-1.1 COMBINATORICS

Suppose we have a set of 3 spheres, each of a different color: red, blue and green - and want to order all of them in a row. How many different ways do we have of organizing the spheres? Figure -1.1 shows all of the options.

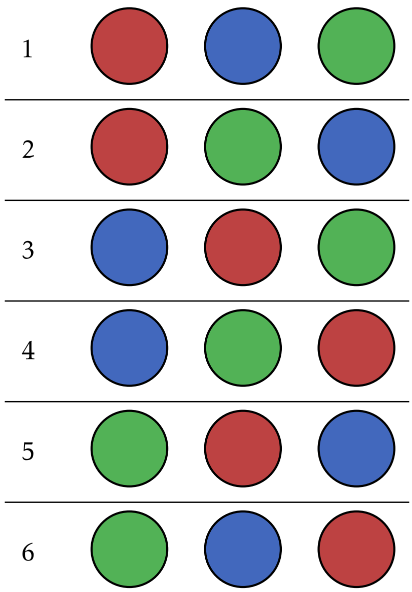


Figure -1.1 All the ways of arranging a set of 3 differently colored spheres.

Going about the options systematically, we can describe them as follows: we have three options for placing the first sphere: red, blue or green. Once we have chosen the first sphere, we're left with only two option for the second sphere: if we chose red, then we're left with choosing between blue and green. The choice of last sphere is then dictated by the previous choices: if for the second sphere we chose green, then we are left with only the blue sphere for the third position (as in option 2 above). We call each of these ways to organize the spheres a **combination**.

Quantitatively the number of ways we have of organizing the spheres is

$$k = 3 \times 2 \times 1 = 6. \qquad (-1.1.1)$$

We can expand this logic to however many $n \in \mathbb{N}$ different spheres we wish: for n different spheres we have n options for placing the first sphere, then $n-1$ options for placing the second sphere, then $n-2$ options for placing the third sphere... all the way to the last sphere. The number of total combinations is therefore

$$k = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1. \qquad (-1.1.2)$$

The function used to represent k in the above general form is called the **factorial**, and is denoted using an exclamation mark:

$$k = n! \tag{-1.1.3}$$

A somewhat more rigouros (and quite intereseting) way of defining the factorial is as follows:

$$n! := \begin{cases} 1 & \text{if } n = 1, \\ n \times (n-1)! & \text{if } n > 1. \end{cases} \tag{-1.1.4}$$

This kind of definition is called a **recursive** definition, since it uses itself in its own definition. For example, for 3! we have $3 > 1$, and thus $3! = 3 \times 2!$. Then $2 > 1$ and thus $2! = 2 \times 1!$, but since $1 = 1$ we get $1! = 1$, and altogether we get $3! = 3 \times 2 \times 1 = 6$.

Going forward we can ask the following question: given that we have a set of 5 spheres, 2 of them red and the rest are blue. How many combinations are there to sort the spheres, assuming there's no way to distinguish spheres of the same color? All the possible combinations are shown in Figure -1.2.

We can again go about solving this by directly counting all possible combinations. We place a red sphere in the 1st spot, and then count all such possible combinations, each time placing the other red sphere in a different place (2nd, 3rd, etc.). This gives us 5 combinations (numbered 1-4 in the table). We then place a red sphere in the 2nd spot, and placing the other red sphere in all possible spots: 3rd, 4th and 5th (combinations 5-7 in the table). Note that we don't count the combination where the red spheres are in the 1st and 2nd spots since this was already counted (combination 1).

We then proceed to puting a red sphere in the 3rd spot, and counting all the possible combinations, i.e. the other red sphere being in the 4th or 5th spots (combinations 8 and 9). Again, all other options were already counted. We're then left with only a single combination: the red spheres being in the 4th and 5th spots. Altogether we have counted 10 distinct combinations.

In the general case, we want to know how many combinations are there of arrainging n spheres in a row, when k of them are red and the other $n-k$ blue. A convinient way of representing this number is using the **binomial coefficients**, which are defined as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{-1.1.5}$$

Note -1.1 Binom pronounciation

Reading the binom notation outloud we say "n choose k", since we are choosing k objects out of a total of n objects.

!

We expect then that $\binom{5}{2} = 10$, since this is what we got in the example above. Let's verify this:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{120}{2 \cdot 3!} = \frac{120}{12} = 10.$$

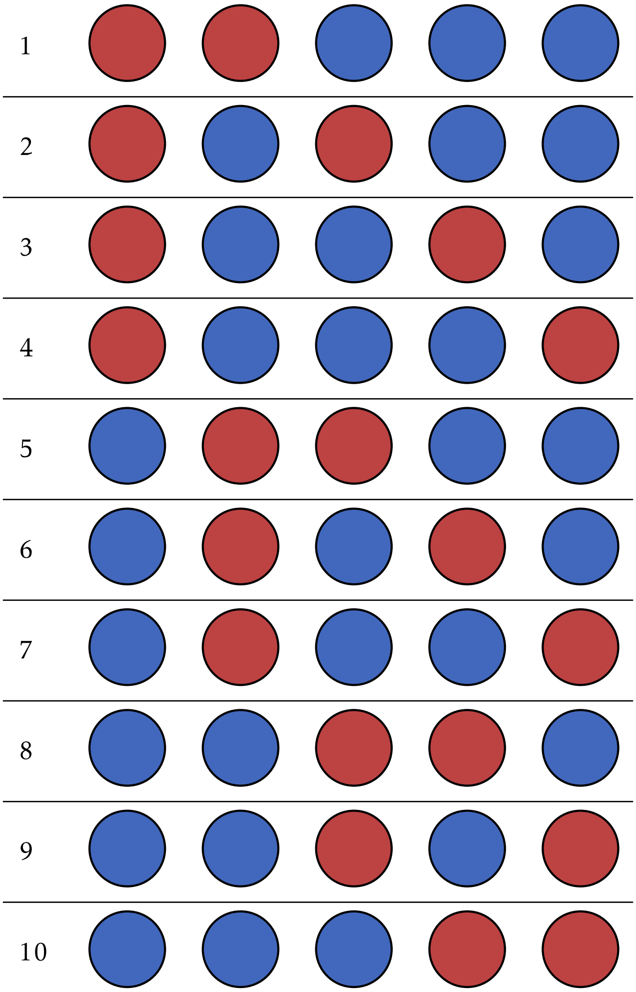


Figure -1.2 All possible combinations of two red spheres and three blue spheres, where spheres of the same color are indistinguishable.

The binomial coefficients are symmetric: note how in the denominator $k!$ is multiplied by $(n-k)!$, i.e. k is always multiplied by the difference between it and n. Thus, if we change our choice of k to be $n-k$ (e.g. for the case of $\binom{5}{2}$ we instead use $\binom{5}{3}$) the result stays the same. This can be visualized in the above example: instead of thinking of two red spheres out of five total spheres, think of three blue spheres out of a total of five spheres. The result doesn't change because we are talking about the same exact problem.

Thus, the following always holds:

$$\binom{n}{k} = \binom{n}{n-k}. \tag{-1.1.6}$$

Note -1.2

Some important values of the binomial coefficianets: Given n objecst,

-1.1. there is a single combination of choosing 0 objects, and also a single combination of choosing n objects. Therefore

$$\binom{n}{0} = \binom{n}{n} = 1.$$

-1.2. there are n combinations for choosing a single object, and also n combination of choosing $n-1$ objects. Therefore

$$\binom{n}{1} = \binom{n}{n-1} = n.$$

!

We can visualize the binomial coefficients using **Pascal's triangle**, in which the value of each cell is the result of adding the values of the two cells above it (Figure -1.3). For example: to find the value of $\binom{4}{2}$ we go to the 4-th row (purple color in the figure) and then start from left starting with 0. The value of the resulting cell is 6, and indeed $\binom{4}{2} = 6$.

Why are the binomial coefficients called that way? The name comes from the **binomial expansion**. You probably had to memorize the following expansion in school:

$$(a+b)^2 = a^2 + 2ab + b^2. \tag{-1.1.7}$$

We can use the binomial coefficients to easily expand any expression of the form $(a+b)^n$, where $n \in \mathbb{N}$. As always, we start with a basic example - this time the above expansion when $n = 2$: when we expand this expression we first get (color coded for clarity)

$$(a+b)^2 = (a+b)(a+b).$$

Then, to calculate all possible products we multiply each possible pair of numbers systematically:

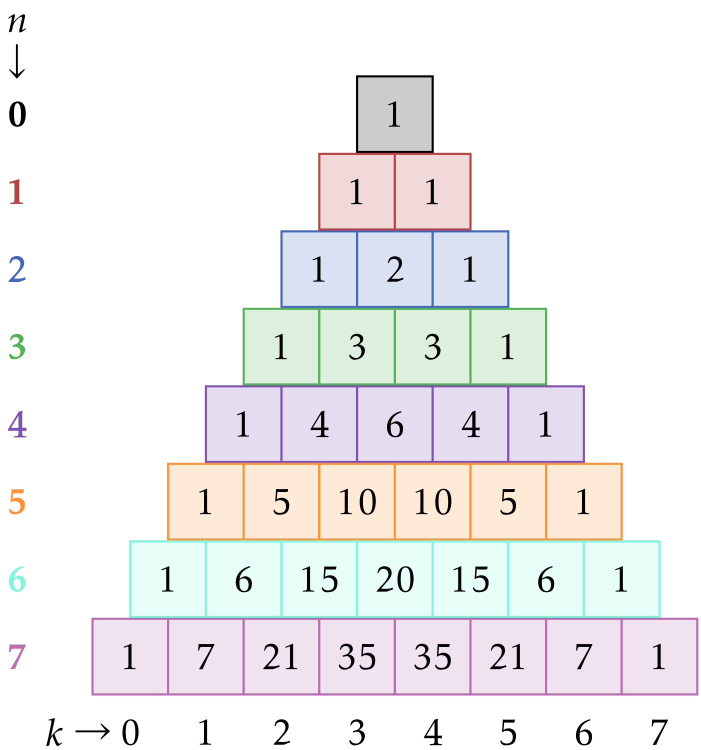


Figure -1.3 Pascal's triangle: the value in each cell is the sum of the values in the two cells above it (empty cells are considered to be with value 0). Each row is enumerated starting from 0: see for example the indeces below the last row, which go from 0 to 7.

$a \cdot a$: $$(a+b)(a+b)$$

$a \cdot b$: $$(a+b)(a+b)$$

$b \cdot a$: $$(a+b)(a+b)$$

$b \cdot b$: $$(a+b)(a+b)$$

Altogether we therefore get

$$(a+b)^2 = (a+b)(a+b) = a \cdot a + ab + ba + b \cdot b = a^2 + 2ab + b^2.$$

(since for real numbers $ab = ba$)

Notice the pattern here: the coefficients of the different terms in $a^2 + 2ab + b^2$ correspond to the binomial coefficients when $n = 2$ (i.e. $1, 2, 1$). This is because there's only a single combination of numbers yielding a^2 (namely $a \cdot a$), two combinations that yield ab (namely $a \cdot b$, and $b \cdot a$), and again a single combination that yields b^2 ($b \cdot b$).

The powers of a and b also follow a pattern: the power of a decreases in each term - starting with 2 (i.e. a^2), then 1 ($2ab$) and finally 0 (b^2). The power of b in each term increases in the same way from 0 to 2. If we call the power of b in each term k, then the power of a is always $2 - k$.

The pattern appears in $(a+b)^3$ as well: (we again color code the numbers)

$$(a+b)^3 = (a+b)(a+b)(a+b).$$

The expansion is then

$$\begin{aligned}
(a+b)^3 &= (a+b)(a+b)(a+b) \\
&= a \cdot a \cdot a + a \cdot a \cdot b + a \cdot b \cdot a + a \cdot b \cdot b + b \cdot a \cdot a + b \cdot a \cdot b + b \cdot b \cdot a + b \cdot b \cdot b \\
&= a^3 + 3a^2b + 3ab^2 + b^3.
\end{aligned}$$

Again, we see that the coefficients in the expansion are exactly the binomial coefficients for $n = 3$ (namely $1, 3, 3, 1$), and that the powers of a and b decrease and increase, respectively, just as for $n = 2$: a goes from 3 to 0, and b from 0 to 3. This time the power of a is always $3 - k$ (where k is the power of b).

Therefore, in the most general case (i.e. $(a+b)^n$) we expect each term to be of the following structure:

$$a^k b^{n-k},$$

and the coefficients being the binomial coefficients $\binom{n}{k}$. Putting this into a formula:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \tag{-1.1.8}$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Example -1.1 Exapnding $(a+b)^4$ using binomial coefficients

Let's use the above patterns to expand $(a+b)^4$. Looking at Pascal's triangle, the coefficients in row $n=4$ are $1, 4, 6, 4, 1$. Starting with the powers a^4 and $b^0 = 1$ and then decreasing and increasing the powers of a and b, respectively, we write all terms:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

You can check this expansion manually and see that it is indeed correct.

-1.2 EXERCISES

-1.2.1 Problems

-1.1. Write the following sets explicitly:

(i) $\{x \in \mathbb{N} \mid 1 < x \leq 7\}$

(ii) $\{x \in \mathbb{Z} \mid x < 5\}$

(iii) $\left\{x \in \mathbb{R} \mid x^2 = -1\right\}$

(iv) $\{x \in \mathbb{N} \wedge x \in \mathbb{Q}\}$

(v) $\left\{x \in \mathbb{R} \mid x^2 - 3x - 4 = 0\right\}$

(vi) $\{x \in \mathbb{R} \mid x < 5 \wedge x \geq 2\}$

-1.2. Determine the relation between the sets:

(i) $A = \{1,\ 2,\ 3\},\ B = \{1,\ 2\}$

(ii) $A = \varnothing,\ B = \{2,\ -5,\ \pi\}$

(iii) $A = \mathbb{Z},\ B = \{\pm x \mid x \in \mathbb{N} \cup \{0\}\}$

(iv) $A = \left\{\pi, \mathrm{e}, \sqrt{2}\right\},\ B = \mathbb{Q}$

-1.3. Write all elements in $S^2 \times W$, where $S = \{\alpha, \beta, \gamma\}$ and $W = \{x, y, z\}$. Find a condition that guarantees $S^2 \times W = W \times S^2$.

-1.4. How many different **bijective** functions exist between a set with 2 elements and another set with 2 elements (e.g. $f : \{1,2\} \to \{\alpha, \beta\}$)? How many exist between two sets, each with 3 elements? Between two sets each with n elements?

-1.5. For each of the real functions below, find a set on which it is surjective (use a graphing calculator if you are not familiar with the shape of a function):

$$x^2,\ x^3 - 5,\ \mathrm{e}^{-x^2/2},\ \sin(x),\ \sin(x) + \cos(x),\ x\mathrm{e}^x.$$

-1.6. Given two sets A, B such that $|A| \neq |B|$, can a bijective function $f : A \to B$ exist? Explain your answer.

-1.7. Find all real roots of the following polynomial function:

$$f(x) = x^3 + x^2 - 6x.$$

-1.8. Given a real $b > 0$ and k, prove that for any real $x > 0$

$$\log_b\left(x^k\right) = k \log_b(x).$$

-1.9. Show that for any positive real x, b

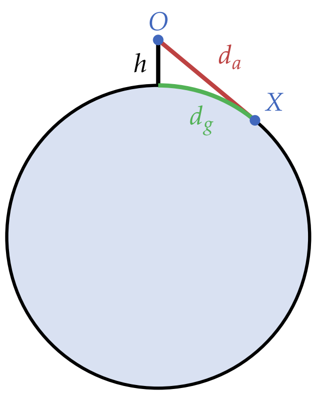
$$\log_b\left(\frac{1}{x}\right) = -\log_b(x).$$

-1.10. Solve the following equation for any real $x > 0$: (CHECK SOLUTION! MIGHT BE WRONG)

$$\log_2(x) + \log_4(x-1) = \log_{16}\left(x^3\right).$$

-1.11. During the second age of Middle-earth, 20 rings of power were forged. The following poem describes their distribution amongs the different peoples of the land:

-1.12. The horizon on a spherical planet such as the earth[1] is defined as the distance from an observer to the point where the ground disappears behind the planet's curve. Following is a 2-dimensional depiction, where $\boldsymbol{O}$ is the observer, $\boldsymbol{h}$ its height above the planet surface, $\boldsymbol{X}$ the horizon point and $\boldsymbol{d_a}$ the air-distance from the observer to the horizon and $\boldsymbol{d_g}$ the ground-distance from the observer to the horizon:



[1] yes.

(i) Find an expression for the air-distance d_a and ground-distance d_g to the horizon as a function of the radius $\boldsymbol{R}$ and height $\boldsymbol{h}$. (**hint**: find a relevant right triangle containing d_a and the radius of the planet)

(ii) Given that the Earth's radius is about 6371km (6.371×10^6m) and an average person is 1.75m tall - what is the distance to the horizon for a person standing at sea-level (both air- and ground-distances)? What would these distances be at the following heights: 165m (Eiffel tower's observation deck), 9.1km (average cruising altitude of a passenger jet) and 408km (average altitude of the International Space Station)?

(iii) How many degrees does the horizon drops from eye-level as function of $\boldsymbol{h}$? (*eye-level* in this context means the direction tangent to the planet's surface)

-1.13. Calculate the following complex product - first using the algebraic form and then the polar form, showing that the result is the same in both cases:

$$z = z_1^2 z_2 = \left(\sqrt{3} + \mathrm{i}\right)^2 \left(-2 + \sqrt{12}\mathrm{i}\right).$$

-1.14. Prove that the **sum** of all the roots of the complex equation $z^n = 1$ is always zero when $n \geq 2$, i.e. if $w_0, w_1, \dots, w_{n-1}$ are the roots of the equation, then

$$\sum_{k=0}^{n-1} w_k = 0.$$

Hint: for $|r| \neq 1$,

$$\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}.$$

-1.15. MORE EXERCISES TO BE WRITTEN...

-1.2.2 Solutions

-1.1. For each of the sets we first write how to read the notation in words, followed by its explicit form:

(i) Any **natural number** such that it is bigger than 1 and smaller or equal to 7. These are of course the numbers

$$\{2, 3, 4, 5, 6, 7\}.$$

(ii) Any **integer** such that it is smaller than 5. These are the numbers

$$\{4, 3, 2, 1, 0, -1, -2, -3, \dots\}.$$

(iii) Any **real number** x such that $x^2 = -1$. Since for any $x \in \mathbb{R}$, $x^2 \geq 0$ - there is no such real number x whose square equals -1. Therefore this definition describes the empty set, i.e. $\emptyset$.

(iv) Any **natural number** that is also a rational number. Since any natural number is also a rational number (e.g. $4 = \frac{4}{1} = \frac{8}{2}$, etc.) the definition actually simply describes the set of natural numbers, $\mathbb{N}$. This fact can also be written as

$$\mathbb{N} \cap \mathbb{Q} = \mathbb{N}.$$

(v) Any **real number** such that it solves the equation $x^2 - 3x - 4 = 0$. The solutions can be found using the quadratic formula:

$$\frac{3 \pm \sqrt{3^2 + 4 \cdot 4}}{2} = \frac{3 \pm \sqrt{9 + 16}}{2} = \frac{3 \pm 5}{2} = 4, -1.$$

Therefore the set described by the definition is simply

$$\{4, -1\}.$$

(vi) Any **real number** that is smaller than 5 **and** is bigger than or equal to 2. This definition describes the half-open interval

$$[2, 5).$$

-1.2. Relations between sets:

(i) All the elements in the set B are also in the set A $(1, 2)$, but there's an element in A which is not in B (namely 3). Therefore, B is a subset of A:

$$B \subset A.$$

(ii) The empty set is a subset of any set (and a proper subset of any set except itself), therefore

$$A \subset B.$$

(iii) The set B is defined as all the natural numbers, their negatives and zero. This is exactly the definition of the integers $\mathbb{Z}$, which set A in this case. Therefore

$$A = B.$$

(iv) All of the elements in A are irrational numbers. The set B is the set of **rational numbers**, and therefore the sets are disjoined:

$$A \cap B = \emptyset.$$

-1.3. S^2 is a Cartesian product of S with itself:

$$S^2 = \{(\alpha, \alpha), (\alpha, \beta), (\alpha, \gamma), (\beta, \alpha), (\beta, \beta), (\beta, \gamma), (\gamma, \alpha), (\gamma, \beta), (\gamma, \gamma)\}.$$

Therefore, to form the Cartesian product $S^2 \times W$ we simply take each of the elements is S^2 and add to it an element from W:

$$\begin{aligned} S^2 \times W = \{&(\alpha, \alpha, x), (\alpha, \beta, x), (\alpha, \gamma, x), (\beta, \alpha, x), (\beta, \beta, x), (\beta, \gamma, x), (\gamma, \alpha, x), (\gamma, \beta, x), (\gamma, \gamma, x) \\ &(\alpha, \alpha, y), (\alpha, \beta, y), (\alpha, \gamma, y), (\beta, \alpha, y), (\beta, \beta, y), (\beta, \gamma, y), (\gamma, \alpha, y), (\gamma, \beta, y), (\gamma, \gamma, y) \\ &(\alpha, \alpha, z), (\alpha, \beta, z), (\alpha, \gamma, z), (\beta, \alpha, z), (\beta, \beta, z), (\beta, \gamma, z), (\gamma, \alpha, z), (\gamma, \beta, z), (\gamma, \gamma, z)\}. \end{aligned}$$

Note that the number of elements in S is 3, and so the number of elements in S^2 is $3 \times 3 = 9$. The number of elements in W is also 3, and so the number of elements in $S^2 \times W$ is $9 \times 3 = 27$.

The Cartesian product $W \times S^2$ has the same structure as $S^2 \times W$, except that the elements from W are now on the left (remember that the order of elements is important in tuples, unlike with sets):

$$
\begin{aligned}
S^2 \times W = \{&(x,\alpha,\alpha),(x,\alpha,\beta),(x,\alpha,\gamma),(x,\beta,\alpha),(x,\beta,\beta),(x,\beta,\gamma),(x,\gamma,\alpha),(x,\gamma,\beta),(x,\gamma,\gamma)\\
&(y,\alpha,\alpha),(y,\alpha,\beta),(y,\alpha,\gamma),(y,\beta,\alpha),(y,\beta,\beta),(y,\beta,\gamma),(y,\gamma,\alpha),(y,\gamma,\beta),(y,\gamma,\gamma)\\
&(z,\alpha,\alpha),(z,\alpha,\beta),(z,\alpha,\gamma),(z,\beta,\alpha),(z,\beta,\beta),(z,\beta,\gamma),(z,\gamma,\alpha),(z,\gamma,\beta),(z,\gamma,\gamma)\}.
\end{aligned}
$$

One way of ensuring that $S^2 \times W = W \times S^2$ is by making all tuples equal, i.e. if

$$\alpha = \beta = \gamma = x = y = z,$$

then

$$S^2 \times W = \{(\alpha,\alpha,\alpha)\} = W \times S^2.$$

-1.4. We start by counting the number of possible bijective functions $f_2 : \{1,2\} \to \{\alpha,\beta\}$. For each element in the domain of f_2 there are two options to connect to an element in the function's image: either 1 or 2. So we can have

$$
\begin{aligned}
&1 \mapsto \alpha, \text{ or}\\
&1 \mapsto \beta.
\end{aligned}
$$

(recall that the symbol $x \mapsto y$ means that the element x is mapped by the function to the element y)

For each of the above options, there is only a single option left for the element 2:

$$
\begin{aligned}
&2 \mapsto \beta \text{ if } 1 \mapsto \alpha,\\
&2 \mapsto \alpha \text{ if } 2 \mapsto \beta.
\end{aligned}
$$

Therefore, we have two possible choices for mapping 1, each of which dictates the choice for mapping 2. Altogether there are two such bijective functions:

We can use the same logic for the case of 3 elements in each set. Let's define the function as

$$f_3 : \{1,2,3\} \to \{\alpha,\beta,\gamma\}.$$

We start with all the possibilities for connecting 1:

$$1 \mapsto \alpha, \text{ or}$$
$$1 \mapsto \beta, \text{ or}$$
$$1 \mapsto \gamma.$$

For each of the above choices, we have two remaining choices for connecting 2 (since one element in the image of f_3 is already decided by our choice for 1). Then, for the last element 3 we remain with a single option only, since two elements in the image of f_3 are already connected to by 1 and 2. Altogether we therefore have

$$3 \times 2 \times 1 = 6$$

total bijective functions f_3.

You probably already noticed the pattern: for a function

$$f_n : \{n \text{ elements}\} \to \{n \text{ elements}\},$$

we have n choices for connecting the first element, then $n-1$ options for connecting the second element, then $n-2$ options for connecting the the third element... and by the last element we have only a single option. Therefore, the number of bijective functions f_n is

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 = n!$$

Note that indeed $2! = 2$ and $3! = 6$, which agrees with the results we got for f_2 and f_3, respectively.

-1.5. solution...

-1.6. A function is bijective if and only if it is both a injective and surjective. There are two cases for $|A| \neq |B|$:

- -1.6.1. $|A| > |B|$, in which case there is at least one element in A which is not connected to any element in B: otherwise, there are at least two elements in A that connect to the same element in B. In the first case the relation is not a function, and in the second it is not injective and therefore not bijective.
- -1.6.2. $|A| < |B|$, in which case there must be at least one element in B that is not connected to by any element from A (by the defintion of a function, there cannot be any element in A that is connected to more than a single element in B). Therefore such a function is not surjective and thus not bijective.

-1.7. The polynomial $f(x)$ can be re-written as

$$f(x) = x\left(x^2 + x - 6\right).$$

Therefore one of its roots are when $x = 0$, and the other when $x^2 + x - 6 = 0$. Using the quadratic formula we get that $x^2 + x - 6 = 0$ when

$$x_{1,2} = \frac{-1 \pm \sqrt{1-4(-6)}}{2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3.$$

Thus, altogether the roots of f are $\{-3, 0, 2\}$.

-1.8. As with most logarithm-related proofs, we transform the problem into the realm of exponents: let us set $m = \log_b(x)$. We then get that $x = b^m$. If we raise both by to the k-th power, we get

$$\begin{aligned} x^k &= (b^m)^k \\ &= b^{mk}. \end{aligned}$$

Taking the logarithm in base b of both sides of the above relation gives

$$\begin{aligned} \log_b\left(x^k\right) &= \log_b\left(b^{mk}\right) \\ &= mk \\ &= k\log_b(x). \end{aligned}$$

The last step results from our original defintion that $m = \log_b(x)$.

-1.9. Using the relation proved in the previous question and setting $k = -1$ we get

$$\log_b\left(\frac{1}{x}\right) = \log_b\left(x^{-1}\right) = -1 \cdot \log_b(x) = -\log_b(x).$$

-1.10. Using the logarithm base-change rule (**??**), we set all logarithms to same base ($b = 16$):

$$\log_2(x) = \log_{16}(x) \cdot \log_2(16) = 4\log_{16}(x).$$
$$\log_4(x-1) = \log_{16}(x-1) \cdot \log_4(16) = 2\log_{16}(x-1).$$

Therefore, the expression is equivalent to

$$4\log_{16}(x) = 2\log_{16}(x-1) + \log_{16}(x^3).$$

We can now bring the coefficients 4 and 2 inside the logarithm expression:

$$\log_{16}(x^4) = \log_{16}([x-1]^2) + \log_{16}\left(x^3\right).$$

Using the logarithmic addition law, we get:

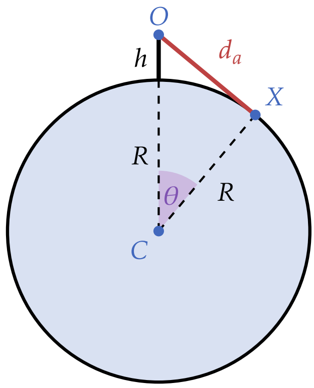
$$\log_{16}(x^4) = \log_{16}(x^3\,[x-1]^3).$$

We can now discard $\log_{16}$ on both sides, and we're left with a simple equation to solve:

$$x^4 = x^3\,(x-1)^2\,,$$

the solutions of which are $x_1 = 0$ and $x_{2,3} = \frac{3\pm\sqrt{5}}{2}$, of which only $x_2 = \frac{3+\sqrt{5}}{2}$ is valid: x_1 isn't valid since $x > 0$, and x_3 isn't valid since $x_3 - 1 < 0$, and thus $\log_b(x_3 - 1)$ isn't defined over the real numbers.

-1.11. (i) We start with drawing two radial lines from the center of the planet $\boldsymbol{C}$: one to the point on the surface where the observer meets the ground, and one to the horizon. The triangle $\triangle COX$ is a right triangle: the angle $\angle CXO = 90°$:



Using the Pythagorean theorem (with $R+h$ as the hypotenuse) we can calculate d_a:

$$d_a^2 + R^2 = (R+h)^2.$$

By expanding the right-hand side, cancelling R^2 and rearranging we get

$$d_a = \sqrt{2Rh + h^2}.$$

To get d_g we need to find the angle θ between the lines CX and CO. For that purpose we can use the law of sines (**??**):

$$\frac{d_a}{\sin(\theta)} = \frac{R+h}{\sin(90°)} = R+h.$$

(since $\sin(90°) = 1$)
Isolating $\sin(\theta)$ and subtituting the value of d_a as function of R and h yields:

$$\sin(\theta) = \frac{d_a}{R+h} = \frac{\sqrt{2Rh+h^2}}{R+h}.$$

Therefore

$$\theta = \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

When θ is given in radians, the length d_g then simply becomes

$$d_g = R\theta = R \cdot \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

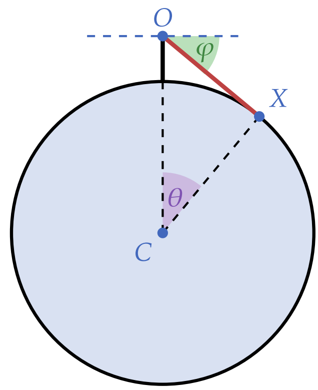
(ii) For an average person on Earth ($h = 1.75\text{m},\ R = 6.371 \times 10^6\text{m}$), standing at sea level, the air-distance to the horizon is therefore

$$d_a = \sqrt{2Rh+h^2} = \sqrt{2 \cdot 6.371 \times 10^6 \cdot 1.75 + 1.75^2} \approx 4722\text{m} = 4.722\text{km}.$$

The ground-distance, on the other hand, is

$$\begin{aligned} d_g &= R \cdot \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right) \\ &= 6.371 \times 10^6\text{m} \cdot \arcsin\left(\frac{4722\text{m}}{6.371 \times 10^6\text{m} + 1.75\text{m}}\right) \\ &\approx 4722\text{m}. \end{aligned}$$

(iii) Let us call the angle representing the drop of the horizon from eye-level φ:



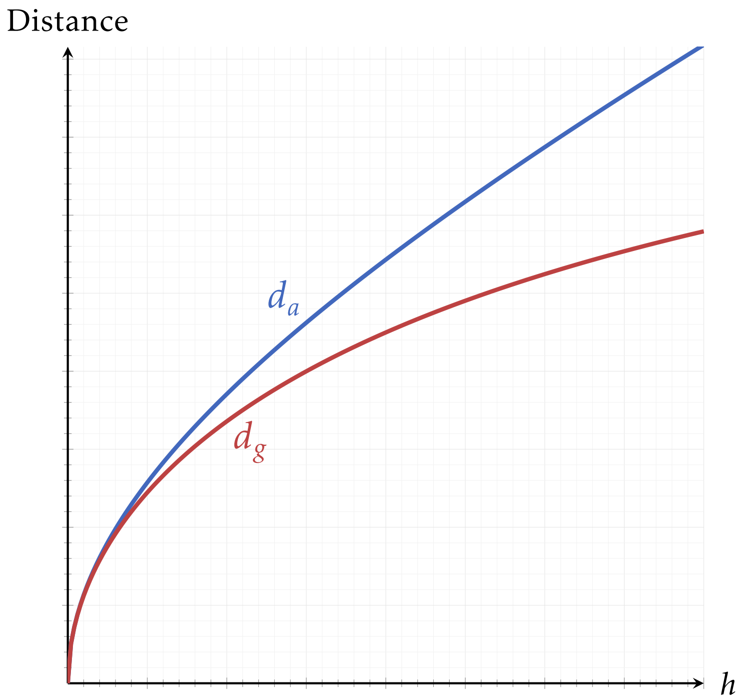
Since $\triangle COX$ is a right triangle ($\angle CXO$ being the right angle), we know θ from previously and all angles in a triangle sum up to deg 180, the angle $\angle COX$ is equal to $90° - \theta$. This in turn means that φ is equal to

$$90° - (90° - \theta) = \theta = \arcsin\left(\frac{\sqrt{2Rh + h^2}}{R + h}\right).$$

The following table sums up all the (approximate) air- and ground-distances to the horizon and the drop of the horizon from eye-level for each of the heights mentioned in the exercise:

Position	Height [m]	d_a[km]	d_g[km]	θ[°]
Person at sea level	1.75	4.7	4.7	0.04
Eiffel Tower observation	165	45.8	45.8	0.41
Average cruising altitude	9100	340.6	340.3	3.06
Internation Space Station	408000	2316	2221	19.98

Note that as the height h grows, the difference between d_a and d_g grows too. We can see this clearly when plotting $d_a(h)$ and $d_g(h)$ in the same graph (disregarding the units and values for now, since we are only interested in the qualitative behaviour of both distances):



For small values of h the two functions are very close to eachother, and as h grows they grow apart, with $d_a > d_g$.

-1.12. • **Algebraic form**: we simply expand all parantheses and multiply everything:

$$\begin{aligned}
\left(\sqrt{3}+\mathrm{i}\right)^2\left(-2+\sqrt{12}\mathrm{i}\right) &= \left(\sqrt{3}+\mathrm{i}\right)\left(\sqrt{3}+\mathrm{i}\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\
&= \left(3+\sqrt{3}\mathrm{i}+\sqrt{3}\mathrm{i}-1\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\
&= \left(2+2\sqrt{3}\mathrm{i}\right)\left(-2+\sqrt{12}\mathrm{i}\right)\\
&= 2\left(1+\sqrt{3}\mathrm{i}\right)\left(-2+\sqrt{4\cdot 3}\mathrm{i}\right)\\
&= 2\left(1+\sqrt{3}\mathrm{i}\right)\left(-2+2\sqrt{3}\mathrm{i}\right)\\
&= 4\left(1+\sqrt{3}\mathrm{i}\right)\left(-1+\sqrt{3}\mathrm{i}\right)\\
&= 4\left(-1+\sqrt{3}\mathrm{i}-\sqrt{3}\mathrm{i}-3\right)\\
&= 4(-4)\\
&= -16.
\end{aligned}$$

• **Polar form**: first we use **??** to find the polar form of the two complex numbers:

$$z_1 = \sqrt{3}+\mathrm{i} \Rightarrow \begin{cases} r_1 = \sqrt{3+1} = 2,\\ \theta_1 = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}. \end{cases}$$

$$z_2 = -2+\sqrt{12} \Rightarrow \begin{cases} r_2 = \sqrt{4+12} = 4,\\ \theta_2 = \arctan\left(-\frac{\sqrt{12}}{2}\right) = \frac{2\pi}{3}. \end{cases}$$

Therefore, $z_1^2 z_2$ in polar form is

$$\begin{aligned} z_1^2 z_2 &= \left(r_1 e^{\theta_1 \mathrm{i}}\right)^2 r_2 e^{\theta_2 \mathrm{i}} \\ &= r_1^2 r_2 e^{(2\theta_1 + \theta_2)\mathrm{i}} \\ &= 2^2 \cdot 4 e^{\left(\frac{2\pi}{6} + \frac{2\pi}{3}\right)\mathrm{i}} \\ &= 16 e^{\pi \mathrm{i}}. \end{aligned}$$

Since $e^{\pi \mathrm{i}} = -1$ (**??**), we get that indeed

$$z_1^2 z_2 = -16,$$

just as we got in the algebraic form.

-1.13. It is easy to see that for even values of n the statement holds: for each w_k theres an oposing w_m ($m \neq k$) such that $w_k + w_m = 0$. See for example $n = 4$ and $n = 6$ in **??**.

For a more general proof which includes the odd values of n we must work a bit harder. Recall that the k-th root of the equation $z^n = 1$ has the following form (**??**):

$$w_k = e^{\frac{2\pi}{n} \mathrm{i} k}.$$

We can re-write the sum of the roots as

$$\sum_{k=0}^{n-1} \left(e^{\frac{2\pi \mathrm{i}}{n}}\right)^k,$$

(since $x^{ab} = (x^a)^b$)

Using the hint we note that in this sum $r = e^{\frac{2\pi \mathrm{i}}{n}}$, and thus

$$\sum_{k=0}^{n-1} \left(e^{\frac{2\pi \mathrm{i}}{n}}\right)^k = \frac{1 - \left(e^{\frac{2\pi \mathrm{i}}{n}}\right)^n}{1 - e^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - e^{\frac{2\pi \mathrm{i}}{n} n}}{1 - e^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - e^{2\pi \mathrm{i}}}{1 - e^{\frac{2\pi \mathrm{i}}{n}}} = \frac{1 - 1}{1 - e^{\frac{2\pi \mathrm{i}}{n}}} = 0.$$