

MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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$$\begin{aligned}a^b &= e^{b \log(a)} & (a+b)^n &= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \\ \binom{n}{k} &= \frac{n!}{k!(n-k)!} & T(\alpha \vec{u} + \beta \vec{v}) &= \alpha T(\vec{u}) + \beta T(\vec{v}) \\ R(\theta) &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} & A &= Q \Lambda Q^{-1} \\ e^{\pi i} + 1 &= 0 & \frac{df}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ \langle \hat{e}_i, \hat{e}_j \rangle &= \delta_{ij} & \Gamma(z) &= \int_0^\infty t^{z-1} e^{-t} dt \\ \int_a^b f(x) dx &= F(b) - F(a) & \vec{v} &= \sum_{i=1}^n \alpha_i \hat{e}_i \\ \cos(x) &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} x^{2n}\end{aligned}$$



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CHAPTER

0



INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.



CHAPTER

1



REAL CALCULUS IN 1D

1.1 SEQUENCES AND SERIES

A **sequence** is an indexed collection of **elements**. By *indexed* we mean that the order of the elements in a sequence matters (unlike with sets): changing the order of any element changes the sequence as a whole. The following are some examples of sequences composed of real numbers:

- $1, -3, 0, -7, 2, 1.5, 4, 0, 1, -0.35, \sqrt{2}.$
- $0, 1, 2, 1, 1, -1, 0.$
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$
- $0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, \dots$

The examples above present two more properties of sequences:

- Elements may repeat (unlike in the case of sets), and
- sequences can be either **finite** (as in the first two examples), or **infinite** (as in the latter two examples).

The number of elements in a sequence is called its **length**. In the case of infinite sequences we say that their length equals ∞ (infinity). The elements of a sequence a are usually indexed using a subscript, such that a_1 is the first element in the sequence, a_2 is the second element in the sequence, etc. - and generally a_i is the i -th element in the sequence, where $i \in \mathbb{N}$.

We can therefore define a sequence somewhat more formally as a function from a subset of the natural numbers to the real numbers:

$$a : N \rightarrow \mathbb{R}, \quad (1.1.1)$$

where $N \subseteq \mathbb{N}$.

Example 1.1 Sequences as functions

The following 9-element sequence a

$$\begin{array}{ccccccccccccc} 3, & 4, & \frac{1}{2}, & 0, & 2, & 6, & -\frac{2}{3}, & 0, & -1. \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ a(1) & a(2) & a(3) & a(4) & a(5) & a(6) & a(7) & a(8) & a(9) \end{array}$$

can be viewed as a function

$$a : \{1, 2, \dots, 9\} \rightarrow \mathbb{R},$$

or more precisely as a function

$$a : \{1, 2, \dots, 9\} \rightarrow \left\{ -1, -\frac{2}{3}, 0, \frac{1}{2}, 2, 3, 4, 6 \right\}.$$

The follow infinite sequence b

$$\begin{array}{ccccccccccc} 1, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \frac{1}{6}, & \frac{1}{7}, & \dots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ b(1) & b(2) & b(3) & b(4) & b(5) & b(6) & b(7) & \end{array}$$

can be viewed as a function

$$b : \mathbb{N} \rightarrow (0, 1].$$



Since sequences can be viewed as functions, they can be defined using formulas: for

example, the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

can be defined using the simple formula

$$a_n = \frac{1}{n}.$$

Example 1.2 Some sequences defined using formulas

$$(-1)^n \Rightarrow -1, 1, -1, 1, -1, 1, -1, 1, \dots$$

$$3n + 4 \Rightarrow 7, 10, 13, 16, 19, 22, \dots$$

$$(n + 1)^2 \Rightarrow 4, 9, 16, 25, 36, 49, \dots$$

$$\begin{cases} 2n+1 & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases} \Rightarrow 3, 1, 7, 3, 11, 5, 15, 7, \dots$$



Sequences can also be defined using **recursion**, where the value of an element is defined using previous values and a **starting value**. For example:

$$a_n = a_{n-1}^2 - 2,$$

with the starting value $a_1 = 3$. We then get that

$$a_2 = a_1^2 - 2 = 3^2 - 2 = 7,$$

and thus

$$a_3 = a_2^2 - 2 = 7^2 - 2 = 47,$$

etc.

Example 1.3 The Fibonacci sequence

The **Fibonacci sequences** is a well-known sequence defined using the following recursive rule:

$$F_n = F_{n-1} + F_{n-2},$$

with $F_1 = F_2 = 1$. The first few elements of the sequence are therefore

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \dots$$

See [Figure 1.1](#) for a graphical representation of the Fibonacci sequence.



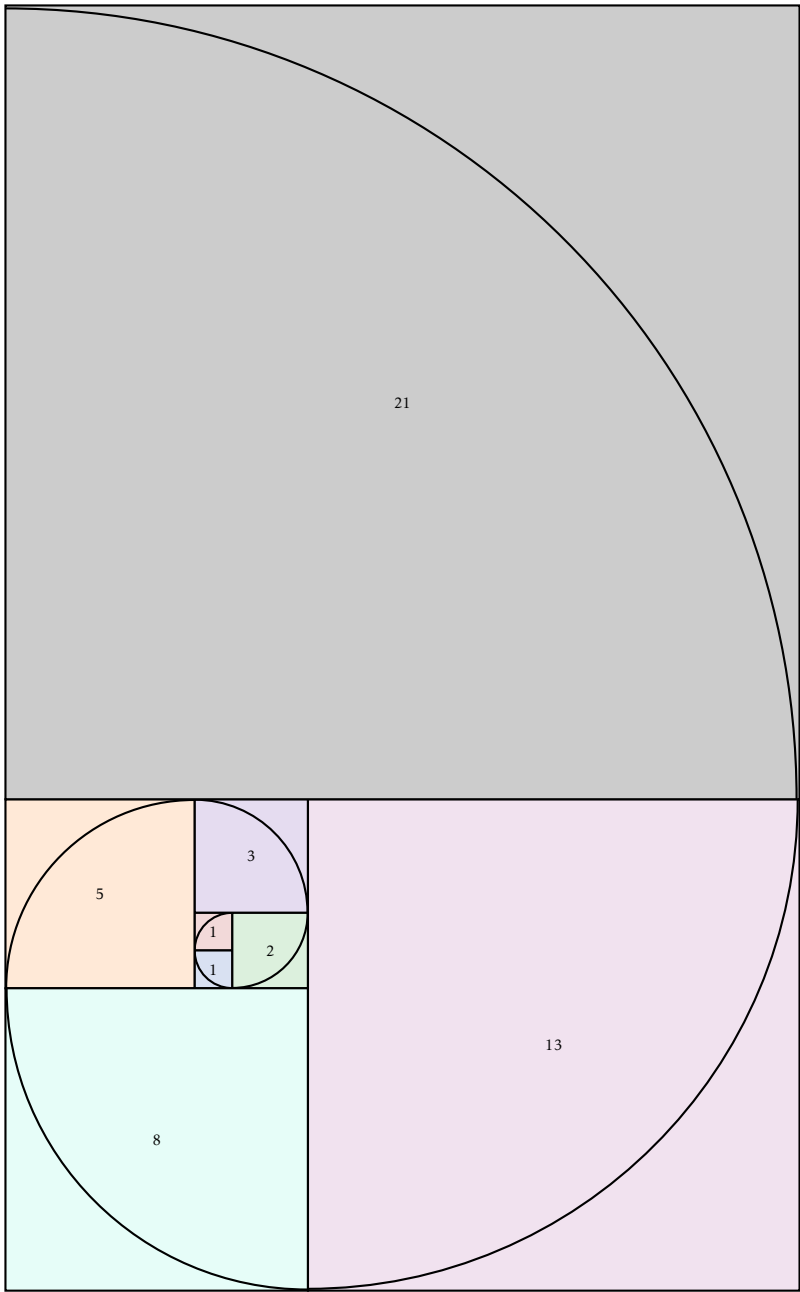


Figure 1.1 A graphical representation of the Fibonacci sequence: two squares of side 1 are placed adjacent to each other on the plane. In each subsequent step a new square is placed such that its side is equal to the combined sides of the previous two squares. This way, the side of each square in the sequence follows the Fibonacci sequence. In each square we draw a quarter circle centered on one of the vertices, such that we get the famous **golden ratio** helix.