-1.1 EXERCISES

-1.1.1 Problems

-1.1. Write the following sets explicitly:

(i) $\{x \in \mathbb{N} \mid 1 < x \leq 7\}$

(ii) $\{x \in \mathbb{Z} \mid x < 5\}$

(iii) $\left\{x \in \mathbb{R} \mid x^2 = -1\right\}$

(iv) $\{x \in \mathbb{N} \wedge x \in \mathbb{Q}\}$

(v) $\left\{x \in \mathbb{R} \mid x^2 - 3x - 4 = 0\right\}$

(vi) $\{x \in \mathbb{R} \mid x < 5 \wedge x \geq 2\}$

-1.2. Determine the relation between the sets:

(i) $A = \{1,\ 2,\ 3\}$, $B = \{1,\ 2\}$

(ii) $A = \varnothing$, $B = \{2,\ -5,\ \pi\}$

(iii) $A = \mathbb{Z}$, $B = \{\pm x \mid x \in \mathbb{N} \cup \{0\}\}$

(iv) $A = \left\{\pi, \mathrm{e}, \sqrt{2}\right\}$, $B = \mathbb{Q}$

-1.3. Write all elements in $S^2 \times W$, where $S = \{\alpha, \beta, \gamma\}$ and $W = \{x, y, z\}$. Find a condition that guarantees $S^2 \times W = W \times S^2$.

-1.4. How many different **bijective** functions exist between a set with 2 elements and another set with 2 elements (e.g. $f : \{1, 2\} \to \{\alpha, \beta\}$)? How many exist between two sets, each with 3 elements? Between two sets each with n elements?

-1.5. For each of the real functions below, find a set on which it is surjective (use a graphing calculator if you are not familiar with the shape of a function):

$$x^2,\ x^3 - 5,\ \mathrm{e}^{-x^2/2},\ \sin(x),\ \sin(x) + \cos(x),\ x\mathrm{e}^x.$$

-1.6. Given two sets A, B such that $|A| \neq |B|$, can a bijective function $f : A \to B$ exist? Explain your answer.

-1.7. Find all real roots of the following polynomial function:

$$f(x) = x^3 + x^2 - 6x.$$

-1.8. Given a real $b > 0$ and k, prove that for any real $x > 0$

$$\log_b\left(x^k\right) = k \log_b(x).$$

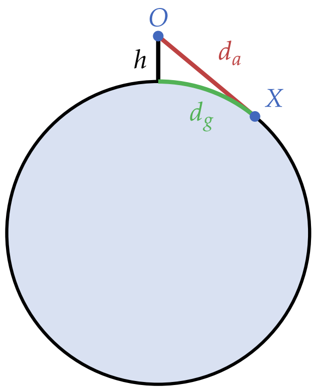
-1.9. Show that for any positive real x, b

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x).$$

-1.10. Solve the following equation for any real $x > 0$: (CHECK SOLUTION! MIGHT BE WRONG)

$$\log_2(x) + \log_4(x-1) = \log_{16}\left(x^3\right).$$

-1.11. The horizon on a spherical planet such as the earth[1] is defined as the distance from an observer to the point where the ground disappears behind the planet's curve. Following is a 2-dimensional depiction, where $\boldsymbol{O}$ is the observer, $\boldsymbol{h}$ its height above the planet surface, $\boldsymbol{X}$ the horizon point and $\boldsymbol{d_a}$ the air-distance from the observer to the horizon and $\boldsymbol{d_g}$ the ground-distance from the observer to the horizon:



(i) Find an expression for the air-distance d and ground-distance D to the horizon as a function of the radius R and height h.

(ii) Given that the Earth's radius is about 6371km (6.371×10^6m) and an average person is 1.75m tall - what is the distance to the horizon for a person standing at sea-level (both air- and ground-distances)? What would these distances be at the following heights: 165m (Eiffel tower's observation deck), 9.1km (average cruising altitude of a passenger jet) and 408km (average altitude of the International Space Station)?

(iii) How many degrees does the horizon "drop" from a striaght-forward view as function of h? ("striaght-forward view" in this context means looking at a direction tangent to the planet's surface)

-1.12. MORE EXERCISES TO BE WRITTEN...

-1.1.2 Solutions

-1.1. For each of the sets we first write how to read the notation in words, followed by its explicit form:

(i) Any **natural number** such that it is bigger than 1 and smaller or equal to 7. These are of course the numbers

$$\{2, 3, 4, 5, 6, 7\}.$$

[1] It is.

(ii) Any **integer** such that it is smaller than 5. These are the numbers

$$\{4,3,2,1,0,-1,-2,-3,\dots\}.$$

(iii) Any **real number** x such that $x^2 = -1$. Since for any $x \in \mathbb{R}$, $x^2 \geq 0$ - there is no such real number x whose square equals -1. Therefore this definition describes the empty set, i.e. $\emptyset$.

(iv) Any **natural number** that is also a rational number. Since any natural number is also a rational number (e.g. $4 = \frac{4}{1} = \frac{8}{2}$, etc.) the definition actually simply describes the set of natural numbers, $\mathbb{N}$. This fact can also be written as

$$\mathbb{N} \cap \mathbb{Q} = \mathbb{N}.$$

(v) Any **real number** such that it solves the equation $x^2 - 3x - 4 = 0$. The solutions can be found using the quadratic formula:

$$\frac{3 \pm \sqrt{3^2 + 4 \cdot 4}}{2} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} = 4, -1.$$

Therefore the set described by the definition is simply

$$\{4, -1\}.$$

(vi) Any **real number** that is smaller than 5 **and** is bigger than or equal to 2. This definition describes the half-open interval

$$[2,5).$$

-1.2. Relations between sets:

(i) All the elements in the set B are also in the set A $(1,2)$, but there's an element in A which is not in B (namely 3). Therefore, B is a subset of A:

$$B \subset A.$$

(ii) The empty set is a subset of any set (and a proper subset of any set except itself), therefore

$$A \subset B.$$

(iii) The set B is defined as all the natural numbers, their negatives and zero. This is exactly the definition of the integers $\mathbb{Z}$, which set A in this case. Therefore

$$A = B.$$

(iv) All of the elements in A are irrational numbers. The set B is the set of **rational numbers**, and therefore the sets are disjoined:

$$A \cap B = \emptyset.$$

-1.3. S^2 is a Cartesian product of S with itself:

$$S^2 = \{(\alpha,\alpha),(\alpha,\beta),(\alpha,\gamma),(\beta,\alpha),(\beta,\beta),(\beta,\gamma),(\gamma,\alpha),(\gamma,\beta),(\gamma,\gamma)\}.$$

Therefore, to form the Cartesian product $S^2 \times W$ we simply take each of the elements is S^2 and add to it an element from W:

$$\begin{aligned} S^2 \times W = \{&(\alpha,\alpha,x),(\alpha,\beta,x),(\alpha,\gamma,x),(\beta,\alpha,x),(\beta,\beta,x),(\beta,\gamma,x),(\gamma,\alpha,x),(\gamma,\beta,x),(\gamma,\gamma,x)\\ &(\alpha,\alpha,y),(\alpha,\beta,y),(\alpha,\gamma,y),(\beta,\alpha,y),(\beta,\beta,y),(\beta,\gamma,y),(\gamma,\alpha,y),(\gamma,\beta,y),(\gamma,\gamma,y)\\ &(\alpha,\alpha,z),(\alpha,\beta,z),(\alpha,\gamma,z),(\beta,\alpha,z),(\beta,\beta,z),(\beta,\gamma,z),(\gamma,\alpha,z),(\gamma,\beta,z),(\gamma,\gamma,z)\}. \end{aligned}$$

Note that the number of elements in S is 3, and so the number of elements in S^2 is $3 \times 3 = 9$. The number of elements in W is also 3, and so the number of elements in $S^2 \times W$ is $9 \times 3 = 27$.

The Cartesian product $W \times S^2$ has the same structure as $S^2 \times W$, except that the elements from W are now on the left (remember that the order of elements is important in tuples, unlike with sets):

$$\begin{aligned} S^2 \times W = \{&(x,\alpha,\alpha),(x,\alpha,\beta),(x,\alpha,\gamma),(x,\beta,\alpha),(x,\beta,\beta),(x,\beta,\gamma),(x,\gamma,\alpha),(x,\gamma,\beta),(x,\gamma,\gamma)\\ &(y,\alpha,\alpha),(y,\alpha,\beta),(y,\alpha,\gamma),(y,\beta,\alpha),(y,\beta,\beta),(y,\beta,\gamma),(y,\gamma,\alpha),(y,\gamma,\beta),(y,\gamma,\gamma)\\ &(z,\alpha,\alpha),(z,\alpha,\beta),(z,\alpha,\gamma),(z,\beta,\alpha),(z,\beta,\beta),(z,\beta,\gamma),(z,\gamma,\alpha),(z,\gamma,\beta),(z,\gamma,\gamma)\}. \end{aligned}$$

One way of ensuring that $S^2 \times W = W \times S^2$ is by making all tuples equal, i.e. if

$$\alpha = \beta = \gamma = x = y = z,$$

then

$$S^2 \times W = \{(\alpha,\alpha,\alpha)\} = W \times S^2.$$

-1.4. We start by counting the number of possible bijective functions $f_2 : \{1,2\} \to \{\alpha,\beta\}$. For each element in the domain of f_2 there are two options to connect to an element in the function's image: either 1 or 2. So we can have

$$\begin{aligned} 1 &\mapsto \alpha, \text{ or}\\ 1 &\mapsto \beta. \end{aligned}$$

(recall that the symbol $x \mapsto y$ means that the element x is mapped by the function to the element y)

For each of the above options, there is only a single option left for the element 2:

$$\begin{aligned} 2 \mapsto \beta \text{ if } 1 \mapsto \alpha,\\ 2 \mapsto \alpha \text{ if } 2 \mapsto \beta. \end{aligned}$$

Therefore, we have two possible choices for mapping 1, each of which dictates the choice for mapping 2. Altogether there are two such bijective functions:

We can use the same logic for the case of 3 elements in each set. Let's define the function as

$$f_3 : \{1,2,3\} \to \{\alpha, \beta, \gamma\}.$$

We start with all the possibilities for connecting 1:

$$\begin{aligned} 1 &\mapsto \alpha, \text{ or} \\ 1 &\mapsto \beta, \text{ or} \\ 1 &\mapsto \gamma. \end{aligned}$$

For each of the above choices, we have two remaining choices for connecting 2 (since one element in the image of f_3 is already decided by our choice for 1). Then, for the last element 3 we remain with a single option only, since two elements in the image of f_3 are already connected to by 1 and 2. Altogether we therefore have

$$3 \times 2 \times 1 = 6$$

total bijective functions f_3.

You probably already noticed the pattern: for a function

$$f_n : \{n \text{ elements}\} \to \{n \text{ elements}\},$$

we have n choices for connecting the first element, then $n-1$ options for connecting the second element, then $n-2$ options for connecting the the third element... and by the last element we have only a single option. Therefore, the number of bijective functions f_n is

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1 = n!$$

Note that indeed $2! = 2$ and $3! = 6$, which agrees with the results we got for f_2 and f_3, respectively.

-1.5. solution...

-1.6. A function is bijective if and only if it is both a injective and surjective. There are two cases for $|A| \neq |B|$:

-1.6.1. $|A| > |B|$, in which case there is at least one element in A which is not connected to any element in B: otherwise, there are at least two elements in A that connect to the same element in B. In the first case the relation is not a function, and in the second it is not injective and therefore not bijective.

-1.6.2. $|A| < |B|$, in which case there must be at least one element in B that is not connected to by any element from A (by the defintion of a function, there cannot be any element in A that is connected to more than a single element in B). Therefore such a function is not surjective and thus not bijective.

-1.7. The polynomial $f(x)$ can be re-written as

$$f(x) = x\left(x^2 + x - 6\right).$$

Therefore one of its roots are when $x = 0$, and the other when $x^2 + x - 6 = 0$. Using the quadratic formula we get that $x^2 + x - 6 = 0$ when

$$x_{1,2} = \frac{-1 \pm \sqrt{1-4(-6)}}{2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3.$$

Thus, altogether the roots of f are $\{-3, 0, 2\}$.

-1.8. As with most logarithm-related proofs, we transform the problem into the realm of exponents: let us set $m = \log_b(x)$. We then get that $x = b^m$. If we raise both by to the k-th power, we get

$$\begin{aligned} x^k &= (b^m)^k \\ &= b^{mk}. \end{aligned}$$

Taking the logarithm in base b of both sides of the above relation gives

$$\begin{aligned} \log_b\left(x^k\right) &= \log_b\left(b^{mk}\right) \\ &= mk \\ &= k\log_b(x). \end{aligned}$$

The last step results from our original defintion that $m = \log_b(x)$.

-1.9. Using the relation proved in the previous question and setting $k = -1$ we get

$$\log_b\left(\frac{1}{x}\right) = \log_b\left(x^{-1}\right) = -1 \cdot \log_b(x) = -\log_b(x).$$

-1.10. Using the logarithm base-change rule (**??**), we set all logarithms to same base ($b = 16$):

$$\begin{aligned} \log_2(x) &= \log_{16}(x) \cdot \log_2(16) = 4\log_{16}(x). \\ \log_4(x-1) &= \log_{16}(x-1) \cdot \log_4(16) = 2\log_{16}(x-1). \end{aligned}$$

Therefore, the expression is equivalent to

$$4\log_{16}(x) = 2\log_{16}(x-1) + \log_{16}(x^3).$$

We can now bring the coefficients 4 and 2 inside the logarithm expression:

$$\log_{16}(x^4) = \log_{16}([x-1]^2) + \log_{16}\left(x^3\right).$$

Using the logarithmic addition law, we get:

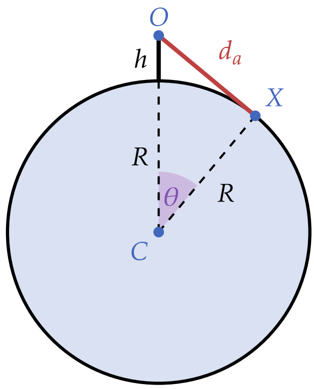
$$\log_{16}(x^4) = \log_{16}(x^3\,[x-1]^3).$$

We can now discard $\log_{16}$ on both sides, and we're left with a simple equation to solve:

$$x^4 = x^3\,(x-1)^2\,,$$

the solutions of which are $x_1 = 0$ and $x_{2,3} = \frac{3\pm\sqrt{5}}{2}$, of which only $x_2 = \frac{3+\sqrt{5}}{2}$ is valid: x_1 isn't valid since $x > 0$, and x_3 isn't valid since $x_3 - 1 < 0$, and thus $\log_b(x_3 - 1)$ isn't defined over the real numbers.

-1.11. (i) We start with drawing two radial lines from the center of the planet $\boldsymbol{C}$: one to the point on the surface where the observer meets the ground, and one to the horizon. The triangle $\triangle COX$ is a right triangle: the angle $\angle CXO = 90°$:



Using the Pythagorean theorem (with $R+h$ as the hypotenuse) we can calculate d_a:

$$d_a^2 + R^2 = (R+h)^2.$$

By expanding the right-hand side, cancelling R^2 and rearranging we get

$$d_a = \sqrt{2Rh + h^2}.$$

To get d_g we need to find the angle θ between the lines CX and CO. For that purpose we can use the law of sines (**??**):

$$\frac{d_a}{\sin(\theta)} = \frac{R+h}{\sin(90°)} = R+h.$$

(since $\sin(90°) = 1$)
Isolating $\sin(\theta)$ and subtituting the value of d_a as function of R and h yields:

$$\sin(\theta) = \frac{d_a}{R+h} = \frac{\sqrt{2Rh+h^2}}{R+h}.$$

Therefore

$$\theta = \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

When θ is given in radians, the length d_g then simply becomes

$$d_g = R\theta = R \cdot \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right).$$

(ii) For an average person on Earth ($h = 1.75\text{m},\ R = 6.371 \times 10^6\text{m}$), standing at sea level, the air-distance to the horizon is therefore

$$d_a = \sqrt{2Rh+h^2} = \sqrt{2 \cdot 6.371 \times 10^6 \cdot 1.75 + 1.75^2} \approx 4722\text{m} = 4.722\text{km}.$$

The ground-distance, on the other hand, is

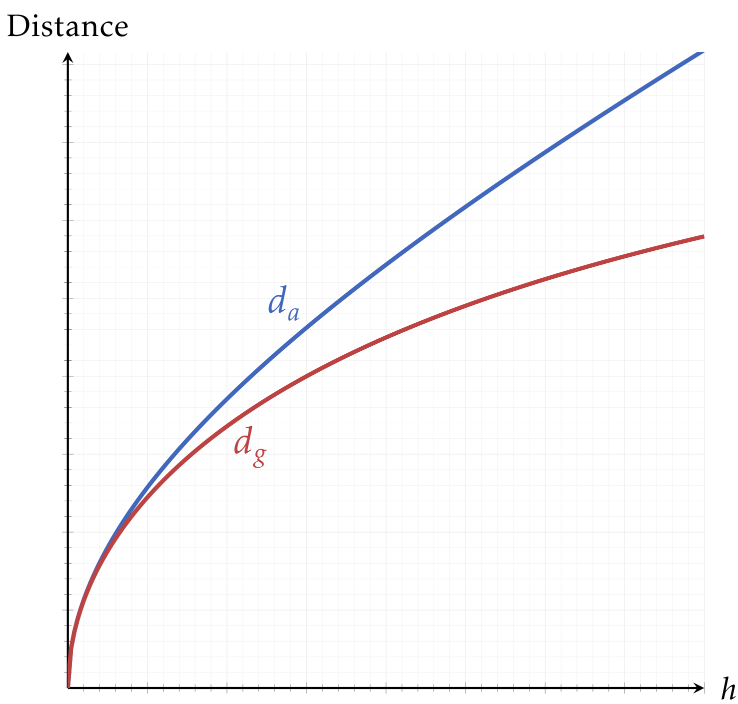
$$
\begin{aligned}
d_g &= R \cdot \arcsin\left(\frac{\sqrt{2Rh+h^2}}{R+h}\right) \\
&= 6.371 \times 10^6\text{m} \cdot \arcsin\left(\frac{4722\text{m}}{6.371 \times 10^6\text{m} + 1.75\text{m}}\right) \\
&\approx 4722\text{m}.
\end{aligned}
$$

(iii) horizon drop...

The following table sums up all the (approximate) air- and ground-distances to the horizon for each of the heights mentioned in the exercise:

Position	Height [m]	d_a[km]	d_g[km]
Person at sea level	1.75	4.7	4.7
Eiffel Tower observation	165	45.8	45.8
Average cruising altitude	9100	340.6	340.3
Internation Space Station	408000	2316	2221

Note that as the height h grows, the difference between d_a and d_g grows too. We can see this clearly when plotting $d_a(h)$ and $d_g(h)$ in the same graph (disregarding the units and values for now, since we are only interested in the qualitative behaviour of both distances):



For small values of h the two functions are very close to eachother, and as h grows they grow apart, with $d_a > d_g$.