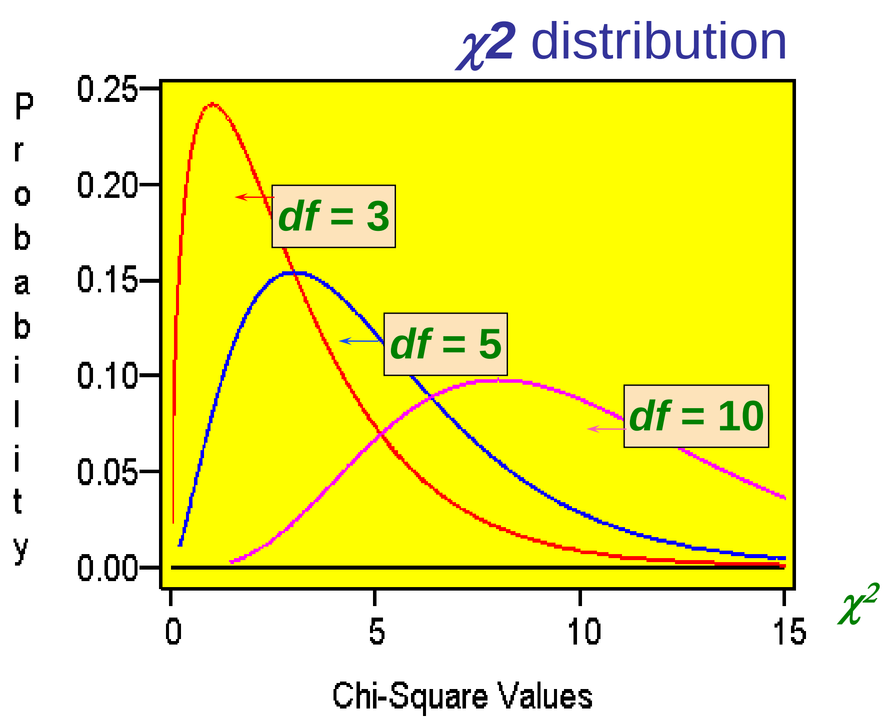
Chi-Square Tests

Chi-Square Test of Independence

- Chi-square test enable us to test whether more than two population proportions can be considered equal.
- Chi-square test allows us to do a lot more than just test for the equality of several proportions. If we classify a population into several categories with respect to two attributes (such as age and job performance), we can then use a chi-square test to determine whether the two attributes are independent of each other.
- The row and columns of a chi-square contingency table must be mutually exclusive categories that exhaust all of the possibilities of the sample.

Chi-Square Test of Independence

- Hypothesis:
  - H_0 : All proportions are equal
  - H_1 : At least two proportions are not equal
- The major characteristics of the chi-square distribution are:
  - It is positively skewed
  - It is non-negative
  - There is a family of chi-square distributions



Procedure of Chi-Square Test

- Describe a contingency table
- Setting up the problem symbolically
- Determining expected frequencies
- Comparing expected and observed frequencies
- Reasoning intuitively about chi-square tests
- Calculating the chi-square statistics
- Interpreting the chi-square statistics

Contingency Table Example

- Used to classify sample observations according to two or more characteristics
- Also called a cross-classification table.

Left-Handed vs. Gender

Dominant Hand: Left vs. Right

Gender: Male vs. Female

- 2 categories for each variable, so called a 2 x 2 table
- Suppose we examine a sample of size 300

Contingency Table Example

Sample results organized in a contingency table:

sample size = n = 300:

120 Females, 12 were left handed

180 Males, 24 were left handed

Gender	Hand Preference		
	Left	Right	
Female	12	108	120
Male	24	156	180
	36	264	300

χ^2 Test for the Difference Between Two Proportions

H_0: $\pi_1 = \pi_2$ (Proportion of females who are left handed is equal to the proportion of males who are left handed)

H_1: $\pi_1 \neq \pi_2$ (The two proportions are not the same – Hand preference is not independent of gender)

- If H_0 is true, then the proportion of left-handed females should be the same as the proportion of left-handed males
- The two proportions above should be the same as the proportion of left-handed people overall

The Chi-Square Test Statistic

The Chi-square test statistic is:

$$\chi^2 = \sum_{\text{all cells}} \frac{(f_o - f_e)^2}{f_e}$$

- where:

  f_o = observed frequency in a particular cell

  f_e = expected frequency in a particular cell if H_0 is true

  χ^2 for the 2 x 2 case has 1 degree of freedom

(Assumed: each cell in the contingency table has expected frequency of at least 5)

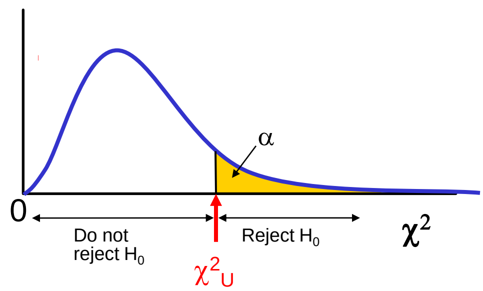
The Chi-Square Test Statistic

- To use the chi-square test, we must calculate the number of degrees of freedom in the contingency table by applying
  - **Number of degree of freedom = (number of rows - 1)*(number of columns -1)**
- A chi-square of zero, on the other hand, indicates that the observed frequencies exactly match the expected frequencies.
- The value of chi-square can never be negative because the differences between the observed and expected frequencies are always squared.

Decision Rule

The χ^2 test statistic approximately follows a chi-squared distribution with one degree of freedom

Decision Rule:
If $\chi^2 > \chi^2_U$, reject H_0, otherwise, do not reject H_0



Example

- In an antimalarial campaign in certain area quinine was administered to 812 persons out of total population of 3248. The number of fever cases is shown below:

Treatment	Fever	No Fever	Total
Quinine	20	792	812
No Quinine	220	2216	2436
Total	240	3008	3248

- Discuss the usefulness of quinine in checking malaria.
- (Given: For χ^2 at 0.05)

Example

- Let us take the following hypotheses:
- Null Hypothesis H_0: Quinine is not effective in checking malaria.
- Alternative Hypothesis Ha: Quinine is effective in checking malaria.
- Applying χ^2 test:

Expectation (E_{11}) column wise (first column) first element of the above table = $\frac{240 \times 812}{3248} = 60$

Expectation (E_{21}) column wise (first column) second element of the above table = $\frac{240 \times 2436}{3248} = 180$

Example

- Expected Frequency is

60	752	812
180	2256	2436
240	3008	3248

Using the formula for chi-square test

O	E	$(0 - E)^2$	$(O - E)^2/E$
20	60	1600	26.667
220	180	1600	8.889
792	752	1600	2.128
2216	2256	1600	0.709
			38.393

Example

- The chi square is

$$\chi^2 = \sum \frac{(O-E)^2}{E} \quad = 38.393$$

and degrees of freedom = (r – 1)(c – 1) = (2-1)(2-1) = 1

Table value of χ^2 for degrees of freedom 1 at 5% level of significance is 3.84. Since the calculated value is greater than table value so the hypothesis is rejected. Hence we conclude that quinine is effective in checking malaria.

TABLE F

χ^2 distribution critical values

	Tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.32	1.64	2.07	2.71	3.84	5.02	5.41	6.63	7.88	9.14	10.83	12.12
2	2.77	3.22	3.79	4.61	5.99	7.38	7.82	9.21	10.60	11.98	13.82	15.20
3	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27	17.73
4	5.39	5.99	6.74	7.78	9.49	11.14	11.67	13.28	14.86	16.42	18.47	20.00
5	6.63	7.29	8.12	9.24	11.07	12.83	13.39	15.09	16.75	18.39	20.51	22.11
6	7.84	8.56	9.45	10.64	12.59	14.45	15.03	16.81	18.55	20.25	22.46	24.10
7	9.04	9.80	10.75	12.02	14.07	16.01	16.62	18.48	20.28	22.04	24.32	26.02
8	10.22	11.03	12.03	13.36	15.51	17.53	18.17	20.09	21.95	23.77	26.12	27.87
9	11.39	12.24	13.29	14.68	16.92	19.02	19.68	21.67	23.59	25.46	27.88	29.67
10	12.55	13.44	14.53	15.99	18.31	20.48	21.16	23.21	25.19	27.11	29.59	31.42
11	13.70	14.63	15.77	17.28	19.68	21.92	22.62	24.72	26.76	28.73	31.26	33.14
12	14.85	15.81	16.99	18.55	21.03	23.34	24.05	26.22	28.30	30.32	32.91	34.82
13	15.98	16.98	18.20	19.81	22.36	24.74	25.47	27.69	29.82	31.88	34.53	36.48
14	17.12	18.15	19.41	21.06	23.68	26.12	26.87	29.14	31.32	33.43	36.12	38.11
15	18.25	19.31	20.60	22.31	25.00	27.49	28.26	30.58	32.80	34.95	37.70	39.72
16	19.37	20.47	21.79	23.54	26.30	28.85	29.63	32.00	34.27	36.46	39.25	41.31
17	20.49	21.61	22.98	24.77	27.59	30.19	31.00	33.41	35.72	37.95	40.79	42.88
18	21.60	22.76	24.16	25.99	28.87	31.53	32.35	34.81	37.16	39.42	42.31	44.43
19	22.72	23.90	25.33	27.20	30.14	32.85	33.69	36.19	38.58	40.88	43.82	45.97
20	23.83	25.04	26.50	28.41	31.41	34.17	35.02	37.57	40.00	42.34	45.31	47.50
21	24.93	26.17	27.66	29.62	32.67	35.48	36.34	38.93	41.40	43.78	46.80	49.01
22	26.04	27.30	28.82	30.81	33.92	36.78	37.66	40.29	42.80	45.20	48.27	50.51
23	27.14	28.43	29.98	32.01	35.17	38.08	38.97	41.64	44.18	46.62	49.73	52.00
24	28.24	29.55	31.13	33.20	36.42	39.36	40.27	42.98	45.56	48.03	51.18	53.48
25	29.34	30.68	32.28	34.38	37.65	40.65	41.57	44.31	46.93	49.44	52.62	54.95
26	30.43	31.79	33.43	35.56	38.89	41.92	42.86	45.64	48.29	50.83	54.05	56.41
27	31.53	32.91	34.57	36.74	40.11	43.19	44.14	46.96	49.64	52.22	55.48	57.86
28	32.62	34.03	35.71	37.92	41.34	44.46	45.42	48.28	50.99	53.59	56.89	59.30
29	33.71	35.14	36.85	39.09	42.56	45.72	46.69	49.59	52.34	54.97	58.30	60.73
30	34.80	36.25	37.99	40.26	43.77	46.98	47.96	50.89	53.67	56.33	59.70	62.16
40	45.62	47.27	49.24	51.81	55.76	59.34	60.44	63.69	66.77	69.70	73.40	76.09
50	56.33	58.16	60.35	63.17	67.50	71.42	72.61	76.15	79.49	82.66	86.66	89.56
60	66.98	68.97	71.34	74.40	79.08	83.30	84.58	88.38	91.95	95.34	99.61	102.7
80	88.13	90.41	93.11	96.58	101.9	106.6	108.1	112.3	116.3	120.1	124.8	128.3
100	109.1	111.7	114.7	118.5	124.3	129.6	131.1	135.8	140.2	144.3	149.4	153.2

Example

"Which pet do you prefer?“ The significance at 0.05

	Cat	Dog
Men	207	282
Women	231	242

Example

- The two **hypotheses** are.
- Gender and preference for cats or dogs are **independent**.
- Gender and preference for cats or dogs are **not independent**.

Lay the data out in a table:

	Cat	Dog
Men	207	282
Women	231	242

Add up rows and columns:

	Cat	Dog	
Men	207	282	489
Women	231	242	473
	438	524	962

Calculate "Expected Value" for each entry:

Multiply each row total by each column total and divide by the overall total:

	Cat	Dog	
Men	$\frac{489\times438}{962}$	$\frac{489\times524}{962}$	489
Women	$\frac{473\times438}{962}$	$\frac{473\times524}{962}$	473
	438	524	962

Which gives us:

	Cat	Dog	
Men	222.64	266.36	489
Women	215.36	257.64	473
	438	524	962

Subtract expected from observed, square it, then divide by expected:

In other words, use formula $\frac{(O-E)^2}{E}$ where

- O = **Observed** (actual) value
- E = **Expected** value

	Cat	Dog	
Men	$\frac{(207-222.64)^2}{222.64}$	$\frac{(282-266.36)^2}{266.36}$	489
Women	$\frac{(231-215.36)^2}{215.36}$	$\frac{(242-257.64)^2}{257.64}$	473
	438	524	962

Which gets us:

	Cat	Dog	
Men	1.099	0.918	489
Women	1.136	0.949	473
	438	524	962

Now add up those calculated values:

$$1.099 + 0.918 + 1.136 + 0.949 = 4.102$$

Chi-Square is 4.102

From Chi-Square to p

Degrees of Freedom

First we need a "Degree of Freedom"

$$\text{Degree of Freedom} = (\text{rows} - 1) \times (\text{columns} - 1)$$

For our example we have 2 rows and 2 columns:

$$\mathbf{DF} = (2-1)(2-1) = 1\times1 = \mathbf{1}$$

The rest of the calculation is look it up in a table

The result is:

p = 3.84 (significance level 0.05)

- **Conclusion:**
- $\chi^2 = 4.102 > \chi^2_\alpha = 3.84$ so **reject H_0** and conclude that Gender and preference for cats or dogs are **independent**.

ANOVA

- ANOVA is used for testing the significance of the differences among more than two sample means.
- Assumptions
  - Each sample is randomly drawn from normal population
  - Each of these population have same variance
- Analysis of variance (ANOVA)is based on comparison of two different estimates of the variance σ^2 ,of overall population.
- Hypothesis:
  - H_0 : All means are equal
  - H1 : At least two means are not equal.

Inferences About a Population Variance

- Sometimes decision makers are interested about the variability in a population.
- Chi-square test can be used to test the variability in a population.
- Assumption:
  - The distribution of data in the underlying population from which the sample is derived is normal.
  - The sample has been randomly selected from the population it represents.

Inferences About Two Population Variance

- Two population variances can be tested by F-test.
- Assumptions
  - Each sample has been randomly selected from the population it represents.
  - The distribution of data in the underlying population from which each of the samples is derived is normal; and
  - homogeneity of variance assumption, states that the variances of the both populations are equal.
- Hypothesis
  - H_0: Both sample have equal variance
  - H_1: Both sample have unequal variance

F Test or The Variance Ratio Test :

- The F test is named in honor of the great statistician R. A. Fisher
- The object of the F test is to find out whether the two independent estimates of population variance differ significantly or whether the two samples may be regarded as drawn from the normal populations having the same variance
- F is defined as
- F = larger estimate of variance / smaller estimate of variance
- $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$

F Test or The Variance Ratio Test :

- v_1 = d. f. for sample having larger variance
- v_2 = d. f. for sample having smaller variance
- The calculated value of F is compared with the table value for v_1 and v_2 at 5% or 1% level of significance
- If calculated value of F is greater than the table value then the F ratio is considered significant and null hypothesis is rejected
- If calculated value of F is smaller than the table value then the F ratio is considered insignificant and null hypothesis is accepted

F Test or The Variance Ratio Test :

- It is inferred that both samples have come from the population having same variance
- Since the F test is based on the ratio of two variances, it is also known as the Variance Ratio Test

F Test or The Variance Ratio Test :

- Two random samples were drawn from the two normal populations and their values are :
- A:66 67 75 76 82 84 88 90 92
- B:64 66 74 78 82 85 87 92 93 95 97
- Test whether the two populations have the same variance at the 5% level of significance

F Test or The Variance Ratio Test :

- H_0: the two populations have the same variance

$$\overline{x_1} = 80 \ , \ s_1^2 = 91.75$$

$$\overline{x_2} = 83 \ , \ s_2^2 = 129.8$$

$$F = 1.415$$

- $F = (s_1^2)/(s_2^2)$
- For $v_1 = 10$ and $v_2 = 8$
- $F_{0.05} = 3.34$
- $F_{0.01} = 5.82$
- H_o is accepted

An insurance company sells health insurance and motor insurance policies. Premiums are paid by customers for these policies. The CEO of the insurance company wonders if premiums paid by either of insurance segments (health insurance and motor insurance) are more variable as compared to another. He finds the following data for premiums paid:

	A	B	C	D
1		Health Insurance	Motor Insurance	
2	Variance	$200	$50	
3	Sample Size	11	51	
4				

Conduct a two-tailed F-test with a level of significance of 10%.

Solution:

- **Step 1:** Null Hypothesis H_0: $\sigma_1^2 = \sigma_2^2$

Alternate Hypothesis H_a: $\sigma_1^2 \neq \sigma_2^2$

- **Step 2:** F statistic = F Value = σ_1^2 / σ_2^2 = 200/50 = **4**
- **Step 3:** $df_1 = n_1 - 1 = 11-1 = 10$

$df_2 = n_2 - 1 = 51-1 = 50$

- **Step 4:** Since it is a two-tailed test, alpha level = 0.10/2 = 0.050. The F value from the F Table with degrees of freedom as 10 and 50 is 2.026.
- **Step 5:** Since F statistic (4) is more than the table value obtained (2.026), we reject the null hypothesis.

Denominator DF	Numerator DF 1	2	3	4	5	6	7	8	9	10
1	161.448	199.500	215.707	224.583	230.162	233.986	236.768	238.883	240.543	241.882
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385	19.396
3	10.128	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637
8	5.318	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753
13	4.667	3.806	3.411	3.179	3.025	2.915	2.832	2.767	2.714	2.671
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348
21	4.325	3.467	3.072	2.840	2.685	2.573	2.488	2.420	2.366	2.321

Note: There are different F Tables for different levels of significance. Above is the F table for alpha = .050.

28	4.196	3.340	2.947	2.714	2.558	2.445	2.359	2.291	2.236	2.190
29	4.183	3.328	2.934	2.701	2.545	2.432	2.346	2.278	2.223	2.177
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165
31	4.160	3.305	2.911	2.679	2.523	2.409	2.323	2.255	2.199	2.153
32	4.149	3.295	2.901	2.668	2.512	2.399	2.313	2.244	2.189	2.142
33	4.139	3.285	2.892	2.659	2.503	2.389	2.303	2.235	2.179	2.133
34	4.130	3.276	2.883	2.650	2.494	2.380	2.294	2.225	2.170	2.123
35	4.121	3.267	2.874	2.641	2.485	2.372	2.285	2.217	2.161	2.114
36	4.113	3.259	2.866	2.634	2.477	2.364	2.277	2.209	2.153	2.106
37	4.105	3.252	2.859	2.626	2.470	2.356	2.270	2.201	2.145	2.098
38	4.098	3.245	2.852	2.619	2.463	2.349	2.262	2.194	2.138	2.091
39	4.091	3.238	2.845	2.612	2.456	2.342	2.255	2.187	2.131	2.084
40	4.085	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077
41	4.079	3.226	2.833	2.600	2.443	2.330	2.243	2.174	2.118	2.071
42	4.073	3.220	2.827	2.594	2.438	2.324	2.237	2.168	2.112	2.065
43	4.067	3.214	2.822	2.589	2.432	2.318	2.232	2.163	2.106	2.059
44	4.062	3.209	2.816	2.584	2.427	2.313	2.226	2.157	2.101	2.054
45	4.057	3.204	2.812	2.579	2.422	2.308	2.221	2.152	2.096	2.049
46	4.052	3.200	2.807	2.574	2.417	2.304	2.216	2.147	2.091	2.044
47	4.047	3.195	2.802	2.570	2.413	2.299	2.212	2.143	2.086	2.039
48	4.043	3.191	2.798	2.565	2.409	2.295	2.207	2.138	2.082	2.035
49	4.038	3.187	2.794	2.561	2.404	2.290	2.203	2.134	2.077	2.030
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026