \ldots, k_5$. If

$$k_1 > k_2 > k_3 = k_4 \geq k_5.$$

then we know that

$$k_1 > k_5.$$

This might seem obvious - but is worth noting before we move on, as we will be using similar chains of inequalities in up-coming proofs.

!

Of course, for negative infinity the behaviour is very similar: a sequence a_n with an upper boundry $\overline{M}$ and no lower bound $\underline{M}$ is said to be **going to negative infinity**, since for any $R \in \mathbb{R}$ there exist an $n_R \in \mathbb{N}$ for which if $n > n_R$ then $a_n < R$.

Generally speaking, proving that a sequence goes to either positive or negative infinity follows a certain pattern, which we will examplify using the sequence $a_n = \frac{n}{2}$ (Figure 1.7). It should be clear that the sequence goes to positive infinity as n increases, since we can make the values of a_n as large as we want by substituting a respective n into $\frac{n}{2}$: for example, given $R = 1000$ we can subtitute $n = 2000$, yielding $a_{2000} = \frac{2000}{2} = 1000$, and thus any a_n where $n > 2000$ will be bigger than $R = 1000$. For $R = 1{,}000{,}000$ we can substitute $n = 2{,}000{,}000$ and so forth.

To show that this is true for any $R \in \mathbb{R}$ we should do the following: given a real number R find an index n_0 such that $a_{n_0} \geq R$. Since a_n is a strictly increasing sequence, that would mean that for any $n > n_0$, $a_n > a_{n_0} \geq R$, or simply $a_n > R$. In the case of $a_n = \frac{n}{2}$ we can simply choose the closest integer to $2R$ that is also bigger than $2R$ (i.e. if $R = 2.3$ we choose $n_0 = 3$, if $R = 100.7$ we choose $n_0 = 101$, etc.).

To always get an integer *equal to or bigger than* R we can use the **ceiling** operator. For any given $x \in \mathbb{R}$, its ceiling (denoted $\lceil x \rceil$) is the closest integer which is bigger than or equal to x, or more formally:

Definition 1.4 Ceiling and floor operators

Let $x \in \mathbb{R}$. Then

$$\lceil x \rceil = \min\left(\{n \in \mathbb{N} | n \geq x\}\right). \tag{1.1.3}$$

$$\lfloor x \rfloor = \max\left(\{n \in \mathbb{N} | n \leq x\}\right). \tag{1.1.4}$$

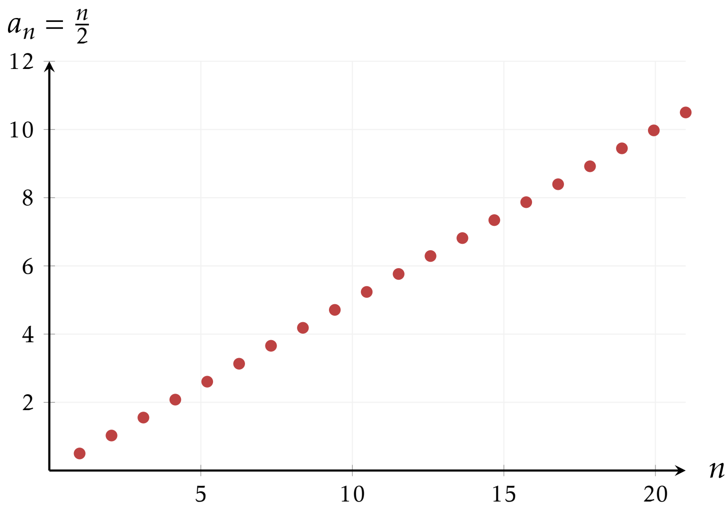


Figure 1.7 The sequence $a_n = \frac{n}{2}$ goes to positive infinity as n increases.

i.e. - $\lceil x\rceil$ is the **minimal** integer n that is **bigger than or equal** to x, and $\lfloor x\rfloor$ is the **maximal** integer n that is **smaller than or equal** to x.

π

Using the value $n_0 = \lceil 2R\rceil$ we can show, step by step, that indeed $\lim\limits_{n\to\infty} = \infty$ by using several subtitutions: for any $n > n_0$ we get that

$$a_n = \frac{n}{2} \overset{\text{since } n > n_0}{>} \frac{n_0}{2} = \frac{\lceil 2R\rceil}{2} \underset{\text{since } \lceil 2R\rceil \geq 2R}{\geq} \frac{2R}{2} = R,$$

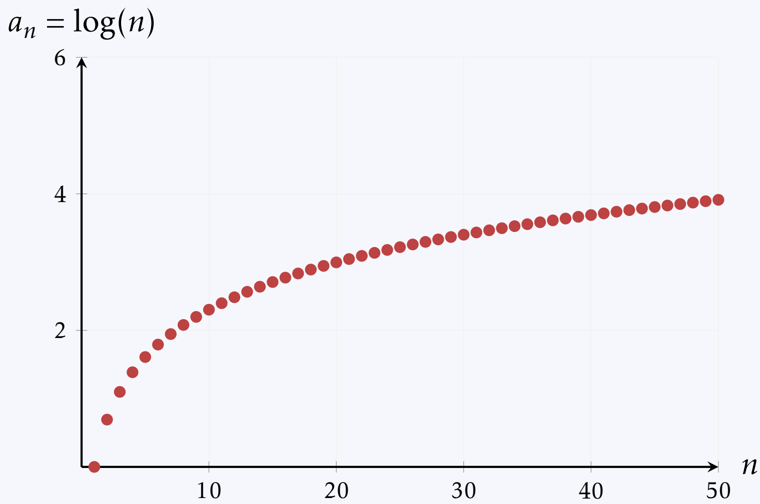
or simply $a_n > R$.

Example 1.7 The sequence $\log(n)$

Let's show that the sequence $a_n = \log(n)$ (see graph below) goes to infinity as n increases: first, we note that $\log(n)$ is a strictly increasing sequence bounded from below by $\underline{M} = 0$. Now, given some positive $R \in \mathbb{R}$ we chose $n_0 = \lceil e^R\rceil$ and thus get that for any $n > n_0$

$$a_n > a_{n_0} = \log(n_0) = \log\left(\lceil e^R\rceil\right) \geq \log\left(e^R\right) = R.$$

altogther we get $a_n > R$, and therefore $\lim\limits_{n\to\infty} \log(n) = \infty$.



Note: this proof works because $\log(n)$ is a *strictly* increasing sequence. If it were only an increasing sequence we would not be guaranteed that $n > n_0$ means that $a_n > a_{n_0}$, and the entire process would not yield that $\log(n)$ always passes any given real number R. Indeed, by naively looking at the graph above we may be mistaken to think that $\log(n)$ actually approaches some finite number, say 4.5 or so, and doesn't go to infinity. This is of course not the case.

Example 1.8 A sequence which goes to negative infinity

Let us now show that the sequence $a_n = -\sqrt{n}$ goes to negative infinity. We first note that $-\sqrt{n}$ is always negative, is decreasing and bouned from above by $M = 0$. For any given negative $R \in \mathbb{R}$ we choose $n_0 = \lfloor R^2 \rfloor$, and thus for any $n > n_0$ we get

$$a_n < a_{n_0} = -\sqrt{n_0} = -\sqrt{\lfloor R^2 \rfloor} \leq -\sqrt{R^2} = -|R| = R.$$

! **To be written**: is this example actually necessery? It seems reduntant unless we show somethin
!

The next type of sequences we ananlyze are those sequences that converge to a real number L as n increases - i.e. as n increases, the terms of the sequence get closer and closer to L. A classical example for such a sequence is $a_n = \frac{1}{n}$ (Figure 1.3): as n increases, the terms $\frac{1}{n}$ become smaller and smaller (while always being positive) and the sequence as a whole approaches $L = 0$.

Sequences don't have to approach a limit from above: some sequences approach a limit from below (Figure 1.8), while others may **oscillate** around the limit (Figure 1.9).

We define convergence in a similar way to how we defined that a sequence goes to ∞: in that case we had to show that for any $R \in \mathbb{R}$ the sequence eventually surpasses R and no

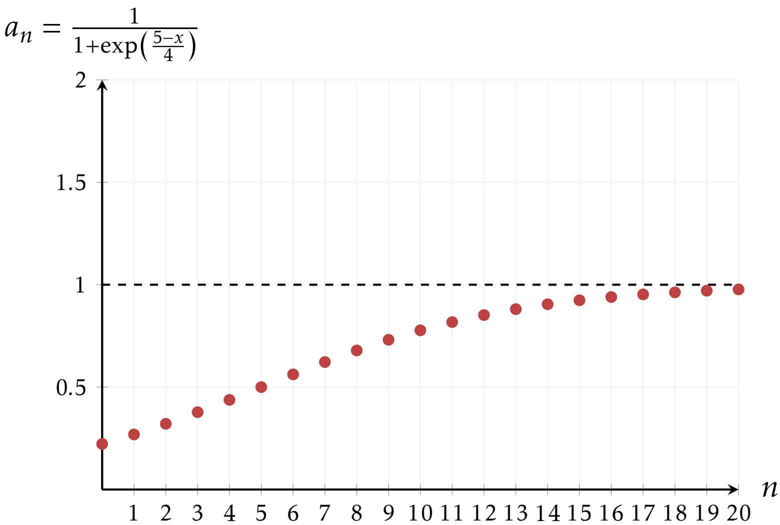


Figure 1.8 The logistic sequence $a_n = \frac{1}{1+\exp\left(\frac{5-x}{4}\right)}$ approaches the value $L = 1$ from below as $n \to \infty$.

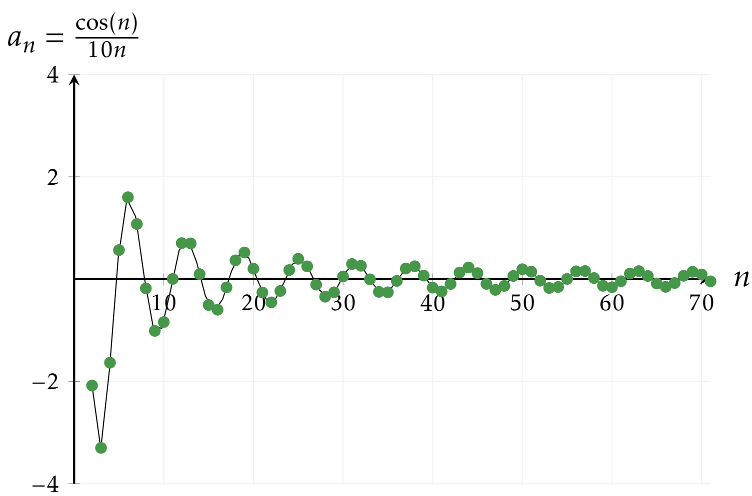


Figure 1.9 The sequence $a_n = \frac{\cos(n)}{10n}$ approaches, with oscillations, the limit $L = 0$. A line connecting the elements is drawn to help see the progression of the sequence.

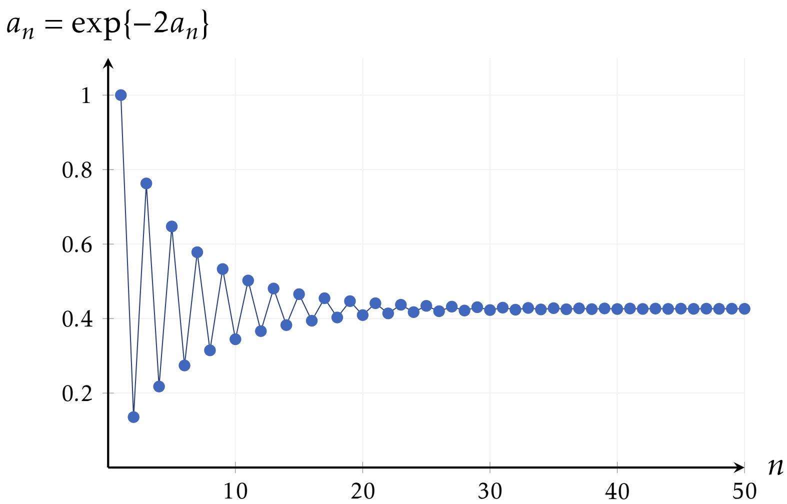


Figure 1.10 The sequence defined by the recursive formula $a_{n+1} = \exp(-2a_n)$ with $a_1 = 1$ converges to the limit $L \approx 0.4263027510068627$, or more precisely $\frac{1}{2}W(2)$, where W is the Lambert W function.

element is ever again equal to or smaller than R. In the case of convergence to some finite number $L \in \mathbb{R}$ we will show that for any distance ε from L - no matter how small! - the sequence eventually stays within ε of L.

Let us use the simplest convergence example to explain this idea: $a_n = \frac{1}{n}$, which converges to $L = 0$. Given a small number ε, say $\varepsilon = \frac{1}{10}$, eventually the sequence stays at most within $\frac{1}{10}$ of $L = 0$. This happens starting from $n_0 = 10$: any element thereafter is smaller than $\frac{1}{10}$, which means that it is withing $\varepsilon = \frac{1}{10}$ of $L = 0$ (see Figure 1.11).

We can repeat this for different value of ε: given $\varepsilon = \frac{1}{100}$, for any $n > 100$ the elements a_n are guaranteed to be within $\pm\frac{1}{100}$ of $L = 0$. Given $\varepsilon = \frac{1}{5000}$, for any $n > 5000$ the elements a_n are within $\pm\frac{1}{5000}$ of $L = 0$, etc. In general, given any real ε, no matter how small, we can set $n_0 = \lceil \frac{1}{\varepsilon} \rceil$, and then for any $n > n_0$ we get that

$$\overset{\text{since } a_n \text{ is strictly decreasing}}{a_n < a_{n_0}} = \frac{1}{n_0} = \frac{1}{\lceil \frac{1}{\varepsilon} \rceil} \underset{\text{since } \lceil \frac{1}{x} \rceil \geq \frac{1}{x}}{\leq} \frac{1}{\frac{1}{\varepsilon}} = \varepsilon, \tag{1.1.5}$$

i.e. altogther

$$a_n < \varepsilon, \tag{1.1.6}$$

and therefore a_n is within $\pm\varepsilon$ of $L = 0$.

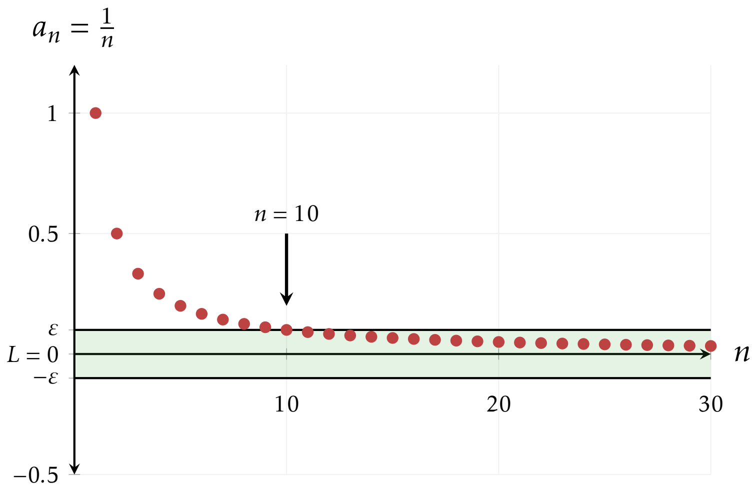


Figure 1.11 The sequence $a_n = \frac{1}{n}$. For any $n > 10$, the element a_n is withing $\varepsilon = \frac{1}{10}$ of the limit $L = 0$. The interval $(-\varepsilon, \varepsilon) = \left(-\frac{1}{10}, \frac{1}{10}\right)$ on the y-axis is highlighted in green.

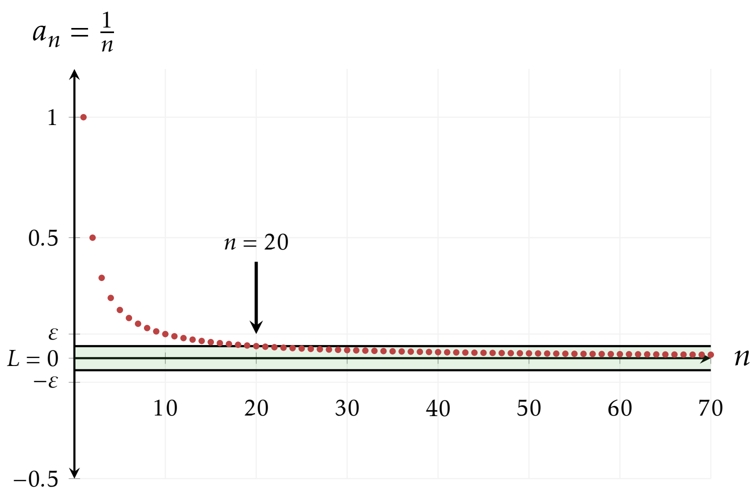


Figure 1.12 Same as Figure 1.11 except here $\varepsilon = 0.05$, and thus $n_0 = \lceil \frac{1}{0.05} \rceil = 20$. Therefore, starting fom $n = 20$ all the elements of the sequence are within the interval $(-0.05, 0.05)$. Note: the values of the sequence are drawn with smaller filled circles to prevent overlap between subsequent points.

Example 1.9 Convergence

Let's prove that the sequence $a_n = \frac{n+1}{n^2}$ converges to $L = 0$ as $n \to \infty$. For any $\varepsilon \in \mathbb{R}$ we chose $n_0 = \lceil \frac{\varepsilon}{2} \rceil$. Since a_n is a strictly decreasing sequence[a], we get that for any $n > n_0$:

$$a_n < a_{n_0} = \frac{n_0 + 1}{n_0^2}.$$

Note the following: for any $x > 1$, $x^3 > x > 1$. Therefore if we replace both n_0 and 1 in the numerator of the expression $\frac{n_0+1}{n_0^2}$ by n_0^3 we are guaranteed to get a number bigger than $\frac{n_0+1}{n_0^2}$. We can therefore continue with the substitution:

$$\frac{n_0 + 1}{n_0^2} < \frac{n_0^3 + n_0^3}{n_0^2} = \frac{2n_0^3}{n_0^2} = 2n_0 = \cancel{2}\frac{\varepsilon}{\cancel{2}} = \varepsilon,$$

and therefore altogther we get that

$$a_n < \varepsilon,$$

and thus a_n converges to $L = 0$ as n increases to ∞.

[a] Show that!

1.2 LIMITS OF REAL FUNCTIONS

1.3 DERIVATIVES

1.4 INTEGRALS

1.5 ANALYZING REAL FUNCTIONS

1.6 TAYLOR SERIES

Say we wish to calculate the value of $\cos(x)$ at some non-trivial value, e.g. $x = \frac{\pi}{7}$, or $x = 1.0423$. Of course today we can simply use a computer or a calculator - but how do these even calculate such values? Moreover, a lot of these values are non-algebraic, meaning that we can't express them as decimal fractions to an infinite precision. What if for some important calculation we need to know the value of $e^{3.153}$ down to 15 digits, but our calculator only shows 10 digits after the period?

We can answer all these questions by using **approximations** instead of precise values. For example, we know values such as $\sin\left(\frac{\pi}{2}\right)$ with infinite precision (it's exactly 1), but in the case of less trivial values such as the ones discussed above, we can instead settle for an approximation, as long as it is *close enough* to the actual value for our needs. By "close enough" we essentially mean *within some given error range*, e.g. in the case of $e^{3.153}$ above, 15 digits of precision means that we want that the value we get is no more than 10^{-15} away from the actual value. Let's write this mathematically: if we denote a as our approximation of $e^{3.153}$, then

$$|a - e^{3.153}| \leq 10^{-15}.$$

More generally, given a value y and and we want to know it to within an **error range** Δ, then for an approximation a to be acceptable, the following must be true:

$$|a - y| \leq \Delta. \tag{1.6.1}$$

A good method to approximate functions would allow us to get as precise as we wish, given that we put enough "work" into finding an approximated value, e.g. we could use it to get an approximation of $e^{3.153}$ up to $\Delta = 10^{-15}$, but we could also use it to get an approximation up to $\Delta = 10^{-20}$ or $\Delta = 10^{-30}$ or any other value - we will simply have to do more calculations to reach such low error ranges. Generaly, the more precise we want our approximation to be, the more calculation we would need to carry out.

Unlike most real functions, polynomials are relatively easy to calculate for any real value x, since we simply need to perform the following operations: addition, subtraction, muliplication and raising by an integer power, all operations that are easy for both humans and computers to perform. Ideally, we would like to use polynomials to approximate all functions, e.g. given the function $f(x) = \cos(x)$ it would be great if we could find some polynomial $P(x)$ of a finite order n for which $P(x) = \cos(x)$. Unfortunately such a polynomial does not exist, nor does such polynomials exist for $\sin(x), \sqrt{x}, \exp(x), a^{x}\ (a > 0)$ or any other of the so-called fundamental functions and their compositions (except, of course, polynomials). The reason for this unfortunate reality is rather complicated, but in short it lies in the fact that these functions are *non-algebric*[1].

However, as mentioned before, for each of these functions, we know at least some values with inifinite precision. For example, we know that $\cos(0) = 1$ and $\cos\left(\frac{1}{2}\pi\right) = 0$. From symmetry we thus know that $\cos(\pi) = -1$ and $\cos\left(\frac{3}{4}\pi\right) = 0$. Since $\cos(x)$ is periodic, i.e. $\cos(x + 2\pi k) = \cos(x)$ for any $k \in \mathbb{Z}$, we actually know infinitely many values of the function. For $\exp(x)$ we know that $\exp(0) = 1$. Table 1.1 lists some known values of common fundamental functions.

We can use this knowledge to constract a polynomial with n terms, which approximates the function's value for any $x \in \mathbb{R}$ to whatever precision we wish, given that a specific

[1] technically $\sqrt{x}$ is algebraic, and there are indeed many methods to calculate its values - but non of these methods are as simple as calculating a polynomial... except the one we discuss in this section.

Table 1.1 Some known values of common fundamental functions. Each such value is known to an infinite precision.

Function	x	$f(x)$
$\sqrt{x}$	4	2
$\sin(x)$	0	0
$\cos(x)$	0	1
$\tan(x)$	0	0
$\exp(x)$	0	1
$\log(x)$	1	0

condition is met. We will describe this condition later, but first let us use an example function to construct such a polynomial: $\exp(x)$. Since we only know $\exp(0)$ with infinite precision, we can use it as a simple approximation of $\exp(x)$, which we denote $T_0(x)$:

$$T_0(x) = 1. \tag{1.6.2}$$

(see Figure 1.13)

We call $T_0(x)$ the **0th-order approximation** of $\exp(x)$. Obviously, this is not a particularly good approximation: it's perhaps ok-ish for values *really* close to $x = 0$, but rapidly diverges from the actual value of $\exp(x)$ as we change x. More precisely, the error $\Delta_0(x)$ is

$$\Delta_0(x) = |\exp(x) - T_0(x)| = |\exp(x) - 1|, \tag{1.6.3}$$

i.e. the error essentialy grows as $\exp(x)$ which is... not great.

Ok, then perhaps we can add another term to the approximation? After all, our notation pretty much suggests that there are more terms we can use[2]. We know that close to a point a a differential function $f(x)$ behaves pretty much like its *derivative* at that point, $f'(a)$. We also know the value of the derivative of $\exp(x)$ at $x = 0$, i.e. $\exp'(0) = 1$, so we can add a line of slope $m = 1$ to our approximation, yielding:

$$\exp(x) = 1 + x \cdot \exp'(0) = 1 + x. \tag{1.6.4}$$

Unsurprisingly, we call $T_1(x)$ the **1st-order approximation** of $\exp(x)$. Looking at Figure 1.14 we see that this approximation is better than the previous one, at least for values near $x = 0$. In other words, the error Δ_1 behaves a bit better than Δ_0:

$$\Delta_1(x) = |\exp(x) - T_1(x)| = |\exp(x) - x - 1|. \tag{1.6.5}$$

Similarly to $\Delta_0(x)$, $\Delta_1(x)$ also generally grows as $\exp(x)$. However, closer to $x = 0$ it behaves like $\exp(x) - x$, so still not great - but definitely an improvement over $\Delta_0(x)$. In practical terms, this means that by using $T_!(x)$ we would generally get better approximations than $T_0(x)$, at least close to $x = 0$. But we can do better!

[2] more like *screams* it.

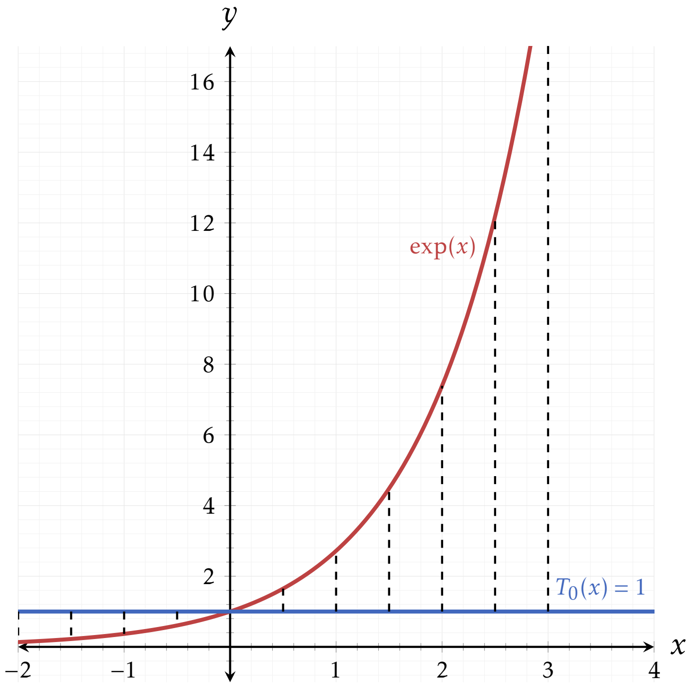


Figure 1.13 The 0th-order approximation $\exp(x) = 1$. The dashed lines show $\Delta_0(x)$ as x diverges from 0.

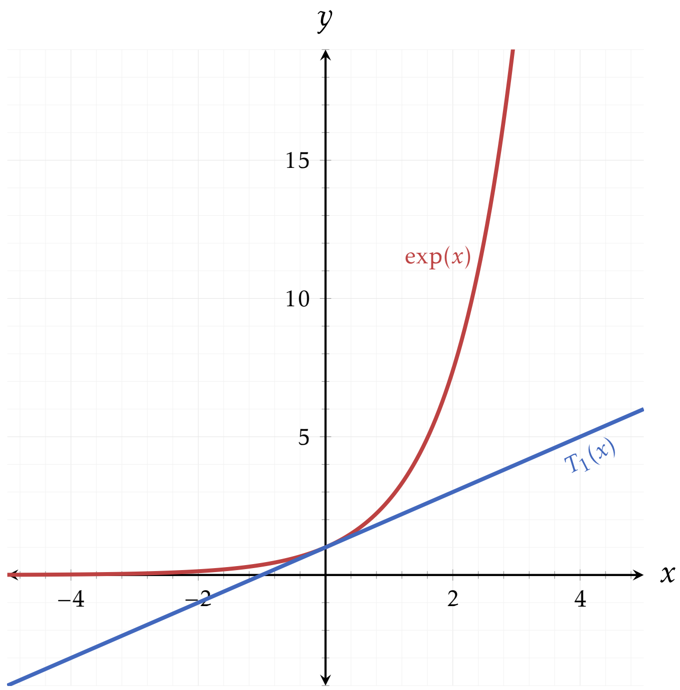


Figure 1.14 The 1st-order approximation $T_1(x) = 1 + x$.

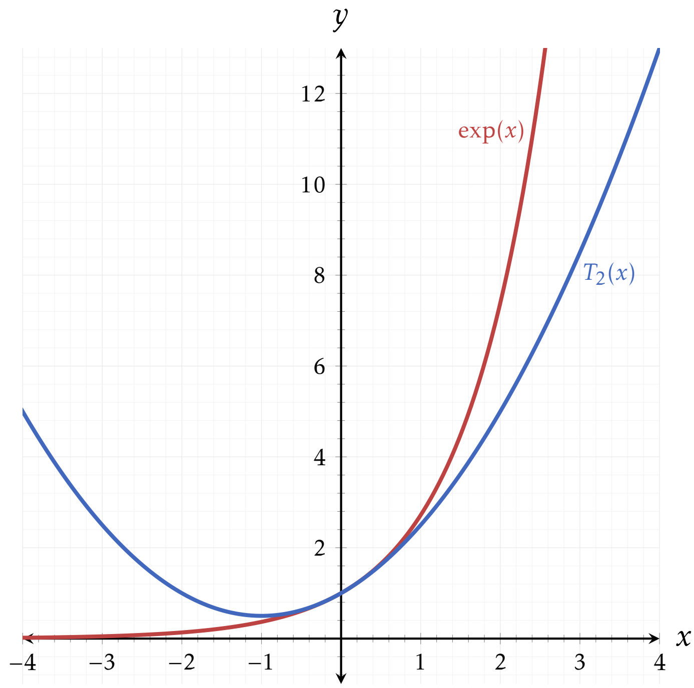


Figure 1.15 The 2nd-order approximation $T_2(x) = 1 + x + \frac{x^2}{2}$.

The next step is of course to add an x^2 term. The second derivative of $\exp(x)$ is also $\exp(x)$, and at $x = 0$ also equal to 1. This time we will divide the term by 2 (for know just accept it as is, it would be explained when we formalize the method). Altogether, we get

$$T_2(x) = 1 + x + \frac{x^2}{2}. \tag{1.6.6}$$

(see Figure 1.15)

Definitely an improvement, but why stop here? We know all the derivatives of $\exp(x)$ at $x = 0$: they are all 1, since $\exp'(x) = \exp(x)$. We can continue adding power terms, yielding the following general approximation:

$$T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}. \tag{1.6.7}$$

(again, for now you should just accept the coefficients $\frac{1}{n!}$, they would be explained soon)

The higher n is, the more term are in the approximation and it gets more precise (see **??**). Of course this means that we need to carry out more calculations, as suggested earlier.

This kind of approximation, where we use a polynomial to approximate a real function, is called a **Taylor series** (sometimes also **Taylor expansion**). To calculate the Taylor series of a real function $f(x)$, we must first select a value of $a \in \mathbb{R}$ for which we know $f(a)$ with infinite precision (e.g. $x = 0$ in the case of $\exp(x)$). The function f must also be infinitely differentiable at a, i.e. the k-th order derivative $f^{(k)}(a)$ must exist for any $k \in \mathbb{N}$, and we must know its value with inifinite precision. If these conditions are met, then the Taylor series of order n of $f(x)$ at $x = a$ is given by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{1.6.8}$$

Example 1.10 Taylor exansion of $\exp(x)$

Let us verify that using Equation 1.6.8 indeed yields Equation 1.6.7: since we're using $a = 0$, we need to know the value $\exp^{(k)}(0)$ for any $k \in \mathbb{N}$. This is pretty simple: the k-th derivative of $\exp(x)$ is $\exp(x)$, and therefore $\exp^{(k)}(0) = 1$ for any $k \in \mathbb{N}$. Substituting $a = 0$ and $f^{(k)}(x) = 1$ to Equation 1.6.8 gives back Equation 1.6.7.

Example 1.11 Taylor series of $\sin(x)$ and $\cos(x)$

Now let's calculate the Taylor series of $\sin(x)$ and $\cos(x)$, denoting them as $T_n^{\mathrm{s}}(x)$ and $T_n^{\mathrm{c}}(x)$, respectively. For both the functions we can use $a = 0$ since we know the values of the functions at these points: $\sin(0) = 0$, $\cos(0) = 1$. The k-th order derivative of $\sin(x)$ depends on the value of k modulo 4, i.e. the derivatives are a repeating sequence of 4 element:

$$\begin{tikzcd}
\sin(x) \arrow[r, "\frac{\mathrm{d}}{\mathrm{d}x}"] & \cos(x) \arrow[r, "\frac{\mathrm{d}}{\mathrm{d}x}"] & -\sin(x) \arrow[r, "\frac{\mathrm{d}}{\mathrm{d}x}"] & -\cos(x) \arrow[lll, bend left, "\frac{\mathrm{d}}{\mathrm{d}x}"]
\end{tikzcd}$$

Since each **even** order derivative of $\sin(x)$ is $\pm\sin(x)$, and $\pm\sin(0) = 0$, we can ignore these terms, since they will vanish from the sum. We therefore need to "run" the sum only for **odd** values, which we can write as $m = 2k+1$. It's important to notice that the values of the derivatives "jump" between positive and negative valiues, i.e. $+\cos(0), -\cos(0), +\cos(0), -\cos(0), \cdots = +1, -1, +1, -1, \ldots$. We therefore set the derivative term to be $(-1)^k$. Altogether we get

$$T_n^{\mathrm{s}}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

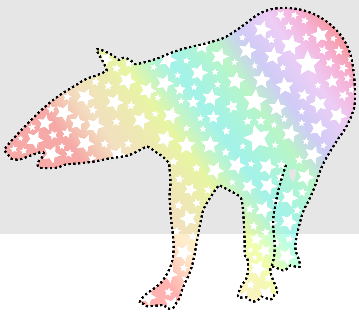
Similarly, the Taylor series for $\cos(x)$ would have only half of its terms non-zero, but the parity is opposite in comparison to $\sin(x)$: this time, the odd terms disappear, and we're left with only the even terms. Therefore we use $m = 2k$, and altogether get

$$T_n^{\mathrm{c}}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}.$$

!
To be written: When reaching the proof that in the limit $n \to \infty$ the Taylor series equals the functio
!

1.7 EXERCISES

CHAPTER 2



LINEAR ALGEBRA
(INTUITIVE APPROACH)

Linear algebra is one of the most important and often used fields, both in theoretical and applied mathematics. It brings together the analysis of systems of linear equations and the analysis of linear functions (in this context usually called linear transformations), and is employed extensively in almost any modern mathematical field, e.g. approximation theory, vector analysis, signal analysis, error correction, 3-dimensional computer graphics and many, many more.

In this book, we divide our discussion of linear algebra into to chapters: the first (this chapter) deals with a wider, birds-eye view of the topic: it aims to give an intuitive understanding of the major ideas of the topic. For this reason, in this chapter we limit ourselves almost exclusively to discussing linear algebra using 2- and 3-dimensional analysis (and higher dimensions when relevant) using real numbers only. This allows us to first create an intuitive picture of what is linear algebra all about, and how to use correctly the tools it provides us with.

The next chapter takes the opposite approach: it builds all concepts from the ground-up, defining precisely (almost) all basic concepts and proving them rigorously, and only then using them to build the next steps. This approach has two major advantages: it guarantees that what we build has firm foundations and does not fall apart at any future point, and it also allows us to generalize the ideas constructed during the process to such extent that they can be used as foundation to build ever newer tools we can apply in a wide range of cases.

2.1 VECTORS

2.1.1 Basics

Vectors are the fundamental objects of linear algebra: the entire field revolves around manipulation of vectors. In this chapter we deal with the so-called **real vectors**, which can be be defined in a geometric way:

Definition 2.1 Real vectors

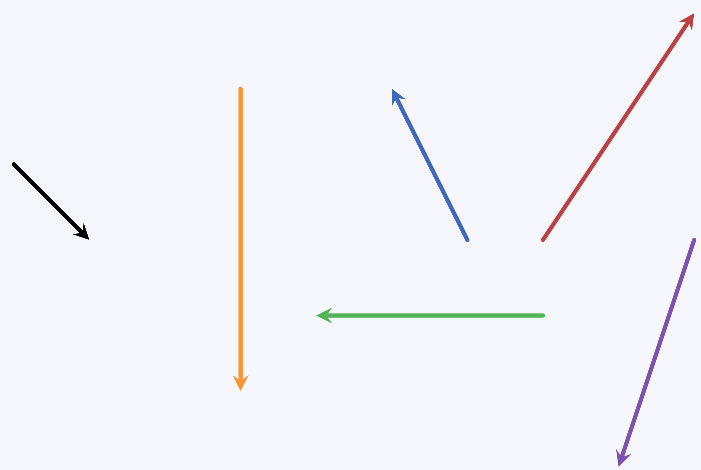
A *real vector* is an object with a **magnitude** (also called **norm**) and a **direction**.

π

In this chapter we refer to real vectors simply as *vectors*.

Example 2.1 Real vectors

The following are all vectors in 2-dimensional space depicted as arrows:



Vectors are usually denoted in one of the following ways:

- **Arrow above letter**: $\vec{u}, \vec{v}, \vec{x}, \vec{a}, \ldots$

- **Bold letter**: $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}, \boldsymbol{a}, \ldots$

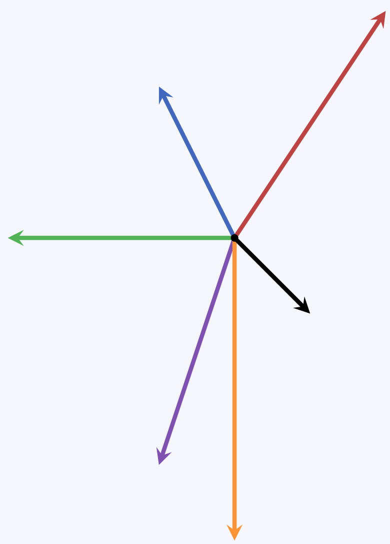
- **Bar below letter**: $\underline{u}, \underline{v}, \underline{x}, \underline{a}, \ldots$

In this book we use the first notation style, i.e. an arrow above the letter. In addition vectors will almost always be denoted using lowercase Lating script.

When discussing vectors in a single context, we always consider them starting at the same point, called the **origin**, and **translating** (moving) vectors around in space does not change their properties: only their norms and directions matter.

Example 2.2 Real vectors

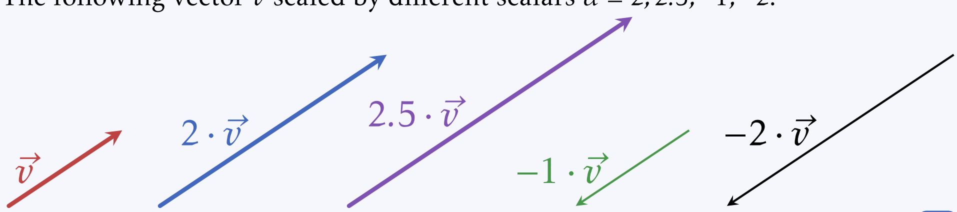
The vectors from the previous translated (moved) such that their origins all lie on the same point:



A vector can be scaled by a real number α: when this happens, its norm is multiplied by α while its direction stays the same. We call α a **scalar**.

Example 2.3 Scaling vectors

The following vector $\vec{v}$ scaled by different scalars $\alpha = 2, 2.5, -1, -2$:



Note 2.1 Negative scale

As can be seen in the example above, when scaling a vector by a negative amount its direction reverses. However, we consider two opposing direction (i.e. directions that are 180° apart) as being the same direction.

In this chapter we use the following notation for the norm of a vector $\vec{v}$: $\left\|\vec{v}\right\|$.

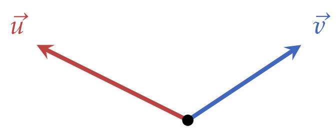
A vector $\vec{v}$ with norm $\left\|\vec{v}\right\| = 1$ is called a **unit vector**, and is usually denoted by replacing the arrow symbol by a hat symbol: $\hat{v}$. Any vector (except $\vec{0}$) can be scaled into a unit vector by scaling the vector by 1 over its own norm, i.e.

$$\hat{v} = \frac{1}{\left\|\vec{v}\right\|}\vec{v}. \tag{2.1.1}$$

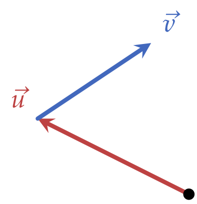
The result of normalization is a vector of unit norm which points in the same direction of the original vector.

Two vectors can be added together to yield a third vector: $\vec{u} + \vec{v} = \vec{w}$. To find $\vec{w}$ we use the following procedure (depicted in Figure 2.1):

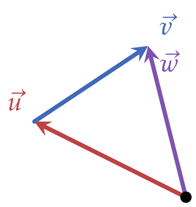
1. Move (translate) $\vec{v}$ such that its origin lies on the head of $\vec{u}$.
2. The vector $\vec{w}$ is the vector drawn from the origin of $\vec{u}$ to the head of $\vec{v}$.



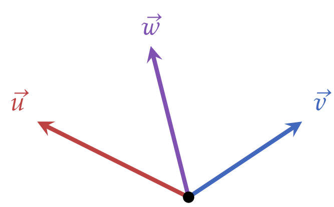
1 The vectors $\vec{u}$ and $\vec{v}$.



2 Translating $\vec{v}$ such that its origin lies at the head of $\vec{u}$.



3 Drawing the vector $\vec{w}$ from the origin to the head of $\vec{v}$.



4 Showing all three vectors.

Figure 2.1 Vector addition.

The addition of vectors as depicted here is commutative, i.e. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. This can be seen by using the **parallogram law of vector addition** as depicted in Figure 2.2:

drawing the two vectors $\vec{u},\vec{v}$ and their translated copies (each such that its origin lies on the other vector's head) results in a parallelogram.

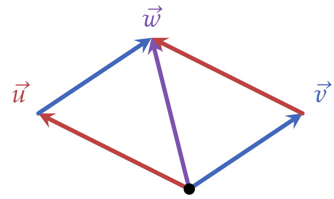


Figure 2.2 The parallogram law of vector addition.

An important vector is the **zero-vector**, denoted as $\vec{0}$. The zero-vector has a unique property: it is neutral in respect to vector addition, i.e. for any vector $\vec{v}$,

$$\vec{v}+\vec{0}=\vec{v}. \tag{2.1.2}$$

(we also say that $\vec{0}$ is the **additive identity** in respect to vectors.)

Any vector $\vec{v}$ always has an **opposite** vector, denoted $-\vec{v}$. The addition of a vector and its opposite always result in the zero-vector, i.e.

$$\vec{v}+\left(-\vec{v}\right)=\vec{0}. \tag{2.1.3}$$

2.1.2 Components

Vectors can be decomposed to their components, the number of which depends on the dimension of space we're using: 2-dimensional vectors can be decomposed into 2 components, 3-dimensional vectors can be decomposed into 3 components, etc. To decompose a vector, say $\vec{v}$, we first choose a coordinate system: the most commonly used system, and the one we will use for most of this chapter, is the Cartesian coordinate system. We place the vector in the coordinate system such that its origin lies at the origin of the system. We then draw a perpendicular line from its head to each of the axes in the system (see Figure 2.3), the point of interception on each axis is the component of the vector in that axis (we label these points v_x, v_y, v_z in the case of 2- or 3-dimensional spaces, and generally $v_1, v_2, v_3, \dots$). The vector can then be written as a column using these components:

$$\vec{v}=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \tag{2.1.4}$$

Note 2.2 Order of components

The order of the components of a vector is important, and should always be consistent. In the case of 2- and 3-dimensional the order is always v_x, v_y, v_z.

!

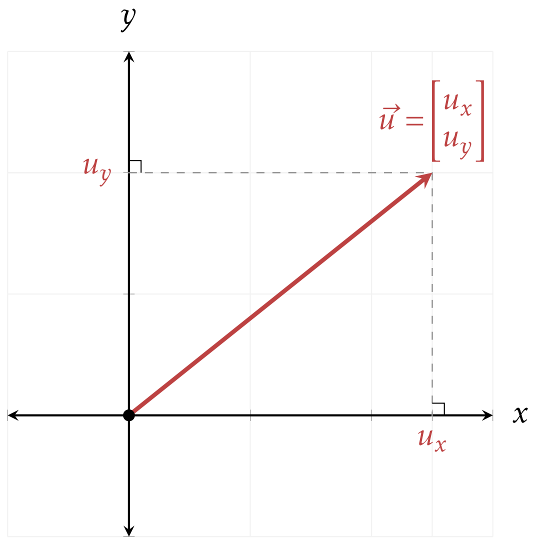
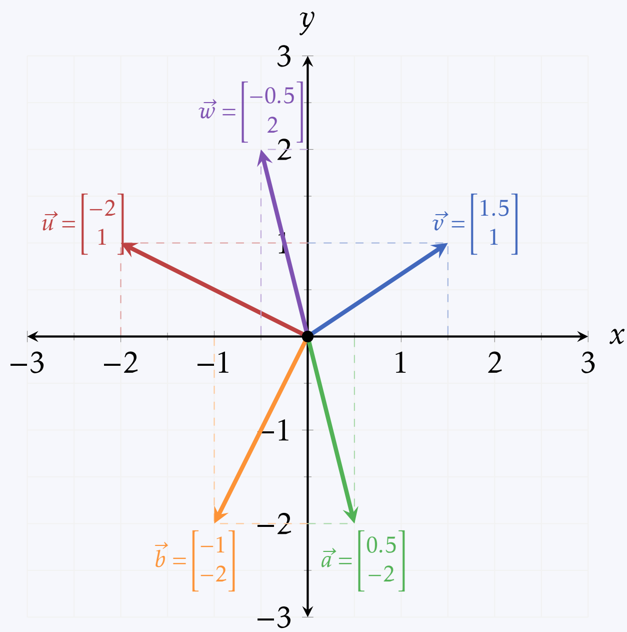


Figure 2.3 Placing a 2-dimensional vector $\vec{u}$ on the 2-dimensional Cartesian coordinate system, showing its x- and y-components.

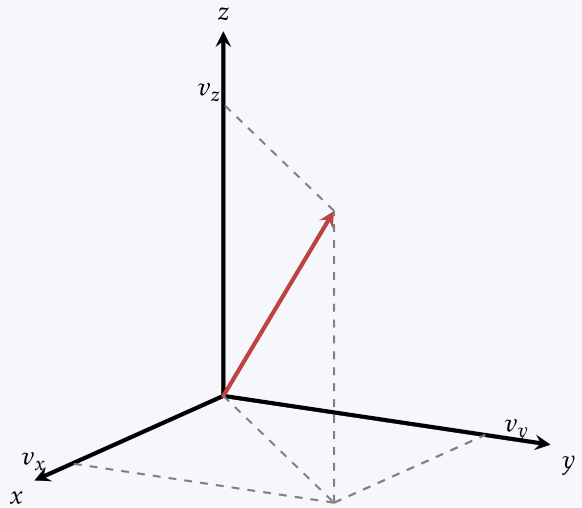
Example 2.4 Vector components in two dimensions

The following five 2-dimensional vectors are decomposed each into its x- and y-components:



Example 2.5 Vector components in three dimensions

The following 3-dimensional vector is decomposed into its x-, y- and z-components: (THIS NEEDS TO BE IMPROVED AND FINISHED)



The column form of a vector is essentially equivalent to an order list of n real numbers, i.e. $(v_1, v_2, \ldots, v_n)$. Why then are we using the column form and not the list form (mostly known as **row vectors**)? In fact, we could use either form - and even using both interchangeably - and with only minor adjusments the entire chapter would stay the same as it is now. However, there are some advantages of using only a single form, and consider the other form as a different object altogether. This idea will become clearer in Section 2.8 and will be used to its fullest extent in future chapters when discussing **covariant vectors**, **contravarient vectors**, and in general **tensors**. For now, we stick with the column form of vectors to stay consistent with common notation.

However, the row form of vectors hints at the space in which they exist: n-dimensional vectors live in a space we call $\mathbb{R}^n$. Recall from Chapter 0 that the set $\mathbb{R}^n$ is a Cartesian product made up of n times the set of real numbers, i.e.

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n}. \tag{2.1.5}$$

Each member of this set is a list of n real numbers, and their order inside the list matters - very similar to vectors, be they in row or column form. For this reason, we refer to $\mathbb{R}^n$ as the space of n-dimensional real vectors. As mentioned, in this chapter we use $\mathbb{R}^2$ (the 2-dimensional real space) and $\mathbb{R}^3$ (the 3-dimensional real space) for most ideas and examples.

2.1.3 Norm, polar coordinates and spherical coordinates

Looking at vectors in $\mathbb{R}^2$, it is rather straight-forward to calculate their norm: since the origin, the head of the vector and the point v_x form a right triangle (see Figure 2.4), we can use the Pythagorean theorem to calculate the norm of the vector, which is equal to the hypotenous of said triangle:

$$\left\|\vec{v}\right\| = \sqrt{v_x^2 + v_y^2}. \tag{2.1.6}$$

Much like complex numbers, vectors in $\mathbb{R}^2$ can be expressed using **polar coordinates**, i.e. using the norm of the vector and its angle θ relative to the x-axis (cf. Equation 0.6.1 and Equation 0.7.11). The relation between the cartesian and polar coordinates is

$$\begin{aligned} v_x &= \left\|\vec{v}\right\| \cos(\theta), \\ v_y &= \left\|\vec{v}\right\| \sin(\theta). \end{aligned} \tag{2.1.7}$$

To calculate θ from v_x and v_y we use the definition of $\tan(\theta)$ (see Section 0.6), and get that

$$\tan(\theta) = \frac{v_y}{v_x}, \tag{2.1.8}$$

i.e.

$$\theta = \arctan\left(\frac{v_y}{v_x}\right). \tag{2.1.9}$$

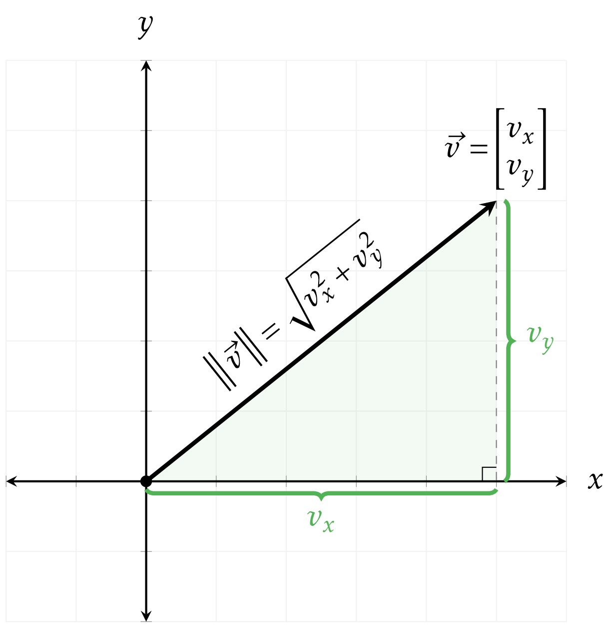


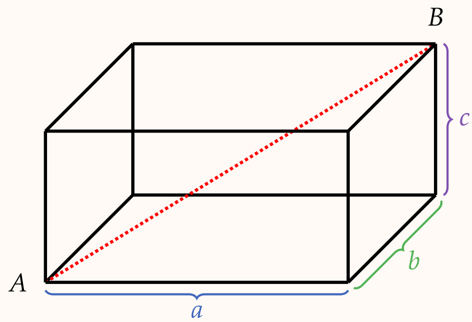
Figure 2.4 Calculating the norm of a 2-dimensional column vector.

In $\mathbb{R}^3$ the norm of a vector $\vec{v}$ is similarily

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}. \tag{2.1.10}$$

Challenge 2.1 Norm of a 3D vector

Show why Equation 2.1.10 is valid, by calculating the length AB in the following figure, depicting a box of sides $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$:



?

Generalizing the vector norms in $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$ yields the following form:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}. \tag{2.1.11}$$

Note 2.3 Other norms

The norm shown here is called the 2-norm. There are other possible norm that can be defined, and are used in different situations, such as the 1-norm (also the called **taxicab norm**), general p-norm where $p \geq 1$ is a real number, the zero-norm, the max-norm, and many others. However, for the purpose of this chapter we use only the standard 2-norm, since it is the most useful for describing basic concepts of linear algebra and its uses.

!

$\mathbb{R}^3$ has its own version of polar coordinates, sometimes refered to as **cylindrical coordinates**. These coordinates are similar to the polar coordinates in $\mathbb{R}^2$, with an additional "height" component: the three coordinates are ρ, φ and z, where

- ρ is the norm of the projection of $\vec{v}$ onto the xy-plane[1],
- φ is the angle between the projection of $\vec{v}$ and the x-axis.
- z is the distance between the head of $\vec{v}$ to the xy-plane.

[1] ρ is used instead of r to prevent confusion with the polar coordinates in $\mathbb{R}^2$

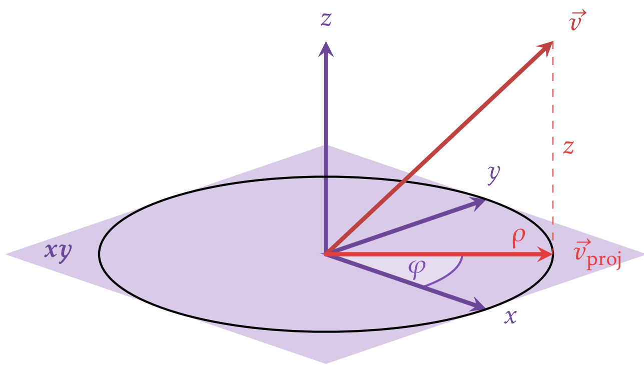


Figure 2.5 The cylindrical coordinates ρ, φ, z.

The conversion between cylindrical and cartesian coordinates is given by

$$\begin{aligned} x &= \rho\cos\left(\varphi\right), \\ y &= \rho\sin\left(\varphi\right), \\ z &= z. \end{aligned} \tag{2.1.12}$$

Yet another useful set of coordinates in $\mathbb{R}^3$ are the **spherical coordinates**. Given a vector $\vec{v}$, instead of using two length coordinates, the spherical coordinate system uses two angles φ and θ: φ is the angle between the projection of $\vec{v}$ onto the xy-plane, and θ the angle between $\vec{v}$ and the z-axis. The third coordinate is then the norm of $\vec{v}$, denoted r. See **??** for a graphical representation.

! **To be written**: a nice figure for spherical coordinates !

2.1.4 Operations

Scaling a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ by a real number α is done by multiplying each of its components by α, i.e.

$$\alpha\vec{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}. \tag{2.1.13}$$

We can prove Equation 2.1.13 by directly calculating the norm of a scaled vector $\vec{w} = \alpha\vec{v}$:

Proof 2.1 Scaling a column vector

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}$, where $\alpha \in \mathbb{R}$. Then $\vec{w}$ has the following norm:

$$\begin{aligned}
\left\|\vec{w}\right\| &= \sqrt{\sum_{i=1}^{n} (\alpha v_i)^2} \\
&= \sqrt{(\alpha v_1)^2 + (\alpha v_2)^2 + \cdots + (\alpha v_1)^2} \\
&= \sqrt{\alpha^2 v_1^2 + \alpha^2 v_2^2 + \cdots + \alpha^2 v_n^2} \\
&= \sqrt{\alpha^2 \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)} \\
&= \alpha \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\
&= \alpha \left\|\vec{v}\right\|.
\end{aligned}$$

This shows that indeed $\vec{w} = \alpha\vec{v}$.

QED

Another idea we can prove in column form is vector normalization (Equation 2.1.1), by showing that dividing each component of a vector by its norm gives a vector of unit norm:

Proof 2.2 Norm of a vector

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Its norm is then $\left\|\vec{v}\right\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. Scaling $\vec{v}$ by $\frac{1}{\|\vec{v}\|}$ yields

$$\hat{v} = \frac{1}{\left\|\vec{v}\right\|} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \frac{1}{\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The norm of $\hat{v}$ is therefore

$$\begin{aligned}
\|\hat{v}\| &= \sqrt{\frac{v_1^2}{v_1^2 + v_2^2 + \cdots + v_n^2} + \frac{v_2^2}{v_1^2 + v_2^2 + \cdots + v_n^2} + \cdots + \frac{v_n^2}{v_1^2 + v_2^2 + \cdots + v_n^2}} \\
&= \sqrt{\frac{1}{v_1^2 + v_2^2 + \cdots + v_n^2} \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)} \\
&= \sqrt{1} = 1,
\end{aligned}$$

i.e. $\hat{v}$ is indeed a unit vector.

QED

Example 2.6 Normalizing a vector

Let's normalize the vector $\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}$. Its norm is

$$\left\|\vec{v}\right\| = \sqrt{0^2 + 4^2 + (-3)^2} = \sqrt{0 + 16 + 9} = \sqrt{25} = 5.$$

Therefore $\hat{v}$ (the normalized $\vec{v}$) is

$$\hat{v} = \begin{bmatrix} 0 \\ \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}.$$

By calculating the norm of $\hat{v}$ directly, we can see that it is indeed a unit vector:

$$\|\hat{v}\| = \sqrt{0^2 + \frac{4^2}{5^2} + \frac{3^2}{5^2}} = \sqrt{\frac{0^2 + 4^2 + 3^2}{5^2}} = \sqrt{\frac{16+9}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1.$$

The addition of two column vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is done by adding their respective components together, i.e.

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}. \tag{2.1.14}$$

! **To be written**: how this addition is the same as the one shown in Figure 2.1. !

Note 2.4 No addition of vectors of different number of components!

Two vectors can only be added together if they have the same number of components. The addition of vectors with different number of components is undefined.

2.1.5 Linear combinations, spans and linear dependency

As seen above, scaling a vector by a scalar results in a vector that has the same number of dimensions as the original vector. The same is true for adding two vectors: both of them must be of the same dimension, and the result is also a vector of the same dimension. Therefore, any combination of scaling and addition of vectors results in a vector of the same dimension as the original vector(s). This kind of combination is called a **linear combination**.

Let's define linear combinations a little more formaly:

Definition 2.2 Linear combinations

A linear combination of n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of the same dimension, using n scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, is an expression of the form

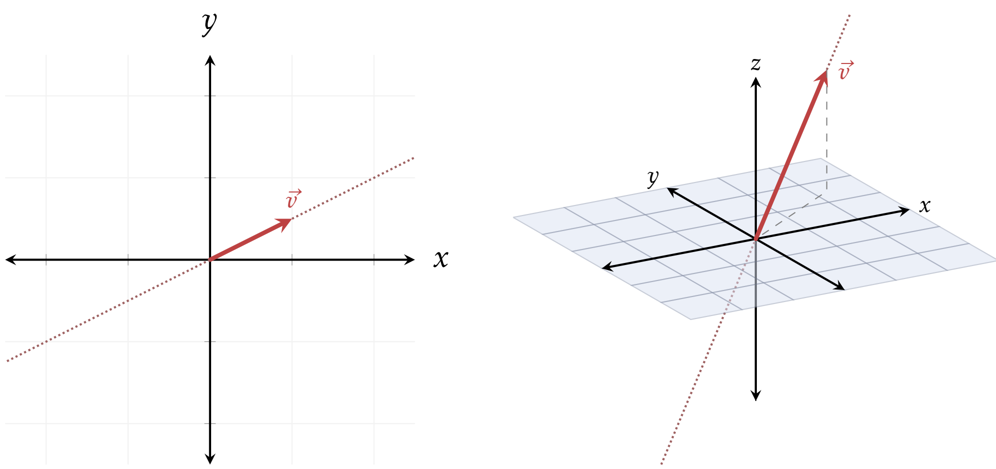
$$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = \sum_{i=1}^{n} \alpha_i \vec{v}_i. \tag{2.1.15}$$

π

Linear combinations of real vectors have geometric meaningsc: we start with the set of all linear combinations of a single vector $\vec{v} \in \mathbb{R}^n$, i.e.

$$V = \left\{ \alpha \vec{v} \mid \alpha \in \mathbb{R} \right\}. \tag{2.1.16}$$

The set V represents a line in the direction of $\vec{v}$ going through the origin (see Figure 2.6). The set V is itself a vector space of dimension 1, and as such a **subspace** of $\mathbb{R}^n$. We say that it is the **span** of the vector $\vec{v}$ (i.e. the vector $\vec{v}$ **spans** the subspace V).



1 $\mathbb{R}^2$ 2 $\mathbb{R}^3$

Figure 2.6 The span of a single vector $\vec{v}$, shown as a dashed line: in $\mathbb{R}^2$ (left) and $\mathbb{R}^3$ (right).

Similarly, the set of all linear combinations of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ that are not scales of each other (i.e. there is no such $\alpha \in \mathbb{R}$ for which $\vec{v} = \alpha \vec{u}$),

$$V = \left\{ \alpha \vec{u} + \beta \vec{v} \mid \alpha, \beta \in \mathbb{R} \right\}, \tag{2.1.17}$$

is a plane that goes through the origin (see Figure 2.7). Such vectors are also said to be **non-collinear**.

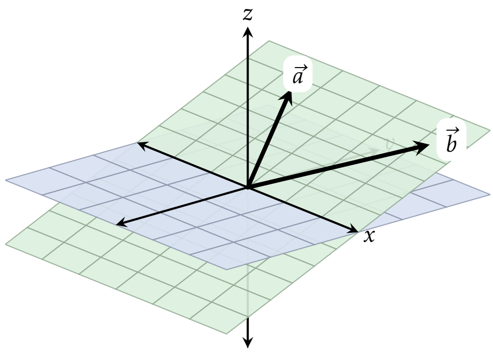


Figure 2.7 Two vectors $\vec{a}$ and $\vec{b}$ span a plane (colored green) in $\mathbb{R}^3$. The xy-plane (i.e. $z = 0$) is shown in blue for emphasis.

Example 2.7 Spanning $\mathbb{R}^2$ using two non-collinear vectors

Since any two non-collinear vectors span a 2-dimensional subspace of $\mathbb{R}^n$, in $\mathbb{R}^2$ this means that any vector $\vec{w}$ can be written as a linear combination of any two vectors $\vec{u}, \vec{v}$ that are not a scale of each other. For example, we can take the vector

$$\vec{w} = \begin{bmatrix} 7 \\ -1 \end{bmatrix},$$

and write it as a linear combination of any two non-collinear vectors, say

$$\vec{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

The equation which forces the relation is

$$\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 5 \end{bmatrix},$$

and we should solve it for α and β. This is possible since the equation above is actually a system of two equations in two variables (namely α and β):

$$\begin{cases} 7 = 2\alpha, \\ -1 = -3\alpha + 5\beta. \end{cases}$$

The solution for the system is $\alpha = 3.5$ and $\beta = 1.9$, and therefore

$$\begin{bmatrix} 7 \\ -1 \end{bmatrix} = 3.5 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + 1.9 \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Generalizing the example above, any vector $\vec{w} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$ can be written as a linear combination of two vectors $\vec{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$, as long as $\vec{u}$ and $\vec{v}$ are non-collinear. Let's prove this:

Proof 2.3 $\mathbb{R}^2$ is spanned by any two non-collinear vectors in $\mathbb{R}^2$

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ be two non-collinear vectors. Their non-collinearity means that the equation

$$\vec{u} = \alpha \vec{v} \tag{2.1.18}$$

has no solution, i.e. the system

$$\begin{cases} u_x = \alpha v_x \\ u_y = \alpha v_y \end{cases} \tag{2.1.19}$$

has no solution. The system has solution only when $u_x v_y = u_y v_x$, and so the restriction is translated to the simple equation

$$u_x v_y \neq u_y v_x. \tag{2.1.20}$$

The system which defines $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$ is

$$\begin{cases} w_x = \alpha u_x + \beta v_x \\ w_y = \alpha u_y + \beta v_y \end{cases} \tag{2.1.21}$$

Isolating α using the first equation yields

$$\alpha = \frac{w_x - \beta v_x}{u_x}, \tag{2.1.22}$$

and subtituting it into the second equation yields

$$\beta = \frac{w_y - \alpha u_y}{v_y} = \frac{w_y - \frac{w_x - \beta v_x}{u_x}}{v_y}, \tag{2.1.23}$$

which rearranges into

$$\beta = \frac{u_x w_y - u_y w_x}{u_x v_y - u_y v_x}, \tag{2.1.24}$$

and thus

$$\alpha = \frac{-v_x w_y + v_y w_x}{u_x v_y - u_y v_x}. \tag{2.1.25}$$

We can see that α and β exist iff $u_x v_y \neq u_y v_x$, which is guaranteed by Equation 2.1.20. Therefore, α and β always exist when $\vec{u}$ and $\vec{v}$ are non-collinear, and thus any vector in $\mathbb{R}^2$ can be written as a linear combination of any two non-collinear vectors in $\mathbb{R}^2$, i.e. any two non-collinear vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$.

QED

Going a step further, any three vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ that are not coplanar span a 3-dimensional subspace of $\mathbb{R}^n$ going through the origin. To generalize the notion of collinear and coplanar vectors to higher dimensions we introduct the concept of **linear dependency** of a set of vectors:

Definition 2.3 Linear dependent set of vectors

A set of n vectors

$$S = \left\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\right\} \tag{2.1.26}$$

is said to be linearly dependent if there exist a linear combination

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0}, \tag{2.1.27}$$

and **at least** one the coefficients $\alpha_i \neq 0$.

π

The following examples shows that the definition above reduces to colinarity and coplanary in the case of 2 and 3 vectors:

Example 2.8 Linear dependency of 2 vectors

Let $\vec{u}$ and $\vec{v}$ be two linearly dependent vectors in $\mathbb{R}^n$. Then there exist a linear combination

$$\alpha\vec{u} + \beta\vec{v} = \vec{0},$$

with either $\alpha \neq 0$ or $\beta \neq 0$ (or both). We can look at the different possible cases:

- $\alpha \neq 0,\ \beta = 0$: in this case $\alpha\vec{u} = \vec{0}$, i.e. $\vec{u} = 0$.
- $\alpha = 0,\ \beta \neq 0$: in this case $\beta\vec{v} = \vec{0}$, i.e. $\vec{v} = 0$.
- $\alpha \neq 0,\ \beta \neq 0$: in this case we can rearrange the equation and get

  $$\vec{u} = -\frac{\beta}{\alpha}\vec{v},$$

  i.e. $\vec{u}$ and $\vec{v}$ are scales of each other and thus are collinear.

What we learn from this is that two vectors form a linearly dependent set if at least one of the is the zero vector, or if they are collinear.

Example 2.9 Linear dependency of 3 vectors

Now, let $\vec{u}, \vec{v}$ and $\vec{w}$ be three linearly dependent vectors in $\mathbb{R}^n$. Then there exists a linear combination

$$\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w} = \vec{0},$$

with either $\alpha \neq 0$ or $\beta \neq 0$ or $\gamma \neq 0$ or any combination where two of the coefficients are non-zero, or all of the coefficients are non-zero. Again, we look at all the possible cases:

- $\alpha \neq 0,\ \beta = \gamma = 0$: we get $\alpha\vec{u} = \vec{0}$, thus $\vec{u} = \vec{0}$.
- $\alpha = 0,\ \beta \neq 0,\ \gamma = 0$: we get $\beta\vec{v} = \vec{0}$, thus $\vec{v} = \vec{0}$.

- $\alpha = \beta = 0,\ \gamma \neq 0$: we get $\gamma \vec{w} = \vec{0}$, thus $\vec{w} = \vec{0}$.
- $\alpha \neq 0,\ \beta \neq 0, \gamma = 0$: we get that $\vec{u}$ and $\vec{v}$ are collinear, since this is exactly as the case for two linearly dependent vectors.
- $\alpha \neq 0,\ \beta = 0, \gamma \neq 0$: similar to the previous case, this time $\vec{u}$ and $\vec{w}$ are collinear.
- $\alpha = 0,\ \beta \neq 0, \gamma \neq 0$: similar to the previous case, this time $\vec{v}$ and $\vec{w}$ are collinear.
- $\alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0$: by rearranging we get

$$\vec{w} = -\frac{1}{\gamma}\left(\alpha \vec{u} + \beta \vec{v}\right),$$

i.e. $\vec{w}$ lies on the the plane spanned by $\vec{u}$ and $\vec{v}$. If we isolate $\vec{u}$ or $\vec{v}$ instead, we get the same result: the isolated vector is a lienar combination of the other two vectors, and thus lies on the plan spanned by these vectors.

From this example we learn that three vectors form a linearly dependent set if one or more of the vectors is the zero vector, or if any two vectors in the set are collinear, or if all three vectors are coplanar.

Just like the case of 2 and 3 vectors seen above, any set of $m \leq n$ vectors in $\mathbb{R}^n$ that are **not** linearly dependent span an m-dimensional subspace of $\mathbb{R}^n$ (which goes throught the origin) - i.e. any vector $\vec{v} \in \mathbb{R}^n$ can be written as a linear combination of these vectors. We call such a set a **basis set** of $\mathbb{R}^n$.

Example 2.10 Basis sets in n dimensions

The following three vectors are non coplanar (i.e. they are linearly independent), and thus form a basis set of $\mathbb{R}^3$:

$$B = \left\{ \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} \right\}.$$

This means that any vector in $\mathbb{R}^3$ can be written as a linear combination of these vectors. We can show this by writing a generic vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ as a linear combination of the vectors:

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix},$$

which can be expanded to the system of equations

$$\begin{cases} x = \cancel{0\alpha} + 4\beta + 1\gamma, \\ y = 4\alpha + 2\beta + \cancel{0\gamma}, \\ z = 5\alpha - 2\beta - 5\gamma. \end{cases}$$

The solution of the above system gives the coefficients of the linear combination to yield any vector in $\mathbb{R}^3$:

$$\begin{aligned} \alpha &= -\frac{5x}{31} + \frac{9y}{31} - \frac{z}{31}, \\ \beta &= \frac{10x}{31} - \frac{5y}{62} + \frac{2z}{31}, \\ \gamma &= -\frac{9x}{31} + \frac{10y}{31} - \frac{8z}{31}. \end{aligned}$$

For example, to yield the vector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ we sustitute $x = 1,\ y = -1,\ z = 0$ into the above solutions, and get that the following coefficients are needed:

$$\alpha = -\frac{28}{62},\ \beta = \frac{25}{62},\ \gamma = -\frac{38}{62},$$

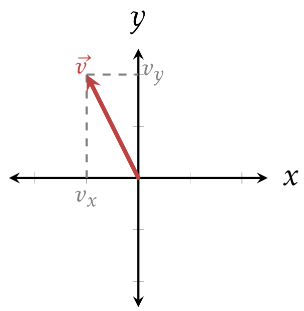
i.e.

$$-\frac{28}{62}\begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} + \frac{25}{62}\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} - \frac{38}{62}\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

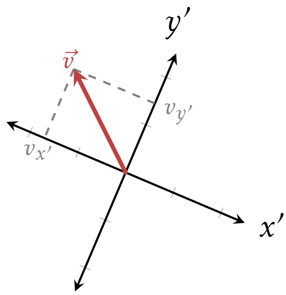
(you, the reader, should verify this!)

The way we described what a basis set is, while being accurate and general, does not give us any particular intuition about what basis sets actually *do*, and why do we even bother with them. To understand this, consider some vector $\vec{v} \in \mathbb{R}^2$. Without defining some frame of reference, as far as we're considered $\vec{v}$ is merely some arrow floating in space:

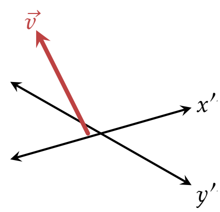
Note that $\vec{v}$ still has all the properties any other general vector has: it a norm and a direction. However, we can't say anything meaningful about this direction, except maybe that it is roughly pointing up and to left. In order to make any sense of $\vec{v}$ we have to choose some frame of reference, i.e. two axes. We can of course use the usual horizontal and vertical directions (which we usually call x and y):



Having this frame of reference, i.e. the x- and y-axes, we can calculate the components of $\vec{v}$ **in relation to these axes** by droping two perpendicular lines, one for each axis. But there's nothing really special about these axes, they are just convinent to draw on a flat paper. We could use any other two non-colinear directions, for example the following x' and y':



Notice how $\vec{v}$ stays the same, the only difference is how we will describe its components using the x', y' axes system. We are of course not restricted to having two perpendicular axes, e.g. the following x'', y'' axes:



! **To be written**: improve the above figure !

The axis system we use as a reference is the basis set we use to describe vectors, except one detail: a basis set also tells us what is the unit of measurement in the direction of each basis vector. This is of course the norm of that basis vector.

! **To be written**: show a vector drawn from integer amounts of 2 basis vectors !

Having described basis sets in somewhat general terms, we can now define them a bit more precisely:

Definition 2.4 Basis sets

Let B be a **linearly independent set** of vectors in $\mathbb{R}^n$. If any vector $\vec{v} \in \mathbb{R}^{\propto}$ can be written as a linear combination of the vectors in B, then B is called a basis set of $\mathbb{R}^n$. The **dimension** of B is the number of vectors in B.

π

The dimension of a basis set B of $\mathbb{R}^n$ is always n. In fact, in a later chapter we will see that the dimension of a vector space is defined by the dimension of its basis sets, i.e. given a vector space V and a basis set $B \subseteq V$, the dimension of V is equal to $|B|$, or mathematically

$$\dim(V) = |B|. \tag{2.1.28}$$

It can be easily shown that any set of vectors in $\mathbb{R}^n$ which has more than n vectors must be a linearly dependent set:

Proof 2.4 Sets with more than n vectors in $\mathbb{R}^n$

Let S be a set of $m \in \mathbb{N}$ vectors in $\mathbb{R}^n$, where $m > n$. Given a vector $\vec{v} \in S$ and the set of all vectors in S except $\vec{v}$ (call this set $\tilde{S}$), there are two possibilities:

- $\tilde{S}$ is a linearly dependent set in $\mathbb{R}^n$. In this case, the addition of $\vec{v}$ doesn't change this fact, i.e. the set S as a whole is linearly dependent.
- The set $\tilde{S}$ is linearly independent, and since it has n vectors it forms a basis set of $\mathbb{R}^n$. Therefore, $\vec{v}$ can be written as a linear combination of the vectors in $\tilde{S}$, and thus the inclusion of $\vec{v}$ in S makes S a linearly dependent set.

QED

Let us now take a vector, for example $\vec{v} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$, and span it by three different basis sets:

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad B_2 = \left\{ \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\}, \quad B_3 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

As can be seen in Figure 2.8, for each basis set the coefficients (colored) are different. In this context we call the coefficients the **coordinates** of $\vec{v}$ in that basis set. In the basis set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ the coordinates of $\vec{v}$ are $(1,-3,7)$ (as we will see next, it is not a

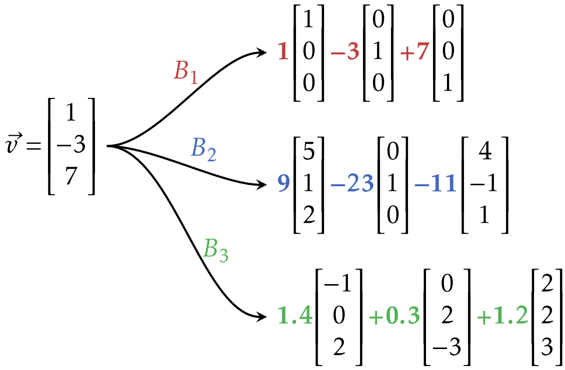


Figure 2.8 The vector $\vec{v} = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$ spanned in three different basis sets.

coincidense that these are equal to its components as a column vector), and in the basis set $\left\{ \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} \right\}$ its coordinates are $(9, -23, -11)$.

Changing the coordinates of a vector between different basis sets is called **basis transformation**, and is generally done using **matrices**. We will discuss this in more details in the next sections of this chapter. For now, let's look at a graphical representation of a vector being expressed in a different basis set (Figure 2.9): in the figure, we see that the vector $\vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ can be written in the basis set $B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \end{bmatrix} \right\}$ using the coefficients 2 and $\frac{1}{2}$, i.e.

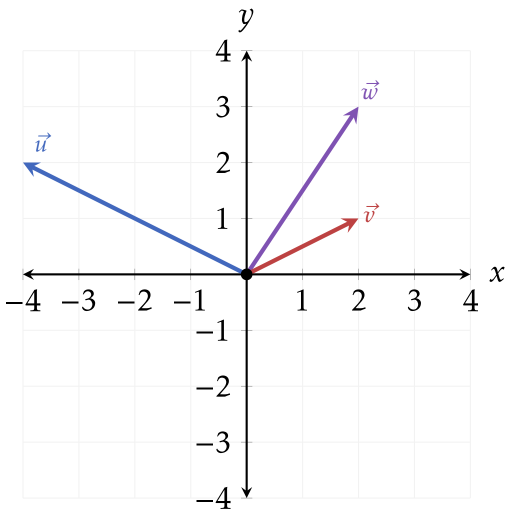
$$\vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -4 \\ 2 \end{bmatrix}.$$

Therefore, in the basis set B, the coordinates of $\vec{w}$ are $\left(2, \frac{1}{2}\right)$.

A basis set B in which all vectors are **orthogonal** (i.e. are at 90°) to each other is called a **orthogonal basis set**. If all vectors are unit vectors as well, i.e. their norms all equal to 1, the basis set is then an **orthonormal basis set**.

Example 2.11 Orthogonal and orthonormal basis sets

The vectors $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are linearly independent and thus form a basis set of $\mathbb{R}^2$. We can calculate their respective angles in relation to the x-axis (θ_a and θ_b) to find the angle between them (φ):



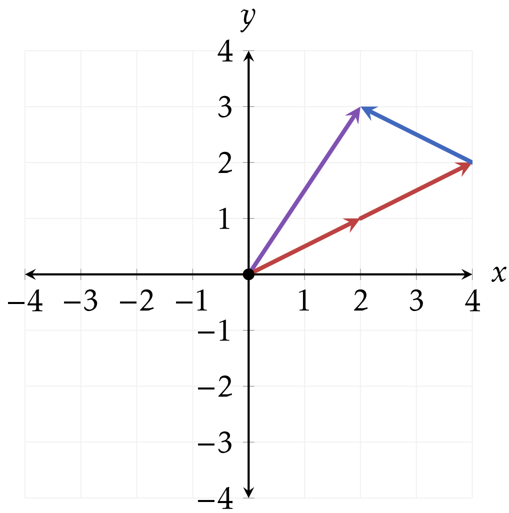
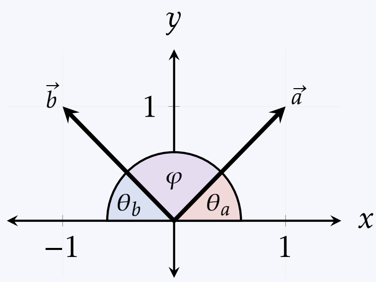


Figure 2.9 The vector $\vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is spanned using the vectors $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, yielding the coordinates $\left(2, \frac{1}{2}\right)$ in the basis set B.



The angle of $\vec{a}$ is

$$\theta_a = \arctan\left(\frac{a_y}{a_x}\right) = \arctan(1) = \frac{\pi}{4} \ (= 45°).$$

Similarily, the angle α_b also equals $\frac{\pi}{4}$. Therefore, $\varphi = 2\frac{\pi}{4} = \frac{\pi}{2}$ $(= 90°)$ - i.e. $\vec{a}$ and $\vec{b}$ are orthogonal, and thus form an orthogonal basis set of $\mathbb{R}^2$.
To get a similar *orthonormal* basis set we can simply normalize the two vectors. We start with $\vec{a}$: its norm is

$$\left\|\vec{a}\right\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Thus, the vector $\hat{a} = \frac{1}{\sqrt{2}}\vec{a} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ is a unit vector. The same argument is valid for $\vec{b}$, i.e. $\hat{b} = \frac{1}{2}\vec{b} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. We therefore get that

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

is an orthonormal basis set of $\mathbb{R}^2$.

Challenge 2.2 Orthonormal basis sets of $\mathbb{R}^2$

Show that all orthonormal basis sets of $\mathbb{R}^2$ are rotations of the set

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

as a whole (i.e. each rotation angle is applied to both vectors).

See example below for such sets in $\mathbb{R}^2$ and $\mathbb{R}^3$.

One common orthonormal basis set in any $\mathbb{R}^n$ is the so-called **standard basis set**. We saw the standard basis set in $\mathbb{R}^3$ in Figure 2.8: it is the set $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Note how in this set, each vector has a special structure: one of its components is 1 while the rest are 0. In the first basis vector the non-zero component is the first component of the vector, in the second basis vector it is the second component, and in the third basis vector it is the third component. In $\mathbb{R}^2$ the standard basis set is simply $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, and generally in $\mathbb{R}^n$ it is

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}, \tag{2.1.29}$$

i.e. in the n-th basis vector the n-th component is 1 while the rest are 0. The standard basis vectors are generally labeled as $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n$ - they get the "hat" symbol since they are all unit length.

In $\mathbb{R}^2$ and $\mathbb{R}^3$ we give $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ special notations: $\hat{x}$, $\hat{y}$ and $\hat{z}$, respectively (obviously $\hat{z}$ doesn't exists in $\mathbb{R}^2$). For historical reasons, these vectors are sometimes denoted in physics textbooks as $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.

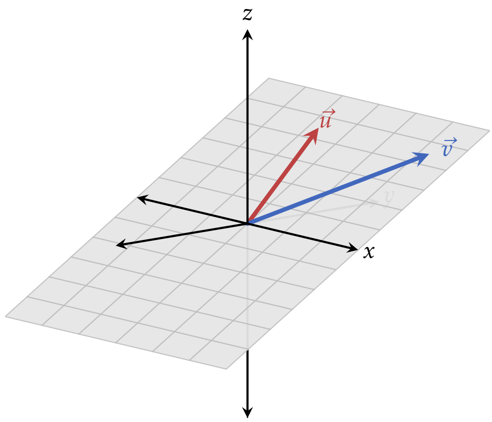


Figure 2.10 The angle between two linearly independent vectors lies on the plane spanned by the vectors.

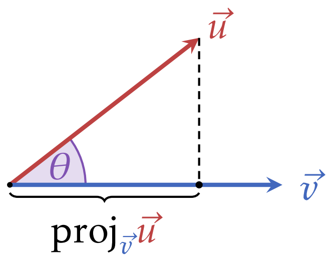


Figure 2.11 The projection of a vector $\vec{u}$ onto another vector $\vec{v}$ in the plane spanned by the two vectors.

2.1.6 The scalar product

When given two vectors $\vec{u},\vec{v} \in \mathbb{R}^n$ it is often useful to know the angle between them: if the two vectors are linearly dependent then the angle is either $\theta = 0$ if they point in the same direction, or $\theta = \pi$ if the point in opposite directions (remember: we measure angles in radians). Otherwise, the angle θ can take any value in $(0,\pi)$. Angles are always measured on a plane, and in the case of two linearly independent vectors that plane is of course the one spanned by the two vectors (Figure 2.10).

If considering only the plane the vectors span, we can rotate it such that one of the vectors, say $\vec{u}$, lies horizotally (see Figure 2.11). We then drop a perpendicular line from the head of the $\vec{u}$ to the horizontal vector $\vec{v}$. We call the length from the origin to the intersection point of $\vec{v}$ and the perpendicular line the **projection** of $\vec{u}$ onto $\vec{v}$, and denote it as $\text{proj}_{\vec{v}}\vec{u}$.

Since the origin, the head of $\vec{u}$ and the intersection point of the perpendicular line with $\vec{v}$ form a right triangle, using basic trigonometry we find that the cosine of the angle θ is

$$\cos(\theta) = \frac{\text{proj}_{\vec{v}}\vec{u}}{\|\vec{u}\|}. \tag{2.1.30}$$

We can now use this construct to define a product between $\vec{u}$ and $\vec{v}$: their **scalar product**. We define it as following:

$$\vec{u} \cdot \vec{v} = \text{proj}_{\vec{v}} \vec{u} \cdot \left\| \vec{v} \right\|. \tag{2.1.31}$$

Subtituting Equation 2.1.30 into Equation 2.1.31 gives a very nice relation between the scalar product of two vectors and the angle between them:

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\left\| \vec{u} \right\| \left\| \vec{v} \right\|}. \tag{2.1.32}$$

The angle between the two vectors is then isolated by applying the arccos function on the right-hand side of Equation 2.1.32. A common form of this equation is the following:

$$\vec{u} \cdot \vec{v} = \left\| \vec{u} \right\| \left\| \vec{v} \right\| \cos(\theta). \tag{2.1.33}$$

Note that the scalar product returns a number, i.e. in the terms of linear algebra - a scalar, and hence its name. Since it is commonly denoted with a dot between the two vectors, it is sometimes refered to as the **dot product**. A common notation for the scalar product is the so-called **bracket notation**:

$$\langle \vec{a}, \vec{b} \rangle.$$

Sometimes the comma in the notation is replaced by a vertical separator line:

$$\langle \vec{a} | \vec{b} \rangle.$$

This notation is very common in physics, and especially quantum physics where it is very useful and helps in simplifying many calculations. This will be discussed in more details in chapter/section TBD.

Later in the section we will examine some common properties of the scalar product, and see how we can calculate it directly from the vectors in their column form. Beofre we do that, let's use what we learned about the scalar product so far to solve some easy problems in the examples below.

Example 2.12 Angle between two vectors

Find the scalar product of the vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Solution:
As seen in Example 2.11, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$. Therefore, their scalar product is

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \left\| \vec{a} \right\| \left\| \vec{b} \right\| \cos(\theta) \\ &= \sqrt{2}\sqrt{2}\cos\left(\frac{\pi}{2}\right) \end{aligned}$$

$$= 2 \cdot 0 = 0.$$

Example 2.13 Scalar product of two vectors

Calculate the scalar product of the two vectors $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, given that the angle between them is $\theta \approx 2.069 \approx 118.561°$.

Solution:
The norms of the two vectors are

$$\left\|\vec{u}\right\| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4+9+1} = \sqrt{14} \approx 3.742,$$
$$\left\|\vec{v}\right\| = \sqrt{(-1)^2 + 0^2 + 2^2} = \sqrt{1+4} = \sqrt{5} \approx 2.236.$$

Therefore, their scalar product is

$$\vec{u} \cdot \vec{v} \approx \sqrt{14}\sqrt{5}\cos(2.069) \approx -4.$$

The scalar product of any two vectors $\vec{u}, \vec{v}$ has two important properties:

- It is commutative, i.e. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
- Scalars can be taken out of the product, i.e. $\left(\alpha\vec{v}\right) \cdot \vec{u} = \vec{v} \cdot \left(\alpha\vec{u}\right) = \alpha\left(\vec{u} \cdot \vec{v}\right)$.
- It equals zero in only one of two cases:
  1. One of the vectors (or both) is the zero vector, or
  2. The angle θ between the vectors is $\frac{\pi}{2}$, since then $\cos(\theta) = \cos\left(\frac{\pi}{2}\right) = 0$.

When the angle between two vectors is $\frac{\pi}{2}$ (remember: this is equivalent to 90°), we say that the two vectors are **orthogonal** to eacth other. Note that in the special case of 2- and 3-dimensional we say that the vectors are **perpendicular** to each other.

This is such an important fact that we will put effort into framing it nicely, so you (the reader) could memorize it well. How well should you memorize this? Such that if someone wakes you up in the middle of the night and asked you, you could easily repeat it[2].

[2] For a humble fee, I'm willing to do this - just write me an email and we can discuss the terms ;)

$$\vec{u} \cdot \vec{v} = 0$$

$$\Updownarrow$$

$\vec{u}$ and $\vec{v}$ are orthogonal

Calculating the scalar product of two vectors in $\mathbb{R}^n$ using their column form is extremely straight-forward: it is nothing more than the sum of the component-wise product of the two vectors, i.e. given

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix},$$

the scalar product $\vec{u} \cdot \vec{v}$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{k=1}^{n} u_i v_i. \tag{2.1.34}$$

Example 2.14 Angle between two vectors

Calculate the scalar product of the two vectors $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ using the above formula (Equation 2.1.34).

Solution:
We simply substitute $\vec{a}$ and $\vec{b}$ into the equation:

$$\vec{a} \cdot \vec{b} = 1 \cdot (-1) + 1 \cdot 1 = -1 + 1 = 0,$$

which is exactly the result we got using the previous method.

Example 2.15 Scalar product of two vectors - algebraicly

Calculate the scalar product $\vec{u} \cdot \vec{v}$ from Example 2.13 using Equation 2.1.34.

Solution:

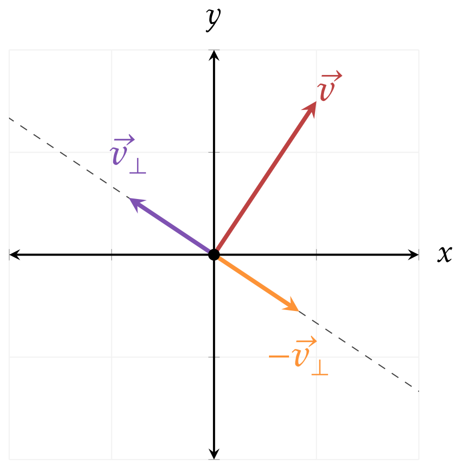


Figure 2.12 A vector $\vec{v}$ and its orthogonal direction, signified by a dashed line. Two vectors $\vec{v}^\perp$ and $\vec{v}^\perp$ are drawn on the orthogonal direction.

$$\vec{u}\cdot\vec{v} = 2\cdot(-1) + 3\cdot 0 + (-1)\cdot 2 = -2 - 2 = -4,$$

exactly the result we got in Example 2.13.

For any given a 2-dimensional vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ there is only a single orthogonal direction (Figure 2.12). We can use Equation 2.1.34 to find a general formula for a vector $\vec{v}^\perp$ representing this direction:

$$0 = \vec{v}\cdot\vec{v}^\perp = \begin{bmatrix} x \\ y \end{bmatrix}\cdot\begin{bmatrix} a \\ b \end{bmatrix} = xa + yb.$$

The solution for the above equation is the vector

$$\vec{v}^\perp = \begin{bmatrix} -y \\ x \end{bmatrix}. \tag{2.1.35}$$

The norm of a vector can be calculated using the scalar product: given a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$,

$$\vec{v}\cdot\vec{v} = v_1 v_1 + v_2 v_2 + \cdots + v_n v_n = v_1^2 + v_2^2 + \cdots + v_n^2 = \left\|\vec{v}\right\|^2. \tag{2.1.36}$$

We therefore usually define the norm in terms of the scalar product:

$$\left\|\vec{v}\right\| = \sqrt{\vec{v}\cdot\vec{v}}. \tag{2.1.37}$$

This might seem unconsequential at the moment, but it will become very useful when we generalize linear algebra to more abstract vector spaces (Chapter 3).

Any vector can be **decomposed** into its projections on n orthogonal directions. In fact, this is exactly what we do when we write a vector as a linear combination of the vectors of an orthogonal basis: consider for example the vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

It can be written as the linear combination

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + \cdots + v_n\hat{e}_n = \sum_{i=1}^{n} v_i\hat{e}_i,$$

where in turn any element v_i is the projection of $\vec{v}$ on the basis vector $\hat{e}_i$:

$$v_i = \mathrm{proj}_{\hat{e}_i}\vec{v}, \tag{2.1.38}$$

and thus the component $v_i\hat{e}_i = \left(\mathrm{proj}_{\hat{e}_i}\vec{v}\right)\hat{e}_i$ is itself a vector of norm v_i pointing at the direction $\hat{e}_i$. In general, given an orthogonal basis set $B = \left\{\vec{b}_1,\ \vec{b}_2,\ \cdots,\ \vec{b}_n\right\}$, any vector in $\mathbb{R}^n$ can be decomposed as follows:

$$\vec{v} = \sum_{i=1}^{n}\left(\mathrm{proj}_{\hat{b}_i}\vec{v}\right)\hat{b}_i. \tag{2.1.39}$$

In the case where B is an orthonormal basis set, we know that each of its vector is a unit vector (i.e. $\left\|\vec{b}_i\right\| = 1$), and using Equation 2.1.31 we can re-write Equation 2.1.39 as

$$\vec{v} = \sum_{i=1}^{n}\left(\vec{v}\cdot\hat{b}_i\right)\hat{b}_i. \tag{2.1.40}$$

Example 2.16 Decomposing a vector

EXAMPLE TBD

2.1.7 The cross product

Another commonly used product of two vectors is the so-called **cross product**. Unlike the scalar product, it is only really valid in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^7$, of which we will focus on $\mathbb{R}^3$ and touch a bit on its uses in $\mathbb{R}^2$. Also in contrast to the scalar product, the cross product in $\mathbb{R}^3$ results in a vector rather than a scalar - therefore the product is sometimes known as the **vector product**. The cross product uses the notation $\vec{a}\times\vec{b}$, from which it derives its name.

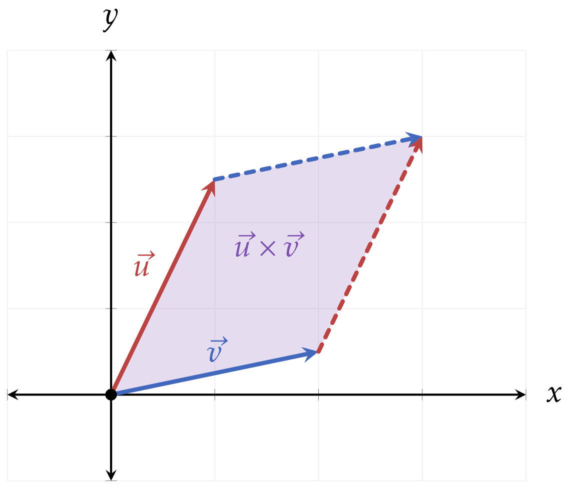


Figure 2.13 The cross product in $\mathbb{R}^2$ of two vectors $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ as the signed area of the parallelogram defined by the vectors.

We start with the definition of the cross product in $\mathbb{R}^2$: the cross product of two vectors $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} c \\ d \end{bmatrix}$ is the (signed) area of the parallelogram defined by the two vectors (see Figure 2.13).

The value of the parallelogram defined by $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin(\theta), \tag{2.1.41}$$

where θ is the angle between the vectors. This is extremely similar to the scalar product, and we can use this fact to find how to calculate the cross product from vectors in column form: if we replace $\vec{u}$ by a vector orthogonal to it, denoted by $\vec{u}^{\perp}$, the cross product is then

$$\vec{u} \times \vec{v} = \|\vec{u}^{\perp}\| \|\vec{v}\| \sin\left(\theta + \frac{\pi}{2}\right), \tag{2.1.42}$$

since the angle between $\vec{u}^{\perp}$ and $\vec{v}$ is $\frac{\pi}{2}$ more than that between $\vec{u}$ and $\vec{v}$. Using the fact that $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$, we get the equality

$$\begin{aligned} \vec{u} \times \vec{v} &= \|\vec{u}^{\perp}\| \|\vec{v}\| \sin\left(\theta + \frac{\pi}{2}\right) \\ &= \|\vec{u}^{\perp}\| \|\vec{v}\| \cos(\theta) \\ &= \vec{u}^{\perp} \cdot \vec{v}. \end{aligned} \tag{2.1.43}$$

In $\mathbb{R}^2$, any vector $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ has two vectors orthogonal to it: $\begin{bmatrix} -b \\ a \end{bmatrix}$ and $\begin{bmatrix} b \\ -a \end{bmatrix}$. Choosing the former gives

$$\vec{u} \times \vec{v} = \begin{bmatrix} -b \\ a \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = -bc + ad, \tag{2.1.44}$$

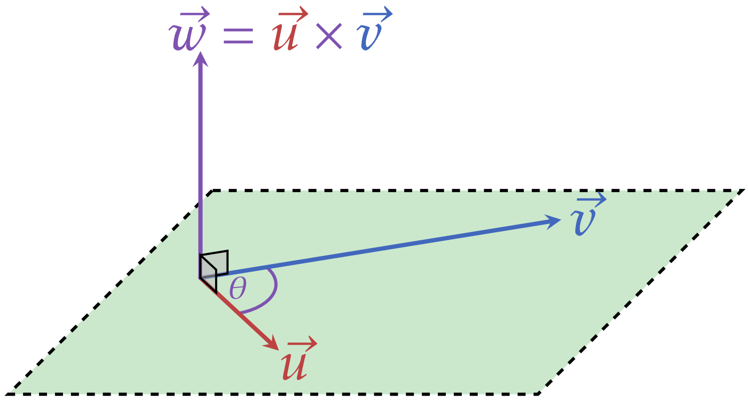


Figure 2.14 The cross product of the vectors $\vec{u}$ and $\vec{v}$ relative to the plane spanned by the two vectors.

while choosing the latter gives

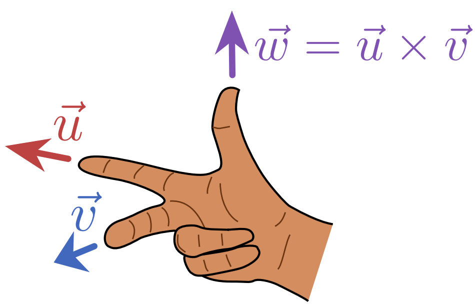
$$\vec{u} \times \vec{v} = \begin{bmatrix} b \\ -a \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = bc - ad. \tag{2.1.45}$$

These two forms are the opposite of each other - i.e. if one yields the value 4, the other yields the value −4. We will see which one is used in a moment.

On to $\mathbb{R}^3$: geometrically, the cross product of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^3$ is defined as a **vector** $\vec{w} \in \mathbb{R}^3$ which is **orthogonal to both** $\vec{u}$ and $\vec{v}$, and with norm of the same magnitude as the product would have in $\mathbb{R}^2$, i.e.

$$\left\|\vec{w}\right\| = \left\|\vec{u}\right\| \left\|\vec{v}\right\| \sin(\theta). \tag{2.1.46}$$

The direction of $\vec{u} \times \vec{v}$ is determined by the **right-hand rule**: using a person's right hand, when $\vec{u}$ points in the direction of their index finger and $\vec{v}$ points in the direction of their middle finger, then vector $\vec{w} = \vec{u} \times \vec{v}$ points in the direction of their thumb:



The cross product is **anti-commutative**, i.e. changing the order of the vectors results in inverting the product:

$$\vec{u} \times \vec{v} = -\left(\vec{v} \times \vec{u}\right).$$

When the vectors are given as column vectors $\vec{u}=\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$, $\vec{v}=\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$, the resulting cross product is

$$\vec{u}\times\vec{v}=\begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix} \tag{2.1.47}$$

Note 2.5 The cross product of the standard basis vectors

The cross product of two of the standard basis vectors in $\mathbb{R}^3$ is the third basis vector. Its sign ($\pm$) is determined by a cyclic rule:

$$\operatorname{sign}\left(\hat{e}_i\times\hat{e}_j\right)=\begin{cases} 1 & \text{if } (i,j)\in\left\{(1,2),\ (2,3),\ (3,1)\right\}, \\ -1 & \text{if } (i,j)\in\left\{(3,2),\ (2,1),\ (1,3)\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

!

Challenge 2.3 Orthogonalily of the cross product

Using component calculation and utilizing the dot product, show that $\vec{a}\times\vec{v}$ is indeed orthogonal to both $\vec{a}$ and $\vec{b}$.

?

2.1.8 The Gram–Schmidt process

While all basis sets of a given space are equaly good at spanning that space[3], as humans we sometimes prefer using orthonormal basis sets due to their nice properties. One such property of orthonormal basis sets, which we will use in a later section, is that the scalar product of any two vectors of the set is the Kronecker's delta - i.e. given an orthonormal basis set

$$B=\left\{\vec{b}_1,\vec{b}_2,\dots,\vec{b}_n\right\},$$

for any two basis vectors $\vec{b}_i$ and $\vec{b}_j$,

$$\vec{b}_i\cdot\vec{b}_j=\delta_{ij}=\begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$$

However, most basis sets are not orthonormal[4]. How can we construct an orthonormal basis set from a given basis set?

[3] they're good basis sets Brent

[4] since there are countlessly infinitely many basis sets for any space, the meaning of "most" in this context is that the probability that a random basis set is not orthonormal is greater than the probability that it is orthonormal.

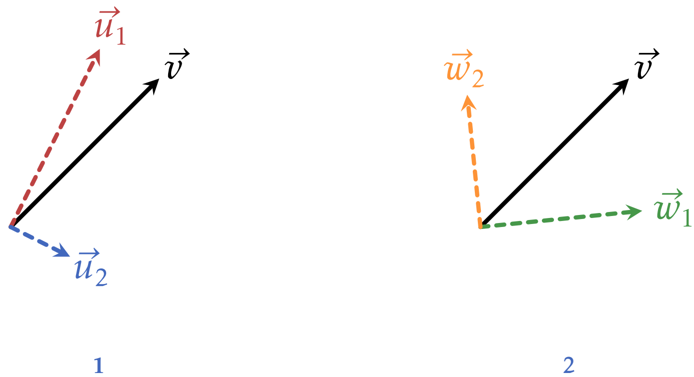


Figure 2.15 The same vector $\vec{v}$ decomposed into two sets of orthogonal components: (a) u_1 and u_2; (b) w_1 and w_2. There are infinitely many such orthogonal sets on any plane containing $\vec{v}$.

In the unlikely case that the given basis set is orthogonal, the answer is simple: normalize each of the basis vectors. When the given basis set in not orthogonal we can use the **Gram-Schmidt process** (GSP), which takes a basis set and transforms it into an orthonormal basis set.

In order to understand the GSP, one must first understand the following fact: given a vector $\vec{v}$ and a plane P which contains the vector, we can always write $\vec{v}$ as the sum of any two orthogonal vectors $\vec{a}$ and $\vec{b}$ in P, i.e.

$$\vec{v} = \vec{a} + \vec{b}, \tag{2.1.48}$$

where all the above vectors lie on the same plane (see Figure 2.15).

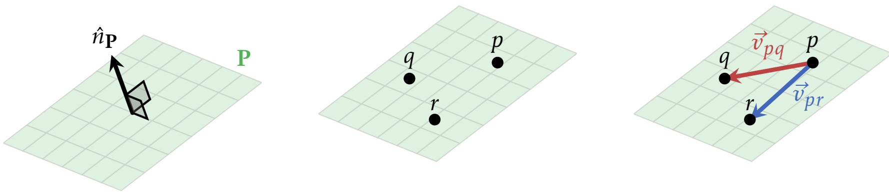
Rearranging Equation 2.1.48, we get that

$$\begin{aligned}\vec{a} &= \vec{v} - \vec{b}, \text{ and}\\ \vec{b} &= \vec{v} - \vec{a}.\end{aligned} \tag{2.1.49}$$

! **To be written**: Finish this subsection !

2.1.9 Normal vectors

A special kind of vector in $\mathbb{R}^3$ is the so-called **normal vector** to a plane $\mathbf{P}$: this vector, usually denoted as $\hat{n}_{\mathbf{P}}$, is pointing at the orthogonal direction to any vector of the plane (see XXX). Given one knows three points on the plane, its normal vector can be calculated: say the following three points in $\mathbf{P}$ are given (for visualizing the following



1 The normal vector to **P**.

2 Finding three points on the plane.

3 Finding two vectors on the plane.

Figure 2.16 A normal vector $\hat{n}_\mathbf{P}$ to the plane **P**.

steps see YYY): ! **To be written**: Change XXX and YYY to the right refs !

$$
\begin{aligned}
p &= (p_x, p_y, p_z) \\
q &= (q_x, q_y, q_z) \\
r &= (r_x, r_y, r_z),
\end{aligned}
\tag{2.1.50}
$$

We can get two vectors lying on the plane by first considering the points as vectors, i.e.

$$
\vec{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \ \vec{q} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}, \ \vec{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}.
\tag{2.1.51}
$$

Then, we calculate two vectors on the plane by subtraction, e.g.

$$
\begin{aligned}
\vec{v}_{pq} &= \vec{q} - \vec{p} = \begin{bmatrix} q_x - p_x \\ q_y - p_y \\ q_z - p_z \end{bmatrix}, \\
\vec{v}_{pr} &= \vec{r} - \vec{p} = \begin{bmatrix} r_x - p_x \\ r_y - p_y \\ r_z - p_z \end{bmatrix}.
\end{aligned}
\tag{2.1.52}
$$

The normal vector $\hat{n}_p$ must be orthogonal to both $\vec{v}_{pq}$ and $\vec{v}_{pr}$ - and so we use the cross product to find its direction:

$$
\vec{n}_\mathbf{P} = \vec{v}_{pq} \times \vec{v}_{pr} = \begin{bmatrix} (q_y - p_y)(r_z - p_z) - (r_y - p_y)(q_z - p_z) \\ (p_x - q_x)(r_z - p_z) - (r_x - p_x)(q_z - p_z) \\ (q_x - p_x)(r_y - p_y) - (r_x - p_x)(p_y - q_y) \end{bmatrix}.
\tag{2.1.53}
$$

Normalizing $\vec{n}_\mathbf{P}$ will then yield the normal vector $\hat{n}_\mathbf{P}$[5].

[5] I leave this as a challenge to the reader, because I'm lazy.

Note 2.6 Sign of normal vectors

The vector $\vec{m} = -\hat{n}_{\mathbf{P}}$ has all the properties of $\hat{n}_{\mathbf{P}}$, and is indeed a normal vector to $\mathbf{P}$. The choice of which of the two vectors to use depends on the application. For now, we do not elaborate on this further.

2.1.10 Examples

To wrap up the vectors section, we solve some problems which cover the material presented in the section:

Example 2.17 Vector form of gravitational force

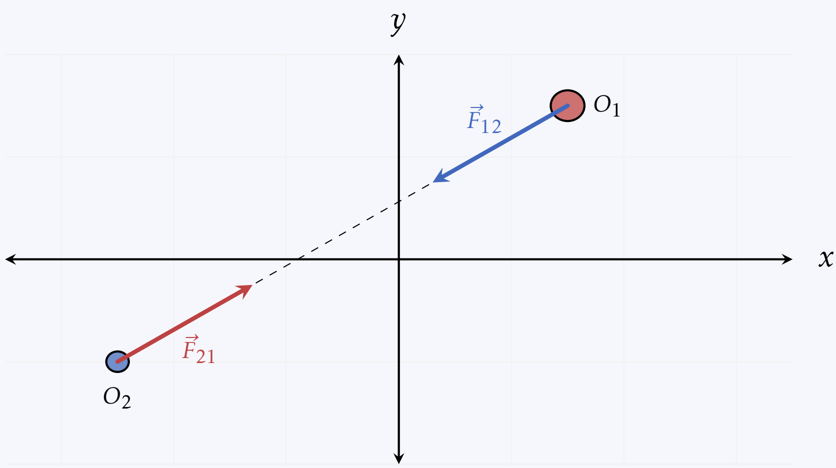
According to Newton's law of gravity, given two objects O_1 and O_2 with masses m_1 and m_2 respectively, each of them would feel a gravitational force of attraction **in the direction of the other object** with the following magnitude:

$$F = G\frac{m_1 m_2}{r^2},$$

where G is a universal constant and r is the distance between the two objects.

Say we put O_1 and O_2 on an axis system such that their positions are $\vec{r}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{r}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, respectively (see below figure for a 2-dimensions representation).
The system therefore has two forces:

- $\vec{F}_{12}$: the gravitational force acting on object O_1 as a result of O_2. It points from O_1 towards O_2.
- $\vec{F}_{21}$: the gravitational force acting on O_2 as a result of O_1. It points from O_2 towards O_1.



Find the explicit form of each of the two force vectors, i.e. using only the parameters G, m_1 and m_2, and the positions $\vec{r}_1$ and $\vec{r}_2$.

Solution
The two vectors $\vec{F}_{12}$ and $\vec{F}_{21}$ both lie on the line connecting O_1 and O_2. Therefore, their orientations are exactly opposite, and since their magnitudes have to be equal (see the force definition above), the two vectors are simply the opposite of eachother, i.e.

$$\vec{F}_{12} = -\vec{F}_{21}.$$

We therefore need to calculate only a single force vector $\vec{F}$ and we automatically get the other force vector as $-\vec{F}$. We will thus first find the explicit form of the vector $\vec{F} = \vec{F}_{12}$, and using this form easily find $\vec{F}_{21}$ as $-\vec{F}$.
To get the explicit form of $\vec{F}$ we should find its magnitude $\alpha \in \mathbb{R}$ and direction $\hat{v} \in \mathbb{R}3$, and then we can write

$$\vec{F} = \alpha \hat{v}.$$

We start with the magnitude: since the magnitude of the gravitational force is given by

$$\alpha = \left\| \vec{F} \right\| = G \frac{m_1 m_2}{r^2},$$

and since we are given G, m_1 and m_2 as parameters, we are only left with expressing r^2 as a function of the positions $\vec{r}_1$ and $\vec{r}_2$ of the two objects. Given any two vectors, we can find their relative distance by simply *subtracting* one vector from the other. The result is a vector connecting the two given vectors, and its norm would be the distance between the vectors. Therefore, we find that

$$r^2 = \left\| \vec{r}_1 - \vec{r}_2 \right\|^2 = \left\| \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} \right\|^2 = (x_2 - x_1)^2 + \left(y_2 - y_1\right)^2 + (z_2 - z_1)^2.$$

(note that the order of subtraction forces the resulting vector to point from O_1 towards O_2. This will become handy soon)

We can therefore write the magnitude of $\vec{F}$ as

$$F = G \frac{m_1 m_2}{\left\| \vec{r}_2 - \vec{r}_1 \right\|^2}.$$

The direction of $\vec{F}$ is the same direction as $\vec{\Delta r}$, i.e. from O_1 to O_2. By normalizing $\vec{\Delta r}$ we get a vector pointing in the direction we want, with norm 1:

$$\hat{r} = \frac{\vec{r}_2 - \vec{r}_1}{\left\| \vec{r}_2 - \vec{r}_1 \right\|}.$$

Altogether we get that $\vec{F}$ has the form

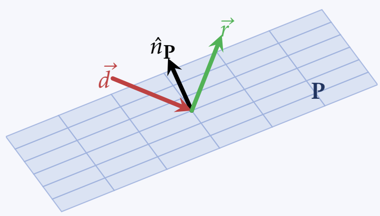
$$\vec{F}_{12} = \vec{F} = \alpha \hat{v} = \frac{G m_1 m_2}{\left\| \vec{r}_2 - \vec{r}_1 \right\|^3} \left(\vec{r}_2 - \vec{r}_1 \right),$$

and also

$$\vec{F}_{21} = -\vec{F} = -\frac{Gm_1m_2}{\left\|\vec{r}_2 - \vec{r}_1\right\|^3}\left(\vec{r}_2 - \vec{r}_1\right) = \frac{Gm_1m_2}{\left\|\vec{r}_2 - \vec{r}_1\right\|^3}\left(\vec{r}_1 - \vec{r}_2\right).$$

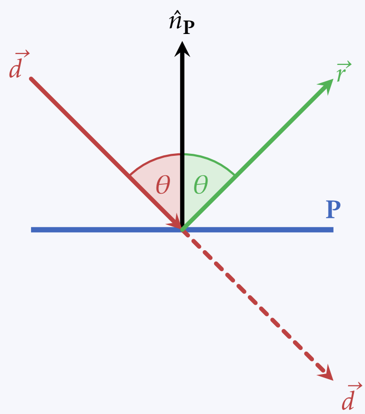
Example 2.18 Reflection of light rays

A ray light hits a mirror, modelled by the plane **P** which is defined by the normal vector $\hat{n}_{\mathbf{P}}$. The direction of the light ray is given by $\vec{d}$. What is the direction of the reflected light ray $\vec{r}$? Recall that both the incident and reflected rays are at the same angle in respect to the normal vector of $\hat{n}_{\mathbf{P}}$, and that the incident ray lie on the plane defined by $\vec{d}$ and $\hat{n}_{\mathbf{P}}$.



Solution

We can rotate our viewpoint of the problem, looking at **P** from the side and in such a way that we look head-on at the plane spanned by $\hat{n}_{\mathbf{P}}$ and $\vec{d}$:



(the dashed red vector in the above figure represents the vector incident ray, $\vec{d}$, moved such that its origin lies at the origin of the other vectors)

As with any vector, we can decompose $\vec{d}$ to its projections on the vectors of an orthonormal basis set (Equation 2.1.40). Since we reduced the problem to two dimensions, we need a basis of two orthonormal directions: we choose one to be $\hat{n}_{\mathbf{P}}$, and the other orthogonal to it (in the figure above it is in the horizontal direction) which we call $\hat{p}$. The decomposition of $\vec{d}$ then reads:

$$\vec{d} = \left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}} + \left(\vec{d} \cdot \hat{p}\right)\hat{p}.$$

Since there are only two vectors in the basis set $\left\{\hat{n}_{\mathbf{P}}, \hat{p}\right\}$, we can actually write the component $\left(\vec{d} \cdot \hat{p}\right)\hat{p}$ as $\vec{d} - \left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}}$, yielding a rather silly looking expression for $\vec{d}$:

$$\vec{d} = \left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}} + \left[\vec{d} - \left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}}\right].$$

However, in closer inspection the above expression is not at all silly, and is actually very similar to the reflected vector $\vec{r}$: since they are both of same norm and oposing directions with respect to the direction $\hat{n}_{\mathbf{P}}$, we can write $\vec{r}$ as

$$\vec{r} = -\left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}} + \left[\vec{d} - \left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}}\right].$$

From the above expressions for $\vec{d}$ and $\vec{r}$ we can isolate an expression for $\vec{r}$ as a function of $\vec{d}$ and $\hat{n}_{\mathbf{P}}$:

$$\begin{aligned}\vec{r} &= d - \left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}} - \left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}} \\ &= d - 2\left(\vec{d} \cdot \hat{n}_{\mathbf{P}}\right)\hat{n}_{\mathbf{P}}.\end{aligned}$$

! **To be written**: discussion about right- and left-handed spaces/orientations !

2.2 LINEAR TRANSFORMATIONS

In the previous section we introduced real vectors and their most important properties. In this section we explore a special set of operations that can act on vectors, namely **linear transformations**.

Linear transformations are a special set of transformations that are relatively easy to analyze. Their use is extremely widespread all throughout different fields of mathematics and its application, e.g. (just to name a few): in computer graphics, machine learning, quantum physics and engineering.

2.2.1 Definition

As mentioned in Chapter 0, a “transformation“ is simply another name for a function. Thus in our context, linear transformations are some functions that act on vectors: a linear transformation T takes a vector as an input, and outputs another vector, possibly of a different dimension, i.e.

$$T : \mathbb{R}^n \to \mathbb{R}^m. \tag{2.2.1}$$

What makes linear transformations more “special” than other functions is their property of **linearity**, which entails the following two properties:

- Scalability: for any scalar α and vector $\vec{v}$,

$$T\left(\alpha\vec{v}\right) = \alpha T\left(\vec{v}\right).$$

- Additivity: for any two vectors $\vec{u}, \vec{v}$

$$T\left(\vec{u}+\vec{v}\right) = T\left(\vec{u}\right) + T\left(\vec{v}\right).$$

Example 2.19 A linear transformation

Claim: the following $\mathbb{R}^3 \to \mathbb{R}$ transformation is linear:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 2x + 3y - z.$$

Proof: We can show this using the properties of linear transformations.

- Scalability: given a scalar $\alpha \in \mathbb{R}$,

$$T\left(\begin{bmatrix} \alpha x \\ \alpha y \\ \alpha z \end{bmatrix}\right) = 2(\alpha x) + 3(\alpha y) - (\alpha z) = \alpha\left(2x + 3y - z\right) = \alpha T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right).$$

- Additivity: given two vectors $\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$,

$$\begin{aligned} T\left(\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}\right) &= T\left(\begin{bmatrix} u_x + v_x \\ u_y + v_y \\ u_z + v_z \end{bmatrix}\right) \\ &= 2\left(u_x + v_x\right) + 3\left(u_y + v_y\right) - \left(u_z + v_z\right) \\ &= T\left(\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}\right) + T\left(\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}\right). \end{aligned}$$

Example 2.20 A non-linear transformation

Claim: the following $\mathbb{R}^3 \to \mathbb{R}$ transformation is **not** a linear transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 2x^2 + 3y - z.$$

Proof: this time we only need to show a single case where linearity breaks - let's choose *scalability*. Given the vector $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$, on one hand

$$T\left(\alpha \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}\right) = 2\left(\alpha u_x\right)^2 + 3\alpha u_y - \alpha u_z = 2\alpha^2 u_x^2 + 3\alpha u_y - \alpha u_z.$$

On the other hand

$$\alpha T\left(\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}\right) = \alpha\left(2u_x^2 + 3u_y - u_z\right) = 2\alpha u_x^2 + 3\alpha u_y - \alpha u_z.$$

For any $a \notin \{0, 1\}$ we get that $T\left(\alpha\vec{v}\right) \neq \alpha T\left(\vec{v}\right)$. Therefore, T is not linear.

2.2.2 Developing intuition

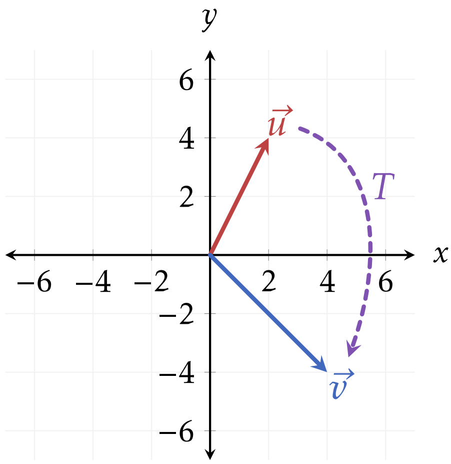
Before moving on to explore the algebraic properties of linear transformations, we first shift our focus to gain some intuition about them. Much like in the last section, we do this using graphical representations of linear transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$. We start with a single vector under transformation: let $\vec{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ -y \end{bmatrix}. \tag{2.2.2}$$

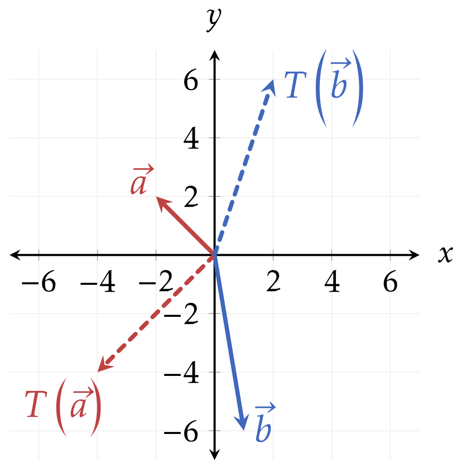
(to the reader: verify that this transformation is indeed linear)

Applying T to $\vec{u}$ yields the vector $\vec{v} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ (see Figure 2.171), i.e. it scales the x-component of $\vec{u}$ by 2 and flips over its y-component.

If we take other vectors, e.g. $\vec{a} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$ we see that T transforms them in the exact same manner: it scales their x-components by 2 and flipps over their y-components (Figure 2.172). This is a fundamental aspect of linear transformations: they always transform all vectors in the exact same manner. We can use this fact to help



1 The vector $\vec{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is transformed by T yielding the vector $\vec{v} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$.



2 The vectors $\vec{a} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\vec{b}$ are transformed by the same T.

visualize transformations, by looking at how they transform the entire space. For example, we can draw all grid lines and observe how they are transformed.

In Figure 2.18 a schematic of $\mathbb{R}^2$ is shown before and after the application of a linear transformation T, by placing a transformed grid (blue) ontop of an untouched grid (gray). In this view, one can see how each point in space is transformed: assuming for example that each two adjacent grid points are 1 unit apart, the gray point at $(-2,2)$ is transformed to where the blue point is, i.e. $(-1,1)$ when measured using the original axes.

For comparison, Figure 2.19 shows a non linear transformation applied to $\mathbb{R}^2$.

Figure 2.18 shows some important properties of linear transformation (cf. Figure 2.19):

1. The origin stays at the same place after application of the transformation, i.e. $T\left(\vec{0}\right) = \vec{0}$.
2. Parallel lines remain parallel after application of the transformation.
3. All areas are scaled by the same amount.

It is rather easy to prove the first two properties.

Proof 2.5 Two properties of linear transformations

1. Let T be a transformation that does not perserve the origin, i.e.

$$T\left(\vec{0}\right) = \vec{v} \neq \vec{0}.$$

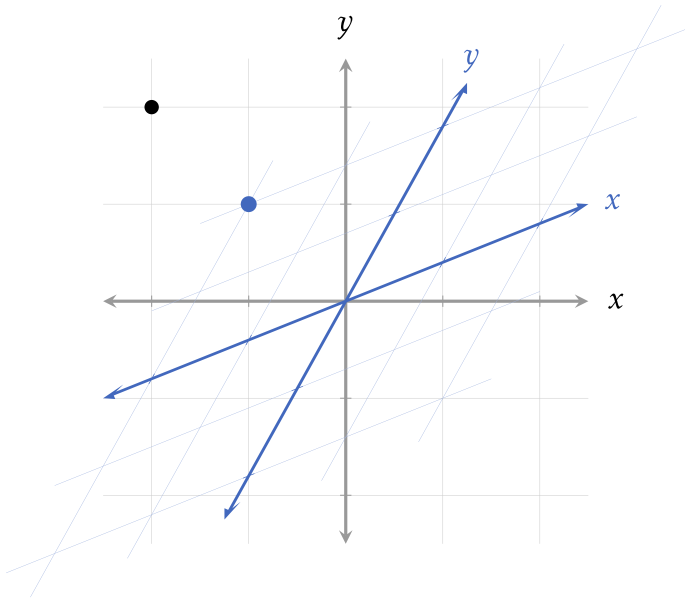


Figure 2.18 $\mathbb{R}^2$ after application of a linear transformation (blue), placed ontop of $\mathbb{R}^2$ before the transformation (gray). Note the black point at the top left at $(-2,2)$ transforming into the blue point at $(-1,1)$.

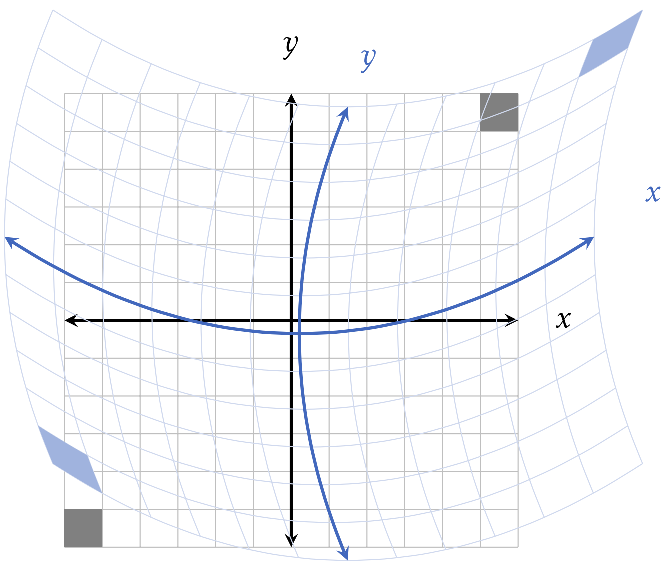


Figure 2.19 A non linear transformation applied to $\mathbb{R}^2$ for comparison.

We can scale $\vec{0}$ by a scalar $\alpha \neq 0$, which would yield

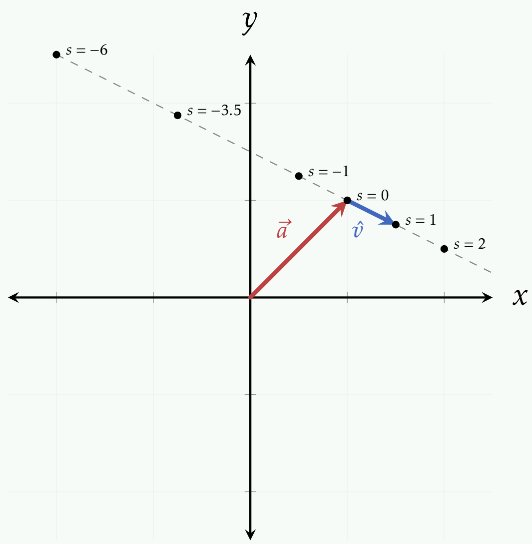
$$T\left(\alpha\vec{0}\right) = T\left(\vec{0}\right) = \vec{v}.$$

However, for T to be linear we expect (due to scalability)

$$T\left(\alpha\vec{0}\right) = \alpha\vec{v},$$

but since $\alpha \neq 0$ and $\vec{v} \neq \vec{0}$ this does not happen. Therefore, T can not be linear - and in turn linear transformations must preserve the origin.

2. A line is defined using a point $\vec{a}$, and a direction $\hat{v}$ as the set of all the points $\left\{x = \vec{a} + s\hat{v}, s \in \mathbb{R}\right\}$:



Parallel lines have the same direction $\hat{s}$, i.e. $x_1 = \vec{a}_1 + s_1\hat{v}$ and $x_2 = \vec{a}_2 + s_2\hat{v}$ are parallel lines. Applying a linear transformation T to these lines yields (using the two defining properties of linear transformations)

$$\begin{aligned} T(x_1) &= T\left(\vec{a}_1\right) + s_1 T(\hat{v}), \\ T(x_2) &= T\left(\vec{a}_2\right) + s_2 T(\hat{v}). \end{aligned}$$

We can see that the two right-hand side equations represent two new lines with the same direction, i.e. $T(\hat{v})$. Therefore parallel lines remain parallel under a linear transformation.

QED

We will prove the the third property (all areas are scaled by the same amount) later in the chapter.

All linear transformations in $\mathbb{R}^2$ can be created by composing transformations from a set of linear transformation which we will refer to as the **basic linear transformations**[6]. To visualize these basic transformations we apply them on a figure of a tapir[7]:

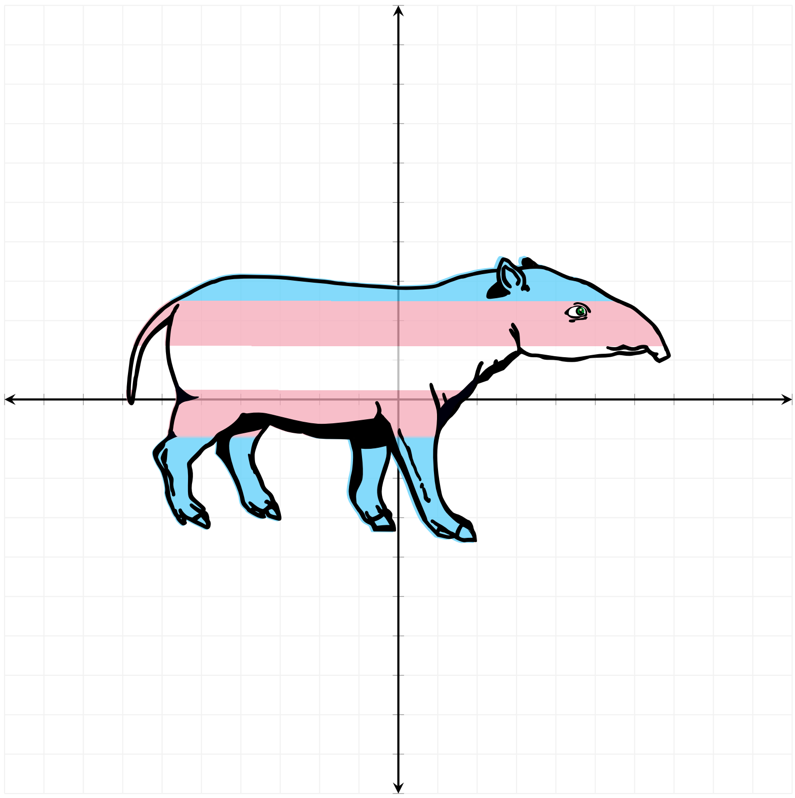
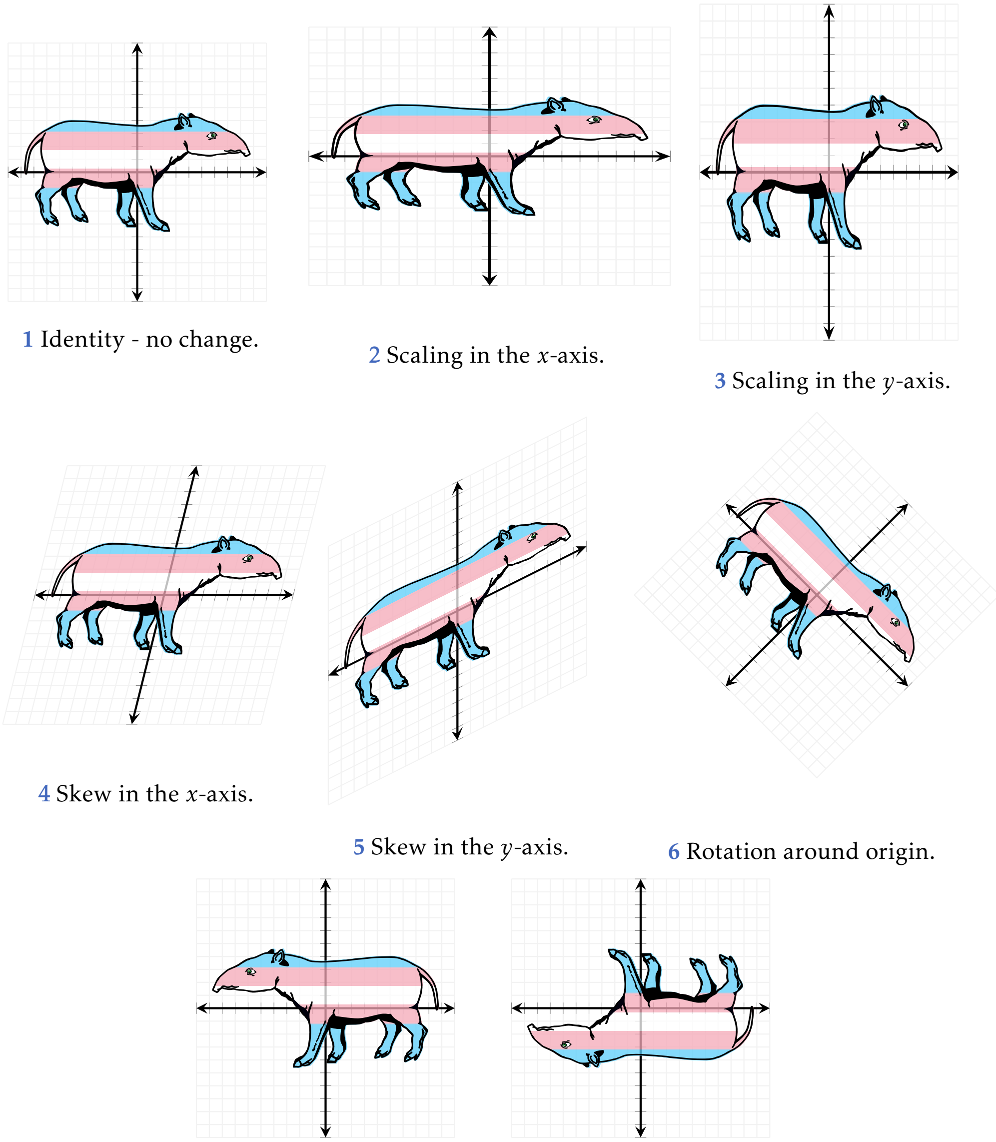


Figure 2.20 shows the basic linear transformations applied to our happy tapir.

[6] not an official name.
[7] They are here, they are a trans tapir. Get used to it.



1 Identity - no change.

2 Scaling in the x-axis.

3 Scaling in the y-axis.

4 Skew in the x-axis.

5 Skew in the y-axis.

6 Rotation around origin.

7 Reflection about line.

8 Reflection through origin.

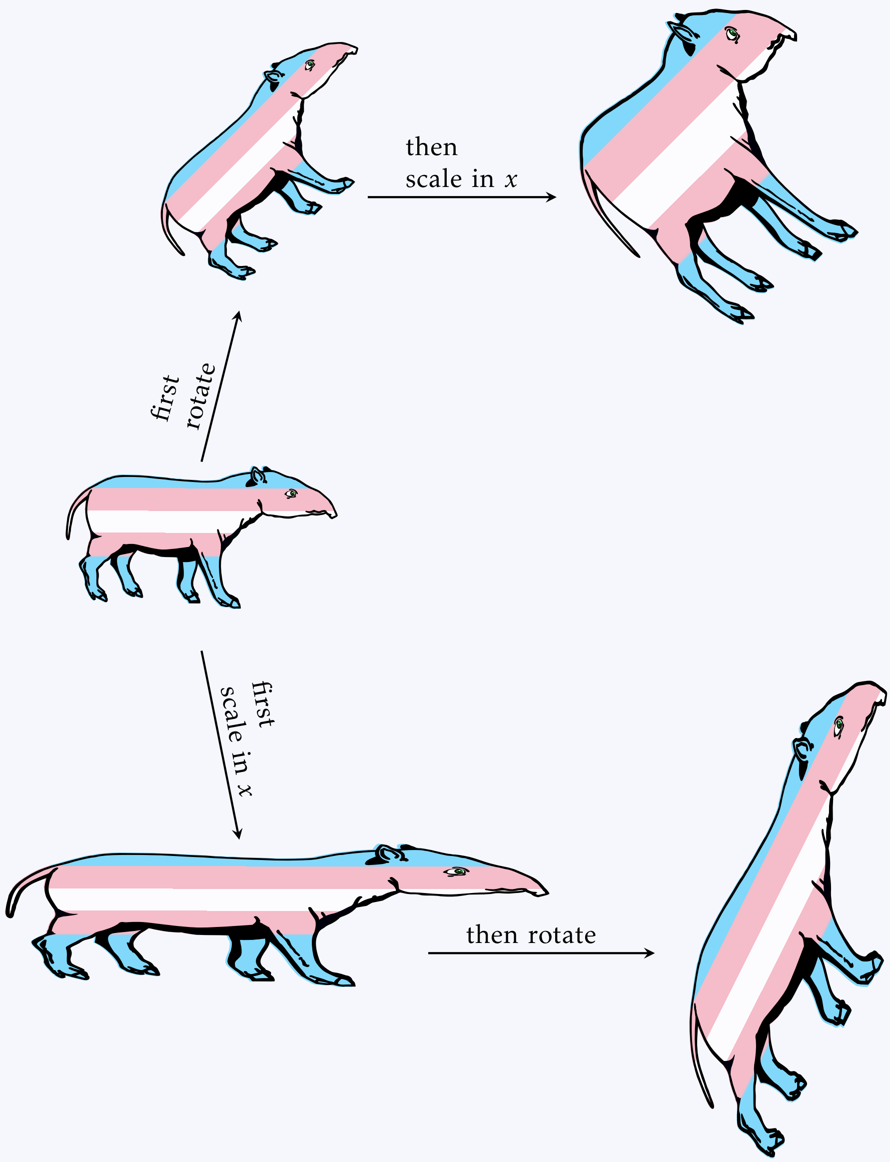
Figure 2.20 The basic linear transformations, exemplified using a very happy tapir.

Example 2.21 Composing basic linear transformations

Given the following two linear transformations:

1. Scale by 1.5 in the x-direction,
2. Rotate by $\frac{\pi}{4}$ anti-clockwise around the origin,

two composite linear transformations can be created: first scale then rotate, and first rotate then scale. As can be seen in the figure bellow, changing the order of composition results in a different linear transformations all together:



This is not a suprising result: in Chapter 0 we learned that function composition is not a commutative operation.

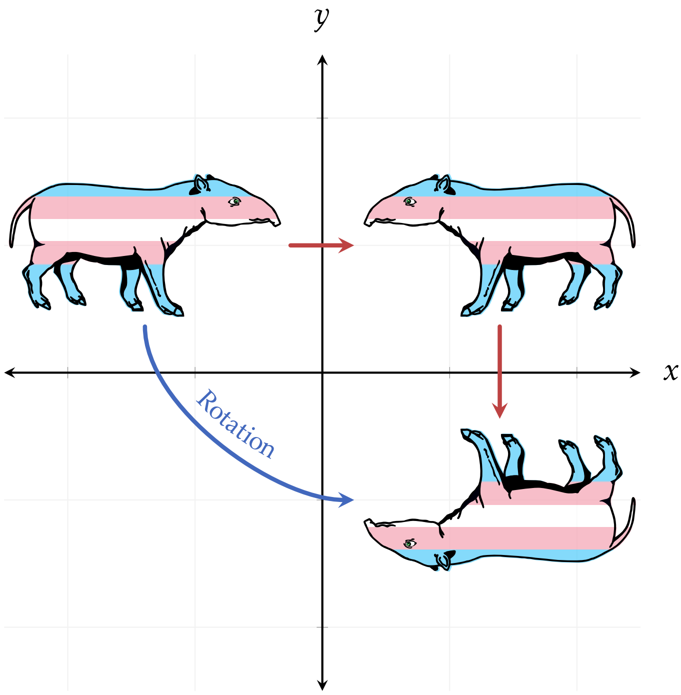


Figure 2.21 Reflection across the y-axis (red arrow) followed by a reflection across the x-axis (red arrow) results in a rotation by 180° around the origin (blue arrow).

Some of the basic linear transformations can be created as compositions of other basic linear transformations. For example, the composition of reflection across the y-axis followed by reflection across the x-axis results in a 180° rotation around the origin (see Figure 2.21).

2.2.3 3D Linear transformations

The basic linear transformations in $\mathbb{R}^3$ are very much similar to those in $\mathbb{R}^2$ with some small differences worth mentioning. For a start, scaling and skewing can be done in three different directions instead of just two directions (namely, x, y and z instead of just x and y). In addition, there are infinetly many axes of rotation: in $\mathbb{R}^2$ there is just a single axis (actually a point) of rotation - the origin. In $\mathbb{R}^3$ any line that goes through the origin can be an axis of rotation (see Figure 2.22). Lastly, there are three types of reflections: about the origin (a point), across a line going through the origin, and across a plane (see fig?).

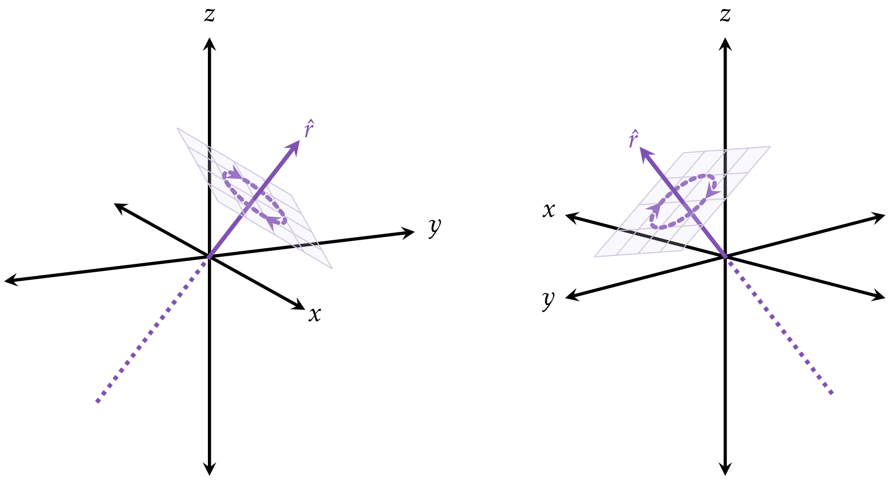


Figure 2.22 Rotation around a direction $\hat{r}$ from two different viewpoints, showing an example of a plane orthogonal to $\hat{r}$ on which the rotation happens.

2.3 MATRICES

In the previous section we described linear transformations in a rather abstract way: what they are, how they behave qualitatively and how they look like in 2- and 3-dimensions. In this section we introduce a numerical method of representing linear transformations: matrices.

2.3.1 Linear transformation of basis vectors

Recall that any vector $\vec{v} \in \mathbb{R}^n$ can be written as a linear combination of basis vectors $\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n$:

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \vec{b}_i = \alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \cdots + \alpha_n \vec{b}_n. \tag{2.3.1}$$

Applying a linear transformation T on $\vec{v}$ yields, using the properties of linear transformations,

$$\begin{aligned} T\left(\vec{v}\right) &= T\left(\alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \cdots + \alpha_n \vec{b}_n\right) \\ \text{additivity} \longrightarrow &= T\left(\alpha_1 \vec{b}_1\right) + T\left(\alpha_2 \vec{b}_2\right) + \cdots + T\left(\alpha_n \vec{b}_n\right) \\ \text{scalability} \longrightarrow &= \alpha_1 T\left(\vec{b}_1\right) + \alpha_2 T\left(\vec{b}_2\right) + \cdots + \alpha_n T\left(\vec{b}_n\right). \end{aligned} \tag{2.3.2}$$

This result is pretty neat: it means that by knowing how a linear transformation T changes the basis vectors, we know exactly how any vector is transformed by T. This true for any basis, and thus specifically to the standard basis, where the coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ are actually the components of the vector, i.e. $v_1, v_2, \dots, v_n$. Thus in the standard basis:

$$T\left(\vec{v}\right) = v_1 T\left(\hat{e}_1\right) + v_2 T\left(\hat{e}_2\right) + \cdots + v_n T\left(\hat{e}_n\right). \tag{2.3.3}$$

Example 2.22 Vector transformation via a basis

Applying the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined as

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y-2z \\ 2x+z \\ -x-y-z \end{bmatrix}$$

on the vector $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ yields the following vector:

$$T\left(\vec{v}\right) = T\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2+(-1)-2\cdot3 \\ 2\cdot2+3 \\ -2-(-1)-3 \end{bmatrix} = \begin{bmatrix} 2-1-6 \\ 4+3 \\ -2+1-3 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ -4 \end{bmatrix}.$$

Now, let us apply T first to the three standard basis vectors $\hat{x}, \hat{y}, \hat{z}$:

$$T\left(\hat{x}\right) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0-\cancel{2\cdot0} \\ 2\cdot1+0 \\ -1-0-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},$$

$$T\left(\hat{y}\right) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0+1-\cancel{2\cdot0} \\ \cancel{2\cdot0}+0 \\ -0-1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$T\left(\hat{z}\right) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0+0-2\cdot1 \\ \cancel{2\cdot0}+1 \\ -0-0-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

Taking these results and applying Equation 2.3.2 yields

$$\begin{aligned} T\left(\vec{v}\right) &= 2T\left(\hat{x}\right) - T\left(\hat{y}\right) + 3T\left(\hat{z}\right) \\ &= 2\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 2-1-6 \\ 4-0+3 \\ -2-(-1)+(-3) \end{bmatrix} \\ &= \begin{bmatrix} -5 \\ 7 \\ -4 \end{bmatrix}, \end{aligned}$$

which is indeed what we got when we applied T directly to $\vec{v}$.

2.3.2 From transformations to matrices

The most general linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ has the following form:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}, \tag{2.3.4}$$

where $a,b,c,d \in \mathbb{R}$. If we apply this transformation to $\hat{x}$ and $\hat{y}$ we get, respectively,

$$T\left(\hat{x}\right) = \begin{bmatrix} a \\ c \end{bmatrix}, \quad T\left(\hat{y}\right) = \begin{bmatrix} b \\ d \end{bmatrix}. \tag{2.3.5}$$

We can now collect these two vectors to form a new structure, which we call a **matrix** (in this specific casr a 2×2 matrix):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{2.3.6}$$

We then define the product of M with a vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ to yield $T\left(\vec{v}\right)$, i.e.

$$A\vec{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}. \tag{2.3.7}$$

This defintion can be re-written as following:

$$A\vec{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A_1 \cdot \vec{v} \\ A_2 \cdot \vec{v} \end{bmatrix}, \tag{2.3.8}$$

i.e. the i-th component of the resulting vector is the scalar product of the i-th **row** of the matrix with the vector $\vec{v}$.

Example 2.23 Matrix-vector product

Some matrix-vector products:

$$\begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot(-3)+(-2)\cdot 2 \\ \cancel{0\cdot(-3)}+5\cdot 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 10 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 1\cdot 5+2\cdot(-4) \\ 1\cdot 5+2\cdot(-4) \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + \cancel{0\cdot(-2)} \\ \cancel{0\cdot 2} + 2\cdot(-2) \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}.$$

Challenge 2.4 Proof of linearity

Prove that the transformation T in Equation 2.3.4 is indeed linear.

The most general form of a linear transformation is $T:\mathbb{R}^n \to \mathbb{R}^m$, i.e. a transformation which takes n-dimensional vectors as input and returns m-dimensional vectors as output:

$$\mathbb{R}^n \ni \qquad T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}, \qquad \in \mathbb{R}^m \tag{2.3.9}$$

where $a_{ij} \in \mathbb{R}$, $i = 1,2,3,\dots,m$ and $j = 1,2,3,\dots,n$.

Challenge 2.5 Proof of linearity

Prove that the above transformation T is indeed linear.

Respectively, we define an $m\times n$ matrix (m rows by n columns) by collecting all the coefficients a_{ij} into a single structure:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{2.3.10}$$

The product $M\vec{v}$ (where $\vec{v} \in \mathbb{R}^n$) is then defined as

$$A\vec{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}. \tag{2.3.11}$$

Again, note that the i-th component of the resulting vector is the scalar product $A_i \cdot \vec{v}$.

Note 2.7 When is a matrix-vector product defined

In order for a matrix-vector product to be defined, the vector must be of the same dimension as the number of **columns** in the matrix - i.e. given an $a\times b$ matrix, a vector must be b-dimensional for the product to be defined.

Example 2.24 Some matrix-vector products

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Etiam lobortis facilisis sem. Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Etiam lobortis facilisis sem. Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper.

The structure of an $m \times n$ matrix A has a nice property: given that the transformation in represented in some basis $B = \left\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\right\}$, the i-th column of the matrix always shows how $\vec{b}_i$ is transformed by the product $A\vec{b}_n$. This is easy to see in the case of the standard basis, which we anyway use througout this chapter:

$$\begin{array}{cc} & \begin{matrix} T(\hat{e}_1) & T(\hat{e}_2) & & T(\hat{e}_n) \\ \downarrow & \downarrow & & \downarrow \end{matrix} \\ A = & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \end{array}$$

Example 2.25 Matrices

The product of the following matrix A with each of the vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ (i.e. $\hat{x}, \hat{y}$ and $\hat{z}$, respectively) returns the respective column of the matrix:

$$A\hat{e}_1 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + \cancel{2 \cdot 0} + \cancel{0 \cdot 0} \\ -1 \cdot 1 + \cancel{3 \cdot 0} + \cancel{4 \cdot 0} \\ \cancel{0 \cdot 1} + \cancel{1 \cdot 0} + \cancel{3 \cdot 0} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

$$A\hat{e}_2 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{1 \cdot 0} + 2 \cdot 1 + \cancel{0 \cdot 0} \\ -\cancel{1 \cdot 0} + 3 \cdot 1 + \cancel{4 \cdot 0} \\ \cancel{0 \cdot 0} + 1 \cdot 1 + \cancel{3 \cdot 0} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix},$$

$$A\hat{e}_3 = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cancel{1 \cdot 0} + \cancel{2 \cdot 0} + \cancel{0 \cdot 1} \\ -\cancel{1 \cdot 0} + \cancel{3 \cdot 0} + 4 \cdot 1 \\ \cancel{0 \cdot 0} + \cancel{1 \cdot 0} + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}.$$

2.3.3 Matrix representation of the basic linear transformations (2D)

We can now represent all of the basic linear transformations in $\mathbb{R}^2$ mentioned in the previous section (Figure 2.20) as 2×2 matrices. We do this by observing how the basis vectors $\hat{x}$ and $\hat{y}$ change after the application of each transformation.

- **Identity**: both basis vectors remain the same: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Therefore the matrix I representing the identity transformation is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.3.12}$$

- **Scaling by s in the x-direction**: the basis vector $\hat{x}$ is streched by s: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} s \\ 0 \end{bmatrix}$. The basis vector $\hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ stays the same. Therefore the matrix S_x representing the transformation is

$$S_x = \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.3.13}$$

- **Scaling by s in the y-direction**: much like with S_x, now the basis vector $\hat{y}$ is the one getting streched, by β: $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ s \end{bmatrix}$. The basis vector $\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ stays the same. Therefore the matrix S_y representing the transformation is

$$S_y = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}. \tag{2.3.14}$$

- **Rotating by θ counter-clockwise about the origin**: Figure 2.23 shows how do $\hat{x}$ and $\hat{y}$ transformed by the rotation. In the case of $\hat{x}$, the resulting vector is $R_\theta(\hat{x}) = \left[\cos(\theta), \sin(\theta)\right]$, since thiese are the respective sides of a right triangle of hypotenous 1 and angle θ. The components of $R_\theta\left(\hat{y}\right)$ can be calculated by rotating $\hat{x}$ by $\theta + \frac{\pi}{2}$ $(\theta + 90°)$: $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin(\theta)$, and $\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta)$.

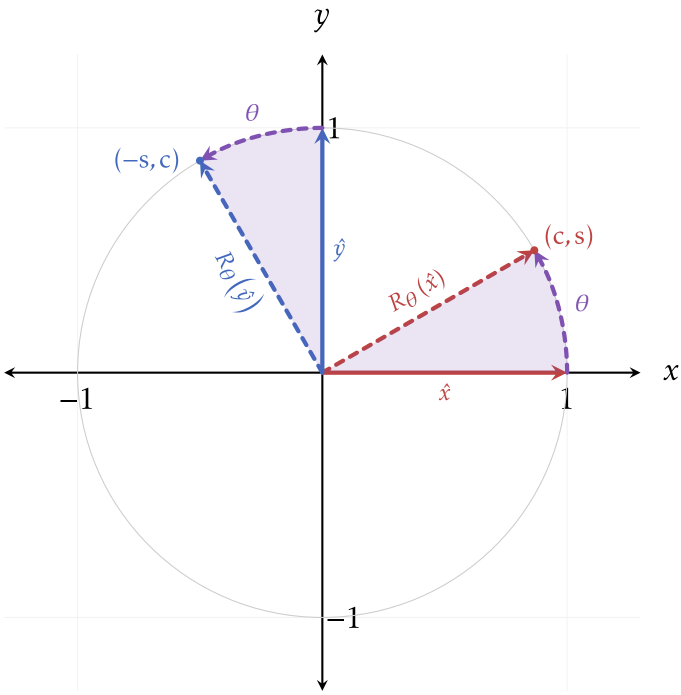


Figure 2.23 Rotation of $\hat{x}$ and $\hat{y}$ by an angle θ counter-clockwise about the origin. The notations c, s stand for $\cos(\theta)$ and $\sin(\theta)$, respectively.

Therefore we get

$$\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}. \tag{2.3.15}$$

Altogether the rotation matrix R_θ is

$$R_\theta = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}. \tag{2.3.16}$$

- **Skew by k in the x-direction**: what differentiates this transformation from scaling in the x-direction is that a skew changes only $\hat{y}$ by adding to it some horizontal displacement $\vec{K} = k\hat{x}$ (see Figure 2.24). Therefore $\hat{x}$ remains the same while $\hat{y}$ is transformed as $\hat{y} \to \hat{y} + \vec{k} = \hat{y} + k\hat{x} = \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}k\\0\end{bmatrix} = \begin{bmatrix}k\\1\end{bmatrix}$, and altogether the matrix is

$$K_x = \begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}. \tag{2.3.17}$$

- **Skew by k in the y-direction**: same idea, except the roles of the axes are revered:

$$\begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}1\\k\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\1\end{bmatrix}.$$

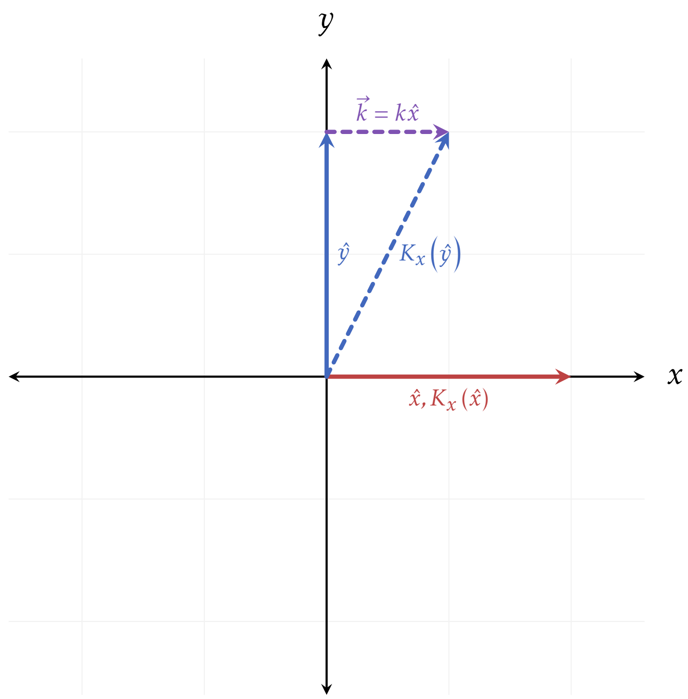


Figure 2.24 Skew in the x-direction.

Thus the matrix is

$$K_y = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}. \tag{2.3.18}$$

- **Reflections across a line going through the origin**: in the case of reflections across the x-axis, $\hat{x}$ stays the same, while $\hat{y}$ is flipped (see Figure 2.251), i.e. $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to -\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Therefore the matrix is

$$\mathrm{Ref}_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.3.19}$$

Similarily, a reflection across the y-axis flipps $\hat{x}$ while keeping $\hat{y}$ the same (see Figure 2.252), i.e.

$$\mathrm{Ref}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.3.20}$$

Another special case of these kinds of reflections is done across the line rotated by $\frac{\pi}{4} = 45°$ relative to the x-axis, i.e the line $y = x$. In this case $\hat{x}$ and $\hat{y}$ are swapped, giving

$$\mathrm{Ref}_{\frac{\pi}{4}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.3.21}$$

The most general reflection is made across a line of angle θ relative to the x-axis (see Figure 2.253):

$$\mathrm{Ref}_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}. \tag{2.3.22}$$

A way to calculate this matrix will be shown later in the chapter.

We can translate the matrix to be based on the slope m of the line instead of its angle θ relative to the x-axis by using the relation $m = \tan(\theta)$ and the two trigonomentric identities for double angles (Equation 0.6.18):

$$\begin{aligned} \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} &= \begin{bmatrix} \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)} & \frac{2\tan(\theta)}{1+\tan^2(\theta)} \\ \frac{2\tan(\theta)}{1+\tan^2(\theta)} & \frac{\tan^2(\theta)-1}{1+\tan^2(\theta)} \end{bmatrix} \\ &= \frac{1}{1+\tan^2(\theta)} \begin{bmatrix} 1-\tan^2(\theta) & 2\tan(\theta) \\ 2\tan(\theta) & \tan^2(\theta)-1 \end{bmatrix} \\ &= \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}. \end{aligned}$$

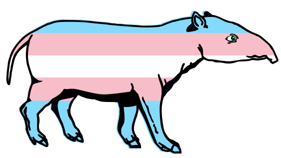
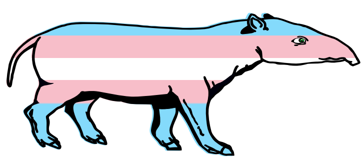
- **Reflection across the origin**: in this case both $\hat{x}$ and $\hat{y}$ are flipped, i.e.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

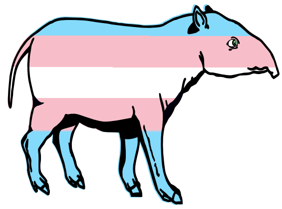
and the matrix is essentially a rotation by π (180°) around the origin:

$$R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.3.23}$$

Table **??** summarizes all the matrices of the basic linear transformations.

Transformation	Trans Tapir	$T(\hat{x})$	$T(\hat{y})$	Matrix
Identity		$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Scale in x		$\begin{bmatrix} s \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}$

Scale in y

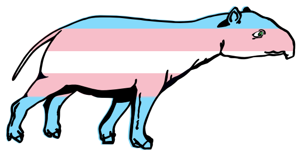


$$\begin{bmatrix}1\\0\end{bmatrix} \quad \begin{bmatrix}0\\s\end{bmatrix} \quad \begin{bmatrix}1 & 0\\0 & s\end{bmatrix}$$

Rotation

$$\begin{bmatrix}\mathrm{c}\\-\mathrm{s}\end{bmatrix} \quad \begin{bmatrix}\mathrm{s}\\\mathrm{c}\end{bmatrix} \quad \begin{bmatrix}\mathrm{c} & -\mathrm{s}\\\mathrm{s} & \mathrm{c}\end{bmatrix}$$

Skew in x

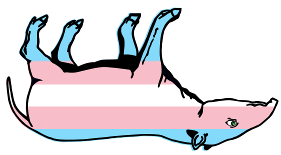


$$\begin{bmatrix}1\\0\end{bmatrix} \quad \begin{bmatrix}k\\1\end{bmatrix} \quad \begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$$

Skew in y

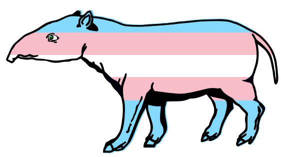
$$\begin{bmatrix}1\\k\end{bmatrix} \quad \begin{bmatrix}0\\1\end{bmatrix} \quad \begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$$

Reflection by x



$$\begin{bmatrix}1\\0\end{bmatrix} \quad \begin{bmatrix}0\\-1\end{bmatrix} \quad \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

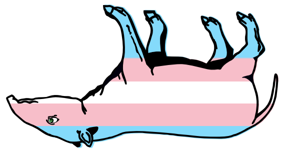
Reflection by y



$$\begin{bmatrix}-1\\0\end{bmatrix} \quad \begin{bmatrix}0\\1\end{bmatrix} \quad \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

Reflection by line

$$\begin{bmatrix}\mathrm{c}_2\\\mathrm{s}_2\end{bmatrix} \quad \begin{bmatrix}\mathrm{s}_2\\-\mathrm{c}_2\end{bmatrix} \quad \begin{bmatrix}\mathrm{c}_2 & \mathrm{s}_2\\\mathrm{s}_2 & -\mathrm{c}_2\end{bmatrix}$$

Reflection about origin		$\begin{bmatrix}-1\\0\end{bmatrix}$ $\begin{bmatrix}0\\-1\end{bmatrix}$ $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

2.3.4 Matrix representation of the basic linear transformations (3D)

In 3-dimensions, the respective matrices are very similar. For example, the matrix for scaling by α in the x-direction, β in the y-direction and γ in the z-direction is

$$S=\begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&\gamma\end{bmatrix}. \tag{2.3.24}$$

As mentioned in the previous section, in 3-dimensions there are infinitely many rotations: the axis of rotation can be any line going through the origin (i.e. any vector except $\vec{0}$ can represent an axis of rotation). Let us start with constructing rotations around the three axes x,y and z first. When rotating around the x axis it stays stationary, while the rotation itself is done in the yz-plane. This means that we can take the 2×2 rotation matrix (Equation 2.3.16) and expand it such that it affects only the yz-plane:

$\hat{x}$ doesn't change

$$R^{x}_{\theta}=\begin{bmatrix}\boxed{\begin{matrix}1\\0\\0\end{matrix}}&\begin{matrix}0&0\\\boxed{\begin{matrix}\cos(\theta)&-\sin(\theta)\\\sin(\theta)&\cos(\theta)\end{matrix}}\end{matrix}\end{bmatrix}. \tag{2.3.25}$$

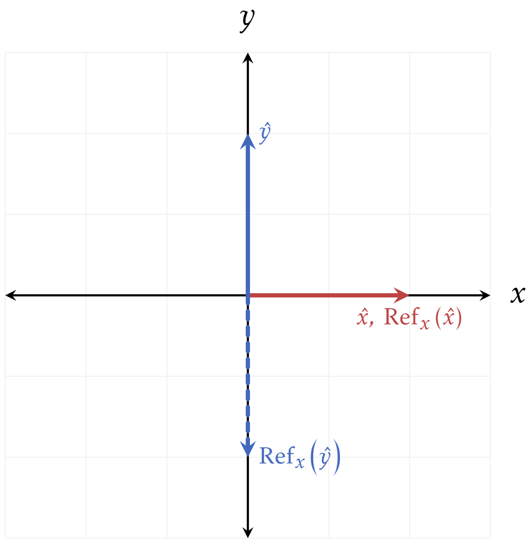
2D rotation matrix

A graphical representation of the rotation can be seen in Figure 2.26.

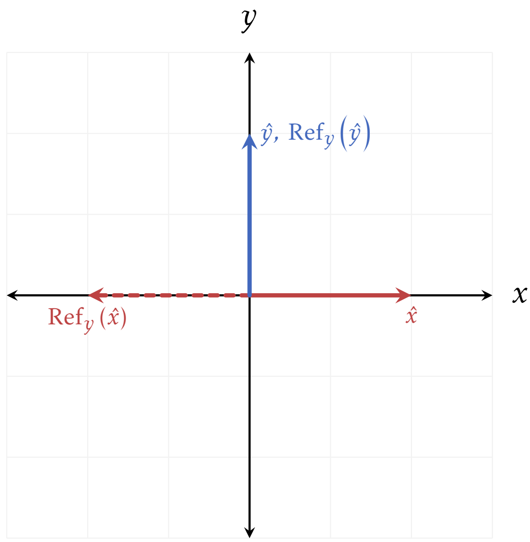
The rotation matrices around the y- and z-axes follow the same structure:

$$R^{y}_{\varphi}=\begin{bmatrix}\cos\left(\varphi\right)&0&\sin\left(\varphi\right)\\0&1&0\\-\sin\left(\varphi\right)&0&\cos\left(\varphi\right)\end{bmatrix}, \tag{2.3.26}$$

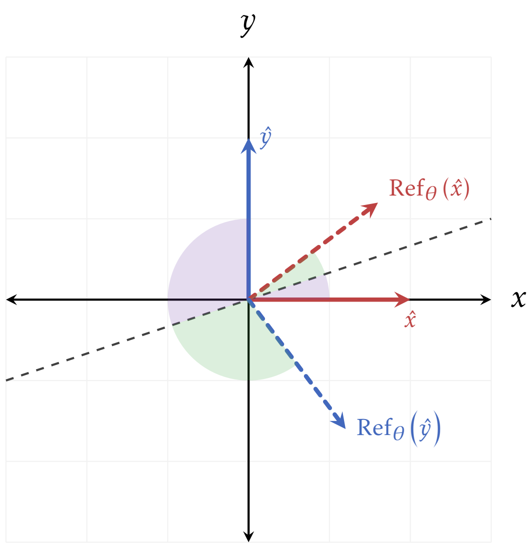
$$R^{z}_{\psi}=\begin{bmatrix}\cos\left(\psi\right)&-\sin\left(\psi\right)&0\\\sin\left(\psi\right)&\cos\left(\psi\right)&0\\0&0&1\end{bmatrix}. \tag{2.3.27}$$



1 Reflection across the x-axis.



2 Reflection across the y-axis.



3 Reflection across a line going through the origin. Notice how in both cases the purple and green angles are the same: this shows that both $\hat{x}$ and $\hat{y}$ are reflected across the line.

Figure 2.25 Reflections across different lines going through the origin.

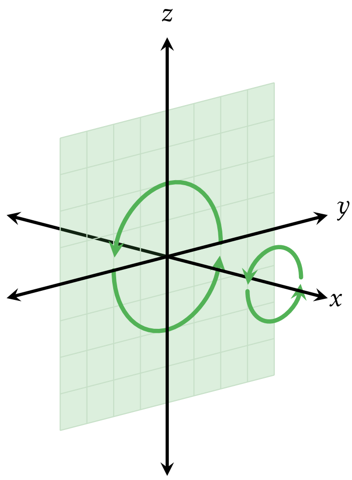


Figure 2.26 In $\mathbb{R}^3$, rotation around the x-axis is a rotation in the yz-plane (i.e. $x = 0$).

Note 2.8 Direction of the y-axis

The signs of $\sin(\varphi)$ in R^y_φ are flipped compared to R^x_θ and R^z_ψ, for the same reason a similar thing happens in the y-component of the cross product: it is due to the use of a right-handed system.

!

The most general rotation in $\mathbb{R}^3$, i.e. around an axis represented by the unit vector $\hat{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$ counter-clockwise by an angle θ, is given in matrix form as

$$R_\theta = \begin{bmatrix} \cos(\theta) + u_x^2\left[1-\cos(\theta)\right] & u_x u_y\left[1-\cos(\theta)\right] - u_z\sin(\theta) & u_x u_z\left[1-\cos(\theta)\right] + u_y\sin(\theta) \\ u_y u_x\left[1-\cos(\theta)\right] + u_z\sin(\theta) & \cos(\theta) + u_y^2\left[1-\cos(\theta)\right] & u_y u_z\left[1-\cos(\theta)\right] - u_x\sin(\theta) \\ u_z u_x\left[1-\cos(\theta)\right] - u_y\sin(\theta) & u_z u_y\left[1-\cos(\theta)\right] + u_x\sin(\theta) & \cos(\theta) + u_z^2\left[1-\cos(\theta)\right] \end{bmatrix}. \tag{2.3.28}$$

For the moment the derivation of this matrix is not presented.

! **To be written**: REFLECTIONS IN 3D. !

2.3.5 Matrix operations

An important operation that can be performed on a matrix is the **transpose**: this operation ”rotates” all rows of the matrix to columns, and wise-versa:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \xrightarrow{\text{transpose}} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}. \tag{2.3.29}$$

Mathematically, the transpose takes any element a_{ij} of the matrix and exchanges its indeces, yielding a_{ji}. If the original matrix has dimensions $m \times n$, then the transposed matrix has dimensions $n \times m$. The notation for the transpose of a matrix A is $A^{\top}$.

Example 2.26 Transposing matrices

The following presents three matrices each with its transpose. The elements in each matrix on the left hand side are highlighted column-wise, and these colors remain with the elements after the transpose. That way, the effect of the transpose is clear: columns in the original matrix become rows in the transposed matrix and vice-versa. In addition, the dimensions of each matrix are written below it.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^{\top}}_{3 \times 3} = \underbrace{\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}_{3 \times 3}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 5 \end{bmatrix}^{\top}}_{2 \times 3} = \underbrace{\begin{bmatrix} 0 & 2 \\ 1 & -3 \\ -1 & 5 \end{bmatrix}}_{3 \times 2}$$

$$\underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 7 \\ -4 \end{bmatrix}^{\top}}_{6 \times 1} = \underbrace{\begin{bmatrix} 1 & 2 & -1 & 0 & 7 & -4 \end{bmatrix}}_{1 \times 6}$$

Since for the main diagonal elements of a matrix the row and column have equal indeces, the transpose operation does not affect their position in the matrix, i.e. $a_{ii} \xrightarrow{\text{transpose}} a_{ii}$. This means that $\operatorname{tr} A = \operatorname{tr} A^{\top}$. Also, diagonal matrices are not affected by a transpose. The transpose of a transposed matrix is the original matrix, i.e. $\left(A^{\top}\right)^{\top} = A$.

Much like vectors, a matrix can be scaled by a real number, and two matrices can be added together if their dimensions are identical. The rules for scaling a matrix by a scalar and the addition of two matrices are the same as with vectors, namely everything is dome element wise:

- **Scaling**: given a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and a scalar $\gamma \in \mathbb{R}$, their product is

$$\gamma A = \begin{bmatrix} \gamma \cdot a_{11} & \gamma \cdot a_{12} & \cdots & \gamma \cdot a_{1n} \\ \gamma \cdot a_{21} & \gamma \cdot a_{22} & \cdots & \gamma \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma \cdot a_{m1} & \gamma \cdot a_{m2} & \cdots & \gamma \cdot a_{mn} \end{bmatrix}. \tag{2.3.30}$$

- **Addition**: given two matrices,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

their sum is

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}. \tag{2.3.31}$$

Note 2.9 Matrix addition

Since matrix addition is done **element wise** it is comutative, i.e. for any two $m \times n$ matrices A and B,

$$A + B = B + A.$$

!

2.3.6 Types of matrices

Any matrix A which represents a transformation of the type $\mathbb{R}^n \to \mathbb{R}^n$ (i.e. from a space onto itself) has the same number of rows and columns (i.e. its dimension is $n \times n$):

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}. \tag{2.3.32}$$

Due to their shape, such matrices are called **square matrices**. The elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ of a square matrix jointly form its **main diagonal** (also: **principal diagonal**):

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}. \tag{2.3.33}$$

The sum of the main diagonal elements is called the **trace** of the matrix:

$$\mathrm{tr}\,(A) = \sum_{i=1}^{n} a_{ii}. \tag{2.3.34}$$

Triangular matrices are matrices in which the elements above or below the main diagonal are all zeros, e.g.

$$U = \begin{bmatrix} 1 & 6 & 6 & -3 \\ 0 & 2 & 7 & 1 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 5 & 1 & -5 & 0 \\ -4 & 1 & 2 & -3 \end{bmatrix}.$$

upper triangular lower triangular

A somewhat formal way of defining the elements ”above” the main diagonal is all elements a_{ij} for which $j < i$. Similarily, when $j > i$ the element a_{ij} is ”below” the main diagonal. Note that the transpose of an upper triangular matrix is a lower triangular matrix and vice-versa.

Challenge 2.6 Upper/lower triangular matrices

Show that if A is an upper triangular matrix then $A^{\top}$ is a lower triangular matrix, and if B is a lower triangular matrix then $B^{\top}$ is an upper triangular matrix.

?

A **diagonal matrix** A is a matrix in which all the non-main diagonal elements, i.e. a_{ij} where $i \neq j$, equal zero. These matrices can be thought of as scaling matrices: each entry a_{ii} tells us how the sapce is scaled in the i-th dimension.

Example 2.27 Diagonal matrices

Text.

As we saw in the cases of $\mathbb{R}^2$ and $\mathbb{R}^3$, diagonal matrices are **scaling matrices**: each entry a_{ii} tells us by how much space is scaled in the i-th direction.

A very important family of **square** matrices are the **identity matrices**. These matrices have a unique structure: their main diagonal elements are all 1, while the rest of the elements (the **off-diagonal elements**) are all 0:

$$I_n = \begin{array}{c} \overbrace{\hphantom{\begin{matrix}1 & 0 & 0 & \cdots & 0\end{matrix}}}^{n \text{ columns}} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \end{array} \left.\vphantom{\begin{matrix}1\\0\\0\\\vdots\\0\end{matrix}}\right\} n \text{ rows} \tag{2.3.35}$$

Sometimes for clarity large areas of zero-value elements in a matrix are depicted together. In that form, the identity matrix is written as

$$I_n = \begin{bmatrix} 1 & & & \text{\huge 0} \\ & 1 & & \\ & & 1 & \\ \text{\huge 0} & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

In such a depiction, the off-diagonal elements are each written using a single zero. This kind of notation will come in handy in later sections. Yet another way of defining the identity matrix is by using the **Kronecker delta**, which takes two integers i, j and returns 1 if they are equal, otherwise it returns 0:

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \tag{2.3.36}$$

Using the Kronecker delta, each element a_{ij} of the identity matrix I_n simply equals δ_{ij}.

An identity matrix of dimension n represents the identity transformation in $\mathbb{R}^n$: each standard basis vector $\hat{e}_i$ is left unchanged by the transformation.

Example 2.28 Identity matrices

The following are the identity matrices of $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^6$, where in each matrix the main diagonal is highlighted:

$$\underset{I_2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \quad \underset{I_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{I_4}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \quad \underset{I_5}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}} \quad \underset{I_6}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}$$

In the next section we will see the importance of the identity matrices.

Another important family of matrices are the **orthogonal matrices** (also **orthonormal matrices**): we say that a matrix Q is an orthogonal matrix if all of its columns, when viewed as column vectors, form an orthonormal set. For example, the identity matrices are all orthogonal matrices. Another orthogonal matrix is the matrix

$$B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{2.3.37}$$

since both $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are unit vectors, and they are orthogonal to each other (as seen in REF).

A **symmetric matrix** is a square matrix for which

$$A^\top = A. \tag{2.3.38}$$

"Graphically", the symmetry of such matrices can be seen in respect to their main diagonal: if we imagine placing a mirror on the main diagonal, each element a_{ij} would be "reflected" across the mirror, and thus be equal to a_{ji} (see example below).

Example 2.29 Symmetric matrix

The following matrix S is a symmetric 4×4 matrix, in which the elements a_{ij}, a_{ji} are higlighted with the same color:

$$S = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 0 & 1 & 3 \\ 5 & 1 & 4 & 2 \\ 7 & 3 & 2 & 6 \end{bmatrix}$$

Note 2.10 Transpose of a symmetric matrix

A symmetric matrix is its own transpose, i.e. if A is a symmetric matrix then $A^\top = A$.

A rather non-interesting family of matrices are the **zero matrices**: these are matrices which have only zero-elements, i.e.

$$\mathbf{0}_n = \overbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}^{n \text{ columns}} \left.\vphantom{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}\right\} m \text{ rows}. \tag{2.3.39}$$

The zero matrices are called that way since for a given matrix A,

$$A + \mathbf{0} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} a_{11}+0 & a_{12}+0 & \cdots & a_{1n}+0 \\ a_{21}+0 & a_{22}+0 & \cdots & a_{2n}+0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+0 & a_{m2}+0 & \cdots & a_{mn}+0 \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = A.
\end{aligned} \tag{2.3.40}
$$

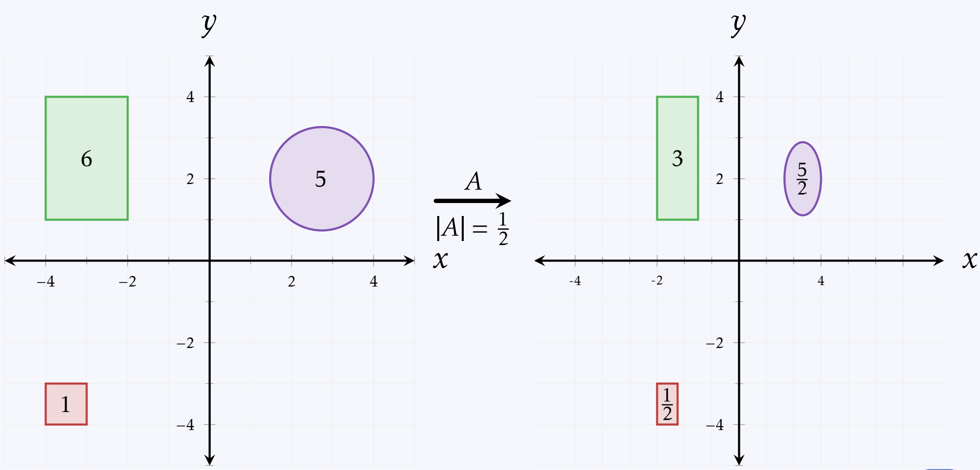
I.e. much like the number zero and the zero vector, the zero matrix is neutral in respect to addition.

2.3.7 The determinant

As mentioned in subsection 2.2.2, linear transformation scale all volumes by the same amount[8]. This scaling factor is encapsulated in the matrix representing the transformation by a number called the **determinant** of the matrix. The determinant of a matrix A is written as $|A|$ and sometimes $\det(A)$.

Example 2.30 The determinant as a scaling factor

In the following transformation, represented by the matrix $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, areas are scaled by a factor of $\frac{1}{2}$ and therefore $|A| = \frac{1}{2}$ (the number inside each shapes is its area):



Since there is not much sense in discussing volume changes between different spaces (e.g. $\mathbb{R}^5 \to \mathbb{R}^7$), only square matrices, which as you recall represent linear

[8]remember that 2-dimensional volumes are areas.

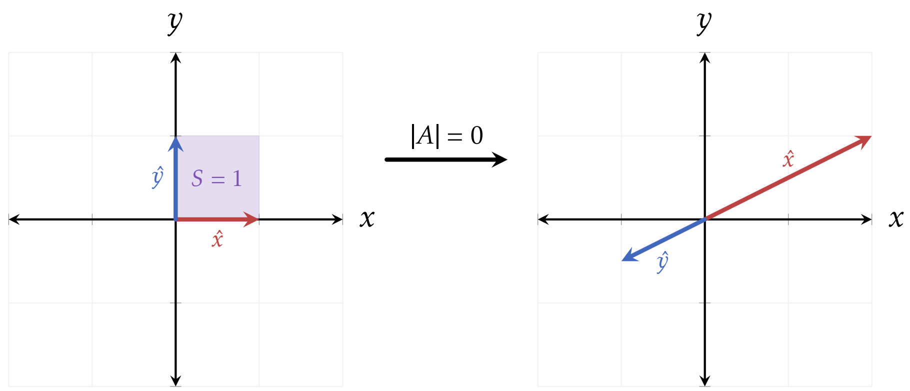


Figure 2.27 A transformation which "squashes" all areas into a line is represented by a matrix A with $|A| = 0$. Note how the unit volume defined by $\hat{x}$ and $\hat{y}$ is transformed into a shape of zero area: a line going through the origin. Also note that $\hat{x}$ and $\hat{y}$ are linearly dependent, since they lie on the same line. Cf. Figure 2.28.

transformations from a space onto itself, have determinants. Determinants can take any real number as values, including zero and negative numbers.

What does a zero determinant mean? Since in $\mathbb{R}^2$ determinants tells us the scaling factor of areas by the transformation, if the matrix representing the linear transformation has a zero determinant, it means that somehow all areas are "squashed" by the transformation to zero. There are two possible relevant shapes of zero area: a line going through the origin, or the origin itself which is a point. See Figure 2.27 for a visualization.

Similarly, in $\mathbb{R}^3$ the determinant tells us how volumes are scaled by a linear transformation, and thus a 3×3 matrix with zero determinant means that all the transformation represented by the matrix "squashes" all volumes to one of three relevant shapes with zero volume: a plane going through the origin, a line going through the origin, or the origin point itself. See **??** for a visualization.

DISCUSSION OF NEGATIVE DETERMINANTS...

To calculate the determinant of a matrix, we start with the simplest case: 2×2 matrices. Since all areas are equaly scaled by a linear transformation, we look at the unit sqaure defined by $\hat{x}$ and $\hat{y}$ (see Figure 2.28). After the application of the transformation represented by the generic matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ (where $a, b, c, d \in \mathbb{R}$), these basis vectors are transformed into the vectors

$$A\hat{x} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } A\hat{y} = \begin{bmatrix} c \\ d \end{bmatrix}, \tag{2.3.41}$$

respectively.

The unit square defined by $\hat{x}$ and $\hat{y}$ is therefore transformed into the parallelogram defined by $A\hat{x}$ and $A\hat{y}$. Equation 2.1.44 tells us that the area of the parallelogram is $S = ad - bc$. Therefore, the determinant - which equals the change in area after

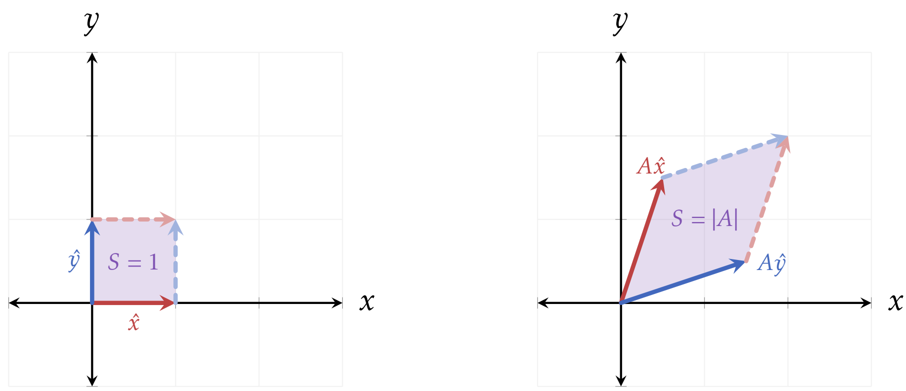


Figure 2.28 Unit area defined by the vectors $\hat{x}$ and $\hat{y}$ before application of a linear transformation represented by the matrix A (left) and the parallelogram defined by the vectors $A\hat{x}$ and $A\hat{y}$ after application of the transformation (right).

application of A, is

$$|A| = ad - bc \tag{2.3.42}$$

as well.

Example 2.31 Determinants of 2×2 matrices

Some 2×2 matrices and their determinants:

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \longrightarrow 1 \cdot 3 - (-2) \cdot 0 = 3.$$

$$\begin{bmatrix} 1 & 5 \\ 1 & 3 \end{bmatrix} \longrightarrow 1 \cdot 3 - 1 \cdot 5 = -2.$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \longrightarrow 1 \cdot 2 - 1 \cdot 2 = 0.$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \longrightarrow 1 \cdot 4 - 2 \cdot 2 = 0.$$

$$\begin{bmatrix} 0 & 7 \\ 0 & -3 \end{bmatrix} \longrightarrow \cancel{0 \cdot (-3)} - \cancel{0 \cdot 7} = 0.$$

Calculating the determinant of a 3×3 matrix is based on the calculation of the determinant of a 2×2 matrix. First, we should define an idea called a **minor** of a matrix. The ij-minor of a 3×3 matrix A is the determinant of the 2×2 matrix resulting by the removal of the i-th row and j-th column of A, e.g. let

$$A = \begin{bmatrix} 2 & -5 & 4 \\ -3 & 0 & 2 \\ 3 & 3 & 2 \end{bmatrix},$$

then Table 2.2 shows all the minors of A.

Using the its minors, the determinant of a 3×3 matrix can be calculated using the following formula:

$$|A| = a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13}, \tag{2.3.43}$$

where a_{ij} and m_{ij} are the elements and minors of the matrix, respectively. For example, using the above matrix A, we get that

$$\begin{aligned}
|A| &= a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13} \\
&= 2\begin{vmatrix} 0 & 2 \\ 3 & 2 \end{vmatrix} - (-5)\begin{vmatrix} -3 & 2 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} -3 & 0 \\ 3 & 3 \end{vmatrix} \\
&= 2 \cdot (-6) - (-5) \cdot (-12) + 4 \cdot (-9) \\
&= -12 - 60 - 36 = -108.
\end{aligned}$$

Example 2.32 Determinants of 3×3 matrices

...

The determinant of a 4×4 matrix follows the same pattern, i.e.

$$|A| = a_{11}m_{11} - a_{12}m_{12} + a_{13}m_{13} - a_{14}m_{14}, \tag{2.3.44}$$

where again a_{ij} and m_{ij} are, respectively, the elements and minors of a matrix. Much like with the case of a minor of a 3×3 matrix being a determinant of a 2×2 matrix, the minor of a 4×4 matrix is a determinant of a 3×3 matrix, itself calculated using determinants of 2×2 matrices. This pattern continues to higher dimensions, i.e. the calculation of the determinant of a 5×5 matrix uses determinants of 5 4×4 matrices, the calculation of the determinant of a 6×6 matrix uses 6 determinants of 5×5 matrices, and so forth. Therefore, the total number of 2×2 determinants needed for the calculation of the determinant of an $n \times n$ matrix is

$$d = n \times (n-1) \times (n-2) \times \cdots \times 5 \times 4 \times 3 = \frac{n!}{2}. \tag{2.3.45}$$

Some properties of determinants:

- In any case where the columns of a matrix form a linearly dependent set, the determinant is zero. This is due to the loss of dimensionality (i.e. at least one basis vector is mapped to a vector which can be written as a linear combination of the other vectors). One obvious case is where there is one or more columns of zeros in the matrix.
- The determinant of the transpose of a matrix is the same as the determinant of the original matrix, i.e. $|A| = |A^\top|$. This is due to the fact that all ideas discussed here can be applied directly to row vectors (as mentioned in the previous sections), and the transpose operation essentially switches between these forms: the columns of the original matrix become the rows in its transposed format. Therefore, the previous property applies to the rows of a matrix as well: e.g. a row of zeros means that the determinant is zero.

Table 2.2 All the minors of the matrix A.

i	j	3×3-matrix	2×2 determinant	value
1	1	$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 0 & 2 \\ \blacksquare & 3 & 2 \end{bmatrix}$	$\begin{vmatrix} 0 & 2 \\ 3 & 2 \end{vmatrix}$	-6
1	2	$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ -3 & \blacksquare & 2 \\ 3 & \blacksquare & 2 \end{bmatrix}$	$\begin{vmatrix} -3 & 2 \\ 3 & 2 \end{vmatrix}$	-12
1	3	$\begin{bmatrix} \blacksquare & \blacksquare & \blacksquare \\ -3 & 0 & \blacksquare \\ 3 & 3 & \blacksquare \end{bmatrix}$	$\begin{vmatrix} -3 & 0 \\ 3 & 3 \end{vmatrix}$	-9
2	1	$\begin{bmatrix} \blacksquare & -5 & 4 \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & 3 & 2 \end{bmatrix}$	$\begin{vmatrix} -5 & 4 \\ 3 & 2 \end{vmatrix}$	-22
2	2	$\begin{bmatrix} 2 & \blacksquare & 4 \\ \blacksquare & \blacksquare & \blacksquare \\ 3 & \blacksquare & 2 \end{bmatrix}$	$\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$	-8
2	3	$\begin{bmatrix} 2 & -5 & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ 3 & 3 & \blacksquare \end{bmatrix}$	$\begin{vmatrix} 2 & -5 \\ 3 & 3 \end{vmatrix}$	21
3	1	$\begin{bmatrix} \blacksquare & -5 & 4 \\ \blacksquare & 0 & 2 \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$	$\begin{vmatrix} -5 & 4 \\ 0 & 2 \end{vmatrix}$	-10
3	2	$\begin{bmatrix} 2 & \blacksquare & 4 \\ -3 & \blacksquare & 2 \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$	$\begin{vmatrix} 2 & 4 \\ -3 & 2 \end{vmatrix}$	16
3	3	$\begin{bmatrix} 2 & -5 & \blacksquare \\ -3 & 0 & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$	$\begin{vmatrix} 2 & -5 \\ -3 & 0 \end{vmatrix}$	-15

2.3.8 Matrix-vector products

As discussed in the first part of this section, matrices represent linear transformations - in fact, we define a matrix in such a way that its product with a vector gives the result of applying the transformation the matrix represents on the vector (see Equation 2.3.11). Let us now review this idea and elaborate a bit on the process of calculating matrix-vector products.

Given an $m \times n$ matrix A and an n-dimensional vector $\vec{v}$, the product $A\vec{v}$ is an m-dimensional vector, in which each element v_i is the scalar product between the i-th **row** of A (interpreted as a vector) and the vector $\vec{v}$ itself. To illustrate this, we use the following 4×3 matrix A and 3-dimensional vector $\vec{v}$:

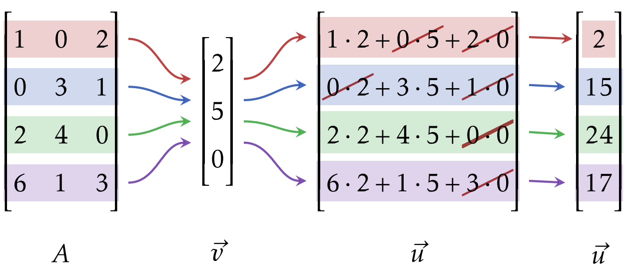
$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & -1 \\ 2 & 4 & 0 \\ 6 & 1 & -3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

(note that the number of **columns** in A is the same as the number of elements of $\vec{v}$, namely 3).

The resulting vector $\vec{u}$ is then given by the following formula:

$$u_i = \underset{\text{i-th row of A}}{A^i} \cdot \vec{v}.$$

In the following illustration, each row of A is scalar multiplied with the vector $\vec{v}$, yielding the respective element of $\vec{u}$. The respective rows of A and elements of $\vec{v}$ are color-coded for clarity.



Example 2.33 Matrix-vector products

The following are some examples of matrix-vector products. Note how in each product the number of columns in the matrix is the same as the number of elements of the vector.

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + \cancel{0 \cdot (-7)} + 2 \cdot 4 \\ 3 \cdot 2 - 1 \cdot (-7) + 5 \cdot 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 33 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 4 & -5 \\ 4 & 6 & -2 \\ -2 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} \cancel{0 \cdot 0} + 4 \cdot (-3) - 5 \cdot (-2) \\ \cancel{4 \cdot 0} + 6 \cdot (-3) - 2 \cdot (-2) \\ -\cancel{2 \cdot 0} + 2 \cdot (-3) + \cancel{0 \cdot (-2)} \end{bmatrix} = \begin{bmatrix} -2 \\ -14 \\ -6 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\cancel{2 \cdot 0} - 1 \cdot (-1) \\ -\cancel{2 \cdot 0} + 1 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\begin{bmatrix} 6 & -1 \\ 5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \cdot 2 - 1 \cdot 2 \\ 5 \cdot 2 + \cancel{0 \cdot 2} \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & -1 & -2 & -2 \\ 2 & 0 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \cancel{2 \cdot 0} - 1 \cdot (-2) - 2 \cdot 2 - 2 \cdot 4 \\ \cancel{2 \cdot 0} + \cancel{0 \cdot (-2)} + \cancel{0 \cdot 2} + 5 \cdot 4 \end{bmatrix} = \begin{bmatrix} -10 \\ 20 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 4 & -1 & 1 \\ -2 & 5 & 6 & -2 \\ 4 & 5 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 3 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 6 + 4 \cdot 3 - \cancel{1 \cdot 0} + 1 \cdot 4 \\ -2 \cdot 6 + 5 \cdot 3 + \cancel{6 \cdot 0} - 2 \cdot 4 \\ 4 \cdot 6 + 5 \cdot 3 - \cancel{1 \cdot 0} + 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 34 \\ -5 \\ 51 \end{bmatrix}.$$

2.3.9 Matrix-matrix products

Since the product of a matrix and a vector is itself a vector, one can take the resulting vector and multiply it by another matrix, i.e. given the matrices A, B and a vector $\vec{v}$, the expression

$$B \cdot \left(A \cdot \vec{v}\right)$$

is a vector as well.

Of course, the dimensions of all participating objects must align for the products to be properly defined: if A is an $m \times n$ matrix, then $\vec{v}$ must be an n-dimensional vector. The result of the product $A \cdot \vec{v}$ is then an m-dimensional vector which we can call $\vec{u}$. Thus, for the product $B \cdot \vec{u}$ to be properly defined, B must have the same number of columns as $\vec{u}$ has elements, namely m columns. The number of rows is free, and can be any natural number k. Therefore, B is a $k \times n$ matrix, and the product $B \cdot \vec{u}$ is a k-dimensional vector.

Example 2.34 Multiple matrix-vector products

Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 5 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2 \\ 6 & -7 \\ -1 & 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix},$$

then

$$\vec{u} = A \cdot \vec{v} = \begin{bmatrix} 2 \cdot 1 - 1 \cdot 2 + \cancel{0 \cdot 5} \\ 5 \cdot 1 + 4 \cdot 2 + 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 28 \end{bmatrix}.$$

The product $B \cdot \vec{u}$ is defined, since $\vec{u}$ has the same number of elements as B has columns (namely 2). Its result is the 3-dimensional vector

$$\vec{w} = B \cdot \vec{u} = \begin{bmatrix} \cancel{5 \cdot 0} + 2 \cdot 28 \\ \cancel{6 \cdot 0} - 7 \cdot 28 \\ \cancel{-1 \cdot 0} + \cancel{0 \cdot 28} \end{bmatrix} = \begin{bmatrix} 56 \\ -196 \\ 0 \end{bmatrix}.$$

Multiple matrix-vector product therefore represent application of multiple linear transformations on an initial vector, in the order the matrix-vector products are performed. For example, consider the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Rotating $\vec{v}$ by $\frac{\pi}{2}$ $(= 90°)$ counter-clockwise around the origin and then scaling the result by 2 should yield the vector $\vec{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{rotation by } \frac{\pi}{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \xrightarrow{\text{scaling by } 2} \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

Using the respective matrix representation of each transformation, we get the following:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

(note that the matrix-vector products are performed from right to left)

More generally, there could be several products made successionally, i.e.

$$A_n \left(A_{n-1} \left(A_{n-2} \cdots \left(A_2 \left(A_1 \vec{v} \right) \right) \right) \right).$$

Matrix-vector products are most commonly written without the paranthesis nor the dot symbol, i.e. as

$$A_n A_{n-1} A_{n-2} \cdots A_2 A_1 \vec{v},$$

and the order of multiplication is from left to right, i.e. A_1 is the first matrix to be multiplied by $\vec{v}$, then A_2 is multiplied by the result of the product $A_1 \vec{v}$, then A_3 is multiplied by the result of $A_2 A_1 \vec{v}$ and so on.

At this point one should wonder whether instead of doing this long chain of products on each individual vector, perhaps the matrices themselves could be multiplied first, yielding a matrix representing the total transformation applied to a vector, as a

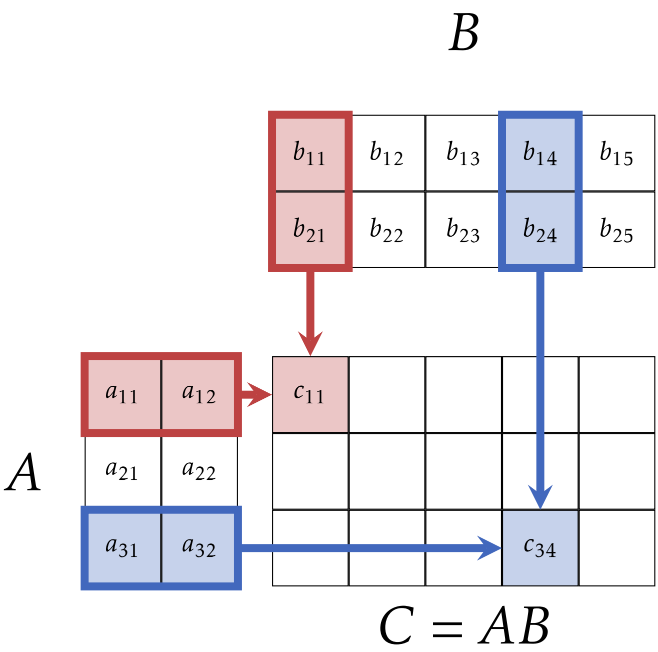


Figure 2.29 The element c_{ij} of the matrix $C = AB$ is the scalar product of the i-th row of A and the j-th column of B.

composition of the separate transformations in the correct order. The answer is of course yes![9]

Let us define the product of two matrices: given the an $m \times n$ matrix A and an $n \times k$ matrix B, the product $C = AB$ is itself a matrix, having the dimension $m \times k$, in which every element c_{ij} is the scalar product of the row A^i and the column B_j, i.e.

$$c_{ij} = A^i \cdot B_j = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}. \tag{2.3.46}$$

Figure 2.29 illustrates this idea graphically.

Using the previous example, instead of calculating the $\frac{pi}{2}$ rotation of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and then scaling it by 2, we calculate the product of the two matrices representing these transformations, and then apply them to the vector:

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \cancel{0 \cdot 2} - \cancel{1 \cdot 0} & \cancel{0 \cdot 0} - 1 \cdot 2 \\ 1 \cdot 2 + \cancel{0 \cdot 0} & \cancel{1 \cdot 0} + \cancel{0 \cdot 2} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

We then apply C to the vector and get the expected result:

$$C\vec{v} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cancel{0 \cdot 1} - 2 \cdot 1 \\ 2 \cdot 1 + \cancel{0 \cdot 1} \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

[9]otherwise this subsection would not be called ”Matrix-matrix products”, after all.

Example 2.35 Matrix-matrix products

! **To be written**: write example !

Recall that the order of composition of linear transformations matters: $T_1 \circ T_2 \neq T_2 \circ T_1$. Therefore, matrix-matrix products, which represent such compositions, are non-commutative, i.e.

$$AB \neq BA. \tag{2.3.47}$$

Example 2.36 Non-commutativity of matrix-matrix products

Of course, there are special cases where $AB = BA$, but these are the exception and not the norm. An example is the the rotation and scaling of the vector $\begin{bmatrix}1\\1\end{bmatrix}$ we saw above: if we flip the order of application of the two linear transformations we get the same result. This is true for any vector:

$$\begin{tikzcd}
& \begin{bmatrix}-y\\x\end{bmatrix} \arrow[dr, "\text{scale}"] & \\
\begin{bmatrix}x\\y\end{bmatrix} \arrow[ur, "\text{rotate}"] \arrow[dr, "\text{scale}"'] & & \begin{bmatrix}-2y\\2x\end{bmatrix} \\
& \begin{bmatrix}2x\\2y\end{bmatrix} \arrow[ur, "\text{rotate}"'] &
\end{tikzcd}$$

We can see that fact by multiplying the two matrices directly, in both directions:

$$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}\cancel{0\cdot 2} - \cancel{1\cdot 0} & \cancel{0\cdot 0} - 1\cdot 2\\ 1\cdot 2 + \cancel{0\cdot 0} & \cancel{1\cdot 0} + \cancel{0\cdot 2}\end{bmatrix} = \begin{bmatrix}0 & -2\\2 & 0\end{bmatrix},$$
$$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} = \begin{bmatrix}\cancel{2\cdot 0} + \cancel{0\cdot 1} & 2\cdot(-1) + \cancel{0\cdot 0}\\ 1\cdot 2 + \cancel{0\cdot 0} & \cancel{1\cdot 0} + \cancel{0\cdot 2}\end{bmatrix} = \begin{bmatrix}0 & -2\\2 & 0\end{bmatrix}.$$

Later in this chapter we will analyze the conditions for such commutativity to occur.

The determinant of a matrix-matrix product AB equals the product of the determinants of the separate matrices, i.e.

$$|AB| = |A| \times |B|. \tag{2.3.48}$$

The reason is that the change in volume after application of two consecutive transformations is the product of the change in volume for each separate transformation. This also mean that $|AB| = |BA|$. The trace of a matrix-matrix product behaves the same as well:

$$\operatorname{tr}(AB) = \operatorname{tr}(BA). \tag{2.3.49}$$

Proof 2.6 Trace of a matrix-matrix product

Prove the above behaviour of the trance operator.

QED

On the other hand, the traspose operator doesn't behave so "nicely":

$$(AB)^{\top} = B^{\top}A^{\top}. \tag{2.3.50}$$

2.3.10 Inverse matrices

Some linear transformations are invertible. For example, given a transformation which rotates any vector in $\mathbb{R}^2$ by θ **counter-clockwise** around the origin, its inverse transformation is one that rotates any vector in $\mathbb{R}^2$ by θ **clockwise** around the origin (or equivalently by $-\theta$ counter-clockwise). Figure 2.30 illustrates such transformation.

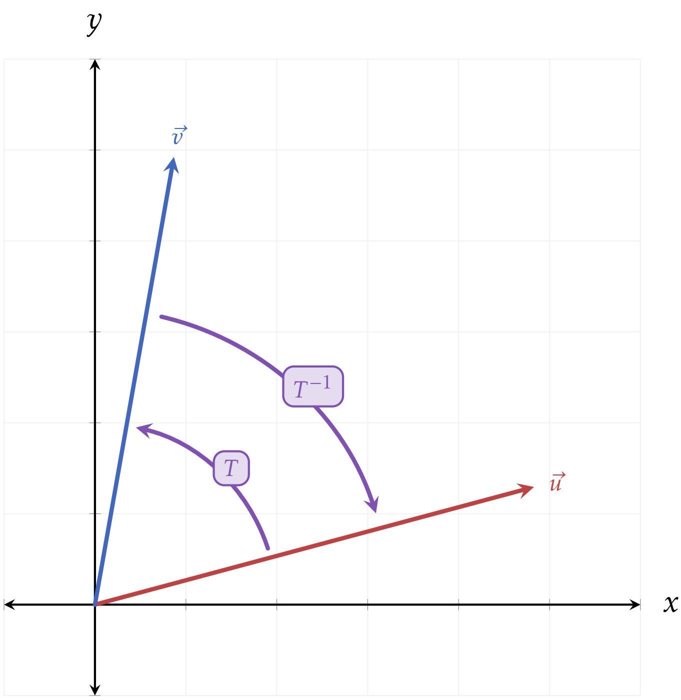


Figure 2.30 The transformation T rotates a vector $\vec{u}$ by θ counter-clockwise, turning it into the vector $\vec{v}$. Its inverse, T^{-1}, turns the vector $\vec{v}$ back into $\vec{u}$.

Recall that a transformations (function) is only invertible if and only if it is bijective (Section 0.2). Specifically to our case, a non-bijective linear transformation is a transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ for which some two vectors $\vec{u}, \vec{v} \in \mathbb{R}^m$ are mapped to the same vector $\vec{w} \in \mathbb{R}^n$.

Example 2.37 Two vectors mapped to the same vector

Let T be a linear transformation represented by the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The two vectors $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are mapped by T to the same vector:

$$A\vec{u} = \begin{bmatrix} 1 \cdot 2 + \cancel{0 \cdot 0} + \cancel{2 \cdot 0} \\ \cancel{0 \cdot 2} + \cancel{1 \cdot 0} + \cancel{0 \cdot 0} \\ \cancel{0 \cdot 2} + \cancel{0 \cdot 0} + \cancel{0 \cdot 0} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

$$A\vec{v} = \begin{bmatrix} \cancel{1 \cdot 0} + \cancel{0 \cdot 0} + 2 \cdot 1 \\ \cancel{0 \cdot 0} + \cancel{1 \cdot 0} + \cancel{0 \cdot 1} \\ \cancel{0 \cdot 0} + \cancel{0 \cdot 0} + \cancel{0 \cdot 2} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

In fact, when a linear transformation maps two vectors in its domain to a single vector in is image - it actually maps **infinitely many** vectors to that point: let T be a linear transformation and $\vec{u}, \vec{v}$ two vectors in its domain that are mapped to the same output $\vec{w}$, i.e.

$$T\left(\vec{u}\right) = T\left(\vec{v}\right) = \vec{w}.$$

Due to the properties of linear transformations, on one side we get

$$T\left(\alpha\vec{v} + \beta\vec{u}\right) = T\left(\alpha\vec{u}\right) + T\left(\beta\vec{v}\right) \tag{2.3.51}$$

$$= \alpha T\left(\vec{u}\right) + \beta T\left(\vec{w}\right) \tag{2.3.52}$$

$$= \alpha\vec{w} + \beta\vec{w} \tag{2.3.53}$$

$$= (\alpha + \beta)\vec{w}. \tag{2.3.54}$$

(for any $\alpha \in \mathbb{R}$)

Thus, for example, the linear combination $2\vec{u} + 6\vec{v}$ will be mapped to the same output as $6\vec{u} + 2\vec{v}$, $3\vec{u} + 9\vec{v}$, $0.5\vec{u} + 11.5\vec{v}$, etc. - the linear combination using any two coefficients α, β which add up to 12 would be mapped to $12\vec{w}$. This is of course true for any real number, and so for each scale of $\vec{w}$ there are infinitely many linear combinations that are mapped to it.

In fact, the only way in which a linear transformation can be non-bijective is by "loosing" a dimension, i.e. when it maps $\mathbb{R}^3$ to a single plane/line/point. As we saw earlier, these kind of transformations are represented by matrices for which the determinant is zero, i.e.

$$|A| = 0 \Leftrightarrow \nexists A^{-1}. \tag{2.3.55}$$

which in words mean that if (and only if!) the determinant of a matrix is zero then it has no iverse, and vice-versa (the symbol $\nexists$ means ”does not exist”). On the other hand, if the determinant isn’t zero, then the inverse must exist.

The product of a matrix and its inverse is the respective identity matrix, since the composition of the respective linear transformations represented by the matrices is the identity transformation. In mathematic terms:

$$AA^{-1} = A^{-1}A = I. \tag{2.3.56}$$

Example 2.38 Inverse matrices

! **To be written**: give a 3×3 matrix, show that its determinant is non-zero, present its inverse and
!

Matrices with zero determinant are also known as a **singular** or **degenerate** matrix. For obvious reason, we will not use the latter term in this book[10].

While the identity matrix is its own inverse (since $I \cdot I = I$), it is not the only matrix which shows such behaviour. For example, consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Geometrically, A takes a vector in $\mathbb{R}^3$ and flips its x-component. In other words it mirrors any vector across the yz-plane (see **??**). If we take a vector $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and multiply it by A we get the vector $\vec{v} = \begin{bmatrix} -x \\ y \\ z \end{bmatrix}$ (tip: calculate the product $A\vec{v}$ directly for practice). If we take $\vec{v}$ and multiply it by A, we get the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ - i.e. we retrieved back $\vec{u}$. Since this is true for any vector in $\mathbb{R}^3$, A is its own inverse. We can check this directly by calculating the product of A with itself:

$$\begin{aligned} A \cdot A &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \cdot (-1) + \cancel{0 \cdot 0} + \cancel{0 \cdot 0} & \cancel{-1 \cdot 0} + \cancel{0 \cdot 1} + \cancel{0 \cdot 0} & \cancel{-1 \cdot 0} + \cancel{0 \cdot 0} + \cancel{0 \cdot 1} \\ \cancel{0 \cdot (-1)} + \cancel{0 \cdot 0} + \cancel{0 \cdot 0} & \cancel{0 \cdot 1} + 1 \cdot 1 + \cancel{0 \cdot 0} & \cancel{0 \cdot 0} + \cancel{0 \cdot 1} + \cancel{0 \cdot 0} \\ \cancel{0 \cdot (-1)} + \cancel{0 \cdot 0} + \cancel{0 \cdot 1} & \cancel{0 \cdot 0} + \cancel{0 \cdot 1} + \cancel{0 \cdot 0} & \cancel{0 \cdot 0} + \cancel{0 \cdot 0} + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

[10] nor should you use it in general, in my opinion.

$$= I_3.$$

A matrix which is its own inverse is known as an **involutory matrix**. Such matrices must be square (challange to the reader: why is that true?).

How do we calculate the inverse of a matrix? In general, non-square matrices don't have so-called "complete" inverses, so we focus on calculating the inverses of square matrices only. The general formula for an inverse matrix A^{-1} is

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A), \tag{2.3.57}$$

where adj(A) is the **adjugate** of the matrix A. The adjugate of a matrix is calculated using its minors, as follows: let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

then its adjugate is the matrix

$$\operatorname{adj}(A) = \begin{bmatrix} +m_{11} & -m_{12} & \cdots & \pm m_{1n} \\ -m_{21} & +m_{22} & \cdots & \mp m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \pm m_{m1} & \mp m_{m2} & \cdots & \pm m_{mn} \end{bmatrix}^{\top}$$

(where m_{ij} is the ij-minor of A).

Example 2.39 Adjugate of a matrix

Let

$$C = \begin{bmatrix} 1 & 7 & -3 \\ 2 & 5 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

then

$$\begin{aligned}\operatorname{adj}(C) &= \begin{bmatrix} +m_{11} & -m_{12} & +m_{13} \\ -m_{21} & +m_{22} & -m_{23} \\ +m_{31} & -m_{32} & +m_{33} \end{bmatrix}^{\top} \\ &= \begin{bmatrix} +\begin{vmatrix} 5 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} 7 & -3 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 7 \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} 7 & -3 \\ 5 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 7 \\ 2 & 5 \end{vmatrix} \end{bmatrix}^{\top}\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} 5\cdot 1 - \cancel{1\cdot 0} & -2\cdot 1 + \cancel{0\cdot 0} & 2\cdot 1 - \cancel{0\cdot 5} \\ -(7\cdot 1 + 1\cdot 3) & 1\cdot 1 + \cancel{3\cdot 0} & -(1\cdot 1 - \cancel{7\cdot 0}) \\ \cancel{7\cdot 0} + 3\cdot 5 & -(\cancel{1\cdot 0} + 3\cdot 2) & 1\cdot 5 - 2\cdot 7 \end{bmatrix}^\top \\
&= \begin{bmatrix} 5 & -2 & 2 \\ -10 & 1 & -1 \\ 15 & -6 & -9 \end{bmatrix}^\top \\
&= \begin{bmatrix} 5 & -10 & 15 \\ -2 & 1 & -6 \\ 2 & -1 & -9 \end{bmatrix}.
\end{aligned}
$$

We can use the above example to calculate the inverse of the matrix C.

Example 2.40 Inverse of a matrix

The determinant of C is

$$|C| = m_{11} - 7m_{12} - 3m_{13} = 5 - 14 - 6 = -15.$$

Therefore the inverse of C is

$$
\begin{aligned}
C^{-1} &= \frac{1}{|C|}\operatorname{adj}(C) \\
&= -\frac{1}{15}\begin{bmatrix} 5 & -10 & 15 \\ -2 & 1 & -6 \\ 2 & -1 & -9 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & -1 \\ \frac{2}{15} & -\frac{1}{15} & \frac{6}{15} \\ -\frac{2}{15} & \frac{1}{15} & \frac{3}{5} \end{bmatrix}.
\end{aligned}
$$

In the case of 2×2 matrices, the adjugate has a rather simple formula:

$$\operatorname{adj}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{2.3.58}$$

Note 2.11 Minors of 2×2 matrices

The minors of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are simply

$$
\begin{aligned}
m_{11} &= d, \; m_{12} = c, \\
m_{21} &= b, \; m_{22} = a.
\end{aligned}
$$

Therefore, the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{2.3.59}$$

We can check that Equation 2.3.59 is correct by directly calculating the products AA^{-1} and $A^{-1}A$:

$$\begin{aligned}
AA^{-1} &= \frac{1}{|A|}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
&= \frac{1}{ad-cb}\begin{bmatrix} ad-bc & \cancel{-ab+ba} \\ \cancel{cd-dc} & -cb+da \end{bmatrix} \\
&= \begin{bmatrix} \frac{ad-bc}{ad-cb} & 0 \\ 0 & \frac{da-cb}{ad-bc} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I_2.
\end{aligned}$$

$$\begin{aligned}
A^{-1}A &= \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \frac{1}{ad-cb}\begin{bmatrix} da-bc & \cancel{db-bd} \\ \cancel{-ca+ac} & -cb+ad \end{bmatrix} \\
&= \begin{bmatrix} \frac{da-bc}{ad-cb} & 0 \\ 0 & \frac{ad-cb}{ad-cb} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I_2.
\end{aligned}$$

Example 2.41 Inverting a 2×2 matrix

The inverse of the matrix

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$$

is

$$\begin{aligned}
A^{-1} &= \frac{1}{|A|}\begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} \\
&= \frac{1}{1 \cdot 0 - (-3) \cdot 2}\begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}.
\end{aligned}$$

Example 2.42 Inverse of the 2-dimensional rotation matrix

We can use Equation 2.3.59 to calculate the inverse of the 2-dimensional rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Since the rotation matrix rotates all vector by θ counter-clockwise around the origin, we expect its inverse to rotate all vectors by $-\theta$ counter-clockwise around the origin. Let's show that this is indeed the case: we first calculate the determinant of R:

$$|R| = \cos(\theta)\cdot\cos(\theta) + \sin(\theta)\sin(\theta) = \cos(\theta)^2 + \sin(\theta)^2 = 1.$$

This result makes sense - rotation does not change areas. We can now use $|R|$ to calculate R^{-1}:

$$\begin{aligned} R^{-1}(\theta) &= \frac{1}{|R|}\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}. \end{aligned}$$

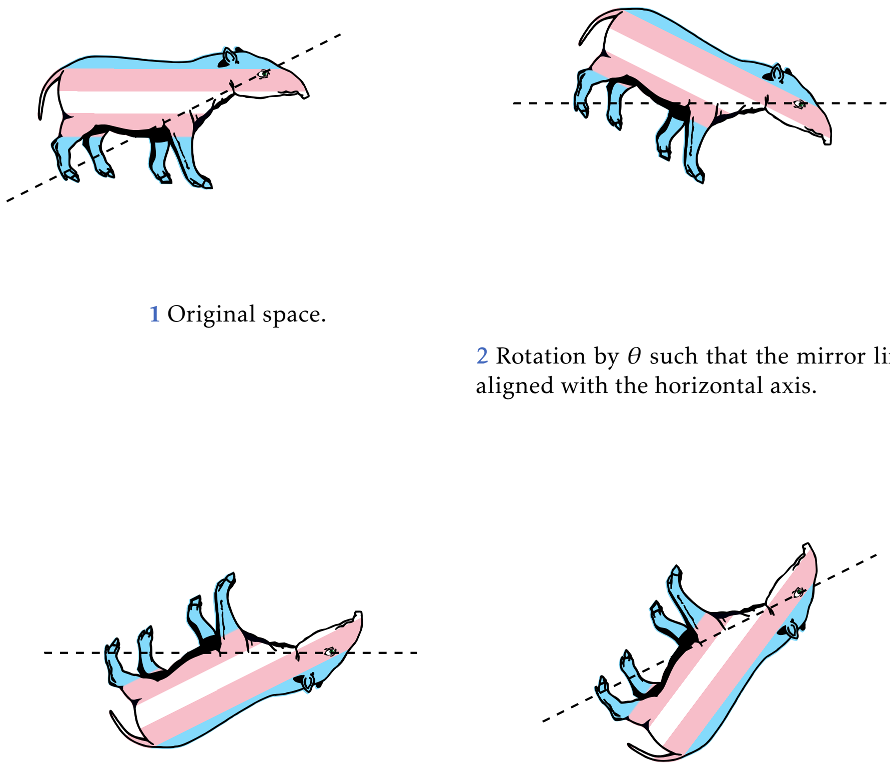
Since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, the above matrix is exactly a rotation by $-\theta$ counter-clockwise around the origin, as we expected.

We can use the rotation matrix and its inverse to calculate the general 2×2 reflection matrix around a line going through the origin (Equation 2.3.22), represented by its angel θ. We do this by taking the following steps (see Figure 2.31 for a graphical illustration):

1. Rotate space by $-\theta$ such that the reflection line aligns with the horizontal axis.
2. Reflect space across the horizontal line.
3. Rotate space by θ to bring back space to its original orientation.

The order of application of the above transformations is 1-2-3. Therefore, in matrix form we write them from **left to right**, i.e. **3 · 2 · 1**:

$$\begin{aligned} \mathrm{Ref}(\theta) &= \overbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}^{\text{3. rotate back}} \cdot \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}^{\text{2. flip vertically}} \cdot \overbrace{\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}}^{\text{1. rotate}} \\ &= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{bmatrix}. \end{aligned}$$



1 Original space.

2 Rotation by θ such that the mirror line is aligned with the horizontal axis.

3 Mirroring across the mirror line.

4 Rotating back to the original orientation (i.e. by $-\theta$).

Figure 2.31 Constructing a general 2×2 reflection matrix as the composition of rotating, mirroring and rotating back.

2.3.11 Kernel and null space

While the determinant of a matrix tells us whether the matrix is reversible or not, it doesn't quantify the loss in dimensionality resulting from applying it: a matrix which "squishes" $\mathbb{R}^3$ in to a plane has the same determinant as a matrix which "squishes" it into a line or the the a single point (the origin): 0. To find a measurement that does quantify this characteristic, we can look at the set of all vectors which the matrix (and thus the transformation it represents) map to $\vec{0}$. We call this set the **kernel** of the matrix/transformation (denoted $\ker(A)$ or $\ker(T)$, respectively). More formally - given an $m \times n$ matrix representing a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$,

$$\ker(A) = \left\{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \vec{0}_m \right\}, \tag{2.3.60}$$

and equivalently,

$$\ker(T) = \left\{ \vec{v} \in \mathbb{R}^n \mid T\left(\vec{v}\right) = \vec{0}_m \right\} \tag{2.3.61}$$

In the language of matrices, the kernel is sometimes refered to as their **null space**.

Note 2.12 Matrix/Transformation duality

From now on in this subsection we will discuss matrices only, however everything discussed here can be applied directly to the transformations they represent. We also use the term kernel instead of null space, as these two concepts are practially equivalent in our context.

!

Example 2.43 Kernel of a matrix

The vector

$$\vec{v} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

is in the kernel of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 6 \\ 2 & 2 & 4 \end{bmatrix},$$

since

$$\begin{aligned} A\vec{v} &= \begin{bmatrix} 0 & 1 & 2 \\ -1 & 3 & 6 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot 0 + 1 \cdot (-2) + 2 \cdot 1 \\ -1 \cdot 0 + 3 \cdot (-2) + 6 \cdot 1 \\ 2 \cdot 0 + 2 \cdot (-2) + 4 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2 + 2 \\ 0 - 6 + 6 \\ 0 - 2 + 4 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Two properties of the kernel of a matrix are relatively straight-forward:

- The (dimentially correct) zero vector is always in the kernel of any matrix. This is because the zero vector of a transformation's domain is always mapped to the zero vector in its image (stated in Section 2.2 as the fact that linear transformations preserve the origin).
- Any linear combination of vectors in the kernel of a matrix is also in the kernel of the matrix. This is easily proved using the basic properties of linear transformations (which we show in matrix form):

Proof 2.7 Linear combinations of kernel vectors

Let A be a matrix, and $\vec{u}, \vec{v}$ two vectors in its kernel. Then

$$\begin{aligned} A \cdot \left(\alpha \vec{u} + \beta \vec{v}\right) &= A \cdot \left(\alpha \vec{u}\right) + A \cdot \left(\beta \vec{v}\right) \\ &= \alpha A \vec{u} + \beta A \vec{v} \\ &= \alpha \vec{0} + \beta \vec{0} \\ &= \vec{0} + \vec{0} \\ &= \vec{0}. \end{aligned}$$

Therefore $\vec{w} = \alpha \vec{u} + \beta \vec{v}$ is also in the kernel of A.

QED

The kernel of a matrix forms a subspace of its domain (see Figure 2.32). Therefore, we can quantify the dimension of the kernel, sometimes called its **nullity**:

$$\mathrm{Null}(A) = \dim(\ker(A)). \tag{2.3.62}$$

Example 2.44 Kernel space of a matrix

The matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{bmatrix}$$

has the following two vectors in its kernel:

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

These two vectors are linearly independant, and span the kernel of A, having

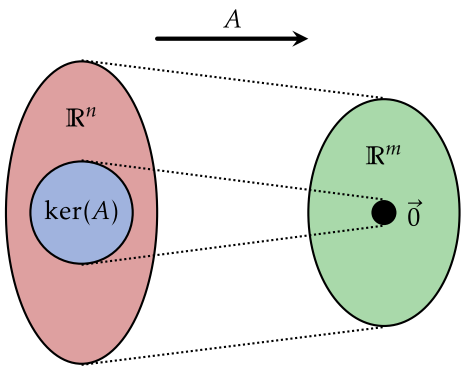


Figure 2.32 The kernel of a matrix A is a subspace of its domain $\mathbb{R}^n$. It is mapped exclusively to $\vec{0}_m$ in its image. The set $\mathbb{R}^n \setminus \ker(A)$, i.e. $\mathbb{R}^n$ minus the kernel (shown in red), is in fact the column space of the matrix.

dimension $\dim(V) = 2$. All the vectors in $\ker(A)$ are of the form

$$\vec{w} = \alpha\vec{u} + \beta\vec{v},$$

i.e. the linear combinations of $\vec{u}$ and $\vec{v}$.

All the vectors which are not in $\ker(A)$ also span a subspace of its domain, called the **column space** of A. The dimension of the column space is called the **rank** of A, $\operatorname{rank}(A)$. The kernel and column space of a matrix are complementary: together they span $\mathbb{R}^n$.

This means that we can split the domain of a matrix to two separate subspaces, which together give us a lot of information about the image of the matrix: in one subspace are all the vectors that will be "squished" by the transformation into the origin, and the other subspace is composed of all the vectors that are transformed to a non-zero vector. If we limit the domain of the transformation to its column space, it becomes reversible - no two vectors in the column space are mapped to a single vector. On the other hand, the transformation on the kernel is always inversible.

Alltogether, the nullity and rank of a matrix add up to the dimension of its domain, i.e.

$$\operatorname{Null}(A) + \operatorname{rank}(A) = n. \tag{2.3.63}$$

TBD: figure to illustrate the kernel and column space of a matrix as complementary.

If the rank of an $m \times n$ matrix A is equal to n, its kernel space must have the dimension

$$\operatorname{Null}(A) = n - \operatorname{rank}(A) = n - n = 0, \tag{2.3.64}$$

i.e. the matrix has no vectors mapped to $\vec{0}$ (except the zero vector itself), and thus the matrix is invertible (non-singular), and so its determinant is non-zero. This is true in the other direction: a matrix with non-zero determinant is invertible, and thus its kernel contains only $\vec{0}$ and the nullity of the matrix is 0. This means that the column space of the matrix must equal n. Altogether, these facts can be written succinctly as

$$\operatorname{rank}(A) = n \Leftrightarrow |A| \neq 0. \tag{2.3.65}$$

!
To be written: how matrix-matrix product changes the right-side matrix (i.e. exchanging rows/colu
!

2.3.12 Matrices as basis change

2.4 SYSTEMS OF LINEAR EQUATIONS

2.4.1 Defintions

Everything we learned so far about vectors and matrices can be used to solve and characterise a family of equations known as **linear equations**. You're probably already very familiar with linear equations: they are equations in which the **variables** appear directly, without any power or other functions acting on them. For example, the simple equation

$$y = ax + b, \tag{2.4.1}$$

where x, y are both variables and a, b are both constant real numbers is a linear equation. Equation 2.4.1 can be re-written as

$$ax - y + b = 0, \tag{2.4.2}$$

where now a is the **coefficient** of the variable x, while the variable y has the coefficient -1 and b is a so-called **free coefficient**. In general, a linear equation of two variables has the form

$$a_0 + a_x x + a_y y = 0, \tag{2.4.3}$$

i.e. we changed the name of a to a_x and b to a_0, and gave y the coefficient a_y. We can also rename x and y to x_1 and x_2, respectively, and name their coefficients accordingly:

$$a_0 + a_1 x_1 + a_2 x_2 = 0. \tag{2.4.4}$$

The form shown in Equation 2.4.4 can be easily expanded into n variables:

$$a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_{n-1} x_{n-1} + a_n x_n = 0, \tag{2.4.5}$$

where $x_1, x_2, \ldots, x_n$ are the variables of the equation, and $a_0, a_1, \ldots, a_n$ are its coefficients. We say that n is the **order** (also: **degree**) of the equation.

Note 2.13 Number set used for linear equations

As with other topics, in the context of this section both the variables and coefficients are all **real numbers**, however almost anything we discuss here can genrally be applied to complex numbers or other structures.

Example 2.45 Linear equations

The following is a linear equation of order 3, using the variables x, y, z:

$$3x + 2y - z + 4 = 0.$$

The coefficients of the equation are

$$\begin{aligned} a_0 &= 4, \\ a_x = a_1 &= 3, \\ a_y = a_2 &= 2, \\ a_z = a_3 &= -1. \end{aligned}$$

Another linear equation of the same three variables is

$$5x - 2y + 1 = 0.$$

In this case the coefficient $a_z = a_3 = 0$. Depending on the context, this equation can be considered as either an equation of order 3 or an equation of order 2.

In $\mathbb{R}^2$ linear equations represent a line, which doesn't necesserally go through the origin (and thus isn't necesserally a subspace of $\mathbb{R}^n$). For a line to go through the origin, the free coefficient a_0 must equal zero (see Figure 2.33).

In $\mathbb{R}^3$ linear equations represent planes. Much like with the lines in $\mathbb{R}^2$, these planes don't necesserally go through the origin. The trend continues with increasing dimensions: in $\mathbb{R}^4$ linear equations represent 3-dimensional spaces, in $\mathbb{R}^5$ linear equations represent 4-dimensional spaces, etc. When the free coefficient is equal to zero, these spaces become subspaces of the respective $\mathbb{R}^n$.

2.4.2 Systems and matrix form

A **system of linear equations** is a set of linear equations using the same variables. For example, the three equations

$$\begin{cases} 2x - 5y + 4z + 2 = 0 \\ -3x - 2y + 1 = 0 \\ 5x + 4z + -3 = 0 \end{cases}$$

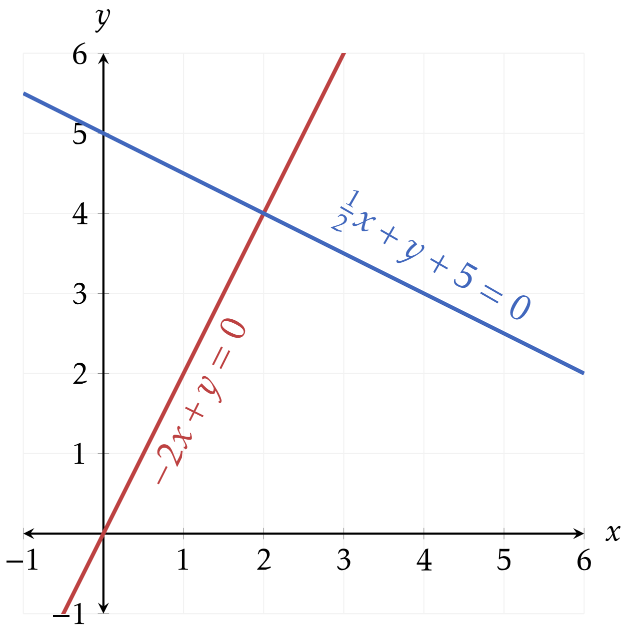


Figure 2.33 Two linear equations represented as lines in $\mathbb{R}^2$. Note how in the red equation the free coefficient is zero, and so the line goes through the origin.

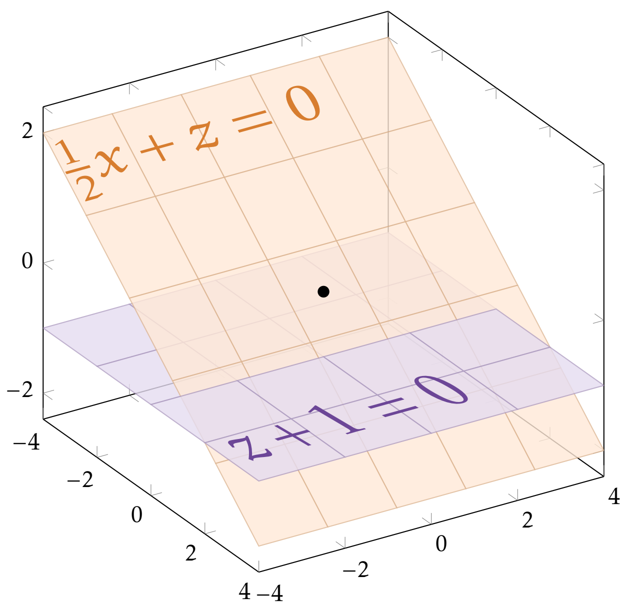


Figure 2.34 Two intersecting planes in $\mathbb{R}^3$ with their corresponding equations.

form together a system of 3 linear equations with 3 variables (x, y and z). Systems of linear equations can be written together in matrix form: in the above example, the system can be represented as the equation

$$\begin{bmatrix} 2 & -5 & 4 \\ -3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

since performing the matrix-vector product and vector addition yields back the system of equations. We call the matrix the **coefficients matrix** of the equation.

Note 2.14

In practice, many times the vector representing the free coefficients is moved to the right hand side of the equation. In the case of the above system this yields the simple equation

$$\begin{bmatrix} 2 & -5 & 4 \\ -3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}.$$

!

In the most general form, a system of m equations in n variavles $x_1, x_2, x_3, \dots, x_n$ can be represented as the product of an $m \times n$ coefficient matrix and the variables vector, yielding the free-coefficient vector:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m \times n \text{ coefficients}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{n \text{ variables}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{m \text{ free coefficients}}, \tag{2.4.6}$$

which can be written succinctly as

$$A\,x = b. \tag{2.4.7}$$

2.4.3 Solutions

A system of linear equations can have either one, infinitely many, or no **solutions**. A solution of a system of linear equations is a tuple

$$s = (s_1, s_2, \dots, s_n)$$

such that if we substitute each s_i into the respective variable x_i all of the equations become **true** statements.

Example 2.46 Solutions of a system of linear equations

The following linear system

$$\begin{cases} -4x + 2y = 0 \\ x - y + 3 = 0 \end{cases}$$

has the solution

$$s = (-1, 2).$$

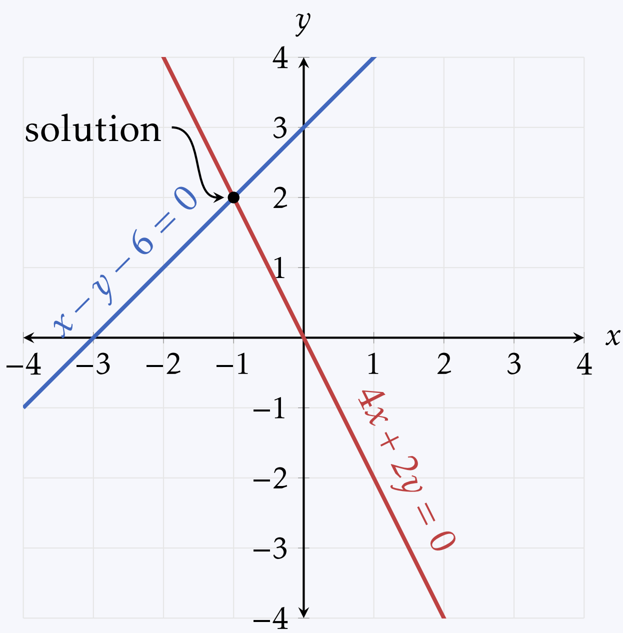
Indeed if we substitute $x = -1$ and $y = 2$ into the system we get

$$\begin{cases} -4 \cdot (-1) + 2 \cdot (2) = -4 + 4 = 0 \;\Rightarrow\; \textbf{true} \\ -1 - 2 + 3 = -3 + 3 = 0 \;\Rightarrow\; \textbf{true} \end{cases}$$

In the graphical representation of linear equations, the solutions of a system are the points where the respective graphs representing the equation (line, plane, etc.) intersect.

Example 2.47 Solutions of a system of linear equations - graph

The linear system from the previous example can be represented by the following graph:



Not all systems have single solutions only, nor do all systems even have any solutions.

For example, if we add to the system in Example 2.47 the equation

$$x - 3y + 3 = 0,$$

the resulting system

$$\begin{cases} -4x + 2y = 0 \\ x - y + 3 = 0 \\ x - 3y + 3 = 0 \end{cases}$$

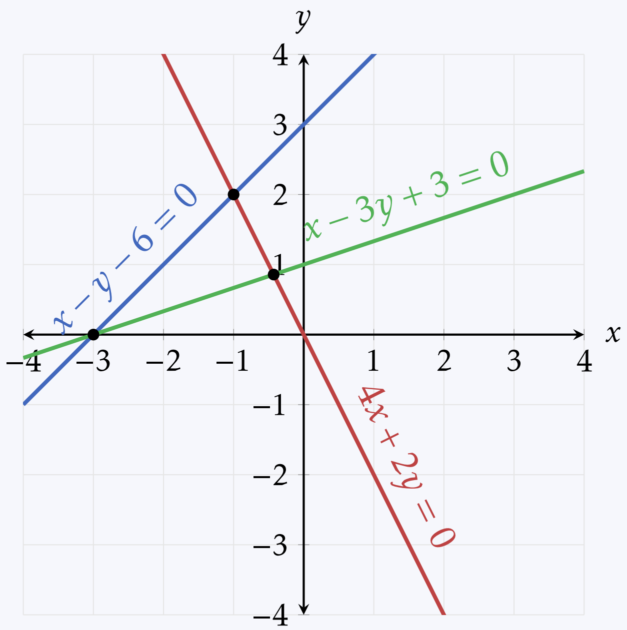
has no solutions (see Example 2.48).

Example 2.48 System with no solutions

When the system of linear equations from Example 2.47 is supplemented with the equation

$$x - 3y + 3 = 0,$$

it has no solution. However, any two equations of the system do have solutions (represented below as black points).



Let us now explore when a set of linear equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ has a single solution, infinitely many solutions or no solutions. In $\mathbb{R}^2$, the lines representing two linear equations can be either parallel or non-parallel. If they are non-parallel then the two equations have a single solution - the intercept of both lines (as seen in the previous two examples). If the lines are parallel, there are two cases: either the two lines are identical, in which case there are infinitely many solutions (all the points on the line), or they are parallel yet distince, in which case there are no solutions to the system (see Figure 2.35).

In the case of more than two linear equations there can be, again, either a single

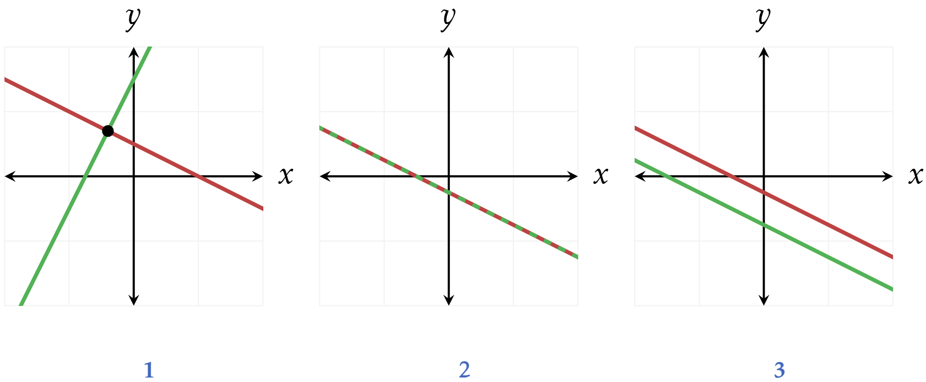


Figure 2.35 Three possible cases for two linear equations in $\mathbb{R}^2$: (a) non-parallel and thus a single solution, (b) parallel and identical and thus infinitely many solutions, and (c) parallel but not identical and thus no solutions.

solution, infinitely many solutions or no solutions. The difference is that in this case zero solutions can happen even when all of the lines representing the equations are non-parallel (see Figure 2.36).

In $\mathbb{R}^3$... ! **To be written**: the rest... later, because right now I'm lazy af !

2.4.4 Finding solutions

Several method to solve systems of linear equations were established over the years. One of the most well-known is **Gauss elimination method**. The goal of this method is to use a set of pre-defined operations to bring the system into a form which is easy to solve. This form essentially exposes the rank of the coefficient matrix, from which one can deduce some important properties of the system such as the existance of a solution (or lack thereof), the degrees of freedom in the values which can be substituted into the system and more.

We start by describing the **row-echelon form** of a matrix. For now we simply assume that such matrices exist, and later we will use a set of pre-defined operations to convert any matrix to this form. The following matrix is in row-echelon form:

$$A = \begin{bmatrix} 1 & 3 & -5 & 7 & 0 \\ 0 & 0 & 2 & 2 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We note three properties of the above matrix:

- The bottom row is all zeros.
- When looking at each row (left to right), all values are zero until we reach a non-zero element. For example, the first row starts with a non-zero element, but

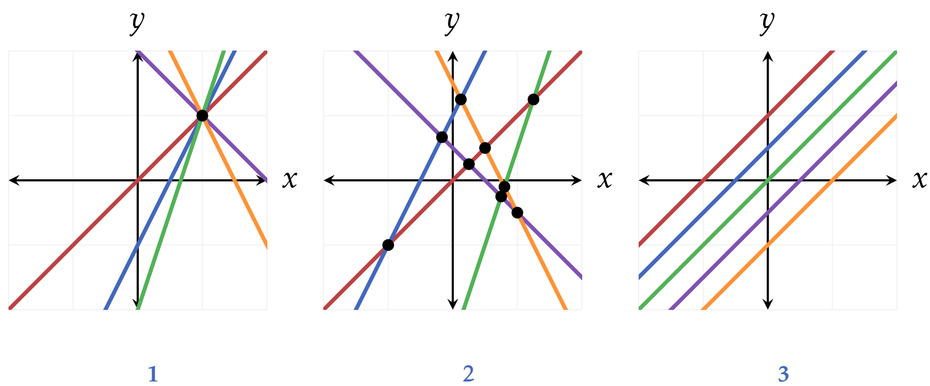


Figure 2.36 Three possible cases for the number of solutions of several linear equations in $\mathbb{R}^2$: (a) non-parallel but all lines intercept at single point and thus the system has a solution, (b) non-parallel but no single interception point and thus no solution, and (c) parallel and thus no solutions. The case where all lines are identical and thus there are infinitely many solutions is ommited from the figure, and looks identical to Figure 2.352.

the second row has two non zero elements followed by the number 2. These first non-zero elements are called **leading coefficients** or **pivots**, and each one is strictle to the **right** of the leading coefficient of the row above it.

- The leading coefficient of each row has only zeros below it in its column.

Any matrix that has the same three properies is said to be in its row-echelon form.

Example 2.49 Row-echelon form

All of the matrices below are in their row-echelon form, and the leading zeros in each row are highlighted:

$$\begin{bmatrix} 0 & 5 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & 2 \\ 0 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 3 & -6 & 9 & -3 \\ 0 & 0 & 0 & 0 & 2 & 6 \end{bmatrix}$$

The following matrices are all **not** in their row-echelon form:

$$\begin{bmatrix} 0 & 5 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & 2 \\ 3 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 7 & -3 & 4 & 4 \\ 0 & 1 & 0 & 0 & 2 & 3 \end{bmatrix}$$

Using the row-echelon form we define the **reduced row-echelon form**. A matrix in a reduced row-echelon form is a matrix that is already in its row-echelon form, and in addition:

- The leading coefficients are all equal to 1 (and are thus called **leading ones**).

- All the elements in the **column** of a leading one are equal to zero (except the leading one itself).

Example 2.50 Reduced row-echelon form

The following matrices are the reduced row-echelon forms of the row-echelon matrices in Example 2.49, with theleading ones in each row highlighted:

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & -2 & 0 & -10 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

To bring a matrix to its reduced row-echelon form, we use a sequence of operations from the following set:

- Scaling a row by a non-zero real number. For example:

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ -3 & 2 & 1 & 1 \\ 5 & 1 & 0 & -4 \\ 2 & 2 & -1 & 7 \end{bmatrix} \xrightarrow{R_2 \to 3\, R_2} \begin{bmatrix} 1 & 0 & -3 & 2 \\ -9 & 6 & 3 & 3 \\ 5 & 1 & 0 & -4 \\ 2 & 2 & -1 & 7 \end{bmatrix}$$

  here we take the 2nd row of the matrix (denoted as R_2) and scale it by 3. The notation above the arrow tells us exactly that: R_2 is transformed into $2 \times R_2$.

- Exchanging two rows. For example:

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ -9 & 6 & 3 & 3 \\ 5 & 1 & 0 & -4 \\ 2 & 2 & -1 & 7 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 5 & 1 & 0 & -4 \\ -9 & 6 & 3 & 3 \\ 0 & 2 & 6 & 3 \end{bmatrix}$$

- Adding one scaled row to a different row (where the scalar is not zero). For example:

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 5 & 1 & 0 & -4 \\ -9 & 6 & 3 & 3 \\ 2 & 2 & -1 & 7 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 2\, R_1} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 5 & 1 & 0 & -4 \\ -9 & 6 & 3 & 3 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

  (note that R_1 itself is not changed by the operation)

Example 2.51 Row operations

Let us apply a sequence of 6 row opertions on a matrix until it reaches its reduced row-echelon form:

$$\begin{bmatrix} 1 & 5 & 0 \\ -2 & 3 & 1 \\ -3 & -8 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 3R_1} \begin{bmatrix} 1 & 5 & 0 \\ -2 & 3 & 1 \\ 0 & 7 & 0 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{7} R_3} \begin{bmatrix} 1 & 5 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$
$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 5R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

To use row operations to solve a system of linear equations, we first write the **augmented matrix** form of the system. This for looks as follows: suppose we have the system

$$\begin{cases} 2x + y - 3z = -5 \\ 4x - 2y + 6z = 2 \\ x + y - z = 0 \end{cases}$$

Then the augmenten matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & -5 \\ 4 & -2 & 6 & 2 \\ 1 & 1 & -1 & 0 \end{array}\right],$$

i.e. it is simpy the coefficients matrix *augmented* with the free coefficients vector. We draw a vertical line where the matrix and vector were "stitched" together, to remind us which elements belong to what object.

To solve the system, we bring the augmented matrix to its reduced row-echelon form. In our case, this would be the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

which means that the solution to the system is

$$x = -1, \; y = 3, \; z = 2.$$

Example 2.52 Solving a system of linear equations

Let us solve the following system of linear equations using the Gaussian elimination method:

$$\begin{cases} -3x+6y+6z-2w=10 \\ 7x-5y+7z-6w=61 \\ 5x-5y-4z-5w=18 \\ -8x+2y+2z+7y=-40 \end{cases}$$

The augmented matrix for the system is

$$A=\left[\begin{array}{cccc|c} -3 & 6 & 6 & -2 & 10 \\ 7 & -5 & 7 & -6 & 61 \\ 5 & -5 & -4 & -5 & 18 \\ -8 & 2 & 2 & 7 & -40 \end{array}\right].$$

Applying a sequence of row operations to A:

$$\left[\begin{array}{cccc|c} -3 & 6 & 6 & -2 & 10 \\ 7 & -5 & 7 & -6 & 61 \\ 5 & -5 & -4 & -5 & 18 \\ -8 & 2 & 2 & 7 & -40 \end{array}\right] \xrightarrow{R_1\to R_1-3R_4} \left[\begin{array}{cccc|c} 21 & 0 & 0 & -23 & 130 \\ 7 & -5 & 7 & -6 & 61 \\ 5 & -5 & -4 & -5 & 18 \\ -8 & 2 & 2 & 7 & -40 \end{array}\right]$$

$$\xrightarrow{R_2\to R_2-R_3} \left[\begin{array}{cccc|c} 21 & 0 & 0 & -23 & 130 \\ 2 & 0 & 11 & -1 & 43 \\ 5 & -5 & -4 & -5 & 18 \\ -8 & 2 & 2 & 7 & -40 \end{array}\right] \xrightarrow{R_3\to R_3+2R_4} \left[\begin{array}{cccc|c} 21 & 0 & 0 & -23 & 130 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 9 & -62 \\ -8 & 2 & 2 & 7 & -40 \end{array}\right]$$

$$\xrightarrow{R_4\to R_4+2R_3} \left[\begin{array}{cccc|c} 21 & 0 & 0 & -23 & 130 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 9 & -62 \\ -30 & 0 & 2 & 25 & -164 \end{array}\right] \xrightarrow{R_4\to R_4+15R_2} \left[\begin{array}{cccc|c} 21 & 0 & 0 & -23 & 130 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 9 & -62 \\ 0 & 0 & 167 & 10 & 481 \end{array}\right]$$

$$\xrightarrow{R_1\to R_1-10.5R_2} \left[\begin{array}{cccc|c} 0 & 0 & -115.5 & -12.5 & -321.5 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 9 & -62 \\ 0 & 0 & 167 & 10 & 481 \end{array}\right]$$

$$\xrightarrow{R_1\to R_1-1.25R_4} \left[\begin{array}{cccc|c} 0 & 0 & 93.25 & 0 & 279.75 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 9 & -62 \\ 0 & 0 & 167 & 10 & 481 \end{array}\right]$$

$$\xrightarrow{R_1\to \frac{1}{93.25}R_1} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 9 & -62 \\ 0 & 0 & 167 & 10 & 481 \end{array}\right] \xrightarrow{R_4\to R_4-167R_1} \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 9 & -62 \\ 0 & 0 & 0 & 10 & -20 \end{array}\right]$$

$$\xrightarrow{R_4\to\frac{1}{10}R_4}\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 9 & -62 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\xrightarrow{R_3\to R_3-9R_4}\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 2 & 0 & 11 & -1 & 43 \\ -11 & -1 & 0 & 0 & -44 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2\to R_2-11R_1}\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 2 & 0 & 0 & -1 & 10 \\ -11 & -1 & 0 & 0 & -44 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\xrightarrow{R_2\to R_2-R_4}\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 2 & 0 & 0 & 0 & 8 \\ -11 & -1 & 0 & 0 & -44 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2\to\frac{1}{2}R_2}\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 0 & 4 \\ -11 & -1 & 0 & 0 & -44 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\xrightarrow{R_3\to R_3+11R_2}\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 0 & 4 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3\to -R_3}\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]\xrightarrow{R_1\leftrightarrow R_2,\ R_2\leftrightarrow R_3}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Reduced row echelon form

The solution of the system is therefore

$$x = 4,\ y = 0,\ z = 3,\ w = -2.$$

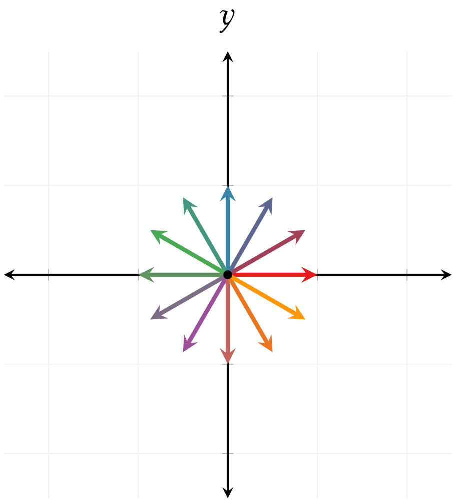
MORE TEXT WILL BE HERE

2.5 EIGENVECTORS AND EIGENVALUES

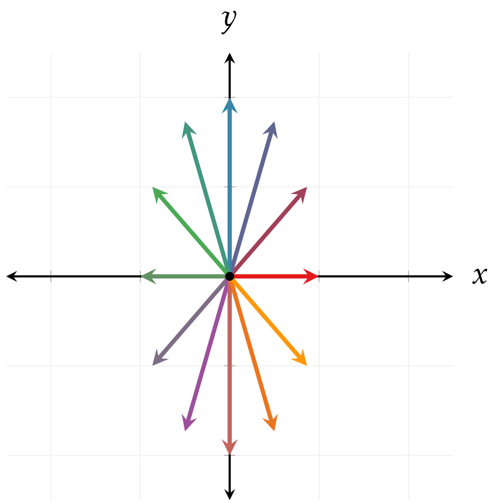
2.5.1 Definition

Some linear transformations have special directions which only scale by the application of the transformation and are not mapped to different directions. Take for example the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, which scales space by 2 in the y-direction. All vectors pointing in the y-direction get scaled by T (namely by a factor of 2) and still point in the y-direction after the application of T. All vectors pointing in the x-direction do not change at all (i.e. they are ”scaled” by a factor of 1), and of course still point in the x-direction after the application of T. Any other vector - i.e. those that have both components different than zero - change their direction after the application of T (see Figure 2.37).

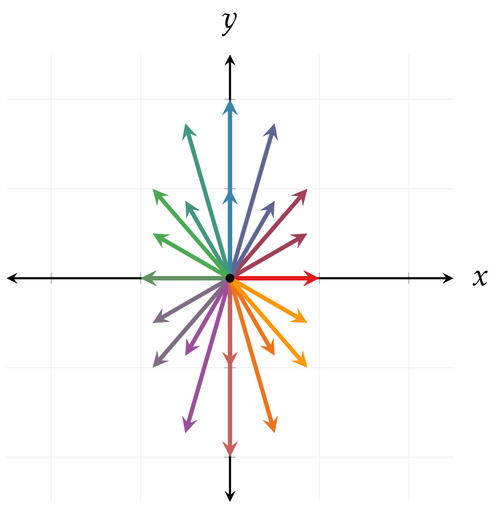
We call such vectors the **eigenvectors** of the transformation. The amount by which the



1 Some vectors.



2 Same vectors after the application of T.



3 The vectors before and after the application of T layered on top of eachother.

Figure 2.37 Some vectors before and after application of the y-scaling transformation T. Note how only the vectors pointing in the direction of the x- and y-axes stay in the same direction, while all the other vectors change their directions.

are scaled is then their respective **eigenvalues**.

Note 2.15 Pronounciation

The word *eigen* is a German word meaning ”own” (as in ”own rules”), or ”self” (as in self-made). We will see how this meaning fits the concept later in the section. The *ei* part it pronounced the same as the English word ”eye”, and the *g* is pronounced like the *g* in the English word dog (i.e. unlike the *g* in *generation*).

Example 2.53 Eigenvectors and eigenvalues

Text here

In matrix form, a vector $\vec{v}$ is an eigenvector of a transformation represented by the matrix A, if

$$A\vec{v} = \lambda\vec{v}, \tag{2.5.1}$$

where $\lambda \in \mathbb{R},\ \lambda \neq 0$. This kind of equation is typically called an **eigenvector equation**. When there are several eigenvectors for a transformation, each with its distinct eigenvalue, we simply add indeces to all relevant parts:

$$A\vec{v}_i = \lambda_i \vec{v}_i, \tag{2.5.2}$$

where again $\lambda_i \in \mathbb{R}$ and $\lambda_i \neq 0$.

Note 2.16 The zero-vector

Although technically the zero vector is indeed ”scaled” by any linear transformation - by infinitely many scalars - it is **not** considered an eigenvector, exactly because of the fact it has no unique eigenvalue.

Before continuing to explore some more examples of eigenvectors, there are two properties[11] of eigenvectors that are important to mention. Given a linear transformation T,

- A scale of any eigenvector $\vec{v}$ of T is also an eigenvector of T, with the same eigenvalue.

Proof 2.8 Eigenvector scale

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation represented by the square matrix A, with eigenvector $\vec{v}$ and its respective eigenvalue λ. Then

$$A\vec{v} = \lambda\vec{v}.$$

Replacing $\vec{v}$ with a scale of itself, i.e. $\vec{u} = \alpha\vec{v}$, then applying A to $\vec{u}$ gives us

$$A\vec{u} \overset{(1)}{=} A\left(\alpha\vec{v}\right) \overset{(2)}{=} \alpha A\vec{v} \overset{(3)}{=} \alpha\lambda\vec{v} \overset{(4)}{=} \lambda\alpha\vec{v} \overset{(5)}{=} \lambda\vec{u}.$$

[11] actually one property and one non-property

where

(1) Substitution of $\vec{u}$ by its definition $\vec{u} = \alpha\vec{v}$.

(2) Due to the linearity of A we can bring α out of the product.

(3) Resulting due to $\vec{v}$ being an eigenvector of A.

(4) The product of real numbers is commutative.

(5) Substituting back $\alpha\vec{v} = \vec{u}$.

Therefore, $\vec{u}$ is also an eigenvector of A (and thus T) with the same eigenvalue λ as $\vec{v}$.

QED

Since a linear transformation never has just a single eigenvector but infinitely many (i.e. its entire span), we will refer from now on the **families** of eigenvectors, all pointing in the same direction, represented by a single vector (usually a unit vector, but not necesseraly).

- The linear combination of two eigenvectors of T is **not neccessarily an eigenvector of T!** For example, consider the above transformation which scales all vectors by 2 in the y-direction: as we saw, any vector in the x-direction is an eigenvector of the transformation, and so does any vector in the y-direction. Specifically, the vectors $\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}0\\1\end{bmatrix}$ are two separate eigenvectors of the transformation (with eigenvalues 1 and 2, respectively), however the vector

$$\vec{c} = \vec{a} + \vec{b} = \begin{bmatrix}1\\1\end{bmatrix}$$

is **NOT** an eigenvector of the transformation, since

$$\begin{bmatrix}1\\1\end{bmatrix} \overset{T}{\mapsto} \begin{bmatrix}1\\2\end{bmatrix} \neq 2\begin{bmatrix}1\\1\end{bmatrix}.$$

2.5.2 Some examples

Example 2.54 Eigenvectors and eigenvalues of common linear transformations

Let's find the eigenvectors and their respective eigenvalues for the some basic linear transformations in 2-dimensions.

- **Skew (shear) in the x-direction**: any vector pointing in the x-direction is unchanged (i.e. scaled by a factor of 1), while any other vector changes its direction. Thus, the transformation has only a single family of eigenvectors, which we will represent by the vector $\hat{x}$, and their respective eigenvalue $\lambda = 1$.

- **Rotation around the origin**: no vector (except the zero-vector) is scaled by rotation, and therefore rotations have no eigenvectors. However, in 3-dimensions any rotation transformation does have an eigenvector, with eigenvalue $\lambda = 1$: it is of course the family of vectors pointing in the direction of the axis of rotation. We will discuss this further later in the section.

- **Reflection across the x-axis**: in this case, any vector pointing in the x-axis is not being changed - an eigenvector with eigenvalue $\lambda = 1$, and any vector pointing in the y-direction is reflected verically, i.e.

$$\begin{bmatrix} 0 \\ \beta \end{bmatrix} \overset{T}{\mapsto} \begin{bmatrix} 0 \\ -\beta \end{bmatrix} = -\begin{bmatrix} 0 \\ \beta \end{bmatrix},$$

  and is therefore an eigenvector with eigenvalue $\lambda = -1$. The transformation has no other eigenvectors.

- **Reflection across a line going through the origin**: much like the previous case, any vector lying on the reflection line will not change ($\lambda = 1$), and any vector pointing in an orthogonal direction to the line will flip ($\lambda = -1$). No other eigenvectors exist for this transformation.

2.5.3 Calulating eigenvectors

Calculating the eigenvectors of a given transformation, and their respective eigenvalues, is a rather easy procedure to perform once one has the transformation in matrix form. However, in order to understand *why* the procedure works it is useful to derive it first, which is what we'll do now.

We take the eigenvector equation (Equation 2.5.1) and rearrange it slightly:

$$A\vec{v} - \lambda\vec{v} = \vec{0}. \tag{2.5.3}$$

We can then group together all parts which include $\vec{v}$, but we must be careful: A is a matrix while λ is a scalar. This means that the term $A - \lambda$ has no meaning, since we haven't defined how to add or subtract matrices and real numbers. We therefore change Equation 2.5.3 a bit without changing its validity, by replacing $\lambda\vec{v}$ with $\lambda I\vec{v}$: i.e. instead of scaling $\vec{v}$ by a scalar, we scale it using the matrix λI, which yields the same result:

$$A\vec{v} - \lambda I\vec{v} = \vec{0}. \tag{2.5.4}$$

Note 2.17 That one weird trick

If you are not convinced the above trick works, consider the following:

$$\lambda = 3, \vec{v} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}.$$

Then the direct scaling of $\vec{v}$ by λ is

$$\lambda \vec{v} = \begin{bmatrix} 3 \\ 15 \\ -21 \end{bmatrix},$$

and scaling it using λI yields

$$\lambda I \vec{v} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ -7 \end{bmatrix},$$

i.e. we get exactly the same result.

Grouping together the parts with $\vec{v}$ gives:

$$(A - \lambda I)\vec{v} = \vec{0}. \tag{2.5.5}$$

Note that $A - \lambda I$ is a matrix, with the following form:

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} \\ &= \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}. \end{aligned} \tag{2.5.6}$$

Equation 2.5.5 tells us that $A - \lambda I$ sends the vector $\vec{v}$ to $\vec{0}$, and therefore $\vec{v}$ is in the kernel of $(A - \lambda I)$. Sinve by the definition of an eigenvector $\vec{v} \neq \vec{0}$, this means that the kernel of $A - \lambda I$ has more then just the zero vector, and thus

$$|A - \lambda I| = 0. \tag{2.5.7}$$

(this is derived from Equation 2.3.64 and Equation 2.3.65)

Therefore, if we solve Equation 2.5.7 for λ, we will get all the values of λ for which the eigenvector equation holds, and in turn we get all the eigenvalues λ_i of the linear transformation represented by A. We can then substitute each λ_i into the eigenvector equation and find its respective eigenvector family.

Example 2.55 Eigenvectors and eigenvalues of a 2×2 matrix

The matrix representing the y-scaling transformation discussed in the beginning of this section is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, in order to find the eigenvectors of A we solve the equation

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda).$$

In this case there are two solutions: $\lambda_1 = 1$ and $\lambda_2 = 2$. To find their corresponding eigenvectors, we subtitute them into the eigenvectors equation. First λ_1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

which corresponds to the following system of equations:

$$\begin{cases} 1 \cdot x + 0 \cdot y = x, \\ 0 \cdot x + 2 \cdot y = y, \end{cases}$$

for which the solution is $x \in \mathbb{R}$ and $y = 0$ - i.e. any vector pointing in the x-direction. Now for $\lambda_2 = 2$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix},$$

i.e. the following system of equations:

$$\begin{cases} 1 \cdot x + 0 \cdot y = 2x, \\ 0 \cdot x + 2 \cdot y = 2y. \end{cases}$$

The solution is of course $x = 0$ and $y \in \mathbb{R}$, meaning any vector pointing in the y-direction.

Example 2.56 Eigenvectors and eigenvalues of a 3×3 matrix

Let us now calculate the eigenvectors and their respective eigenvalues for the following 3×3 matrix:

$$A = \begin{bmatrix} 5 & -6 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

We start by calculating the determinant $|A - \lambda I|$:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 5-\lambda & -6 & -3 \\ 0 & -1-\lambda & 0 \\ 0 & 2 & 2-\lambda \end{vmatrix} \\ &= (5-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} - (-6)\cancel{\begin{vmatrix} 0 & 0 \\ 0 & 2-\lambda \end{vmatrix}} + 3(-3)\cancel{\begin{vmatrix} 0 & -1-\lambda \\ 0 & 2 \end{vmatrix}} \\ &= (5-\lambda)(-1-\lambda)(2-\lambda) \end{aligned}$$

The solutions of $|A - \lambda I| = 0$ are therefore

$$\lambda_1 = 5, \ \lambda_2 = -1, \ \lambda_3 = 2.$$

- $\boldsymbol{\lambda_1 = 5}$: solving $A\vec{v} = 5\vec{v}$ yields

$$\begin{bmatrix} 5 & -6 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 5\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \\ 5z \end{bmatrix},$$

  i.e.

$$\begin{cases} 5x - 6y - 3z &= 5x, \\ 0x - 1y + 0z &= 5y, \\ 0x + 2y + 2z &= 5z, \end{cases}$$

  for which the solution is

$$x \in \mathbb{R}, \ y = 0, \ z = 0.$$

  The first family of eigenvectors are the vectors pointing in the x-direction, and their respective eigenvalue is $\lambda = 5$. We can verify this:

$$\begin{bmatrix} 5 & -6 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x - 6\cdot 0 - 3\cdot 0 \\ 0x - 1\cdot 0 + 0\cdot 0 \\ 0x + 2\cdot 0 + 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 5x \\ 0 \\ 0 \end{bmatrix} = 5\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}.$$

- $\boldsymbol{\lambda_2 = -1}$: solving $A\vec{v} = -\vec{v}$ yields

$$\begin{bmatrix} 5 & -6 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ -z \end{bmatrix},$$

i.e.

$$\begin{cases} 5x - 6y - 3z &= -x, \\ 0x - 1y + 0z &= -y, \\ 0x + 2y + 2z &= -z, \end{cases}$$

for which the solution is

$$x = \frac{2}{3}y, \; y \in \mathbb{R}, \; z = -x.$$

Verifying the solution using the representative vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$:

$$\begin{bmatrix} 5 & -6 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \cdot 2 - 6 \cdot 3 + 3 \cdot 2 \\ 0 \cdot 2 - 1 \cdot 3 + 0 \cdot (-2) \\ 0 \cdot 2 + 2 \cdot 3 + 2 \cdot (-2) \end{bmatrix} = \begin{bmatrix} 10 - 18 + 6 \\ -3 \\ 6 - 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} = -\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}.$$

- $\lambda_3 = 2$: solving $A\vec{v} = 2\vec{v}$ yields

$$\begin{bmatrix} 5 & -6 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix},$$

i.e.

$$\begin{cases} 5x - 6y - 3z &= 2x, \\ 0x - 1y + 0z &= 2y, \\ 0x + 2y + 2z &= 2z, \end{cases}$$

for which the solution is

$$x = z, \; y = 0.$$

Verifying the solution using the representative vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix} 5 & -6 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \cdot 1 - 6 \cdot 0 - 3 \cdot 1 \\ 0 \cdot 1 - 1 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 2 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

To summarize: the linear transformation represented by the matrix

$$A = \begin{bmatrix} 5 & -6 & -3 \\ 0 & -1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

has three families of eigenvectors:

Eigenvalue	Eigenvector
$\lambda_1 = 5$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
$\lambda_2 = -1$	$\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$
$\lambda_3 = 2$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Example 2.57 The eigenvectors of reflections in $\mathbb{R}^2$, polar form

In $\mathbb{R}^2$, The reflection transformation across the x-axis has two families of eigenvectors: vectors pointing in x-direction (with eigenvalue $\lambda = 1$), and vectors pointing in the y-direction (with eigenvalue $\lambda = -1$). The reflection transformation across the y-axis has the same families, but their eigenvalues are flipped: the vectors pointing in the x-direction have eigenvalue $\lambda = -1$, and those pointing in the y-direction have eigenvalu $\lambda = 1$.

Generalizing to reflections in any direction, we expect to see a similar result: one family of eigenvectors being vectors pointing in the direction of the reflection line with eigenvalue $\lambda = 1$, and the other family of vectors pointing in the orthogonal direction, with eigenvalue $\lambda = -1$. Let us now calculate this directly from the matrix representations of the transformation. We start with the angle-based representation, where the reflection line is represented by its angle θ relative to the x-axis (Equation 2.3.22):

$$\mathrm{Ref}_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}.$$

Using **??**, we expect the vectors with eigenvalue $\lambda = 1$ to have the following form:

$$\begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix},$$

since this is a structure of a vector pointing in the direction with angle θ relative to the x-axis. The ratio of the components of these vectors is therefore expected to be

$$\frac{x}{y} = \frac{\cos(\theta)}{\sin(\theta)}.$$

The vectors with eigenvalue $\lambda = -1$ should be orthogonal to the previous family, i.e. have the structure

$$\begin{bmatrix} -r\sin(\theta) \\ r\cos(\theta) \end{bmatrix},$$

or component ratio of

$$\frac{x}{y} = -\frac{\sin(\theta)}{\cos(\theta)}.$$

Now that we know what to expect, let's start with the actual calculations:

$$\begin{aligned}
0 &= \left|\mathrm{Ref}_\theta - \lambda I\right| \\
&= (\cos(2\theta) - \lambda)(-\cos(2\theta) - \lambda) - \sin^2(\theta) \\
&= -\cos^2(2\theta) - \cancel{\lambda\cos(2\theta)} + \cancel{\lambda\cos(2\theta)} + \lambda^2 - \sin^2(\theta) \\
&= \lambda^2 - 1.
\end{aligned}$$

(the last equality stems from Equation 0.6.3)
We see that the solutions for the above equation in λ are $\lambda_1 = 1$ and $\lambda_2 = -1$, as expected. Substituting each of these eigenvalues into the eigenvector equation yields:

- $\lambda = 1$: the eigenvector equation

  $$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

  gives rise to the system

  $$\begin{cases} \cos(2\theta)x + \sin(2\theta)y &= x, \\ \sin(2\theta)x - \cos(2\theta)y &= y. \end{cases}$$

  The solution of the system is

  $$\begin{aligned}
  y &= \frac{x\sqrt{1-\cos^2(2\theta)}}{\cos(2\theta)+1} \\
  &= \frac{x\sin(2\theta)}{\cos(2\theta)+1} \\
  &= \frac{\cancel{2}x\sin(\theta)\cancel{\cos(\theta)}}{\cancel{2}\cos^{\cancel{2}}(\theta)} \\
  &= \frac{x\sin(\theta)}{\cos(\theta)}.
  \end{aligned}$$

  Rearranging this yields

  $$\frac{x}{y} = \frac{\cos\theta}{\sin\theta},$$

  as expected.

- $\lambda = -1$: the eigenvector equation

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

gives rise to the system

$$\begin{cases} \cos(2\theta)x + \sin(2\theta)y & = -x, \\ \sin(2\theta)x - \cos(2\theta)y & = -y. \end{cases}$$

The solution of the system is

$$\begin{aligned} y &= \frac{x\sqrt{1-\cos^2(2\theta)}}{\cos(2\theta)-1} \\ &= \frac{x\sin(2\theta)}{\cos(2\theta)-1} \\ &= \frac{2x\sin(\theta)\cos(\theta)}{\cancel{1} - 2\sin^2(\theta) - \cancel{1}} \\ &= -\frac{\cancel{2}x\cancel{\sin(\theta)}\cos(\theta)}{\cancel{2}\sin^{\cancel{2}}(\theta)} \\ &= -\frac{x\cos(\theta)}{\sin(\theta)}. \end{aligned}$$

And rearranging this yields

$$\frac{x}{y} = -\frac{\sin(\theta)}{\cos(\theta)},$$

also as expected.

Example 2.58 The eigenvectors of reflections in $\mathbb{R}^2$, slope form

In this example we will again calculate the eigenvectors and eigenvalues of the general reflection transformation, this time by using the slope m of the reflection line, and the matrix representation

$$\mathrm{Ref}_m = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Again we expect the two eigenvalues $\lambda = \pm 1$. The eigenvector family corresponding to $\lambda = 1$ should have a component ratio $\frac{y}{x} = m$, i.e. $y = mx$ (the line equation) - and the eigenvector family corresponding to the eigenvalue $\lambda = -1$ should be orthogonal to the other family, i.e. has component ratio $\frac{y}{x} = -\frac{1}{m}$ (i.e. $y = -\frac{1}{m}x$).

<ins>The calculation</ins>: the determinant $\left|\mathrm{Ref}_m - \lambda\right|$ is then

$$\begin{aligned}
\left|\mathrm{Ref}_m - \lambda\right| &= \begin{vmatrix} \frac{1-m^2}{1+m^2} - \lambda & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} - \lambda \end{vmatrix} \\
&= \left(\frac{1-m^2}{1+m^2} - \lambda\right)\left(\frac{m^2-1}{1+m^2} - \lambda\right) - \frac{4m^2}{\left(1+m^2\right)^2} \\
&= \frac{\left(1-m^2\right)\left(m^2-1\right)}{\left(1+m^2\right)^2} - \lambda\frac{1-m^2}{1+m^2} - \lambda\frac{m^2-1}{1+m^2} + \lambda^2 - \frac{4m^2}{\left(1+m^2\right)^2}.
\end{aligned}$$

Setting $a = 1 - m^2$ and $b = 1 + m^2$ we get that $m^2 - 1 = -a$, and we continue:

$$\begin{aligned}
\ldots &= \frac{a(-a)}{b^2} - \cancel{\lambda\frac{a}{b}} - \cancel{\lambda\frac{-a}{b}} + \lambda^2 - \frac{4m^2}{b^2} \\
&= \frac{-a^2 - 4m^2}{b^2} + \lambda^2 \\
&= \frac{-(1-m^2) - 4m^2}{\left(1+m^2\right)^2} + \lambda^2 \\
&= \frac{-\left(1 - 2m^2 + m^4\right) - 4m^2}{\left(1+m^2\right)^2} + \lambda^2 \\
&= \frac{-1 + 2m^2 - m^4 - 4m^2}{\left(1+m^2\right)^2} + \lambda^2 \\
&= \frac{-1 - 2m^2 - m^4}{\left(1+m^2\right)^2} + \lambda^2 \\
&= -\frac{1 + 2m^2 + m^4}{\left(1+m^2\right)^2} + \lambda^2 \\
&= -\frac{\left(1+m^2\right)^2}{\left(1+m^2\right)^2} + \lambda^2 \\
&= -1 + \lambda^2,
\end{aligned}$$

i.e. $\lambda^2 = 1$ and thus $\lambda = \pm 1$ as expected. We now subtitute these eigenvalues in the eigenvector equation:

- $\lambda = 1$: the matrix equation is

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

for which the system of equations is

$$\begin{cases} \frac{1-m^2}{1+m^2}x + \frac{2m}{1+m^2}y &= x, \\ \frac{2m}{1+m^2}x + \frac{m^2-1}{1+m^2}y &= y, \end{cases}$$

and its solution is

$$y = mx,$$

i.e. $\frac{y}{x} = m$ as expected.

- $\lambda = -1$: the matrix equation is

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix},$$

for which the system of equations is

$$\begin{cases} \frac{1-m^2}{1+m^2}x + \frac{2m}{1+m^2}y &= -x, \\ \frac{2m}{1+m^2}x + \frac{m^2-1}{1+m^2}y &= -y, \end{cases}$$

and its solution is

$$y = -\frac{x}{m},$$

i.e. $\frac{y}{x} = -\frac{1}{m}$ as expected.

2.5.4 Characteristic polynomial

Did you notice that in all of the above examples the expression $|A - \lambda I|$ is a polynomial in λ? This is not a coincidence: any expression of such form is a polynomial in λ, its degree depending on the form of A. We call this polynomial the **characteristic polynomial** of A. As we saw in the examples, the roots of the characteristic polynomial are the eigenvectors of A. A more precise definition of the characteristic polynomial is given below.

Definition 2.5 Characteristic polynomial of a matrix A

Let A be a square matrix representing some transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Then

$$P(\lambda) = |A - \lambda I|, \tag{2.5.8}$$

is called the characteristic polynomial of A (and T). The roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of P are the eigenvalues of A (and T).

Let us examine some values and coefficients of the characteristic polynomial $P(\lambda)$.

Subtituting $\lambda = 0$ into P produces

$$P(0) = |A - 0I| = |A|, \tag{2.5.9}$$

i.e. the value $P(0)$ is always the determinant of A. Recall that for a polynomial, this value is the free coefficient of P. We can see this fact clearly when we consider a generic 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Its characteristic polynomial is then

$$\begin{aligned} P(\lambda) &= |A - \lambda I| \\ &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= ad - a\lambda - d\lambda + \lambda^2 - bc \\ &= \underbrace{\boxed{(ad-bc)}}_{|A|} + \underbrace{\boxed{(a+d)}}_{\mathrm{Tr}(A)}\lambda + \lambda^2. \end{aligned}$$

We see that not only is the free coefficient of P equal to the determinant of A, but also that the coefficient of λ is the trace of A. This is not just the case in 2×2 matrices, but all square matrices: the coefficient of λ^{n-1} is the trace of the matrix (up to a sign, i.e. $\pm\mathrm{Tr}(A)$). For example, the characteristic polynomial of a generic 3×3 matrix[12]

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

is

$$\begin{aligned} P(\lambda) = |B - \lambda I| &= \begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} \\ &= (a-\lambda)\Big((e-\lambda)(i-\lambda) - fh\Big) - b\Big(d(i-\lambda) - fg\Big) + c\Big(dh - (e-\lambda)g\Big) \\ &= (a-\lambda)\Big(ei - e\lambda - \lambda i + \lambda^2 - fh\Big) - b\Big(di - d\lambda - fg\Big) + c\Big(dh - eg + \lambda g\Big) \\ &= aei - ae\lambda - ai\lambda + a\lambda^2 - afh - ei\lambda - i\lambda^2 - \lambda^3 - fh\lambda - bdi + bd\lambda + bfg + cdh - ceg + cg\lambda \\ &= aei - ae\lambda - afh - ai\lambda + a\lambda^2 - bdi + bd\lambda + bfg + cdh - ceg + cg\lambda - ei\lambda + e\lambda^2 + fh\lambda \\ &+ i\lambda^2 - \lambda^3 \\ &= \boxed{aei - afh - bdi + bfg - cdh - ceg} + (-ae - ai - ei - fh + bd + cg - ei + fh)\lambda \\ &+ \Big(\boxed{a+e+i}\Big)\lambda^2 - \lambda^3. \end{aligned} \tag{2.5.10}$$

We see that highlighted parts are exactly $\boxed{|B|}$ and $\boxed{\mathrm{Tr}(B)}$, respectively.

[12] note that e and i are just some real numbers, and **not** the constants e and i, respectively.

2.5.5 Special cases

In the case of upper- or lower-triangular matrices, the eigenvalues are simply the main diagonal elements, e.g. the eigenvalues of the matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \tag{2.5.11}$$

are

$$\lambda = \{a_{11}, a_{22}, a_{33}, \dots, a_{nn}\},$$

i.e. written simply,

$$\lambda_i = a_{ii}. \tag{2.5.12}$$

! **To be written**: rest of the section !

2.6 DECOMPOSITIONS

! **To be written**: the entire section. !

2.7 FUNCTIONS AS VECTORS

Note 2.18 Calculus ahead!

This section uses ideas discussed in the chapter about 1-dimensional real calculus (Chapter 1). While strict knowledge of calculus is not necessary, some concepts would be rather difficult to understand without it.
In addition, this section is merely a brief introduction to a broader idea, and is not meant in any way to be an exhaustive analysis of the topic.

!

2.7.1 Properties of vectors

Up until now we used a rather informal definition for vectors (2.1). While the formal definition will be discussed in the next chapter, it is worth while to review the basic properties of vectors we saw so far. We start with the scaling of vectors:

- **It is 'close'**: the result of scaling a vector is itself a vector.
- **The scalar 1 is neutral to scaling**: i.e. $1 \cdot \vec{v} = \vec{v}$.
- **It is associative**: for any $\alpha, \beta \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$,

$$\alpha \cdot \left(\beta \vec{v}\right) = \left(\alpha \cdot \beta\right) \vec{v}.$$

And for vector addition:

- **It is close**: the sum of any two vectors is also a vector.
- **It is commutative**: the order of addition does not change its result.
- **It is associative**: while not discussed directly, it is obvious that when adding together any three vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, we can change the order of addition and that too will not change its result: calculating $\vec{u} + \vec{v}$ first and then adding $\vec{w}$ to the result is the same as calculating $\vec{v} + \vec{w}$ first and then adding $\vec{u}$ to the result, i.e.

$$\left(\vec{u} + \vec{v}\right) + \vec{w} = \vec{u} + \left(\vec{v} + \vec{w}\right).$$

- **$\vec{0}$ is neutral to addition**: i.e. the addition of $\vec{0}$ to any vector $\vec{v}$ results in $\vec{v}$.
- **Any vector has an additive inverse**: given a vector $\vec{u} \in \mathbb{R}^n$, there is always a vector $\vec{v} \in \mathbb{R}^n$ such that

$$\vec{u} + \vec{v} = \vec{0},$$

  namely the vector $\vec{u} = -\vec{v}$.

The two operations also have two important properties together:

- **Vector addition is distributive**: for any $\alpha \in \mathbb{R}$ and $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$\alpha \left(\vec{u} + \vec{v}\right) = \alpha \vec{u} + \alpha \vec{v}.$$

- **Scalar multiplication is distributive**: for any $\alpha, \beta \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$,

$$\left(\alpha + \beta\right) \vec{v} = \alpha \vec{v} + \beta \vec{v}.$$

All these properties seem rather obvious, but not all mathematical structures actually have them: for example, in a later chapter in the book we learn about **groups**, which do not have a scaling operator and don't necessarily have some of these properties, e.g. not all of them are commutative under their own operation.

Real functions, on the other hand, do have all these properties. We can therefore apply to real functions all the stuff we learned about vectors in this chapter. While we're not promised that everything will *look* the same, their general behavior is identical to that of vectors: the two obvious examples are scaling and addition:

- **Scaling**: given a real function f we can always scale it by multiplying its output by any real number α: if $f(x) = y$, then
$$\alpha f(x) = \alpha y.$$
For example, we can scale the polynomial $P(x) = x^3 - 2x + 5$ by a factor of 7, yielding
$$7P(x) = 7 \cdot x^3 - 7 \cdot 2x + 7 \cdot 5 = 7x^3 - 14x + 35.$$
And indeed we see that the result is itself a real function.

- **Addition**: any two real functions f, g can be added together. For example, given the functions $f(x) = \mathrm{e}^x$ and $g(x) = 5\sin(x)$, their sum is then
$$\left[f + g\right](x) = \mathrm{e}^x + 5\sin(x),$$
which is indeed a real function by itself.

You should go over all the above properties of vectors and verify for yourself that real functions do indeed posses them.

2.7.2 Smooth functions

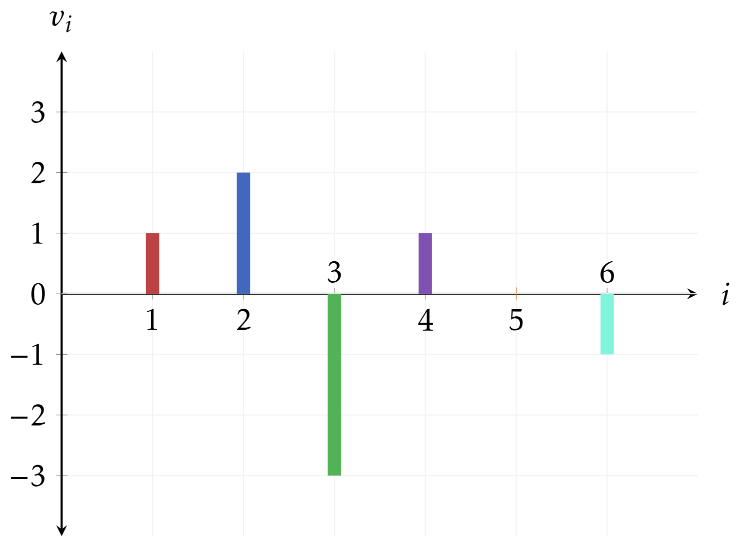
Before we move on, we should limit our further discussion to a specific set of real functions in order to avoid some annoying details which might rise later, and are not critical for the understanding of the idea of functions as vectors. Thus, from now on in the section we will only discuss functions of the set C^∞. **Reminder**: a function f which is continuous on an interval $I \subseteq \mathbb{R}$ is said to be in C^k $(k \in \mathbb{N})$ if $f', f'', f''', \dots, f^{(k)}$ are all continuous on I as well. The set C^∞ is thus the set of all functions which are infinitely many times differentiable in some interval $I \subseteq \mathbb{R}$.

2.7.3 Function values as vector components

The first important thing to do when using vectors is to choose a basis set to represent them. For any $\mathbb{R}^n$ it is rather easy to understand how this representation looks like: we simply write the vector with n components. How can we do that with functions? Well, consider the following vector in $\mathbb{R}^6$:

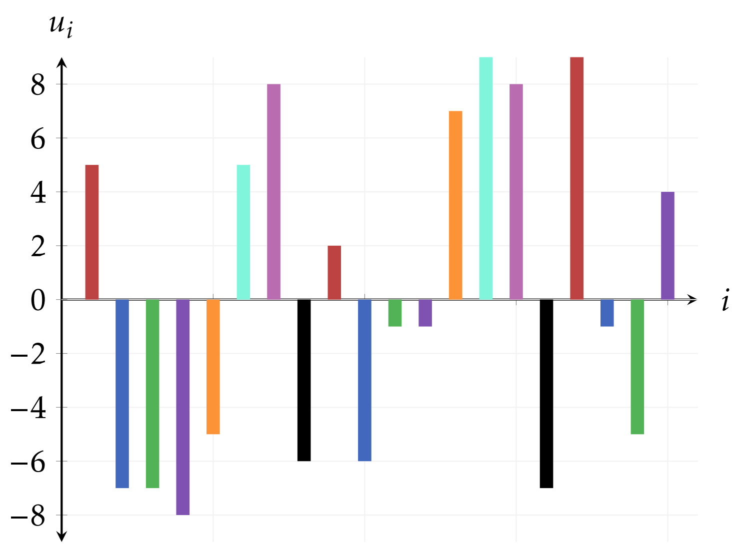
$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \tag{2.7.1}$$

While drawing 6-dimensional spaces is rather difficult, we can draw as a bar chart each of the components v_i as a function of the component index i:

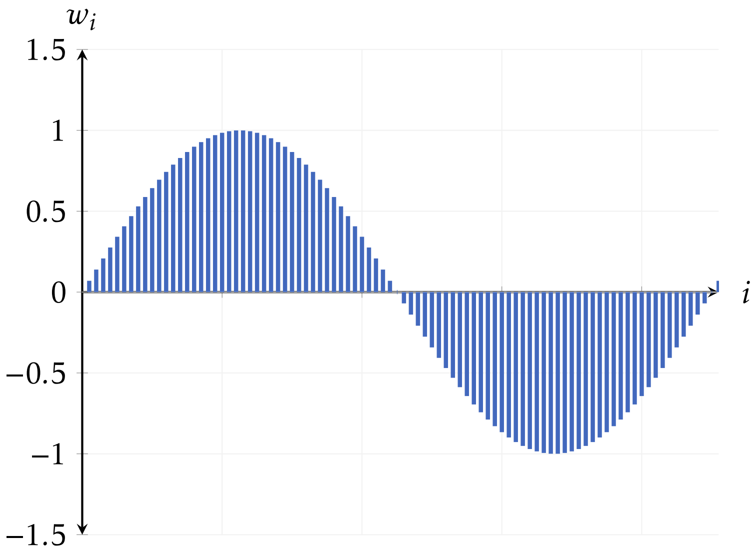


Now let's do the same with a vector in $\mathbb{R}^{20}$, e.g.

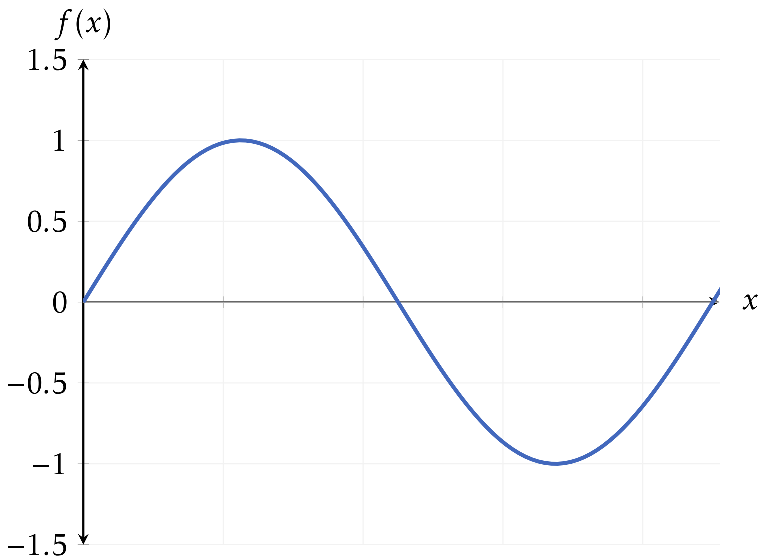
$$\vec{u} = \left[5,\ -7,\ -7,\ -8,\ -5,\ 5,\ 8,\ -6,\ 2,\ -6,\ -1,\ -1,\ 7,\ 9,\ 8,\ -7,\ 9,\ -1,\ -5,\ 4\right]^\top : \tag{2.7.2}$$



...and for some vector $\vec{w} \in \mathbb{R}^{90}$:



What we see here is that we can use this method for any natural number. We can also easily generalize it for any integer by shifting our indexing to include negative integers. In fact, we can expand it to any real number by taking this idea to the continuous limit: given any $x \in \mathbb{R}$, we assign it some $f(x) \in \mathbb{R}$ and say that $f(x)$ is the component corresponding to the index x... and we just defined a real function! For example, the function $f(x) = \sin(x)$:



We saw in the comparison between vectors in functions how scaling and addition behave in functions in a similar way to their behavior in vectors. Now we will describe how to view these operations in a similar way to their component-wise depiction in vectors (Equation 2.1.13 and Equation 2.1.14):

- **Scaling**: given a scalar $\alpha \in \mathbb{R}$ and a function f which is smooth on an interval I, for each component $x \in I$, if x is mapped by f to $y \in \mathbb{R}$, then the function αf maps x to αy, i.e.

$$x \overset{f}{\mapsto} y \Rightarrow x \overset{\alpha f}{\mapsto} \alpha y.$$

Example: for $f(x) = x^2$, $x \overset{f}{\mapsto} x^2$. Therefore for αf, $x \overset{\alpha f}{\mapsto} \alpha x^2$.

- **Addition**: given two functions f, g which are smooth on an interval I, for any $x \in I$ if $x \overset{f}{\mapsto} y_f$ and $x \overset{g}{\mapsto} y_g$, then

$$x \overset{f+g}{\mapsto} y_f + y_g.$$

Example: for $f(x) = 3x^2$ and $g(x) = -7x$, $x \overset{f+g}{\mapsto} 3x^2 - 7x$.

2.7.4 Scalar product, norm

Given two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, the scalar product $\vec{u} \cdot \vec{v}$ is defined using the respective components of the vectors as

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i.$$

The natural extension of this sum of products to the continuous realm is via an integral (see **??**), i.e. given two functions f, g which are smooth on the interval (a, b) (where $a, b \in \mathbb{R}$), then

$$f \cdot g = \int_a^b f(x) g(x) \, \mathrm{d}x. \tag{2.7.3}$$

Example 2.59 Scalar product of two functions

Given the functions $f(x) = x^3 + 3x^2$ and $g(x) = \frac{1}{x}$, the scalar product $f \cdot g$ in the interval $I = [-1, 1]$ is

$$\begin{aligned} f \cdot g &= \int_{-1}^{1} f(x) g(x) \, \mathrm{d}x \\ &= \int_{-1}^{1} \frac{x^3 + 3x^2}{x} \, \mathrm{d}x \\ &= \int_{-1}^{1} \left(x^2 + 3x\right) \mathrm{d}x \\ &= \left.\left(\frac{1}{3}x^3 + \frac{3}{2}x^2\right)\right|_{-1}^{1} \end{aligned}$$

$$= \frac{1}{3} + \frac{3}{2} - \frac{-1}{3} - \frac{3}{2}$$
$$= \frac{2}{3}.$$

This brings up an interesting question: what does it mean when the scalar product of two functions is 0? For example, consider the two functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Their scalar product $f \cdot g$ on the interval $I = (-\pi, \pi)$ is

$$\begin{aligned} f \cdot g &= \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} \sin(x)\cos(x)\,\mathrm{d}x = \left.\frac{\sin^2(x)}{2}\right|_{-\pi}^{\pi} \\ &= \frac{1}{2}\left(\sin^2(\pi) - \sin^2(-\pi)\right) = \frac{1}{2}(0-0) = 0. \end{aligned}$$

Much like with vectors in $\mathbb{R}^n$ we say that these two functions are orthogonal. In turn, this means that we can use them to form an orthonormal basis set to express all smooth functions on the same interval. In fact - a very similar idea, i.e. using infinite series of trigonometric functions to construct other functions, is used to form the **Fourier series**, which we explore in greater depth in **??**).

The **Legendre polynomial** are an infinite set of orthogonal polynomials defined on the interval $[-1, 1]$, which arise in many different situations throughout science such as the wave functions of the electrons in a Hydrogen-like atom. The first few Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right), \\ P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right), \\ P_4(x) &= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \\ P_5(x) &= \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right), \\ P_6(x) &= \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right), \\ &\vdots \end{aligned} \tag{2.7.4}$$

with the general formula for the n-th Legendre polynomial being

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(x-1)^{n-k}(x+1)^k. \tag{2.7.5}$$

Let's check two combinations from the first 7 Legendre polynomials to see that they functions are indeed orthogonal:

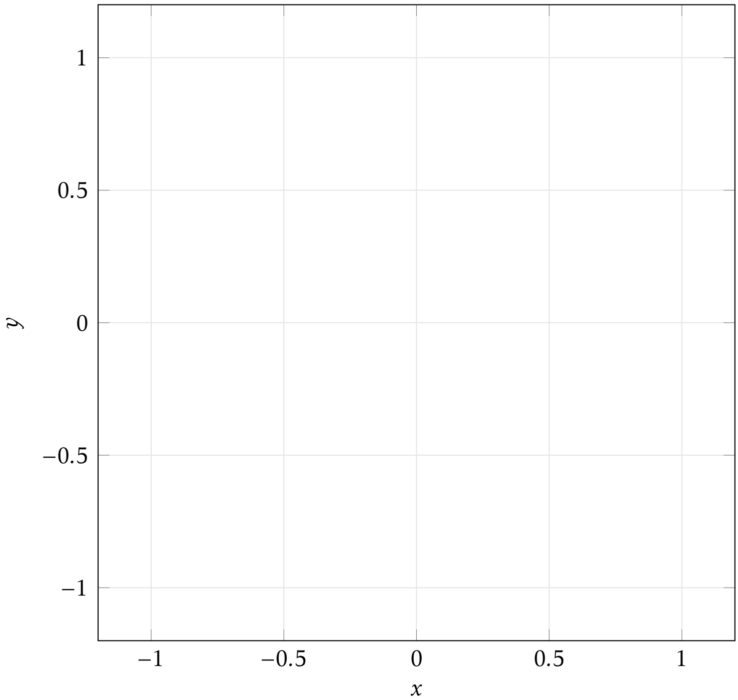


Figure 2.38 The first 6 Legendre polynomials.

- <u>P_1 and P_3:</u>

$$P_1 \cdot P_3 = \frac{1}{2}\int_{-1}^{1} \left(5x^4 - 3x^2\right) \mathrm{d}x = \frac{1}{2}\left(x^5 - x^3\right)\Big|_{-1}^{1} = 1 - 1 - 1 + 1 = 0.$$

- <u>P_2 and P_5:</u>

$$\begin{aligned} P_2 \cdot P_5 &= \frac{1}{16}\int_{-1}^{1} \left(3x^2 - 1\right)(63x^5 - 70x^3 + 15x)\,\mathrm{d}x \\ &= \frac{1}{16}\int_{-1}^{1} \left(189x^7 - 210x^5 + 45x^3 - 63x^5 + 70x^3 - 15x\right) \mathrm{d}x \\ &= \frac{1}{16}\left(27x^8 - 42x^6 + 15x^4 - 12x^6 + 23x^4 - 7.5x^2\right)\Big|_{-1}^{1} \\ &= \frac{1}{16}\left[(27 - 42 + 15 - 12 + 23 - 7.5) - (27 - 42 + 15 - 12 + 23 - 7.5)\right] \\ &= 0. \end{aligned}$$

2.7.5 Basis sets

2.7.6 Operators

In the context of vectors, linear transformations are functions that obey the two properties of scalability and additivity (Section 2.2). With smooth real functions we can introduce a similar idea: **linear operators**. In our context, an operator is itself a function, however instead of being applied to vectors it is applied to smooth functions in some interval $[a,b]$, i.e.

$$O : C^{\infty} \to C^{\infty}. \tag{2.7.6}$$

Example 2.60 Operator

Let O be an operator $O : C^{\infty} \to C^{\infty}$ defined as

$$O\big(f(x)\big) = 2\big[f(x)\big]^2.$$

The following table shows some results of applying O to different smooth functions:

$f(x)$	$O\big(f(x)\big)$
x	$2x^2$
x^2	$2x^4$
x^2+1	$2\left(x^2+1\right)^2$
$\sqrt{\frac{x}{2}}$	x
e^x	$2e^{2x}$
$\sin(x)$	$2\sin^2(x)$

A **linear operator** is therefore an operator which behaves like a linear transformation, i.e. given the operator $O : C^{\infty} \to C^{\infty}$ it is linear if it obeys the following two criteria:

- **Scalability**: for any $\alpha \in \mathbb{R}$ and $f \in C^{\infty}$,

$$O\big(\alpha f(x)\big) = \alpha O\big(f(x)\big).$$

- **Additivity**: for any $f, g \in C^{\infty}$,

$$O\big(f(x) + g(x)\big) = O\big(f(x)\big) + O\big(g(x)\big).$$

One very well known operator which obeys these two criteria is the derivative operator: recall that for any real number α and smooth functions f, g,

$$f'(\alpha x) = \alpha f'(x),$$

$$\left(f(x)+g(x)\right)' = f'(x)+g'(x). \tag{2.7.7}$$

We can therefore apply to derivatives all the general ideas we learned about linear transformations! This is very useful, and will be used in later chapters.

In fact, in some cases we can even express the derivative opeator as a matrix: consider the following set of real smooth functions:

$$B = \left\{1, x, x^2, x^3, \dots\right\}. \tag{2.7.8}$$

We can write any smooth function as an infinite sum of the functions in B, i.e. a Maclaurin expansion (**??**). Therefore, the set spans C^{∞}. For example,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

Since all the terms of the series are needed in order to express all the possible functions in C^{∞}, it is also has a minimal set which spans C^{∞} - and thus B is a basis set of C^{∞}.

Note 2.19 Orthogonality of B

B is not an orthogonal set on $\mathbb{R}$, since e.g.

$$x \cdot x^2 = \int_{-\infty}^{\infty} x^3 \, dx = \frac{1}{4}x^4\Big|_{-\infty}^{\infty} = \infty + \infty = \infty.$$

!

We can represent any function spanned by B as an infinite column vector, where each component is the respective coefficient of $1, x, x^2, \dots$ in the function's Maclaurin expansion. For example,

$$e^x = \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \\ \frac{1}{3!} \\ \frac{1}{4!} \\ \vdots \end{bmatrix}$$

The basis functions in B can then be written as

$$1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad x^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \quad x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix}, \quad \dots$$

Recall that the first derivative of a general n-degree polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

is

$$P'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na^n x^{n-1}.$$

(**??**)

We can describe the derivative as follows: for any n we set the coefficient of x^{n-1} in the derivative to na_n, and set the coefficient of x^n to be zero:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 4a_4 \\ \vdots \end{bmatrix},$$

and the effect on the basis vector is then

$$\begin{array}{ccccc} P(x) & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \\ & \mapsdown & \mapsdown & \mapsdown & \mapsdown \\ P'(x) & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} & \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ \vdots \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ \vdots \end{bmatrix} \end{array}$$

Now we can easily write the matrix representation of the derivative in this basis:

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.7.9}$$

Since the derivative transforms the first basis vector into the zero vector (i.e. the derivative of all constants is always zero), D has a column of zeros. Having a zero column means that the determinant of the derivative is zero. This makes sense: the derivative "looses" a dimension, i.e. it isn't an injection: as mentioned, all constants are transformed into zero - and thus the derivative isn't invertible. The closest we get to recover an original function given its derivative is by using the anti-derivative (i.e. indefinite integrals) - but this gives us only a family of functions and not the any specific one we used.

Note 2.20 Infinite dimensions

It is important to note that we didn't define what infinite column vectors and matrices are, and a-priori there is no guarantee that we can even perform any of the operations we use for finite-dimensional vectors and matrices. However, what we showed here is correct - but a more advanced and rigorous linear algebra is needed in order to prove it.

We can use the Gram-Schmidt process to orthogonalize the basis B. In order to avoid infinite scalar products we use the interval $I=[-1,1]$. First, let's quickly calculate the first three square norms of B:

$$\begin{aligned}
\|1\|^2 &= \int_{-1}^{1} 1\cdot 1\,\mathrm{d}x = x\Big|_{-1}^{1} = 2,\\
\|x\|^2 &= \int_{-1}^{1} x\cdot x\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3},\\
\left\|x^2\right\|^2 &= \int_{-1}^{1} x^4\cdot x\,\mathrm{d}x = \frac{1}{5}x^5\Big|_{-1}^{1} = \frac{2}{5},\\
&\vdots\\
\left\|x^n\right\|^2 &= \int_{-1}^{1} x^{2n}\cdot x\,\mathrm{d}x = \frac{1}{2n+1}x^{2n}\Big|_{-1}^{1} = \frac{2}{2n+1}.
\end{aligned}$$

The following table summarizes the dot product of any two of the basis functions $1, x, x^2$ and x^3 on $[-1,1]$:

	1	x	x^2	x^3
1	2	0	$\frac{2}{3}$	0
x	0	$\frac{2}{3}$	0	$\frac{2}{5}$
x^2	$\frac{2}{3}$	0	$\frac{2}{5}$	0
x^3	0	$\frac{2}{5}$	0	$\frac{2}{7}$

Using these values we can now apply the GSP (we use the braket notation as to not confuse the scalar product with the regular product):

$$\begin{aligned}
1 &\longrightarrow 1,\\
x &\longrightarrow x - \frac{\langle 1, x\rangle}{2} = x - 0 = x,
\end{aligned}$$

$$
\begin{aligned}
x^2 &\longrightarrow x^2 - \frac{\langle 1, x^2 \rangle}{2} - \frac{\langle x, x^2 \rangle}{\frac{2}{3}} x = x^2 - \frac{1}{3}, \\
x^3 &\longrightarrow x^3 - \frac{\langle 1, x^3 \rangle}{2} - \frac{\langle x, x^3 \rangle}{\frac{2}{3}} x - \frac{\langle x^2, x^3 \rangle}{\frac{2}{5}} x^2 = x^3 - \frac{\frac{2}{5}}{\frac{2}{3}} x = x^3 - \frac{3}{5} x.
\end{aligned}
$$

The resulting orthogonal basis set is exactly the Legendre polynomials, up to scaling! This is not really suprising, as the Legendre polynomials are orthogonal, after all.

2.7.7 Eigenfunctions

Some operators have **eigenfunctions**: functions that are scaled when the operator is applied to them. In the case of the derivative operator, for example, one such function is the exponent function, since

$$\left[\mathrm{e}^{\alpha x}\right]' = \alpha \mathrm{e}^{\alpha x}. \tag{2.7.10}$$

The corresponding eigenvalue of $\mathrm{e}^{\alpha x}$ is therefore α.

2.8 DUAL VECTORS

2.8.1 Functionals

In the context of linear algebra, a **functional** is a linear function which takes a vector and returns a scalar, e.g. in the case of $\mathbb{R}^n$,

$$\varphi : \mathbb{R}^n \to \mathbb{R}. \tag{2.8.1}$$

The linearity of the functional means that given two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and two scalars $\alpha, \beta \in \mathbb{R}$,

$$\varphi\left(\alpha \vec{u} + \beta \vec{v}\right) = \alpha \varphi\left(\vec{u}\right) + \beta\left(\vec{v}\right). \tag{2.8.2}$$

Example 2.61 A functional over $\mathbb{R}^n$

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x - y.$$

Then e.g.

$$
\begin{aligned}
\varphi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= 2 \cdot 1 - 0 = 2, \\
\varphi\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) &= 2 \cdot 1 - (-4) = 6,
\end{aligned}
$$

$$\varphi\left(\begin{bmatrix}-3\\-2\end{bmatrix}\right)=2\cdot(-3)-(-2)=-4,$$

etc. Let us show that this is indeed a linear function. Given two vectors

$$\vec{u}=\begin{bmatrix}u_x\\u_y\end{bmatrix}\quad\vec{v}=\begin{bmatrix}v_x\\v_y\end{bmatrix},$$

and the scalars α and β,

$$\begin{aligned}\varphi\left(\alpha\vec{u}+\beta\vec{v}\right)&=\varphi\left(\begin{bmatrix}\alpha u_x+\beta v_x\\\alpha u_y+\beta v_y\end{bmatrix}\right)\\&=2\left(\alpha u_x+\beta v_x\right)-\left(\alpha u_y+\beta v_y\right)\\&=2\alpha u_x+2\beta v_x-\alpha u_y-\beta v_y\\&=\left(2\alpha u_x-\alpha u_y\right)+\left(2\beta v_x-\beta v_y\right)\\&=\alpha\left(2u_x-u_y\right)+\beta\left(2v_x-v_y\right)\\&=\alpha\varphi\left(\vec{u}\right)+\beta\varphi\left(\vec{v}\right).\end{aligned}$$

Since functionals are linear, the most general functional over $\mathbb{R}^n$, when acting on a general vector

$$\vec{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}\tag{2.8.3}$$

gives the following output:

$$\varphi\left(\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}\right)=\alpha_1v_1+\alpha_2v_2+\cdots+\alpha_nv_n=\sum_{i=1}^{n}\alpha_iv_i.\tag{2.8.4}$$

2.8.2 Duality

By a closer examination of Equation 2.8.4 we notice that it can be written as a scalar product of the vector

$$\vec{\alpha}=\begin{bmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{bmatrix}\tag{2.8.5}$$

and the vector $\vec{v}$ as defined in Equation 2.8.3. This means that in a sense, applying a functional to a vector $\vec{v}$ is identical to performing a scalar product of some coefficient vector and $\vec{v}$. In fact, we can actually define the functional φ as defined in Equation 2.8.4 via the scalar product, i.e.

$$\varphi\left(\vec{v}\right) = \vec{\alpha} \cdot \vec{v}.$$

Example 2.62 Functional as vector

The functional defined in Example 2.61 can be defined via the vector

$$\vec{\alpha} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

since for a general vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

the dot product $\vec{\alpha} \cdot \vec{v}$ yields precisely the same result:

$$\vec{\alpha} \cdot \vec{v} = 2x - y = \varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Since over $\mathbb{R}^n$ a functional can be represented by a scalar product of a vector with specific components, we call the functional a **dual vector**, and the space of all dual vectors the **dual space** of $\mathbb{R}^n$.

Generally speaking, any vector space V has a dual space, which is denoted by V^*. In the case of $\mathbb{R}^n$ this dual space is actually $\mathbb{R}^n$ itself, since any vector in $\mathbb{R}^n$ can be used as a dual vector, and any dual vector is essentially a list of n real components - i.e. is equivalent to a vector in $\mathbb{R}^n$.

Note 2.21 When a functional has less than n coefficients

onsider the following functional over $\mathbb{R}^3$:

$$\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 5x + 3z.$$

It seems to have only two coefficients, namely $\alpha_1 = 5$ and $\alpha_2 = 3$ - however, in order to represent it as a vector in $\mathbb{R}^3$ we need three coefficients, i.e. we say that $\alpha_2 = 0$ and $\alpha_3 = 3$ - essentially, we use the vector

$$\vec{\alpha} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}.$$

to represent it.

2.8.3 Row vectors and the outer product

In order to keep things consistent, we actually don't represent dual-vectors as column vectors but instead as row vectors. This allows us to treat the scalar product between two vectors similarily to the product of two matrices: we always make sure that the vector on the left is written as a row vector, and the vector on the right is written as a column vector, i.e. given

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

we write the scalar product between the two vectors as

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \begin{bmatrix} u_1, & u_2, & \dots, & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \\ \vec{v} \cdot \vec{u} &= \begin{bmatrix} v_1, & v_2, & \dots, & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}. \end{aligned} \tag{2.8.6}$$

(note how the value of the scalar product of $\vec{u}$ and $\vec{v}$ doesn't depend on the order of the vectors, since real numbers are commutative) The "matrices" in question have the dimensions $1 \times n$ and $n \times 1$ respectively, and thus we can calculate their product, which would be a 1×1 matrix that we interpret as a scalar. This gives rise to the question of what happens if we put a column vector on the left and a row vector on the right? In that case, treating them as "matrices" would mean that we are calculating the product of an $n \times 1$ matrix and a $1 \times n$ matrix, and thus the result should be an $n \times n$ matrix.

Indeed, this is what we call the **outer product** of two vectors. Let us examine how the outer product is structured with a simple case:

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 5, & -1 \end{bmatrix}.$$

Then

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 5, & -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 & 2 \cdot (-1) \\ 3 \cdot 5 & 3 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 10 & -2 \\ 15 & -3 \end{bmatrix}. \tag{2.8.7}$$

And generaly, for any two vectors in $\mathbb{R}^n$:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1, & v_2, & \dots, & v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & \dots & u_n v_n \end{bmatrix}, \tag{2.8.8}$$

i.e. the value of any element a_{ij} in the outer product is equal to

$$a_{ij} = u_i v_j. \tag{2.8.9}$$

Note 2.22 The inner vs. outer product

Recall that the scalar product is sometimes called the inner product. If we consider scalars as having some rank equal to 0, vectors having a rank of 1 and matrices a rank of 2 - the inner product reduces the rank of the two components from 1 to 0, while the outer product increases them from 1 to 2.

In that sense, the normal product of two real numbers and the matrix-matrix product are found somewhere inbetween the inner and outer products, as the ranks of their outputs are equal to the ranks of their inputs.

!

2.8.4 A bit about more general vector spaces and their duals

! **To be written**: the subsection. !

2.9 THE BRA-KET NOTATION

! **To be written**: this section needs some rework regarding dual vectors !

In this section we introduce a special vector notation widely used in physics: the **Bra-ket notation**, also known as **Dirac's notation**. This notation helps simplify many aspects of linear algebra, and make its use more streamlined.

Note 2.23 Importance of this section

A person can have a pretty good grasp of linear algebra without ever learning about the bra-ket notation. This section, while interesting and providing a useful tool in working with linear algebra - is not obligatory, especially to people who do not intend to ever learn topics such as quantum physics, relativity theory, statistical mechanics, etc. It is however recommended even for those readers.

!

2.9.1 Definition

Up until now we presented the theory of linear algebra based on real numbers: all of the vectors we used were real vectors, i.e. of the form $\vec{v} \in \mathbb{R}^n$, where $n \in \mathbb{N}$. All of the matrices used were also made up of real components - and so were of course the scalars themselves, which we defined simply as real numbers.

However, it is aparently useful in many cases to use linear algebra in the context of complex numbers: instead of working with spaces of the form $\mathbb{R}^n$ we can use spaces of the form $\mathbb{C}^n$, e.g. a vector in $\mathbb{C}^3$ can be the following:

$$\vec{v} = \begin{bmatrix} 1-i \\ 2+3i \\ \pi+\sqrt{2}i \end{bmatrix}.$$

When using complex numbers instead of real numbers, a small change must be made to the way we conceptualize row vs. column vectors. Before we said that essentially both forms can be used interchangeably without affecting the outcome. However now we define row vectors a bit differently: given the column vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \tag{2.9.1}$$

we can get its row form by transposing it, i.e. we look at $\vec{v}^\top$. However, when doing this we must change all the components of $\vec{v}$ to their respective complex conjugates, i.e.

$$\vec{v}^\top = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^\top = \begin{bmatrix} \overline{v_1}, \overline{v_2}, \dots, \overline{v_n} \end{bmatrix}. \tag{2.9.2}$$

$$\vec{v}^\top = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^\top = \begin{bmatrix} v_1^*, v_2^*, \dots, v_n^* \end{bmatrix}. \tag{2.9.3}$$

To be consistent with the usual notation used in physics (and have more succisent, we introduce the following changes:

- The complex conjugate of the number $z \in \mathbb{C}$ is changed to C^*.
- The star notation is also used for transpose (this is normaly called **conjugate transpose**, and is sometimes denoted by $u^\dagger$).
- The arrow is dropped from the vector notation.

Applying these changes, Equation 2.9.3 has the form

$$v^* = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^* = \begin{bmatrix} v_1^*, v_2^*, \dots, v_n^* \end{bmatrix}. \tag{2.9.4}$$

Example 2.63 Bra/Row vectors

The bra form of the vector

$$v = \begin{bmatrix} 1+2i \\ 3-i \\ \sqrt{2}+5i \\ 4 \\ -3i \end{bmatrix}$$

is

$$v^* = \begin{bmatrix} 1+2i \\ 3-i \\ \sqrt{2}+5i \\ 4 \\ -3i \end{bmatrix}^* = \begin{bmatrix} 1-2i, & 3+i, & \sqrt{2}-5i, & 4, & 3i \end{bmatrix}.$$

Next, we make sure that the scalar product between any two vectors u, v is such that the left vector is a row vector, and the right vector is a column vector, i.e. given $u, v \in \mathbb{C}^n$ such that

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix},$$

their scalar product is

$$u \cdot v = u^* \cdot v = \begin{bmatrix} u_1^*, & u_2^*, & \dots, & u_n^* \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n. \tag{2.9.5}$$

Recall that a common notation for the scalar product of two vectors uses triangular brakets, i.e.

$$u \cdot v = \langle u | v \rangle.$$

We can use Equation 2.9.5 and "separate" this product into two parts: a **bra** $\langle u|$ and a **ket** $|v\rangle$, define as

$$\langle u| = \begin{bmatrix} u_1^*, & u_2^*, & \dots, & u_n^* \end{bmatrix},$$

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \tag{2.9.6}$$

2.9.2 Norm and products

The norm of a vector v can be calculated by taking the square root of its scalar product with itself (Equation 2.1.37). Using the bra-ket notation this becomes

$$\left\| \vec{v} \right\| = \sqrt{\langle v|v\rangle}. \tag{2.9.7}$$

Let us write the properties of the scalar product adjusted to the bra-ket notation:

- **Non-negative norm**: for any vector $v \in \mathbb{C}^n$, $\langle v|v\rangle \geq 0$.
- **Uniqeness of zero**: if $\langle v|v\rangle = 0$, then $v = 0$.
- **Conjugate commutativity**: for any two vectors $u, v \in \mathbb{C}^n$, $\langle u|v\rangle = \langle v|u\rangle^*$.
- **Distributivity**: Given three vectors $u, v, w \in \mathbb{C}^n$ and two scalars $\alpha, \beta \in \mathbb{C}$,

$$\langle u|\Big(\alpha\,|v\rangle + \beta\,|w\rangle\Big) = \alpha\,\langle u|v\rangle + \beta\,\langle u|w\rangle.$$

Note 2.24 Hilbert spaces

enerally speaking, any vector space that is "equiped" with a norm complying with these properties is called a **Hilbert space**. We will discuss such spaces in more details later in the book.

!

There is an interesting way one can interpret the scalar product: instead of as an operation acting on two vectors, we can view a bra as an operator acting on a ket and returning a scalar. Mathematically this is written as

$$\langle\bigcirc| : \mathbb{C}^n \to \mathbb{C}. \tag{2.9.8}$$

(the empty circle signifies that that the symbol representing the bra is placed inside the bra symbol)

! **To be written**: this is the dual space of $\mathbb{C}^n$, etc. !

Another product that is easily defined usin the bra-ket notation is the **exterior product** of two vectors (recall that the scalar product is also called the *inner product*). The exterior product arises when we multiply two vectors in the opposite order compared to the scalar product, i.e. instead of $\langle u|v\rangle$ we calculate $|u\rangle\langle v|$:

$$|u\rangle\langle v| = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1^*, & v_2^*, & \dots, & v_n^* \end{bmatrix} = \begin{bmatrix} u_1 v_1^* & u_1 v_2^* & \dots & u_1 v_n^* \\ u_2 v_1^* & u_2 v_2^* & \dots & u_2 v_n^* \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1^* & u_n v_2^* & \dots & u_n v_n^* \end{bmatrix}, \tag{2.9.9}$$

i.e. the result of the outer product is a matrix, in which any element a_{ij} is equal to

$$a_{ij} = u_i v_j^*. \tag{2.9.10}$$

Note 2.25 Bra-ket vs. ket-bra

Using the bra-ket notation, the names for the inner and outer products make sense: in the inner product the vectors stay inside the brackets, while in the outer product they are outside of it. This is just a small demonstration for the "power" of the bra-ket notation in simplifying mathematical expressions.

Example 2.64 Inner and outer products

Given the two kets

$$|u\rangle = \begin{bmatrix} 1+\mathrm{i} \\ 3 \end{bmatrix}, \; |v\rangle = \begin{bmatrix} -2+3\mathrm{i} \\ 5-\mathrm{i} \end{bmatrix},$$

let us calculate the following:

$$\langle u|v\rangle, \; \langle v|u\rangle, \; |u\rangle\langle v|, \; |v\rangle\langle u|.$$

We start by writing $\langle u|$ and $\langle v|$:

$$\langle u| = \begin{bmatrix} 1-\mathrm{i}, & 3 \end{bmatrix}, \; \langle v| = \begin{bmatrix} -2-3\mathrm{i}, & 5+\mathrm{i} \end{bmatrix}.$$

Then, the 4 requested products are easy to calculate:

$$\begin{aligned} \langle u|v\rangle &= (1-\mathrm{i})(-2+3\mathrm{i}) + 3(5-\mathrm{i}) = -2+3\mathrm{i}+2\mathrm{i}+3+15-3\mathrm{i} = 16+2\mathrm{i}. \\ \langle v|u\rangle &= (1+\mathrm{i})(-2-3\mathrm{i}) + 3(5+\mathrm{i}) = -2-3\mathrm{i}-2\mathrm{i}+3+15+3\mathrm{i} = 16-2\mathrm{i} = \langle u|v\rangle^*. \end{aligned}$$

$$\begin{aligned} |u\rangle\langle v| &= \begin{bmatrix} (1+\mathrm{i})(-2-3\mathrm{i}) & (1+\mathrm{i})(5+\mathrm{i}) \\ 3(-2-3\mathrm{i}) & 3(5+\mathrm{i}) \end{bmatrix} = \begin{bmatrix} 1-5\mathrm{i} & 4+6\mathrm{i} \\ -6-9\mathrm{i} & 15+3\mathrm{i} \end{bmatrix}. \\ |v\rangle\langle u| &= \begin{bmatrix} (-2+3\mathrm{i})(1-\mathrm{i}) & 3(-2+3\mathrm{i}) \\ (5-\mathrm{i})(1-\mathrm{i}) & 3(5-\mathrm{i}) \end{bmatrix} = \begin{bmatrix} 1+5\mathrm{i} & -6+9\mathrm{i} \\ 4-6\mathrm{i} & 15-3\mathrm{i} \end{bmatrix}. \end{aligned}$$

2.9.3 Scalar multiplication and addition of vectors

Scaling a vector in the bra-ket notation is represented by simply putting the scalar on the left or right of the vector, i.e. given a ket

$$|u\rangle = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

its scaled version by the scalar $\alpha \in \mathbb{C}$ is

$$\alpha\,|u\rangle = |u\rangle\,\alpha = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{bmatrix}. \tag{2.9.11}$$

The bra version is essentialy the same:

$$\alpha\,\langle u| = \langle u|\,\alpha = \left[\alpha u_1^*,\ \alpha u_2^*,\ \ldots,\ \alpha u_n^*\right]. \tag{2.9.12}$$

Normaly, scaling a ket is written with the scalar on the left, and scaling a bra is written with scalar on the right. However, in some instances the other forms are used (we will see such case soon).

Adding two kets or two bras is also quote simple:

$$\begin{aligned} |u\rangle + |v\rangle = |u+v\rangle &= \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}, \\ \langle u| + \langle v| = \langle u+v| &= \left[(u_1+v_1)^*,\ (u_2+v_2)^*,\ \ldots,\ (u_n+v_n)^*\right]. \end{aligned} \tag{2.9.13}$$

Note 2.26 Addition of a bra and a ket

Of course, a bra and a ket cannot be added together.

!

2.9.4 Linear combinations and basis sets

Linear combinations are easily represented using the bra-ket notation: let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be complex numbers, and $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$ kets in $\mathbb{C}^n$. Then

$$|w\rangle = \alpha_1\,|v_1\rangle + \alpha_2\,|v_2\rangle + \cdots + \alpha_n\,|v_n\rangle = \sum_{i=1}^{n} \alpha_i\,|v_i\rangle \tag{2.9.14}$$

is of course itself a vector in $\mathbb{C}^n$. If the set $B = \left\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\right\}$ is a basis set of $\mathbb{C}^n$ then any vector $\vec{v} \in \mathbb{C}^n$ can be written as a linear combination of the vectors in B, i.e.

$$\vec{w} = \sum_{i=1}^{n} \beta_i\,|b_i\rangle, \tag{2.9.15}$$

such that not all $\beta_i = 0$ (i.e. at least one of them is non-zero). If in addition

$$\left\langle b_i \middle| b_j \right\rangle = \delta_{ij}, \tag{2.9.16}$$

the basis set is orthonormal: each vector has unit norm (since $\langle b_i|b_i\rangle = \delta_{ii} = 1$), and they are all orthogonal to each other (since for $i \neq j$, $\left\langle b_i \middle| b_j \right\rangle = \delta_{i\neq j} = 0$).

Sometimes, and depending on the context, basis vectors are written as their indeces only inside a bra or a ket. This is common for example with the standard basis vectors, e.g. in $\mathbb{C}^3$

$$\hat{x} = |1\rangle, \ \hat{y} = |2\rangle, \ \hat{z} = |3\rangle. \tag{2.9.17}$$

and more generaly

$$\hat{e}_i = |i\rangle. \tag{2.9.18}$$

Let us now take a general ket in $|v\rangle \in \mathbb{C}^n$ and write it as a linear combination of some orthonormal basis set B:

$$|v\rangle = \sum_{i=1}^{n} \alpha_i \left|b_i\right\rangle. \tag{2.9.19}$$

We now choose one of the basis vectors and write it in its bra form: $\left\langle b_j\right|$, where $j \in \{1, 2, \dots, n\}$. We can easily write the inner product of $\left\langle b_j\right|$ with $|v\rangle$:

$$\begin{aligned}\left\langle b_j \middle| v\right\rangle &= \left\langle b_j\right| \sum_{i=1}^{n} \alpha_i \left|b_i\right\rangle \\ &= \left\langle b_j\right| \alpha_1 \left|b_1\right\rangle + \left\langle b_j\right| \alpha_2 \left|b_2\right\rangle + \cdots + \left\langle b_j\right| \alpha_n \left|b_n\right\rangle. \end{aligned}\tag{2.9.20}$$

Since α_i are scalars, we can move each one of them to the left of $\left\langle b_j\right|$, yielding

$$\left\langle b_j \middle| v\right\rangle = \alpha_1 \left\langle b_j \middle| b_1\right\rangle + \alpha_2 \left\langle b_j \middle| b_2\right\rangle + \cdots + \alpha_n \left\langle b_j \middle| b_n\right\rangle. \tag{2.9.21}$$

Since the basis vectors are all orthonormal, $\left\langle b_j \middle| b_i\right\rangle = \delta_{ji}$, or zero for all i except when $i = j$, in which case the product equals 1. Thus all the terms where $i \neq j$ disappear, and we are left with a single term:

$$\left\langle b_j \middle| v\right\rangle = \alpha_j \left\langle b_j \middle| b_j\right\rangle = \alpha_j. \tag{2.9.22}$$

This shows is that we can calculate the coefficient α_j for any $j \in \{1, 2, \dots, n\}$ by simply taking the dot product of the j-th basis vector with $|u\rangle$. This is very easy compared to using the vector notation we used so far.

We can now start again with spanning $|u\rangle$ using B, but this time we write the scalar coefficients on the right side:

$$|u\rangle = \sum_{i=1}^{n} \left|b_i\right\rangle \alpha_i. \tag{2.9.23}$$

Subtituting Equation 2.9.22 into Equation 2.9.23 we get

$$|u\rangle = \sum_{i=1}^{n} \left|b_i\right\rangle \left\langle b_i \middle| u\right\rangle. \tag{2.9.24}$$

Since $|u\rangle$ is found in all the terms of the sum, we can put parenthesis around the summation excluding $|u\rangle$:

$$|u\rangle = \left(\sum_{i=1}^{n} \left|b_i\right\rangle \left\langle b_i\right| \right) |u\rangle. \tag{2.9.25}$$

Recall that the outer product return a matrix. Thus, Equation 2.9.25 tells us that the matrix $A = \sum\limits_{i=1}^{n} \left|b_i\right\rangle\left\langle b_i\right|$ is a matrix that doesn't change any vector $|u\rangle$ (remember that there is nothing special about $|u\rangle$, we didn't even write it explicitly). This matrix is of course the identity matrix, i.e.

$$I_n = \sum_{i=1}^{n} \left|b_i\right\rangle\left\langle b_i\right|. \tag{2.9.26}$$

This result is called the **completeness relation**. It essentially tells us that given an orthonormal basis set, the sum of the outer product of each basis vector with itself is the identity matrix. For example, given the standard basis set in $\mathbb{C}^3$,

$$|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = I_3. \tag{2.9.27}$$

Example 2.65 The completeness relation in $\mathbb{C}^2$

As we saw in Example 2.11, the following basis set is orthonormal:

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ |2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The outer products of each of the two vectors with itself are

$$\begin{aligned} |1\rangle\langle 1| &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \\ |2\rangle\langle 2| &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Thus,

$$|1\rangle\langle 1| + |2\rangle\langle 2| = \frac{1}{2}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

! **To be written**: change of basis, matries inside brakets, eigenvectors, etc. !

2.10 SOME REAL LIFE USES OF LINEAR ALGEBRA

! **To be written**: you guessed it - the entire section. !

2.10.1 Formulating plant nutrient solutions

Theoretical background

In order to grow and thrive, plants require **nutrients**. These nutrients are chemical elements which the plants use in different processes to create new roots, stems, leaves, fruits, etc. and to sustain those that already exist. Generaly speaking, there are 17 such elements, and they can be roughly divided in to three categories:

- **Nutrients from air**: carbon (C), oxygen (O) and hydrogen (H). Plants get these from water (H_2O), carbon dioxide (CO_2) and atmospheric oxygen (O_2), and use them together with light to build carbohydrates (such as glucose, fructose and cellulose), in a process known as photosynthesis.

- **Macronutrients from ground**: nitrogen (N), phosphorus (P), potassium (K), calcium (Ca), magnesium (Mg) and sulfur (S). These nutrients are absorbed mostly through the roots of the plant, and are needed in relatively large quantities.

- **Micronutrients from groud**: chlorine (Cl), iron (Fe), manganese (Mn), zinc (Zn), copper (Cu), molybdenum (Mo), boron (B) and nickel (Ni). These are also absorbed mainly via the roots, and are needed in relatively smaller quantities than the macronutrients.

Table 2.3 describes what each of the above elements is used for by plants.

! **To be written**: add data and format the table !

Different plants require different nutrient concentrations for optimal growth: not only do the actuall concentrations change, but the ratio between the different elements. See for example Table 2.4, which compares between the nutritional needs of a basil plant and a tomato plant - the ratio of the calcium vs. nitrogen requirements is 30% higher in tomatoes compared to basil (this is especially true during the fruiting stage of tomatoes).

! **To be written**: rest of subsection !

2.10.2 Primary components analisis

2.10.3 Hunter-pray population growth

2.11 EXERCISES

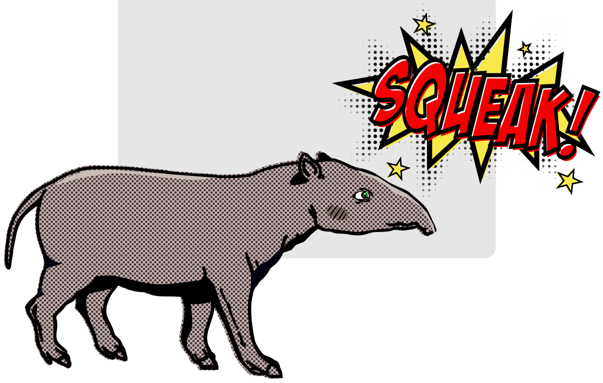
Table 2.3 The 17 essential elements for plants and their uses. The column marked [ppm] shows the mass fraction of the element in dried plant material ($1[\text{ppm}] = 1 \times 10^{-6}$).

	Element	Symbol	[ppm]	Main uses
Air	Oxygen	O		Practically all organic mulecules
	Carbon	C		Carbohydrates and protein synthesis
	Hydrogen	H		Practically all organic mulecules
Macronutrients	Nitrogen	N	15,000	Proteins
	Phosphorus	P	2,000	Phospholipids, ATP
	Potassium	K	10,000	Ion balance
	Calcium	Ca	5,000	Ion balance
	Magnesium	Mg	2,000	Chlorophyl
	Sulfur	S	1,000	Proteins, vitamins, chloroplast
Micronutrients	Chlorine	Cl	100	Ion balance
	Iron	Fe	100	Enzyme cofactors, photosynthesis
	Manganese	Mn	50	Chloroplast
	Zinc	Zn	20	DNA transcriptase
	Copper	Cu	6	Photosynthesis
	Molybdenum	Mo	0.1	Enzyme cofactors
	Boron	B	20	Metabolism
	Nickel	Ni	0.1	Nitrogen metabolism

Table 2.4 Target concentrations of macronutrients and micronutrients in aqueous feeding solutions for basil and tomato plants. Amounts given in mg/L (ppm).

	Basil	**Tomato**
N	280	308
P	25	31
K	274	313
Ca	280	400
Mg	61	109
S	112	218
Cl	< 284	< 284
Fe	1.4	1.96
Mn	0.27	0.28
Zn	0.46	0.46
Cu	0.06	0.04
Mo	0.05	0.05
B	0.54	0.54

CHAPTER 3



LINEAR ALGEBRA
(RIGOROUS APPROACH)

Something about formalism, theorems, proofs, etc.

> **Note 3.1 Why present rigorous mathematics in this book?**
>
> Rigorous mathematics is rarely necessary for those who are interested in the tools mathematics provides us with, rather than the full and deep understanding of the concepts these tools are based on. However, it can be useful to students of scientific fields to experience rigorous mathematics at least once in their course of study. Usually, the choice for the topic to be analyzed rigorously is between linear algebra and calculus - for this book the latter was chosen.
>
> !

3.1 FIELDS

We begin our dive into the rigorous analysis of linear algebra by defining an algebraic construction call a **field**, which we need in order to properly define vector spaces later. In essence, a field has most of the important properties of the real numbers, namely the closure, commutativity, associativity, identity and inverse of addition and multiplication of any two elements in the field (except the product inverse of the field equivalent object for the number 0). In a later section we will use fields to construct the general notion of **vector spaces**.

Definition 3.1 Field

A field $\mathbb{F}$ is a set of objects together with two operations called **addition** and **multiplication** (denoted $+$ and $\cdot$, respectively), for which the following axioms hold:

- **Closure of under addition and multiplication**: for any $a, b \in \mathbb{F}$,
  1. $(a+b) \in \mathbb{F}$,
  2. $(a \cdot b) \in \mathbb{F}$.
- **Commutativity under addition multiplication**: for any $a, b \in \mathbb{F}$,
  1. $a + b = b + a$,
  2. $a \cdot b = b \cdot a$.
- **Associativity under addition and multiplication**: for any $a, b, c \in \mathbb{F}$,
  1. $a + (b + c) = (a + b) + c$,
  2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- **Additive and multiplicative identity**: there exist an element in $\mathbb{F}$ called the *additive identity* and denoted by 0, for which $a + 0 = a$ for any $a \in \mathbb{F}$.

  Similarity, there exists an element in $\mathbb{F}$ called the *multiplicative identity* and denoted by 1, for which $a \cdot 1 = a$ for any $a \in \mathbb{F}$.
- **Additive and multiplicative inverses**: for any element $a \in \mathbb{F}$ (except the additive identity) there exists:
  1. $b \in \mathbb{F}$ such that $a + b = 0$, and
  2. $c \in \mathbb{F}$ such that $a \cdot c = 1$.

  (usually b is denoted as $-a$, while c is denoted as a^{-1})
- **Distributivity of multiplication over addition**: for any $a, b, c \in \mathbb{F}$,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

π

3.1.1 Infinite fields

We start with one of the most obvious examples of a field: the real numbers together with the standard addition and product.

Theorem 3.1 $\mathbb{R}$ as a field

The set of real numbers $\mathbb{R}$ forms a field together with the standard addition and product.

We leave the proof of 3.1 to the reader, as it is pretty straight forward using the known properties of the standard addition and product over $\mathbb{R}$ (and rather uninteresting). Instead, we jump forward to using 3.1 for proving the same idea about the complex numbers:

Theorem 3.2 $\mathbb{C}$ as a field and more more more

The set of complex numbers $\mathbb{C}$ forms a field together with the addition and product operations as defined in Section 0.7 (namely Equation 0.7.3, Equation 0.7.4 and Equation 0.7.13).

Proof 3.1 $\mathbb{C}$ as a field

(note: in the following proof, equalities marked with ! use the respective property of the real numbers)

- **Closure under both operations**: for any two complex numbers $z_1 = a + ib$ and $z_2 = c + id$,
  - Addition: since addition in $\mathbb{R}$ is closed, $(a+c) \in \mathbb{R}$ and $(b+d) \in \mathbb{R}$. Therefore
  $$z = z_1 + z_2 = a + c + (b + d)\mathrm{i}$$
  is also a complex number with $\mathrm{Re}(z) = a + c$ and $\mathrm{Im}(z) = b + d$.
  - Multiplication: since multiplication in $\mathbb{R}$ is also closed, $(ac - bd) \in \mathbb{R}$ and $(ad + bc) \in \mathbb{R}$. Therefore
  $$z = z_1 \cdot z_2 = ac - bd + (ad + bc)\mathrm{i}$$
  is a complex number with $\mathrm{Re}(z) = ac - bdc$ and $\mathrm{Im}(z) = ad + bc$.
- **Commutativity of both operation**: for any two complex numbers $z_1 = a + ib$ and $z_2 = c + id$,
  - Addition: since addition in $\mathbb{R}$ is commutative, $a + c = c + a$ and $b + d = d + b$. Therefore
  $$z_1 + z_2 = a + c + (b + d)\mathrm{i} \stackrel{!}{=} c + a + (d + b)\mathrm{i} = z_2 + z_2.$$

• Multiplication: since multiplication in $\mathbb{R}$ is also commutative, $ac - bd = ca - db$ and $ad + bc = da + cb$. Therefore

$$z_1 \cdot z_2 = ac - bd + (ad + bc)\mathrm{i} \overset{!}{=} ca - db + (da + cb)\mathrm{i} = z_2 \cdot z_1.$$

• **Associativity of both operation**: for any three complex numbers $z_1 = a + \mathrm{i}b$, $z_2 = c + \mathrm{i}d$ and $z_3 = g + \mathrm{i}h$ (where $a, b, c, d, g, h \in \mathbb{R}$[a]),

• Addition: since addition in $\mathbb{R}$ is associative, $a + (c + g) = (a + c) + g$ and $b + (d + h) = (b + d) + h$. Therefore

$$z_1 + (z_2 + z_3) = a + (c + g) + [b + (d + h)]\mathrm{i} \overset{!}{=} (a + c) + g + [(b + d) + h]\mathrm{i} = (z_1 + z_2) + z_3.$$

• Multiplication: since multiplication in $\mathbb{R}$ is also associative, the following equalities apply:

$$\begin{aligned}
a \cdot (c \cdot g) &= (a \cdot c) \cdot g, \\
b \cdot (c \cdot h) &= (b \cdot c) \cdot h, \\
a \cdot (d \cdot h) &= (a \cdot d) \cdot h, \\
b \cdot (d \cdot g) &= (b \cdot d) \cdot g, \\
a \cdot (c \cdot h) &= (a \cdot c) \cdot h, \\
a \cdot (d \cdot g) &= (a \cdot d) \cdot g, \\
b \cdot (c \cdot g) &= (b \cdot c) \cdot g, \\
b \cdot (d \cdot h) &= (b \cdot d) \cdot h.
\end{aligned}$$

Therefore,

$$\begin{aligned}
z_1 \cdot (z_2 \cdot z_3) &= a \cdot (c \cdot g) - a \cdot (d \cdot h) - b \cdot (c \cdot h) - b \cdot (d \cdot g) \\
&\quad + [a \cdot (c \cdot h) + a \cdot (d \cdot g) + b \cdot (c \cdot g) - b \cdot (d \cdot h)]\mathrm{i} \\
&\overset{!}{=} (a \cdot c) \cdot g - (a \cdot d) \cdot h - (b \cdot c) \cdot h - (b \cdot d) \cdot g \\
&\quad + [(a \cdot c) \cdot h + (a \cdot d) \cdot g + (b \cdot c) \cdot g - (b \cdot d) \cdot h]\mathrm{i} \\
&= (z_1 \cdot z_2) \cdot z_3.
\end{aligned}$$

• **Identity for both operations**:

• Addition: the complex number $0 = 0 + 0i$ is the complex addition identity: for any real number $x \in \mathbb{R}$, $x + 0 = x$. Therefore, for any complex number $z = a + \mathrm{i}b$,

$$z + 0 = a + \mathrm{i}b + 0 + 0i = a + 0 + (b + 0)\mathrm{i} \overset{!}{=} a + \mathrm{i}b.$$

• Multiplication: the complex number $1 = 1 + 0\mathrm{i}$ is the complex multiplication identity: for any real number $x \in \mathbb{R}$, $x \cdot 1 = x$ and $x \cdot 0 = 0$. Therefore, for any complex number $z = a + \mathrm{i}b$,

$$z \cdot 1 = (a + \mathrm{i}b) \cdot (1 + 0\mathrm{i}) \stackrel{!}{=} a \cdot 1 - \cancel{b \cdot 0\mathrm{i}^2} + (\cancel{a \cdot 0\mathrm{i}} + b \cdot 1)\mathrm{i} = a + \mathrm{i}b.$$

- **Inverse for both operations**:

  - Addition: for any complex number $z_1 = a + \mathrm{i}b$, the number $z_2 = -a - \mathrm{i}b$ is also a complex number for which

$$z_1 + z_2 = a + \mathrm{i}b + -a - \mathrm{i}b \stackrel{!}{=} a - a + (b - b)\mathrm{i} = 0 + 0\mathrm{i} = 0.$$

  - Multiplication: for any complex number $z = re^{\theta i}$ where $r \neq 0$, the number $z^{-1} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta}$ is also a complex number for which

$$z \cdot z^{-1} = r\mathrm{e}^{\mathrm{i}\theta} \cdot \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta} \stackrel{!}{=} \frac{r}{r}\mathrm{e}^{\cancel{\mathrm{i}\theta - \mathrm{i}\theta}} = 1 \cdot 1 = 1.$$

  Note: for $z = a + \mathrm{i}b$,

$$z^{-1} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta} = \frac{1}{|z|} \cdot \frac{a - \mathrm{i}b}{|z|} = \frac{1}{|z|} \cdot \frac{\bar{z}}{|z|} = \frac{\bar{z}}{|z|^2}.$$

  Therefore, for any $z \neq 0$, $z^{-1} = \frac{\bar{z}}{|z|^2}$.

- **Distributivity of multiplication over addition**: for any $z_1 = a + \mathrm{i}b$, $z_2 = c + \mathrm{i}d$ and $z_3 = g + \mathrm{i}h$,

$$\begin{aligned} z_1 \cdot (z_2 + z_3) &= (a + \mathrm{i}b) \cdot (c + \mathrm{i}d + g + \mathrm{i}h) = (a + \mathrm{i}b) \cdot (c + g + [d + h]\mathrm{i}) \\ &= ac + ag + (bd)\mathrm{i}^2 + (bh)\mathrm{i}^2 \\ &+ (ad)\mathrm{i} + (ah)\mathrm{i} + (bc)\mathrm{i} + (bg)\mathrm{i} \\ &= ac + ag - bd - bh + (ad + ah + bc + bg)\mathrm{i} \\ &= ac - bd + (ad + bc)\mathrm{i} + ag - bh + (ah + bg)\mathrm{i} \\ &= (z_1 \cdot z_2) + (z_1 \cdot z_3). \end{aligned}$$

[a] The letters g and h are used instead of e and f to avoid confusion with Eurler's constant and the common notation for real functions, respectively.

QED

The sets $\mathbb{R}$ and $\mathbb{C}$ are examples of **infinite fields**, since they each have infinite number of elements. The set $\mathbb{Q}$ (rational numbers) can be shown to also be an infinite field, however unlike $\mathbb{R}$ and $\mathbb{C}$ it has **countable** number of elements, i.e. each number in $\mathbb{Q}$ can be assigned an index $1, 2, 3, \ldots$ [1].

[1] For proof, see ...

Challenge 3.1 $\mathbb{Q}$ as a field

Prove that $\mathbb{Q}$ (together with the usual addition and product operation) is indeed a field.

?

3.1.2 Finite fields

While all three examples of fields we encountered so far have each an infinite number of elements, some fields only have a finite number of elements (called their **order**). For example, consider the set $S = \{0, 1, a, b\}$ and the addition and product operations described using the following tables (left table describes addition, right table describes multiplication):

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

By examining the tables above, several points become clear:

- all the possible combinations of operands in both addition and multiplication give elements from S itself, meaning that the set is closed under both these operations.
- both tables are symmetric around their main diagonal, meaning that both addition and multiplication are commutative operations.
- in the addition table, the first row and first column both show that $x + 0 = x$ for any $x \in S$, meaning that 0 is the additive identity in S.
- in the product table, the second row and second column both show that $x \cdot 1 = x$ for any $x \in S$, meaning that 1 is the multiplicative identity in S.
- in the addition table, the element 0 appears in each row and each column exactly once. This means that every element x has a single additive inverse $y \in S$.
- in the product table, the element 1 appears in each row and each column exactly once, except for the first row and first column. This means that every element $x \neq 0$ has a single multiplicative inverse $z \in S$.

We therefore only need to prove two points to show that S is a field together with the operations described by the above tables: associativity of both operations and distributivity of multiplication over addition. We leave these proofs as a challenge to the reader. Such a field is sometime denoted as $\mathbb{F}_4$. There are, of course, infinitely many finite fields.

3.1.3 Modulo fields

Another example of finite fields are sets of integers of the form $\{0,1,2,3,\ldots,n\}$ where n is a prime, together with **modular addition** and **modular product**. To understand modular arithmetics, we recall the fact that on a circle, an angle can have a negative value but also greater than $360°$ values are possible (see Figure 0.6): $390°$ is equivalent to $30°$, $-30°$ is equivalent to $330°$, etc. The set of integer values $0°,1°,2°,\ldots,359°$ on a circle is an example of a modular set: if for example we add together two angles of values $\deg 100$ and $300°$ we get the equivalent angle $\deg 60$. If we subtract $\deg 300$ from $\deg 100$ the result is an angle of $\deg 160$.

We say that on a circle, the values $360°, 720°, -360°$ etc. are all **congruent** to 0 modulo 360. In mathematical notation we represent this fact as e.g.

$$720 \equiv 0 \pmod{360}. \tag{3.1.1}$$

Note that from this point forward we drop the degrees unit, and deal with pure integers. The notation for the set $\{0,1,2,\ldots,359\}$ is $\mathbb{Z}_{360}$. Generally speaking, the set $\{0,1,2,\ldots,n\}$ is denoted as $\mathbb{Z}_n$.

Note 3.2 About modulo set notation

It is not common to use the notation $\mathbb{Z}_n$ for the modulo-n set, since it is also used for a different algebraic construct, namely the n-adic ring. However, due to the simplicity of the notation, and the fact that we don't discuss rings in this chapter we are using it in this book. Common notations for the set are $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/n$.

!

Addition and multiplication on $\mathbb{Z}_n$ is done by the following rather straight forward definition:

Definition 3.2 Operations in $\mathbb{Z}_n$

In the set $\mathbb{Z}_n$ addition and multiplication are defined as the following:

- **Addition**: for any two elements $a,b \in \mathbb{Z}_n$, $a+b := (a+b) \pmod{n}$.
- **Multiplication**: for any two elements $a,b \in \mathbb{Z}_n$, $a \cdot b := (a \cdot b) \pmod{n}$.

π

Example 3.1 Operations in $\mathbb{Z}_n$

The tables below show addition and multiplication results of numbers in different modulo sets $\mathbb{Z}_n$ for some values of n:

n	$2+3$	$2\cdot 3$
4	1	2
5	0	1
6	5	0
7	5	6
8	5	6
9	5	6
10	5	6
11	5	6

n	$4+7$	$4\cdot 7$
8	3	4
9	2	1
10	1	8
15	11	13
20	11	8
27	11	1
28	11	0
30	11	28

Figure 3.1 shows the equivalency between integers and the elements of $\mathbb{Z}_5$.

$\mathbb{Z}$	$\cdots$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	$\cdots$
$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
$\mathbb{Z}_5$	$\cdots$	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	$\cdots$

Figure 3.1 An example of the periodicity of $\mathbb{Z}_5$: the top numbers are the ordinary integers, each showing their respective congruent modulo 5 below (blue dashed arrow).

Only the sets $\mathbb{Z}_n$ for which n is a prime number are also fields. Let's define this property precisely:

Theorem 3.3 **$\mathbb{Z}_p$ is a field**

Any modulo set $\mathbb{Z}_p$ where p is a prime number greater than 1 is also a field together with the operations as defined in 3.2.

In order to prove 3.3 we use two lemmas: the first is known as **Bézout's lemma**:

Lemma 3.1 **Bézout's lemma**

For any two positive integers a, b there exist two integers x, y such that

$$\gcd(a, b) = xa + yb.$$

Note 3.3 **$\gcd(a, b)$**

$\gcd(a, b)$ is the **greatest common divisor** of the two integers a and b. For example, $\gcd(36, 24) = 12$ since the divisors of 36 are $1, 2, 3, 4, 6, 9, 12, 18, 36$, and the divisors of 24 are $1, 2, 3, 4, 6, 8, 12, 24$.

An example of Bézout's lemma is the following:

Example 3.2 Bézout's lemma in action

For the two positive integers $a = 60,\ y = 114$

$$\gcd(60, 114) = 6.$$

Therefore, Bézout's lemma says that there exist two integers x, y such that

$$6 = 60x + 114y.$$

Indeed, two such integers exist: $x = 2$ and $y = -1$.

(SHOULD WE PROVE THE LEMMA?..)

The second lemma we use is the following:

Lemma 3.2 $\gcd(n, p) = 1$

Given a positive prime number p, then for any positive integer $n < p$,

$$\gcd\left(p, n\right) = 1.$$

Proving the lemma:

Proof 3.2 $\gcd(n, p) = 1$

We assume that $\gcd\left(p, n\right) \neq 1$. Then there exist an integer $a \leq n < p$ which divides both n and p, meaning that p has a divider, contrary to the assumption that p is a prime number. Therefore $\gcd\left(n, p\right)$ must equal 1.

QED

Now we can proceed to the proof of 3.3:

Proof 3.3 $\mathbb{Z}_p$ is a field

- **Closure under both operations**: the definition of the modulo operator limit any $M \pmod{p}$ (where $M \in \mathbb{Z}$) to be in $[0, p-1]$. Therefore the result of using the operators given in 3.2 must be within the same range, and thus in $\mathbb{Z}_p$.

- **Commutativity and associativity of both operations**: for any two numbers $a, b \in \mathbb{Z}_p$ the result $a + b$ and $a \cdot b$ under $\mathbb{Z}$ is both commutative and associative. Therefore the result modulo n is the same no matter the order of operations.

- **Additive identity**: the number $0 \in \mathbb{Z}_p$ is the additive identity, since for each

$a \in \mathbb{Z}_p$, $a + 0 = a$.

- **Multiplicative identity**: the number $1 \in \mathbb{Z}_p$ is the additive identity, since for each $a \in \mathbb{Z}_p$, $a \cdot 1 = a$.

- **Additive inverse**: for each $a \in \mathbb{Z}_p$ the element $n = p - a$ is in $\mathbb{Z}_p$ since $p > a$. Adding n to a results in 0:

$$a + n = a + (p - a) = p \equiv 0 \pmod{p}.$$

- **Multiplicative inverse**: let $a \in \mathbb{Z}_p$ and $a \neq 0$. Since p is a prime, $\gcd\left(a, p\right) = 1$ and from Bézout's theorem we know that there exist two integers x, y such that

$$xa + yp = 1.$$

Rearrangement gives $p = \frac{1-xa}{y}$ meaning that p divides $1 - xa$, and thus

$$xa \equiv 1 \pmod{p}.$$

Therefore x is the multiplicative inverse of a.

- **Distributivity of multiplication over addition**: ...

QED

The only part of the proof that uses the fact that p is a prime number is the multiplicative inverse. When n is not a prime, $\mathbb{Z}_n$ is not a field.

Challenge 3.2 $\mathbb{Z}_n$ is not a field when n is not a prime number

Prove that the modulo set $\mathbb{Z}_n$ where n is **not** a prime number, is not a field. (**hint**: what property of prime numbers is used in the above proof to show that there is always a multiplicative inverse in $\mathbb{Z}_p$ where p is prime?)

?

3.2 VECTOR SPACES

As we've seen in Chapter 2 vectors are found at the heart of linear algebra. We first defined them in a geometric way as objects with magnitude and direction, and later as lists of real numbers, analyzing the connections between these two mostly parallel definitions. We also spoke about vector spaces of the type $\mathbb{R}n$ as the structures vectors exist in. However, we haven't defined vectors nor vector spaces formally - which is exactly what we do in this section, by defining the concept of **vector spaces**.

Note 3.4 $\mathbb{R}n$ as a guide to general vector spaces

While reading the definition below, it is worthwhile to reflect on each of the given axioms as it relates to the familiar vector space $\mathbb{R}n$.

!

Definition 3.3 Vector space

A vector space over a field $\mathbb{F}$ is a set V which, together with two operations described below, fulfils a list of axioms. The two operations are

- Vector addition: an operation which takes two elements of V and returns a single element of V, i.e. $+ : V \times V \to V$.

- Scalar multiplication: an operation which takes a single element of $\mathbb{F}$ and a single element of V and returns a single element of V, i.e. $\cdot : \mathbb{F}, V \to V$.

The axioms to be fulfilled are:

- **Commutativity of vector addition**: for any $u, v \in V$,

$$u + v = v + u.$$

- **Associativity of vector addition**: for any $u, v, w \in V$,

$$u + (v + w) = (u + v) + w.$$

- **Additive identity**: there exist an element $0 \in V$ for which, for any $v \in V$,

$$v + 0 = v.$$

- **Scalar multiplicative identity**: for any $v \in V$

$$1 \cdot v = v,$$

  where 1 is the multiplicative identity in $\mathbb{F}$.

- **Additive inverse**: for any $v \in V$ there exist an element $u \in V$ for which

$$v + u = 0.$$

- **Associativity of scalar multiplication**: for any $\alpha, \beta \in \mathbb{F}$ and $v \in V$

$$\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v,$$

  where $\alpha\beta$ is the multiplication defined for $\mathbb{F}$.

- **Distributivity of vector addition**: for any $\alpha \in \mathbb{F}$ and $u, v \in V$,

$$\alpha \cdot (u + v) = (\alpha \cdot u) + (\alpha \cdot v).$$

- **Distributivity of scalar addition**: for any $\alpha, \beta \in \mathbb{F}$ and $v \in V$,

$$(\alpha + \beta) \cdot v = (\alpha \cdot v) + (\beta \cdot v).$$

The elements of V are then called **vectors**, and the elements of $\mathbb{F}$ are called **scalars**.

π

Since we discussed $\mathbb{R}n$ thoroughly in Chapter 2, let's prove that it is indeed a vector space under the above definition. First, the claim:

Theorem 3.4 $\mathbb{R}n$ is a vector space

The set of elements of the form

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where $v_i \in \mathbb{R}$, forms a vector space over $\mathbb{R}$ together with the following two operations:

- **Vector addition**:

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

- **Scalar multiplication**:

$$\alpha \cdot \vec{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}.$$

The proof itself is pretty easy, based on the fact that $\mathbb{R}$ is a field:

Proof 3.4 $\mathbb{R}n$ is a vector space

Since the results of both operations defined for $\mathbb{R}n$ only depend on the respective components of a vector $v \in \mathbb{R}n$, all the axioms of a vector space apply, since they derive directly from the fact that $\mathbb{R}$ is a field. As an example, we will elaborate on two of the axioms:

- **Additive inverse**: Given a vector $\vec{v} \in \mathbb{R}n$, each of its components v_i has an inverse under $\mathbb{R}$, namely $-v_i$. Therefore,

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix} = \begin{bmatrix} v_1 - v_1 \\ v_2 - v_2 \\ \vdots \\ v_n - v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0},$$

which is the additive identity in $\mathbb{R}n$.

- **Distributivity of vector addition**: for each component of two vectors $\vec{u}, \vec{v} \in \mathbb{R}n$, given the rules for vector addition and scalar multiplication, together with the distributivity of numbers in $\mathbb{R}$:

$$\begin{aligned} \alpha(\vec{u}+\vec{v}) = \alpha\left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) &= \alpha\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \\ &= \begin{bmatrix} \alpha u_1 + \alpha v_1 \\ \alpha u_2 + \alpha v_2 \\ \vdots \\ \alpha u_n + \alpha v_n \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{bmatrix} + \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix} = \alpha\vec{u} + \alpha\vec{v}. \end{aligned}$$

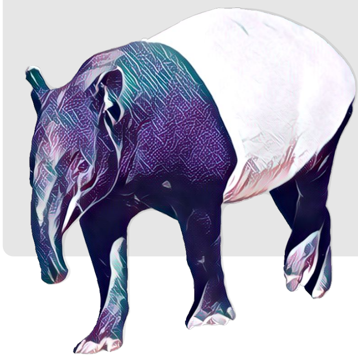
QED

(it is adviceable for the reader to go over the rest of the axioms and prove them for $\mathbb{R}n$)

3.3 EXERCISES

CHAPTER

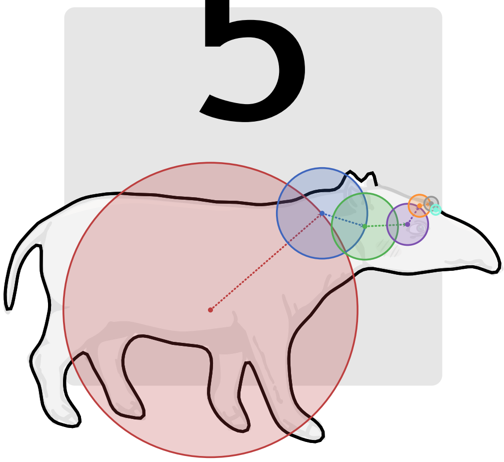
4



DIFFERENTIAL EQUATIONS

4.1 EXERCISES

CHAPTER 5

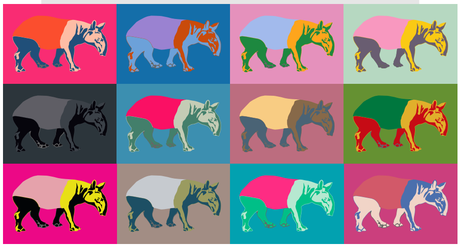


THE FOURIER TRANSFORM

5.1 EXERCISES

CHAPTER

6



SYMMETRY GROUPS

6.1 EXERCISES