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## -1.1 SYSTEMS OF LINEAR EQUATIONS

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### -1.1.1 Defintions

Everything we learned so far about vectors and matrices can be used to solve and characterise a family of equations known as **linear equations**. You're probably already very familiar with linear equations: they are equations in which the **variables** appear directly, without any power or other functions acting on them. For example, the simple equation

$$y = ax + b, \quad (-1.1.1)$$

where  $x, y$  are both variables and  $a, b$  are both constant real numbers is a linear equation. Equation -1.1.1 can be re-written as

$$ax - y + b = 0, \quad (-1.1.2)$$

where now  $a$  is the **coefficient** of the variable  $x$ , while the variable  $y$  has the coefficient  $-1$  and  $b$  is a so-called **free coefficient**. In general, a linear equation of two variables has the form

$$a_0 + a_x x + a_y y = 0, \quad (-1.1.3)$$

i.e. we changed the name of  $a$  to  $a_x$  and  $b$  to  $a_0$ , and gave  $y$  the coefficient  $a_y$ . We can also rename  $x$  and  $y$  to  $x_1$  and  $x_2$ , respectively, and name their coefficients accordingly:

$$a_0 + a_1 x_1 + a_2 x_2 = 0. \quad (-1.1.4)$$

The form shown in Equation -1.1.4 can be easily expanded into  $n$  variables:

$$a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_{n-1} x_{n-1} + a_n x_n = 0, \quad (-1.1.5)$$

where  $x_1, x_2, \dots, x_n$  are the variables of the equation, and  $a_0, a_1, \dots, a_n$  are its coefficients. We say that  $n$  is the **order** (also: **degree**) of the equation.

#### Note -1.1 Number set used for linear equations

As with other topics, in the context of this section both the variables and coefficients are all **real numbers**, however almost anything we discuss here can generally be applied to complex numbers or other structures.



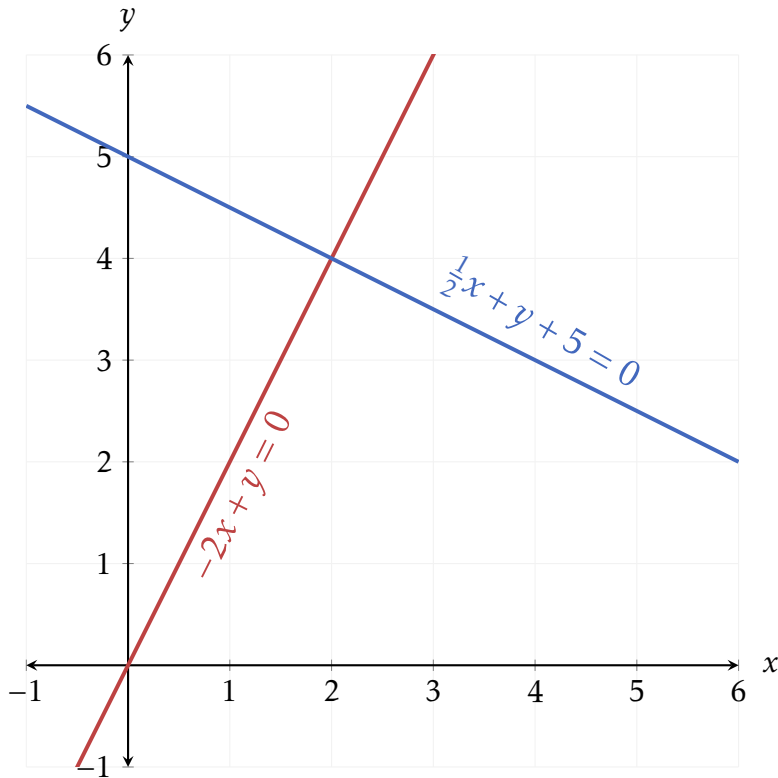
#### Example -1.1 Linear equations

The following is a linear equation of order 3, using the variables  $x, y, z$ :

$$3x + 2y - z + 4 = 0.$$

The coefficients of the equation are

$$a_0 = 4,$$



**Figure -1.1** Two linear equations represented as lines in  $\mathbb{R}^2$ . Note how in the red equation the free coefficient is zero, and so the line goes through the origin.

$$\begin{aligned} a_x &= a_1 = 3, \\ a_y &= a_2 = 2, \\ a_z &= a_3 = -1. \end{aligned}$$

Another linear equation of the same three variables is

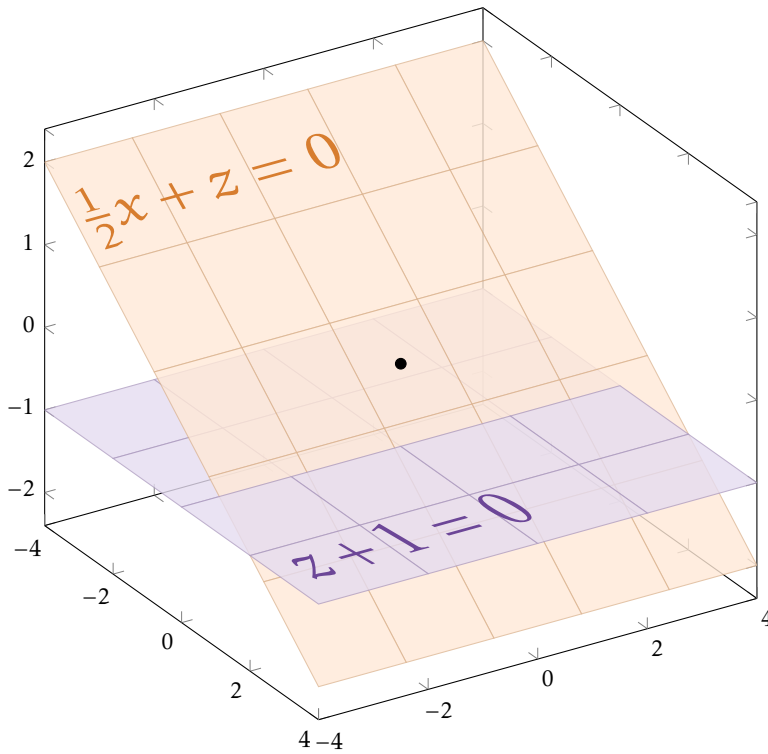
$$5x - 2y + 1 = 0.$$

In this case the coefficient  $a_z = a_3 = 0$ . Depending on the context, this equation can be considered as either an equation of order 3 or an equation of order 2.



In  $\mathbb{R}^2$  linear equations represent a line, which doesn't necessarily go through the origin (and thus isn't necessarily a subspace of  $\mathbb{R}^n$ ). For a line to go through the origin, the free coefficient  $a_0$  must equal zero (see [Figure -1.1](#)).

In  $\mathbb{R}^3$  linear equations represent planes. Much like with the lines in  $\mathbb{R}^2$ , these planes don't necessarily go through the origin. The trend continues with increasing dimensions: in  $\mathbb{R}^4$  linear equations represent 3-dimensional spaces, in  $\mathbb{R}^5$  linear equations



**Figure -1.2** Two intersecting planes in  $\mathbb{R}^3$  with their corresponding equations.

represent 4-dimensional spaces, etc. When the free coefficient is equal to zero, these spaces become subspaces of the respective  $\mathbb{R}^n$ .

### -1.1.2 Systems and matrix form

A **system of linear equations** is a set of linear equations using the same variables. For example, the three equations

$$\begin{cases} 2x - 5y + 4z + 2 = 0 \\ -3x - 2y + 1 = 0 \\ 5x + 4z - 3 = 0 \end{cases}$$

form together a system of 3 linear equations with 3 variables ( $x, y$  and  $z$ ). Systems of linear equations can be written together in matrix form: in the above example, the system can be represented as the equation

$$\begin{bmatrix} 2 & -5 & 4 \\ -3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

since performing the matrix-vector product and vector addition yields back the system of equations. We call the matrix the **coefficients matrix** of the equation.

### Note -1.2

In practice, many times the vector representing the free coefficients is moved to the right hand side of the equation. In the case of the above system this yields the simple equation

$$\begin{bmatrix} 2 & -5 & 4 \\ -3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}.$$



In the most general form, a system of  $m$  equations in  $n$  variables  $x_1, x_2, x_3, \dots, x_n$  can be represented as the product of an  $m \times n$  coefficient matrix and the variables vector, yielding the free-coefficient vector:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad (-1.1.6)$$

$m \times n$  coefficients
 $n$  variables
 $m$  free coefficients

which can be written succinctly as

$$A x = b. \quad (-1.1.7)$$

### -1.1.3 Solutions

A system of linear equations can have one or more **solutions**. A solution is a tuple

$$s = (s_1, s_2, \dots, s_n)$$

such that if we substitute each  $s_i$  into the respective variable  $x_i$  all equation become **true** statements.

#### Example -1.2 Solutions of a system of linear equations

The following linear system

$$\begin{cases} -4x + 2y = 0 \\ x - y + 3 = 0 \end{cases}$$

has the solution

$$s = (-1, 2).$$

Indeed if we substitute  $x = -1$  and  $y = 2$  into the system we get

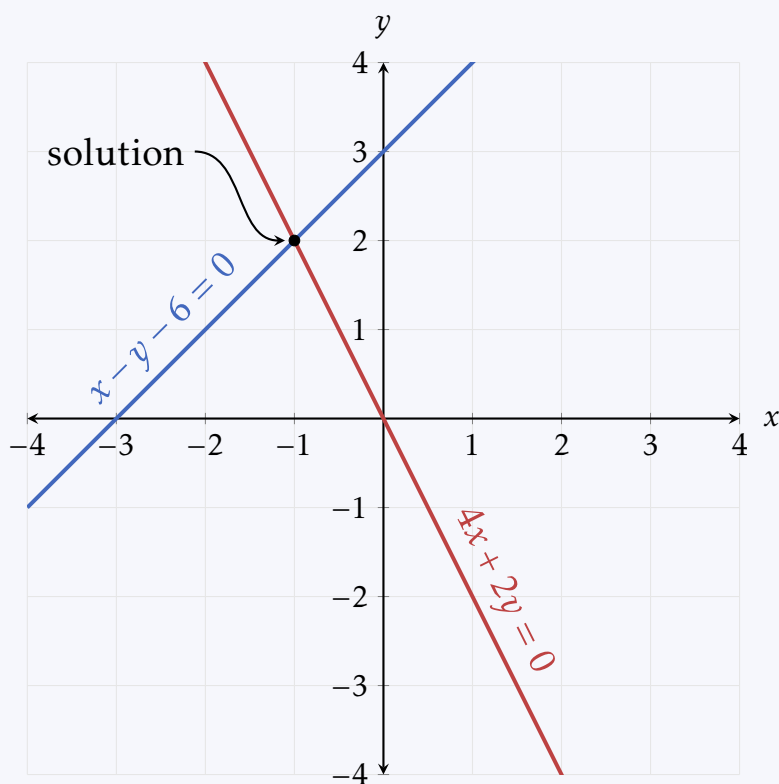
$$\begin{cases} -4 \cdot (-1) + 2 \cdot (2) = -4 + 4 = 0 \Rightarrow \text{true} \\ -1 - 2 + 3 = -3 + 3 = 0 \Rightarrow \text{true} \end{cases}$$



In the graphical representation of linear equations, the solutions of a system are the points where the respective graphs representing the equation (line, plane, etc.) intersect.

### Example -1.3 Solutions of a system of linear equations - graph

The linear system from the previous example can be represented by the following graph:



Not all systems have single solutions only, nor do all systems even have any solutions. For example, if we add to the system in [Example -1.3](#) the equation

$$x - 3y + 3 = 0,$$

the resulting system

$$\begin{cases} -4x + 2y = 0 \\ x - y + 3 = 0 \\ x - 3y + 3 = 0 \end{cases}$$

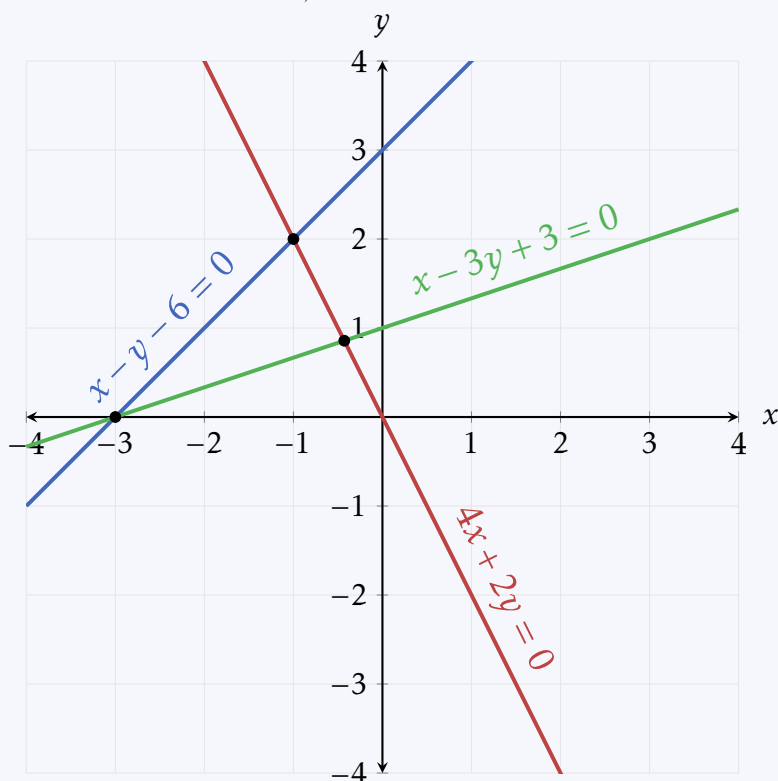
has no solutions (see [Example -1.4](#)).

#### Example -1.4 System with no solutions

When the system of linear equations from [Example -1.3](#) is supplemented with the equation

$$x - 3y + 3 = 0,$$

it has no solution. However, any two equations of the system do have solutions (represented below as black dots).



#### -1.1.4 Finding solutions