MATHEMATICS FOR SCIENCE STUDENTS

An open-source book

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$a^b = e^{b\log(a)}$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$T\left(\alpha\vec{u} + \beta\vec{v}\right) = \alpha T\left(\vec{u}\right) + \beta T\left(\vec{v}\right)$

$A = Q\Lambda Q^{-1}$

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$

$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

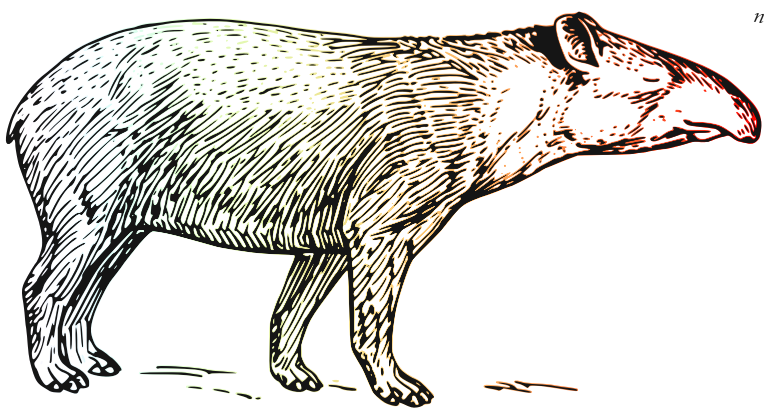
$\langle \hat{e}_i, \hat{e}_j \rangle = \delta_{ij}$

$\vec{v} = \sum_{i=1}^{n} \alpha_i \hat{e}_i$

$e^{\pi i} + 1 = 0$

$\int_a^b f(x)\,dx = F(b) - F(a)$

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$



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CHAPTER 0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explanations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is advised to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concepts presented here.

!

0.1 EIGENVECTORS AND EIGENVALUES

Some linear transformations have special directions which only scale by the application of the transformation and are not mapped to different directions. Take for example the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, which scales space by 2 in the y-direction. All vectors pointing in the y-direction get scaled by T (namely by a factor of 2) and still point in the y-direction after the application of T. All vectors pointing in the x-direction do not change at all (i.e. they are "scaled" by a factor of 1), and of course still point in the x-direction after the application of T. Any other vector - i.e. those that have both components different than zero - change their direction after the application of T (see Figure 0.1).

We call such vectors the **eigenvectors** of the transformation. The amount by which the are scaled is then their respective **eigenvalues**.

Example 0.1 Eigenvectors and eigenvalues

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In matrix form, a vector $\vec{v}$ is an eigenvector of a transformation represented by the matrix A, if

$$A\vec{v} = \lambda\vec{v}, \tag{0.1.1}$$

where $\lambda \in \mathbb{R}$, $\lambda \neq 0$. This kind of equation is typically called an **eigenvector equation**. When there are several eigenvectors for a transformation, each with its distinct eigenvalue, we simply add indeces to all relevant parts:

$$A\vec{v}_i = \lambda_i\vec{v}_i, \tag{0.1.2}$$

where again $\lambda_i \in \mathbb{R}$ and $\lambda_i \neq 0$.

Before continuing to explore some more examples of eigenvectors, there are two properties[1] of eigenvectors that are important to mention. Given a linear transformation T,

- A scale of any eigenvector $\vec{v}$ of T is also an eigenvector of T, with the same eigenvalue.

Proof 0.1 Eigenvector scale

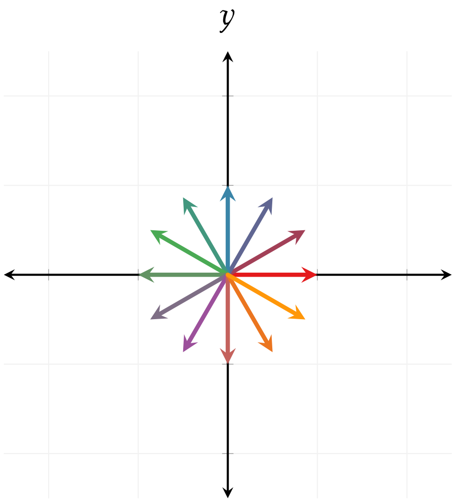
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation represented by the square matrix A, with eigenvector $\vec{v}$ and its respective eigenvalue λ. Then

$$A\vec{v} = \lambda\vec{v}.$$

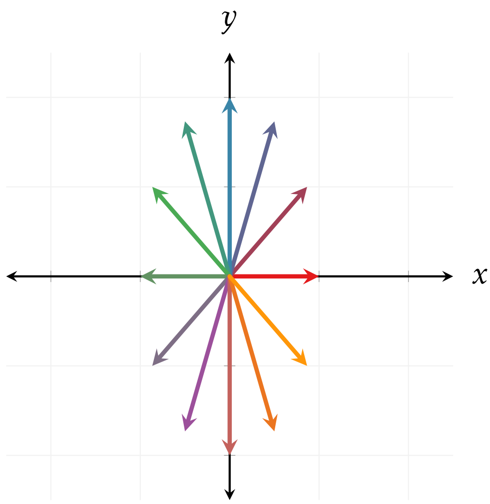
Replacing $\vec{v}$ with a scale of itself, i.e. $\vec{u} = \alpha\vec{v}$, then applying A to $\vec{u}$ gives us

$$A\vec{u} \overset{(1)}{=} A\left(\alpha\vec{v}\right) \overset{(2)}{=} \alpha A\vec{v} \overset{(3)}{=} \alpha\lambda\vec{v} \overset{(4)}{=} \lambda\alpha\vec{v} \overset{(5)}{=} \lambda\vec{u}.$$

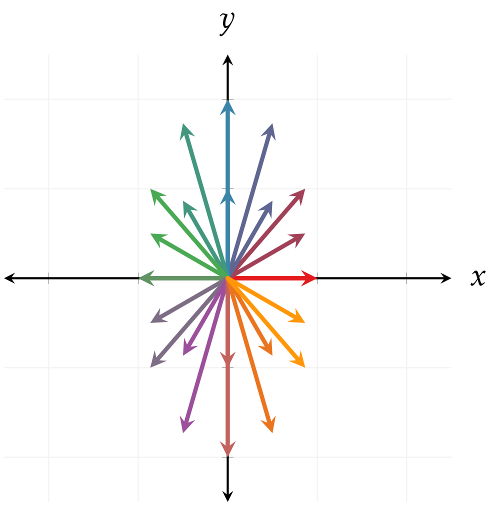
[1] actually one property and one non-property



(a) Some vectors.



(b) Same vectors after the application of T.



(c) The vectors before and after the application of T layered on top of eachother.

Figure 0.1 Some vectors before and after application of the y-scaling transformation T. Note how only the vectors pointing in the direction of the x- and y-axes stay in the same direction, while all the other vectors change their directions.

where

(1) Substitution of $\vec{u}$ by its definition $\vec{u} = \alpha\vec{v}$.

(2) Due to the linearity of A we can bring α out of the product.

(3) Resulting due to $\vec{v}$ being an eigenvector of A.

(4) The product of real numbers is commutative.

(5) Substituting back $\alpha\vec{v} = \vec{u}$.

Therefore, $\vec{u}$ is also an eigenvector of A (and thus T) with the same eigenvalue λ as $\vec{v}$.

QED

- The linear combination of two eigenvectors of T is **not neccessarily an eigenvector of $\boldsymbol{T}$**! For example, consider the above transformation which scales all vectors by 2 in the y-direction: as we saw, any vector in the x-direction is an eigenvector of the transformation, and so does any vector in the y-direction. Specifically, the vectors $\vec{a} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}0\\1\end{bmatrix}$ are two separate eigenvectors of the transformation (with eigenvalues 1 and 2, respectively), however the vector

$$\vec{c} = \vec{a} + \vec{b} = \begin{bmatrix}1\\1\end{bmatrix}$$

is **NOT** an eigenvector of the transformation, since

$$\begin{bmatrix}1\\1\end{bmatrix} \overset{T}{\mapsto} \begin{bmatrix}1\\2\end{bmatrix} \neq 2\begin{bmatrix}1\\1\end{bmatrix}.$$