Iterative Methods

Computer Oriented Numerical and Statistical Methods

Minal Shah

Outline

- Iterative methods :
  - Bisection
  - False-Position
  - Newton-Raphson

Types of Equations

- Linear Equations : The equation in which power of the unknown quantity is one is called linear equation. The equation in which the power of the unknown is two is called quadratic equation.
- Non-linear Equations : Most of the equation having more power of unknowns or involving sin, log function are non-linear equations. Ex. $x^2 - 3x = 15$, $x - \cos x = 4$

Kinds of Equations

- They are classified on the basis of unknown quantity or power.
  - An equation which contains the first power only of an unknown quantity is called simple / linear equation [e.x. $x - 2 = 5$ here the power of x is 1]
  - If the power of the unknown quantity in an equation is 2 then it is called a quadratic equation [e.x. $x^2 - 2x = 15$ here the power of x is 2]
  - Some times two linear equations contains two unknown quantities. Also in order to find two unknown quantities we must have two linear equations. Such equations are called simultaneous equation [e.x $2x + y = 5$; $x + 3y = 8$]

Kinds of Equations

- Non-linear equations
  - Most of non-linear equations can be solved algebraically.
  - The solution obtained by algebraic manipulation is known as algebraic solution or an analytical solution
  - There are many non-linear equation that cannot be solved algebraically for example $2^x - x - 3 = 0$ which seems very simple but cannot be solved algebraically.
  - This solutions will be numerical not algebraic, and are called numerical solutions.
  - Types of non- linear equations are
    - Polynomial
    - Transcendental

Polynomial Equations

- A polynomial has the general form

  $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$ where $a_n \neq 0$

- It is n^{th} degree polynomial in x and has n roots. These roots may be
  - Real and different
  - Real and repeated
  - complex

Transcendental Equations

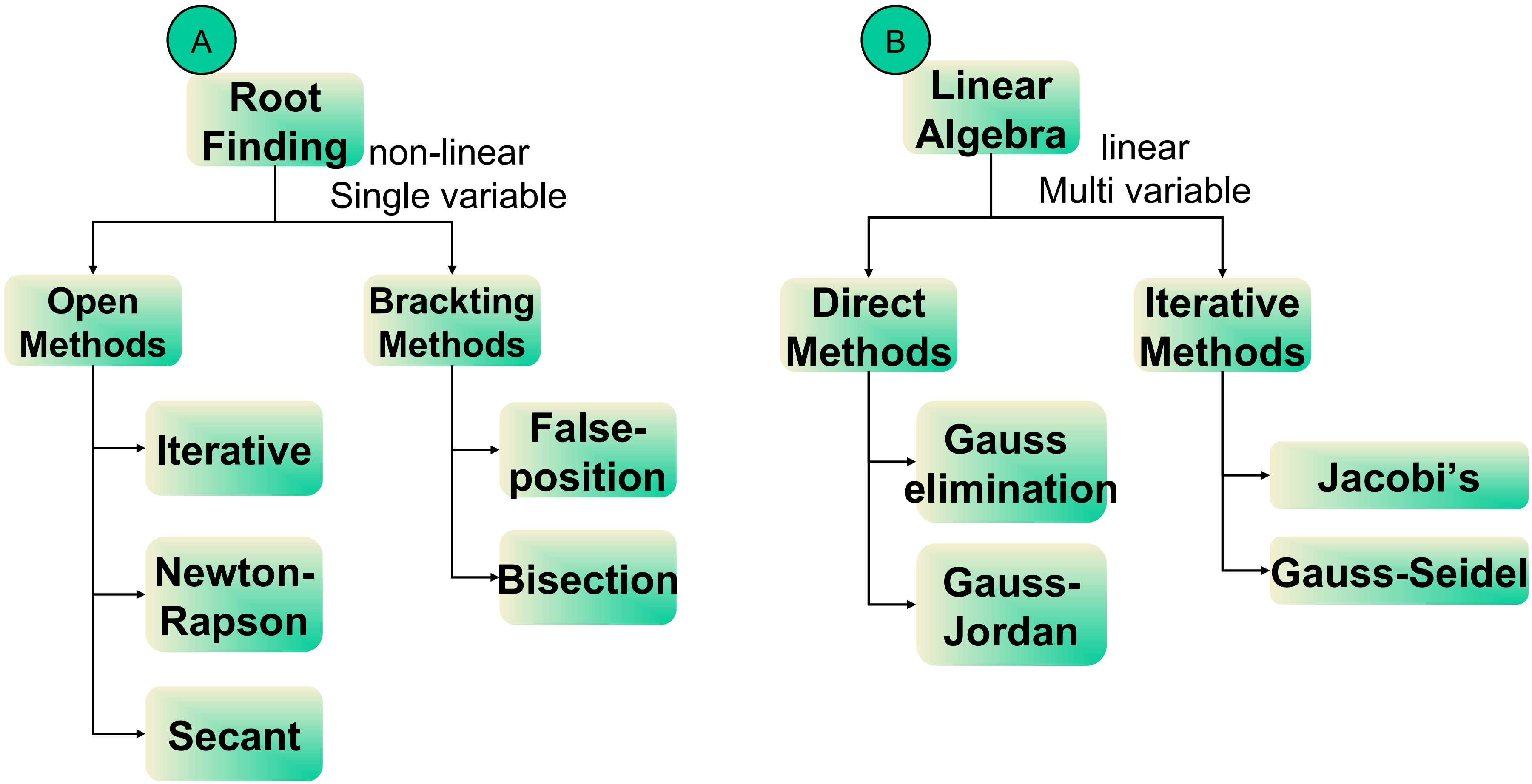
- A non-polynomial equation is called transcendental equations.
- Examples
  - $xe^x - x\sin x = 0$
  - $e^x\cos x - 3x = 0$
  - $2^x - x - 3 = 0$
- A transcendental equation may have finite/ infinite numbers of roots or may not have any roots at all.

Convergence Notation

A sequence $x_1, x_2, \ldots, x_n, \ldots$ is said to **converge** to x if to every $\varepsilon > 0$ there exists N such that :

$$\left| x_n - x \right| < \varepsilon \quad \forall n > N$$

Equation Solving



Iterative Method

- The Latin word ‘Iterate’ means to ‘repeat’
- It is also known as trial and error methods, are based on idea of successive approximations.
- They start with one or more initial approximations to the root and obtain a sequence of approximations by repeating a fixed sequence of steps till the solution with reasonably accuracy is obtained.
- Iterative method generally gives one root at a time.
- Iterative methods are very cumbersome and time-consuming for solving non-linear equations manually.

Iterative Method

- However they are best suited for use on computers, due to following reasons:
  - Iterative methods can be concisely expressed as computational algorithms.
  - It is possible to formulate algorithms that can handle class of similar problems. For e.g. an algorithm can be developed to solve a polynomial equation of degree n
  - Round – off errors are negligible in iterative methods as compared to direct methods.

Steps involved in an Iterative Method

- To develop an algorithm which is a step by step procedure for solving the problem
- An initial guess or initial estimates is made for the variable or variables of the solution.
  - The initial estimates should be reasonable.
  - Success in the solution depends on the choice of proper initial values for the variables.
- Using the algorithm developed, better and better estimates are obtained in the successive iterations.
- The iteration process is stopped when an acceptable solution is obtained, based on some reasonable criteria for stopping the iteration process.

Bracketing / Interpolation Methods

- In bracketing methods, the method starts with an interval that contains the root and a procedure is used to obtain a smaller interval containing the root.
- Two estimates of the roots are made- one giving a positive value for the function f(x) and other a negative value for the function f(x). Since the value of f(x) would be zero at the root.
- It means the root is effectively bracketed between these two values..
- By proper choice, the gap between the two estimates of the roots is reduced further and further so as to arrive at very small gap between the two estimates successively.
- Examples of bracketing methods:
  - Bisection method
  - False position method

OpenEnd / Extrapolation / Successive Approximation Methods

- In the open methods, the method starts with one or more initial guess points. In each iteration, a new guess of the root is obtained.
- Open methods are usually more efficient than bracketing methods.
- They may not converge to a root.
- Examples of open end methods:
  - Netwon –Raphson Method
  - Secant Method.

Bisection / Binary Chopping / Half- Interval/ Midpoint / Bolzano / Interval –Halving Method.

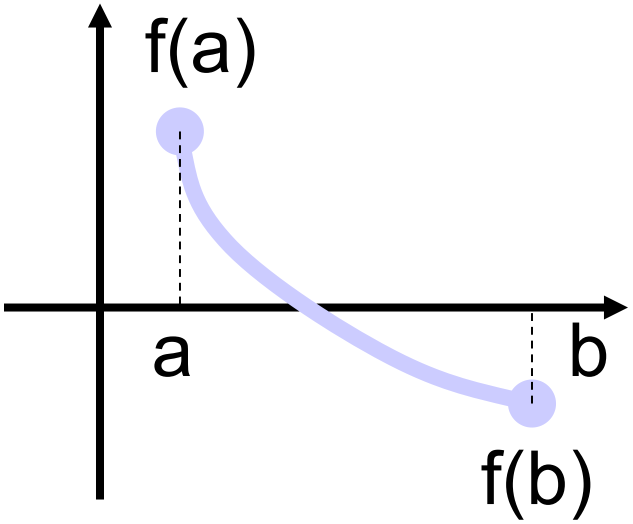
- This method is based on the theorem which states that if a function f(x) is continuous between a and b and f(a) and f(b) are of opposite signs, then there exists atleast one root between a and b
- Here f(a) is negative and f(b) is positive
- The root lies between a and b and its approximation is given by $x_0 = (a + b)/2$
- If $f(x_0) = 0$ then x_0 is the root.

Intermediate Value Theorem

- Let f(x) be defined on the interval [a,b].

- Intermediate value theorem:

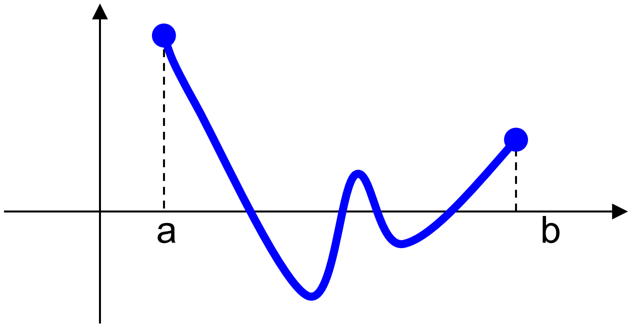
  if a function is continuous and f(a) and f(b) have different signs then the function has at least one zero in the interval [a,b].



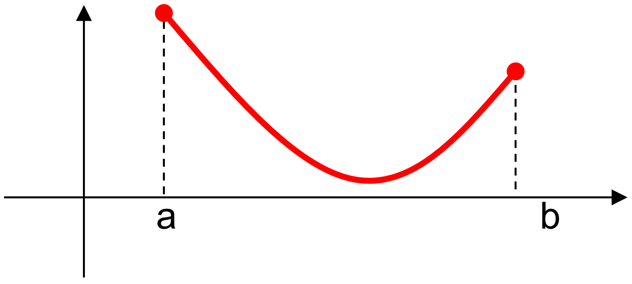
Examples

- If f(a) and f(b) have the same sign, the function may have an even number of real zeros or no real zeros in the interval [a, b].

- Bisection method can not be used in these cases.



The function has four real zeros



The function has no real zeros

Bisection Method

Assumptions:

Given an interval $[a,b]$

$f(x)$ is continuous on $[a,b]$

$f(a)$ and $f(b)$ have opposite signs.

These assumptions ensure the existence of at least one zero in the interval $[a,b]$ and the bisection method can be used to obtain a smaller interval that contains the zero.

Bisection Algorithm

Assumptions:

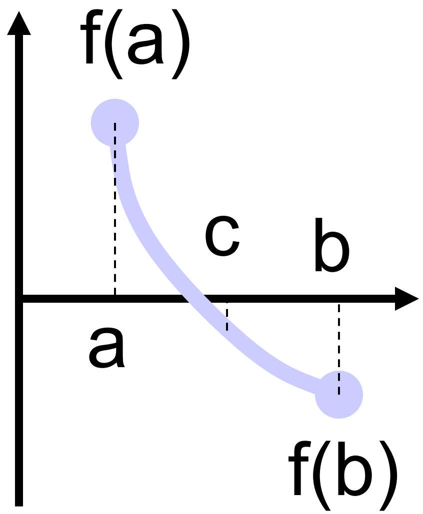
- f(x) is continuous on [a,b]
- f(a) f(b) < 0

Algorithm:

Loop

1. Compute the mid point c=(a+b)/2
2. Evaluate f(c)
3. If f(a) f(c) < 0 then new interval [a, c]
   If f(a) f(c) > 0 then new interval [c, b]

End loop



Stopping Criteria

Two common stopping criteria

1. Stop after a fixed number of iterations
2. Stop when the absolute error is less than a specified value

How are these criteria related?

c_n : is the midpoint of the interval at the n$^{\text{th}}$ iteration

(c_n is usually used as the estimate of the root).

r : is the zero of the function.

After n iterations:

$$\left|error\right| = \left| r - c_n \right| \leq E_a^n = \frac{b-a}{2^n} = \frac{\Delta x^0}{2^n}$$

Bisection Method

Advantages

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- The number of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function does not have to be differentiable

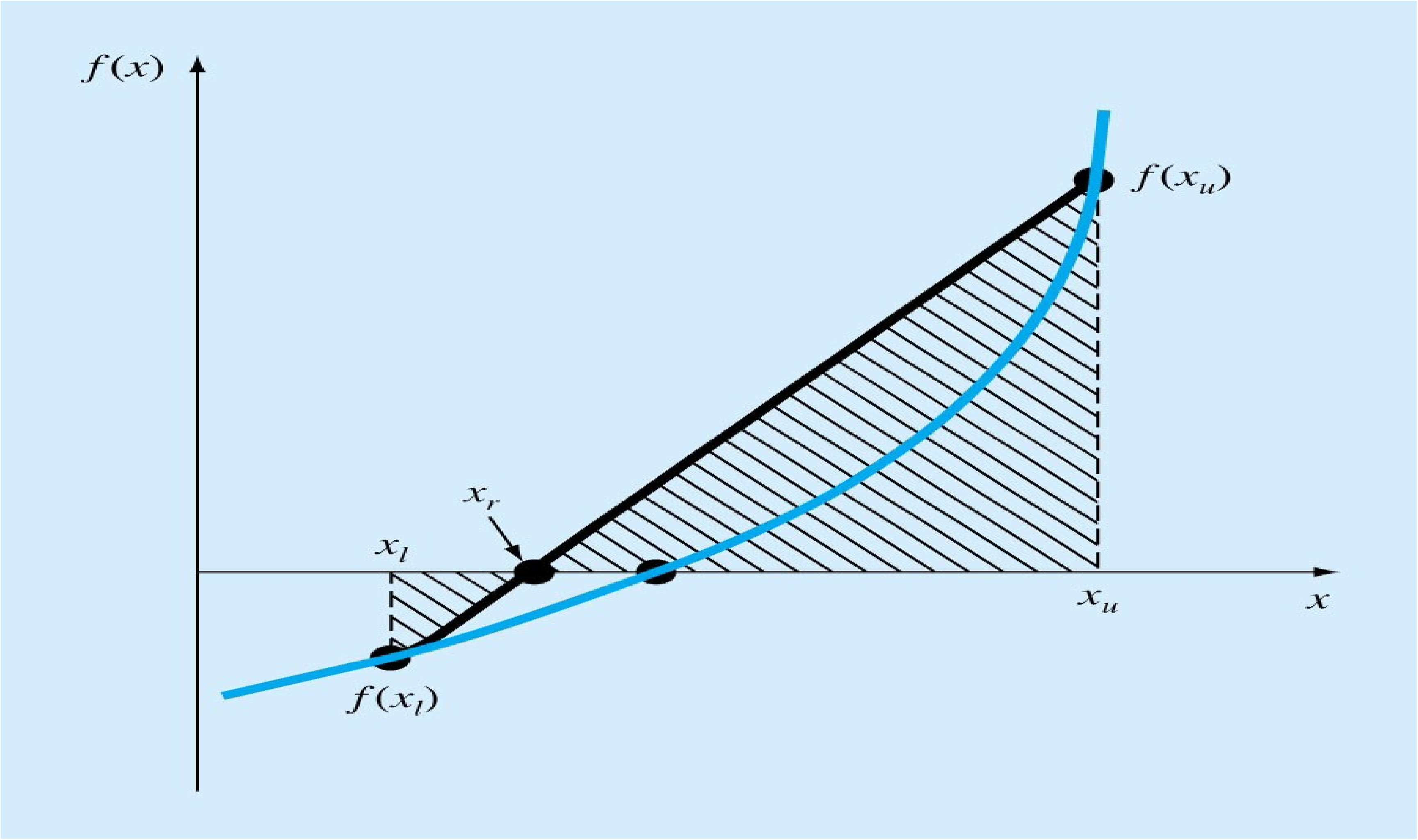
Disadvantage

- Slow to converge
- Good intermediate approximations may be discarded

False- Position /Regula – Falsi / Linear Interpolation / Method Of Chords

- Here, we choose two points x_n and x_{n-1} such that $f(x_n)$ and $f(x_{n-1})$ are of opposite signs.
- Intermediate value property suggests that the graph of $y = f(x)$ crosses the x-axis between these two points and therefore, a root say lies between these two points.
- Thus, to find a real root of $f(x) = 0$ using Regula-Falsi method, we replace the part of the curve between the points $A[x_n, f(x_n)]$ and $B[x_{n-1}, f(x_{n-1})]$ by a chord in that interval and we take the point of intersection of this chord with the x-axis as a first approximation to the root.

False- Position /Regula – Falsi / Linear Interpolation / Method Of Chords



False- Position Method

- Start with two approximation to root say x_1 and x_2 for which f(x) has opposite sign.
- Compute x_3 as

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

- There are 3 possibilities
  - If $f(x_3) = 0$ then we have a root as x_3
  - If $f(x_1)$ and $f(x_3)$ are of opposite sign, then the root lies in the interval $[x_1, x_3]$. Thus x_2 is replaced by x_3 and the iterative procedure is repeated.
  - If $f(x_1)$ and $f(x_3)$ are of same sign, then the root lies in the interval $[x_3, x_2]$. Thus x_1 is replaced by x_3 and the iterative procedure is repeated.
  - Terminate the process when the size of the search interval becomes less than prescribed tolerance.

Bisection Method Vs False- Position Method

- Difference between bisection and regula falsi method

Bisection	Regula -Falsi
If bisect the given interval $[x_1,x_2]$ on x -axis	It intersects the x- axis by a straight line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$
It only depends on the sign change of f(x), but it does not depend on the values of f(x)	It only depends on both the sign change of f(x) and values of f(x)
Usually it obtains a repeated zero (i.e. root) accurately	It cannot obtain a repeated zero accurately
It detects both the zero and the jump discontinuity in a given interval	It detects mainly the zero but it may fail to detect jump discontinuity

Bisection Method Vs False- Position Method

- Similarity between bisection and regula falsi method

1. Activation occurs when there is a sign change in f(x) for the points x_1 and x_2.
2. They do not accurately find the number of repetitions of a root.
3. These methods are applicable only for obtaining real roots of a real function, not for complex roots.
4. When the curve y = f(x) approaches and touches the x- axis, these methods gives an indication of the approach but they fail to detect the root (i.e. the touching point)

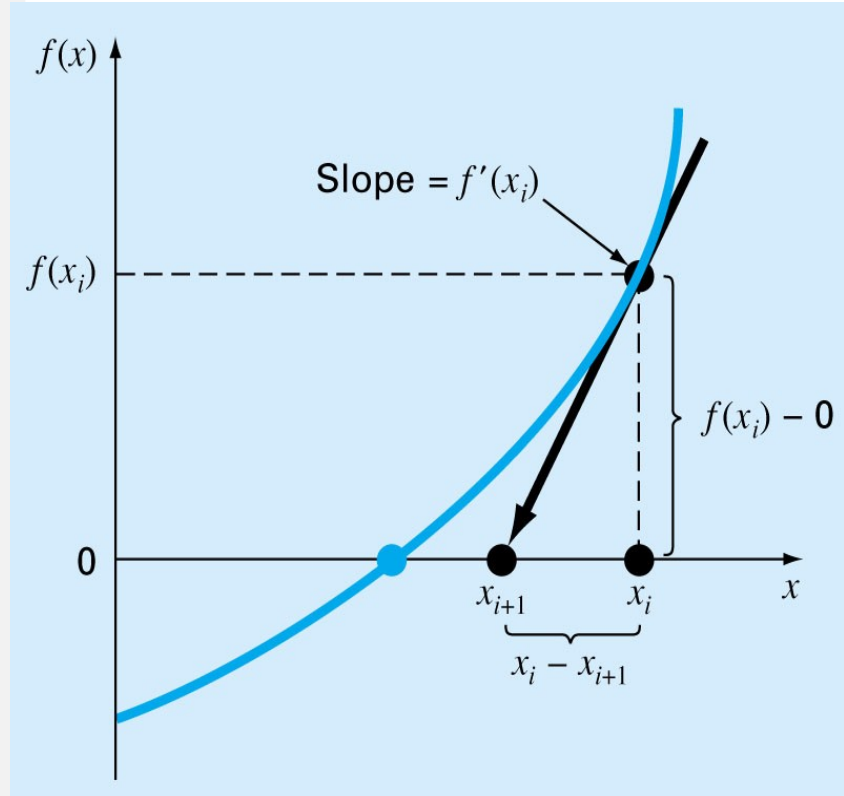
Newton-Raphson Method/ Netwon's Method of Tangents

- Most widely used formula for locating roots.
- It is one of the fastest iterative methods.
- Can be derived using **Taylor series** or the geometric interpretation of the slope in the figure

$$f'(x_i) = \frac{f(x_i) - 0}{(x_i - x_{i+1})}$$

rearrange to obtain :

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$



Newton-Raphson Method

- Given an initial guess of the root $\boldsymbol{x_0}$, Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.
- Assumptions:
  - $\boldsymbol{f(x)}$ is continuous and the first derivative is known
  - An initial guess $\boldsymbol{x_0}$ such that $\boldsymbol{f'(x_0) \neq 0}$ is given

Newton-Raphson Method

- Start with the arbitrary point x0
- Determine $f(x_0)$, $f'(x_0)$
- Determine

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Stop the iterative cycle when two successive values of Xi are nearly equal with a prescribed tolerance
  $|x_0 - x_1| < \varepsilon_1$ or $|f(x_1)| < \varepsilon_2$

Advantage of Newton’s Method

- It as quadratic convergence. It converges fast at the cost of slightly increase labour in less number of iteration.
- Convergence is assured.

Disadvantage of Newton's Method

- For every iteration, $f(x^{(k)}) = f'(x^{(k)})$ have to be evaluated.
- If the initial guess of the root is far from the root the method may not converge.
- Since $f'(x^{(k)})$ occurs in the denominator of the expression for $x^{(k+1)}$ this poses a problem. If $f'(x^{(k)}) = 0$ ***or nearly zero***
- Newton's method converges linearly near multiple zeros { $f(x) = f'(x) = 0$ }. In such a case, modified algorithms can be used to regain the quadratic convergence.


THANKS!


THANKS


Thanks!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