MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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with contributions from others

$a^b = e^{b\log(a)}$

$(a+b)^n = \sum\limits_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$T\left(\alpha\vec{u}+\beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$

$A = Q\Lambda Q^{-1}$

$\Gamma(z) = \int\limits_{0}^{\infty} t^{z-1} e^{-t}\,\mathrm{d}t$

$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim\limits_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

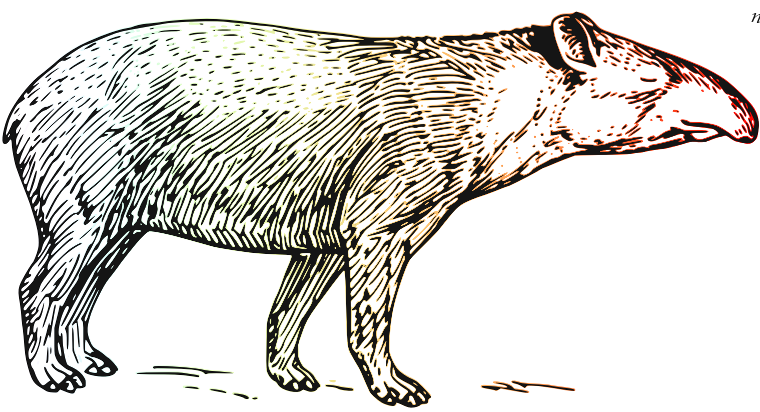
$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$

$\vec{v} = \sum\limits_{i=1}^{n} \alpha_i \hat{e}_i$

$e^{\pi i} + 1 = 0$

$\int\limits_{a}^{b} f(x)\,\mathrm{d}x = F(b) - F(a)$

$\cos(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$



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HERE BE TABLE

CHAPTER

0

INTRODUCTION

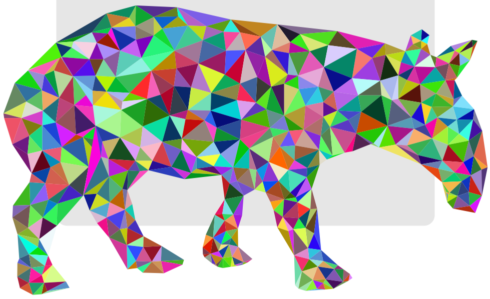
In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

CHAPTER

1



LINEAR ALGEBRA
(INTUITIVE APPROACH)

Linear algebra is one of the most important and often used fields, both in theoretical and applied mathematics. It brings together the analysis of systems of linear equations and the analysis of linear functions (in this context usually called linear transformations), and is employed extensively in almost any modern mathematical field, e.g. approximation theory, vector analysis, signal analysis, error correction, 3-dimensional computer graphics and many, many more.

In this book, we divide our discussion of linear algebra into to chapters: the first (this chapter) deals with a wider, birds-eye view of the topic: it aims to give an intuitive understanding of the major ideas of the topic. For this reason, in this chapter we limit ourselves almost exclusively to discussing linear algebra using 2- and 3-dimensional analysis (and higher dimensions when relevant) using real numbers only. This allows us to first create an intuitive picture of what is linear algebra all about, and how to use correctly the tools it provides us with.

The next chapter takes the opposite approach: it builds all concepts from the ground-up, defining precisely (almost) all basic concepts and proving them rigorously, and only then using them to build the next steps. This approach has two major advantages: it guarantees that what we build has firm foundations and does not fall apart at any future point, and it also allows us to generalize the ideas constructed during the process to such extent that they can be used as foundation to build ever newer tools we can apply in a wide range of cases.

1.1 DUAL VECTORS

1.1.1 Functionals

In the context of linear algebra, a **functional** is a linear function which takes a vector and returns a scalar, e.g. in the case of $\mathbb{R}n$,

$$\varphi : \mathbb{R}^n \to \mathbb{R}. \tag{1.1.1}$$

The linearity of the functional means that given two vectors $\vec{u}, \vec{v} \in \mathbb{R}n$ and two scalars $\alpha, \beta \in \mathbb{R}$,

$$\varphi\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha\varphi\left(\vec{u}\right) + \beta\left(\vec{v}\right). \tag{1.1.2}$$

Example 1.1 A functional over $\mathbb{R}n$

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x - y.$$

Then e.g.

$$\begin{aligned}
\varphi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= 2 \cdot 1 - 0 = 2, \\
\varphi\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) &= 2 \cdot 1 - (-4) = 6, \\
\varphi\left(\begin{bmatrix} -3 \\ -2 \end{bmatrix}\right) &= 2 \cdot (-3) - (-2) = -4,
\end{aligned}$$

etc. Let us show that this is indeed a linear function. Given two vectors

$$\vec{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix},$$

and the scalars α and β,

$$\varphi\left(\alpha\vec{u} + \beta\vec{v}\right) = \varphi\left(\begin{bmatrix} \alpha u_x + \beta v_x \\ \alpha u_y + \beta v_y \end{bmatrix}\right)$$

$$
\begin{aligned}
&= 2\left(\alpha u_x + \beta v_x\right) - \left(\alpha u_y + \beta v_y\right) \\
&= 2\alpha u_x + 2\beta v_x - \alpha u_y - \beta v_y \\
&= \left(2\alpha u_x - \alpha u_y\right) + \left(2\beta v_x - \beta v_y\right) \\
&= \alpha\left(2u_x - u_y\right) + \beta\left(2v_x - v_y\right) \\
&= \alpha\varphi\left(\vec{u}\right) + \beta\varphi\left(\vec{v}\right).
\end{aligned}
$$

Since functionals are linear, the most general functional over $\mathbb{R}^n$, when acting on a general vector

$$
\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \tag{1.1.3}
$$

gives the following output:

$$
\varphi\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \sum_{i=1}^{n} \alpha_i v_i. \tag{1.1.4}
$$

1.1.2 Duality

By a closer examination of Equation 1.1.4 we notice that it can be written as a scalar product of the vector

$$
\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{1.1.5}
$$

and the vector $\vec{v}$ as defined in Equation 1.1.3. This means that in a sense, applying a functional to a vector $\vec{v}$ is identical to performing a scalar product of some coefficient vector and $\vec{v}$. In fact, we can actually define the functional φ as defined in Equation 1.1.4 via the scalar product, i.e.

$$
\varphi\left(\vec{v}\right) = \vec{\alpha} \cdot \vec{v}.
$$

Example 1.2 Functional as vector

The functional defined in Example 1.1 can be defined via the vector

$$
\vec{\alpha} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},
$$

since for a general vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

the dot product $\vec{\alpha} \cdot \vec{v}$ yields precisely the same result:

$$\vec{\alpha} \cdot \vec{v} = 2x - y = \varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Since over $\mathbb{R}^n$ a functional can be represented by a scalar product of a vector with specific components, we call the functional a **dual vector**, and the space of all dual vectors the **dual space** of $\mathbb{R}^n$.

Generally speaking, any vector space V has a dual space, which is denoted by V^*. In the case of $\mathbb{R}^n$ this dual space is actually $\mathbb{R}^n$ itself, since any vector in $\mathbb{R}n$ can be used as a dual vector, and any dual vector is essentially a list of n real components - i.e. is equivalent to a vector in $\mathbb{R}^n$.

Note 1.1 When a functional has less than n coefficients

onsider the following functional over $\mathbb{R}^3$:

$$\varphi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 5x + 3z.$$

It seems to have only two coefficients, namely $\alpha_1 = 5$ and $\alpha_2 = 3$ - however, in order to represent it as a vector in $\mathbb{R}^3$ we need three coefficients, i.e. we say that $\alpha_2 = 0$ and $\alpha_3 = 3$ - essentially, we use the vector

$$\vec{\alpha} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}.$$

to represent it.

1.1.3 Row vectors and the outer product

In order to keep things consistent, we actually don't represent dual-vectors as column vectors but instead as row vectors. This allows us to treat the scalar product between two vectors similarily to the product of two matrices: we always make sure that the vector on the left is written as a row vector, and the vector on the right is written as a column vector, i.e. given

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

we write the scalar product between the two vectors as

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \begin{bmatrix} u_1, \ u_2, \ \dots, \ u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \\ \vec{v} \cdot \vec{u} &= \begin{bmatrix} v_1, \ v_2, \ \dots, \ v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}. \end{aligned} \tag{1.1.6}$$

(note how the value of the scalar product of $\vec{u}$ and $\vec{v}$ doesn't depend on the order of the vectors, since real numbers are commutative) The "matrices" in question have the dimensions $1 \times n$ and $n \times 1$ respectively, and thus we can calculate their product, which would be a 1×1 matrix that we interpret as a scalar. This gives rise to the question of what happens if we put a column vector on the left and a row vector on the right? In that case, treating them as "matrices" would mean that we are calculating the product of an $n \times 1$ matrix and a $1 \times n$ matrix, and thus the result should be an $n \times n$ matrix.

Indeed, this is what we call the **outer product** of two vectors. Let us examine how the outer product is structured with a simple case:

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 5, \ -1 \end{bmatrix}.$$

Then

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 5, \ -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5 & 2 \cdot (-1) \\ 3 \cdot 5 & 3 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 10 & -2 \\ 15 & -3 \end{bmatrix}. \tag{1.1.7}$$

And generaly, for any two vectors in $\mathbb{R}^n$:

$$\begin{bmatrix} u_{1} \\ u_{2} \\ \vdots \\ u_{n} \end{bmatrix} \begin{bmatrix} v_{1}, \ v_{2}, \ \dots, \ v_{n} \end{bmatrix} = \begin{bmatrix} u_{1}v_{1} & u_{1}v_{2} & \dots & u_{1}v_{n} \\ u_{2}v_{1} & u_{2}v_{2} & \dots & u_{2}v_{n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n}v_{1} & u_{n}v_{2} & \dots & u_{n}v_{n} \end{bmatrix}, \tag{1.1.8}$$

i.e. the value of any element a_{ij} in the outer product is equal to

$$a_{ij} = u_i v_j. \tag{1.1.9}$$

Note 1.2 The inner vs. outer product

Recall that the scalar product is sometimes called the inner product. If we consider scalars as having some rank equal to 0, vectors having a rank of 1 and matrices a rank of 2 - the inner product reduces the rank of the two components from 1 to 0, while the outer product increases them from 1 to 2.

In that sense, the normal product of two real numbers and the matrix-matrix product are found somewhere inbetween the inner and outer products, as the ranks of their outputs are equal to the ranks of their inputs.

!

1.1.4 A bit about more general vector spaces and their duals

! **To be written**: the subsection. !

1.2 THE BRA-KET NOTATION

! **To be written**: this section needs some rework on dual vectors !

In this section we introduce a special vector notation widely used in physics: the **Bra-ket notation**, also known as **Dirac's notation**. This notation helps simplify many aspects of linear algebra, and make its use more streamlined.

Note 1.3 Importance of this section

A person can have a pretty good grasp of linear algebra without ever learning about the bra-ket notation. This section, while interesting and providing a useful tool in working with linear algebra - is not obligatory, especially to people who do not intend to ever learn topics such as quantum physics, relativity theory, statistical mechanics, etc. It is however recommended even for those readers.

!

1.2.1 Definition

Up until now we presented the theory of linear algebra based on real numbers: all of the vectors we used were real vectors, i.e. of the form $\vec{v} \in \mathbb{R}^n$, where $n \in \mathbb{N}$. All of the matrices used were also made up of real components - and so were of course the scalars themselves, which we defined simply as real numbers.

However, it is aparently useful in many cases to use linear algebra in the context of complex numbers: instead of working with spaces of the form $\mathbb{R}^n$ we can use spaces of the form $\mathbb{C}^n$, e.g. a vector in $\mathbb{C}^3$ can be the following:

$$\vec{v} = \begin{bmatrix} 1-i \\ 2+3i \\ \pi + \sqrt{2}i \end{bmatrix}.$$

When using complex numbers instead of real numbers, a small change must be made to the way we conceptualize row vs. column vectors. Before we said that essentially both forms can be used interchangeably without affecting the outcome. However now we define row vectors a bit differently: given the column vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \tag{1.2.1}$$

we can get its row form by transposing it, i.e. we look at $\vec{v}^\top$. However, when doing this we must change all the components of $\vec{v}$ to their respective complex conjugates, i.e.

$$\vec{v}^\top = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^\top = \begin{bmatrix} \overline{v_1}, \overline{v_2}, \dots, \overline{v_n} \end{bmatrix}. \tag{1.2.2}$$

$$\vec{v}^\top = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^\top = \begin{bmatrix} v_1^*, v_2^*, \dots, v_n^* \end{bmatrix}. \tag{1.2.3}$$

To be consistent with the usual notation used in physics (and have more succisent, we introduce the following changes:

- The complex conjugate of the number $z \in \mathbb{C}$ is changed to C^*.
- The star notation is also used for transpose (this is normaly called **conjugate transpose**, and is sometimes denoted by $u^\dagger$).
- The arrow is dropped from the vector notation.

Applying these changes, Equation 1.2.3 has the form

$$v^* = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^* = \begin{bmatrix} v_1^*, v_2^*, \dots, v_n^* \end{bmatrix}. \tag{1.2.4}$$

Example 1.3 Bra/Row vectors

The bra form of the vector

$$v = \begin{bmatrix} 1+2i \\ 3-i \\ \sqrt{2}+5i \\ 4 \\ -3i \end{bmatrix}$$

is

$$v^* = \begin{bmatrix} 1+2i \\ 3-i \\ \sqrt{2}+5i \\ 4 \\ -3i \end{bmatrix}^* = \begin{bmatrix} 1-2i, & 3+i, & \sqrt{2}-5i, & 4, & 3i \end{bmatrix}.$$

Next, we make sure that the scalar product between any two vectors u, v is such that the left vector is a row vector, and the right vector is a column vector, i.e. given $u, v \in \mathbb{C}^n$ such that

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix},$$

their scalar product is

$$u \cdot v = u^* \cdot v = \begin{bmatrix} u_1^*, & u_2^*, & \dots, & u_n^* \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n. \tag{1.2.5}$$

Recall that a common notation for the scalar product of two vectors uses triangular brakets, i.e.

$$u \cdot v = \langle u|v \rangle.$$

We can use Equation 1.2.5 and "separate" this product into two parts: a **bra** $\langle u|$ and a **ket** $|v\rangle$, define as

$$\langle u| = \begin{bmatrix} u_1^*, & u_2^*, & \dots, & u_n^* \end{bmatrix},$$

$$|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \tag{1.2.6}$$

1.2.2 Norm and products

The norm of a vector v can be calculated by taking the square root of its scalar product with itself (**??**). Using the bra-ket notation this becomes

$$\left\|\vec{v}\right\| = \sqrt{\langle v|v\rangle}. \tag{1.2.7}$$

Let us write the properties of the scalar product adjusted to the bra-ket notation:

- **Non-negative norm**: for any vector $v \in \mathbb{C}^n$, $\langle v|v\rangle \geq 0$.

- **Uniqeness of zero**: if $\langle v|v\rangle = 0$, then $v = 0$.

- **Conjugate commutativity**: for any two vectors $u, v \in \mathbb{C}^n$, $\langle u|v\rangle = \langle v|u\rangle^*$.

- **Distributivity**: Given three vectors $u, v, w \in \mathbb{C}^n$ and two scalars $\alpha, \beta \in \mathbb{C}$,

$$\langle u|\left(\alpha\,|v\rangle + \beta\,|w\rangle\right) = \alpha\,\langle u|v\rangle + \beta\,\langle u|w\rangle.$$

Note 1.4 Hilbert spaces

enerally speaking, any vector space that is "equiped" with a norm complying with these properties is called a **Hilbert space**. We will discuss such spaces in more details later in the book.

!

There is an interesting way one can interpret the scalar product: instead of as an operation acting on two vectors, we can view a bra as an operator acting on a ket and returning a scalar. Mathematically this is written as

$$\langle\bigcirc| : \mathbb{C}^n \to \mathbb{C}. \tag{1.2.8}$$

(the empty circle signifies that that the symbol representing the bra is placed inside the bra symbol)

! **To be written**: this is the dual space of $\mathbb{C}^n$, etc. !

Another product that is easily defined usin the bra-ket notation is the **exterior product** of two vectors (recall that the scalar product is also called the *inner product*). The exterior product arises when we multiply two vectors in the opposite order compared to the scalar product, i.e. instead of $\langle u|v\rangle$ we calculate $|u\rangle\langle v|$:

$$|u\rangle\langle v| = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1^*, & v_2^*, & \dots, & v_n^* \end{bmatrix} = \begin{bmatrix} u_1 v_1^* & u_1 v_2^* & \dots & u_1 v_n^* \\ u_2 v_1^* & u_2 v_2^* & \dots & u_2 v_n^* \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1^* & u_n v_2^* & \dots & u_n v_n^* \end{bmatrix}, \tag{1.2.9}$$

i.e. the result of the outer product is a matrix, in which any element a_{ij} is equal to

$$a_{ij} = u_i v_j^*. \tag{1.2.10}$$

Note 1.5 Bra-ket vs. ket-bra

Using the bra-ket notation, the names for the inner and outer products make sense: in the inner product the vectors stay inside the brackets, while in the outer product they are outside of it. This is just a small demonstration for the “power” of the bra-ket notation in simplifying mathematical expressions.

Example 1.4 Inner and outer products

Given the two kets

$$|u\rangle = \begin{bmatrix} 1+\mathrm{i} \\ 3 \end{bmatrix},\ |v\rangle = \begin{bmatrix} -2+3\mathrm{i} \\ 5-\mathrm{i} \end{bmatrix},$$

let us calculate the following:

$$\langle u|v\rangle,\ \langle v|u\rangle,\ |u\rangle\langle v|,\ |v\rangle\langle u|.$$

We start by writing $\langle u|$ and $\langle v|$:

$$\langle u| = \begin{bmatrix} 1-\mathrm{i},\ 3 \end{bmatrix},\ \langle v| = \begin{bmatrix} -2-3\mathrm{i},\ 5+\mathrm{i} \end{bmatrix}.$$

Then, the 4 requested products are easy to calculate:

$$\langle u|v\rangle = (1-\mathrm{i})(-2+3\mathrm{i}) + 3(5-\mathrm{i}) = -2+3\mathrm{i}+2\mathrm{i}+3+15-3\mathrm{i} = 16+2\mathrm{i}.$$
$$\langle v|u\rangle = (1+\mathrm{i})(-2-3\mathrm{i}) + 3(5+\mathrm{i}) = -2-3\mathrm{i}-2\mathrm{i}+3+15+3\mathrm{i} = 16-2\mathrm{i} = \langle u|v\rangle^*.$$

$$|u\rangle\langle v| = \begin{bmatrix} (1+\mathrm{i})(-2-3\mathrm{i}) & (1+\mathrm{i})(5+\mathrm{i}) \\ 3(-2-3\mathrm{i}) & 3(5+\mathrm{i}) \end{bmatrix} = \begin{bmatrix} 1-5\mathrm{i} & 4+6\mathrm{i} \\ -6-9\mathrm{i} & 15+3\mathrm{i} \end{bmatrix}.$$
$$|v\rangle\langle u| = \begin{bmatrix} (-2+3\mathrm{i})(1-\mathrm{i}) & 3(-2+3\mathrm{i}) \\ (5-\mathrm{i})(1-\mathrm{i}) & 3(5-\mathrm{i}) \end{bmatrix} = \begin{bmatrix} 1+5\mathrm{i} & -6+9\mathrm{i} \\ 4-6\mathrm{i} & 15-3\mathrm{i} \end{bmatrix}.$$

1.2.3 Scalar multiplication and addition of vectors

Scaling a vector in the bra-ket notation is represented by simply putting the scalar on the left or right of the vector, i.e. given a ket

$$|u\rangle = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix},$$

its scaled version by the scalar $\alpha \in \mathbb{C}$ is

$$\alpha\,|u\rangle = |u\rangle\,\alpha = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{bmatrix}. \tag{1.2.11}$$

The bra version is essentialy the same:

$$\alpha\,\langle u| = \langle u|\,\alpha = \left[\alpha u_1^*,\ \alpha u_2^*,\ \ldots,\ \alpha u_n^*\right]. \tag{1.2.12}$$

Normaly, scaling a ket is written with the scalar on the left, and scaling a bra is written with scalar on the right. However, in some instances the other forms are used (we will see such case soon).

Adding two kets or two bras is also quote simple:

$$\begin{aligned} |u\rangle + |v\rangle = |u+v\rangle &= \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}, \\ \langle u| + \langle v| = \langle u+v| &= \left[(u_1+v_1)^*,\ (u_2+v_2)^*,\ \ldots,\ (u_n+v_n)^*\right]. \end{aligned} \tag{1.2.13}$$

Note 1.6 Addition of a bra and a ket

Of course, a bra and a ket cannot be added together.

!

1.2.4 Linear combinations and basis sets

Linear combinations are easily represented using the bra-ket notation: let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be complex numbers, and $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$ kets in $\mathbb{C}^n$. Then

$$|w\rangle = \alpha_1\,|v_1\rangle + \alpha_2\,|v_2\rangle + \cdots + \alpha_n\,|v_n\rangle = \sum_{i=1}^{n} \alpha_i\,|v_i\rangle \tag{1.2.14}$$

is of course itself a vector in $\mathbb{C}^n$. If the set $B = \left\{|b_1\rangle, |b_2\rangle, \ldots, |b_n\rangle\right\}$ is a basis set of $\mathbb{C}^n$ then any vector $\vec{v} \in \mathbb{C}^n$ can be written as a linear combination of the vectors in B, i.e.

$$\vec{w} = \sum_{i=1}^{n} \beta_i\,|b_i\rangle, \tag{1.2.15}$$

such that not all $\beta_i = 0$ (i.e. at least one of them is non-zero). If in addition

$$\left\langle b_i \middle| b_j \right\rangle = \delta_{ij}, \tag{1.2.16}$$

the basis set is orthonormal: each vector has unit norm (since $\langle b_i | b_i \rangle = \delta_{ii} = 1$), and they are all orthogonal to each other (since for $i \neq j$, $\left\langle b_i \middle| b_j \right\rangle = \delta_{i \neq j} = 0$).

Sometimes, and depending on the context, basis vectors are written as their indeces only inside a bra or a ket. This is common for example with the standard basis vectors, e.g. in $\mathbb{C}^3$

$$\hat{x} = |1\rangle,\ \hat{y} = |2\rangle,\ \hat{z} = |3\rangle. \tag{1.2.17}$$

and more generaly

$$\hat{e}_i = |i\rangle. \tag{1.2.18}$$

Let us now take a general ket in $|v\rangle \in \mathbb{C}^n$ and write it as a linear combination of some orthonormal basis set B:

$$|v\rangle = \sum_{i=1}^{n} \alpha_i \left|b_i\right\rangle. \tag{1.2.19}$$

We now choose one of the basis vectors and write it in its bra form: $\left\langle b_j\right|$, where $j \in \{1, 2, \ldots, n\}$. We can easily write the inner product of $\left\langle b_j\right|$ with $|v\rangle$:

$$\begin{aligned} \left\langle b_j \middle| v\right\rangle &= \left\langle b_j\right| \sum_{i=1}^{n} \alpha_i \left|b_i\right\rangle \\ &= \left\langle b_j\right| \alpha_1 \left|b_1\right\rangle + \left\langle b_j\right| \alpha_2 \left|b_2\right\rangle + \cdots + \left\langle b_j\right| \alpha_n \left|b_n\right\rangle. \end{aligned} \tag{1.2.20}$$

Since α_i are scalars, we can move each one of them to the left of $\left\langle b_j\right|$, yielding

$$\left\langle b_j \middle| v\right\rangle = \alpha_1 \left\langle b_j \middle| b_1\right\rangle + \alpha_2 \left\langle b_j \middle| b_2\right\rangle + \cdots + \alpha_n \left\langle b_j \middle| b_n\right\rangle. \tag{1.2.21}$$

Since the basis vectors are all orthonormal, $\left\langle b_j \middle| b_i\right\rangle = \delta_{ji}$, or zero for all i except when $i = j$, in which case the product equals 1. Thus all the terms where $i \neq j$ disappear, and we are left with a single term:

$$\left\langle b_j \middle| v\right\rangle = \alpha_j \left\langle b_j \middle| b_j\right\rangle = \alpha_j. \tag{1.2.22}$$

This shows is that we can calculate the coefficient α_j for any $j \in \{1, 2, \ldots, n\}$ by simply taking the dot product of the j-th basis vector with $|u\rangle$. This is very easy compared to using the vector notation we used so far.

We can now start again with spanning $|u\rangle$ using B, but this time we write the scalar coefficients on the right side:

$$|u\rangle = \sum_{i=1}^{n} \left|b_i\right\rangle \alpha_i. \tag{1.2.23}$$

Subtituting Equation 1.2.22 into Equation 1.2.23 we get

$$|u\rangle = \sum_{i=1}^{n} \left|b_i\right\rangle \left\langle b_i \middle| u\right\rangle. \tag{1.2.24}$$

Since $|u\rangle$ is found in all the terms of the sum, we can put parenthesis around the summation excluding $|u\rangle$:

$$|u\rangle = \left(\sum_{i=1}^{n} \left|b_i\right\rangle \left\langle b_i\right| \right) |u\rangle. \tag{1.2.25}$$

Recall that the outer product return a matrix. Thus, Equation 1.2.25 tells us that the matrix $A = \sum_{i=1}^{n} |b_i\rangle\langle b_i|$ is a matrix that doesn't change any vector $|u\rangle$ (remember that there is nothing special about $|u\rangle$, we didn't even write it explicitly). This matrix is of course the identity matrix, i.e.

$$I_n = \sum_{i=1}^{n} |b_i\rangle\langle b_i|. \tag{1.2.26}$$

This result is called the **completeness relation**. It essentially tells us that given an orthonormal basis set, the sum of the outer product of each basis vector with itself is the identity matrix. For example, given the standard basis set in $\mathbb{C}^3$,

$$|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| = I_3. \tag{1.2.27}$$

Example 1.5 The completeness relation in $\mathbb{C}^2$

As we saw in **??**, the following basis set is orthonormal:

$$|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix},\ |2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}.$$

The outer products of each of the two vectors with itself are

$$|1\rangle\langle 1| = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix},$$
$$|2\rangle\langle 2| = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}.$$

Thus,

$$|1\rangle\langle 1| + |2\rangle\langle 2| = \frac{1}{2}\left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} + \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2.$$

! **To be written**: change of basis, matries inside brakets, eigenvectors, etc. !