

MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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$$\begin{aligned}a^b &= e^{b \log(a)} & (a+b)^n &= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \\ \binom{n}{k} &= \frac{n!}{k!(n-k)!} & T(\alpha \vec{u} + \beta \vec{v}) &= \alpha T(\vec{u}) + \beta T(\vec{v}) \\ R(\theta) &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} & A &= Q \Lambda Q^{-1} \\ e^{\pi i} + 1 &= 0 & \frac{df}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ \langle \hat{e}_i, \hat{e}_j \rangle &= \delta_{ij} & \Gamma(z) &= \int_0^\infty t^{z-1} e^{-t} dt \\ \int_a^b f(x) dx &= F(b) - F(a) & \vec{v} &= \sum_{i=1}^n \alpha_i \hat{e}_i \\ \cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\end{aligned}$$



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HERE BE TABLE

CHAPTER

0



INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.



0.1 TAYLOR SERIES

Say we wish to calculate the value of $\cos(x)$ at some non-trivial value, e.g. $x = \frac{\pi}{7}$, or $x = 1.0423$. Of course today we can simply use a computer or a calculator - but how do these even calculate such values? Moreover, a lot of these values are non-algebraic, meaning that we can't express them as decimal fractions to an infinite precision. What if for some important calculation we need to know the value of $e^{3.153}$ down to 15 digits, but our calculator only shows 10 digits after the period?

We can answer all these questions by using **approximations** instead of precise values. For example, we know values such as $\sin\left(\frac{\pi}{2}\right)$ with infinite precision (it's exactly 1), but in the case of less trivial values such as the ones discussed above, we can instead settle for an approximation, as long as it is *close enough* to the actual value for our needs. By "close enough" we essentially mean *within some given error range*, e.g. in the case of $e^{3.153}$ above, 15 digits of precision means that we want that the value we get is no more than 10^{-15} away from the actual value. Let's write this mathematically: if we denote a as our approximation of $e^{3.153}$, then

$$|a - e^{3.153}| \leq 10^{-15}.$$

More generally, given a value y and we want to know it to within an **error range** Δ , then for an approximation a to be acceptable, the following must be true:

$$|a - y| \leq \Delta. \tag{0.1.1}$$

A good method to approximate functions would allow us to get as precise as we wish, given that we put enough "work" into finding an approximated value, e.g. we could use it to get an approximation of $e^{3.153}$ up to $\Delta = 10^{-15}$, but we could also use it to get an approximation up to $\Delta = 10^{-20}$ or $\Delta = 10^{-30}$ or any other value - we will simply have to do more calculations to reach such low error ranges. Generally, the more precise we want our approximation to be, we would need to make more calculations.

Unlike most real functions, polynomials are relatively easy to calculate for any real value x . For example, to correctly calculate the value of the polynomial $P(x) = x^2 - 2x + 5$ we simply need to know how to perform the following operations: addition, subtraction, multiplication and raising by an integer power, all operations that are easy for both humans and computers to perform.

Ideally, we would like to use polynomials to approximate all functions, e.g. given the function $f(x) = \cos(x)$ it would be great if we could find some polynomial $P(x)$ of a finite order n for which $P(x) = \cos(x)$. Unfortunately such a polynomial does not exist, nor does such polynomials exist for $\sin(x)$, \sqrt{x} , $\exp(x)$, a^x ($a > 0$) or any other of the so-called fundamental functions and their compositions (except, of course, polynomials). The reason for this unfortunate reality is rather complicated, but in short it lies in the fact that these functions are *non-algebraic*¹.

¹technically \sqrt{x} is algebraic, and there are indeed many methods to calculate its values - but non of these methods are as simple as calculating a polynomial...except the one we discuss in this section.

However, as mentioned before, for each of these functions, we know at least some values with infinite precision. For example, we know that $\cos(0) = 1$ and $\cos(\frac{1}{2}\pi) = 0$. From symmetry we thus know that $\cos(\pi) = -1$ and $\cos(\frac{3}{4}\pi) = 0$. Since $\cos(x)$ is periodic, i.e. $\cos(x + 2\pi k) = \cos(x)$ for any $k \in \mathbb{Z}$, we actually know infinitely many values of the function. For $\exp(x)$ we know that $\exp(0) = 1$. [Table 0.1](#) lists some known values of common fundamental functions.

Table 0.1 Some known values of common fundamental functions. Each such value is known to an infinite precision.

Function	x	$f(x)$
\sqrt{x}	4	2
$\sin(x)$	0	0
$\cos(x)$	0	1
$\tan(x)$	0	0
$\exp(x)$	0	1
$\log(x)$	1	0

We can use this knowledge to construct a polynomial which approximates the function's value for any $x \in \mathbb{R}$ to whatever precision we wish, given that a specific condition is met. We will describe this condition later, but first let's use an example function to construct such a polynomial: $\exp(x)$. Since we only know $\exp(0)$ with infinite precision, we can use it as a simple approximation of $\exp(x)$, which we denote $T_0(x)$:

$$T_0(x) = 1. \quad (0.1.2)$$

(see [Figure 0.1](#))

We call $T_0(x)$ the **0th-order approximation** of $\exp(x)$. Obviously, this is not a particularly good approximation: it's perhaps ok-ish for values *really* close to $x = 0$, but rapidly diverges from the actual value of $\exp(x)$ as we change x . More precisely, the error $\Delta_0(x)$ is

$$\Delta_0(x) = |\exp(x) - T_0(x)| = |\exp(x) - 1|, \quad (0.1.3)$$

i.e. the error essentially grows as $\exp(x)$ which is... not great.

Ok, then perhaps we can add another term to the approximation? After all, our notation pretty much suggests that there are more terms we can use². We know that close to a point a a differential function $f(x)$ behaves pretty much like its *derivative* at that point, $f'(a)$. We also know the value of the derivative of $\exp(x)$ at $x = 0$, i.e. $\exp'(0) = 1$, so we can add a line of slope $m = 1$ to our approximation, yielding:

$$\exp(x) = 1 + x \cdot \exp'(0) = 1 + x. \quad (0.1.4)$$

Unsurprisingly, we call $T_1(x)$ the **1st-order approximation** of $\exp(x)$. Looking at [Figure 0.2](#) we see that this approximation is better than the previous one, at least for values

²more like *screams* it.

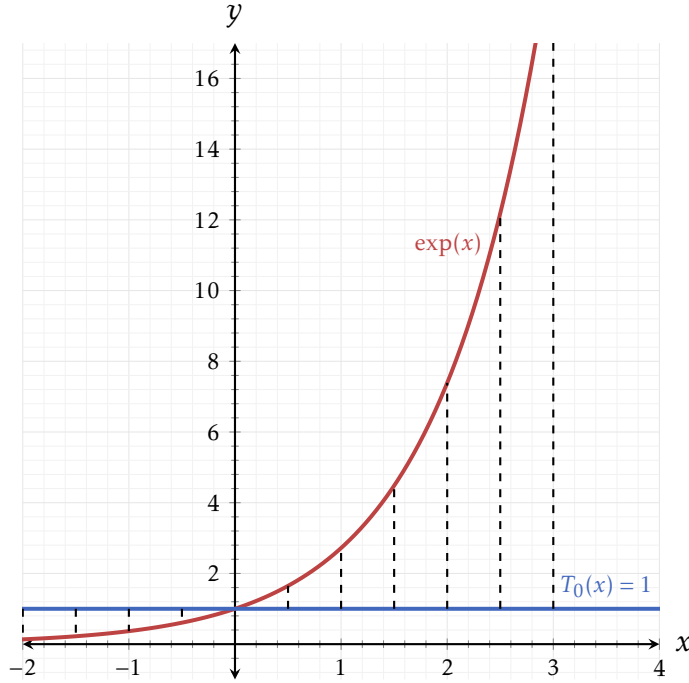


Figure 0.1 The 0th-order approximation $\exp(x) = 1$. The dashed lines show $\Delta_0(x)$ as x diverges from 0.

near $x = 0$. In other words, the error Δ_1 behaves a bit better than Δ_0 :

$$\Delta_1(x) = |\exp(x) - T_1(x)| = |\exp(x) - x - 1|. \quad (0.1.5)$$

Similarly to $\Delta_0(x)$, $\Delta_1(x)$ also generally grows as $\exp(x)$. However, closer to $x = 0$ it behaves like $\exp(x) - x$, so still not great - but definitely an improvement over $\Delta_0(x)$. In practical terms, this means that by using $T_1(x)$ we would generally get better approximations than $T_0(x)$, at least close to $x = 0$. But we can do better!

The next step is of course to add an x^2 term. The second derivative of $\exp(x)$ is also $\exp(x)$, and at $x = 0$ also equal to 1. This time we will divide the term by 2 (for now just accept it as is, it would be explained when we formalize the method). Altogether, we get

$$T_2(x) = 1 + x + \frac{x^2}{2}. \quad (0.1.6)$$

(see Figure 0.3)

Definitely an improvement, but why stop here? We know all the derivatives of $\exp(x)$ at $x = 0$: they are all 1, since $\exp'(x) = \exp(x)$. We can continue adding power terms, giving us

$$T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}. \quad (0.1.7)$$

(again, for now you should just accept the coefficients $\frac{1}{n!}$, they would be explained soon)

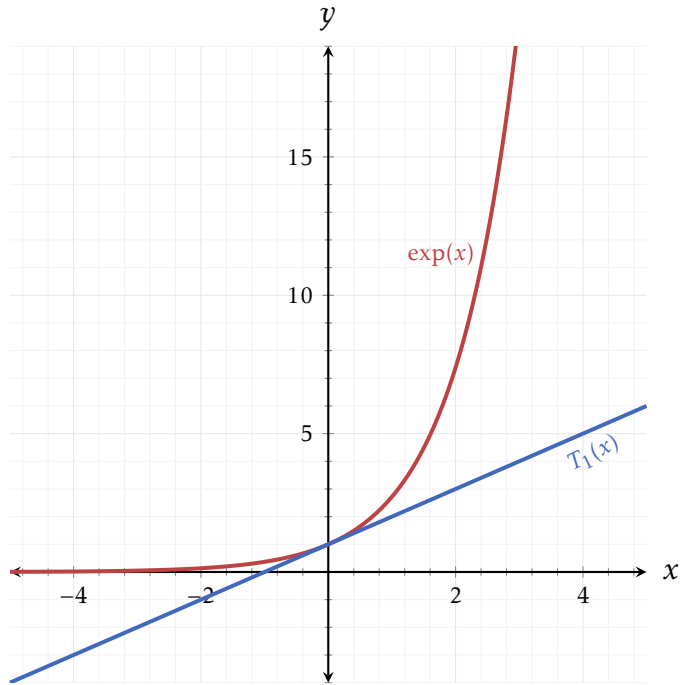


Figure 0.2 The 1st-order approximation $T_1(x) = 1 + x$.

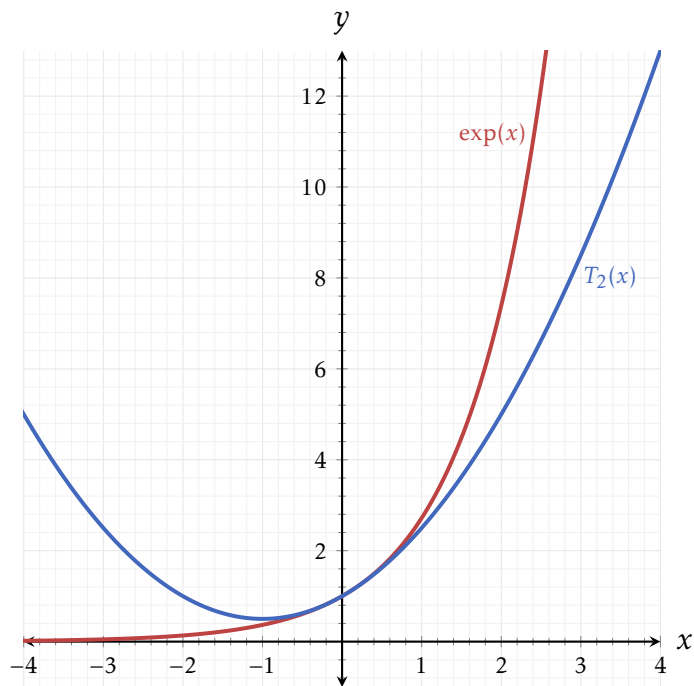


Figure 0.3 The 2nd-order approximation $T_2(x) = 1 + x + \frac{x^2}{2}$.