MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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with contributions from others

$$a^b = e^{b \log(a)}$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$T\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$$

$$A = Q\Lambda Q^{-1}$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t$$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

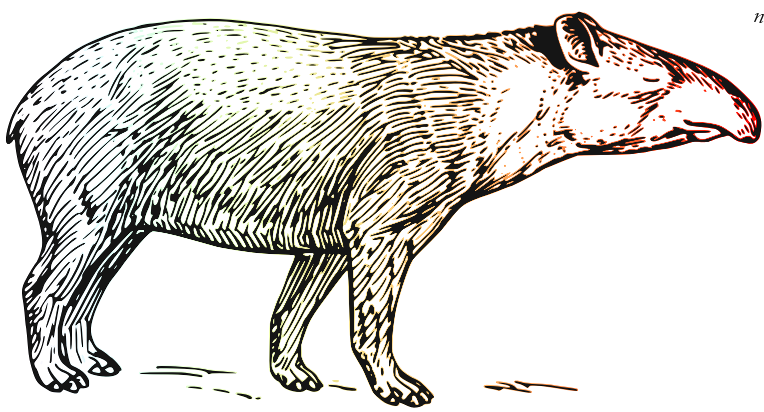
$$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$$

$$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$$

$$e^{\pi i} + 1 = 0$$

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$



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! **To be written/to do**: Rights, lefts, etc. will be written here in the future !

HERE BE TABLE

CHAPTER

0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

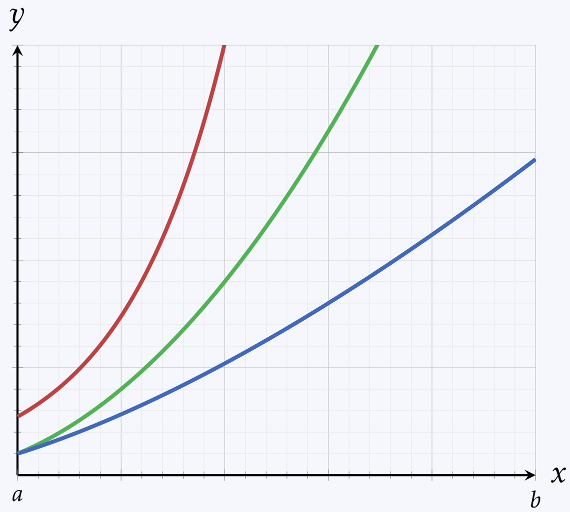
0.1 DERIVATIVES

0.1.1 Introduction

One of the most important tools in analyzing real functions is the ability to quantitatively describe the way they behaves as we change the argument x. At any given point a function can either increase in its value, decrease in its value, stay constant or be undefined. In this section we will explore a method which enables us to quantitatively measure the change in a function's value at (almost) any point in its domain.

Example 0.1 Quantitative measure of change

Compare the following three functions on the domain $x \in [a, b]$:



While all three functions are increasing on $[a, b]$ it is clear that the rate of increase is different in each function: the red function increases faster than the green one, which in turn increases faster than the blue one. In fact, even within each function the increase is not uniform: the more x increases so does the rate of increase of each of the functions.

Any fundamental real function has the following property: if we zoom in enough on some point $\boldsymbol{p} = \left(a, f(a)\right)$ on the function, we would see that it behaves somewhat like a straight line around $\boldsymbol{p}$ (Figure 0.1). In fact, the more we zoom in, the more the function becomes linear around $\boldsymbol{p}$. At the limit where the zoom factor is inifinite, the function is exactly linear around $\boldsymbol{p}$, and has the same direction (i.e. slope) as the tangent line to the function at $\boldsymbol{p}$ (Figure 0.2). We call this slope the **derivative** of f at $x = a$, and denote it as $f'(a)$.

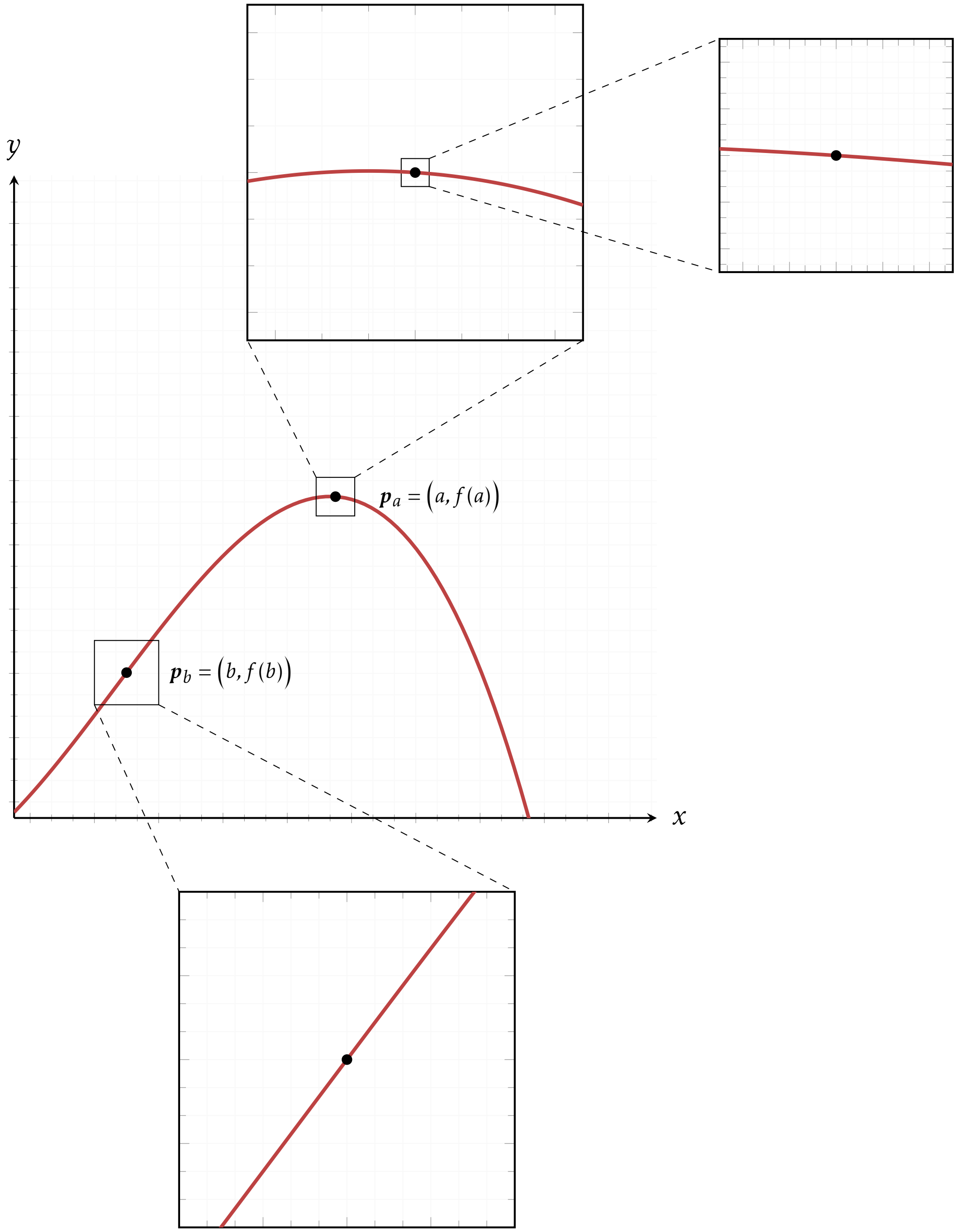


Figure 0.1 Zooming in on a real function f at two points: $\boldsymbol{p}_a = \big(a, f(a)\big)$ (upper right) and $\boldsymbol{p}_b = \big(b, f(b)\big)$ (bottom right). Note how around each of the points, the function looks somewhat linear: this is more pronounced around $\boldsymbol{p}_b$ where the function looks linear in the entire zoomed-in area, while near $\boldsymbol{p}_a$ it looks linear only near the point itself even though the zoom factor is higher.

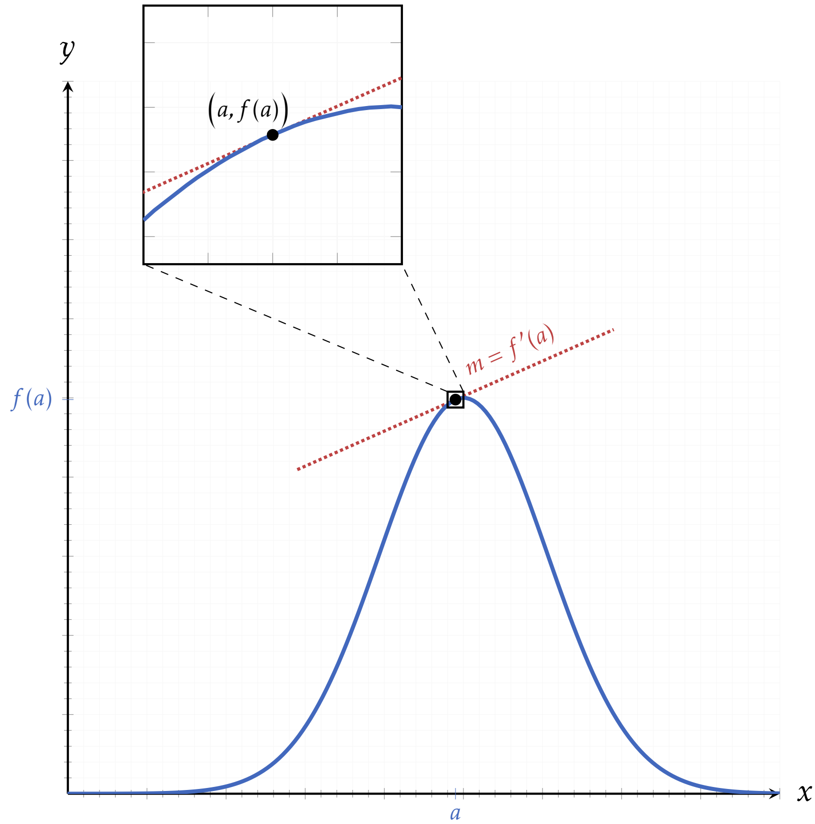


Figure 0.2 The derivative of the function $f(x)$ at $x = a$ is equal to the slope of the tangent line to f at the point $\big(a, f(a)\big)$.

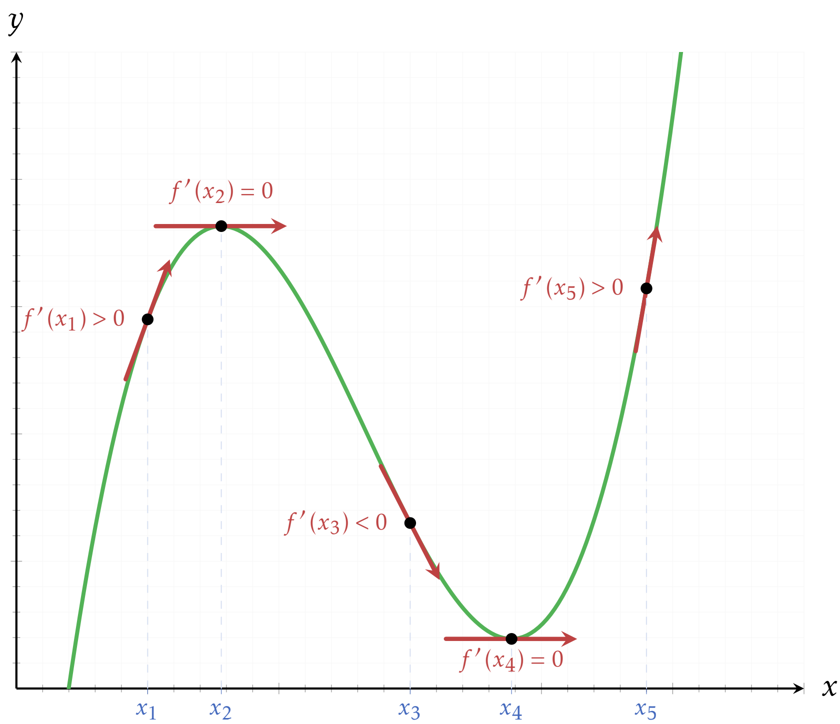


Figure 0.3 The derivative of a function $f(x)$ at some points $\boldsymbol{p}_i = \big(x_i, f(x_i)\big)$ on the function: when f increases the derivative is positive ($\boldsymbol{p}_1, \boldsymbol{p}_5$), when it decreases the derivative is negative ($\boldsymbol{p}_3$), and when it is stationary the derivative equals zero ($\boldsymbol{p}_2, \boldsymbol{p}_4$). The slope of the tangent line to f at each of the points $\boldsymbol{p}_i$ is drawn as an arrow to make its sign (positive, negative or zero) more clear.

0.1.2 Sign and stationary points

The derivative of a function f at the point a can be one of four possible categories (Figure 0.3):

- $f(a) > 0$, meaning that f **increases** at a.
- $f(a) < 0$, meaning that f **decreases** at a.
- $f(a) = 0$, meaning that a is a **stationary point**.
- $f(a)$ is undefined, which can mean different things and which we will address later.

! **To be written/to do**: a graph of a “complicated” function together with its derivative !

! **To be written/to do**: when the derivative isn’t defined !

! **To be written/to do**: differentiability on an interval !

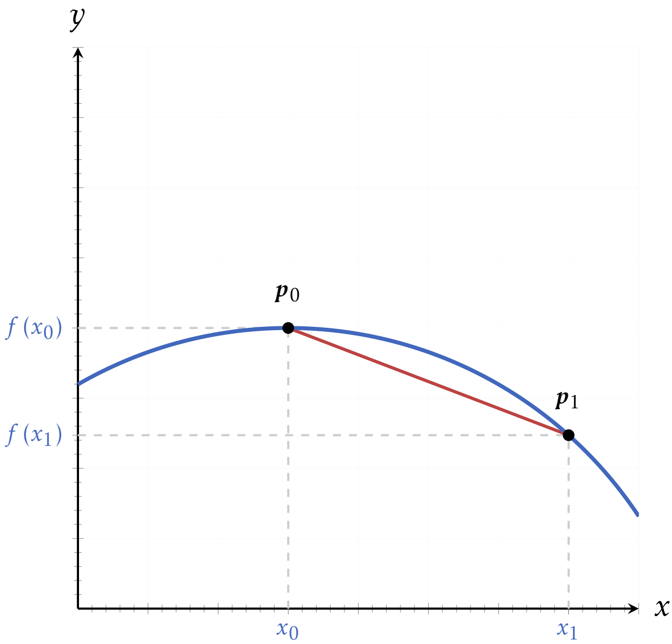


Figure 0.4 Text.

! **To be written/to do**: a summary of what we learned so far !

0.1.3 Calculating the derivative

How can we quantify the derivative? Let us consider some real function f and a point $\boldsymbol{p}_0 = \left(x_0, f(x_0)\right)$ on the function. We can then define another point to the right of x_0: $\boldsymbol{p}_1 = \left(x_1, f(x_1)\right)$. Since x_1 is to the right of x_0 we can write it as $x_1 = x_0 + \Delta x$, where $\Delta x > 0$. We then connect the two points with a line (Figure 0.4). The slope of this line can then be calculated using **??**:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{x_0 + \Delta x - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{0.1.1}$$

We can then take the limit of Equation 0.1.1 as $\Delta x \to 0$ (Figure 0.5):

$$M = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{0.1.2}$$

This limit is defined as **the derivative** of f at the point $x = x_0$, and it tells us, quantitatively, how f locally behaves at x_0, i.e. how much does it increase, decrease or stay the same around x_0.

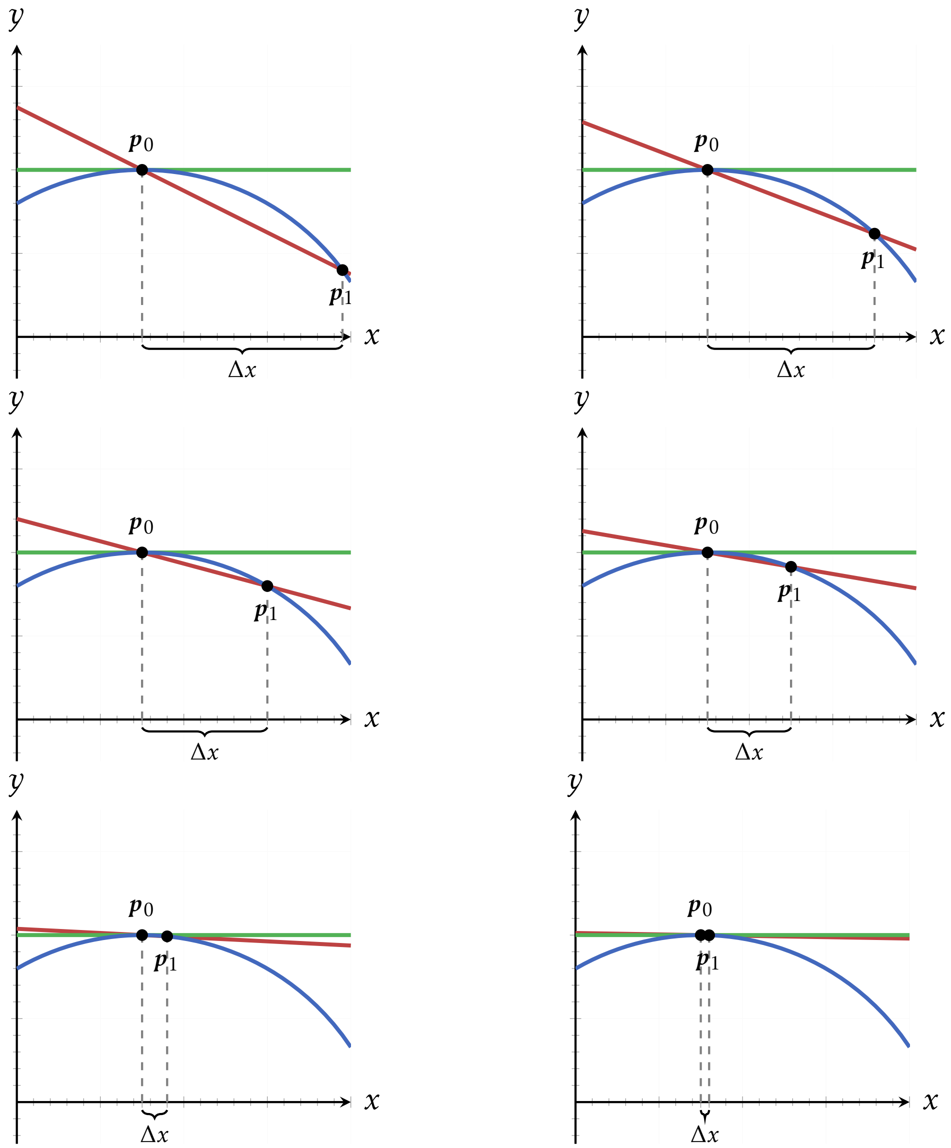


Figure 0.5 As $\Delta x \to 0$, $\boldsymbol{p}_1$ approaches $\boldsymbol{p}_0$ and the slope of the red line connecting the two points approches the slope M of the green line at $\boldsymbol{p}_0$.

Example 0.2 Validation of the derivative using a linear function

Given a linear function $f(x) = mx + b$, we expect that the derivative of f at any point x_0 would equal m, since the entire function is a line connecting all the points on the function itself. Let us check that:

$$\begin{aligned} M &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{m(x_0 + \Delta x) + \cancel{b} - \left(mx_0 + \cancel{b}\right)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cancel{mx_0} + m\Delta x - \cancel{mx_0}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{m\cancel{\Delta x}}{\cancel{\Delta x}} \\ &= m. \end{aligned}$$

Example 0.3 Derivative of x^2

Unlike for a linear function, we shouldn't expect the derivate of $f(x) = x^2$ to be constant at any point. However, we can easily calculate what the derivative would be at some point x_0:

$$\begin{aligned} M &= \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cancel{x_0^2} + 2x_0\Delta x - (\Delta x)^2 - \cancel{x_0^2}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x_0 - \Delta x)}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} 2x_0 - \Delta x \\ &= 2x_0. \end{aligned}$$

I.e. we see that any point x_0 the derivative of $f(x) = x^2$ is simply $2x_0$.
For example, at $x_0 = 3$ the derivative is $M = 6$, and at $x_0 = 0$ the derivative is $M = 0$.

Up until now we have regarded the derivative as a property of some point on a function f. However, since we can calculate the derivative at each point of the function[1], we can collect all these points together to form a new function, which we call the **derivative** of f and denote as f' (read: "f-prime").

In Example 0.2 we saw that the derivative of a linear function at any point gives the same value m (namely the slope of the linear function). Therefore, this derivative is itself a *constant* function $f'(x) = m$. When we calculated the derivative of $f(x) = x^2$ (Example 0.3), we found that it depends on the point where it was calculated, using the

[1] except for some points which we will discuss later.

relation $f'(x) = 2x$, which is a linear function with slope 2 that goes through the origin.

Let us now calculate the derivative of some common functions.

Example 0.4 Derivative of ax^n

The derivative of the function $f(x) = x^n$ (where $a \in \mathbb{R}$ is a constant) is (recall the binomial expansion, **??**):

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\cancel{x^n} + nx^{n-1}\Delta x + \binom{2}{n}x^{n-2}(\Delta x)^2 + \cdots + nx(\Delta x)^{n-1} + (\Delta x)^n - \cancel{x^n}}{\Delta x}. \end{aligned}$$

We can take Δx out of the numerator and cancel it out with the Δx in the denominator:

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\left[nx^{n-1} + \binom{2}{n}x^{n-2}\Delta x + \cdots + nx(\Delta x)^{n-2} + (\Delta x)^{n-1}\right]}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} nx^{n-1} + \binom{2}{n}x^{n-2}\Delta x + \cdots + nx(\Delta x)^{n-2} + (\Delta x)^{n-1}. \end{aligned}$$

Since all expressions except nx^{n-1} have some power of Δx in them, in the limit $\Delta x \to 0$ they all vanish, leaving us with

$$f'(x) = nx^{n-1}.$$

This derivative is commonly described as the power of x being reduced by 1 and the expression gaining a factor of n (i.e. the power before reducing it).

Example 0.5 Derivative of a^x

Calculating the derivative of a^x:

$$\begin{aligned} \left(a^x\right)' &= \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{a^x(a^{\Delta x} - 1)}{\Delta x} \\ &= a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}. \end{aligned}$$

(we take a^x out of the limit expression because it isn't affected by a change in Δx) According to **??**, the limit in the last expression is one of the defintions of $\ln(a)$, and therefore

$$\left(a^x\right)' = a^x \ln(x).$$

In the case where $a = \mathrm{e}$, we get

$$\left(\mathrm{e}^x\right)' = \mathrm{e}^x \ln(\mathrm{e}) = \mathrm{e}^x,$$

i.e. e^x is its own derivative.

Example 0.6 Derivative of $\sin(x)$

Calculating the derivative of $\sin(x)$:

$$\begin{aligned}\sin'(x) &= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\sin(x)\cos(\Delta x)+\cos(x)\sin(\Delta x)-\sin(x)}{\Delta x}.\end{aligned}$$

We can separate the three terms into three limits:

$$\sin'(x) = \lim_{\Delta x\to 0}\frac{\sin(x)\left[\cos(\Delta x)-1\right]}{\Delta x} + \lim_{\Delta x\to 0}\frac{\cos(x)\sin(\Delta x)}{\Delta x}.$$

The second limit equals $\cos(x)$, since $\lim\limits_{c\to 0}\frac{\sin(c)}{c} = 1$ (**??**). Since $\sin(x)$ does not change as we decrease Δx, we can regard it as a constant and take it out of the limit:

$$\sin'(x) = \sin(x)\lim_{\Delta x\to 0}\frac{\cos(\Delta x)-1}{\Delta x} + \cos(x).$$

Using the double angle identity (**??**) on $\cos(\Delta x)$ we get that

$$\cos(\Delta x) = 1-2\sin^2\left(\frac{\Delta x}{2}\right),$$

and by plugging this back into the derivative calculation we get:

$$\begin{aligned}\sin'(x) &= \lim_{\Delta x\to 0}\frac{-2\sin^2\left(\frac{\Delta x}{2}\right)}{\Delta x} + \cos(x)\\ &= \lim_{\Delta x\to 0}\frac{-\cancel{\frac{2}{2}}\sin^2\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} + \cos(x)\\ &= \cos(x),\end{aligned}$$

since $\lim\limits_{h\to 0}\frac{\sin^2(h)}{h} = 0$ (**??**).

! **To be written/to do**: in the limit section show and prove this limit !

Example 0.7 Derivatie of $\sqrt{x}$

The derivative of the function $f(x)=\sqrt{x}$ is:

$$f'(x) = \lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}.$$

We can multiply the numerator and denominator each by $\sqrt{x+\Delta x}+\sqrt{x}$. This would allow us to use the relation $(a-b)(a+b)=a^2-b^2$:

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\cancel{x}+\cancel{\Delta x}-\cancel{x}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x}+\sqrt{x}\right)} \\ &= \frac{1}{\sqrt{x}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

Table 0.1 lists some common functions and their derivatives.

Table 0.1 Some common real functions and their derivatives.

$f(x)$	$f'(x)$	**Remarks**
c	0	$c \in \mathbb{R}$
$mx+b$	m	$m, b \in \mathbb{R}$
x^2	$2x$	
x^n	nx^{n-1}	
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$x \geq 0$
e^x	e^x	
a^x	$a^x \log(a)$	$a > 0$
$\ln(x)$	$\frac{1}{x}$	$x > 0$
$\log_b(x)$	$\frac{1}{x\ln(b)}$	$x, b > 0$
$\sin(x)$	$\cos(x)$	
$\cos(x)$	$\sin(x)$	
$\tan(x)$	$\frac{1}{\cos^2(x)} = 1+\tan^2(x)$	$x \notin \left\{\frac{\pi}{2}+k\right\}, k \in \mathbb{Z}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$	$-1 < x < 1$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$	$-1 < x < 1$
$\arctan(x)$	$\frac{1}{1+x^2}$	
$\sinh(x)$	$\cosh(x)$	
$\cosh(x)$	$\sinh(x)$	
$\tanh(x)$	$1-\tanh^2(x)$	

0.2 LINEARITY AND COMBINED FUNCTIONS

More often than not we need to derive functions which aren't simple fundamental functions. For example, we might want to derive the polynomial

$$P(x) = 2x^4 - 3x^3 - 2x + 7.$$

Using the rule we found in Example 0.4 we can derive each term of P separately, yielding the following:

$$\left(2x^4\right)' = 8x^3, \quad \left(-3x^3\right)' = -9x^2, \quad (-2x)' = -2, \quad (7)' = 0.$$

But how do we derive the entire polynomial? Luckily for us there's an exceedingly simple rule for getting the derivative of any addition of two functions: we simply derive each of the functions separately and add the result. For example, the derivative of P is simply the sum of the derivatives of each of the terms of P:

$$P'(x) = 8x^3 - 9x^2 - 2.$$

Example 0.8 Derivatives of sums

The derivative of the function

$$f(x) = 4x^7 + \sin(x) + 5^x + \log_7(x)$$

is

$$\begin{aligned} f'(x) &= \left(4x^7\right)' + (\sin(x))' + \left(5^x\right)' + \left(\log_7(x)\right)' \\ &= 28x^6 + \cos(x) + 5^x \ln(5) + \frac{1}{x\ln(7)}. \end{aligned}$$

In the most general case, given two real functions f and g which are differentiable over some interval I, then for any $x \in I$

$$\left(\left[f+g\right](x)\right)' = f'(x) + g'(x). \tag{0.2.1}$$

A similar idea also applies to functions which are scaled by a real number α: if f is differentiable on some interval I, then for any $x \in I$

$$\left(\alpha f(x)\right)' = \alpha f'(x). \tag{0.2.2}$$

Example 0.9 Derivative of scaled functions

The derivative of the function

$$f(x) = \frac{1}{2}\cos(x)$$

is simply

$$f'(x) = -\frac{1}{2}\sin(x).$$

! **To be written/to do**: the derivative as a linear operator !