# -1.1 SYSTEMS OF LINEAR EQUATIONS

## -1.1.1 Defintions

Everything we learned so far about vectors and matrices can be used to solve and characterise a family of equations known as **linear equations**. You're probably already very familiar with linear equations: they are equations in which the **variables** appear directly, without any power or other functions acting on them. For example, the simple equation

$$y = ax + b, \tag{-1.1.1}$$

where $x, y$ are both variables and $a, b$ are both constant real numbers is a linear equation. Equation -1.1.1 can be re-written as

$$ax - y + b = 0, \tag{-1.1.2}$$

where now $a$ is the **coefficient** of the variable $x$, while the variable $y$ has the coefficient $-1$ and $b$ is a so-called **free coefficient**. In general, a linear equation of two variables has the form

$$a_0 + a_x x + a_y y = 0, \tag{-1.1.3}$$

i.e. we changed the name of $a$ to $a_x$ and $b$ to $a_0$, and gave $y$ the coefficient $a_y$. We can also rename $x$ and $y$ to $x_1$ and $x_2$, respectively, and name their coefficients accordingly:

$$a_0 + a_1 x_1 + a_2 x_2 = 0. \tag{-1.1.4}$$

The form shown in Equation -1.1.4 can be easily expanded into $n$ variables:

$$a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_{n-1} x_{n-1} + a_n x_n = 0, \tag{-1.1.5}$$

where $x_1, x_2, \ldots, x_n$ are the variables of the equation, and $a_0, a_1, \ldots, a_n$ are its coefficients. We say that $n$ is the **order** (also: **degree**) of the equation.

**Note -1.1 Number set used for linear equations**

As with other topics, in the context of this section both the variables and coefficients are all **real numbers**, however almost anything we discuss here can genrally be applied to complex numbers or other structures.

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**Example -1.1 Linear equations**

The following is a linear equation of order 3, using the variables $x, y, z$:

$$3x + 2y - z + 4 = 0.$$

The coefficients of the equation are

$$a_0 = 4,$$

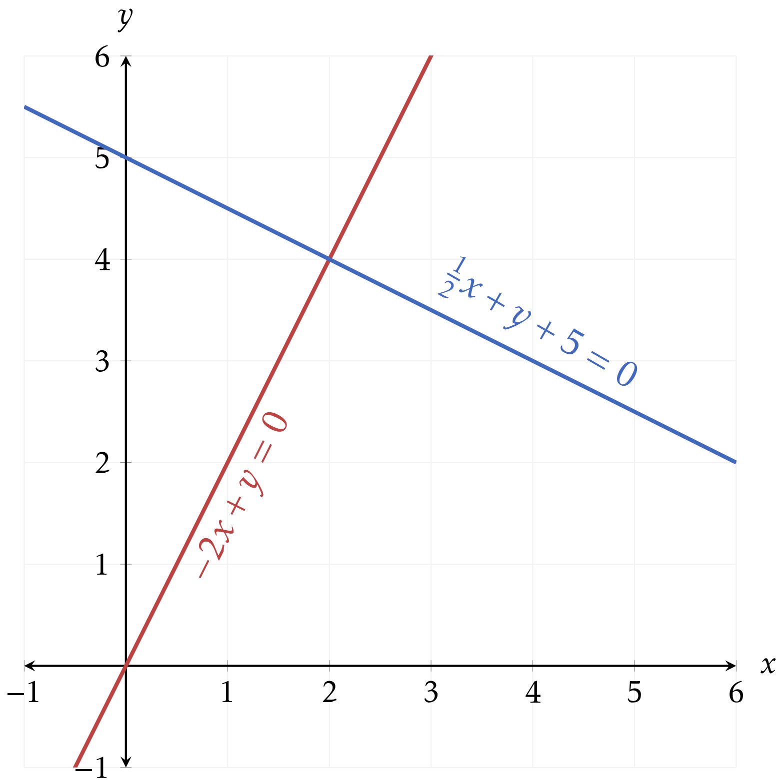



**Figure -1.1** Two linear equations represented as lines in $\mathbb{R}^2$. Note how in the red equation the free coefficient is zero, and so the line goes through the origin.

$$a_x = a_1 = 3,$$
$$a_y = a_2 = 2,$$
$$a_z = a_3 = -1.$$

Another linear equation of the same three variables is

$$5x - 2y + 1 = 0.$$

In this case the coefficient $a_z = a_3 = 0$. Depending on the context, this equation can be considered as either an equation of order 3 or an equation of order 2.

In $\mathbb{R}^2$ linear equations represent a line, which doesn't necesserally go through the origin (and thus isn't necesserally a subspace of $\mathbb{R}^n$). For a line to go through the origin, the free coefficient $a_0$ must equal zero (see Figure -1.1).

In $\mathbb{R}^3$ linear equations represent planes. Much like with the lines in $\mathbb{R}^2$, these planes don't necesserally go through the origin. The trend continues with increasing dimensions: in $\mathbb{R}^4$ linear equations represent 3-dimensional spaces, in $\mathbb{R}^5$ linear equations

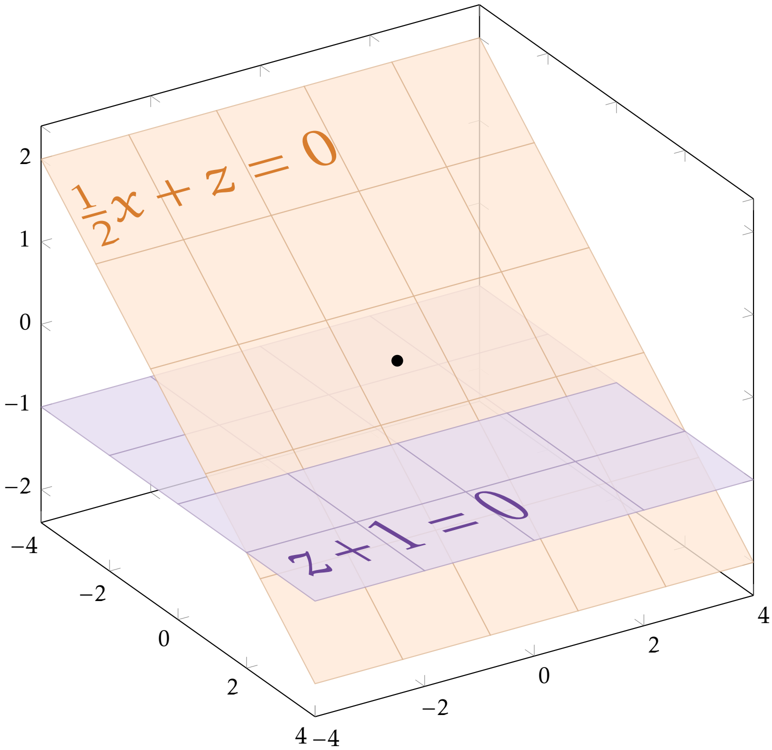



**Figure -1.2** Two intersecting planes in $\mathbb{R}^3$ with their corresponding equations.

represent 4-dimensional spaces, etc. When the free coefficient is equal to zero, these spaces become subspaces of the respective $\mathbb{R}^n$.

### -1.1.2 Systems and matrix form

A **system of linear equations** is a set of linear equations using the same variables. For example, the three equations

$$\begin{cases} 2x - 5y + 4z + 2 = 0 \\ -3x - 2y + 1 = 0 \\ 5x + 4z + -3 = 0 \end{cases}$$

form together a system of 3 linear equations with 3 variables ($x, y$ and $z$). Systems of linear equations can be written together in matrix form: in the above example, the system can be represented as the equation

$$\begin{bmatrix} 2 & -5 & 4 \\ -3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

since performing the matrix-vector product and vector addition yields back the system of equations. We call the matrix the **coefficients matrix** of the equation.

**Note -1.2**

In practice, many times the vector representing the free coefficients is moved to the right hand side of the equation. In the case of the above system this yields the simple equation

$$\begin{bmatrix} 2 & -5 & 4 \\ -3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}.$$

In the most general form, a system of $m$ equations in $n$ variavles $x_1, x_2, x_3, \dots, x_n$ can be represented as the product of an $m \times n$ coefficient matrix and the variables vector, yielding the free-coefficient vector:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m \times n \text{ coefficients}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{n \text{ variables}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{m \text{ free coefficients}}, \tag{-1.1.6}$$

which can be written succinctly as

$$A\, x = b. \tag{-1.1.7}$$

### -1.1.3 Solutions

A system of linear equations can have one or more **solutions**. A solution is a tuple

$$s = (s_1, s_2, \dots, s_n)$$

such that if we substitute each $s_i$ into the respective variable $x_i$ all equation become **true** statements.

**Example -1.2 Solutions of a system of linear equations**

The following linear system

$$\begin{cases} -4x + 2y = 0 \\ x - y + 3 = 0 \end{cases}$$

has the solution

$$s = (-1, 2).$$

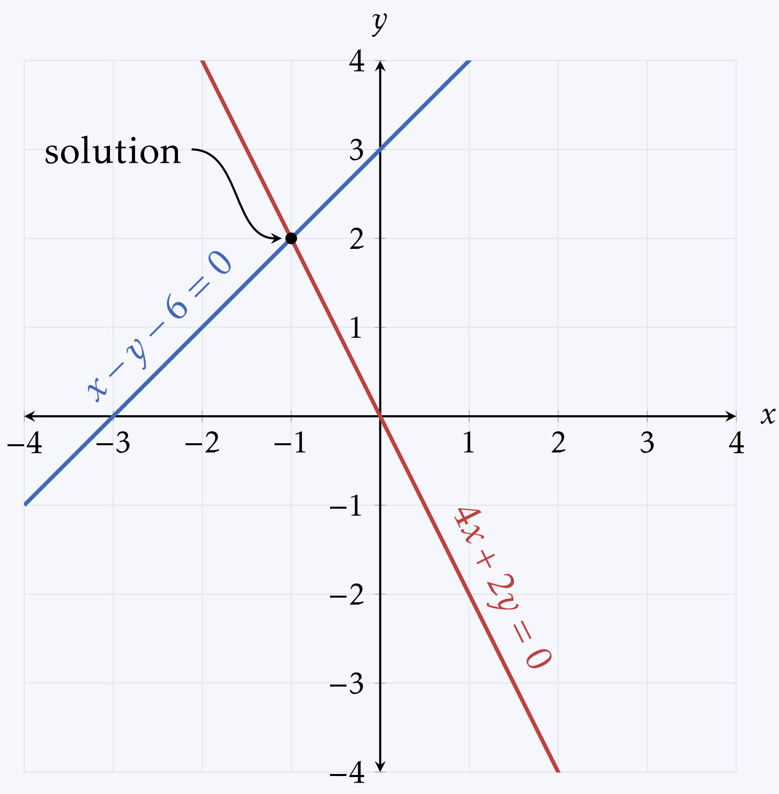
Indeed if we substitute $x = -1$ and $y = 2$ into the system we get

$$\begin{cases} -4 \cdot (-1) + 2 \cdot (2) = -4 + 4 = 0 \;\Rightarrow\; \textbf{true} \\ -1 - 2 + 3 = -3 + 3 = 0 \;\Rightarrow\; \textbf{true} \end{cases}$$

In the graphical representation of linear equations, the solutions of a system are the points where the respective graphs representing the equation (line, plane, etc.) intersect.

**Example -1.3 Solutions of a system of linear equations - graph**

The linear system from the previous example can be represented by the following graph:




Not all systems have single solutions only, nor do all systems even have any solutions. For example, if we add to the system in Example -1.3 the equation

$$x - 3y + 3 = 0,$$

the resulting system

$$\begin{cases} -4x + 2y = 0 \\ x - y + 3 = 0 \\ x - 3y + 3 = 0 \end{cases}$$

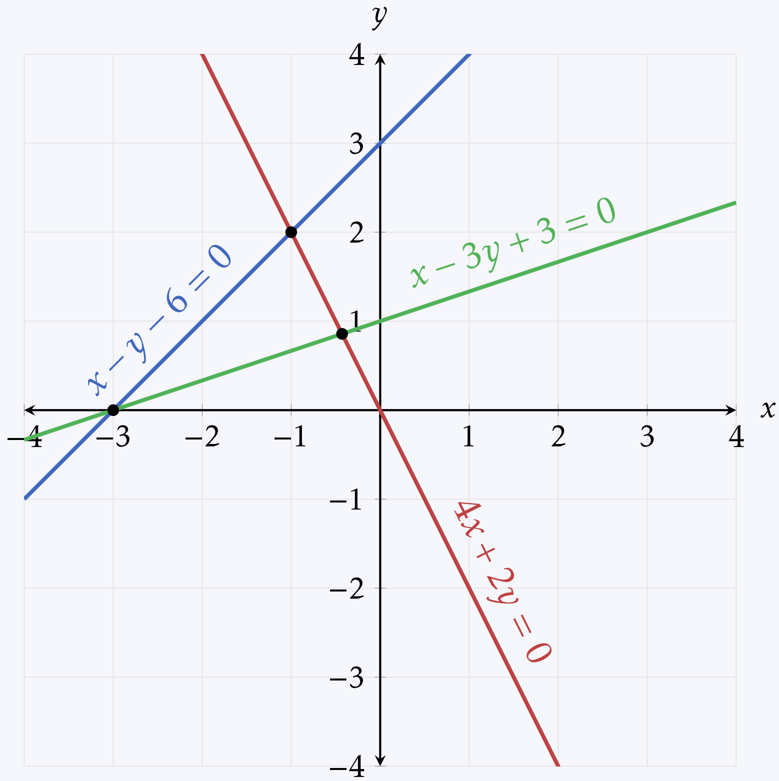
has no solutions (see Example -1.4).

**Example -1.4 System with no solutions**

When the system of linear equations from Example -1.3 is supplemented with the equation

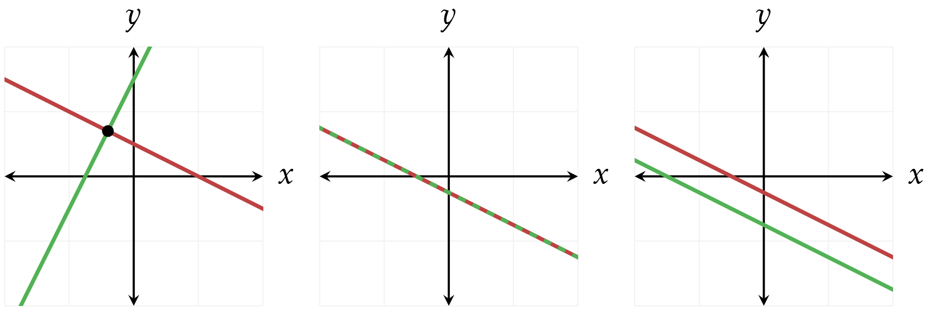
$$x - 3y + 3 = 0,$$

it has no solution. However, any two equations of the system do have solutions (represented below as black points).




Let us now explore when a set of linear equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ has a single solution, infinitely many solutions or no solutions. In $\mathbb{R}^2$, the lines representing two linear equations can be either parallel or non-parallel. If they are non-parallel then the two equations have a single solution - the intercept of both lines (as seen in the previous two examples). If the lines are parallel, there are two cases: either the two lines are identical, in which case there are infinitely many solutions (all the points on the line), or they are parallel yet distince, in which case there are no solutions to the system (see Figure -1.3).

In the case of more than two linear equations there can be, again, either a single solution,




(a)

(b)

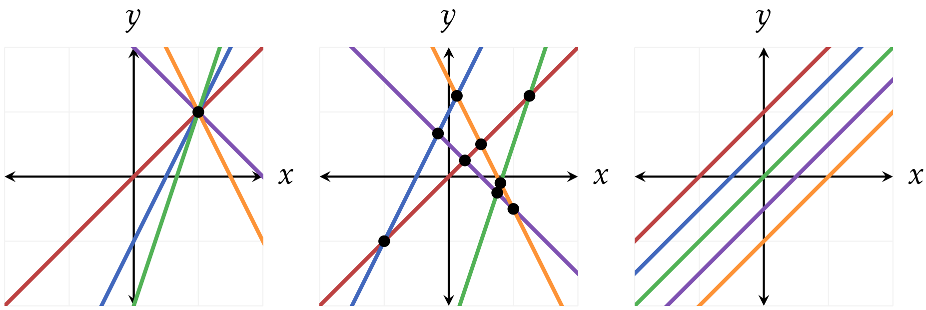
(c)

**Figure -1.3** Three possible cases for two linear equations in $\mathbb{R}^2$: (a) non-parallel and thus a single solution, (b) parallel and identical and thus infinitely many solutions, and (c) parallel but not identical and thus no solutions.

infinitely many solutions or no solutions. The difference is that in this case zero solutions can happen even when all of the lines representing the equations are non-parallel (see Figure -1.4).

In $\mathbb{R}^3$...

### -1.1.4 Finding solutions




(a) (b) (c)

**Figure -1.4** Three possible cases for the number of solutions of several linear equations in $\mathbb{R}^2$: (a) non-parallel but all lines intercept at single point and thus the system has a solution, (b) non-parallel but no single interception point and thus no solution, and (c) parallel and thus no solutions. The case where all lines are identical and thus there are infinitely many solutions is ommited from the figure, and looks identical to Figure -1.3(b).