Interpolation

Computer Oriented Numerical and Statistical Methods

Outline

- Polynomial Interpolation
- Difference Tables
- Netwon's Forward and Backward Interpolation Formula
- Lagrange's Formula
- Divided Difference Formula
- Inverse Interpolation

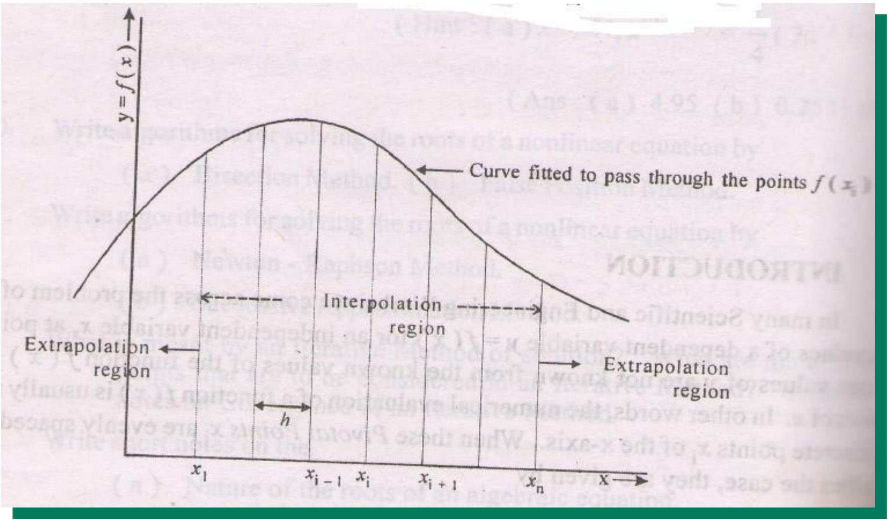
Introduction

- Suppose x and y are two variables and their relation can be expressed as $y = f(x)$; $x_1 \leq x \leq x_n$. Then we say that x is an independent variable and y is a dependent variable.
- When the form of f(x) is known, then the value of y can be computed directly corresponding to any value of x in the range $x_1 \leq x \leq x_n$.
- However, if the form of f(x) is not known, and only a set of values $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ satisfying the relation $y = f(x)$ are known, then the process of estimating the value of independent variable y for a given value of x in the range $x_1 \leq x \leq x_n$ is known as interpolation.

Introduction

- However, if we move in opposite direction i.e. estimate the value of dependent variable x for a given value of independent variable y, the process is known as inverse interpolation.
- The process of estimating the value of independent variable y for a given value of x outside the range $x_1 \leq x \leq x_n$ is known as extrapolation.

Introduction



Methods Of Interpolation

- The decision of using a particular method depends in tabulation of the functions.
- The tabulated points (x_i,y_i) i = 1,2,…,n of function y = f(x), can be equally spaced or unequal spaced.
- Methods for equally spaced functions
  – Netwon's forward interpolation formula
  – Netwon's backward interpolation formula

Methods Of Interpolation

- Methods for unequally spaced functions
  - Netwon’s divided difference interpolation formula
  - Lagrangian interpolation (Lagrange’s)
  - These methods also works well for equally spaced function.

Finite Differences

- The finite differences are either difference between the values of the function or the differences between the past differences.
- There are 3 types of differences
  - Forward Differences
  - Backward Differences
  - Divided Differences

Forward Differences

- If $y_1, y_2, y_3, \ldots, y_n$ denotes the values of the function of type $y = f(x)$ at $x = x_1, x_2, \ldots, x_n$ then $y_2 - y_1, y_3 - y_2, y_4 - y_3, \ldots, y_n - y_{n-1}$ are called the forward differences of y
- These differences are denoted as $\Delta y_1, \Delta y_2, \Delta y_3, \ldots, \Delta y_{n-1}$

$\forall$ $\therefore\ \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$
where Δ is called forward difference operator and $\Delta y_1, \Delta y_2, \Delta y_3, \ldots, \Delta y_{n-1}$ are called first order forward differences.

Forward Differences

- The differences of the first order forward differences are called second order forward differences and are denoted as $\Delta^2 y_1, \Delta^2 y_2, \Delta^2 y_3, \ldots, \Delta^2 y_{n-1}$

$\forall$ $\therefore\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1$

- $\Delta^2 y_2 = \Delta y_3 - \Delta y_2 = y_4 - 2y_3 + y_2$
- In the similar manner, the third order forward differences are

$\forall$ $\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1 = y_4 - 3y_3 + 3y_2 - y_1$

$\forall$ $\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2 = y_5 - 3y_4 + 3y_3 - y_2$

- In general, the first order forward differences at the i^{th} point is $\Delta y_i = y_{i+1} - y_i$ and the j^{th} order forward differences at the i^{th} point is $\Delta^j y_i = \Delta^{j-1} y_{i+1} - \Delta^{j-1} y_i$

Backward Differences

- If $y_1, y_2, y_3, \ldots, y_n$ denotes the values of the function of type $y = f(x)$ at $x = x_1, x_2, \ldots, x_n$ then $y_2 - y_1, y_3 - y_2, y_4 - y_3, \ldots, y_n - y_{n-1}$ are called the backward differences of y
- These differences are denoted as $\nabla y_2, \nabla y_3, \ldots, \nabla y_n$
- $\therefore \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \ldots, \nabla y_n = y_n - y_{n-1}$ where ∇ is called backward difference operator and $\nabla y_2, \nabla y_3, \ldots, \nabla y_n$ are called first order backward differences.

Backward Differences

- The differences of the first order backward differences are called second order backward differences and are denoted as $\nabla^2 y_3, \nabla^2 y_4, \ldots$ etc.
- $\therefore \nabla^2 y_3 = \nabla y_3 - \nabla y_2 = y_3 - 2y_2 + y_1$
- $\nabla^2 y_4 = \nabla y_4 - \nabla y_3 = y_4 - 2y_3 + y_2$
- In the similar manner, the third order backward differences are
- $\nabla^3 y_4 = y_4 - 3y_3 + 3y_2 - y_1$
- $\nabla^3 y_5 = y_5 - 3y_4 + 3y_3 - y_2$
- In general, the first order backward differences at the i^{th} point is $\nabla y_i = y_i - y_{i-1}$ and the j^{th} order backward differences at the i^{th} point is $\nabla^j y_i = \nabla^{j-1} y_i - \nabla^{j-1} y_{i-1}$

Divided Differences

- If $y_1, y_2, y_3, \ldots, y_n$ denotes the values of the function of type $y = f(x)$ at $x = x_1, x_2, \ldots, x_n$ then

$$\frac{y_2 - y_1}{x_2 - x_1}, \quad \frac{y_3 - y_2}{x_3 - x_2}, \quad \frac{y_4 - y_3}{x_4 - x_3}, \ldots, \quad \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

- are called the divided differences of y and are denoted as $\Delta_d y_1, \Delta_d y_2, \Delta_d y_3, \ldots, \Delta_d y_{n-1}$
- $\therefore\ \Delta_d y_1 = (y_2 - y_1) / (x_2 - x_1) = [x_1, x_2]$
- $\Delta_d y_2 = (y_3 - y_2) / (x_3 - x_2) = [x_2, x_3]$
- $\Delta_d y_{n-1} = (y_n - y_{n-1}) / (x_n - x_{n-1}) = [x_{n-1}, x_n]$
- Where Δ_d is called divide difference operator and $\Delta_d y_1, \Delta_d y_2, \ldots, \Delta_d y_n$ are called first order divided differences.

Divided Differences

- The differences of the first order divided differences are called second order divided differences and are denoted as $\Delta^2{}_d y_1, \Delta^2{}_d y_2, \ldots, \Delta^2{}_d y_n$ etc.
- $\therefore\ \Delta^2{}_d y_1 = (\Delta_d y_2 - \Delta_d y_1) / (x_3 - x_1)$
- $\Delta^2{}_d y_2 = (\Delta_d y_3 - \Delta_d y_2) / (x_4 - x_2)$
- In the similar manner, the third order forward differences are
- $\Delta_d{}^3 y_1 = (\Delta_d{}^2 y_2 - \Delta_d{}^2 y_1) / (x_4 - x_1)$
- $\Delta_d{}^3 y_2 = (\Delta_d{}^2 y_3 - \Delta_d{}^2 y_2) / (x_5 - x_2)$
- In general, the first order divided differences at the i^{th} point is $\Delta_d y_i = (y_{i+1} - y_i) / (x_{i+1} - x_i)$ and the j^{th} order forward differences at the i^{th} point is $\Delta_d{}^j y_1 = (\Delta_d{}^{j-1} y_{i+1} - \Delta_d{}^{j-1} y_i) / (x_{i+1} - x_i)$

Differences Tables

- A difference table is a table that lists the differences of the function values and the differences of differences in succession.

Forward Difference Table

- Let us consider the values y_1, y_2, y_3, y_4 of the function type $y = f(x)$ tabulated at equally spaced points x_1, x_2, x_3, x_4. The forward difference table along with tabulated points will look like

Forward Difference Table					
i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
1	x_1	y_1	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$
2	x_2	y_2	$\Delta y_2 = y_3 - y_2$	$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$	
3	x_3	y_3	$\Delta y_3 = y_4 - y_3$		
4	x_4	y_4			

Forward Difference Table

- The forward difference table for function tabulated at n equally spaced points can be represented by a matrix of size (n-1) * (n-1) where j^{th} order frequency at the i^{th} point ($\Delta^j y_i$) is represented by the element d_{ij} of matrix D.
- Note that only the elements in the column 1 to (n – i), for rows i = 1, 2, …., n-1 are of interest.

Backward Difference Table

- Let us consider the values y_1, y_2, y_3, y_4 of the function type y = f(x) tabulated at equally spaced points x_1, x_2, x_3, x_4. The Backward difference table along with tabulated points will look like

Backward Difference Table					
i	x_i	y_i	∇y_i	$\nabla^2 y_i$	$\nabla^3 y_i$
1	x_1	y_1			
2	x_2	y_2	$\nabla y_2 = y_2 - y_1$		
3	x_3	y_3	$\nabla y_3 = y_3 - y_2$	$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
4	x_4	y_4	$\nabla y_4 = y_4 - y_3$	$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$

Divided Difference Table

- Let us consider the values y_1, y_2, y_3, y_4 of the function type $y = f(x)$ tabulated at points x_1, x_2, x_3, x_4 not necessarily equally spaced . The divide difference table along with tabulated points will look like

Divided Difference Table

Divided Difference Table

i	x_i	y_i	$\Delta_d y_i$	$\Delta^2 y_i$	$\Delta^3 y_i$
1	x_1	y_1	$\Delta_d y_1 = (y_2 - y_1) / (x_2 - x_1)$	$\Delta_d^2 y_1 = (\Delta_d y_2 - \Delta_d y_1) / (x_3 - x_1)$	$\Delta_d^3 y_1 = (\Delta_d^2 y_2 - \Delta_d^2 y_1) / (x_4 - x_1)$
2	x_2	y_2	$\Delta_d y_2 = (y_3 - y_2) / (x_3 - x_2)$	$\Delta_d y_2 = (\Delta_d y_3 - \Delta_d y_2) / (x_4 - x_2)$	
3	x_3	y_3	$\Delta_d y_3 = (y_4 - y_3) / (x_4 - x_3)$		
4	x_4	y_4			

Netwon's Methods Of Interpolation

- It is divided into following methods depending on the type of differences being used.

1. Netwon's Forward Difference Interpolation Formula
2. Netwon's Backward Difference Interpolation Formula
3. Netwon's Divided Difference Interpolation Formula

- If the function is tabulated at equal intervals, then we can use either Netwon's Forward Difference Interpolation Formula or Netwon's Backward Difference Interpolation Formula.

Netwon's Forward Difference Interpolation Formula

- Let us assume that the function y(x) is tabulated at (n+1) equally spaced (interval size h) points.
- To derive the formula for netwon's forward difference interpolation assume a polynomial of type
  $y(x) = a_1 + a_2(x-x_1) + a_3(x-x_1)(x-x_2)+ \dots +a_{n+1}(x-x_1)(x-x_2) \dots (x-x_n)$ ……………………..[a]
- It is the n^{th} order polynomial in x.
- It is an interpolation polynomial for the table values x_i and y_i then the polynomial must pass through all points.

∀ ∴ we can obtain y_i by substituting the corresponding x_i for x

Netwon's Forward Difference Interpolation Formula

- At $x = x_1$ $y_1 = a_1$
- $x = x_2$ $y_2 = a_1 + a_2(x_2-x_1)$
- $x = x_3$ $y_3 = a_1 + a_2(x_3-x_1) + a_3(x_3-x_1)(x_3-x_2)$
- And so on.
- Since x_i's are equally spaced therefore we can write $x_{i+1} - x_i = h$ and $x_{i+m} - x_i = mh$ then we can express the function values y_i's in terms of intervals values as

  $y_1 = a_1$

  $y_2 = a_1 + a_2h$

  $y_3 = a_1 + a_2(2h) + a_3(2h)(h)$

  …….

$y_{n+1} = a_1 + a_2(nh) + a_3(nh)((n-1)h)+….+a_{n+1}(nh)((n-1)h)…(h)$

Netwon's Forward Difference Interpolation Formula

- Now for a_1, a_2, a_3, a_{n+1} we get

$$a_1 = y_1$$

$$a_2 = \frac{y_2 - a_1}{h} = \frac{y_2 - y_1}{h}$$

$$a_3 = \frac{y_3 - 2y_2 + y_1}{2!h^2}$$

$$a_{n+1} = \frac{y_{n+1} - ny_n + ... + y_1}{n!h^n}$$

Netwon’s Forward Difference Interpolation Formula

- Using forward difference table we get

$$a_1 = y_1$$

$$a_2 = \frac{\Delta y_1}{h}$$

$$a_3 = \frac{\Delta^2 y_1}{2!h^2}$$

.

.

.

$$a_{n+1} = \frac{\Delta^n y_1}{n!h^n}$$

Netwon’s Forward Difference Interpolation Formula

- Substituting these values of a_1, a_2, a_3, a_{n+1} in equation [a] we get

$$y(x) = y_1 + \frac{\Delta y_1}{h}(x - x_1) +$$

$$\frac{\Delta^2 y_1}{2!h^2}(x - x_1)(x - x_2) +$$

$$\ldots\ldots + \frac{\Delta^n y_1}{n!h^n}(x - x_1)(x - x_2)\ldots(x - x_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [b]$$

Netwon’s Forward Difference Interpolation Formula

- If we use the relation $(x-x_1) / h = u$ then $x - x_1 = hu$
  $x – x_2 = x – (x_1 + h) = h(u-1)$
  $x – x_3 = x – (x_2 + h) = h(u-2)$ …..
  $x – x_n = h(u-(n-1))$
- Substituting these values $(x-x_1)$, $(x-x_2)$, …., $(x – x_n)$ in equation [b] we get

Netwon’s Forward Difference Interpolation Formula

$$y(x) = y_1 + \frac{\Delta y_1}{h} hu +$$

$$\frac{\Delta^2 y_1}{2h^2} hu * h(u-1) +$$

$$\ldots + \frac{\Delta^n y_1}{n h^n} hu * h(u-1) * \ldots * h(u-(n-1))$$

$$\ldots \ldots \ldots [c]$$

Netwon's Forward Difference Interpolation Formula

- Simplifying we get

$$y(x) = y_1 + \Delta y_1 u + \frac{\Delta^2 y_1}{2!} u(u-1) + \ldots + \frac{\Delta^n y_1}{n!} u(u-1) * \ldots * (u-(n-1)) \quad \ldots\ldots\ldots\ldots\ldots [d]$$

- The above derivation assumes that $x_1 < x < x_2$
- This polynomial is called Netwon's Forward Difference Interpolation Formula

Netwon's Backward Difference Interpolation Formula

- Let us assume that the function y(x) is tabulated at (n+1) equally spaced (interval size h) points.
- To derive the formula for netwon's backward difference interpolation assume a polynomial of type $y(x) = a_1 + a_2(x-x_n) + a_3(x-x_n)(x-x_{n-1}) + \ldots + a_{n+1}(x-x_n)(x-x_{n-1}) \ldots (x-x_1)$ ……………………..[a]
- It is the n^{th} order polynomial in x.
- It is an interpolation polynomial for the table values x_i and y_i then the polynomial must pass through all points.

∀ ∴ we can obtain y_i by substituting the corresponding x_i for x

Netwon's Backward Difference Interpolation Formula

- At $x = x_n$ $y_n = a_1$
- $x = x_{n-1}$ $y_{n-1} = a_1 + a_2(x_{n-1} - x_n)$
- $x = x_{n-2}$ $y_{n-2} = a_1 + a_2(x_{n-2} - x_n) + a_3(x_{n-2}-x_n)(x_{n-2}-x_{n-1})$
- And so on.
- Since x_i's are equally spaced therefore we can write $x_i - x_{i-1} = h$ and $x_{i-1} - x_i = -h$ and $x_{i-m} - x_i = -mh$ then we can express the function values y_i's in terms of intervals values as

  $y_n = a_1$

  $y_{n-1} = a_1 + a_2(-h)$

  $y_{n-2} = a_1 + a_2(-2h) + a_3(-2h)(-h)$

  …….

$y_1 = a_1 + a_2(-nh) + a_3(-nh)(-(n-1)h) + \ldots + a_{n+1}(-nh)(-(n-1)h)\ldots(-h)$

Netwon's Backward Difference Interpolation Formula

- Now for a_1, a_2, a_3, a_{n+1} we get

$$a_1 = y_n$$

$$a_2 = \frac{y_{n-1} - a_1}{h} = \frac{y_n - y_{n-1}}{h}$$

$$a_3 = \frac{y_n - 2y_{n-1} + yn-2}{2!h^2}$$

$$a_{n+1} = \frac{y1 - ny_2 + \ldots + y_n}{n!h^n}$$

Netwon's Backward Difference Interpolation Formula

- Using backward difference table we get

$$a_1 = y_n$$

$$a_2 = \frac{\nabla y_n}{h}$$

$$a_3 = \frac{\nabla^2 y_n}{2!h^2}$$

.
.
.

$$a_{n+1} = \frac{\nabla^n y_n}{n!h^n}$$

Netwon's Backward Difference Interpolation Formula

- Substituting these values of a_1, a_2, a_3, a_{n+1} in equation [a] we get

$$y(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) +$$

$$\frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1}) +$$

$$\ldots\ldots + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1})\ldots(x - x_1)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [b]$$

Netwon's Backward Difference Interpolation Formula

- If we use the relation $(x-x_n) / h = u$ then $x - x_n = hu$
  $x - x_{n-1} = x - (x_n - h) = h(u+1)$
  $x - x_{n-2} = x - (x_{n-1} - h) = h(u+2)$ …..
  $x - x_1 = h(u+(n-1))$
- Substituting these values $(x-x_n)$, $(x-x_{n-1})$, …., $(x - x_1)$ in equation [b] we get

Netwon's Backward Difference Interpolation Formula

$$y(x) = y_n + \frac{\nabla y_n}{h} hu +$$

$$\frac{\nabla^2 y_n}{2h^2} hu * h(u+1) +$$

$$\ldots + \frac{\nabla^n y_n}{n h^n} hu * h(u+1) * \ldots * h(u+(n-1))$$

$$\ldots\ldots\ldots\ldots [c]$$

Netwon's Backward Difference Interpolation Formula

- Simplifying we get

$$y(x) = y_n + \nabla y_n u +$$

$$\frac{\nabla^2 y_n}{2!} u(u+1) +$$

$$\ldots + \frac{\nabla^n y_n}{n!} u(u+1) * \ldots * (u + (n-1))$$

$$\ldots\ldots\ldots\ldots\ldots [d]$$

- The above derivation assumes that $x_{n-1} < x < x_n$
- This polynomial is called Netwon's Backward Difference Interpolation Formula

Netwon's Divided Difference Interpolation Formula

- Let us assume that the function y(x) is tabulated at (n+1) equally spaced (interval size h) points.
- To derive the formula for netwon's divided difference interpolation assume a polynomial of type
  $y(x) = a_1 + a_2(x-x_1) + a_3(x-x_1)(x-x_2) + \ldots + a_{n+1}(x-x_1)(x-x_2) \ldots (x-x_n)$ ……………………..[a]
- It is the n^{th} order polynomial in x.
- It is an interpolation polynomial for the table values x_i and y_i then the polynomial must pass through all points.
- ∀ ∴ we can obtain y_i by substituting the corresponding x_i for x

Netwon's Divided Difference Interpolation Formula

- At $x = x_1$ $y_1 = a_1$
- $x = x_2$ $y_2 = a_1 + a_2(x_2-x_1)$
- $x = x_3$ $y_3 = a_1 + a_2(x_3-x_1) + a_3(x_3-x_1)(x_3-x_2)$
- And so on.
- Solving for a_1, a_2, a_3, a_{n+1} we get

Netwon's Divided Difference Interpolation Formula

$$a_1 = y_1$$

$$a_2 = \frac{y_2 - a_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$a_3 = \frac{1}{(x_3 - x_2)} * \left[\frac{(y_3 - y_1)}{(x_3 - x_1)} - \frac{(y_2 - y_1)}{(x_2 - x_1)}\right]$$

and so on

Netwon's Divided Difference Interpolation Formula

- Using divided difference table we get

$$a_1 = y_1$$

$$a_2 = \Delta_d y_1$$

$$a_3 = \Delta_d^2 y_1$$

.

.

.

$$a_{n+1} = \Delta_d^n y_1$$

Netwon's Divided Difference Interpolation Formula

- Substituting these values of a_{1}, a_2, a_3, a_{n+1} in equation [a] we get

$$y(x) = y_1 + \Delta_d y_1 (x - x_1) +$$

$$\Delta_d^2 y_1 (x - x_1)(x - x_2) +$$

$$\ldots + \Delta_d^n y_1 (x - x_1)(x - x_2) \ldots (x - x_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots [b]$$

- This polynomial is called Netwon's Divided Difference Interpolation Formula

Lagrangian / Lagranges Interpolation Formula

- In order to derive a general formula for lagrangian interpolation, we consider a second order polynomial of type.
  $y(x) = a_1(x-x_2)(x-x_3) + a_2(x-x_1)(x-x_3) + a_3(x-x_1)(x-x_2)$ ……………………..[a]
  passing through the points (x_1,y_1), (x_2,y_2), and (x_3,y_3) where a_1, a_2, and a_3 are unknown constants whose values are determined as follows.
- At $x = x_1$ $y(x_1) = a_1(x_1 - x_2)(x_1-x_3)$

Lagrangian / Lagranges Interpolation Formula

$$At\ x = x_1 \quad y(x_1) = a_1(x_1 - x_2)(x_1 - x_3)$$

$$\Rightarrow a_1 = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)}$$

$$At\ x = x_2 \quad y(x_2) = a_2(x_2 - x_1)(x_2 - x_3)$$

$$\Rightarrow a_2 = \frac{y_2}{(x_2 - x_1)(x_2 - x_3)}$$

$$At\ x = x_3 \quad y(x_3) = a_3(x_3 - x_1)(x_3 - x_2)$$

$$\Rightarrow a_3 = \frac{y_3}{(x_3 - x_1)(x_3 - x_2)}$$

Lagrangian / Lagranges Interpolation Formula

- Substituting these values of a_1 , a_2, a_3 in equation [a] we get

$$y(x) = y_1 * \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} +$$

$$y_2 * \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} +$$

$$y_3 * \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Lagrangian / Lagranges Interpolation Formula

$$f(x) = f(x_0) * \frac{(x - x_1)(x - x_2) \ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \ldots (x_0 - x_n)} +$$

$$f(x_1) * \frac{(x - x_0)(x - x_2) \ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \ldots (x_1 - x_n)} + \ldots +$$

$$f(x_n) * \frac{(x - x_0)(x - x_1) \ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})} +$$

- This polynomial is known as the Lagrange's polynomial

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