MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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with contributions from others

$$a^b = e^{b\log(a)}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$T\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$$

$$A = Q\Lambda Q^{-1}$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,\mathrm{d}t$$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

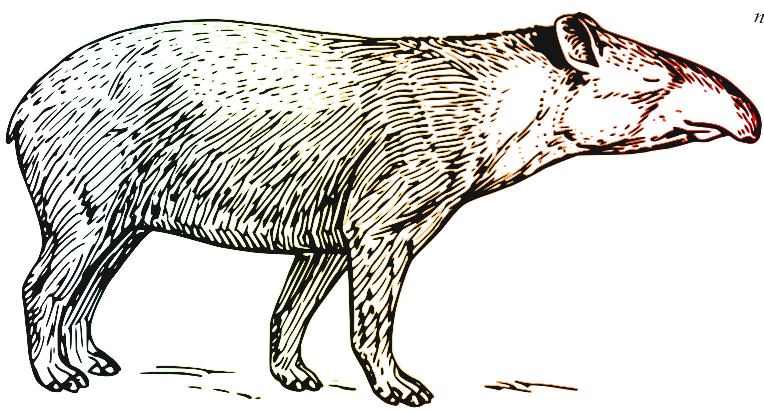
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$e^{\pi i} + 1 = 0$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$



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HERE BE TABLE

CHAPTER
0

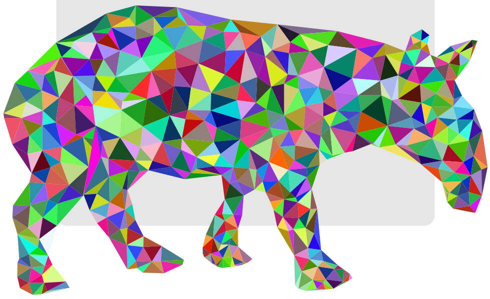
INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

CHAPTER 1



REAL CALCULUS IN 1D

1.1 SEQUENCES AND SERIES

1.1.1 Basics

A **sequence** is an indexed collection of **elements**. By *indexed* we mean that the order of the elements in a sequence matters (unlike with sets): changing the order of any element changes the sequence as a whole. The following are some examples of sequences composed of real numbers:

- $1, -3, 0, -7, 2, 1.5, 4, 0, 1, -0.35, \sqrt{2}$.
- $0, 1, 2, 1, 1, -1, 0$.
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$

- $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots$

The examples above present two more properties of sequences:

- Elements may repeat (unlike in the case of sets), and
- sequences can be either **finite** (as in the first two examples), or **infinite** (as in the latter two examples).

The number of elements in a sequence is called its **length**. In the case of infinite sequences we say that their length equals ∞ (inifinity). The elements of a sequence a are usually indexed using a subscript, such that a_1 is the first element in the sequence, a_2 is the second element in the sequence, etc. - and generally a_i is the i-th element in the sequence, where $i \in \mathbb{N}$.

We can therefore define a sequence somewhat more formally as a function from a subset of the natural numbers to the real numbers:

$$a : N \to \mathbb{R}, \tag{1.1.1}$$

where $N \subseteq \mathbb{N}$.

Example 1.1 Sequences as functions

The following 9-element sequence a

$$\begin{matrix} 3, & 4, & \frac{1}{2}, & 0, & 2, & 6, & -\frac{2}{3}, & 0, & -1. \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a(1) & a(2) & a(3) & a(4) & a(5) & a(6) & a(7) & a(8) & a(9) \end{matrix}$$

can be viewed as a function

$$a : \{1, 2, \ldots, 9\} \to \mathbb{R},$$

or more precisely as a function

$$a : \{1, 2, \ldots, 9\} \to \left\{-1, -\frac{2}{3}, 0, \frac{1}{2}, 2, 3, 4, 6\right\}.$$

The follow infinite sequence b

$$\begin{matrix} 1, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \frac{1}{6}, & \frac{1}{7}, & \cdots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ b(1) & b(2) & b(3) & b(4) & b(5) & b(6) & b(7) & \end{matrix}$$

can be viewed as a function

$$b : \mathbb{N} \to (0, 1].$$

Since sequences can be viewed as functions, they can be defined using formulas: for example, the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

can be defined using the simple formula

$$a_n = \frac{1}{n}.$$

Example 1.2 Some sequences defined using formulas

$$(-1)^n \quad \Rightarrow \quad -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

$$3n + 4 \quad \Rightarrow \quad 7, 10, 13, 16, 19, 22, \ldots$$

$$(n+1)^2 \quad \Rightarrow \quad 4, 9, 16, 25, 36, 49, \ldots$$

$$\begin{cases} \text{2n+1} & \text{if } n \text{ is odd,} \\ \text{n-1} & \text{if } n \text{ is even.} \end{cases} \quad \Rightarrow \quad 3, 1, 7, 3, 11, 5, 15, 7, \ldots$$

Sequences can also be defined using **recursion**, where the value of an element is defined using previous values and a **starting value**. For example:

$$a_n = a_{n-1}^2 - 2,$$

with the starting value $a_1 = 3$. We the get that

$$a_2 = a_1^2 - 2 = 3^2 - 2 = 7,$$

and thus

$$a_3 = a_2^2 - 2 = 7^2 - 2 = 47,$$

etc.

Example 1.3 The Fibonacci sequence

The **Fibonacci sequences** is a well-known sequence defined using the following recursive rule:

$$F_n = F_{n-1} + F_{n-2},$$

with $F_1 = F_2 = 1$. The first few elements of the sequence are therefore

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

See Figure 1.1 for a graphical representation of the Fibonacci sequence.

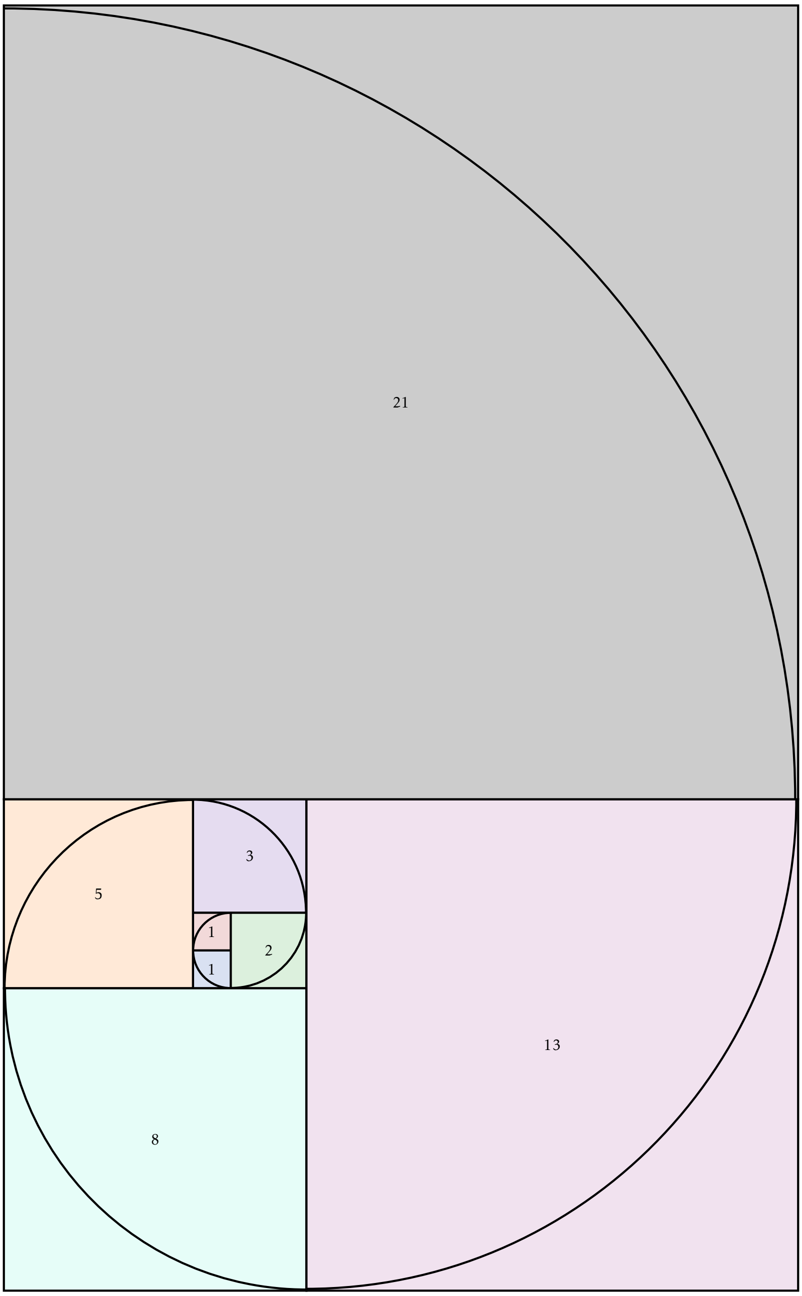


Figure 1.1 A graphical representation of the Fibonacci sequence: two squares of side 1 are placed adjacent to each other on the plane. In each subsequent step a new square is place such that its side is equal to the combined sides of the previous two squares. This way, the side of each square in the sequence follows the Fibonacci sequence. In each square we draw a quarter circle centered on one of the vertices, such that we get the famous **golden ratio** helix.

Note 1.1 Focus of section

From now on in the section we will focus on infinite sequences only.

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1.1.2 Types of sequences

Consider the sequence $a_n = n^2$. Since $n \in \mathbb{N}$, for any n, $a_{n+1} > a_n$, since $(n+1)^2 > n^2$ (see Figure 1.2). We say that such a sequence is **increasing**. In fact, for a sequence to be increasing some sequential elements can be equal: for example, the sequence $c_n = 1,1,1,2,2,2,3,3,3,4,4,4,5,5,5,\dots$ is also a increasing sequence. Thus, the definition of a increasing sequence is the following:

Definition 1.1 increasing sequence

sequence a_n is said to be *increasing* if for any $n \in \mathbb{N}$, $a_{n+1} \geq a_n$.

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If we change the condition to $a_{n+1} > a_n$ we say that such a sequence is **strictly increasing**. In the above examples a_n is a strictly increasing sequence, while c_n is just increasing (since for some indeces n, $c_{n+1} = c_n$).

Similarly, a **decreasing** sequence is a sequence b_n for which for any $n \in \mathbb{N}$, $b_{n+1} \leq b_n$. An example of such sequence is $b_n = \frac{1}{n}$ (see Figure 1.3). ANd of course, if we change the condition to $b_{n+1} < b_n$ then the sequence is **strictly decreasing**.

Generally, a sequence that is either increasin or decreasing is said to be **monotone**. If a sequence is monotone starting only from a certain n, we say that the sequence is **eventually monotone** (i.e. *eventually increasing* or *eventually decreasing*). An example of such sequence is $d_n = (n-5)^2$ (Figure 1.4): for $N \in 1,2,3,4,5$ it is decreasing, but starting from $n = 5$ it is increasing for any n.

As an example of a sequence which isn't monote, consider the sequence $e_n = \sin(n)$: for some values of n, $e_{n+1} > e_n$ and for some other values $e_{n+1} < e_n$ (see Figure 1.5).

The following are two ways to determine whether a sequence a_n is monotone:

- **Difference test**: if $a_{n+1} - a_n \geq 0$ for all $n \in \mathbb{N}$, then the sequence is increasing. If $a_{n+1} - a_n \leq 0$ for all $n \in \mathbb{N}$ then the sequence is decreasing.

- **Ratio test**: if $\frac{a_n+1}{a_n} \geq 1$ for all $n \in \mathbb{N}$ then the sequence is increasing, and if $\frac{a_{n+1}}{a_n} < 1$ for all $n \in \mathbb{N}$ then the sequence is decreasing.

Example 1.4 Difference test

Given the sequence $a_n = \frac{n}{n+1}$, we look at the difference $a_{n+1} - a_n$:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)(n+1) - (n+2)n}{(n+1)(n+2)}$$

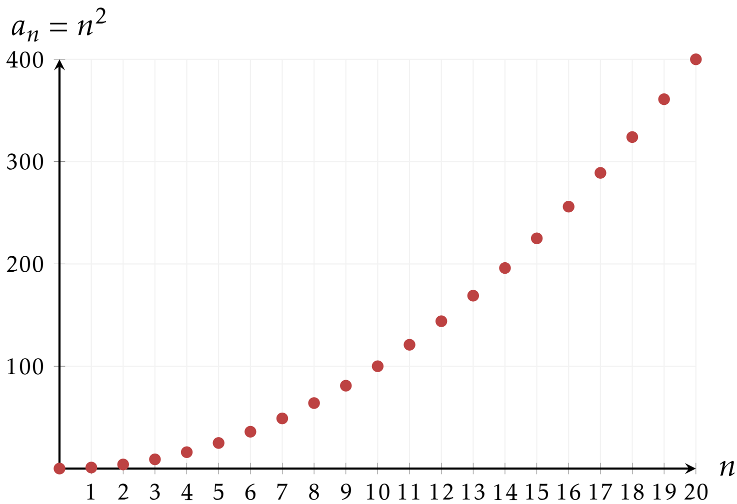

Figure 1.2 The sequence $a_n = n^2$ is increasing, and is in fact *strictly* increasing.

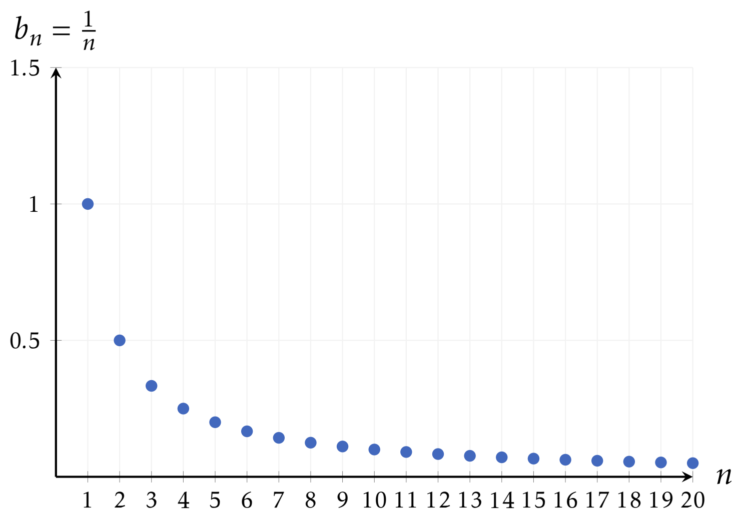

Figure 1.3 The sequence $b_n = \frac{1}{n}$ is decreasing, and is in fact *strictly* decreasing.

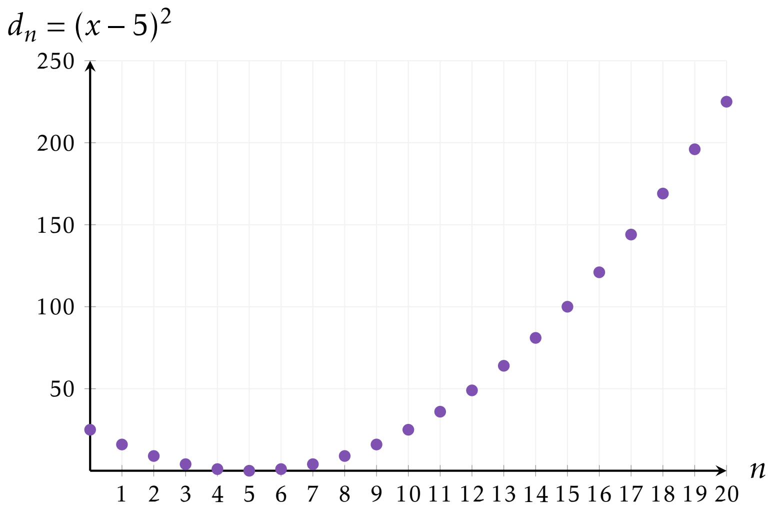

Figure 1.4 The sequence $d_n = (n-5)^2$ starts as a decreasing sequence, but starting from $n = 5$ it is increasing, making it an *eventually increasing sequence*.

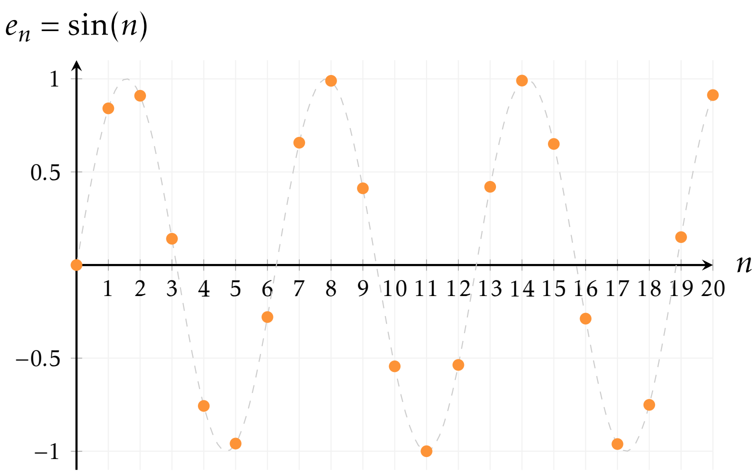

Figure 1.5 The sequence $e_n = \sin(n)$ is neither increasing nor decreasing. For reference, the function $\sin(x)$ is plotted as a dashed line behind e_n.

$$= \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} < 1 \quad \forall n \in \mathbb{N}.$$

The last (in)equality stems from the fact that no matter what n we subtitute into $(n+1)(n+2)$, the result will be greater than 1, and thus $\frac{1}{(n+1)(n+2)}$ is always smaller than 1. Therefore, a_n is a decreasing sequence.

Example 1.5 Ratio test

Given the sequence $b_n = \frac{2^n}{n^2}$, the ratio of a_{n+1} to a_n is

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)^2}}{\frac{2^n}{n^2}} = 2\frac{n^2}{(n+1)^2}.$$

Let's look at the first few approximated values of the ratio $\frac{n^2}{(n+1)^2}$:

n	$\frac{n^2}{(n+1)^2}$
0	0
1	0.25
2	0.44444...
3	0.5625
4	0.64
5	0.69444...
6	0.7346938775510204
7	0.765625
8	0.7901234567901234
9	0.81
10	0.8264462809917356
11	0.840277777...
12	0.8520710059171598
13	0.8622448979591837

We see that for any $n \geq 3$, $\frac{n^2}{(n+1)^2} > \frac{1}{2}$, and therefore $2\frac{n^2}{(n+1)^2} > 1$. Thus, the sequence is eventually increasing.

Some sequences are **bounded** from below: this means that their elements never get smaller than some constant $\underline{M} \in \mathbb{R}$. For example, consider the simple sequence $a_n = n$, where $n = \{1, 2, 3, 4, \dots\}$: there is no element in the sequence that is smaller than 1. Therefore, a_n is bounded from below by 1. Of course, one may argue that b_n is also bounded from below by 0, or -6, or in fact any negative number. This is true, however we are usually interested in the *maximal* number $\underline{M}$ that bounds the sequence from below, which in this case is $\underline{M} = 1$. We call that number the **infimum** of the sequence, and denote it as $\inf a_n$.

Similarly, a sequence a_n can be bounded from above by some number $\overline{M} \in \mathbb{R}$, i.e. there exist no n for which $a_n > \overline{M}$. We call the *minimal* such number the **supremum** of the sequence a_n, denoted $\sup a_n$. For example, the sequence $b_n = \frac{1}{n}$ is bounded from above by any real number $x \geq 1$, and therefore $\sup b_n = 1$. In fact, b_n is also bounded from below by $\underline{M} = 0$, and therefore we say that it is **bounded**. Another example of a sequence that is bounded is $e_n = \sin(n)$, which is bounded from below by $\underline{M} = -1$ and from above by $\overline{M} = 1$.

Example 1.6 Bounded and unbounded sequences

The following table shows some examples of sequences that are bounded from below, from above, or neither:

a_n	**First 5 elements**	$\inf a_n$	$\sup a_n$
$n^2 - n$	$0, 2, 6, 12, 20, \dots$	0	-
$\frac{n}{n+1}$	$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$	$\frac{1}{2}$	1
e^{-n}	$e^{-1}, e^{-2}, e^{-3}, e^{-4}, e^{-5}, \dots$	0	e^{-1}
$\log(n)$	$0, \log(2), \log(3), \log(4), \log(5), \dots$	0	-
$(-1)^n$	$-1, 1, -1, 1, -1, \dots$	-1	1
$(-1)^n n$	$-1, 2, -3, 4, -5, \dots$	-	-
$(-2)^n$	$-2, 4, -8, 16, -32, \dots$	-	-

1.1.3 Subsequences

Given any sequence a_n, we can remove from it any number of its elements (including 0 elements) and get a new sequence b_n which is a **subsequence** of a_n. For example, let $a_n = n^2 - 5n$. We can remove each 2nd element from a_n (i.e. those with indices $2, 4, 6, 8, \dots$) and get the following sequence b_n:

$$a_n = -4, -6, -6, -4, 0, 6, 14, 24, 36, 50, 66, 84, \dots$$

$$b_n = -4, -6, 0, 14, 36, 66, \dots$$

Note 1.2 Order of elements in a subsequence

A subsequence must preserve the order of the original sequence, since all we do in practice is removing elements from the original sequence, without changing the order of the remaining elements.

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Let's look at a more formal defintion of a subsequence, which uses the choice of indices instead of removing elements:

Definition 1.2 Subsequence

A subsequence of a sequence a_n is a sequence a_{n_k}, where n_k is a **strictly increasing** sequence of natural numbers.

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To create a subsequence using the above defintion, one can first create a sequence of indices n_k, and then subtitute only those indices into n in a_n. For example, given the sequence $a_n = n^2 - 5n$ from before, we can define a sequence of indices $n_k = 1, 3, 5, 7, 9, 11, \dots$ which would then yield the subsequence b_n shown before.

The reason we define n_k to be strictly increasing is to avoid changing the order of the elements from the original sequeunce a_n: for example, if we allowed n_k to be "just" increasing, we might end up with a case where there are two subsequent equal indices, e.g. $n_k = 1, 3, 5, 8, 9, 9, 10, \dots$. That would mean that we repeat an element from a_n **twice or more** in the subsequence (in the example this would be a_9), rendering it invalid as a subsequence, since as mentioned before - a subsequence must perserve the order of the original sequence.

Subsequences share all of the above-mentioned properties of the original sequence: if the original sequence is increasing or decreasing - so do all of its subsequences, and if it is bounded from above or below - so do all of its subsequences. Let's prove two of these properties:

Proof 1.1 Rising sequences and their subsequences

Claim: given an **increasing** sequence a_n, all of its subsequences are also increasing sequences themselves.

Proof: using contradiction. Let a_n be an increasing sequence, and b_n a subsequence of a_n which isn't increasing. From the fact that b_n is not an increasing sequence we know that there exist at least two indices k, m such that $k < m$ but $b_k > b_m$. Since any b_n is an element of a_n without change of order, we can substitute $b_k = a_i$ for some index i and $b_m = a_j$ for some index j, such that $i < j$ (since $k < m$ - this is exactly the idea of preserving the order of a_n). We therefore get that $a_i = b_k > b_m = a_j$, or simply $a_i > a_j$ even though $i < j$ - in contradiction to a_n being an increasing sequence. Therefore there can be no subsequence of a_n that isn't an increasing sequence.

QED

Proof 1.2 Bounded sequences and their subsequences

Claim: given a sequence a_n which is **bounded from below**, all of its subsequences are also bounded from below.

Proof: also using contradiction. Let a_n be a sequence bounded from below by $\inf a_n = \underline{M}$. Let b_n be a subsequence of a_n that isn't bounded from below - i.e. there exist an element b_i such that $b_i < \underline{M}$. Since b_n is a subsequence of a_n, $b_i = a_j$ for some index j. Therefore $a_j = b_i < \underline{M}$ in contradiction to $\underline{M}$ being the infimum of a_n. Therefore, a subsequence of a sequence bounded from below can not be unbounded from below.

QED

Challenge 1.1 Further proofs

1. Prove that all subsequences of a decreasing sequence are themselves decreasing.
2. Prove that all subsequences of a sequence bounded from above are themselves bounded from above.
3. Prove that all subsequences of a **strictly** increasing/decreasing sequence are themselves increasing or decreasing, respectively.
4. Given a bounded sequence a_n with $\inf a_n = \underline{M}$, can it have a subseqeunce b_n with $\inf b_n \neq \underline{M}$? If yes - give an example. If no - prove your claim.

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1.1.4 Limits

As you probably noticed by now, some infinite sequences seem to approach a certain value as we increase n. That is to say, the bigger n is, the closer such a sequence a_n gets to a certain value $L \in \mathbb{R}$. For example, the sequence $b_n = \frac{1}{n}$ approaches to $L = 0$ as we increase n (see again Figure 1.3). The sequence $a_n = \frac{1}{n^2+1}$ approaches the value $L = 1$ as we increase n (see Figure 1.6). On the contrary, the sequence $d_n = (n-5)^2$ eventually increase in such a way that it "approaches" $L = \infty$, while $e_n = \sin(n)$ doesn't approach any value and instead endlessly "jumps" around in a repeated manner.

The formal term for the behaviour of a_n and b_n is called **convergance**, and it is one of the most important properties of infinite sequences. In this subsection we will define, analyze, and explain it in detail. To begin, we can divide all infinite sequences into two separate categories:

1. Sequences which converge to a finite number $L \in \mathbb{R}$.
2. Sequences which do not converge to any finite number.

Sequences in the second category are said to be **diverging**, and they can be further split into two separate categories:

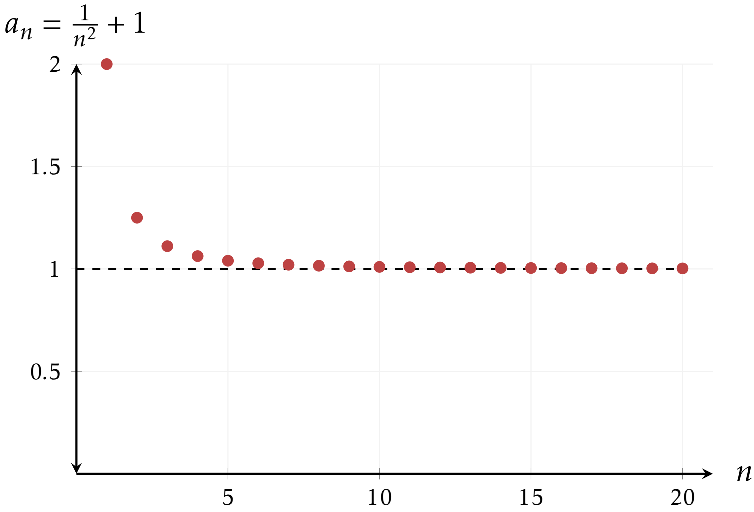

Figure 1.6 The sequence $a_n = \frac{1}{n^2} + 1$ approaches the value $L = 1$ as n increases.

i. Sequences which diverge to either positive or negative infinity.

ii. Sequences which neither converge nor diverge to $\pm\infty$.

Let us start with a more precise analysis of sequences that diverge to $\pm\infty$. In essence, a sequence which diverges to positive infinity is a sequence that is bounded from below but not from above, that is - given any real number R the sequence eventually passes it. In other words, all the elements in the suquence *after a certain value of* n are greater than R, for any R that we choose.

Take for example the sequence $a_n = n^2$. It most certainly has a lower boundry, namely $\inf a_n = 1$. On the other hand, given any $R \in \mathbb{R}$ eventually the values of a_n pass it. For example, given $R = 100$, the elements of a_n pass it after just 10 elements (since $a_{11} = 11^2 = 121 > 100$). The number $R = 1{,}000{,}000$ is passed after 1000 elements, etc. No matter how big R is, **eventually** a_n will pass it. Therefore, we say that a_n *goes to infinity*, and denote it by writing

$$\lim_{n\to\infty} a_n = \infty. \tag{1.1.2}$$

The notation lim is short for **limit**. While it can be argued that a divergent sequence has no limit, sometimes the term is used in the case of divergence to $\pm\infty$.

Note 1.3 Another limit notation

Another common notation used to denote that a sequence a_n is going to infinity as n increases is the following:

$$a_n \xrightarrow{n\to\infty} \infty.$$

A more formal definition of this behaviour is as follows:

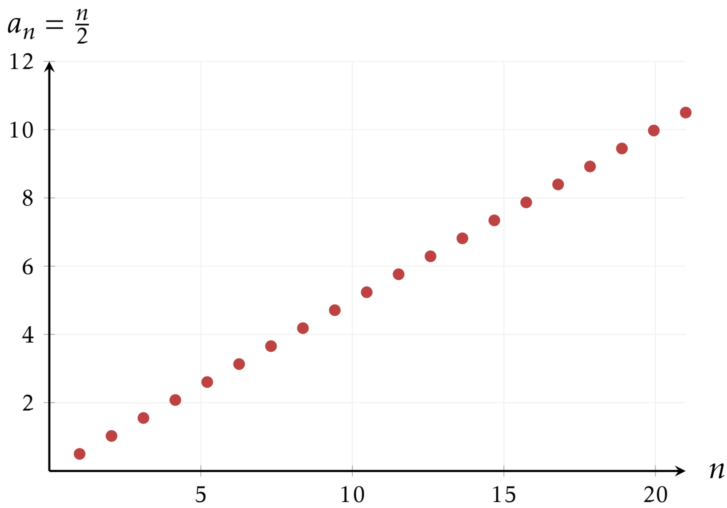

Figure 1.7 The sequence $a_n = \frac{n}{2}$ goes to positive infinity as n increases.

Definition 1.3 Sequence going to inifnity

Let a_n be an infinite sequence. If for any $R \in \mathbb{R}$ there exist $n_R \in \mathbb{N}$ such that for any $n > n_R$, $a_n > R$, the sequence is said to be going to positive infinity. We denote this fact by writing

$$\lim_{n\to\infty} a_n = \infty.$$

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Of course, for negative infinity the behaviour is very similar: a sequence a_n with an upper boundry $\overline{M}$ and no lower bound $\underline{M}$ is said to be **going to negative infinity**, since for any $R \in \mathbb{R}$ there exist an $n_R \in \mathbb{N}$ for which if $n > n_R$ then $a_n < R$.

Generally speaking, proving that a sequence goes to either positive or negative infinity follows a certain pattern, which we will examplify using the sequence $a_n = \frac{n}{2}$ (Figure 1.7). It should be clear that the sequence goes to positive infinity as n increases, since we can make the values of a_n as large as we want by substituting a respective n into $\frac{n}{2}$: for example, given $R = 1000$ we can subtitute $n = 2000$, yielding $a_{2000} = \frac{2000}{2} = 1000$, and thus any a_n where $n > 2000$ will be bigger than $R = 1000$. For $R = 1{,}000{,}000$ we can substitute $n = 2{,}000{,}000$ and so forth.

To show that this is true for any $R \in \mathbb{R}$ we should do the following: given a real number R find an index n_0 such that $a_{n_0} \geq R$. Since a_n is a strictly increasing sequence, that would mean that for any $n > n_0$, $a_n > a_{n_0} \geq R$, or simply $a_n > R$. In the case of $a_n = \frac{n}{2}$ we can simply choose the closest integer to $2R$ that is also bigger than $2R$ (i.e. if $R = 2.3$ we choose $n_0 = 3$, if $R = 100.7$ we choose $n_0 = 101$, etc.).

To always get an integer *equal to or bigger than* R we can use the **ceiling** operator. For any given $x \in \mathbb{R}$, its ceiling (denoted $\lceil x \rceil$) is the closest integer which is bigger than or equal to x, or more formally:

Definition 1.4 Ceiling and floor operators

Let $x \in \mathbb{R}$. Then

$$\lceil x \rceil = \min\Big(\{n \in \mathbb{N} | n \geq x\}\Big). \tag{1.1.3}$$

$$\lfloor x \rfloor = \max\Big(\{n \in \mathbb{N} | n \leq x\}\Big). \tag{1.1.4}$$

i.e. - $\lceil x \rceil$ is the **minimal** integer n that is **bigger than or equal** to x, and $\lfloor x \rfloor$ is the **maximal** integer n that is **smaller than or equal** to x.

Using the value $n_0 = \lceil 2R \rceil$ we can show, step by step, that indeed $\lim\limits_{n\to\infty} = \infty$ by using several subtitutions: for any $n > n_0$ we get that

$$a_n = \frac{n}{2} \overset{\text{since } n > n_0}{>} \frac{n_0}{2} = \frac{\lceil 2R \rceil}{2} \underset{\text{since } \lceil 2R \rceil \geq 2R}{\geq} \frac{2R}{2} = R,$$

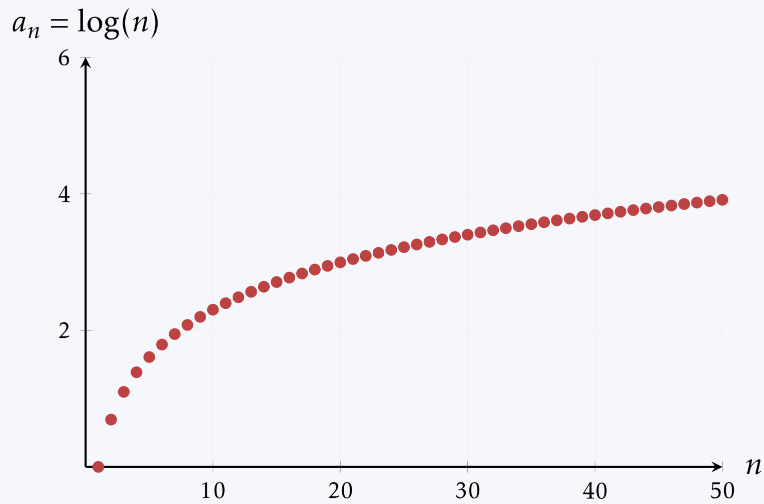
or simply $a_n > R$.

Example 1.7 The sequence $\log(n)$

Let's show that the sequence $a_n = \log(n)$ (see graph below) goes to infinity as n increases: first, we note that $\log(n)$ is a strictly increasing sequence bounded from below by $\underline{M} = 0$. Now, given some positive $R \in \mathbb{R}$ we chose $n_0 = \lceil e^R \rceil$ and thus get that for any $n > n_0$

$$a_n > a_{n_0} = \log(n_0) = \log\left(\lceil e^R \rceil\right) \geq \log\left(e^R\right) = R.$$

altogther we get $a_n > R$, and therefore $\lim\limits_{n\to\infty} \log(n) = \infty$.



Note: this proof works because $\log(n)$ is a *strictly* increasing sequence. If it were only an increasing sequence we would not be guaranteed that $n > n_0$ means that $a_n > a_{n_0}$, and the entire process would not yield that $\log(n)$ always passes any given real number R. Indeed, by naively looking at the graph above we may be mistaken to think that $\log(n)$ actually approaches some finite number, say 4.5 or so, and doesn't go to infinity. This is of course not the case.

Example 1.8 A sequence which goes to negative infinity

Let us now show that the sequence $a_n = -\sqrt{n}$ goes to negative infinity. We first note that $-\sqrt{n}$ is always negative, is decreasing and bouned from above by $\overline{M} = 0$. For any given negative $R \in \mathbb{R}$ we choose $n_0 = \lfloor R^2 \rfloor$, and thus for any $n > n_0$ we get

$$a_n < a_{n_0} = -\sqrt{n_0} = -\sqrt{\lfloor R^2 \rfloor} \leq -\sqrt{R^2} = -|R| = R.$$

! **To be written**: is this example actually necessery? It seems reduntant unless we show somethin
!

The next type of sequences we ananlyze are those sequences that converge to a real number L as n increases - i.e. as n increases, the terms of the sequence get closer and closer to L. A classical example for such a sequence is $a_n = \frac{1}{n}$ (Figure 1.3): as n increases, the terms $\frac{1}{n}$ become smaller and smaller (while always being positive) and the sequence as a whole approaches $L = 0$.

Sequences don't have to approach a limit from above: some sequences approach a limit from below (Figure 1.8), while others may **oscillate** around the limit (Figure 1.9).

We define convergence in a similar way to how we defined that a sequence goes to ∞: in that case we had to show that for any $R \in \mathbb{R}$ the sequence eventually surpasses R and

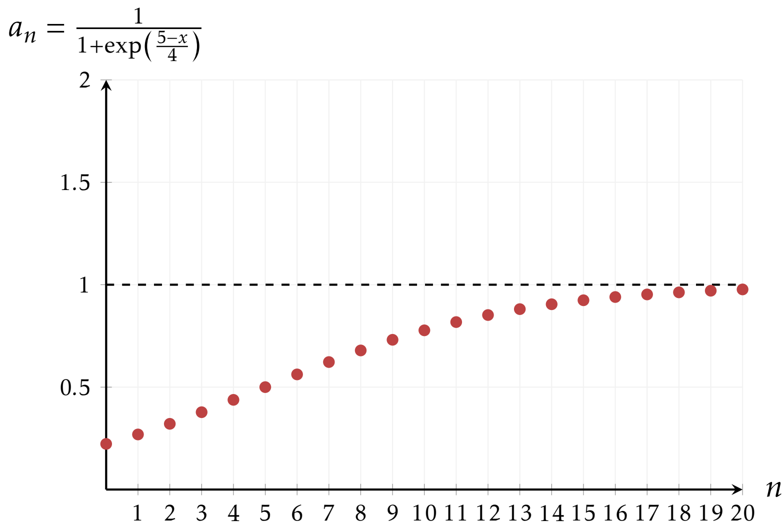


Figure 1.8 The logistic sequence $a_n = \frac{1}{1+\exp\left(\frac{5-x}{4}\right)}$ approaches the value $L = 1$ from below as $n \to \infty$.

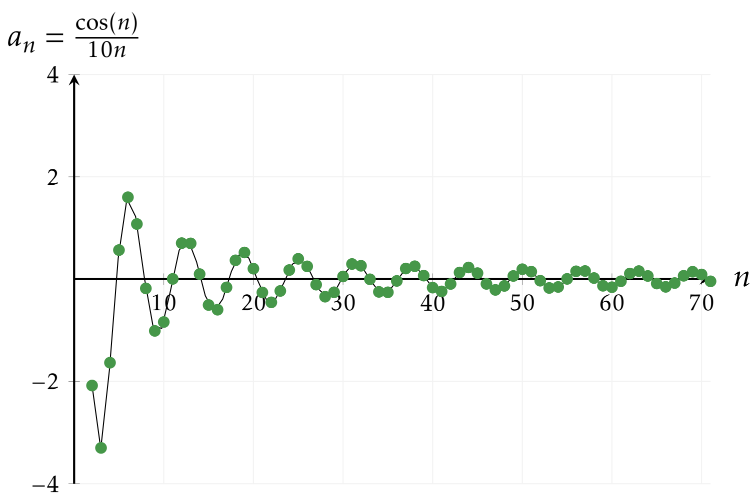


Figure 1.9 The sequence $a_n = \frac{\cos(n)}{10n}$ approaches, with oscillations, the limit $L = 0$. A line connecting the elements is drawn to help see the progression of the sequence.

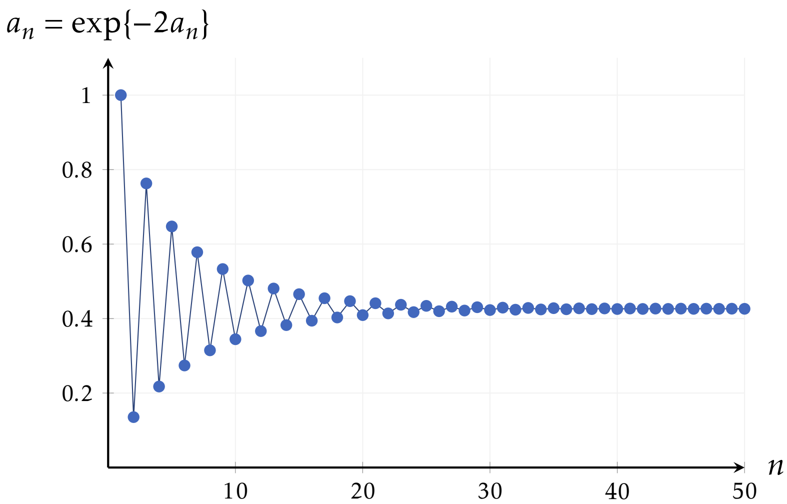


Figure 1.10 The sequence defined by the recursive formula $a_{n+1} = \exp(-2a_n)$ with $a_1 = 1$ converges to the limit $L \approx 0.4263027510068627$, or more precisely $\frac{1}{2}W(2)$, where W is the Lambert W function.

no element is ever again equal to or smaller than R. In the case of convergence to some finite number $L \in \mathbb{R}$ we will show that for any distance ε from L - no matter how small! - the sequence eventually stays within ε of L.

Let us use the simplest convergence example to explain this idea: $a_n = \frac{1}{n}$, which converges to $L = 0$. Given a small number ε, say $\varepsilon = \frac{1}{10}$, eventually the sequence stays at most within $\frac{1}{10}$ of $L = 0$. This happens starting from $n_0 = 10$: any element thereafter is smaller than $\frac{1}{10}$, which means that it is withing $\varepsilon = \frac{1}{10}$ of $L = 0$ (see Figure 1.11).

We can repeat this for different value of ε: given $\varepsilon = \frac{1}{100}$, for any $n > 100$ the elements a_n are guaranteed to be within $\pm\frac{1}{100}$ of $L = 0$. Given $\varepsilon = \frac{1}{5000}$, for any $n > 5000$ the elements a_n are within $\pm\frac{1}{5000}$ of $L = 0$, etc. In general, given any real ε, no matter how small, we can set $n_0 = \lceil \frac{1}{\varepsilon} \rceil$, and then for any $n > n_0$ we get that

$$a_n < a_{n_0} = \frac{1}{n_0} = \frac{1}{\lceil \frac{1}{\varepsilon} \rceil} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon, \tag{1.1.5}$$

i.e. altogther

$$a_n < \varepsilon, \tag{1.1.6}$$

and therefore a_n is within $\pm\varepsilon$ of $L = 0$.

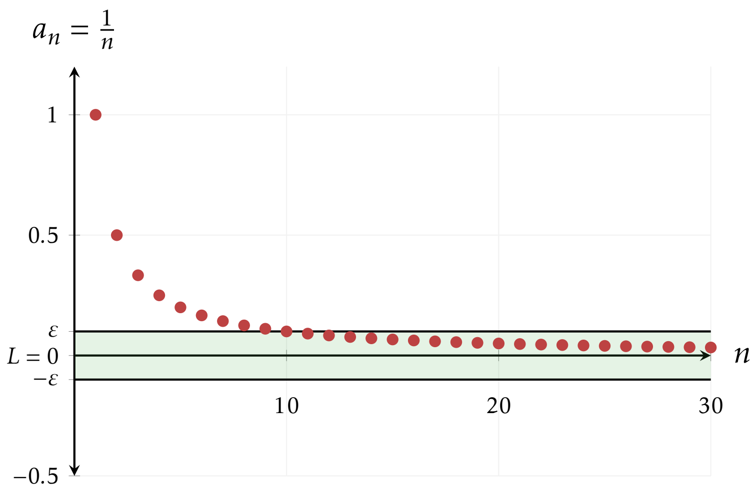


Figure 1.11 The sequence $a_n = \frac{1}{n}$. For any $n > 10$, the element a_n is withing $\varepsilon = \frac{1}{10}$ of the limit $L = 0$. The interval $(-\varepsilon, \varepsilon) = \left(-\frac{1}{10}, \frac{1}{10}\right)$ on the y-axis is highlighted in green.