-1.1 EXERCISES

-1.1.1 Problems

-1.1. Write the following sets explicitly:

(i) $\{x \in \mathbb{N} \mid 1 < x \leq 7\}$
(ii) $\{x \in \mathbb{Z} \mid x < 5\}$
(iii) $\left\{x \in \mathbb{R} \mid x^2 = -1\right\}$
(iv) $\{x \in \mathbb{N} \wedge x \in \mathbb{Q}\}$
(v) $\left\{x \in \mathbb{R} \mid x^2 - 3x - 4 = 0\right\}$
(vi) $\{x \in \mathbb{R} \mid x < 5 \wedge x \geq 2\}$

-1.2. Determine the relation between the sets:

(i) $A = \{1,\ 2,\ 3\},\ B = \{1,\ 2\}$
(ii) $A = \varnothing,\ B = \{2,\ -5,\ \pi\}$
(iii) $A = \mathbb{Z},\ B = \{\pm x \mid x \in \mathbb{N} \cup \{0\}\}$
(iv) $A = \left\{\pi, \mathrm{e}, \sqrt{2}\right\},\ B = \mathbb{Q}$

-1.3. Write all elements in $S^2 \times W$, where $S = \{\alpha, \beta, \gamma\}$ and $W = \{x, y, z\}$. Find a condition that guarantees $S^2 \times W = W \times S^2$.

-1.4. How many different injective functions $f : \{1,2\} \to \{1,2\}$ exist? How many injective functions $f : \{1,2,3\} \to \{1,2,3\}$ exist? How many inject functions $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ exist for a given $n \in \mathbb{N}$?

-1.5. For each of the real functions below, find a set on which it is surjective (use a graphing calculator if you are not familiar with the shape of a function):

$$x^2,\ x^3 - 5,\ \mathrm{e}^{-x^2/2},\ \sin(x),\ \sin(x) + \cos(x),\ x\mathrm{e}^x.$$

-1.6. Given two sets A, B such that $|B| = |A| - 1$, can a bijective function $f : A \to B$ exist? Explain your answer.

-1.7. MORE EXERCISES TO BE WRITTEN...

-1.1.2 Solutions

-1.1. For each of the sets we first write how to read the notation in words, followed by its explicit form:

(i) Any **natural number** such that it is bigger than 1 and smaller or equal to 7. These are of course the numbers

$$\{2,3,4,5,6,7\}.$$

(ii) Any **integer** such that it is smaller than 5. These are the numbers

$$\{4, 3, 2, 1, 0, -1, -2, -3, \dots\}.$$

(iii) Any **real number** x such that $x^2 = -1$. Since for any $x \in \mathbb{R}$, $x^2 \geq 0$ - there is no such real number x whose square equals -1. Therefore this definition describes the empty set, i.e. $\emptyset$.

(iv) Any **natural number** that is also a rational number. Since any natural number is also a rational number (e.g. $4 = \frac{4}{1} = \frac{8}{2}$, etc.) the definition actually simply describes the set of natural numbers, $\mathbb{N}$. This fact can also be written as

$$\mathbb{N} \cap \mathbb{Q} = \mathbb{N}.$$

(v) Any **real number** such that it solves the equation $x^2 - 3x - 4 = 0$. The solutions can be found using the quadratic formula:

$$\frac{3 \pm \sqrt{3^2 + 4 \cdot 4}}{2} = \frac{3 \pm \sqrt{9 + 16}}{2} = \frac{3 \pm 5}{2} = 4, -1.$$

Therefore the set described by the definition is simply

$$\{4, -1\}.$$

(vi) Any **real number** that is smaller than 5 **and** is bigger than or equal to 2. This definition describes the half-open interval

$$[2, 5).$$

-1.2. Relations between sets:

(i) All the elements in the set B are also in the set A $(1, 2)$, but there's an element in A which is not in B (namely 3). Therefore, B is a subset of A:

$$B \subset A.$$

(ii) The empty set is a subset of any set (and a proper subset of any set except itself), therefore

$$A \subset B.$$

(iii) The set B is defined as all the natural numbers, their negatives and zero. This is exactly the definition of the integers $\mathbb{Z}$, which set A in this case. Therefore

$$A = B.$$

(iv) All of the elements in A are irrational numbers. The set B is the set of **rational numbers**, and therefore the sets are disjoined:

$$A \cap B = \emptyset.$$

-1.3. S^2 is a Cartesian product of S with itself:

$$S^2 = \{(\alpha, \alpha), (\alpha, \beta), (\alpha, \gamma), (\beta, \alpha), (\beta, \beta), (\beta, \gamma), (\gamma, \alpha), (\gamma, \beta), (\gamma, \gamma)\}.$$

Therefore, to form the Cartesian product $S^2 \times W$ we simply take each of the elements is S^2 and add to it an element from W:

$$\begin{aligned}S^2 \times W = \{&(\alpha,\alpha,x),(\alpha,\beta,x),(\alpha,\gamma,x),(\beta,\alpha,x),(\beta,\beta,x),(\beta,\gamma,x),(\gamma,\alpha,x),(\gamma,\beta,x),(\gamma,\gamma,x)\\ &(\alpha,\alpha,y),(\alpha,\beta,y),(\alpha,\gamma,y),(\beta,\alpha,y),(\beta,\beta,y),(\beta,\gamma,y),(\gamma,\alpha,y),(\gamma,\beta,y),(\gamma,\gamma,y)\\ &(\alpha,\alpha,z),(\alpha,\beta,z),(\alpha,\gamma,z),(\beta,\alpha,z),(\beta,\beta,z),(\beta,\gamma,z),(\gamma,\alpha,z),(\gamma,\beta,z),(\gamma,\gamma,z)\}.\end{aligned}$$

Note that the number of elements in S is 3, and so the number of elements in S^2 is $3 \times 3 = 9$. The number of elements in W is also 3, and so the number of elements in $S^2 \times W$ is $9 \times 3 = 27$.

The Cartesian product $W \times S^2$ has the same structure as $S^2 \times W$, except that the elements from W (x, y, z) are now on the left (remember that in tuples the order matters):

$$\begin{aligned}S^2 \times W = \{&(x,\alpha,\alpha),(x,\alpha,\beta),(x,\alpha,\gamma),(x,\beta,\alpha),(x,\beta,\beta),(x,\beta,\gamma),(x,\gamma,\alpha),(x,\gamma,\beta),(x,\gamma,\gamma)\\ &(y,\alpha,\alpha),(y,\alpha,\beta),(y,\alpha,\gamma),(y,\beta,\alpha),(y,\beta,\beta),(y,\beta,\gamma),(y,\gamma,\alpha),(y,\gamma,\beta),(y,\gamma,\gamma)\\ &(z,\alpha,\alpha),(z,\alpha,\beta),(z,\alpha,\gamma),(z,\beta,\alpha),(z,\beta,\beta),(z,\beta,\gamma),(z,\gamma,\alpha),(z,\gamma,\beta),(z,\gamma,\gamma)\}.\end{aligned}$$