-1.1 SYSTEMS OF LINEAR EQUATIONS

Everything we learned so far about vectors and matrices can be used to solve and characterise a family of equations known as **linear equations**. You're probably already very familiar with linear equations: they are equations in which the **variables** appear directly, without any power or other functions acting on them. For example, the simple equation

$$y = ax + b, \tag{-1.1.1}$$

where x, y are both variables and a, b are both constant real numbers is a linear equation. Equation -1.1.1 can be re-written as

$$ax - y + b = 0, \tag{-1.1.2}$$

where now a is the **coefficient** of the variable x, while the variable y has the coefficient -1 and b is a so-called **free coefficient**. In general, a linear equation of two variables has the form

$$a_0 + a_x x + a_y y = 0, \tag{-1.1.3}$$

i.e. we changed the name of a to a_x and b to a_0, and gave y the coefficient a_y. We can also rename x and y to x_1 and x_2, respectively, and name their coefficients accordingly:

$$a_0 + a_1 x_1 + a_2 x_2 = 0. \tag{-1.1.4}$$

The form shown in Equation -1.1.4 can be easily expanded into n variables:

$$a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_{n-1} x_{n-1} + a_n x_n = 0, \tag{-1.1.5}$$

where $x_1, x_2, \ldots, x_n$ are the variables of the equation, and $a_0, a_1, \ldots, a_n$ are its coefficients. We say that n is the **order** (also: **degree**) of the equation.

Note -1.1 Number set used for linear equations

As with other topics, in the context of this section both the variables and coefficients are all **real numbers**, however almost anything we discuss here can genrally be applied to complex numbers or other structures.

!

Example -1.1 Linear equations

The following is a linear equation of order 3, using the variables x, y, z:

$$3x + 2y - z + 4 = 0.$$

The coefficients of the equation are

$$\begin{aligned} a_0 &= 4, \\ a_x = a_1 &= 3, \\ a_y = a_2 &= 2, \end{aligned}$$

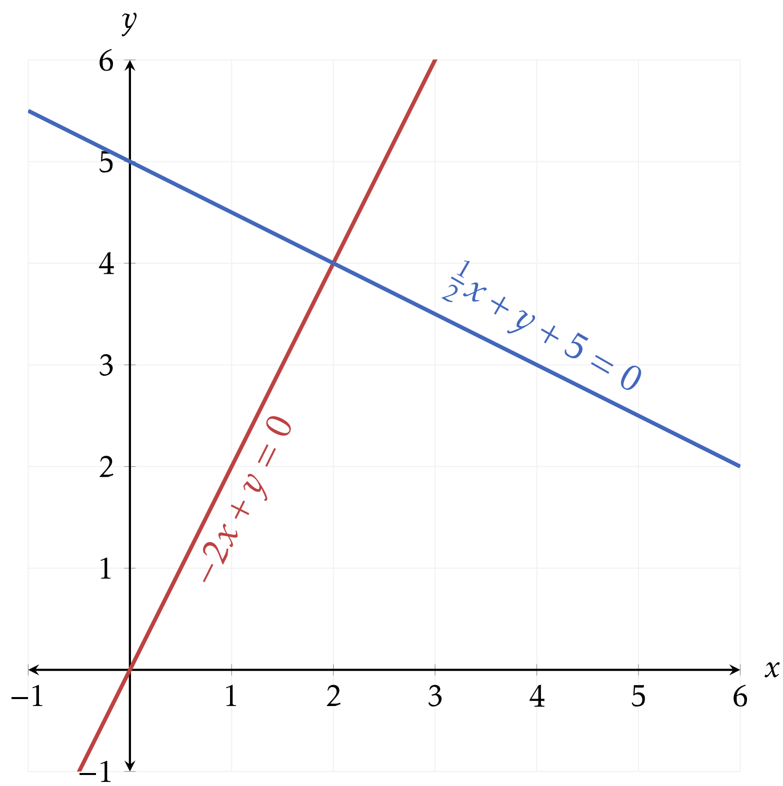

Figure -1.1 Two linear equations represented as lines in $\mathbb{R}^2$. Note how in the red equation the free coefficient is zero, and so the line goes through the origin.

$$a_z = a_3 = -1.$$

Another linear equation of the same three variables is

$$5x - 2y + 1 = 0.$$

In this case the coefficient $a_z = a_3 = 0$. Depending on the context, this equation can be considered as either an equation of order 3 or an equation of order 2.

In $\mathbb{R}^2$ linear equations represent a line, which doesn't necesserally go through the origin (and thus isn't necesserally a subspace of $\mathbb{R}^n$). For a line to go through the origin, the free coefficient a_0 must equal zero (see Figure -1.1).

In $\mathbb{R}^3$ linear equations represent planes. Much like with the lines in $\mathbb{R}^2$, these planes don't necesserally go through the origin.

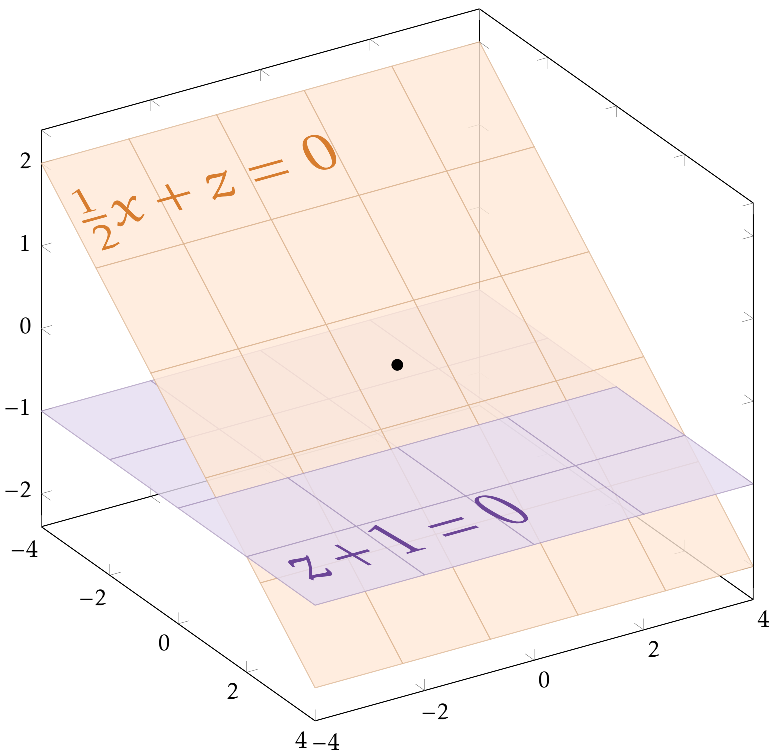

Figure -1.2 Two intersecting planes in $\mathbb{R}^3$ with their corresponding equations.