CHAPTER 0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

0.1 EXERCISES

0.1. Write the following sets explicitly:

(i) $\{x \in \mathbb{N} \mid 1 < x \leq 7\}$

(ii) $\{x \in \mathbb{Z} \mid x < 5\}$

(iii) $\left\{x \in \mathbb{R} \mid x^2 = -1\right\}$

(iv) $\{x \in \mathbb{N} \wedge x \in \mathbb{Q}\}$

(v) $\left\{x \in \mathbb{R} \mid x^2 - 3x - 4 = 0\right\}$

(vi) $\{x \in \mathbb{R} \mid x < 5 \wedge x \geq 2\}$

0.2. Determine the relation between the sets:

(i) $A = \{1,\ 2,\ 3\},\ B = \{1,\ 2\}$

(ii) $A = \varnothing,\ B = \{2,\ -5,\ \pi\}$

(iii) $A = \mathbb{Z},\ B = \{\pm x \mid x \in \mathbb{N} \cup \{0\}\}$

(iv) $A = \left\{\pi, \mathrm{e}, \sqrt{2}\right\},\ B = \mathbb{Q}$

0.3. Write all elements in $S^2 \times W$, where $S = \{\alpha, \beta, \gamma\}$ and $W = \{x, y, z\}$. Find a condition that guarantees $S^2 \times W = W \times S^2$.

0.4. How many different injective functions $f : \{1, 2\} \to \{1, 2\}$ exist? How many injective functions $f : \{1, 2, 3\} \to \{1, 2, 3\}$ exist? How many inject functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ exist for a given $n \in \mathbb{N}$?

0.5. For each of the real functions below, find a set on which it is surjective (use a graphing calculator if you are not familiar with the shape of a function):

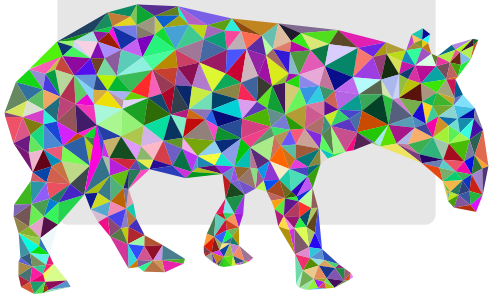
$$x^2,\ x^3 - 5,\ \mathrm{e}^{-x^2/2},\ \sin(x),\ \sin(x) + \cos(x),\ x\mathrm{e}^x.$$

0.6. Given two sets A, B such that $|B| = |A| - 1$, can a bijective function $f : A \to B$ exist? Explain your answer.

0.7. MORE EXERCISES TO BE WRITTEN...

CHAPTER

1



LINEAR ALGEBRA

(INTUITIVE APPROACH)

Linear algebra is one of the most important and often used fields, both in theoretical and applied mathematics. It brings together the analysis of systems of linear equations and the analysis of linear functions (in this context usually called linear transformations), and is employed extensively in almost any modern mathematical field, e.g. approximation theory, vector analysis, signal analysis, error correction, 3-dimensional computer graphics and many, many more.

In this book, we divide our discussion of linear algebra into to chapters: the first (this chapter) deals with a wider, birds-eye view of the topic: it aims to give an intuitive understanding of the major ideas of the topic. For this reason, in this chapter we limit ourselves almost exclusively to discussing linear algebra using 2- and 3-dimensional analysis (and higher dimensions when relevant) using real numbers only. This allows us to first create an intuitive picture of what is linear algebra all about, and how to use correctly the tools it provides us with.

The next chapter takes the opposite approach: it builds all concepts from the ground-up, defining precisely (almost) all basic concepts and proving them rigorously, and only

then using them to build the next steps. This approach has two major advantages: it guarantees that what we build has firm foundations and does not fall apart at any future point, and it also allows us to generalize the ideas constructed during the process to such extent that they can be used as foundation to build ever newer tools we can apply in a wide range of cases.

1.1 VECTORS

1.1.1 Basics

Vectors are the fundamental objects of linear algebra: the entire field revolves around manipulation of vectors. In this chapter we deal with the so-called **real vectors**, which can be be defined in a geometric way:

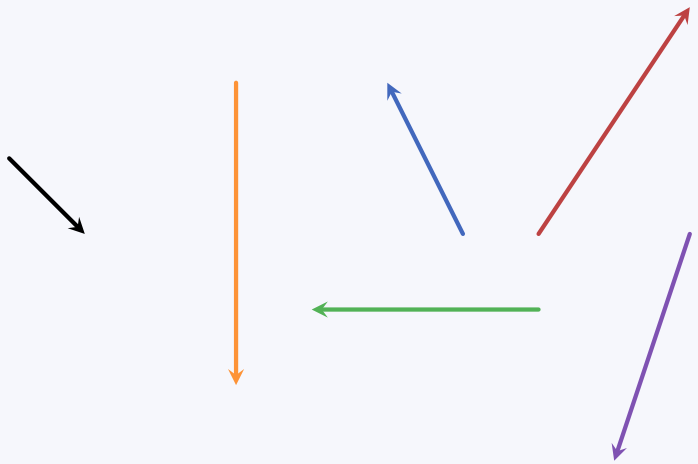
Definition 1.1 Real vectors

A *real vector* is an object with a **magnitude** (also called **norm**) and a **direction**.

In this chapter we refer to real vectors simply as *vectors*.

Example 1.1 Real vectors

The following are all vectors in 2-dimensional space depicted as arrows:



Vectors are usually denoted in one of the following ways:

- **Arrow above letter**: $\vec{u}$, $\vec{v}$, $\vec{x}$, $\vec{a}$, ...

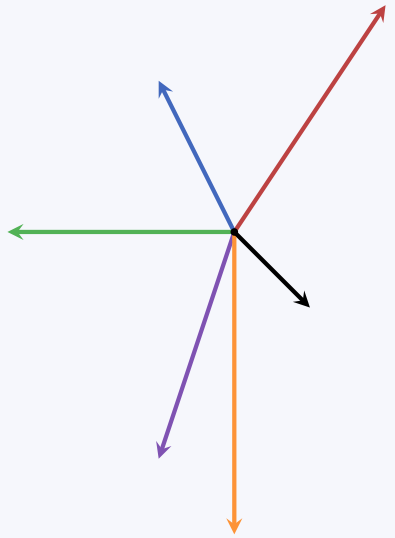
- **Bold letter**: $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{x}$, $\boldsymbol{a}$, ...
- **Bar below letter**: $\underline{u}$, $\underline{v}$, $\underline{x}$, $\underline{a}$, ...

In this book we use the first notation style, i.e. an arrow above the letter. In addition vectors will almost always be denoted using lowercase Lating script.

When discussing vectors in a single context, we always consider them starting at the same point, called the **origin**, and **translating** (moving) vectors around in space does not change their properties: only their norms and directions matter.

Example 1.2 Real vectors

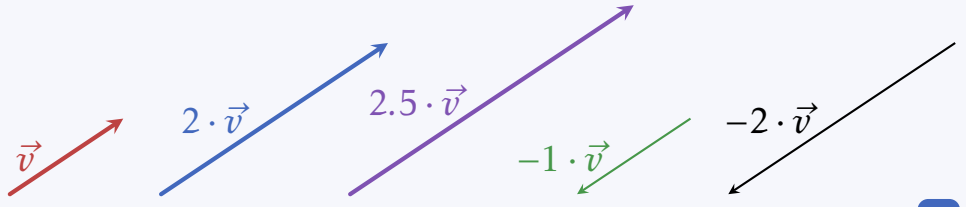
The vectors from the previous translated (moved) such that their origins all lie on the same point:



A vector can be scaled by a real number α: when this happens, its norm is multiplied by α while its direction stays the same. We call α a **scalar**.

Example 1.3 Scaling vectors

The following vector $\vec{v}$ scaled by different scalars $\alpha = 2, 2.5, -1, -2$:



Note 1.1 Negative scale

As can be seen in the example above, when scaling a vector by a negative amount its direction reverses. However, we consider two opposing direction (i.e. directions that are 180° apart) as being the same direction.

!

In this book we use the following notation for the norm of a vector $\vec{v}$: $\left\|\vec{v}\right\|$.

A vector $\vec{v}$ with norm $\left\|\vec{v}\right\| = 1$ is called a **unit vector**, and is usually denoted by replacing the arrow symbol by a hat symbol: $\hat{v}$. Any vector (except $\vec{0}$) can be scaled into a unit vector by scaling the vector by 1 over its own norm, i.e.

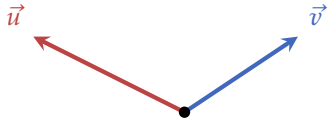
$$\hat{v} = \frac{1}{\left\|\vec{v}\right\|}\vec{v}. \tag{1.1.1}$$

The result of normalization is a vector of unit norm which points in the same direction of the original vector.

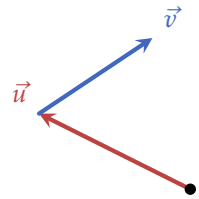
Two vectors can be added together to yield a third vector: $\vec{u} + \vec{v} = \vec{w}$. To find $\vec{w}$ we use the following procedure (depicted in Figure 1.1):

1.1. Move (translate) $\vec{v}$ such that its origin lies on the head of $\vec{u}$.

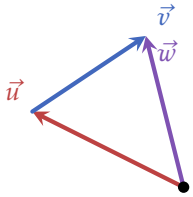
1.2. The vector $\vec{w}$ is the vector drawn from the origin of $\vec{u}$ to the head of $\vec{v}$.



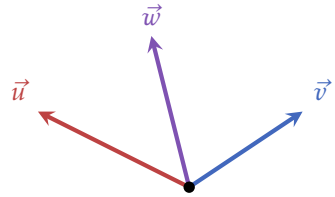
(1) The vectors $\vec{u}$ and $\vec{v}$.



(2) Translating $\vec{v}$ such that its origin lies at the head of $\vec{u}$.



(3) Drawing the vector $\vec{w}$ from the origin to the head of $\vec{v}$.



(4) Showing all three vectors.

Figure 1.1 Vector addition.

The addition of vectors as depicted here is commutative, i.e. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$. This can be seen by using the **parallogram law of vector addition** as depicted in Figure 1.2: drawing

the two vectors $\vec{u}, \vec{v}$ and their translated copies (each such that its origin lies on the other vector's head) results in a parallelogram.

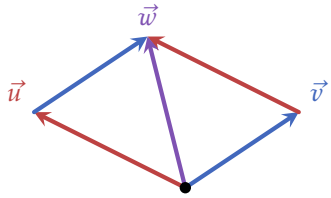


Figure 1.2 The parallogram law of vector addition.

An important vector is the **zero-vector**, denoted as $\vec{0}$. The zero-vector has a unique property: it is neutral in respect to vector addition, i.e. for any vector $\vec{v}$,

$$\vec{v} + \vec{0} = \vec{v}. \tag{1.1.2}$$

(we also say that $\vec{0}$ is the **additive identity** in respect to vectors.)

Any vector $\vec{v}$ always has an **opposite** vector, denoted $-\vec{v}$. The addition of a vector and its opposite always result in the zero-vector, i.e.

$$\vec{v} + (-\vec{v}) = \vec{0}. \tag{1.1.3}$$

1.1.2 Components

Vectors can be decomposed to their components, the number of which depends on the dimension of space we're using: 2-dimensional vectors can be decomposed into 2 components, 3-dimensional vectors can be decomposed into 3 components, etc. To decompose a vector, say $\vec{v}$, we first choose a coordinate system: the most commonly used system, and the one we will use for most of this chapter, is the Cartesian coordinate system. We place the vector in the coordinate system such that its origin lies at the origin of the system. We then draw a perpendicular line from its head to each of the axes in the system (see Figure 1.3), the point of interception on each axis is the component of the vector in that axis (we label these points v_x, v_y, v_z in the case of 2- or 3-dimensional spaces, and generally $v_1, v_2, v_3, \ldots$). The vector can then be written as a column using these components:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \tag{1.1.4}$$

Note 1.2 Order of components

The order of the components of a vector is important, and should always be consistent. In the case of 2- and 3-dimensional the order is always v_x, v_y, v_z.

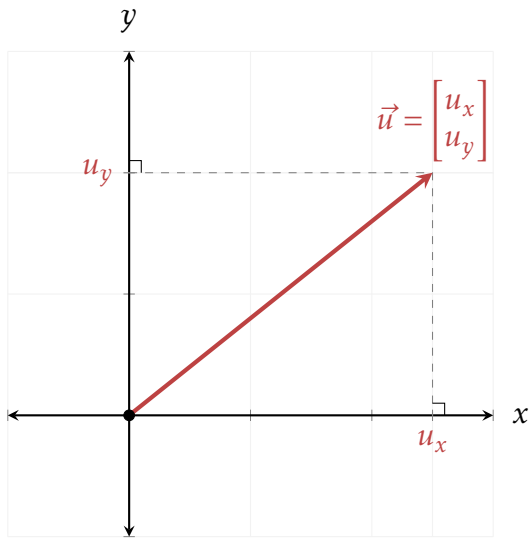
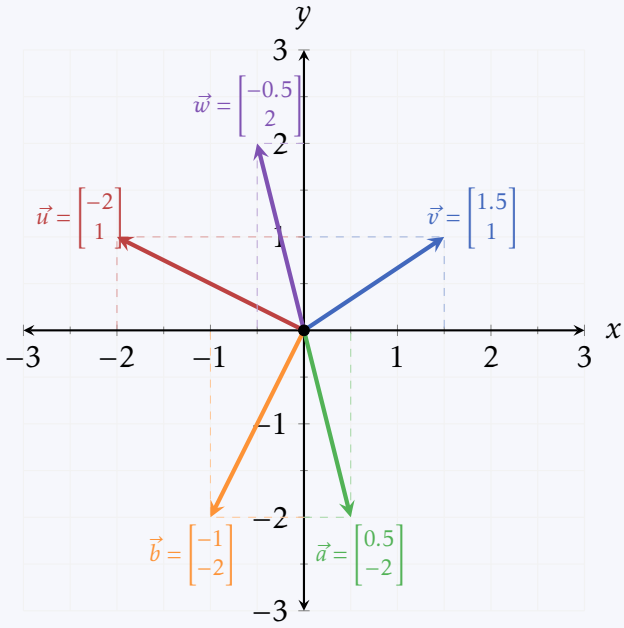


Figure 1.3 Placing a 2-dimensional vector $\vec{u}$ on the 2-dimensional Cartesian coordinate system, showing its x- and y-components.

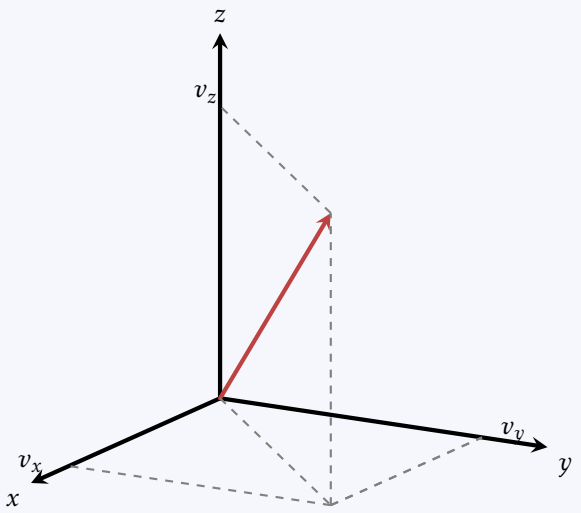
Example 1.4 Vector components in two dimensions

The following five 2-dimensional vectors are decomposed each into its x- and y-components:



Example 1.5 Vector components in three dimensions

The following 3-dimensional vector is decomposed into its x-, y- and z-components: (THIS NEEDS TO BE IMPROVED AND FINISHED)



The column form of a vector is essentially equivalent to an order list of n real numbers, i.e. $(v_1, v_2, \ldots, v_n)$. Why then are we using the column form and not the list form (mostly known as **row vectors**)? In fact, we could use either form - and even using both interchangeably - and with only minor adjusments the entire chapter would stay the same as it is now. However, there are some advantages of using only a single form, and consider the other form as a different object altogether. This idea will become clear in future chapters, when discussing **covariant vectors**, **contravarient vectors**, and **tensors**. For now, we stick with the column form of vectors to stay consistent with common notation.

However, the row form of vectors highlights the space in which they exist: n-dimensional vectors live in a space we call $\mathbb{R}^n$. Recall from Chapter 0 that the set $\mathbb{R}^n$ is a Cartesian product made up of n times the set of real numbers, i.e.

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n}. \tag{1.1.5}$$

Each member of this set is a list of n real numbers, and their order inside the list matters - very similar to vectors, be they in row or column form. For this reason, we refer to $\mathbb{R}^n$ as the space of n-dimensional real vectors. As mentioned, in this chapter we use $\mathbb{R}^2$ (the 2-dimensional real space) and $\mathbb{R}^3$ (the 3-dimensional real space) for most ideas and examples.

Looking at vectors in $\mathbb{R}^2$, it is rather straight-forward to calculate their norm: since the origin, the head of the vector and the point v_x form a right triangle (see Figure 1.4), we can use the Pythagorean theorem to calculate the norm of the vector, which is equal to the hypotenous of said triangle:

$$\left\|\vec{v}\right\| = \sqrt{v_x^2 + v_y^2}. \tag{1.1.6}$$

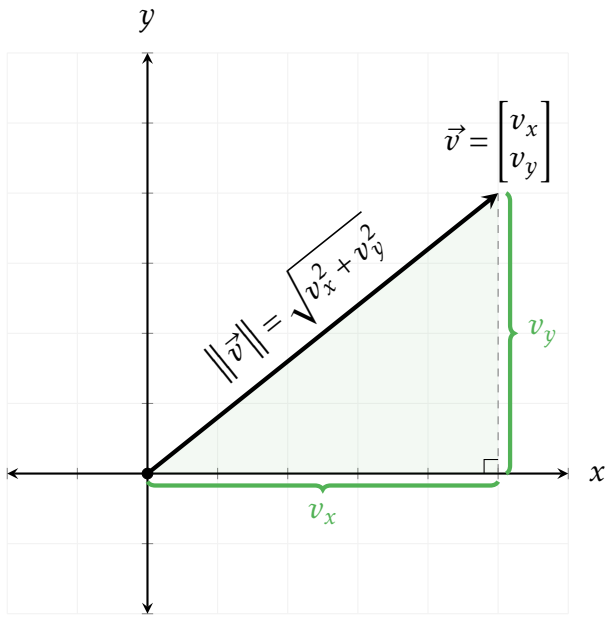


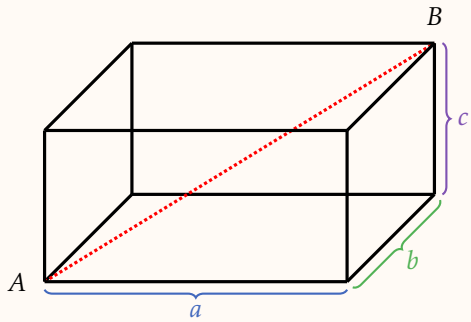
Figure 1.4 Calculating the norm of a 2-dimensional column vector.

In $\mathbb{R}^3$ the norm of a vector $\vec{v}$ is similarily

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}. \tag{1.1.7}$$

Challange 1.1 Norm of a 3D vector

Show why Equation 1.1.7 is valid, by calculating the length AB in the following figure, depicting a box of sides a, b and c:



?

Generalizing the vector norms in $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^n$ yields the following form:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}. \tag{1.1.8}$$

Note 1.3 Other norms

The norm shown here is called the 2-norm. There are other possible norm that can be defined, and are used in different situations, such as the 1-norm (also the called **taxicab norm**), general p-norm where $p \geq 1$ is a real number, the zero-norm, the max-norm, and many others. However, for the purpose of this chapter we use only the standard 2-norm, since it is the most useful for describing basic concepts of linear algebra and its uses.

!

Scaling a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ by a real number α is done by multiplying each of its components by α, i.e.

$$\alpha\vec{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}. \tag{1.1.9}$$

We can prove Equation 1.1.9 by directly calculating the norm of a scaled vector $\vec{w} = \alpha\vec{v}$:

Proof 1.1 Scaling a column vector

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}$, where $\alpha \in \mathbb{R}$. Then $\vec{w}$ has the following norm:

$$\begin{aligned} \left\|\vec{w}\right\| &= \sqrt{\sum_{i=1}^{n} (\alpha v_i)^2} \\ &= \sqrt{(\alpha v_1)^2 + (\alpha v_2)^2 + \cdots + (\alpha v_1)^2} \\ &= \sqrt{\alpha^2 v_1^2 + \alpha^2 v_2^2 + \cdots + \alpha^2 v_n^2} \\ &= \sqrt{\alpha^2 \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)} \\ &= \alpha \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\ &= \alpha \left\|\vec{v}\right\|. \end{aligned}$$

This shows that indeed $\vec{w} = \alpha\vec{v}$.

QED

Another idea we can prove in column form is vector normalization (Equation 1.1.1), by showing that dividing each component of a vector by its norm gives a vector of unit norm:

Proof 1.2 L

t $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Its norm is then $\left\|\vec{v}\right\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. Scaling $\vec{v}$ by $\frac{1}{\left\|\vec{v}\right\|}$ yields

$$\hat{v} = \frac{1}{\left\|\vec{v}\right\|} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \frac{1}{\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The norm of $\hat{v}$ is therefore

$$\begin{aligned} \|\hat{v}\| &= \sqrt{\frac{v_1^2}{v_1^2 + v_2^2 + \cdots + v_n^2} + \frac{v_2^2}{v_1^2 + v_2^2 + \cdots + v_n^2} + \cdots + \frac{v_n^2}{v_1^2 + v_2^2 + \cdots + v_n^2}} \\ &= \sqrt{\frac{1}{v_1^2 + v_2^2 + \cdots + v_n^2}\left(v_1^2 + v_2^2 + \cdots + v_n^2\right)} \\ &= \sqrt{1} = 1, \end{aligned}$$

i.e. $\hat{v}$ is indeed a unit vector.

QED

Example 1.6 Normalizing a vector

Let's normalize the vector $\vec{v} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix}$. Its norm is

$$\left\|\vec{v}\right\| = \sqrt{0^2 + 4^2 + (-3)^2} = \sqrt{0 + 16 + 9} = \sqrt{25} = 5.$$

Therefore $\hat{v}$ (the normalized $\vec{v}$) is

$$\hat{v} = \begin{bmatrix} 0 \\ \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}.$$

By calculating the norm of $\hat{v}$ directly, we can see that it is indeed a unit vector:

$$\|\hat{v}\| = \sqrt{0^2 + \frac{4^2}{5^2} + \frac{3^2}{5^2}} = \sqrt{\frac{0^2 + 4^2 + 3^2}{5^2}} = \sqrt{\frac{16 + 9}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1.$$

The addition of two column vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is done by adding their respective components together, i.e.

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}. \tag{1.1.10}$$

TBW: how this addition is the same as the one shown in Figure 1.1.

Note 1.4 No addition of vectors of different number of components!

Two vectors can only be added together if they have the same number of components. The addition of vectors with different number of components is undefined.

1.1.3 Linear combinations, spans and linear dependency

As seen above, scaling a vector by a scalar results in a vector that has the same number of dimensions as the original vector. The same is true for adding two vectors: both of them must be of the same dimension, and the result is also a vector of the same dimension. Therefore, any combination of scaling and addition of vectors results in a vector of the same dimension as the original vector(s). This kind of combination is called a **linear combination**.

Let's define linear combinations a little more formaly:

Definition 1.2 Linear combinations

A linear combination of n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ of the same dimension, using n scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, is an expression of the form

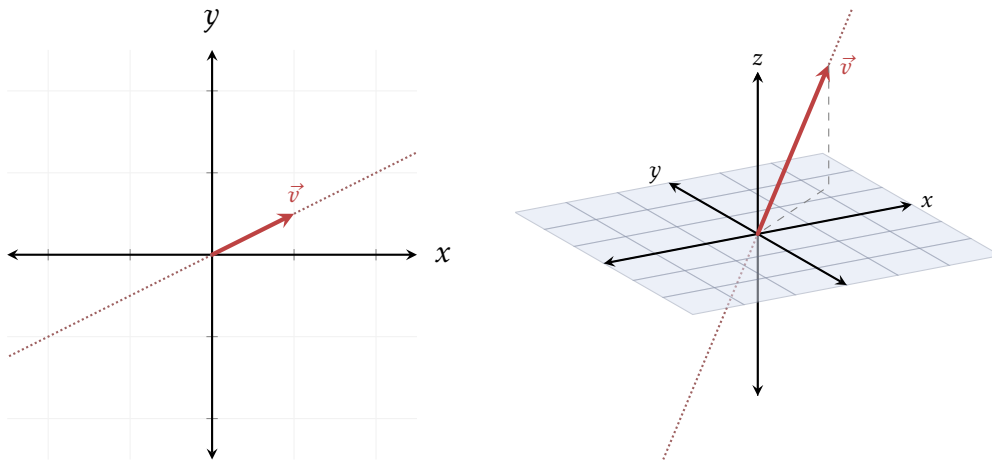
$$\vec{w} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = \sum_{i=1}^{n} \alpha_i \vec{v}_i. \tag{1.1.11}$$

Linear combinations of real vectors have geometric meaningsc: we start with the set of all linear combinations of a single vector $\vec{v} \in \mathbb{R}^n$, i.e.

$$V = \{\alpha \vec{v} \mid \alpha \in \mathbb{R}\}. \tag{1.1.12}$$

The set V represents a line in the direction of $\vec{v}$ going through the origin (see Figure 1.5). The set V is itself a vector space of dimension 1, and as such a **subspace** of $\mathbb{R}^n$. We say that it is the **span** of the vector $\vec{v}$ (i.e. the vector $\vec{v}$ **spans** the subspace V).

Similarily, the set of all linear combinations of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ that are not scales



(1) $\mathbb{R}^2$ (2) $\mathbb{R}^3$

Figure 1.5 The span of a single vector, shown as dashed lines: in $\mathbb{R}^2$ (left) and $\mathbb{R}^3$ (right).

of each other (i.e. there is no such $\alpha \in \mathbb{R}$ for which $\vec{v} = \alpha\vec{u}$),

$$V = \{\alpha\vec{u} + \beta\vec{v} \mid \alpha, \beta \in \mathbb{R}\}, \tag{1.1.13}$$

is a plane that goes through the origin (see Figure 1.6). Such vectors are also said to be **non-collinear**.

Example 1.7 Spanning $\mathbb{R}^2$ using two non-collinear vectors

Since any two non-collinear vectors span a 2-dimensional subspace of $\mathbb{R}^n$, in $\mathbb{R}^2$ this means that any vector $\vec{w}$ can be written as a linear combination of any two vectors $\vec{u}, \vec{v}$ that are not a scale of each other. For example, we can take the vector

$$\vec{w} = \begin{bmatrix} 7 \\ -1 \end{bmatrix},$$

and write it as a linear combination of any two non-collinear vectors, say

$$\vec{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

The equation which forces the relation is

$$\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 5 \end{bmatrix},$$

and we should solve it for α and β. This is possible since the equation above is actually a system of two equations in two variables (namely α and β):

$$\begin{cases} 7 = 2\alpha, \\ -1 = -3\alpha + 5\beta. \end{cases}$$

The solution for the system is $\alpha = 3.5$ and $\beta = 1.9$, and therefore

$$\begin{bmatrix} 7 \\ -1 \end{bmatrix} = 3.5 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + 1.9 \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

As the reader, you should verify for yourself the above equation.

Generalizing the example above, any vector $\vec{w} = \begin{bmatrix} w_x \\ w_y \end{bmatrix}$ can be written as a linear combination of two vectors $\vec{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$, as long as $\vec{u}$ and $\vec{v}$ are non-collinear. Let's prove this:

Proof 1.3 $\mathbb{R}^2$ is spanned by any two non-collinear vectors in $\mathbb{R}^2$

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ be two non-collinear vectors. Their non-collinearity means that the equation

$$\vec{u} = \alpha \vec{v} \tag{1.1.14}$$

has no solution, i.e. the system

$$\begin{cases} u_x = \alpha v_x \\ u_y = \alpha v_y \end{cases} \tag{1.1.15}$$

has no solution. The system has solution only when $u_x v_y = u_y v_x$, and so the restriction is translated to the simple equation

$$u_x v_y \neq u_y v_x. \tag{1.1.16}$$

The system which defines $\vec{w}$ as a linear combination of $\vec{u}$ and $\vec{v}$ is

$$\begin{cases} w_x = \alpha u_x + \beta v_x \\ w_y = \alpha u_y + \beta v_y \end{cases} \tag{1.1.17}$$

Isolating α using the first equation yields

$$\alpha = \frac{w_x - \beta v_x}{u_x}, \tag{1.1.18}$$

and subtituting it into the second equation yields

$$\beta = \frac{w_y - \alpha u_y}{v_y} = \frac{w_y - \frac{w_x - \beta v_x}{u_x}}{v_y}, \tag{1.1.19}$$

which rearranges into

$$\beta = \frac{u_x w_y - u_y w_x}{u_x v_y - u_y v_x}, \tag{1.1.20}$$

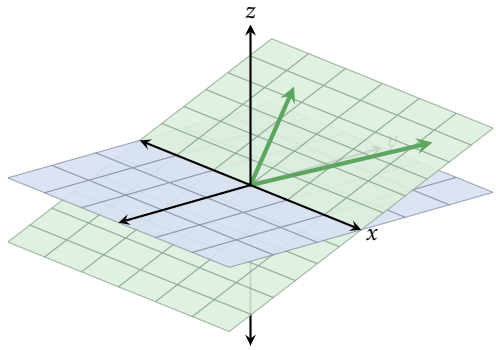


Figure 1.6 Two vectors (colored green) span a plane (also colored green) in $\mathbb{R}^3$. The xy-plane (i.e. $z = 0$) is shown in blue.

and thus

$$\alpha = \frac{-v_x w_y + v_y w_x}{u_x v_y - u_y v_x}. \tag{1.1.21}$$

We can see that α and β exist iff $u_x v_y \neq u_y v_x$, which is guaranteed by Equation 1.1.16. Therefore, α and β always exist when $\vec{u}$ and $\vec{v}$ are non-collinear, and thus any vector in $\mathbb{R}^2$ can be written as a linear combination of any two non-collinear vectors in $\mathbb{R}^2$, i.e. any two non-collinear vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$.

QED

Expanding this idea to higher dimensions, any three vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ that are not co-planar (i.e. that don't lie on the same plane) span a 3-dimensional subspace of $\mathbb{R}^n$.

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1.2 LINEAR TRANSFORMATIONS

1.3 MATRICES

1.4 SYSTEMS OF LINEAR EQUATIONS

1.5 EIGENVECTORS AND EIGENVALUES

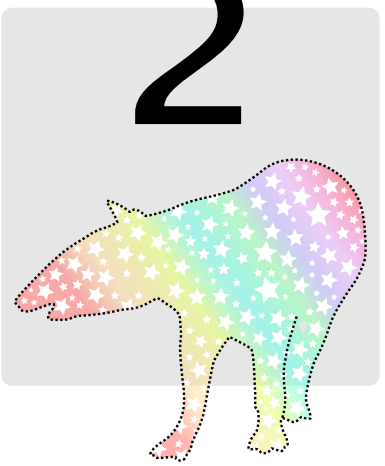
1.6 DECOMPOSITIONS

1.7 SOME REAL LIFE USES OF LINEAR ALGEBRA

1.8 EXERCISES

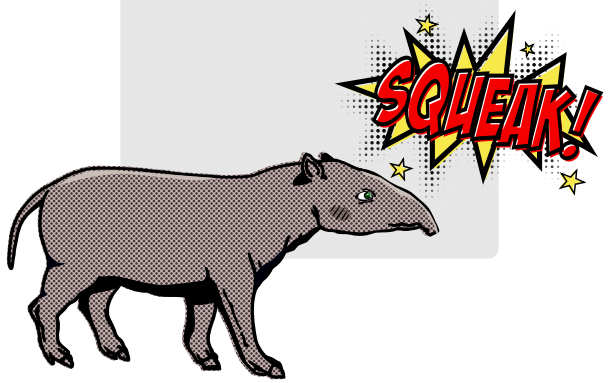
CHAPTER

2



CALCULUS IN 1D

CHAPTER 3



LINEAR ALGEBRA

(RIGOROUS APPROACH)

Something about formalism, theorems, proofs, etc.

Note 3.1 Why present rigorous mathematics in this book?

Rigorous mathematics is rarely necessary for those who are interested in the tools mathematics provides us with, rather than the full and deep understanding of the concepts these tools are based on. However, it can be useful to students of scientific fields to experience rigorous mathematics at least once in their course of study. Usually, the choice for the topic to be analyzed rigorously is between linear algebra and calculus - for this book the latter was chosen.

3.1 FIELDS

We begin our dive into the rigorous analysis of linear algebra by defining an algebraic construction call a **field**, which we need in order to properly define vector spaces later. In essence, a field has most of the important properties of the real numbers, namely the closure, commutativity, associativity, identity and inverse of addition and multiplication of any two elements in the field (except the product inverse of the field equivalent object for the number 0). In a later section we will use fields to construct the general notion of **vector spaces**.

Definition 3.1 Field

A field $\mathbb{F}$ is a set of objects together with two operations called **addition** and **multiplication** (denoted + and ·, respectively), for which the following axioms hold:

- **Closure of under addition and multiplication**: for any $a, b \in \mathbb{F}$,
  1. $(a + b) \in \mathbb{F}$,
  2. $(a \cdot b) \in \mathbb{F}$.

- **Commutativity under addition multiplication**: for any $a, b \in \mathbb{F}$,
  1. $a + b = b + a$,
  2. $a \cdot b = b \cdot a$.

- **Associativity under addition and multiplication**: for any $a, b, c \in \mathbb{F}$,
  1. $a + (b + c) = (a + b) + c$,
  2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

- **Additive and multiplicative identity**: there exist an element in $\mathbb{F}$ called the *additive identity* and denoted by 0, for which $a + 0 = a$ for any $a \in \mathbb{F}$.
  Similarity, there exists an element in $\mathbb{F}$ called the *multiplicative identity* and denoted by 1, for which $a \cdot 1 = a$ for any $a \in \mathbb{F}$.

- **Additive and multiplicative inverses**: for any element $a \in \mathbb{F}$ (except the additive identity) there exists:
  1. $b \in \mathbb{F}$ such that $a + b = 0$, and
  2. $c \in \mathbb{F}$ such that $a \cdot c = 1$.

  (usually b is denoted as $-a$, while c is denoted as a^{-1})

- **Distributivity of multiplication over addition**: for any $a, b, c \in \mathbb{F}$,
  $$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

3.1.1 Infinite fields

We start with one of the most obvious examples of a field: the real numbers together with the standard addition and product.

Theorem 3.1 $\mathbb{R}$ as a field

The set of real numbers $\mathbb{R}$ forms a field together with the standard addition and product.

We leave the proof of 3.1 to the reader, as it is pretty straight forward using the known properties of the standard addition and product over $\mathbb{R}$ (and rather uninteresting). Instead, we jump forward to using 3.1 for proving the same idea about the complex numbers:

Theorem 3.2 $\mathbb{C}$ as a field and more more more

The set of complex numbers $\mathbb{C}$ forms a field together with the addition and product operations as defined in **??** (namely **??**, **??** and **??**).

Proof 3.1 $\mathbb{C}$ as a field

(note: in the following proof, equalities marked with ! use the respective property of the real numbers)

- **Closure under both operations**: for any two complex numbers $z_1 = a + \mathrm{i}b$ and $z_2 = c + \mathrm{i}d$,

  - Addition: since addition in $\mathbb{R}$ is closed, $(a+c) \in \mathbb{R}$ and $(b+d) \in \mathbb{R}$. Therefore

    $$z = z_1 + z_2 = a + c + (b + d)\mathrm{i}$$

    is also a complex number with $\mathfrak{Re}(z) = a + c$ and $\mathfrak{Im}(z) = b + d$.

  - Multiplication: since multiplication in $\mathbb{R}$ is also closed, $(ac - bd) \in \mathbb{R}$ and $(ad + bc) \in \mathbb{R}$. Therefore

    $$z = z_1 \cdot z_2 = ac - bd + (ad + bc)\mathrm{i}$$

    is a complex number with $\mathfrak{Re}(z) = ac - bdc$ and $\mathfrak{Im}(z) = ad + bc$.

- **Commutativity of both operation**: for any two complex numbers $z_1 = a + \mathrm{i}b$ and $z_2 = c + \mathrm{i}d$,

  - Addition: since addition in $\mathbb{R}$ is commutative, $a+c = c+a$ and $b+d = d+b$. Therefore

    $$z_1 + z_2 = a + c + (b + d)\mathrm{i} \overset{!}{=} c + a + (d + b)\mathrm{i} = z_2 + z_2.$$

• Multiplication: since multiplication in $\mathbb{R}$ is also commutative, $ac - bd = ca - db$ and $ad + bc = da + cb$. Therefore

$$z_1 \cdot z_2 = ac - bd + (ad + bc)\mathrm{i} \overset{!}{=} ca - db + (da + cb)\mathrm{i} = z_2 \cdot z_1.$$

• **Associativity of both operation**: for any three complex numbers $z_1 = a + \mathrm{i}b$, $z_2 = c + \mathrm{i}d$ and $z_3 = g + \mathrm{i}h$ (where $a, b, c, d, g, h \in \mathbb{R}$[a]),

• Addition: since addition in $\mathbb{R}$ is associative, $a + (c + g) = (a + c) + g$ and $b + (d + h) = (b + d) + h$. Therefore

$$z_1 + (z_2 + z_3) = a + (c + g) + [b + (d + h)]\mathrm{i} \overset{!}{=} (a + c) + g + [(b + d) + h]\mathrm{i} = (z_1 + z_2) + z_3.$$

• Multiplication: since multiplication in $\mathbb{R}$ is also associative, the following equalities apply:

$$\begin{aligned}
a \cdot (c \cdot g) &= (a \cdot c) \cdot g, \\
b \cdot (c \cdot h) &= (b \cdot c) \cdot h, \\
a \cdot (d \cdot h) &= (a \cdot d) \cdot h, \\
b \cdot (d \cdot g) &= (b \cdot d) \cdot g, \\
a \cdot (c \cdot h) &= (a \cdot c) \cdot h, \\
a \cdot (d \cdot g) &= (a \cdot d) \cdot g, \\
b \cdot (c \cdot g) &= (b \cdot c) \cdot g, \\
b \cdot (d \cdot h) &= (b \cdot d) \cdot h.
\end{aligned}$$

Therefore,

$$\begin{aligned}
z_1 \cdot (z_2 \cdot z_3) &= a \cdot (c \cdot g) - a \cdot (d \cdot h) - b \cdot (c \cdot h) - b \cdot (d \cdot g) \\
&\quad + [a \cdot (c \cdot h) + a \cdot (d \cdot g) + b \cdot (c \cdot g) - b \cdot (d \cdot h)]\mathrm{i} \\
&\overset{!}{=} (a \cdot c) \cdot g - (a \cdot d) \cdot h - (b \cdot c) \cdot h - (b \cdot d) \cdot g \\
&\quad + [(a \cdot c) \cdot h + (a \cdot d) \cdot g + (b \cdot c) \cdot g - (b \cdot d) \cdot h]\mathrm{i} \\
&= (z_1 \cdot z_2) \cdot z_3.
\end{aligned}$$

• **Identity for both operations**:

• Addition: the complex number $0 = 0 + 0i$ is the complex addition identity: for any real number $x \in \mathbb{R}$, $x + 0 = x$. Therefore, for any complex number $z = a + \mathrm{i}b$,

$$z + 0 = a + \mathrm{i}b + 0 + 0i = a + 0 + (b + 0)\mathrm{i} \overset{!}{=} a + \mathrm{i}b.$$

• Multiplication: the complex number $1 = 1 + 0\mathrm{i}$ is the complex multiplication identity: for any real number $x \in \mathbb{R}$, $x \cdot 1 = x$ and $x \cdot 0 = 0$. Therefore, for

any complex number $z = a + \mathrm{i}b$,

$$z \cdot 1 = (a + \mathrm{i}b) \cdot (1 + 0\mathrm{i}) \overset{!}{=} a \cdot 1 - \cancel{b \cdot 0\mathrm{i}^2} + (\cancel{a \cdot 0\mathrm{i}} + b \cdot 1)\mathrm{i} = a + \mathrm{i}b.$$

- **Inverse for both operations**:

  - Addition: for any complex number $z_1 = a + \mathrm{i}b$, the number $z_2 = -a - \mathrm{i}b$ is also a complex number for which

  $$z_1 + z_2 = a + \mathrm{i}b + -a - \mathrm{i}b \overset{!}{=} a - a + (b - b)\mathrm{i} = 0 + 0\mathrm{i} = 0.$$

  - Multiplication: for any complex number $z = re^{\theta i}$ where $r \neq 0$, the number $z^{-1} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta}$ is also a complex number for which

  $$z \cdot z^{-1} = r\mathrm{e}^{\mathrm{i}\theta} \cdot \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta} \overset{!}{=} \frac{r}{r}\mathrm{e}^{\cancel{\mathrm{i}\theta - \mathrm{i}\theta}} = 1 \cdot 1 = 1.$$

  Note: for $z = a + \mathrm{i}b$,

  $$z^{-1} = \frac{1}{r}\mathrm{e}^{-\mathrm{i}\theta} = \frac{1}{|z|} \cdot \frac{a - \mathrm{i}b}{|z|} = \frac{1}{|z|} \cdot \frac{\overline{z}}{|z|} = \frac{\overline{z}}{|z|^2}.$$

  Therefore, for any $z \neq 0$, $z^{-1} = \frac{\overline{z}}{|z|^2}$.

- **Distributivity of multiplication over addition**: for any $z_1 = a + \mathrm{i}b$, $z_2 = c + \mathrm{i}d$ and $z_3 = g + \mathrm{i}h$,

$$\begin{aligned} z_1 \cdot (z_2 + z_3) &= (a + \mathrm{i}b) \cdot (c + \mathrm{i}d + g + \mathrm{i}h) = (a + \mathrm{i}b) \cdot (c + g + [d + h]\mathrm{i}) \\ &= ac + ag + (bd)\mathrm{i}^2 + (bh)\mathrm{i}^2 \\ &+ (ad)\mathrm{i} + (ah)\mathrm{i} + (bc)\mathrm{i} + (bg)\mathrm{i} \\ &= ac + ag - bd - bh + (ad + ah + bc + bg)\mathrm{i} \\ &= ac - bd + (ad + bc)\mathrm{i} + ag - bh + (ah + bg)\mathrm{i} \\ &= (z_1 \cdot z_2) + (z_1 \cdot z_3). \end{aligned}$$

[a] The letters g and h are used instead of e and f to avoid confusion with Eurler's constant and the common notation for real functions, respectively.

QED

The sets $\mathbb{R}$ and $\mathbb{C}$ are examples of **infinite fields**, since they each have infinite number of elements. The set $\mathbb{Q}$ (rational numbers) can be shown to also be an infinite field, however unlike $\mathbb{R}$ and $\mathbb{C}$ it has **countable** number of elements, i.e. each number in $\mathbb{Q}$ can be assigned an index $1, 2, 3, \ldots$ [1].

[1] For proof, see ...

Challange 3.1 $\mathbb{Q}$ as a field

Prove that $\mathbb{Q}$ (together with the usual addition and product operation) is indeed a field.

?

3.1.2 Finite fields

While all three examples of fields we encountered so far have each an infinite number of elements, some fields only have a finite number of elements (called their **order**). For example, consider the set $S = \{0, 1, a, b\}$ and the addition and product operations described using the following tables (left table describes addition, right table describes multiplication):

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

By examining the tables above, several points become clear:

- all the possible combinations of operands in both addition and multiplication give elements from S itself, meaning that the set is closed under both these operations.
- both tables are symmetric around their main diagonal, meaning that both addition and multiplication are commutative operations.
- in the addition table, the first row and first column both show that $x + 0 = x$ for any $x \in S$, meaning that 0 is the additive identity in S.
- in the product table, the second row and second column both show that $x \cdot 1 = x$ for any $x \in S$, meaning that 1 is the multiplicative identity in S.
- in the addition table, the element 0 appears in each row and each column exactly once. This means that every element x has a single additive inverse $y \in S$.
- in the product table, the element 1 appears in each row and each column exactly once, except for the first row and first column. This means that every element $x \neq 0$ has a single multiplicative inverse $z \in S$.

We therefore only need to prove two points to show that S is a field together with the operations described by the above tables: associativity of both operations and distributivity of multiplication over addition. We leave these proofs as a challenge to the reader. Such a field is sometime denoted as $\mathbb{F}_4$. There are, of course, infinitely many finite fields.

3.1.3 Modulo fields

Another example of finite fields are sets of integers of the form $\{0, 1, 2, 3, \ldots, n\}$ where n is a prime, together with **modular addition** and **modular product**. To understand modular arithmetics, we recall the fact that on a circle, an angle can have a negative value but also greater than 360° values are possible (see **??**): 390° is equivalent to 30°, −30° is equivalent to 330°, etc. The set of integer values $0°, 1°, 2°, \ldots, 359°$ on a circle is an example of a modular set: if for example we add together two angles of values deg 100 and 300° we get the equivalent angle deg 60. If we subtract deg 300 from deg 100 the result is an angle of deg 160.

We say that on a circle, the values $360°, 720°, -360°$ etc. are all **congruent** to 0 modulo 360. In mathematical notation we represent this fact as e.g.

$$720 \equiv 0 \pmod{360}. \tag{3.1.1}$$

Note that from this point forward we drop the degrees unit, and deal with pure integers. The notation for the set $\{0, 1, 2, \ldots, 359\}$ is $\mathbb{Z}_{360}$. Generally speaking, the set $\{0, 1, 2, \ldots, n\}$ is denoted as $\mathbb{Z}_n$.

Note 3.2 About modulo set notation

It is not common to use the notation $\mathbb{Z}_n$ for the modulo-n set, since it is also used for a different algebraic construct, namely the n-adic ring. However, due to the simplicity of the notation, and the fact that we don't discuss rings in this chapter we are using it in this book. Common notations for the set are $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/n$.

Addition and multiplication on $\mathbb{Z}_n$ is done by the following rather straight forward definition:

Definition 3.2 Operations in $\mathbb{Z}_n$

In the set $\mathbb{Z}_n$ addition and multiplication are defined as the following:

- **Addition**: for any two elements $a, b \in \mathbb{Z}_n$, $a + b := (a + b) \pmod{n}$.
- **Multiplication**: for any two elements $a, b \in \mathbb{Z}_n$, $a \cdot b := (a \cdot b) \pmod{n}$.

Example 3.1 Operations in $\mathbb{Z}_n$

The tables below show addition and multiplication results of numbers in different modulo sets $\mathbb{Z}_n$ for some values of n:

n	$2+3$	$2\cdot 3$
4	1	2
5	0	1
6	5	0
7	5	6
8	5	6
9	5	6
10	5	6
11	5	6

n	$4+7$	$4\cdot 7$
8	3	4
9	2	1
10	1	8
15	11	13
20	11	8
27	11	1
28	11	0
30	11	28

Figure 3.1 shows the equivalency between integers and the elements of $\mathbb{Z}_5$.

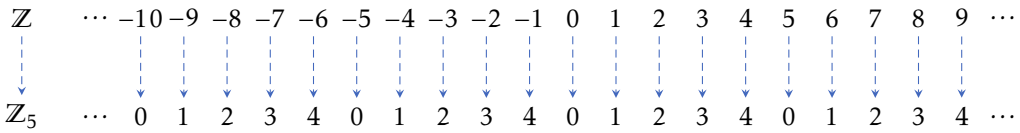


Figure 3.1 An example of the periodicity of $\mathbb{Z}_5$: the top numbers are the ordinary integers, each showing their respective congruent modulo 5 below (blue dashed arrow).

Only the sets $\mathbb{Z}_n$ for which n is a prime number are also fields. Let's define this property precisely:

Theorem 3.3 $\mathbb{Z}_p$ is a field

Any modulo set $\mathbb{Z}_p$ where p is a prime number greater than 1 is also a field together with the operations as defined in 3.2.

In order to prove 3.3 we use two lemmas: the first is known as **Bézout's lemma**:

Lemma 3.1 Bézout's lemma

For any two positive integers a, b there exist two integers x, y such that

$$\gcd(a,b) = xa + yb.$$

Note 3.3 $\gcd(a,b)$

$\gcd(a,b)$ is the **greatest common deviser** of the two integers a and b. For example, $\gcd(36,24) = 12$ since the divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36, and the divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

An example of Bézout's lemma is the following:

Example 3.2 Bézout's lemma in action

For the two positive integers $a = 60,\ y = 114$

$$\gcd(60, 114) = 6.$$

Therefore, Bézout's lemma says that there exist two integers x, y such that

$$6 = 60x + 114y.$$

Indeed, two such integers exist: $x = 2$ and $y = -1$.

(SHOULD WE PROVE THE LEMMA?..)

The second lemma we use is the following:

Lemma 3.2 $\gcd(n, p) = 1$

Given a positive prime number p, then for any positive integer $n < p$,

$$\gcd(p, n) = 1.$$

Proving the lemma:

Proof 3.2 $\gcd(n, p) = 1$

We assume that $\gcd(p, n) \neq 1$. Then there exist an integer $a \leq n < p$ which divides both n and p, meaning that p has a divider, contrary to the assumption that p is a prime number. Therefore $\gcd(n, p)$ must equal 1.

QED

Now we can proceed to the proof of 3.3:

Proof 3.3 $\mathbb{Z}_p$ is a field

- **Closure under both operations**: the definition of the modulo operator limit any $M \pmod p$ (where $M \in \mathbb{Z}$) to be in $[0, p-1]$. Therefore the result of using the operators given in 3.2 must be within the same range, and thus in $\mathbb{Z}_p$.

- **Commutativity and associativity of both operations**: for any two numbers $a, b \in \mathbb{Z}_p$ the result $a + b$ and $a \cdot b$ under $\mathbb{Z}$ is both commutative and associative. Therefore the result modulo n is the same no matter the order of operations.

- **Additive identity**: the number $0 \in \mathbb{Z}_p$ is the additive identity, since for each $a \in \mathbb{Z}_p,\ a + 0 = a$.

- **Multiplicative identity**: the number $1 \in \mathbb{Z}_p$ is the additive identity, since for each $a \in \mathbb{Z}_p$, $a \cdot 1 = a$.

- **Additive inverse**: for each $a \in \mathbb{Z}_p$ the element $n = p - a$ is in $\mathbb{Z}_p$ since $p > a$. Adding n to a results in 0:

$$a + n = a + (p - a) = p \equiv 0 \pmod{p}.$$

- **Multiplicative inverse**: let $a \in \mathbb{Z}_p$ and $a \neq 0$. Since p is a prime, $\gcd(a, p) = 1$ and from Bézout's theorem we know that there exist two integers x, y such that

$$xa + yp = 1.$$

  Rearrangement gives $p = \frac{1-xa}{y}$ meaning that p divides $1 - xa$, and thus

$$xa \equiv 1 \pmod{p}.$$

  Therefore x is the multiplicative inverse of a.

- **Distributivity of multiplication over addition**: ...

QED

The only part of the proof that uses the fact that p is a prime number is the multiplicative inverse. When n is not a prime, $\mathbb{Z}_n$ is not a field.

Challange 3.2 $\mathbb{Z}_n$ is not a field when n is not a prime number

Prove that the modulo set $\mathbb{Z}_n$ where n is **not** a prime number, is not a field. (**hint**: what property of prime numbers is used in the above proof to show that there is always a multiplicative inverse in $\mathbb{Z}_p$ where p is prime?)

?

3.2 VECTOR SPACES

As we've seen in Chapter 1 vectors are found at the heart of linear algebra. We first defined them in a geometric way as objects with magnitude and direction, and later as lists of real numbers, analyzing the connections between these two mostly parallel definitions. We also spoke about vector spaces of the type $\mathbb{R}^n$ as the structures vectors exist in. However, we haven't defined vectors nor vector spaces formally - which is exactly what we do in this section, by defining the concept of **vector spaces**.

Note 3.4 $\mathbb{R}^n$ as a guide to general vector spaces

While reading the definition below, it is worthwhile to reflect on each of the given axioms as it relates to the familiar vector space $\mathbb{R}^n$.

Definition 3.3 Vector space

A vector space over a field $\mathbb{F}$ is a set V which, together with two operations described below, fulfils a list of axioms. The two operations are

• Vector addition: an operation which takes two elements of V and returns a single element of V, i.e. $+ : V \times V \to V$.

• Scalar multiplication: an operation which takes a single element of $\mathbb{F}$ and a single element of V and returns a single element of V, i.e. $\cdot : \mathbb{F}, V \to V$.

The axioms to be fulfilled are:

• **Commutativity of vector addition**: for any $u, v \in V$,

$$u + v = v + u.$$

• **Associativity of vector addition**: for any $u, v, w \in V$,

$$u + (v + w) = (u + v) + w.$$

• **Additive identity**: there exist an element $0 \in V$ for which, for any $v \in V$,

$$v + 0 = v.$$

• **Scalar multiplicative identity**: for any $v \in V$

$$1 \cdot v = v,$$

where 1 is the multiplicative identity in $\mathbb{F}$.

• **Additive inverse**: for any $v \in V$ there exist an element $u \in V$ for which

$$v + u = 0.$$

• **Associativity of scalar multiplication**: for any $\alpha, \beta \in \mathbb{F}$ and $v \in V$

$$\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v,$$

where $\alpha\beta$ is the multiplication defined for $\mathbb{F}$.

• **Distributivity of vector addition**: for any $\alpha \in \mathbb{F}$ and $u, v \in V$,

$$\alpha \cdot (u + v) = (\alpha \cdot u) + (\alpha \cdot v).$$

• **Distributivity of scalar addition**: for any $\alpha, \beta \in \mathbb{F}$ and $v \in V$,

$$(\alpha + \beta) \cdot v = (\alpha \cdot v) + (\beta \cdot v).$$

The elements of V are then called **vectors**, and the elements of $\mathbb{F}$ are called **scalars**.

Since we discussed $\mathbb{R}^n$ thoroughly in Chapter 1, let's prove that it is indeed a vector space under the above definition. First, the claim:

Theorem 3.4 $\mathbb{R}^n$ is a vector space

The set of elements of the form

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where $v_i \in \mathbb{R}$, forms a vector space over $\mathbb{R}$ together with the following two operations:

• **Vector addition**:

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

• **Scalar multiplication**:

$$\alpha \cdot \vec{v} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}.$$

The proof itself is pretty easy, based on the fact that $\mathbb{R}$ is a field:

Proof 3.4 $\mathbb{R}^n$ is a vector space

Since the results of both operations defined for $\mathbb{R}^n$ only depend on the respective components of a vector $v \in \mathbb{R}^n$, all the axioms of a vector space apply, since they derive directly from the fact that $\mathbb{R}$ is a field. As an example, we will elaborate on two of the axioms:

• **Additive inverse**: Given a vector $\vec{v} \in \mathbb{R}^n$, each of its components v_i has an in-

verse under $\mathbb{R}$, namely $-v_i$. Therefore,

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{bmatrix} = \begin{bmatrix} v_1 - v_1 \\ v_2 - v_2 \\ \vdots \\ v_n - v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0},$$

which is the additive identity in $\mathbb{R}^n$.

- **Distributivity of vector addition**: for each component of two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, given the rules for vector addition and scalar multiplication, together with the distributivity of numbers in $\mathbb{R}$:

$$\begin{aligned} \alpha(\vec{u} + \vec{v}) &= \alpha \left(\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right) = \alpha \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \\ &= \begin{bmatrix} \alpha u_1 + \alpha v_1 \\ \alpha u_2 + \alpha v_2 \\ \vdots \\ \alpha u_n + \alpha v_n \end{bmatrix} = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{bmatrix} + \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix} = \alpha \vec{u} + \alpha \vec{v}. \end{aligned}$$

QED

(it is adviceable for the reader to go over the rest of the axioms and prove them for $\mathbb{R}^n$)

3.3 EXERCISES

CHAPTER

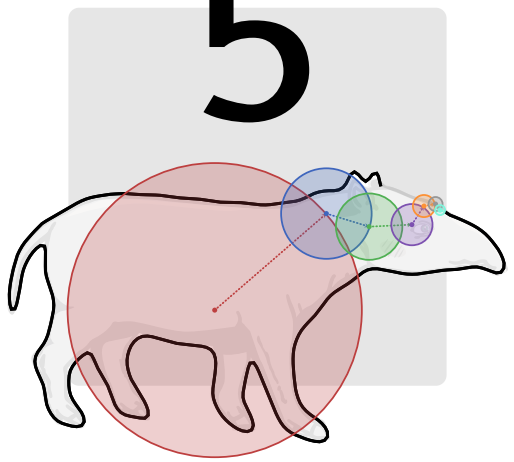
4

DIFFERENTIAL EQUATIONS

4.1 EXERCISES

CHAPTER

5

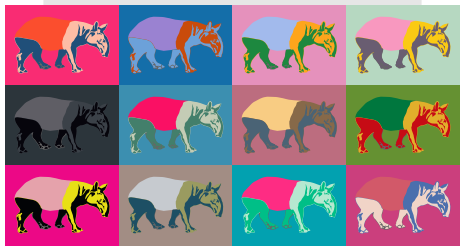


THE FOURIER TRANSFORM

5.1 EXERCISES

CHAPTER

6



SYMMETRY GOURPS

6.1 EXERCISES