-1.1 SYSTEMS OF LINEAR EQUATIONS

-1.1.1 Defintions

Everything we learned so far about vectors and matrices can be used to solve and characterise a family of equations known as **linear equations**. You're probably already very familiar with linear equations: they are equations in which the **variables** appear directly, without any power or other functions acting on them. For example, the simple equation

$$y = ax + b, \tag{-1.1.1}$$

where x, y are both variables and a, b are both constant real numbers is a linear equation. Equation -1.1.1 can be re-written as

$$ax - y + b = 0, \tag{-1.1.2}$$

where now a is the **coefficient** of the variable x, while the variable y has the coefficient -1 and b is a so-called **free coefficient**. In general, a linear equation of two variables has the form

$$a_0 + a_x x + a_y y = 0, \tag{-1.1.3}$$

i.e. we changed the name of a to a_x and b to a_0, and gave y the coefficient a_y. We can also rename x and y to x_1 and x_2, respectively, and name their coefficients accordingly:

$$a_0 + a_1 x_1 + a_2 x_2 = 0. \tag{-1.1.4}$$

The form shown in Equation -1.1.4 can be easily expanded into n variables:

$$a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_{n-1} x_{n-1} + a_n x_n = 0, \tag{-1.1.5}$$

where $x_1, x_2, \ldots, x_n$ are the variables of the equation, and $a_0, a_1, \ldots, a_n$ are its coefficients. We say that n is the **order** (also: **degree**) of the equation.

Note -1.1 Number set used for linear equations

As with other topics, in the context of this section both the variables and coefficients are all **real numbers**, however almost anything we discuss here can genrally be applied to complex numbers or other structures.

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Example -1.1 Linear equations

The following is a linear equation of order 3, using the variables x, y, z:

$$3x + 2y - z + 4 = 0.$$

The coefficients of the equation are

$$a_0 = 4,$$

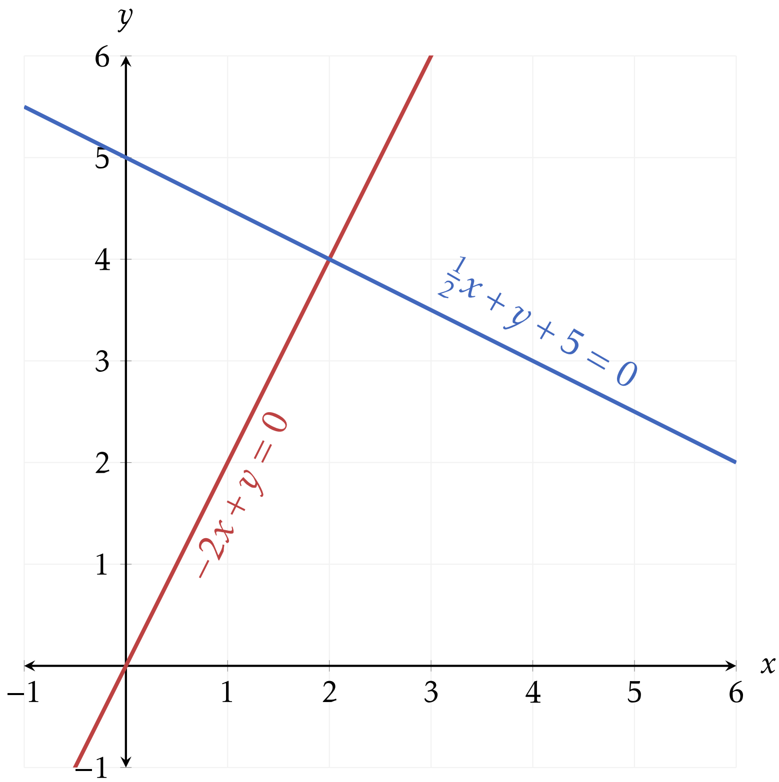

Figure -1.1 Two linear equations represented as lines in $\mathbb{R}^2$. Note how in the red equation the free coefficient is zero, and so the line goes through the origin.

$$\begin{aligned} a_x &= a_1 = 3, \\ a_y &= a_2 = 2, \\ a_z &= a_3 = -1. \end{aligned}$$

Another linear equation of the same three variables is

$$5x - 2y + 1 = 0.$$

In this case the coefficient $a_z = a_3 = 0$. Depending on the context, this equation can be considered as either an equation of order 3 or an equation of order 2.

In $\mathbb{R}^2$ linear equations represent a line, which doesn't necesserally go through the origin (and thus isn't necesserally a subspace of $\mathbb{R}^n$). For a line to go through the origin, the free coefficient a_0 must equal zero (see Figure -1.1).

In $\mathbb{R}^3$ linear equations represent planes. Much like with the lines in $\mathbb{R}^2$, these planes don't necesserally go through the origin. The trend continues with increasing dimensions: in $\mathbb{R}^4$ linear equations represent 3-dimensional spaces, in $\mathbb{R}^5$ linear equations

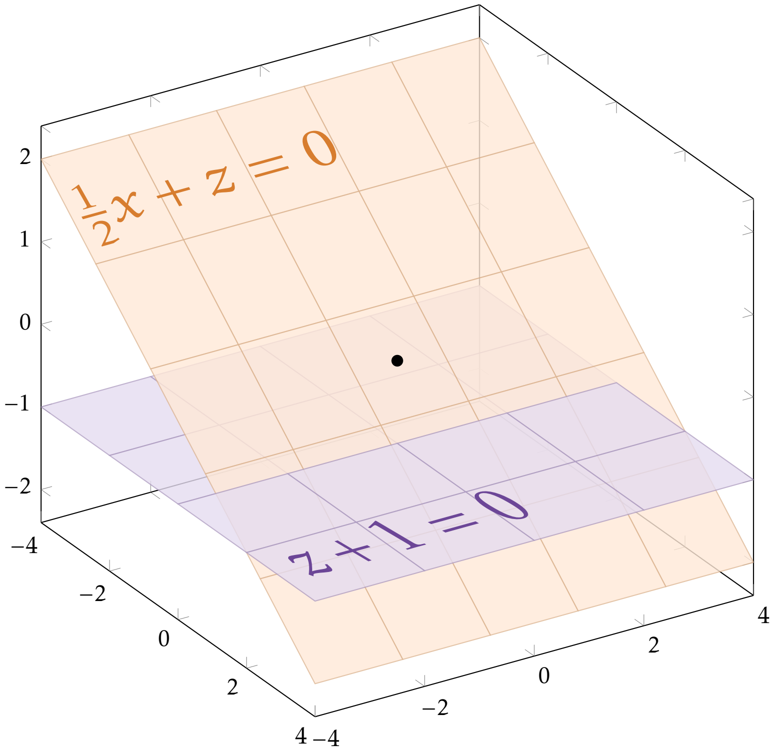

Figure -1.2 Two intersecting planes in $\mathbb{R}^3$ with their corresponding equations.

represent 4-dimensional spaces, etc. When the free coefficient is equal to zero, these spaces become subspaces of the respective $\mathbb{R}^n$.

-1.1.2 Systems and matrix form

A **system of linear equations** is a set of linear equations using the same variables. For example, the three equations

$$\begin{cases} 2x - 5y + 4z + 2 = 0 \\ -3x - 2y + 1 = 0 \\ 5x + 4z + -3 = 0 \end{cases}$$

form together a system of 3 linear equations with 3 variables (x, y and z). Systems of linear equations can be written together in matrix form: in the above example, the system can be represented as the equation

$$\begin{bmatrix} 2 & -5 & 4 \\ -3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

since performing the matrix-vector product and vector addition yields back the system of equations. We call the matrix the **coefficients matrix** of the equation.

Note -1.2

In practice, many times the vector representing the free coefficients is moved to the right hand side of the equation. In the case of the above system this yields the simple equation

$$\begin{bmatrix} 2 & -5 & 4 \\ -3 & -2 & 0 \\ 5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}.$$

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In the most general form, a system of m equations in n variavles $x_1, x_2, x_3, \dots, x_n$ can be represented as the product of an $m \times n$ coefficient matrix and the variables vector, yielding the free-coefficient vector:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m \times n \text{ coefficients}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{n \text{ variables}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{m \text{ free coefficients}}, \tag{-1.1.6}$$

which can be written succinctly as

$$A\,x = b. \tag{-1.1.7}$$

-1.1.3 Solutions

A system of linear equations can have one or more **solutions**. A solution is a tuple

$$s = (s_1, s_2, \dots, s_n)$$

such that if we substitute each s_i into the respective variable x_i all equation become **true** statements.

Example -1.2 Solutions of a system of linear equations

The following linear system

$$\begin{cases} -4x + 2y = 0 \\ x - y + 3 = \end{cases}$$

has the solution

$$s = (-1, 2).$$

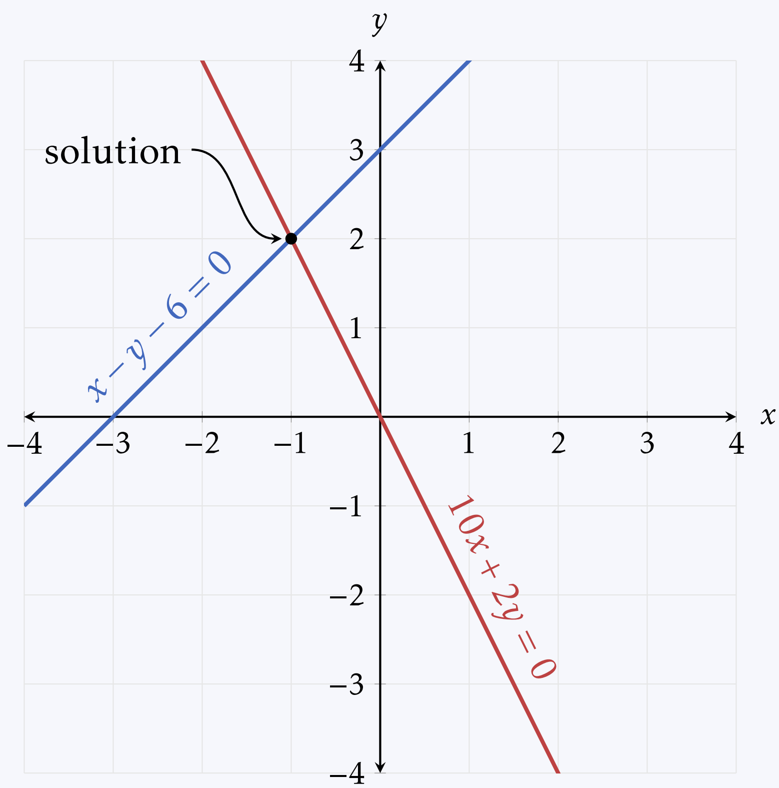
Indeed if we substitute $x = -1$ and $y = 2$ into the system we get

$$\begin{cases} -4\cdot(-1)+2\cdot(2) = -4+4 = 0 \;\Rightarrow\; \textbf{true} \\ -1-2+3 = -3+3 = 0 \;\Rightarrow\; \textbf{true} \end{cases}$$

In the graphical representation of linear equations, the solutions of a system are the points where the respective graph representing the equation (line, plane, etc.) intersect.

Example -1.3 Solutions of a system of linear equations - graph

The linear system from the previous example can be represented by the following graph:



-1.1.4 Finding solutions