CHAPTER 0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explainations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is adviced to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concept presented here. !

0.1 MATHEMATICAL SYMBOLS AND SETS

LOGICAL STATEMENTS AND THEIR TRUTH VALUE

We start our discussion with the simplest mathematical concept: a **proposition**. A proposition is simply a statement that might be either **true** or **false**.

Example 0.1 Truth of propositions

- $3 > 1$ (**true**)
- $-2 = 5 - 7$ (**true**)
- $7 < 5$ (**false**)
- The radius of the earth is bigger than that of the moon. (**true**)
- The word 'House' starts with the letter 'G'. (**false**)

We can group together propositions using **logical operators**. Two of the most common logical operators are **AND** and **OR**.

The **AND** operator returns a **true** statement only if **both** the statements it groups are themselves **true**, otherwise it returns **false**.

Example 0.2 The AND operator

- $2+4=6$ is **true**, $4-2=2$ is **true**. $(2+4=6$ **AND** $4-2=2)$ is therefore **true**.
- $2+4=6$ is **true**, $2>6$ is **false**. $(2+4=6$ **AND** $2>6)$ is therefore **false**.
- $\frac{10}{2}=1$ is **false**, $2^4=16$ is **true**. $\left(\frac{10}{2}=1 \textbf{ AND } 2^4=16\right)$ is therefore **false**.
- $7<5$ is **false**, $10+2=13$ is **false**. $(7<5$ **AND** $10+2=13)$ is therefore **false**.

The **OR** operator returns **true** if **at least** one of the statements it groups is true.

Example 0.3 The OR operator

- $2+4=6$ is **true**, $4-2=2$ is **true**. $(2+4=6$ **OR** $4-2=2)$ is therefore **true**.
- $2+4=6$ is **true**, $2>6$ is **false**. $(2+4=6$ **OR** $2>6)$ is therefore **true**.
- $\frac{10}{2}=1$ is **false**, $2^4=16$ is **true**. $\left(\frac{10}{2}=1 \textbf{ OR } 2^4=16\right)$ is therefore **true**.
- $7<5$ is **false**, $10+2=13$ is **false**. $(7<5$ **OR** $10+2=13)$ is therefore **false**.

The behaviour of both operators can be summarized using a **truth table** (see Table 0.1 below).

Table 0.1 The truth table for the operators **AND** and **OR**.

A	B	A AND B	A OR B
true	**true**	**true**	**true**
true	**false**	**false**	**true**
false	**true**	**false**	**true**
false	**false**	**false**	**false**

When writing, it is convinient to use **notations** to represent operators: the **AND** operator is denoted by $\wedge$, while the **OR** operator is denoted by $\vee$.

Example 0.4 Using the notations for AND and OR

$$\underbrace{(2+2=5)}_{\textbf{false}} \wedge \underbrace{(1-1=0)}_{\textbf{true}} \Rightarrow \textbf{false}$$

$$\underbrace{(2+2=5)}_{\textbf{false}} \vee \underbrace{(1-1=0)}_{\textbf{true}} \Rightarrow \textbf{true}$$

COMMON MATHEMATICAL NOTATIONS

Several more common mathematical notations are given in **Table 0.2** .

Table 0.2 Common Mathematical NotationsUsed in this Book.

Symbol	In words
$\neg a$	**not** a
$a \wedge b$	a **and** b
$a \vee b$	a **or** b
$a \Rightarrow b$	a **implies** b
$a \Leftrightarrow b$	a **is equivalent to** b
$\forall x$	**For all** x (...)
$\exists x$	**There exists** x **such that** (...)
$a := b$	a **is defined to be** b

The notation $\Rightarrow$ need a bit of clarification: implication means that we can directly derive a proposition from another proposition. For example, if $x = 3$ then $x > 2$. The opposite implication can be a **false** statemt, i.e. for the example above $x > 2$ does not imply $x = 3$ (denoted as $x > 2 \not\Rightarrow x = 3$). Sometimes implication is expressed by using the word *if*: in the above example $x > 2$ if $x = 3$, but the other way around is not **true**.

We say that two propositions are **equivalent** when they imply each other. For example: $x = 2$ implies that $\frac{x}{2} = 1$, while $\frac{x}{2} = 1$ implies that $x = 2$. We can write this as

$$\frac{x}{2} = 1 \Leftrightarrow x = 2.$$

Instead of the word *equivalent*, the phrase *if and only if* (sometimes shortened to **iff**) is commonly used, e.g.

$$x = 2 \text{ iff } \frac{x}{2} = 1.$$

SETS AND SUBSETS

The concept of **sets** is perhaps one of the most basic ideas in modern mathematics. Much of the material convered in this book will be built upon sets and their properties. However, as with the rest of the material presented here - our description of sets will not be thorough nor percise.

For our purposes, a set is a collection of **elements**. These elements can be any concept - be it physical (a chair, a bicycle, a tapir) or abstract (a number, an idea). However, we will consider only sets comprised of numbers. Sets can have finite of infinite number of elements in them.

We denote sets by using curly brackets, and if the number of elements in them is not too big - we display the elements, separated by commas, inside the brackets. In other cases we can express the sets as a sentence or a mathematical proposition.

Example 0.5 Simple sets

$$\{1,2,3,4\} \qquad \left\{-4, \frac{3}{7}, 0, \pi, 0.13, -2.5, \frac{e}{3}, 2^{-\pi}\right\} \qquad \{\text{all even numbers}\}$$

Sets have two important properties:

1. Elements in a set do not repeat.
2. The order of elements in a set does not matter.

Example 0.6 Important set properties

Examples demonstrating the two afforementioned important properties of sets:

1. The following is not a proper set:

$$\{1,1,0,1,0,0,-1,0,0,-1,-1,1\}$$

2. The following sets are all identical:

$$\{1,2,3,4\} \qquad \{1,3,2,4\} \qquad \{3,4,1,2\} \qquad \{1,3,2,4\} \qquad \{4,3,2,1\}$$

Sets can be denoted using **conditions**, with the symbol | representing the phrase ”such that”.

Example 0.7 Defining a set using a condition

he following set contains all the odd whole numbers between 0 and 10, including both:

$$\{0 < x < 10 \mid x \text{ is an odd number}\}.$$

The definition of this set can be read as

all numbers x that are bigger than 0 *and are smaller than* 10*, such that x is odd.*

(note that the requirement of x to be an odd number means that it is neccessarily a whole number as well)

This set can be written explicitly as

$$\{1, 3, 5, 7, 9\}.$$

Sets are usually denoted with an uppercase latin letter $(A, B, C, \dots)$, while their elements are denoted as lowercase letters $(a, b, \alpha, \phi, \dots)$. When we want to denote that an element belongs to a set we use the following symbol: $\in$. Conversly, $\notin$ is used to denote that an element *does not* belong to a set.

Example 0.8 Elements in sets

For the two sets

$$A = \{1, 2, 5, 7\}, \quad B = \{\text{even numbers}\},$$

all the following propostions are **true**:

$$\begin{aligned} 1 \in A, \quad 2 \in A, \quad 5 \in A, \quad 7 \in A, \\ 2 \in B, \quad 1 \notin B, \quad 5 \notin B, \quad 7 \notin B. \end{aligned}$$

The number of elements in a set, also called its **cardinality** is denoted using two vertical bars (similar to the way absolute values are denoted).

Example 0.9 Cardinality

For $S = \left\{-3, 0, -2, 7, 1, \frac{1}{2}, 5\right\}$, $|S| = 7$.

An important special set is the **empty set**, which is the set containing no elements. It is denoted by $\emptyset$, and has the unique property that $|\emptyset| = 0$.

INTERSECTION, UNION, DIFFERENCE AND COMPLEMENT SETS

Two sets are equal if they both contain the exact same elements and only these elements, i.e.

$$A = B \iff x \in A \Leftrightarrow x \in B. \tag{0.1.1}$$

This proposition reads 'The sets A and B are equal *if and only if* any element x in A is also in B, and any element x in B is also in A'. When all the elements of a set B are also elements of another set A, we say that B is a **subset** of A, and we denote that as $B \subset A$. In mathematical notation, we write

$$B \subset A \Leftrightarrow \forall x \in B, x \in A. \tag{0.1.2}$$

i.e. B is a subset of A **iff** the following is true: any element in B is also an element in A.

Note 0.2 (not so) Surprising properties of subsets

The definion of a subset (Equation 0.1.2) gives rise to two interesting properties:

- The empty set $\emptyset$ is a subset of any set.
- Any set is a subset of itself.

!

Note 0.3 The uniqeness of $\emptyset$

There is only a single empty set, as any set that has no elements is equivalent to any other set with no elements (i.e. they have the same elements). Due to the way subsets are defined, the empty set is a subset of any set (including itself!).

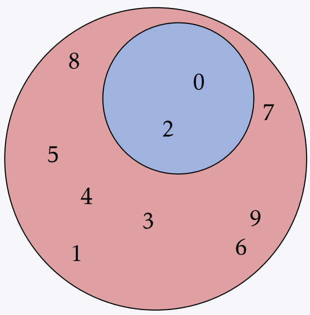
!

Of course, since we have a definition for a subset, the opposite concept also exists: if B is a subset of A, then we say that A is a **superset** of B.

A very useful way of illustrating the relationship beyween two or more sets is by using **Venn diagrams**, where sets are represented by circles (or other 2D shapes).

Example 0.10 Subsets and Venn diagrams

A Venn diagram depicting the set $B = \{0, 2\}$ as a subset of $A = \{0, 1, 2, 3, 4, \ldots, 9\}$:



If for two sets A, B both $A \subset B$ and $B \subset A$, then $A = B$. We can write this fact as a methematical proposition:

$$(A \subset B) \wedge (B \subset A) \Leftrightarrow A = B. \tag{0.1.3}$$

The **intersection** of two sets A and B, denoted $A \cup B$, is the set of all elements x such that

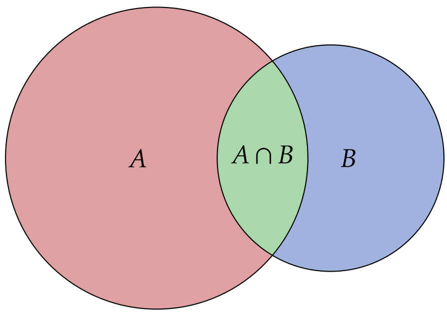
$x \in A$ **AND** $x \in B$:

$$A \cup B = \{x \mid x \in A \land x \in B\}. \tag{0.1.4}$$

Example 0.11 Intersection of sets

The intersection of the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$ is the set $A \cup B = \{3, 4\}$.
The intersection of the sets $C = \{0, 1, 2, 6, 7\}$ and $D = \{3, 9, -4, 5\}$ is the empty set $\emptyset$, since no element is in both sets.

The following Venn diagram depicts the intersection of two sets (the green area):



Note 0.4 Disjoint sets

When the intersection of two sets is the empty set, we say that the set is **disjoint**.

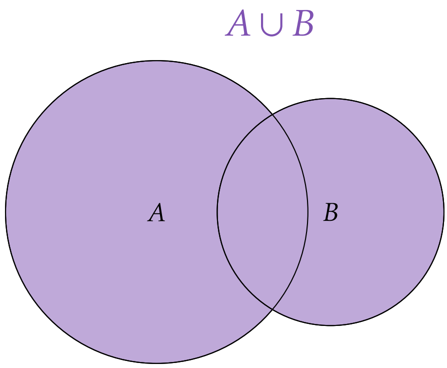
The **union** of two sets (denoted using the symbol $\cup$) is the set composed of all the elements that belong to any of the sets, including elements that are in both sets:

$$A \cup B = \{x \mid x \in A \lor x \in B\}. \tag{0.1.5}$$

Example 0.12 Union of sets

The union of the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$ is the set $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
The union of the sets $C = \{0, 1, 2, 6, 7\}$ and $D = \{3, 9, -4, 5\}$ is the set $C \cup D = \{0, 1, 2, 3, -4, 5, 6, 7, 9\}$.

The following Venn diagram depicts the union of two sets (the purple area):



Naively, the number of elements of a union $A \cup B$ is simply the sum of the number of elements in A and the number of elements in B. However, this naive approach might count the elements in both sets twice: once for A and once for B (see **Figure 0.1**) - this is exactly the set $A \cap B$. We therefore subtract the number of elements in $A \cap B$ and get

$$|A \cup B| = |A| + |B| - |A \cap B|. \tag{0.1.6}$$

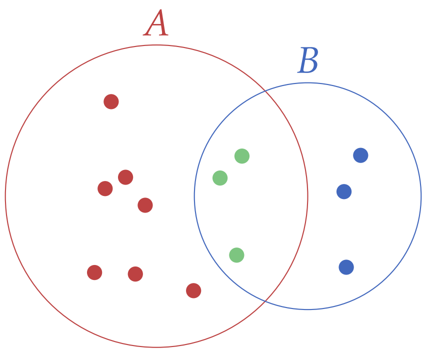


Figure 0.1 Counting the number of elements in the union of two sets: A has 10 elements (red + green dots), while B has 6 elements (blue + green dots). If we count both we get 16 elements, but this counts the joint elements (green dots) twice. Therefore we should subtract the number of joint points, and get that there are only 13 elements in the union.

When two sets A, B are disjoint, then $|A \cap B| = 0$, and so $|A \cup B| = |A| + |B|$.

The definitions of intersections and unions can be easily extended to any whole number of sets.

Example 0.13 Intersection and union of 3 sets

The intersection of 3 sets $A = \{1,2,3,4,5\}$, $B = \{-2,-1,0,1,2\}$ and $C = \{2,3,4,5,6\}$ is the set of all elements that are in A and in B and in C, i.e. the set $A \cap B \cap C = \{2\}$. The union of these sets is the set of all elements that are in either of the sets, i.e. $A \cup B \cup C = \{-2,-1,0,1,2,3,4,5,6\}$.

The most general definition of an intersection of n sets (where n is a whole number), which we will call $A_1, A_2, A_3, \cdots, A_n$ is

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \{x \mid (x \in A_1) \wedge (x \in A_2) \wedge (x \in A_3) \wedge \cdots \wedge (x \in A_n)\}. \tag{0.1.7}$$

the left hand side of Equation 0.1.7 can be written as

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i. \tag{0.1.8}$$

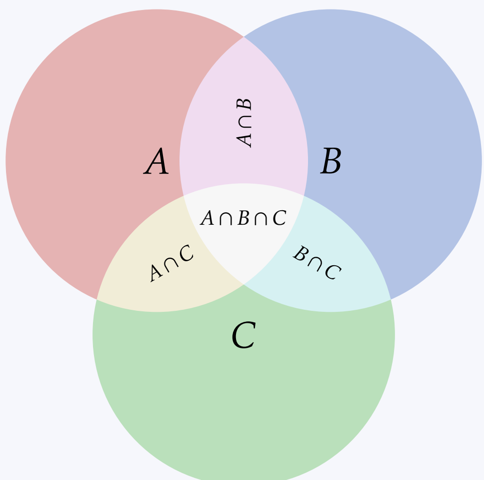
(clarifying the notation? i.e. indexing, etc.)

Similarly, the union of n different sets is defined as

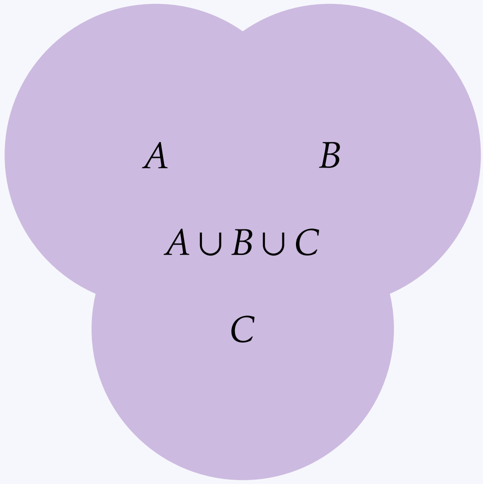
$$\begin{aligned} \bigcup_{i=1}^{n} A_i &= A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \\ &= \{x \mid (x \in A_1) \vee (x \in A_2) \vee (x \in A_3) \vee \cdots \vee (x \in A_n)\}. \end{aligned} \tag{0.1.9}$$

Example 0.14 Venn diagrams: intersection and union of 3 sets

The following Venn diagram shows all possible intersections between three sets:



...and this Venn diagram depicts the union of the same three sets:



TBW: difference, complement.

Given a set A with $|A|$ elements - how many different subsets does it have? We'll start by looking at a practical example: $A = \{1, 2, 3\}$. We can immidetly see that any set which contains just one of the elements of A is a subset of A, i.e. $\{1\}, \{2\}, \{3\}$ are all subsets of A. In addition, any set which contains only two elements from A is a subset of A, i.e. $\{1, 2\}, \{1, 3\}, \{2, 3\}$. Of course, we must not forget the empty set and A itself - both subsets of A (see **Note 0.2**). Thus altogether A has 8 subsets:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}.$$

Generaly, any set A with $|A|$ elements has $2^{|A|}$ different subsets. The set of all these subsets is called the **power set** of A, and is denoted as $P(A)$.

Example 0.15 Power set

The power set of $A = \{1, 2, 3\}$ is

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

IMPORTANT NUMBER SETS

It is now time to introduce some important number sets. We begin with the simplest of these sets: the **natural numbers**, denoted by $\mathbb{N}$. These are the numbers $1, 2, 3, 4, \dots$. Adding the opposites to the natural numbers and adding 0 to the set yields the **integers**, denoted by $\mathbb{Z}$. Loosley speaking, we can define the integers as

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \dots\}. \tag{0.1.10}$$

This makes the integers a superset of the natual numbers, i.e.

$$\mathbb{N} \subset \mathbb{Z}. \tag{0.1.11}$$

One can think of the integers as all the number needed for solving an equation of the form $a + x = b$, where a and b are integers themselves, and x is an unknown. No matter which integer values we put in a and b, the unknown x will always be an integer as well (whether it be positive, negative or zero depends on the values of a and b). However, when one wishes to solve an equation fo the sort $ax = b$, the integers are not longer sufficient: for example, if $a = 2$ and $b = 1$, then x is not an integer.

To solve $ax = b$ (where $a, b \in \mathbb{Z}$) we must introduce the **rational numbers**: numbers with values that are ratios of two integers. We denote the set of rational numbers with the symbol $\mathbb{Q}$, and write

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z} \wedge b \neq 0 \right\}. \tag{0.1.12}$$

(TBW: discuss briefly why $b \neq 0$)

For some combinations of a and b the ratio $\frac{a}{b}$ is an integer. For example: $\frac{3}{1}$, $\frac{8}{4}$, $\frac{-2}{2}$. This makes the integers a subset of the rational numbers, i.e.

$$\mathbb{Z} \subset \mathbb{Q}. \tag{0.1.13}$$

About 2500 years ago it was discovered that some numbers are not rational (and thus also not integers). The most famous example is the number $\sqrt{2}$ - there are not two integers a, b such that $\frac{a}{b} = \sqrt{2}$. We call some of these numbers **algebraic numbers** (denoted by $\mathbb{A}$), and what makes them special is that they are solutions to **polynomial equations**, which we will not define yet (see section xxx). Instead, here is an example for a 2nd order polynomial equation (called a **quadratic equation**):

$$x^2 - 2x - 1 = 0. \tag{0.1.14}$$

Similar to what we saw before, the rational numbers are a subset of the algebraic numbers, i.e.

$$\mathbb{Q} \subset \mathbb{A}. \tag{0.1.15}$$

The algebraic numbers together with other non-rational numbers, such as π and e, form the set of **real numbers**, denoted as $\mathbb{R}$. The definition of real numbers is way beyond the scope of this book, but it is important to understand that the progression we used so far still holds, i.e.

$$\mathbb{A} \subset \mathbb{R}. \tag{0.1.16}$$

The final set of numbers we will touch upon here is the set of **complex numbers**, denoted $\mathbb{C}$, which we can define as

$$\mathbb{C} = \left\{ a + ib \mid a, b \in \mathbb{R},\ i = \sqrt{-1} \right\}. \tag{0.1.17}$$

When $b = 0$, Equation 0.1.17 becomes just a single real number - and so

$$\mathbb{R} \subset \mathbb{C}. \tag{0.1.18}$$

(chapter xxx is dedicated to complex numbers)

Equations 0.1.11-0.1.18 can be merged together to the following single equation:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A} \subset \mathbb{R} \subset \mathbb{C}. \tag{0.1.19}$$

There are more advanced constructions that generalize the complex numbers (i.e. create supersets of the complex number set). These include **quaternions** and **Clifford algebras**. However, as stated before, we will not consider them in this book.

INTERVALS ON THE REAL NUMBER LINE

An important concept that is easily defined over the set $\mathbb{R}$ is an **interval**. A **closed interval** $[a,b]$ is a subset of $\mathbb{R}$ which is defined as

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}. \tag{0.1.20}$$

An **open interval** (a,b) is a subset of $\mathbb{R}$ which is defined as

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\}. \tag{0.1.21}$$

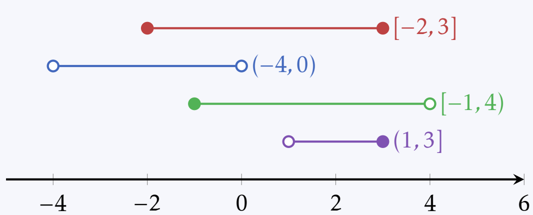
The difference between closed and open intervals is the inclusion and exclusion, respectyively, of the edge point: in a closed interval the points a, b are included, while they are not included in an open interval. Of course, we can also create **half open intervals**, i.e.

$$\begin{aligned} [a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\}, \\ (a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\}, \end{aligned} \tag{0.1.22}$$

where the first interval includes a but not b, and the second interval includes b but not a.

Example 0.16 Intervals

Intervals can be drawn as colored line segments on top of the real number line:



Note how a full point denotes a closed edge, while an empty point denotes an open edge.

In some cases, it is necessary to use intervals that are infinite in one side, i.e. the left or the right edge are at infinity. In these cases, we use the symbol ∞ to denote infinity, and always keep the interval open at that end:

$$\begin{aligned}(-\infty, b) &= \{x \mid x < b\},\\ (-\infty, b] &= \{x \mid x \leq b\},\\ (a, \infty) &= \{x \mid x > a\},\\ [a, \infty) &= \{x \mid x \geq a\}.\end{aligned} \tag{0.1.23}$$

CARTESIAN PRODUCTS

The **Cartesian product** of two sets A, B (denoted $A \times B$) is the set of all possible **ordered** pairs, where the first component is an element of A and the second component is an element of B:

$$A \times B = \{(a, b) \mid a \in A,\ b \in B\}. \tag{0.1.24}$$

Example 0.17 Cartesian products

Consider $A = \{1, 2, 3\}$, $B = \{x, y\}$. Then

$$A \times B = \{(1, x),\ (1, y),\ (2, x),\ (2, y),\ (3, x),\ (3, y)\}$$

The concept of 'ordered pairs' is paramount: if we reverse the order of the elements in a pair the result might not be in the Cartesian product. We therefore say that the Cartesian product is **not commutative**.

Example 0.18 Non-commutivity of the Cartesian product

The elements $(x, 1)$, $(y, 1)$, $(x, 2)$ and so on **are not** in the Cartesian product $A \times B$ as defined in the previous example, since in each one of the pairs the first element is from B and the second element is from A.

The number of elements in a Cartesian product is the product of the number of elements in each of the sets it is composed of, i.e.

$$|A \times B| = |A| \cdot |B|. \tag{0.1.25}$$

Example 0.19 Number of elements in a Cartesian product

The Cartesian product described in the previous two examples has in total $3 \cdot 2 = 6$ elements, as seen in Example 0.17.

As with intersections and unions, the definition of a Cartesian product can be expanded into any natural number of sets:

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_1 \in A_1,\ x_2 \in A_2,\ \ldots, x_n \in A_n\}. \tag{0.1.26}$$

Example 0.20 Cartesian product of three sets

The Cartesian product of the sets $A = \{1,2,3\},\ B = \{x,y\},\ C = \{\alpha,\beta\}$ is

$$\begin{aligned} A\times B\times C = \{&(1,x,\alpha),\ (1,x,\beta),\ (1,y,\alpha),\ (1,y,\beta)\\ &(2,x,\alpha),\ (2,x,\beta),\ (2,y,\alpha),\ (2,y,\beta)\\ &(3,x,\alpha),\ (3,x,\beta),\ (3,y,\alpha),\ (3,y,\beta)\}. \end{aligned}$$

A special case of Cartesian products are those products for which all the sets composing them are the same set. We denote these as the respective integer power, for example the Cartesian product $\mathbb{R}\times\mathbb{R}$ is denoted as $\mathbb{R}^2$, the Cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ is denoted as $\mathbb{R}^3$, etc.

Specifically, the Cartesian product $\mathbb{R}^2$ can be interpreted as the two-dimensional **Euclidean space**, which is the space used to draw graphs in one-dimensional calculus and shapes in two-dimensional analytican geometry. We will explore this idea (and higher dimensional spaces) in more details in upcoming chapters.

0.2 RELATIONS AND FUNCTIONS

BASICS

Cartesian products of two sets can be viewed as describing all possible connections between the elements of the first set to the elements of the second set, and thus any subset of a Cartesian product forms a specific **relation** between the sets.

Example 0.21 Relations as subsets of Cartesian products

Given the following two sets:

$$A = \{1,2,3,4\},\ B = \{\alpha,\beta,\gamma\},$$

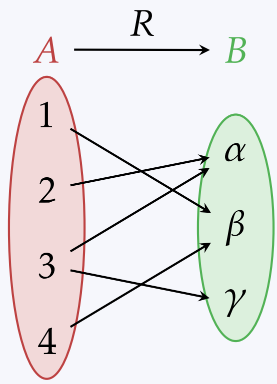
then

$$\begin{aligned} A\times B = \{&(1,\alpha),\ (1,\beta),\ (1,\gamma),\\ &(2,\alpha),\ (2,\beta),\ (2,\gamma),\\ &(3,\alpha),\ (3,\beta),\ (3,\gamma),\\ &(4,\alpha),\ (4,\beta),\ (4,\gamma)\}. \end{aligned}$$

We can choose the following pairs to form a subset of $A\times B$:

$$R = \{(1,\beta),\ (2,\alpha),\ (3,\alpha),\ (3,\beta),(4,\gamma)\}.$$

R is thus a relation between A and B. We can graphically illustrate R as follows:



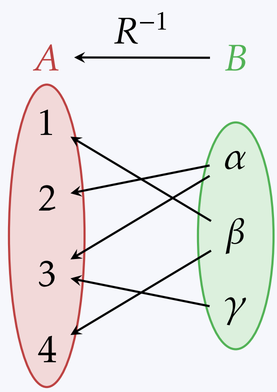
Relations can be inversed by reversing the order of each of its pairs.

Example 0.22 Inverse relation

The inverse relation to the relation in **Example 0.21** is

$$R^{-1} = \{(\beta, 1),\ (\alpha, 2),\ (\alpha, 3),\ (\beta, 3), (\gamma, 4)\}.$$

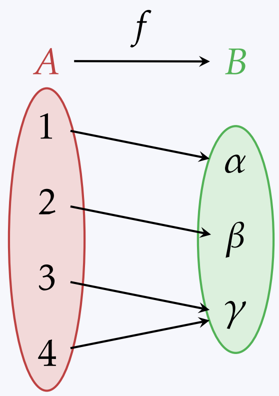
Graphically:

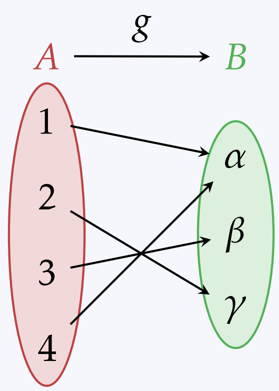


A **function** f from a set A to a set B is a relation for which any element in A is connected to a single element in B.

Example 0.23 Functions

The following are two functions from the set A to the set B defined in **Example 0.21** :





The pairs making up f are $(1,\alpha)$, $(2,\beta)$, $(3,\gamma)$ and $(4,\gamma)$, and the pairs making up g are $(1,\alpha)$, $(2,\gamma)$, $(3,\beta)$ and $(4,\alpha)$.

Note 0.5 Relations which are not functions

Note that the relation in **Example 0.21** is **not** a function, since the element $3\in A$ is connected to more than one element in B, namely α and γ.

Different names are used in some branches of mathematics to describe functions, such as **maps** and **transformations**. Baring context, they all mean the same thing.

A common way to denote that a function f is connecting elements in A to elements in B is

$$f: A \to B. \tag{0.2.1}$$

A is called the **domain** of f, and B its **image**. In this book and many other sources, the following notation is used: $f(x)=y$, which means that when we apply the function f to an element $x\in A$, the result is the element it is connected to, i.e. $y\in B$. We write this as $x\mapsto y$ (the special symbol $\mapsto$ is called a **mapping notation**).

Example 0.24 Value $\mapsto$ value notation for functions

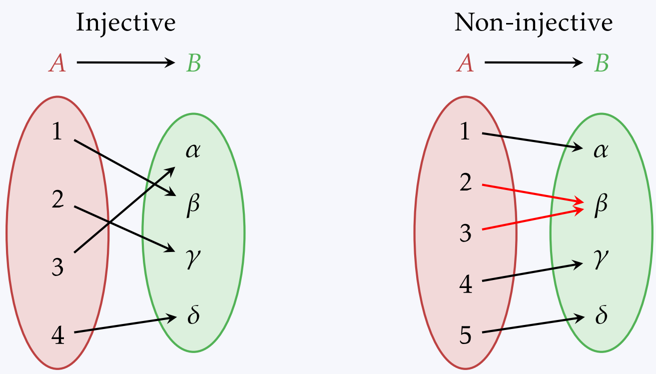
For the functions f, g as defined in **Example 0.23** :

$$f(1) = \alpha,\ f(2) = \beta,\ f(3) = f(4) = \gamma.$$
$$g(1) = g(4) = \alpha,\ g(2) = \gamma,\ g(3) = \beta.$$

INJECTIVE, SURJECTIVE AND BIJECTIVE FUNCTIONS

A function is **injective** if each of the elements in its **image** is connected to by at most a single element in its **domain**. An injective function is also known as an **injection**.

Example 0.25 Injective function



The function on the right in non-injective because the element $\beta \in B$ is connected to by two elements in A (2 and 3, red arrows).

A function is **surjective** if every element in its image is connected to by at least a single element in its domain (see **Example 0.26**). As with injective functions, a surjective function is also known as a **surjection**. A non surjective function can be made into surjective function by excluding from its image any element that is not connected to by any element from its domain (see **Example 0.27**).

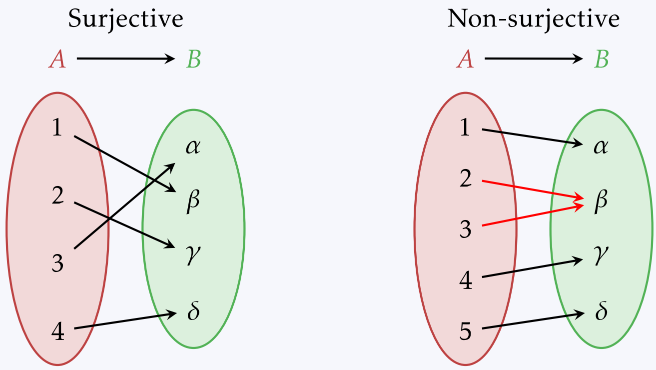
A function $f : A \to B$ that is both surjective and bijective is called a **bijective function** (also a **bijection**). All elements in the image of a bijection are connected to by exactly a single element in its domain. This means that the direction of the connections can be flipped, yielding the **inverse** of the original function (denoted f^{-1}).

The reason only bijective functions have inverses is as follows: Given a function $f : A \to B$,

- if f is non-injective, then there is at least one element $y_1 \in B$ which is connected to by at least two elements from A. We can name these elements x_1 and x_2. When inverted, $f^{-1} : B \to A$ has an element $y_1 \in B$ (note that for f^{-1}, B is its domain), which is connected to two or more elements in A, the image of f^{-1}. These are of course x_1, x_2. This fact disqualifies f^{-1} from being a function.

- If f is non-surgetive, then there exists at least one element $y_2 \in B$ that is not connected to by any element from A. When inverted, y_2 in the domain B of f^{-1} is not connected to any element in its image A. This fact disqualifies f^{-1} from being a function.

Example 0.26 Surjective function



Example 0.27 Making a non-surjective function into a surjection

Given the two sets $A = \{1, 2, 3, 4\}$ and $B = \{\alpha, \beta, \gamma, \delta\}$, the following non-surjective function $f : A \to B$ is defined:

$$f = \{(1, \alpha),\ (2, \beta),\ (3, \gamma),\ (4, \gamma)\}.$$

By removing δ from B, the function f becomes surjective (though it remains non-injective).

TBW: an example of all four possible combinations (injective non-surjective, surjective non-injective, bijective, non-injective non-surjective) and their inverses.

Note 0.6 Other names for bijections !

Bijections are also called **one-to-one correspondences** and **invertible functions**.

REAL FUNCTIONS

In suitable cases, a function is defined via a general mapping rule. This should be very familiar to anyone who learned mathematics in highschool, where many times functions are defined this way, e.g.

$$f(x) = x^2 + 3x - 4. \tag{0.2.2}$$

In mapping notation we can write Equation 0.2.2 as $f : x \mapsto x^2 + 3x - 4$. In highschool mathematics, both the domain and image of such functions is $\mathbb{R}$, although it is almost never specified explicetly. Such functions are commonly refered to as **real functions**, a convention used in this book as well.

Example 0.28 Functions defined using a mapping rule

The following are real functions:

$$f_1(x) = 2x^2 - 5, \quad f_2(x) = \sin\left(\frac{x}{3}\right), \quad f_3(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{\sigma^2}}.$$

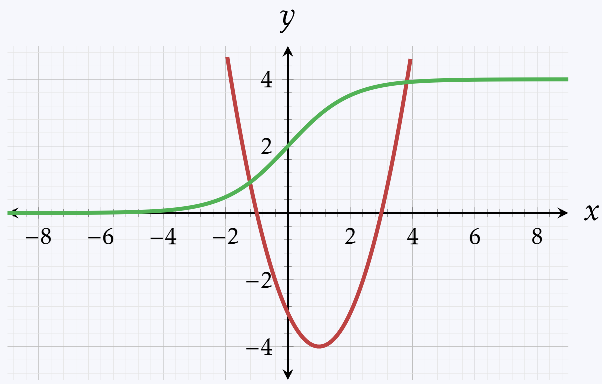
Note that these functions can also be defined using different sets, for example $f_1 : \mathbb{N} \to \mathbb{Z}, \quad f_2 : \mathbb{N} \to [-1, 1], \quad$ etc.

Real functions can be easily plotted in a **Cartesian coordinate system** by drawing all the points $(x, f(x))$ (i.e. all the points (x, y), where $x, y \in \mathbb{R}$ and $x \mapsto y$). We call these points the **graph** of f over $\mathbb{R}$.

Example 0.29 Graphs of real functions

The following two functions are plotted on the domain $[-9, 9]$:

- $f(x) = x^2 - 2x - 3$,
- $g(x) = 4e^x/(e^x + 1)$.

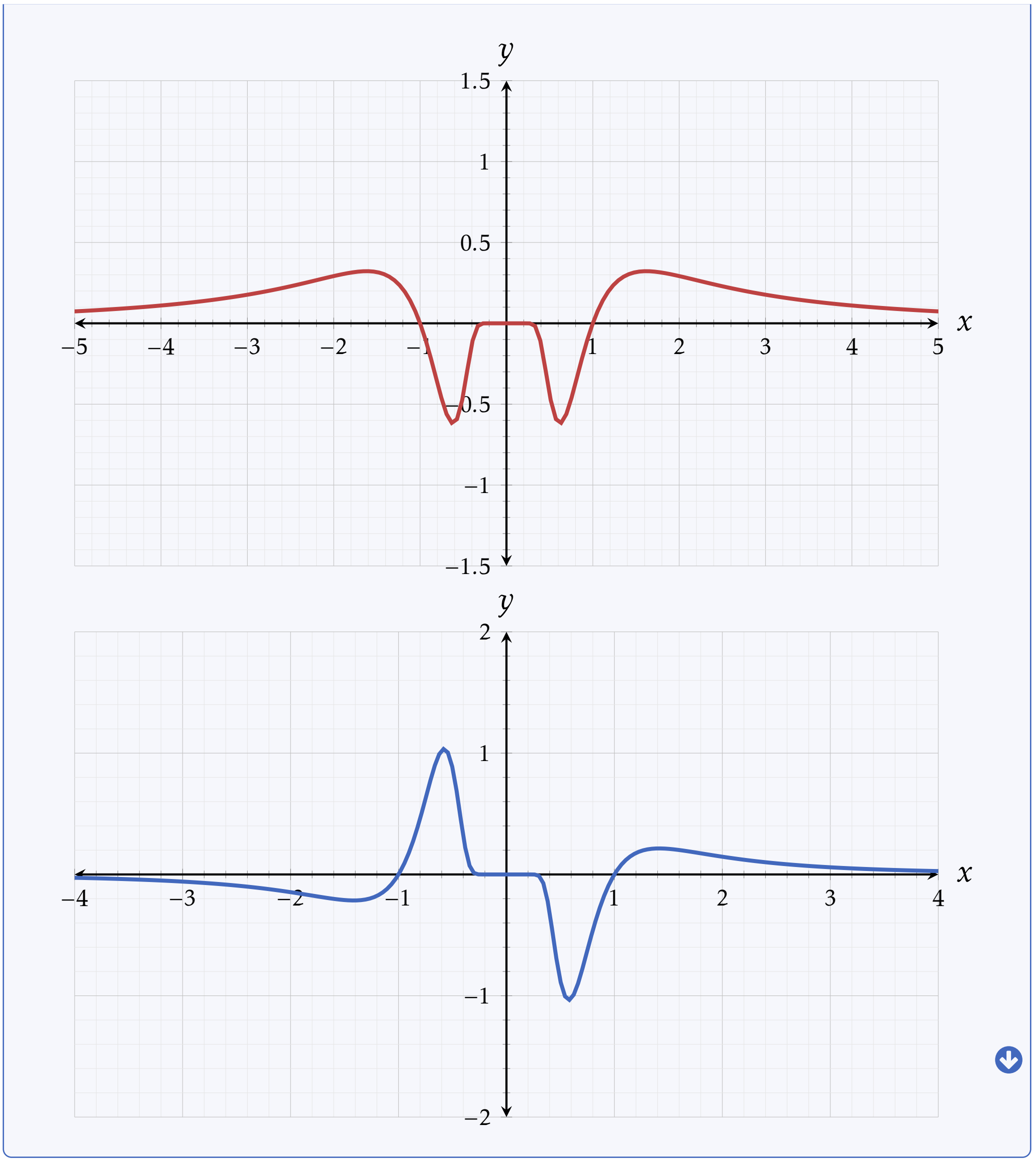


In **Example 0.29**, the function $g(x)$ always increases in value from left to right. Let's give this notion a more formal tone: a function f is said to be **increasing** on an interval I if for any $x_1, x_2 \in I$, if $x_2 > x_1$ then $f(x_2) > f(x_1)$. We can similarily define the idea of **decreasing** on an interval.

A property of some functions which is visualy easy to depict is symmetry. A real function f is said to be **symmetric** if for any $x \in \mathbb{R}$, $f(-x) = f(x)$. This essentially means that the y-axis mirrors the function's plot. If for any $x \in \mathbb{R}$, $f(-x) = -f(x)$, we say that the function is **anti-symmetric**. A function can be neither, but there's only a single function which is both: the zero function, i.e. $f(x) = 0$.

Example 0.30 Symmetric and anti-symmetric functions

In the following graphs, the function on the top is symmetric, while the function on the bottom is anti-symmetric:



(injections/surjections of real functions?)

COMPOSITION OF FUNCTIONS

Functions can be **composed** together, generating new functions. Given two functions $f : A \to B$ and $g : B \to C$, their composition is denoted as $f \circ g$. For the composition to be well defined, the **image** of f must be the same as the **domain** of g, and the resulting composition would have A as its domain and C as its image, i.e. $f \circ g : A \to C$.

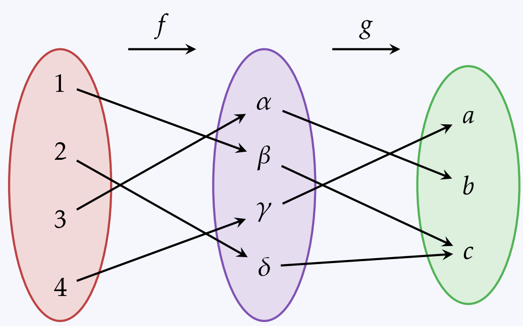
Example 0.31 Composition of functions

Consider the functions
$$f(x) = x^2, \quad g(x) = \sin(x).$$
Using these functions, the two possible compositions are

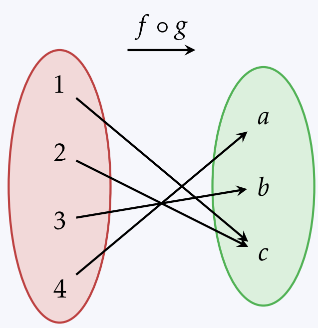
- $f \circ g = f(g(x)) = [\sin(x)]^2$, and
- $g \circ f = g(f(x)) = \sin\left(x^2\right)$.

Example 0.32 Graphical representation of function composition

A graphical representation of composing two functions:
$$f : \{1, 2, 3, 4\} \to \{\alpha, \beta, \gamma, \delta\}, \quad g : \{\alpha, \beta, \gamma, \delta\} \to \{a, b, c\}.$$



The composition results in the following function
$$f \circ g : \{1, 2, 3, 4\} \to \{a, b, c, \}.$$



0.3 POLYNOMIAL FUNCTIONS

A very useful family of real functions can be derived using only three fundamental operations: addition, multiplication and exponentiation: the (real) **polynomial functions**. These are functions of the form

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n, \tag{0.3.1}$$

where $a_0, a_1, \ldots, a_n$ are real numbers called the **coefficients** of the polynomial function. Note that $a_n \neq 0$, i.e. the **degree** of the polynomial function is the index of the highest non-zero coefficient (and thus the highest power in the expression). We also call this the **order** of the polynomial function.

Example 0.33 Polynomial

The following is a polynomial function of degree $n = 6$:

$$P(x) = 4 + 2x - 3x^2 + 7x^4 - x^5 + 3x^6.$$

Breaking down this polynomial to its constituent terms:

$$\begin{array}{ccccccc} P(x) = & 4 & +2x & -3x^2 & +7x^4 & -x^5 & +3x^6 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & a_0 = 4 & a_1 = 2 & a_2 = -3 & a_4 = 7 & a_5 = -1 & a_6 = 3 \end{array}$$

Note that a_3 is missing from the polynomial function (i.e. there is no x^3 term). This means that $a_3 = 0$.

A shorthand way to write the general form of a polynomial function is by using the **summation notation**:

$$P(x) = \sum_{k=0}^{n} a_k x^k. \tag{0.3.2}$$

This notation, called the **Capital-sigma notation**, essentially represents addition of n elements (in the case shown here), each with its own **index of summation**, in this case i. The most general form of the summation notation is

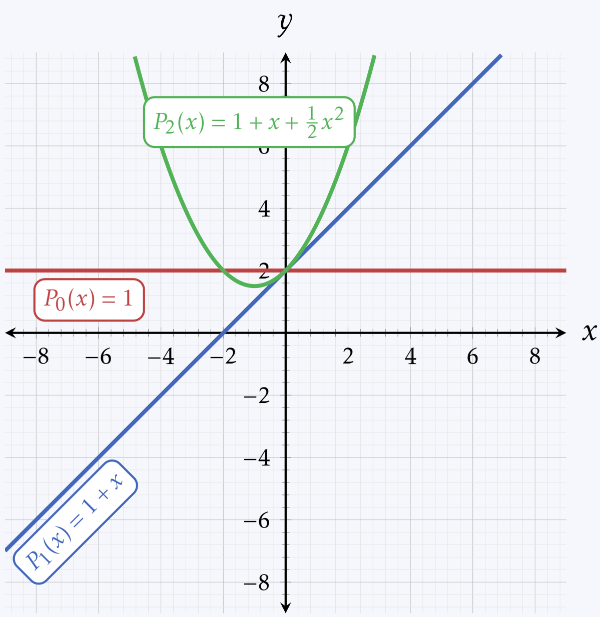
$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + a_{k+2} + \cdots + a_{n-1} + a_n, \tag{0.3.3}$$

i.e. the notation tells us to add those elements a_i for which $k \geq i \geq n$. Note that in the case of Equation 0.3.2, when $k = 0$, $x^k = x^0 = 1$ and the first term of the polynomial function has no x power (i.e. it is simply a_0), and when $k = 1$, $x^k = x^1 = x$ and thus the second term is $a_1 x$. We will encounter the summation notation in more details later in the book.

In the special case $n = 0$, i.e. when $P(x) = a_0$, the function is constant. When $n = 1$ the function $P(x) = a_0 + a_1 x$ is a line, and when $n = 2$, $P(x) = a_0 + a_1 x + a_2 x^2$ is a quadratic function.

Example 0.34 Polynomial functions for $n = 0, 1, 2$

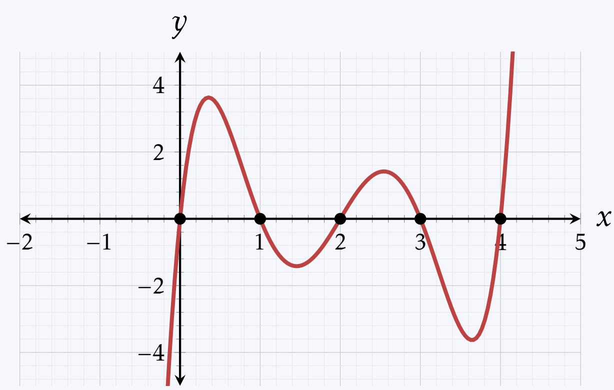
The following graphs represent the polynomial functions of degrees $n = 0, 1, 2$ with coefficients $a_0 = 2$, $a_1 = 1$, $a_2 = \frac{1}{2}$:



The values $x \in \mathbb{R}$ for which $P(x) = 0$ are called the **roots** (also: **zeros**) of the polynomial function.

Example 0.35 Roots of a polynomial function

The polynomial function $P(x) = 24x - 50x^2 + 35x^3 - 10x^4 + x^5$ has the following 5 roots: $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$. In the following graph of $P(x)$ the roots are shown as black dots.



The maximum number of **real** roots of a polynomial function with degree $n \geq 1$ is n, e.g. a polynomial of degree $n = 4$ has at most 4 real roots. This statement is a consequence of a very important theorem called **the fundamental theorem of algebra**, which due to its importance we will mention here without proof:

Theorem 0.1 The fundamental theorem of algebra

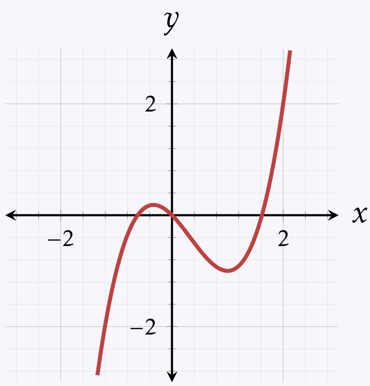
For any $n \geq 1$, the polynomial function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_0, a_1, a_2, \ldots, a_n$ are all **complex numbers** and $a_n \neq 0$, has n complex roots.

Given a polynomial function $P(x)$ with n roots $r_1,\ r_2,\ \cdots,\ r_n$, the function can be written as a product of terms of the form $x - r_i$ (up to a constant), e.g. the polynomial function of degree $n = 3$ with roots $-1, 1, 2$ can be written as

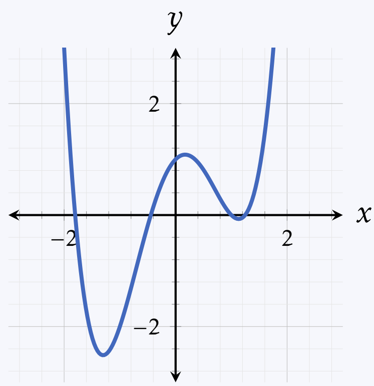
$$P(x) = (x+1)(x-1)(x-2) = x^3 - 2x^2 - x + 2. \tag{0.3.4}$$

Example 0.36 Higher order polynomial functions

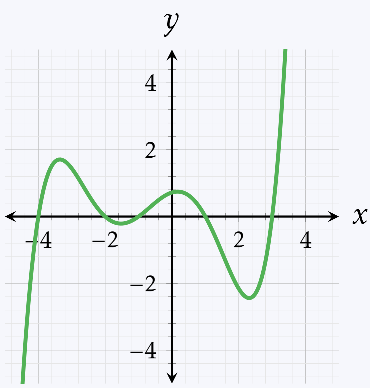
The following are the graphs of high-order polynomial functions ($n = 3, 4, 5, 6$):



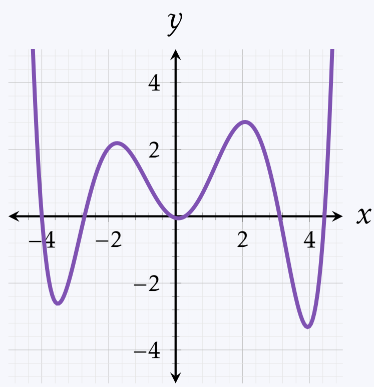
$x^3 - x^2 - x$



$x^4 - 3x^2 + x + 1$



$\frac{3}{100}\left(x^5 + 3x^4 - 11x^3 - 27x^2 + 10x + 24\right)$



$\frac{1}{100}\left(x^6 - x^5 - 26x^4 + 15x^3 + 150x^2 - 25x - 5\right)$

As can be seen in **Example 0.36**, the maximal number of 'bends' in a polynomial function of order n is $n-1$ (i.e. one less than the order of the function).

We will continue to explore polynomial functions in more details in future chapters.

0.4 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In the previous section we dealt with functions composed of integer powers of x. We will now shortly focus on functions where x is in the power itself and their inverse functions.

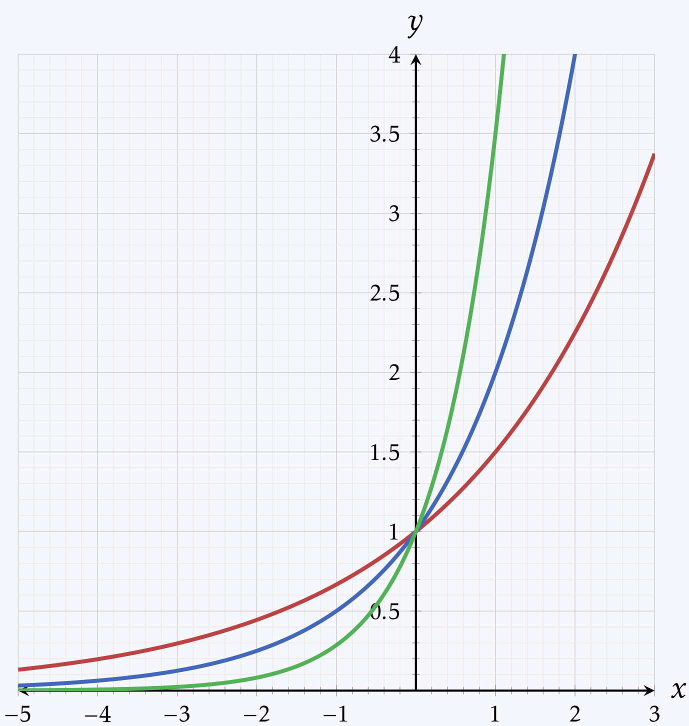
An **exponential function**, or simply an **exponential**, is a real function of the type

$$f(x) = b^x, \tag{0.4.1}$$

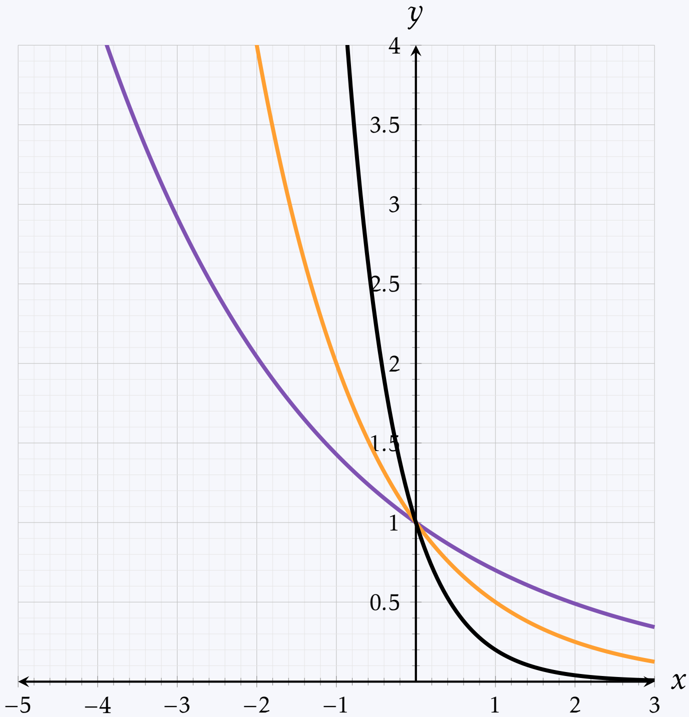
where $b > 0$ is called the **base** of the exponentiation, and x the exponential. All exponents, regardless of base, are always positive. In addition, all exponents pass through the point $(0, 1)$ since $b^0 = 1$ for any real positive number, and through the point $(1, b)$ since $b^1 = b$. When $b > 1$ the function is increasing on $\mathbb{R}$, while for $b < 1$ the function is descending on $\mathbb{R}$.

Example 0.37 Exponential functions

The following are graphs of the exponential functions 1.5^x, 2^x and 3.5^x:



And the following are graphs of the exponential functions 0.7^x, 0.5^x and 0.2^x:



As a reminder, the following are two well known properties of exponents: given a base $b > 0$,

$$b^{-x} = \frac{1}{b^x}, \tag{0.4.2}$$
$$b^x b^y = b^{x+y}. \tag{0.4.3}$$

A special base for exponential functions is the real, non-aglebric number e. This number has many names, among them is **Euler's number**, but in the constant of exponentials it is known as the **natural base**. Its exact value is not entirely important for the moment: it is about 2.718, and in any case it is not possible to write it as there it has infinitely many digits after the period. It is very common across different fields of mathematics and science to write $\exp(x)$ instead of e^x.

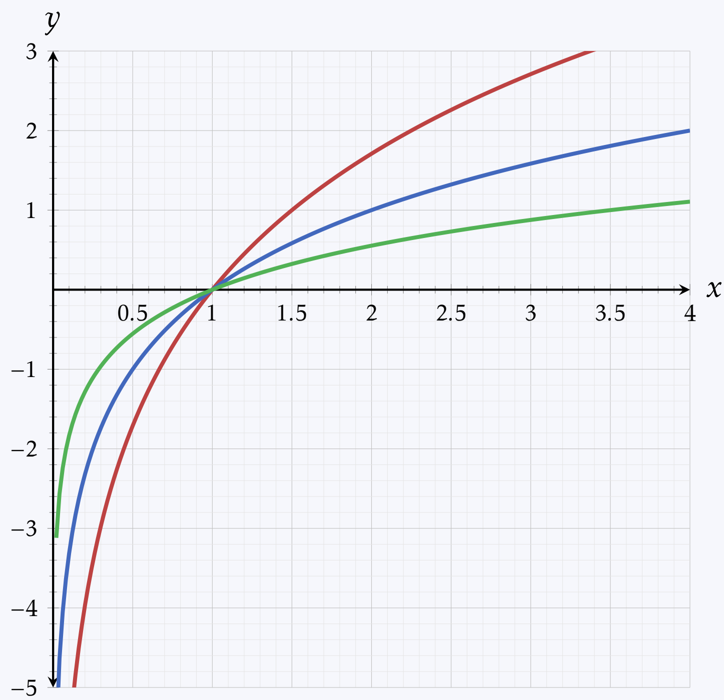
The inverse function to exponentials are the **logatithmic functions** (or simply **logarithms**), i.e. for any real $b > 0,\ b \neq 1$,

$$\log_b\left(b^x\right) = b^{\log_b(x)} = x. \tag{0.4.4}$$

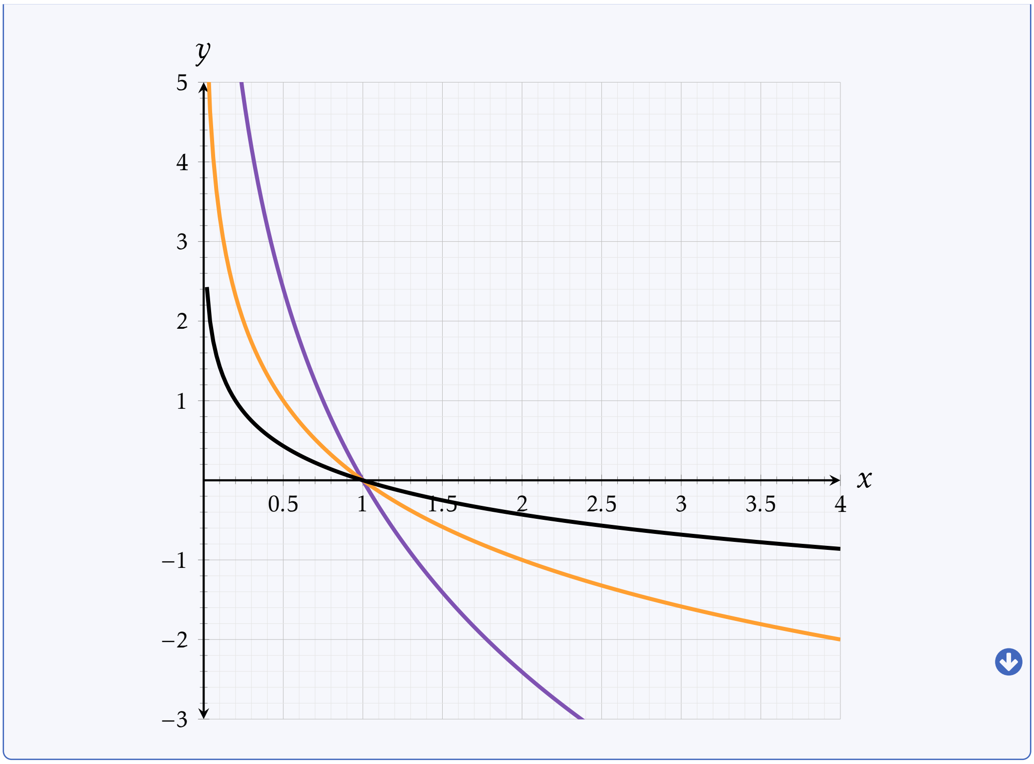
In essence, the logarithm in base b of a number x answers the question *"what is the number a for which $b^a = x$?"*. Being the inverses of exponential functions, all logarithms go through the point $(1, 0)$, and each also passes through its own point $(b, 1)$.

Example 0.38 Logarighmic functions

The following are graphs of the logatithmic functions $\log_{1.5}(x)$, $\log_2(x)$ and $\log_{3.5}(x)$:



...and the following are graphs of the exponential functions $\log_{0.75}(x)$, $\log_{0.5}(x)$ and $\log_{0.2}(x)$:



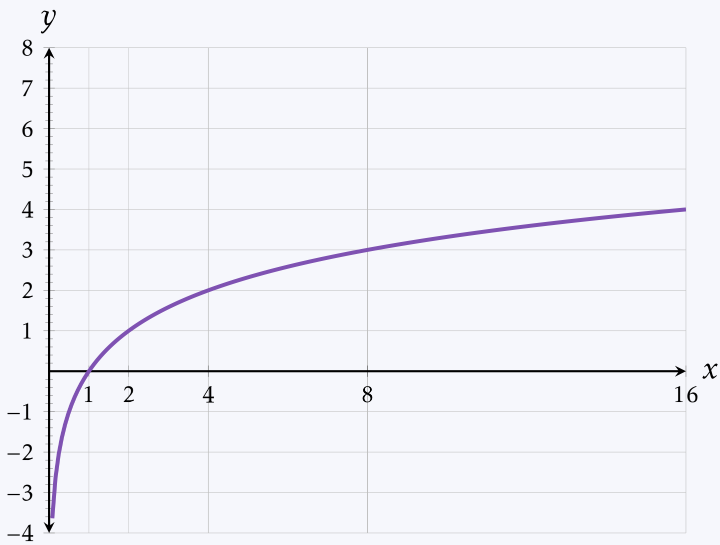
A useful property of logarithms is that they can help reduce ranges spanning several orders of magnitude to numebrs humans can deal with. The easiest way to see this is using $b = 10$: $10^1 = 10$, and so $\log_{10}(10) = 1$. $10^2 = 100$, and so $\log_{10}(100) = 2$. $10^3 = 1000$, and so $\log_{10}(1000) = 3$, etc. The value of the logarithm goes by 1 for each raise in order of magnitude of its argument.

Therefore, if we have some measurement x which can hold values spanning severl orders of magnitude (say $x \in [3, 1500000000]$), then it can sometimes be useful to use instead the logarithmic value of x (which in our case would span the range $\log_{10}(x) \in [0.477, 9.176]$). This is done in many fields of science, for example some definitions of entropy[1], acid disociation constants[2], pH[3] and more.

Example 0.39 Logarithms as eavluating orders of magnitude

In the following graph of $\log_2(x)$, each increase by power of two in x (i.e. $x = 1, 2, 4, 8, 16, \dots$) yields only a single increase in y (i.e. $y = 0, 1, 2, 3, 4, \dots$). This shows how logarithms shift our perspective from absolute values to orders of magnitude.

[1] $S = k_B \log(\Omega)$
[2] $pK_a = -\log(K_{diss})$
[3] $pH = -\log([H^+])$



Using the definition of the logarithmic function $\log_b(x)$ (Equation 0.4.4) and the product rule for exponentials (Equation 0.4.3), a similar rule can be derived for logarithms. Let $x, y > 0$ and $b > 0,\ b \neq 1$ all be real numbers. We define

$$\log_b(x) = M,\ \log_b(y) = N, \tag{0.4.5}$$

which means

$$b^M = x,\ b^N = y. \tag{0.4.6}$$

From Equation 0.4.3 we know that

$$xy = b^M b^N = b^{M+N}, \tag{0.4.7}$$

and by re-applying the definition of logarithmic functions we get that

$$\log_b(xy) = M + N = \log_b(x) + \log_b(y). \tag{0.4.8}$$

Simlilarily to Equation 0.4.8 , division yields subtraction:

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y). \tag{0.4.9}$$

Equations 0.4.8 and 0.4.9 reveal another valueable property of logarithms: they reduce multiplication to addition (and subsequently division to subtraction). While today this property doesn't seem very impressive, in pre-computers days it helped carrying on complicated calculations, using tables of pre-calculated logarithms (called simply **logarithm tables**) - a sight rarely seen today.

Taking one step forward in regards to reduction of operations, logarithms reduce powers to multiplication:

$$\log_b\left(x^k\right) = k\log_b(x). \tag{0.4.10}$$

for any $k \in \mathbb{R}$.

(TBW: proving this will be in the chapter questions to the reader)

Any logarithm $\log_b(x)$ can be expressed using another base, i.e. $\log_a(x)$ (where $a > 0$, $a \neq 1$) using the following formula:

$$\log_a(x) = \log_b(x) \cdot \log_a(b). \tag{0.4.11}$$

(TBW: proving this too will be a question to the reader)

Example 0.40 Changing logarithm base

Expressing $\log_4(x)$ in terms of $\log_2(x)$:

$$\log_4(x) = \log_2(x) \cdot \underbrace{\log_4(2)}_{=\frac{1}{2}} = \frac{1}{2}\log_2(x).$$

Much like with exponentials, the number e plays an important role when it comes to logarithms, for reasons that are discussed in the calculus chapter (ref). For now, we will just mention that $\log_e(x)$ gets a special notation: $\ln(x)$, which stands for **natural logarithm**. This notation is mainly used in applied mathematics and science, while in pure mathematics the notation is simply $\log(x)$, i.e. without mentioning the base [4].

For reason we will see in the calculus chapter, it is relatively simple to calculate both the exponential and logarithm in base e. Therefore, many operations in modern computations are actually done using these functions, for example calculating logarithms in other bases:

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}. \tag{0.4.12}$$

Another operation commonly using both e^x and $\ln(x)$ is raising a real number a to a real power b: using the properties of both exponential and logarithmic functions, any such power can be expressed as

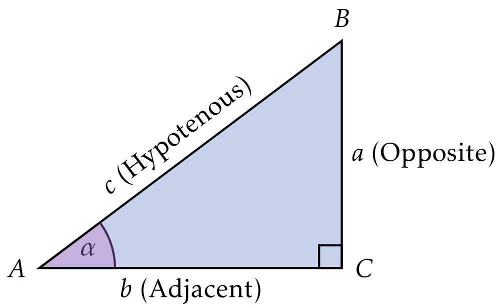
$$a^b = e^{b\ln(a)}. \tag{0.4.13}$$

0.5 TRIGONOMETRIC FUNCTIONS

BASIC DEFINITIONS

Consider a **right triangle** $\triangle ABC$ with sides a, b, and Hypotenous c, where the angle $\angle ACB$ is $90°$, and the angle $\angle BAC$ is denoted as α:

[4] Depending on convention and context, this notation can refer to logarithm in any other base, most commonly $\log_{10}(x)$ and $\log_2(x)$.



We use the ratios between the three sides of the triangle to define three functions of α:

Definition 0.1 The basic triginometric functions

1. The **sine** of the angle α is $\sin(\alpha) = \frac{a}{c}$,
2. the **cosine** of the angle α is $\cos(\alpha) = \frac{b}{c}$, and
3. the **tangent** of the angle α is $\tan(\alpha) = \frac{a}{b}$, which in turn is equal to $\frac{\sin(\alpha)}{\cos(\alpha)}$.

π

We can rearrange the above definitions to yield

$$\begin{aligned} a &= c\sin(\alpha), \\ b &= c\cos(\alpha). \end{aligned} \tag{0.5.1}$$

Normaly, the Hypotenous is the longest side of a right triangle. We will consider here the two edge cases where one of the sides a, b is equal to the Hypotenous (and the other side is thus 0):

- if $a = c$ then $\alpha = 90°$,
- if $b = c$ then $\alpha = 0$.

The posssible length of a is therefore in the range $0 \leq a \leq c$, which means that $0 \leq \frac{a}{c} \leq 1$, or since $\sin(\alpha) = \frac{a}{c}$,

$$0 \leq \sin(\alpha) \leq 1. \tag{0.5.2}$$

The same is of course true for b, and thus

$$0 \leq \cos(\alpha) \leq 1 \tag{0.5.3}$$

as well.

As a reminder, the **Pythagorean theorem**[5] states that for a right triangle like the one

[5] It's worth mentioning that no three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer value of $n > 2$. This can be proven, however the proof is too large to fit in the footnotes.

Table 0.3 Common angles in radians.

Degrees	Radians
$0°$	0
$45°$	$\frac{\pi}{4}$
$90°$	$\frac{\pi}{2}$
$180°$	π
$270°$	$\frac{3\pi}{2}$
$360°$	2π

here,

$$a^2 + b^2 = c^2. \tag{0.5.4}$$

By substituting **Equation** 0.5.1 into the above we get

$$c^2 = a^2 + b^2 = (c\sin(\alpha))^2 + (c\cos(\alpha))^2 = c^2\sin^2(\alpha) + c^2\cos^2(\alpha) = c^2\left[\sin^2(\alpha) + \cos^2(\alpha)\right],$$

and cancelling c^2 on both sides simply yields

$$\sin^2(\alpha) + \cos^2(\alpha) = 1. \tag{0.5.5}$$

THE UNIT CIRCLE

The range of the trigonometric functions can be extended by using the **unit circle**: a circle of radius $R = 1$ is placed such that its center lies at the origin of a 2-dimensional axis system, i.e. at the point $\boldsymbol{O} = (0,0)$. A radius to a point $\boldsymbol{P} = (x, y)$ on the circle's circumference is the drawn. This radius has an angle θ to the x-axis. A line from $\boldsymbol{P}$ perpendicular to the x-axis intersecting at the point $\boldsymbol{D}$ is drawn (see **Figure** 0.3).

The triangle $\triangle OPD$ is a right triangle. Therefore, we can use the trigonometic functions to calculate the coordinates of the point $\boldsymbol{P} = (x, y)$:

$$\begin{aligned} x &= R\cos(\theta) = \cos(\theta), \\ y &= R\sin(\theta) = \sin(\theta). \end{aligned} \tag{0.5.6}$$

We can now define $\cos(\theta)$ and $\sin(\theta)$ as the values of x and y, respectively, as a function of θ.

We will switch to measuring angles in **Radians** instead of degrees: θ radians are equal to the length of an arc on a unit circle, which corresponds to the angle θ (**Figure** 0.4). This allows us to use the same units as x and y: for example, when length is measured in [meter], an angle in radians is measured in [meter] as well. The full circumference of a circle equals 2π radians, and therefore a single radian is equivalent to $\frac{180}{\pi} \approx 57.3°$. **Table** 0.3 shows some common angles in radians.

Another advantage which we gain by defining the trigonometric functions using the unit circle is the extension of their domain to all of $\mathbb{R}$: an angle of size $2\frac{1}{2}\pi$ (equivalent

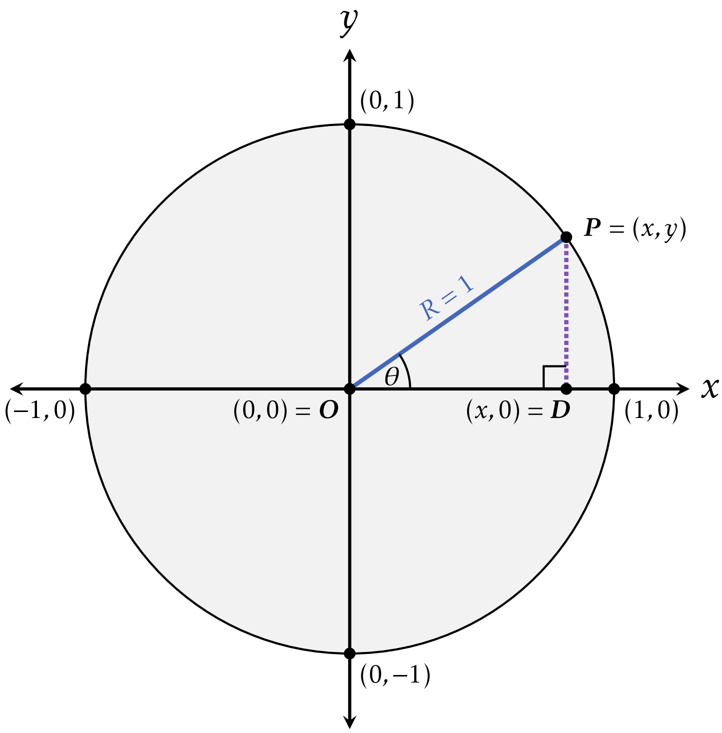


Figure 0.3 A unit circle with (...)

Table 0.4 Text

Quadrant	$\cos(\theta) = x$	$\sin(\theta) = y$
1	$[0,1]$	$[0,1]$
2	$[-1,0]$	$[0,1]$
3	$[-1,0]$	$[-1,0]$
4	$[0,1]$	$[-1,0]$

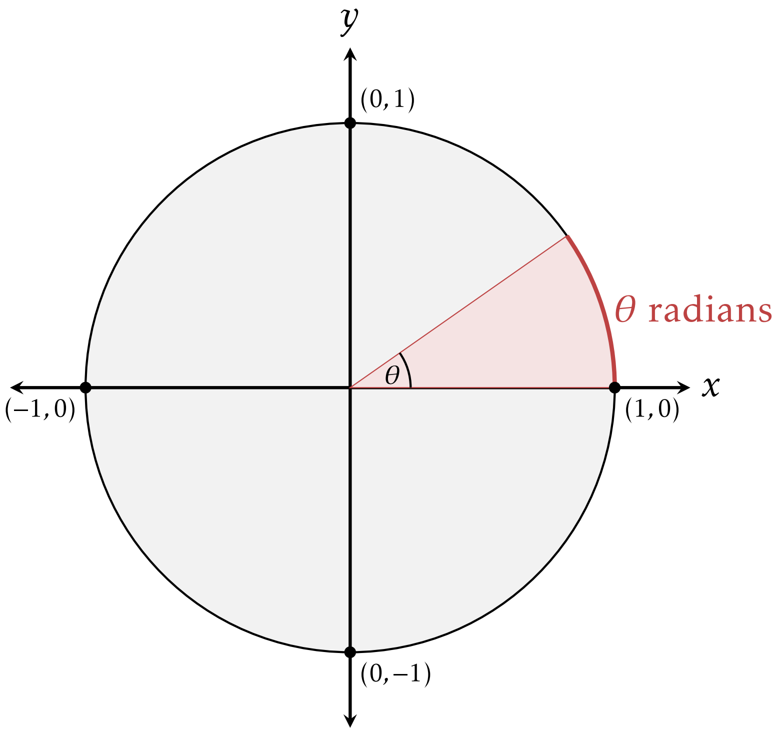


Figure 0.4 Radians

to 450°), for example is the same as an angle of size $\frac{1}{2}\pi$ (90°), and an angle of $-\frac{1}{6}\pi$ (−30°) is the same as $\frac{5}{6}\pi$ (330°).

MORE WILL BE WRITTEN HERE

0.6 COMPLEX NUMBERS

ALGEBRAIC APPROACH

Real numbers, while being extremely useful, are not complete - they can't solve all equations involving numbers. For example, the equation

$$x^2 + 1 = 0 \tag{0.6.1}$$

has no real solutions, since there can be no real number x such that $x^2 = -1$. However, we can choose to define a new number, $i = \sqrt{-1}$ and using it to build a new number system. This system is of course the set of complex numbers, $\mathbb{C}$. It is defined as the set of all z such that

$$z = a + ib, \tag{0.6.2}$$

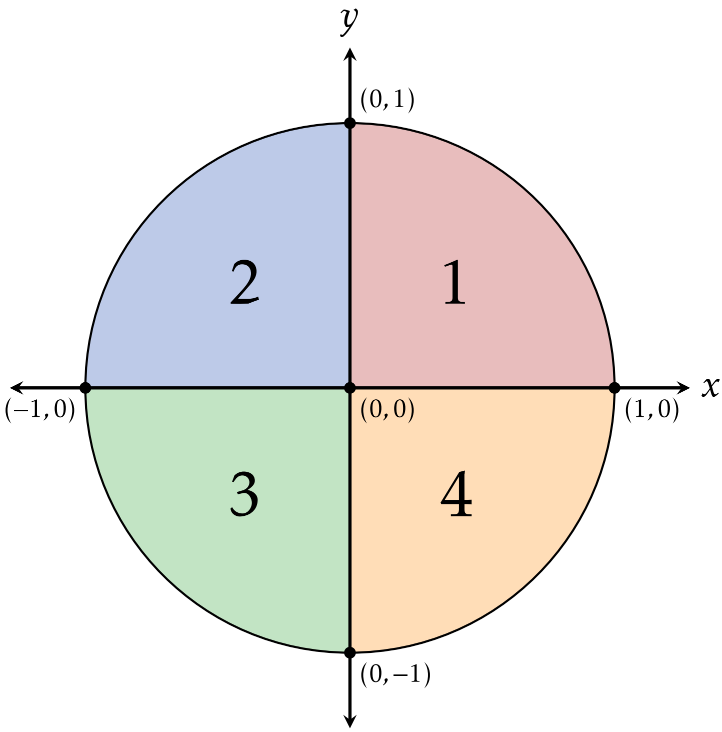


Figure 0.5 The different quadrants of the unit circle.

where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. We call a the **real component** of z or $\mathfrak{Re}(z)$, and b its **imaginary component** or $\mathfrak{Im}(z)$[6]. These numbers appear a lot all throughout the exact sciences (but especially in physics and engineering), so we must at the very least learn their basic properties.

It is not so obvious that we can add two different kinds of numbers together, but it works (the linear algebra chapter sheds more light on this idea). What is important is that we always keep these two parts separated. We see this when we add together two complex numbers z_1, z_2:

$$z = z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i. \tag{0.6.3}$$

The real part of z is therefore $a_1 + b_1$, and its imaginary part is $b_1 + b_2$.

What happens when we multiply two complex numbers? Let's check:

$$\begin{aligned} z = z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2 \\ &= a_1 a_2 + i a_1 b_2 + i a_2 b_1 - b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1). \end{aligned} \tag{0.6.4}$$

We see that we can still separate the real part and imaginary part of the result. What happens in the case of two real numbers? For real numbers $b = 0$, and thus Equation

[6]There is nothing more "real" about real numbers than imaginary numbers, but unfortunately that's the terminology we're stuck with ¯_(ツ)_/¯

0.6.4 devolves to $z = a_1 a_2 \in \mathbb{R}$, which is exactly what we expect: multiplying two real numbers yields their product, which is a real number. Notice that this doesn't happen with purely imaginary numbers: multiplying together two imaginary numbers (i.e. numbers for which $a = 0$) results in a real number. Will get to understand why this happens very soon.

When discussing real numbers sometimes we like to refer to their *magnitude*, i.e. their absolute value. With complex numbers this is defined as

$$|z| = \sqrt{a^2 + b^2}, \tag{0.6.5}$$

i.e. in a sense, to get the magnitude of a complex number we imagine its two components as being perpendicular and calculate the length of the resulting hypotenous (cf. the Pythagorean theorem). In fact, this is one very useful interpertation of complex numbers, which we will explore in depth in the next subsection.

A very important operation that can be applied to complex numbers is **conjugation**. The conjugate of a complex number $z = a + bi$ is defined as

$$\bar{z} = a - bi, \tag{0.6.6}$$

i.e. conjugating a number is simply negating its imaginary part. When we multiply a complex number by its own complex conjugate we get

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2 i^2 = a^2 + b^2, \tag{0.6.7}$$

i.e. $z\bar{z} = |z|^2$. The inverse of a complex number can be expressed as

$$z^{-1} = \frac{\bar{z}}{|z|^2}. \tag{0.6.8}$$

GEOMETRIC APPROACH

As alluded to in the previous subsection, we can interpret a complex number $z = a + bi$ as two components in a 2-dimensional space (called the **complex plane**), in which the horizontal axis represents real components, and the vertical access represents imaginary components:

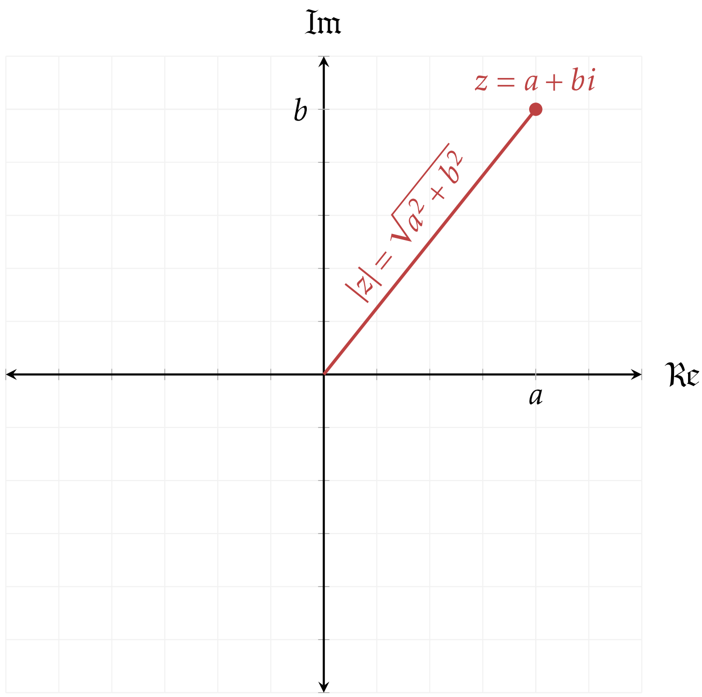


Figure 0.6 Text.

Drawing a line from z to a (on the real axis) creates a right triangle. We can then define θ to be the angle near the origin and r the length of the hypotenous:

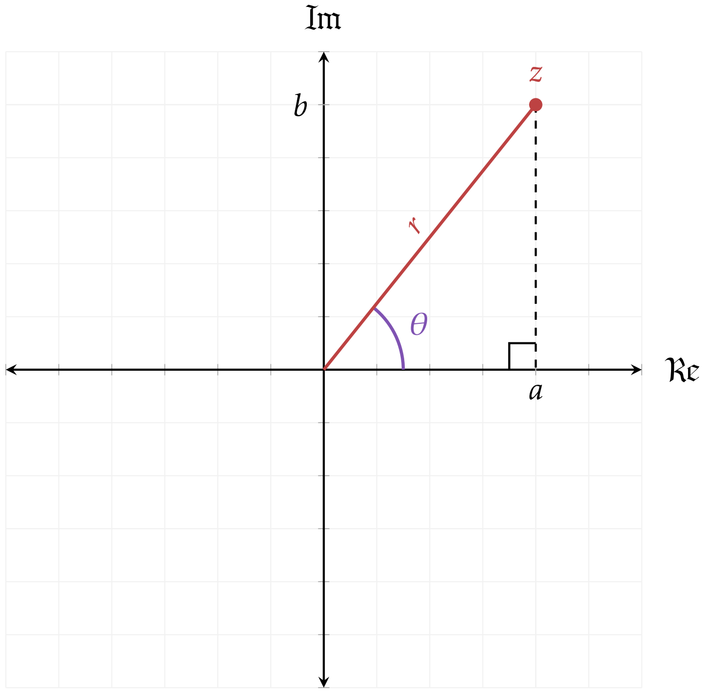


Figure 0.7 Text.

We call r the **magnitude** of z, and θ its **argument**.

Using Equation 0.5.6 the real and imaginary components of z are

$$\begin{aligned} a &= r\cos(\theta), \\ b &= r\sin(\theta), \end{aligned} \tag{0.6.9}$$

and z can be re-written as

$$z = r\left(\cos(\theta) + i\sin(\theta)\right). \tag{0.6.10}$$

Which we call the **polar form** of z (contrasted with $z = a + ib$ being the **Cartesian form** of z).

Inverting the relations in Equation 0.6.9 yields the relations

$$\begin{aligned} r &= a^2 + b^2, \\ \theta &= \arctan\left(\frac{b}{a}\right). \end{aligned} \tag{0.6.11}$$

Let's examine the same properties of complex numbers shown in Equations 0.6.3, 0.6.4 and 0.6.6, and verify that they work in the polar form of complex numbers. We start with addition (Equation 0.6.3):

$$\begin{aligned} z_1 + z_2 &= r_1\left[\cos(\theta_1) + i\sin(\theta_1)\right] + r_2\left[\cos(\theta_2) + i\sin(\theta_2)\right] \\ &= \underbrace{r_1\cos(\theta_1)}_{a_1} + \underbrace{r_2\cos(\theta_2)}_{a_2} + i\,\underbrace{r_1\sin(\theta_1)}_{b_1} + i\,\underbrace{r_2\sin(\theta_2)}_{b_2} \\ &= (a_1 + a_2) + i\,(b_1 + b_2). \end{aligned} \tag{0.6.12}$$

We see that indeed, the polar form of complex numbers adheres to the addition rule in Equation 0.6.3 . Next is the product rule:

$$\begin{aligned} z_1 z_2 &= r_1\left[\cos(\theta_1) + i\sin(\theta_1)\right] \cdot r_2\left[\cos(\theta_2) + i\sin(\theta_2)\right] \\ &= r_1 r_2\left[\cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right] \\ &= r_1\cos(\theta_1)\,r_2\cos(\theta_2) - r_1\sin(\theta_1)\,r_2\sin(\theta_2) + i\left[r_1\cos(\theta_1)\,r_2\sin(\theta_2) + r_1\sin(\theta_1)\,r_2\cos(\theta_2)\right] \\ &= (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1), \end{aligned} \tag{0.6.13}$$

which is indeed the result seen in Equation 0.6.4 . We can also develope further the second row of Equation 0.6.13 using some trigonometry (specifically the trigonometric identities xx and yy):

$$\begin{aligned} z_1 z_2 &= r_1 r_2\left[\cos(\theta_1)\cos(\theta_2) + i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right] \\ &= r_1 r_2\left[\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\left[\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)\right]\right] \\ &= r_1 r_2\left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right]. \end{aligned} \tag{0.6.14}$$

This is a very important result: it shows that multiplying a complex number z_1 by another complex number z_2 gives a complex number with magnitude $r_1 r_2$, i.e. the product of the magnitudes of the two complex numbers, and argument $\theta_1 + \theta_2$, i.e. the argument of z_1 rotated by the argument of z_2 (or vice-versa). We will consider this result in more detail soon.

In the polar form the complex conjugate of a number $z = r\left[\cos(\theta) + i\sin(\theta)\right]$ can be brought about by substituting $-\theta$ into the arguments of the trigonometric functions:

$$\begin{aligned} \bar{z} &= r\left[\cos(-\theta) + i\sin(-\theta)\right] \\ &= r\left[\cos(\theta) - i\sin(\theta)\right] \\ &= r\cos(\theta) - ir\sin(\theta) \\ &= a - ib. \end{aligned} \tag{0.6.15}$$

Lastly, let's show that Equation 0.6.7 can be derived in the polar form:

$$\begin{aligned} z\bar{z} &= r\left[\cos(\theta) + i\sin(\theta)\right] \cdot r\left[\cos(\theta) - i\sin(\theta)\right] \\ &= r^2\left[\cos^2(\theta) - \cancel{i\cos(\theta)\sin(\theta)} + \cancel{i\sin(\theta)\cos(\theta)} + \sin^2(\theta)\right] \\ &= r^2\left[\sin^2(\theta) + \cos^2(\theta)\right] \\ &= r^2 = a^2 + b^2. \end{aligned} \tag{0.6.16}$$

In 1748 Leonhard Euler published his famous work *Introductio in analysin infinitorum*[7]. In it he introduced the following relation, called **Euler's formula**:

$$e^{ix} = \sin(x) + i\cos(x). \tag{0.6.17}$$

[7]Latin for **Introduction to the Analysis of the Infinite**.

Using Euler's formula a complex number z can be written as

$$z = re^{i\theta}. \tag{0.6.18}$$

In **Table 0.5** we can see some usefull complex exponentials e^{xi}. Specifically, setting $x = \pi$ yields the famous **Eurler's identity**, considered by many to be one of the most beautiful equations in mathematics, as it binds together five important numbers, namely $0, 1, \pi, e$ and i:

$$e^{\pi i} + 1 = 0. \tag{0.6.19}$$

Table 0.5 Values of e^{ix} for some useful values of x (cf. xxxx for the values of $\sin(\theta)$ and $\cos(\theta)$).

x	$\cos(x)$	$\sin(x)$	$z = e^{ix}$
$\frac{\pi}{2}$	0	1	i
π	-1	0	-1
$\frac{3\pi}{2}$	0	-1	$-i$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}\left(1 + i\sqrt{3}\right)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}(1 + i)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{2}\left(\sqrt{3} + i\right)$

Table 0.5 also shows us the integer behaviours of i:

$$i^2 = -1,\ i^3 = -i, i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i,\ \ldots \tag{0.6.20}$$

COMPLEX ROOTS OF POLYNOMIAL FUNCTIONS

We began this section discussing an equation that can be solved over the real numbers (Equation 0.6.1). Let us now use similar equations to find roots of complex numbers, focusing on equtions of the type

$$z^n = 1, \tag{0.6.21}$$

where $n \in \mathbb{N}$. Over the real numbers, depending on the parity of n (i.e. whether it is even or odd) there are either two solutions (namely $z_{1,2} = \pm 1$) or a single solution (namely $z = 1$). However over the complex numbers there are always exactly n solutions to the equation. Since we already know the solutions to $z^2 = 1$, we move on to $z^3 = 1$. One solution to the equation is pretty obvious: $z_1 = 1$. To find the other two solutions we switch to the polar form, in which 1 has magnitude $r = 1$ and argument $\theta = 0$, i.e.

$$1 = \cos(0) + i\sin(0). \tag{0.6.22}$$

We saw earlier that multiplying two complex numbers $z_1 z_2$ together results in a number which has magnitude that is the product of the magnitudes of z_1 and z_2, and an argument which is the sum of the arguments of z_1 and z_2 (Equation 0.6.14). This means that the two remaining solutions to $z^3 = 1$ must have magnitude $r = 1$, and arguments θ such that $3\theta = 0$. Recall however that we are dealing with a circle of radius 1, meaning that there are infinitely many angles which corresponds to a full circle, namely $\theta = 2\pi k$ where $k \in \mathbb{Z}$. There are exactly two angles in $[0, 2\pi)$ which when multipled by 3 result in a complete circle: $\theta_2 = \frac{2\pi}{3}$ and $\theta_3 = \frac{4\pi}{3}$.

Therefore, the three solutions to $z^3 = 1$ are (see **Figure 0.8**):

$$
\begin{aligned}
z_1 &= \cos(0) + i\sin(0),\\
z_2 &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right),\\
z_3 &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right),
\end{aligned} \tag{0.6.23}
$$

which correspond to the following Cartesian forms:

$$
\begin{aligned}
z_1 &= 1,\\
z_2 &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\\
z_3 &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i.
\end{aligned} \tag{0.6.24}
$$

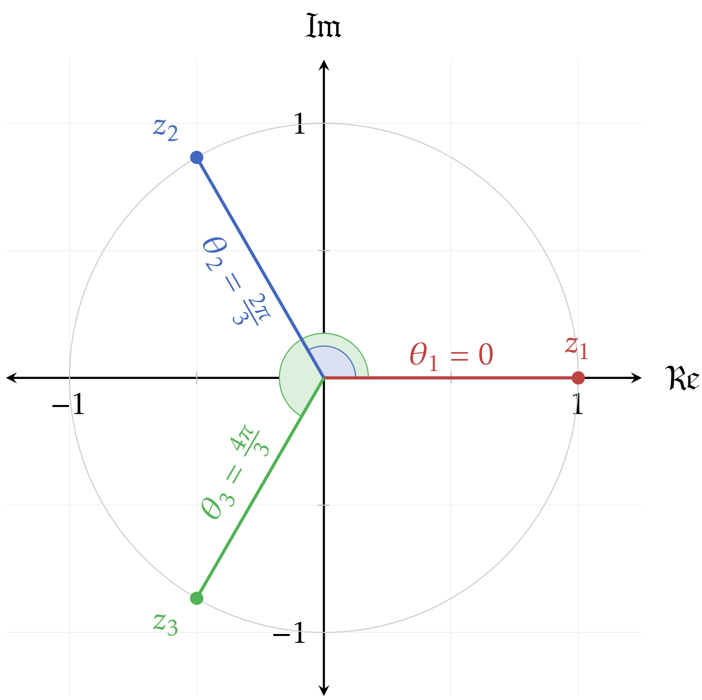


Figure 0.8 The three complex solutions to $z^3 = 1$.

We can verify that z_2 indeed solves the equation[8]:

$$\begin{aligned}
\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)^3 &= -\frac{1}{8}+3\frac{\sqrt{3}}{8}i-3\frac{3}{8}i^2+\frac{3\sqrt{3}}{8}i^3 \\
&= -\frac{1}{8}+3i\cancel{\frac{\sqrt{3}}{8}}+3\frac{3}{8}-3i\cancel{\frac{\sqrt{3}}{8}} \\
&= -\frac{1}{8}+\frac{9}{8} \\
&= 1. \qquad (0.6.25)
\end{aligned}$$

Verfying that z_3 solves the equation as well is left as an excercise to the reader.

[8] Using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.