Numerical Solution Of Ordinary Differential Equations

Computer Oriented Numerical and Statistical Methods

Outline

- Introduction
- Euler method
- Runge – Kutta (RK) method

Introduction

- The subject of ordinary differential equations is not only fascinating part of mathematics but also an essential tool for modeling many physical processes.
- Most scientific laws are expressed in terms of differential equations.
  - Thermodynamics $dT/dt = -0.27(u-60)^{5/4}$
  - Probability $dPr/dt = (r+1)(n-r)P_{r+1} – r(n-r+1)Pr$
  - Mechanics $mdv/dt = mf – kv^2$
  - Economics $dx/dt = Sf(x) – g(x)$.

Solving A Differential Equations

- Formulation of differential equation is simple but difficult to solve it.
- Use numerical solution i.e. instead of finding an algebraic (analytical) solution we compute (approximately) the numerical values taken by solution.
- Therefore, as a solution of differential equations, instead of finishing up with an expression.
- This is known as the numerical solution of a differential equations.

Basic Terminology Of Differential Equations

- Differential Equation : A differential equation is an equation containing an unknown function and its derivatives. $$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+ay=0$$

  y is dependent variable and ***x*** is independent variable, and this is an ordinary differential equations [i.e. involves only one independent]
- Ordinary Derivative : If y is a function of x i.e. y = f(x) , then dy/dx is called the ordinary derivative. Physically it means the rate of change of the dependent variable with respect to the independent variable.

Basic Terminology Of Differential Equations

- Partial Derivative : If u is a function of x and y i.e. u = f(x,y) then $\left.\frac{\partial u}{\partial x}\right|_y$ is called the partial derivative with respect to x keeping y constant, and $\left.\frac{\partial u}{\partial y}\right|_x$ is called the partial derivative with respect to y keeping x constant. Physically it means the rates of change of dependent variable with respect to one of the independent variable keeping others fixed.

Basic Terminology Of Differential Equations

- Ordinary Differential Equations : An ordinary differential equation is an equation involving only ordinary derivative of one or more function with respect to a single independent variables.

Examples:.

1. $$\frac{dy}{dx} = 2x + 3$$
2. $$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + ay = 0$$
3. $$\frac{d^3 y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

Basic Terminology Of Differential Equations

- Partial Differential Equations : A partial differential equation is an equation involving a single independent variables .
- Examples:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial t^4} = 0 \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$$

Order of Differential Equation

The order of the differential equation is order of the highest derivative in the differential equation.

Differential Equation	ORDER
$\frac{dy}{dx} = 2x + 3$	1
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0$	2
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$	3

Degree of Differential Equation

The degree of a differential equation is power of the highest order derivative term in the differential equation.

Differential Equation	Degree
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ay = 0$	1
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$	1
$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + 3 = 0$	3

Basic Terminology Of Differential Equations

- Solution of A Differential Equations : Consider the first order ordinary differential equation of type $\frac{dy}{dx} = f(x,y)$

  which can also be written as y'=f(x,y) where the function f(x,y) may be a general non-linear function of x and y or in the form of a table of values.
- The solution of such an ordinary differential equation is a 2-D curve of (x,y) in the xy plane whose slope at every point (x,y) is the specified region is given by the equation $\frac{dy}{dx} = f(x,y)$

Basic Terminology Of Differential Equations

- Initial value problem : if the parameters of ordinary differential equation is determined based on some given initial values, i.e. initial condition then this system is known as an initial value problem.

Linear Differential Equation

A differential equation is linear, if

1. **dependent variable and its derivatives are of degree one,**
2. **coefficients of a term does not depend upon dependent variable.**

Example: 1. $$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+9y=0.$$

is linear.

Example: 2. $$\frac{d^3y}{dx^3}+\left(\frac{dy}{dx}\right)^4+6y=3$$

is non - linear because in 2nd term is not of degree one.

Minal Shah

Linear Differential Equation

Example: 3. $$x^2\frac{d^2y}{dx^2}+y\frac{dy}{dx}=x^3$$

is non - linear because in 2nd term coefficient depends on *y*.

Example: 4. $$\frac{dy}{dx}=\sin y$$

is non - linear because $\sin y = y-\frac{y^3}{3!}+-$ is non – linear

Minal Shah

Numerical Solution Of Differential Equations

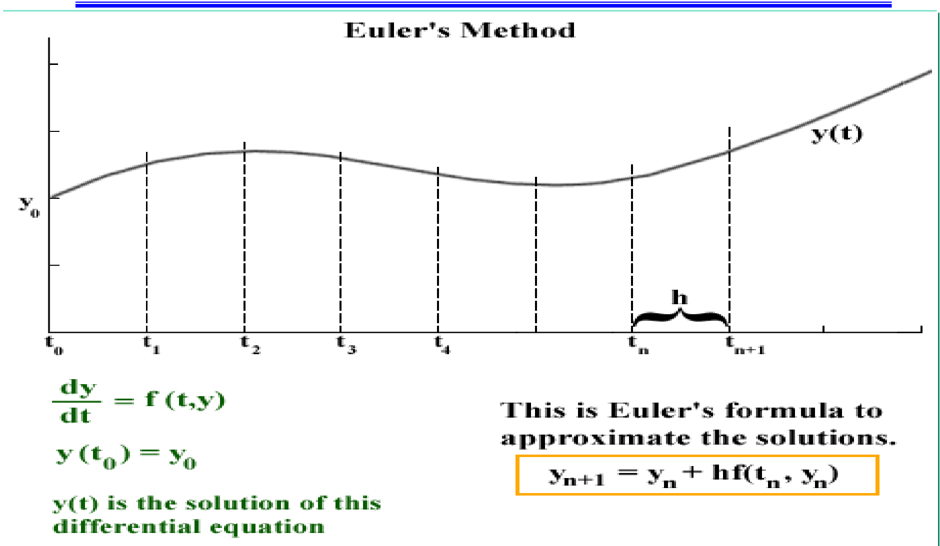
- To describe various numerical methods for the solution of ordinary differential equation we consider the general first order differential equations $\frac{dy}{dx} = f(x, y)$

  with the initial condition $y(x_0) = y_0$
- Numerical solution of differential are classified into two types.
  - A series of y in terms of power of x, from which the value of y can be obtained by direct substitution. These methods are : Taylor series, Picard's Method.
  - A set of tabulated values of x and y. the method are : Euler's method, Runge – Kutta method, Adam- Bashforth method.

Euler's Method

- The Euler's method is one of the oldest and the simplest method.
- It can be described as a techniques of developing a piecewise linear approximation to the solution
- In the initial value problem, the starting point of the solution curve and the slope of the curve at the starting point are given.

Euler’s Method



Euler’s Method

- Consider again the following first order ordinary differential equation $\frac{dy}{dx} = f(x,y)$ with initial conditions y = y_0 for x = x_0 and h is a positive increment of x. $x_1=x_0+h$
- Divide $l - x_0$ into n equal parts. Length of each part is equal to h. So $x_1=x_0+h$, $x_2=x_1+h$, …..
- The mean value theorem

$$y'(c) = \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$

- If we substitute c = x_0 and h = x_1 - x_0 in the above equation can be written as $y(x_1) - y(x_0) = hy'(x_0)$

Euler's Method

- Now $\frac{dy}{dx} = f(x, y)$

- $\therefore$ $y'(x_0) = f(x_0,y_0)$
  $y(x_1) - y(x_0) = h\, f(x_0,y_0)$
  $y(x_1) = y(x_0) + h\, f(x_0,y_0)$
  $y_1 = y_0 + h\, f(x_0,y_0)$ [because $y(x_0) = y_0$]
- Using this equation, we can find the second point on the solution curve as (x_1,y_1)
- Similarly, taking (x_1,y_1) as the starting point, we get
  $y_2 = y_1 + h\, f(x_1,y_1)$

Euler's Method

- In general, the $(i+1)^{th}$ point of the solution curve is obtained from the i^{th} point using the following formula.
  $y_{i+1} = y_i + h\, f(x_i,y_i)$ which is Euler's method
- *[The process to find the solution using this method is too slow, and to obtain the reasonable accuracy we must take a very small value of the h.]*

Runge - Kutta (RK) Method

- The basic objectives of R-K methods are as follows:

1. The method propagate a solution over an interval by combining the information from several Euler-style steps. Here each step is evaluating the function f with different parameters.
2. Using this information obtained to match a Taylor series expansion up to some higher orders.

- Euler's method is less efficient in practical problems since it requires h to be small for obtaining reasonably accuracy.

Runge - Kutta (RK) Method

- The R-K methods are designed to give greater accuracy and they possess the advantage of requiring only the function values at some selected points on the subintervals.
- There are different Runge – Kutta formulae of various orders and methods:
  - R- K second order method
  - R-K fourth order method

Runge – Kutta Second Method (R-K 2nd order method)

- The R-K second order methods are actually a family of methods, each of that matches the Taylor series method up to the second terms in h, where h is the step size.
- In these methods the interval $[x_1, x_f]$ is divided into subintervals and a weighted average of derivatives (slopes) at these intervals is used to determine the value of the dependent variable.
- One advantage of these methods is that they, like to evaluate y_{i+1} we need information only at the preceding point (x_i, y_i)

Runge – Kutta Second Method (R-K 2nd order method)

- The second order method can be expressed as follows
- $y_1 = y_0 + h/2(f(x_0,y_0) + f(x_1,y_1))$
- Substitute $y_1 = y_0 + hf(x_0,y_0)$ in the above equation
- $y_1 = y_0 + h/2[f(x_0,y_0) + f(x_1, y_0 + hf(x_0,y_0))]$
- $y_1 = y_0 + h/2[f_0 + f(x_0 + h, y_0 + hf_0)]$ where $f_0 = f(x_0, y_0)$
- We can write $k_1 = hf_0$ and

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$y_1 = y_0 + ½(k_1 + k_2)$$

Runge – Kutta Second Method (R-K 2nd order method)

- In similar we can find y_2, y_3, …y_{n+1}
- $y_{n+1} = y_n + ½(k_1+k_2)$
  where $k_1 = hf(x_n,y_n)$ and
  $k_2 = hf(x_n +h, y_n+k_1)$
- Which is R-K 2nd order formula

Runge – Kutta Fourth Method (R-K 4th Order Method)

- The fourth order method can be expressed as follows
- $y_1 = y_0 + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$
- Where $k_1 = hf(x_{0,} y_0)$
  $k_2 = hf(x_0 +h/2, y_0+k_1/2)$
  $k_3 = hf(x_0 +h/2, y_0+k_2/2)$
  $k_4 = hf(x_0 +h, y_0+k_3)$
- In similar we can find y_2, y_3, …y_{n+1}

Runge – Kutta Fourth Method (R-K 4th Order Method)

- $y_{n+1} = y_n + 1/6(k_1+2k_2+2k_3+k_4)$
- Where $k_1 = hf(x_{n,} y_n)$
  $k_2 = hf(x_n + h/2, y_n+k_1/2)$
  $k_3 = hf(x_n + h/2, y_n+k_2/2)$
  $k_4 = hf(x_n + h, y_n+k_3)$
- Which is R-K 4th order formula