
-1.1 EXERCISES

-1.1.1 Problems

-1.1. Write the following sets explicitly:

- (i) $\{x \in \mathbb{N} \mid 1 < x \leq 7\}$
- (ii) $\{x \in \mathbb{Z} \mid x < 5\}$
- (iii) $\{x \in \mathbb{R} \mid x^2 = -1\}$
- (iv) $\{x \in \mathbb{N} \wedge x \in \mathbb{Q}\}$
- (v) $\{x \in \mathbb{R} \mid x^2 - 3x - 4 = 0\}$
- (vi) $\{x \in \mathbb{R} \mid x < 5 \wedge x \geq 2\}$

-1.2. Determine the relation between the sets:

- (i) $A = \{1, 2, 3\}, B = \{1, 2\}$
- (ii) $A = \emptyset, B = \{2, -5, \pi\}$
- (iii) $A = \mathbb{Z}, B = \{\pm x \mid x \in \mathbb{N} \cup \{0\}\}$
- (iv) $A = \{\pi, e, \sqrt{2}\}, B = \mathbb{Q}$

-1.3. Write all elements in $S^2 \times W$, where $S = \{\alpha, \beta, \gamma\}$ and $W = \{x, y, z\}$. Find a condition that guarantees $S^2 \times W = W \times S^2$.

-1.4. How many different **bijective** functions exist between a set with 2 elements and another set with 2 elements (e.g. $f : \{1, 2\} \rightarrow \{\alpha, \beta\}$)? How many exist between two sets, each with 3 elements? Between two sets each with n elements?

-1.5. For each of the real functions below, find a set on which it is surjective (use a graphing calculator if you are not familiar with the shape of a function):

$$x^2, x^3 - 5, e^{-x^2/2}, \sin(x), \sin(x) + \cos(x), xe^x.$$

-1.6. Given two sets A, B such that $|A| \neq |B|$, can a bijective function $f : A \rightarrow B$ exist? Explain your answer.

-1.7. Find all real roots of the following polynomial function:

$$f(x) = x^3 + x^2 - 6x.$$

-1.8. Given a real $b > 0$ and k , prove that for any real $x > 0$

$$\log_b(x^k) = k \log_b(x).$$

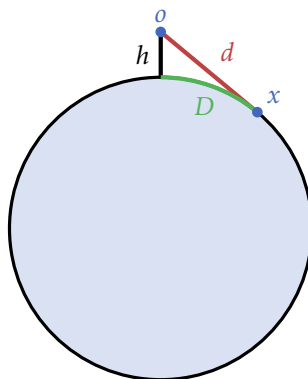
-1.9. Show that for any positive real x, b

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x).$$

- 1.10. Solve the following equation for any real $x > 0$: (CHECK SOLUTION! MIGHT BE WRONG)

$$\log_2(x) + \log_4(x-1) = \log_{16}(x^3).$$

- 1.11. The horizon on a spherical planet such as the earth¹ is defined as the distance from an observer to the point where the ground disappears behind the planet's curve. Following is a 2-dimensional depiction, where o is the observer, h its height above the planet surface, x the horizon point and d the air-distance from the observer to the horizon and D the ground-distance from the observer to the horizon:



Find an expression for the air-distance d and ground-distance D to the horizon as a function of the planet's radius R and the observer's height h . Given that the Earth's radius is about 6371km and an average person is 1.75m tall - what is the distance to the horizon for a person standing at sea-level (both air- and ground-distances)?

- 1.12. MORE EXERCISES TO BE WRITTEN...

-1.1.2 Solutions

- 1.1. For each of the sets we first write how to read the notation in words, followed by its explicit form:

- (i) Any **natural number** such that it is bigger than 1 and smaller or equal to 7. These are of course the numbers

$$\{2, 3, 4, 5, 6, 7\}.$$

- (ii) Any **integer** such that it is smaller than 5. These are the numbers

$$\{4, 3, 2, 1, 0, -1, -2, -3, \dots\}.$$

- (iii) Any **real number** x such that $x^2 = -1$. Since for any $x \in \mathbb{R}$, $x^2 \geq 0$ - there is no such real number x whose square equals -1 . Therefore this definition describes the empty set, i.e. \emptyset .

¹It is.

- (iv) Any **natural number** that is also a rational number. Since any natural number is also a rational number (e.g. $4 = \frac{4}{1} = \frac{8}{2}$, etc.) the definition actually simply describes the set of natural numbers, \mathbb{N} . This fact can also be written as

$$\mathbb{N} \cap \mathbb{Q} = \mathbb{N}.$$

- (v) Any **real number** such that it solves the equation $x^2 - 3x - 4 = 0$. The solutions can be found using the quadratic formula:

$$\frac{3 \pm \sqrt{3^2 + 4 \cdot 4}}{2} = \frac{3 \pm \sqrt{9 + 16}}{2} = \frac{3 \pm 5}{2} = 4, -1.$$

Therefore the set described by the definition is simply

$$\{4, -1\}.$$

- (vi) Any **real number** that is smaller than 5 **and** is bigger than or equal to 2. This definition describes the half-open interval

$$[2, 5).$$

-1.2. Relations between sets:

- (i) All the elements in the set B are also in the set A ($1, 2$), but there's an element in A which is not in B (namely 3). Therefore, B is a subset of A :

$$B \subset A.$$

- (ii) The empty set is a subset of any set (and a proper subset of any set except itself), therefore

$$A \subset B.$$

- (iii) The set B is defined as all the natural numbers, their negatives and zero. This is exactly the definition of the integers \mathbb{Z} , which set A in this case. Therefore

$$A = B.$$

- (iv) All of the elements in A are irrational numbers. The set B is the set of **rational numbers**, and therefore the sets are disjointed:

$$A \cap B = \emptyset.$$

-1.3. S^2 is a Cartesian product of S with itself:

$$S^2 = \{(\alpha, \alpha), (\alpha, \beta), (\alpha, \gamma), (\beta, \alpha), (\beta, \beta), (\beta, \gamma), (\gamma, \alpha), (\gamma, \beta), (\gamma, \gamma)\}.$$

Therefore, to form the Cartesian product $S^2 \times W$ we simply take each of the elements in S^2 and add to it an element from W :

$$S^2 \times W = \{(\alpha, \alpha, x), (\alpha, \beta, x), (\alpha, \gamma, x), (\beta, \alpha, x), (\beta, \beta, x), (\beta, \gamma, x), (\gamma, \alpha, x), (\gamma, \beta, x), (\gamma, \gamma, x), \\ (\alpha, \alpha, y), (\alpha, \beta, y), (\alpha, \gamma, y), (\beta, \alpha, y), (\beta, \beta, y), (\beta, \gamma, y), (\gamma, \alpha, y), (\gamma, \beta, y), (\gamma, \gamma, y), \\ (\alpha, \alpha, z), (\alpha, \beta, z), (\alpha, \gamma, z), (\beta, \alpha, z), (\beta, \beta, z), (\beta, \gamma, z), (\gamma, \alpha, z), (\gamma, \beta, z), (\gamma, \gamma, z)\}.$$

Note that the number of elements in S is 3, and so the number of elements in S^2 is $3 \times 3 = 9$. The number of elements in W is also 3, and so the number of elements in $S^2 \times W$ is $9 \times 3 = 27$.

The Cartesian product $W \times S^2$ has the same structure as $S^2 \times W$, except that the elements from W are now on the left (remember that the order of elements is important in tuples, unlike with sets):

$$S^2 \times W = \{(x, \alpha, \alpha), (x, \alpha, \beta), (x, \alpha, \gamma), (x, \beta, \alpha), (x, \beta, \beta), (x, \beta, \gamma), (x, \gamma, \alpha), (x, \gamma, \beta), (x, \gamma, \gamma), \\ (y, \alpha, \alpha), (y, \alpha, \beta), (y, \alpha, \gamma), (y, \beta, \alpha), (y, \beta, \beta), (y, \beta, \gamma), (y, \gamma, \alpha), (y, \gamma, \beta), (y, \gamma, \gamma), \\ (z, \alpha, \alpha), (z, \alpha, \beta), (z, \alpha, \gamma), (z, \beta, \alpha), (z, \beta, \beta), (z, \beta, \gamma), (z, \gamma, \alpha), (z, \gamma, \beta), (z, \gamma, \gamma)\}.$$

One way of ensuring that $S^2 \times W = W \times S^2$ is by making all tuples equal, i.e. if

$$\alpha = \beta = \gamma = x = y = z,$$

then

$$S^2 \times W = \{(\alpha, \alpha, \alpha)\} = W \times S^2.$$

- 1.4. We start by counting the number of possible bijective functions $f_2 : \{1, 2\} \rightarrow \{\alpha, \beta\}$. For each element in the domain of f_2 there are two options to connect to an element in the function's image: either 1 or 2. So we can have

$$1 \mapsto \alpha, \text{ or} \\ 1 \mapsto \beta.$$

(recall that the symbol $x \mapsto y$ means that the element x is mapped by the function to the element y)

For each of the above options, there is only a single option left for the element 2:

$$2 \mapsto \beta \text{ if } 1 \mapsto \alpha, \\ 2 \mapsto \alpha \text{ if } 1 \mapsto \beta.$$

Therefore, we have two possible choices for mapping 1, each of which dictates the choice for mapping 2. Altogether there are two such bijective functions:



We can use the same logic for the case of 3 elements in each set. Let's define the function as

$$f_3 : \{1, 2, 3\} \rightarrow \{\alpha, \beta, \gamma\}.$$

We start with all the possibilities for connecting 1:

$1 \mapsto \alpha$, or
 $1 \mapsto \beta$, or
 $1 \mapsto \gamma$.

For each of the above choices, we have two remaining choices for connecting 2 (since one element in the image of f_3 is already decided by our choice for 1). Then, for the last element 3 we remain with a single option only, since two elements in the image of f_3 are already connected to by 1 and 2. Altogether we therefore have

$$3 \times 2 \times 1 = 6$$

total bijective functions f_3 .

You probably already noticed the pattern: for a function

$$f_n : \{n \text{ elements}\} \rightarrow \{n \text{ elements}\},$$

we have n choices for connecting the first element, then $n-1$ options for connecting the second element, then $n-2$ options for connecting the the third element... and by the last element we have only a single option. Therefore, the number of bijective functions f_n is

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 = n!$$

Note that indeed $2! = 2$ and $3! = 6$, which agrees with the results we got for f_2 and f_3 , respectively.

-1.5. solution...

-1.6. A function is bijective if and only if it is both a injective and surjective. There are two cases for $|A| \neq |B|$:

-1.6.1. $|A| > |B|$, in which case there is at least one element in A which is not connected to any element in B : otherwise, there are at least two elements in A that connect to the same element in B . In the first case the relation is not a function, and in the second it is not injective and therefore not bijective.

-1.6.2. $|A| < |B|$, in which case there must be at least one element in B that is not connected to by any element from A (by the definition of a function, there cannot be any element in A that is connected to more than a single element in B). Therefore such a function is not surjective and thus not bijective.

-1.7. The polynomial $f(x)$ can be re-written as

$$f(x) = x(x^2 + x - 6).$$

Therefore one of its roots are when $x = 0$, and the other when $x^2 + x - 6 = 0$. Using the quadratic formula we get that $x^2 + x - 6 = 0$ when

$$x_{1,2} = \frac{-1 \pm \sqrt{1 - 4(-6)}}{2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3.$$

Thus, altogether the roots of f are $\{-3, 0, 2\}$.

- 1.8. As with most logarithm-related proofs, we transform the problem into the realm of exponents: let us set $m = \log_b(x)$. We then get that $x = b^m$. If we raise both by to the k -th power, we get

$$\begin{aligned} x^k &= (b^m)^k \\ &= b^{mk}. \end{aligned}$$

Taking the logarithm in base b of both sides of the above relation gives

$$\begin{aligned} \log_b(x^k) &= \log_b(b^{mk}) \\ &= mk \\ &= k \log_b(x). \end{aligned}$$

The last step results from our original definition that $m = \log_b(x)$.

- 1.9. Using the relation proved in the previous question and setting $k = -1$ we get

$$\log_b\left(\frac{1}{x}\right) = \log_b(x^{-1}) = -1 \cdot \log_b(x) = -\log_b(x).$$

- 1.10. Using the logarithm base-change rule (??), we set all logarithms to same base ($b = 16$):

$$\begin{aligned} \log_2(x) &= \log_{16}(x) \cdot \log_2(16) = 4 \log_{16}(x). \\ \log_4(x-1) &= \log_{16}(x-1) \cdot \log_4(16) = 2 \log_{16}(x-1). \end{aligned}$$

Therefore, the expression is equivalent to

$$4 \log_{16}(x) = 2 \log_{16}(x-1) + \log_{16}(x^3).$$

We can now bring the coefficients 4 and 2 inside the logarithm expression:

$$\log_{16}(x^4) = \log_{16}([x-1]^2) + \log_{16}(x^3).$$

Using the logarithmic addition law, we get:

$$\log_{16}(x^4) = \log_{16}(x^3 [x-1]^2).$$

We can now discard \log_{16} on both sides, and we're left with a simple equation to solve:

$$x^4 = x^3 (x-1)^2,$$

the solutions of which are $x_1 = 0$ and $x_{2,3} = \frac{3 \pm \sqrt{5}}{2}$, of which only $x_2 = \frac{3 + \sqrt{5}}{2}$ is valid: x_1 isn't valid since $x > 0$, and x_3 isn't valid since $x_3 - 1 < 0$, and thus $\log_b(x_3 - 1)$ isn't defined over the real numbers.