MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

Written, illustrated and typeset (mostly) by

PELEG BAR SAPIR

with contributions from others

$$a^b = \mathrm{e}^{b\log(a)}$$

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$T\left(\alpha\vec{u}+\beta\vec{v}\right) = \alpha T\left(\vec{u}\right)+\beta T\left(\vec{v}\right)$$

$$A = Q\Lambda Q^{-1}$$

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t$$

$$R(\theta) = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

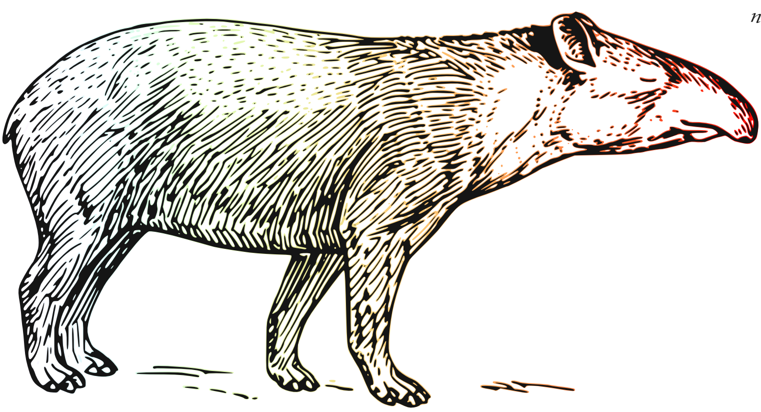
$$\langle\hat{e}_i,\hat{e}_j\rangle = \delta_{ij}$$

$$\vec{v} = \sum_{i=1}^{n}\alpha_i\hat{e}_i$$

$$\mathrm{e}^{\pi\mathrm{i}}+1 = 0$$

$$\int_a^b f(x)\,\mathrm{d}x = F(b)-F(a)$$

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$



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! **To be written/to do**: Rights, lefts, etc. will be written here in the future !

HERE BE TABLE

CHAPTER 0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

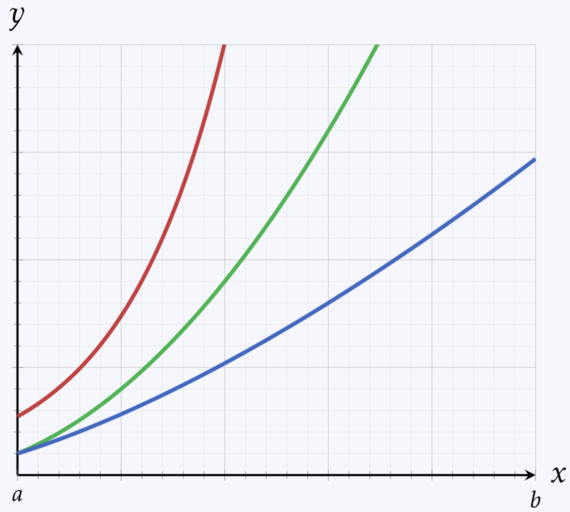
It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

0.1 DERIVATIVES

One of the most important tool in analyzing a function $f : \mathbb{R} \to \mathbb{R}$ is the ability to quantitatively describe the way it behaves as we change its argument x. At any given point a function can either increase in its value, decrease in its value, or stay constant. We would like to develop a method that is able to tell us exactly how such function changes at any given point.

Example 0.1 Quantitative measure of change

Compare the following three functions on the domain $x \in [a, b]$:



While all three functions are increasing on $[a, b]$ it is clear that the rate of increase between the functions is different: the red function increases faster than the green one, which in turn increases faster than the blue one. In fact, even within each function the increase is not uniform: the more x increases so does the rate of increase of the function.

To start developing a method to quantitatively measure the rate of change in a real function, we can first notice a property of some functions: if we zoom in on some point of the function $\boldsymbol{p} = \left(a, f(a)\right)$ we would see that the more we zoom in the more the the function behaves like a straight line around $\boldsymbol{p}$ (Figure 0.1). We can define the limit of this zoom (i.e. when the zoom factor goes to infinity) as the slope of the function at the $\boldsymbol{p}$.

How can we quantify this idea? Let us consider some real function f and a point $\boldsymbol{p}_0 = \left(x_0, f(x_0)\right)$ on the function. We can then define another point to the right of x_0: $\boldsymbol{p}_1 = \left(x_1, f(x_1)\right)$. Since x_1 is to the right of x_0 we can write it as $x_1 = x_0 + \Delta x$, where $\Delta x > 0$. We then connect the two points with a line (Figure 0.2). The slope of this line can then be

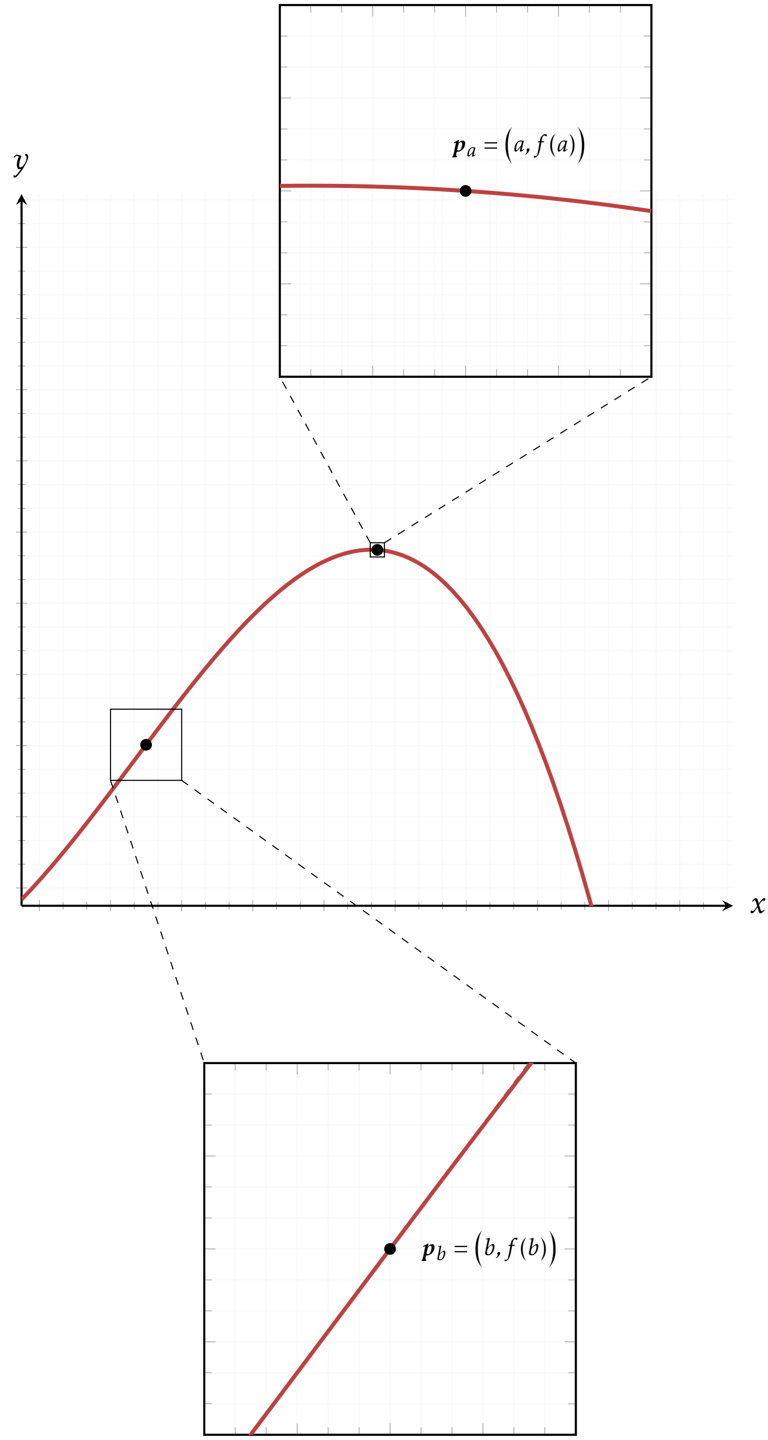


Figure 0.1 Zooming in on a real function f at two points: $\boldsymbol{p}_a = \big(a, f(a)\big)$ (upper right) and $\boldsymbol{p}_b = \big(b, f(b)\big)$ (bottom right). Note how around each of the points, the function looks somewhat linear: this is more pronounced around $\boldsymbol{p}_b$ where the function looks linear in the entire zoomed-in area, while near $\boldsymbol{p}_a$ it looks linear only near the point itself even though the zoom factor is higher.

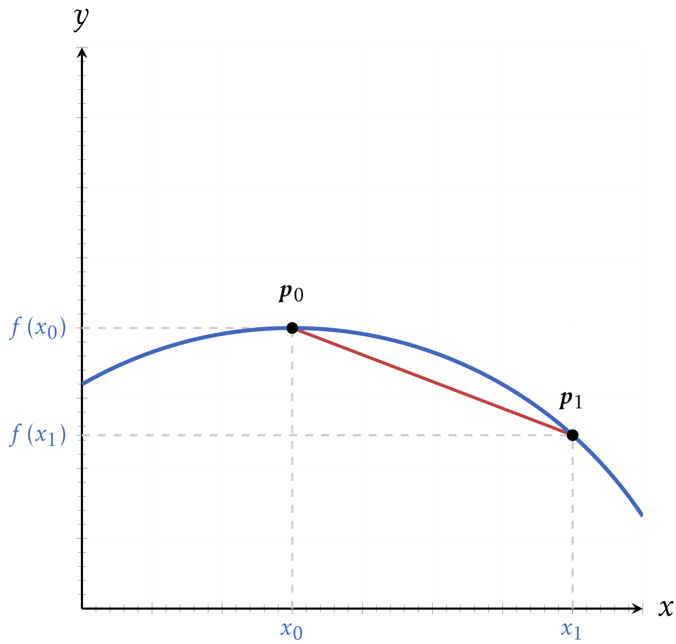


Figure 0.2 Text.

calculated using **??**:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{x_0 + \Delta x - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{0.1.1}$$

We can then take the limit of Equation 0.1.1 as $\Delta x \to 0$ (Figure 0.3):

$$M = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{0.1.2}$$

This limit is defined as **the derivative** of f at the point $x = x_0$, and it tells us, quantitatively, how f locally behaves at x_0, i.e. how much does it increase, decrease or stay the same around x_0.

Example 0.2 Validation of the derivative using a linear function

Given a linear function $f(x) = mx + b$, we expect that the derivative of f at any point x_0 would equal m, since the entire function is a line connecting all the points on the function itself. Let us check that:

$$\begin{aligned}
M &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{m(x_0 + \Delta x) + \cancel{b} - \left(mx_0 + \cancel{b}\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{mx_0} + m\Delta x - \cancel{mx_0}}{\Delta x}
\end{aligned}$$

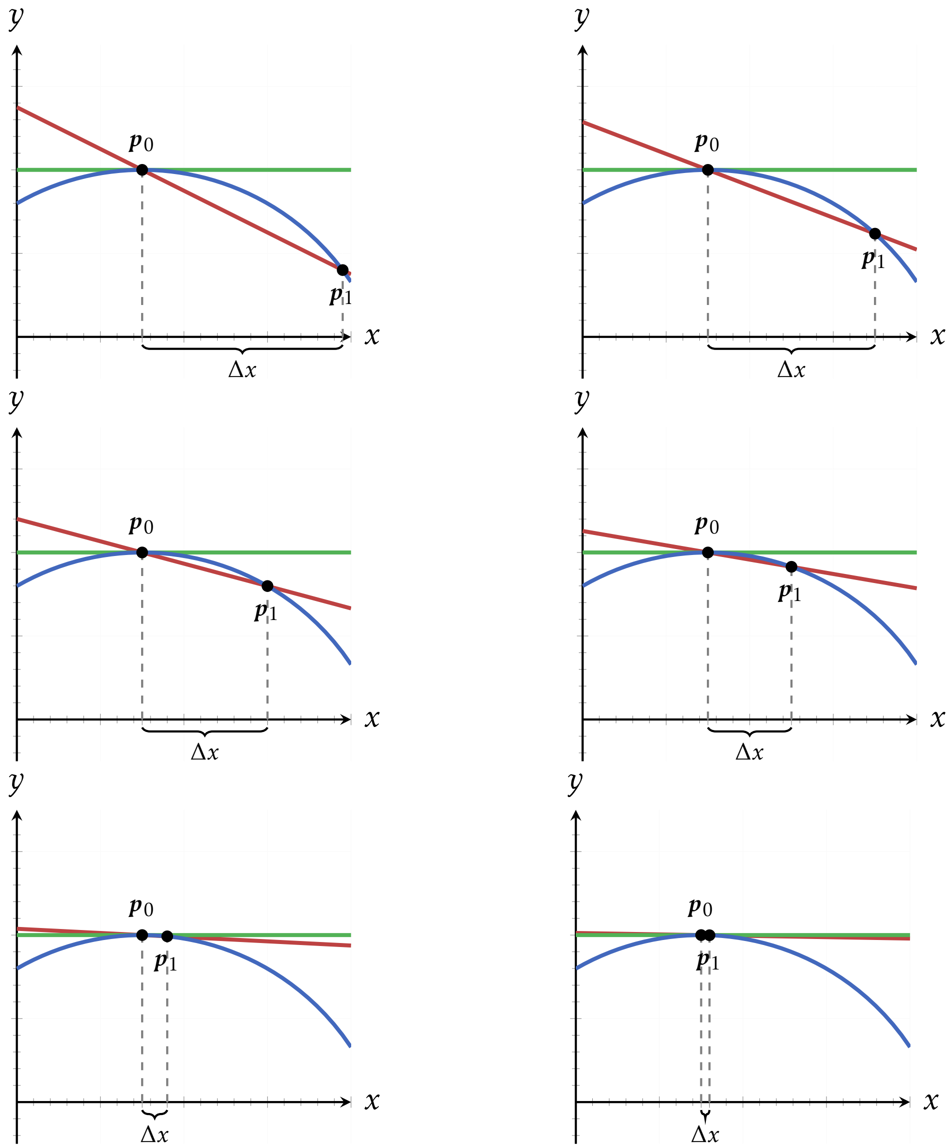


Figure 0.3 As $\Delta x \to 0$, $\boldsymbol{p}_1$ approaches $\boldsymbol{p}_0$ and the slope of the red line connecting the two points approches the slope M of the green line at $\boldsymbol{p}_0$.

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{m\cancel{\Delta x}}{\cancel{\Delta x}} \\
&= m.
\end{aligned}
$$

Example 0.3 Derivative of x^2

Unlike for a linear function, we shouldn't expect the derivate of $f(x) = x^2$ to be constant at any point. However, we can easily calculate what the derivative would be at some point x_0:

$$
\begin{aligned}
M &= \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{x_0^2} + 2x_0\Delta x - (\Delta x)^2 - \cancel{x_0^2}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x_0 - \Delta x)}{\cancel{\Delta x}} \\
&= \lim_{\Delta x \to 0} 2x_0 - \Delta x \\
&= 2x_0.
\end{aligned}
$$

I.e. we see that any point x_0 the derivative of $f(x) = x^2$ is simply $2x_0$.
For example, at $x_0 = 3$ the derivative is $M = 6$, and at $x_0 = 0$ the derivative is $M = 0$.

Up until now we have regarded the derivative as a property of some point on some function f. However, since we can calculate the derivative at each point of the function[1], we can collect all these points together to form a new function, which we call the **derivative** of f and denote as f' (read: "f-prime").

In Example 0.2 we saw that the derivative of a linear function at any point gives the same value m (namely the slope of the linear function). Therefore, this derivative is itself a *constant* function $f'(x) = m$. When we calculated the derivative of $f(x) = x^2$ (Example 0.3), we found that it depends on the point where it was calculated, using the relation $f'(x) = 2x$, which is a linear function with slope 2 that goes through the origin.

Let us now calculate the derivative of some common functions.

Example 0.4 Derivative of ax^n

The derivative of the function $f(x) = ax^n$ (where $a \in \mathbb{R}$ is a constant) is (recall the binomial expansion, **??**):

$$
f'(x) = \lim_{\Delta x \to 0} \frac{a(x + \Delta x)^n - ax^n}{\Delta x}
$$

[1] except for some points which we will discuss later.

$$= \lim_{\Delta x \to 0} \frac{a\left[\cancel{x^n} + nx^{n-1}\Delta x + \binom{2}{n}x^{n-2}(\Delta x)^2 + \cdots + nx(\Delta x)^{n-1} + (\Delta x)^n\right] - \cancel{ax^n}}{\Delta x}.$$

We can take Δx out of the numerator and cancel it out with the Δx in the denominator:

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{a\cancel{\Delta x}\left[nx^{n-1} + \binom{2}{n}x^{n-2}\Delta x + \cdots + nx(\Delta x)^{n-2} + (\Delta x)^{n-1}\right]}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} a\left[nx^{n-1} + \binom{2}{n}x^{n-2}\Delta x + \cdots + nx(\Delta x)^{n-2} + (\Delta x)^{n-1}\right]. \end{aligned}$$

Since all expressions except nx^{n-1} have some power of Δx in them, in the limit $\Delta x \to 0$ they all vanish, leaving us with

$$f'(x) = anx^{n-1}.$$

This derivative is commonly described as the power of x being reduced by 1 and the expression gaining a factor of n (i.e. the power before reducing it).

Example 0.5 Derivative of a^x

! **To be written/to do**: Note ! :

Use the fact that $\lim\limits_{h\to 0} \frac{a^h-1}{h} = \log(a)$. Therefore we also get that $\lim\limits_{h\to 0} \frac{e^h-1}{h} = \log(e) = 1$.

Example 0.6 Derivative of $\sin(x)$

Calculating the derivative of $\sin(x)$:

$$\begin{aligned} \sin'(x) &= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x}. \end{aligned}$$

We can separate the three terms into three limits:

$$\sin'(x) = \lim_{\Delta x \to 0} \frac{\sin(x)\left[\cos(\Delta x) - 1\right]}{\Delta x} + \lim_{\Delta x \to 0} \frac{\cos(x)\sin(\Delta x)}{\Delta x}.$$

The second limit equals $\cos(x)$, since $\lim\limits_{c\to 0} \frac{\sin(c)}{c} = 1$ (**??**). Since $\sin(x)$ does not change as we decrease Δx, we can regard it as a constant and take it out of the limit:

$$\sin'(x) = \sin(x) \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} + \cos(x).$$

Using the double angle identity (**??**) on $\cos(\Delta x)$ we get that

$$\cos(\Delta x) = 1 - 2\sin^2\left(\frac{\Delta x}{2}\right),$$

and by plugging this back into the derivative calculation we get:

$$\begin{aligned}
\sin'(x) &= \lim_{\Delta x \to 0} \frac{-2\sin^2\left(\frac{\Delta x}{2}\right)}{\Delta x} + \cos(x) \\
&= \lim_{\Delta x \to 0} \frac{-\cancel{\frac{2}{2}}\sin^2\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} + \cos(x) \\
&= \cos(x),
\end{aligned}$$

since $\lim\limits_{h \to 0} \frac{\sin^2(h)}{h} = 0$ (**??**).

! **To be written/to do**: in the limit section show and prove this limit !

Example 0.7 Derivatie of $\sqrt{x}$

The derivative of the function $f(x) = \sqrt{x}$ is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}.$$

We can multiply the numerator and denominator each by $\sqrt{x + \Delta x} + \sqrt{x}$. This would allow us to use the relation $(a - b)(a + b) = a^2 - b^2$:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{x} + \cancel{\Delta x} - \cancel{x}}{\cancel{\Delta x}\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$