CHAPTER 0

INTRODUCTION

In this chapter we introduce key concepts that will be used in later chapters. For this reason, unlike other chapters it contains many statements, sometimes given without thorough explainations or reasoning. While all of these statements are grounded in deep ideas and can be formulated in a rigorous manner, it is adviced to first get an intuitive understanding of the ideas before diving into their more formal construction.

Note 0.1 In case you are already familiar with the topics

It is recommended for readers who are familiar with the topics to at least gloss over this chapter and make sure they know and understand all the concept presented here. !

0.1 COMPLEX NUMBERS

ALGEBRAIC APPROACH

Real numbers, while being extremely useful, are not complete - they can't solve all equations involving numbers. For example, the equation

$$x^2 + 1 = 0 \tag{0.1.1}$$

has no real solutions, since there can be no real number x such that $x^2 = -1$. However, we can choose to define a new number, $i = \sqrt{-1}$ and using it to build a new number system. This system is of course the set of complex numbers, $\mathbb{C}$. It is defined as the set of all z such that

$$z = a + ib, \tag{0.1.2}$$

where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. We call a the **real component** of z or $\mathfrak{Re}(z)$, and b its **imaginary component** or $\mathfrak{Im}(z)$[1]. These numbers appear a lot all throughout the exact sciences (but especially in physics and engineering), so we must at the very least learn their basic properties.

It is not so obvious that we can add two different kinds of numbers together, but it works (the linear algebra chapter sheds more light on this idea). What is important is that we always keep these two parts separated. We see this when we add together two complex numbers z_1, z_2:

$$z = z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i. \tag{0.1.3}$$

The real part of z is therefore $a_1 + b_1$, and its imaginary part is $b_1 + b_2$.

What happens when we multiply two complex numbers? Let's check:

$$\begin{aligned} z = z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= a_1 a_2 + i a_1 b_2 + i a_2 b_1 + i^2 b_1 b_2 \\ &= a_1 a_2 + i a_1 b_2 + i a_2 b_1 - b_1 b_2 \\ &= (a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1). \end{aligned} \tag{0.1.4}$$

We see that we can still separate the real part and imaginary part of the result. What happens in the case of two real numbers? For real numbers $b = 0$, and thus Equation 0.1.4 devolves to $z = a_1 a_2 \in \mathbb{R}$, which is exactly what we expect: multiplying two real numbers yields their product, which is a real number. Notice that this doesn't happen with purely imaginary numbers: multiplying together two imaginary numbers (i.e. numbers for which $a = 0$) results in a real number. Will get to understand why this happens very soon.

When discussing real numbers sometimes we like to refer to their *magnitude*, i.e. their absolute value. With complex numbers this is defined as

$$|z| = \sqrt{a^2 + b^2}, \tag{0.1.5}$$

[1]There is nothing more "real" about real numbers than imaginary numbers, but unfortunately that's the terminology we're stuck with ¯_(ツ)_/¯

i.e. in a sense, to get the magnitude of a complex number we imagine its two components as being perpendicular and calculate the length of the resulting hypotenous (cf. the Pythagorean theorem). In fact, this is one very useful interpertation of complex numbers, which we will explore in depth in the next subsection.

A very important operation that can be applied to complex numbers is **conjugation**. The conjugate of a complex number $z = a + bi$ is defined as

$$\overline{z} = a - bi, \tag{0.1.6}$$

i.e. conjugating a number is simply negating its imaginary part. When we multiply a complex number by its own complex conjugate we get

$$z\overline{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2 i^2 = a^2 + b^2, \tag{0.1.7}$$

i.e. $z\overline{z} = |z|^2$. The inverse of a complex number can be expressed as

$$z^{-1} = \frac{\overline{z}}{|z|^2}. \tag{0.1.8}$$

GEOMETRIC APPROACH

As alluded to in the previous subsection, we can interpret a complex number $z = a + bi$ as two components in a 2-dimensional space (called the **complex plane**), in which the horizontal axis represents real components, and the vertical access represents imaginary components:

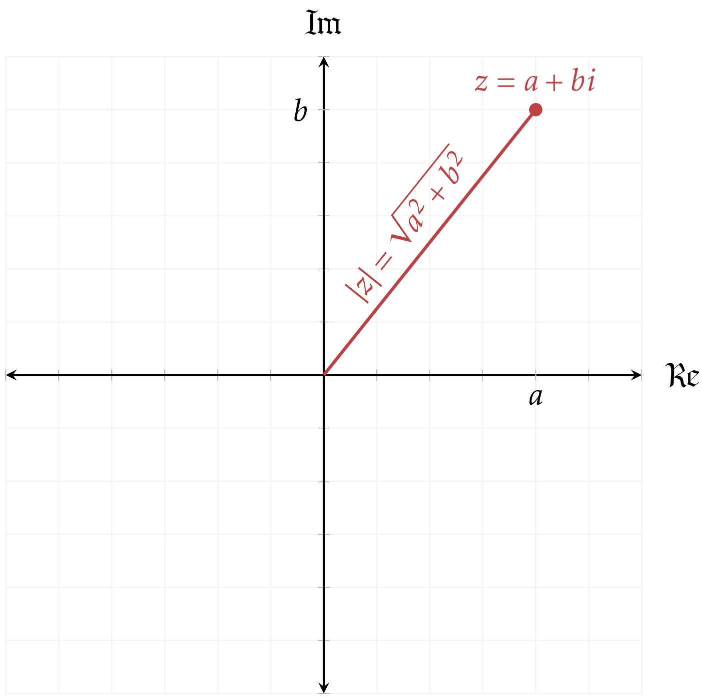

Figure 0.1 Text.

Drawing a line from z to a (on the real axis) creates a right triangle. We can then define θ to be the angle near the origin and r the length of the hypotenous:

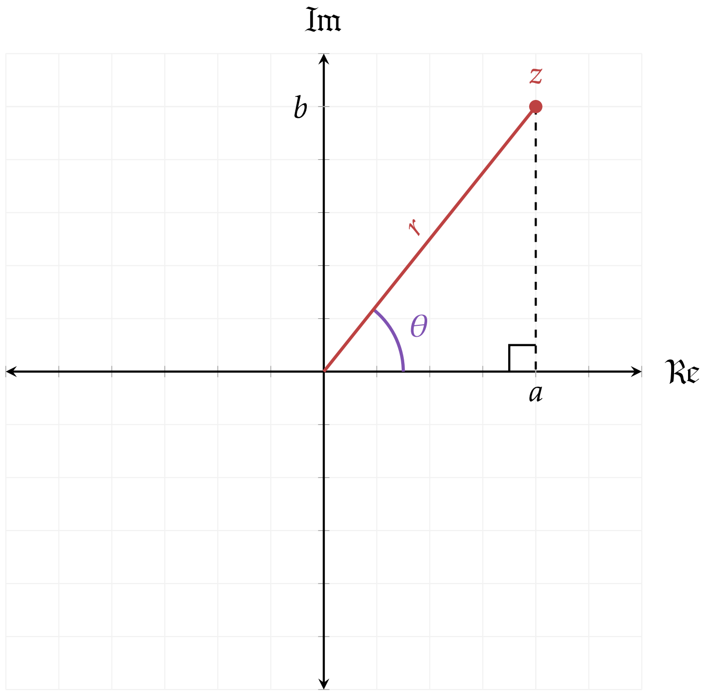

Figure 0.2 Text.

We call r the **magnitude** of z, and θ its **argument**.

Using Equation **??** the real and imaginary components of z are

$$\begin{aligned} a &= r\cos(\theta), \\ b &= r\sin(\theta), \end{aligned} \tag{0.1.9}$$

and z can be re-written as

$$z = r\left(\cos(\theta) + i\sin(\theta)\right). \tag{0.1.10}$$

Which we call the **polar form** of z (contrasted with $z = a + ib$ being the **cartesian form** of z).

Inverting the relations in Equation 0.1.9 yields the relations

$$\begin{aligned} r &= a^2 + b^2, \\ \theta &= \arctan\left(\frac{b}{a}\right). \end{aligned} \tag{0.1.11}$$

Let's examine the same properties of complex numbers shown in Equations 0.1.3, 0.1.4 and 0.1.6, and verify that they work in the polar form of complex numbers. We start with addition (Equation 0.1.3):

$$\begin{aligned}
z_1+z_2 &= r_1\left[\cos(\theta_1)+i\sin(\theta_1)\right]+r_2\left[\cos(\theta_2)+i\sin(\theta_2)\right]\\
&= \underbrace{r_1\cos(\theta_1)}_{a_1}+\underbrace{r_2\cos(\theta_2)}_{a_2}+i\,\underbrace{r_1\sin(\theta_1)}_{b_1}+i\,\underbrace{r_2\sin(\theta_2)}_{b_2}\\
&= (a_1+a_2)+i(b_1+b_2). \qquad (0.1.12)
\end{aligned}$$

We see that indeed, the polar form of complex numbers adheres to the addition rule in Equation 0.1.3 . Next is the product rule:

$$\begin{aligned}
z_1z_2 &= r_1\left[\cos(\theta_1)+i\sin(\theta_1)\right]\cdot r_2\left[\cos(\theta_2)+i\sin(\theta_2)\right]\\
&= r_1r_2\left[\cos(\theta_1)\cos(\theta_2)+i\cos(\theta_1)\sin(\theta_2)+i\sin(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)\right]\\
&= r_1\cos(\theta_1)r_2\cos(\theta_2)-r_1\sin(\theta_1)r_2\sin(\theta_2)+i\left[r_1\cos(\theta_1)r_2\sin(\theta_2)+r_1\sin(\theta_1)r_2\cos(\theta_2)\right]\\
&= (a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1), \qquad (0.1.13)
\end{aligned}$$

which is indeed the result seen in Equation 0.1.4 . We can also develope further the second row of Equation 0.1.13 using some trigonometry (specifically the trigonometric identities xx and yy):

$$\begin{aligned}
z_1z_2 &= r_1r_2\left[\cos(\theta_1)\cos(\theta_2)+i\cos(\theta_1)\sin(\theta_2)+i\sin(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)\right]\\
&= r_1r_2\left[\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)+i\left[\cos(\theta_1)\sin(\theta_2)+\sin(\theta_1)\cos(\theta_2)\right]\right]\\
&= r_1r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]. \qquad (0.1.14)
\end{aligned}$$

This is a very important result: it shows that multiplying a complex number z_1 by another complex number z_2 gives a complex number with magnitude r_1r_2, i.e. the product of the magnitudes of the two complex numbers, and argument $\theta_1+\theta_2$, i.e. the argument of z_1 rotated by the argument of z_2 (or vice-versa). We will consider this result in more detail soon.

In the polar form the complex conjugate of a number $z=r\left[\cos(\theta)+i\sin(\theta)\right]$ can be brought about by substituting $-\theta$ into the arguments of the trigonometric functions:

$$\begin{aligned}
\bar{z} &= r\left[\cos(-\theta)+i\sin(-\theta)\right]\\
&= r\left[\cos(\theta)-i\sin(\theta)\right]\\
&= r\cos(\theta)-ir\sin(\theta)\\
&= a-ib. \qquad (0.1.15)
\end{aligned}$$

Lastly, let's show that Equation 0.1.7 can be derived in the polar form:

$$\begin{aligned}
z\bar{z} &= r\left[\cos(\theta)+i\sin(\theta)\right]\cdot r\left[\cos(\theta)-i\sin(\theta)\right]\\
&= r^2\left[\cos^2(\theta)-\cancel{i\cos(\theta)\sin(\theta)}+\cancel{i\sin(\theta)\cos(\theta)}+\sin^2(\theta)\right]\\
&= r^2\left[\sin^2(\theta)+\cos^2(\theta)\right]\\
&= r^2=a^2+b^2. \qquad (0.1.16)
\end{aligned}$$

In 1748 Leonhard Euler published his famous work *Introductio in analysin infinitorum*[2]. In it he introduced the following relation, called **Euler's formula**:

$$e^{ix}=\sin(x)+i\cos(x). \qquad (0.1.17)$$

[2] Latin for **Introduction to the Analysis of the Infinite**.

Using Euler's formula a complex number z can be written as

$$z = re^{i\theta}. \tag{0.1.18}$$

In Table 0.1 we can see some usefull complex exponentials e^{xi}. Specifically, setting $x = \pi$ yields the famous **Eurler's identity**, considered by many to be one of the most beautiful equations in mathematics, as it binds together five important numbers, namely $0, 1, \pi, e$ and i:

$$e^{\pi i} + 1 = 0. \tag{0.1.19}$$

Table 0.1 Values of e^{ix} for some useful values of x (cf. xxxx for the values of $\sin(\theta)$ and $\cos(\theta)$).

x	$\cos(x)$	$\sin(x)$	$z = e^{ix}$
$\frac{\pi}{2}$	0	1	i
π	-1	0	-1
$\frac{3\pi}{2}$	0	-1	$-i$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}\left(1 + i\sqrt{3}\right)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}(1 + i)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{2}\left(\sqrt{3} + i\right)$

Table 0.1 also shows us the integer behaviours of i:

$$i^2 = -1,\ i^3 = -i, i^4 = 1,\ i^5 = i,\ i^6 = -1,\ i^7 = -i,\ \ldots \tag{0.1.20}$$

COMPLEX ROOTS OF POLYNOMIAL FUNCTIONS

Consider the simple equation $z^3 = 1$. We know the solution when z is a real number: $z_1 = 1$. However, on $\mathbb{C}$ there are actually two more solutions: $z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. We can verify that z_2 indeed solves the equation[3]:

$$\begin{aligned}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^3 &= -\frac{1}{8} + 3\frac{\sqrt{3}}{8}i - 3\frac{3}{8}i^2 + \frac{3\sqrt{3}}{8}i^3 \\ &= -\frac{1}{8} + 3i\cancel{\frac{\sqrt{3}}{8}} + 3\frac{3}{8} - 3i\cancel{\frac{\sqrt{3}}{8}} \\ &= -\frac{1}{8} + \frac{9}{8} \\ &= 1.\end{aligned} \tag{0.1.21}$$

Verfying that z_3 solves the equation as well is left as an excercise to the reader. Let's view all three solutions in polar form: z_1 has magnitude $r = 1$ and argument $\theta = 0$. z_2 also has magnitude $r = 1$ and argument $\theta = \frac{\pi}{4}$, and z_3 has the same magnitude, and argument

[3] Using $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

$\theta = \frac{7\pi}{4}$. Notice how the arguments of z_1, z_2, z_3 are equaly distributed in a circle of radius $r = 1$ (**Figure 0.4**).

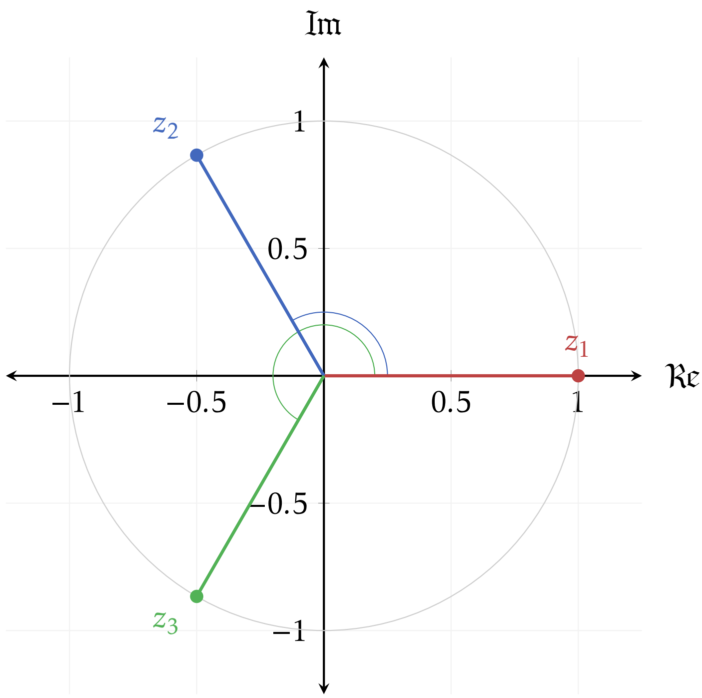

Figure 0.3 Text.

A similar thing happens for any $z^n = 1$, where $n \in \mathbb{N}$: all solutions lie equidistant from each otheron a circle with $r = 1$.

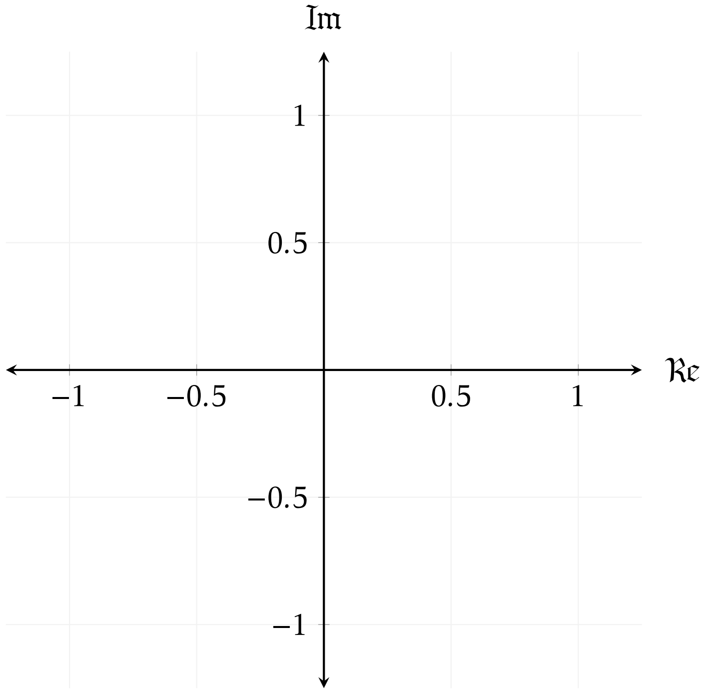

Figure 0.4 Text.