Numerical Integration

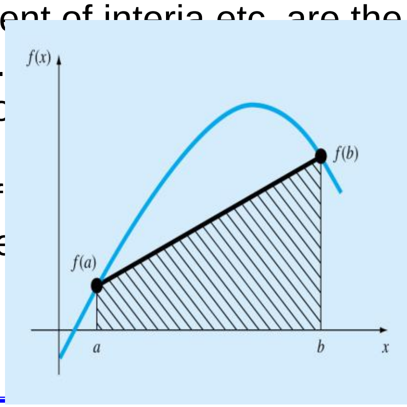
Computer Oriented Numerical and Statistical Methods

Outline

- Introduction
- Trapezoidal Rule
- Simpson's Rule

Introduction

- The concept of definite integral is essential for this, the calculation regarding area of regions bounded by curve, volume of solid figures, centre of gravity, length of arc, work, velocity, movement of interia etc. are the problems of definite integrals.
- Suppose f is a positive functio and increasing in the interval
- Hence for a ≤ x ≤ b we have f the graph of y = f(x) in the inte continuous curve.



Introduction

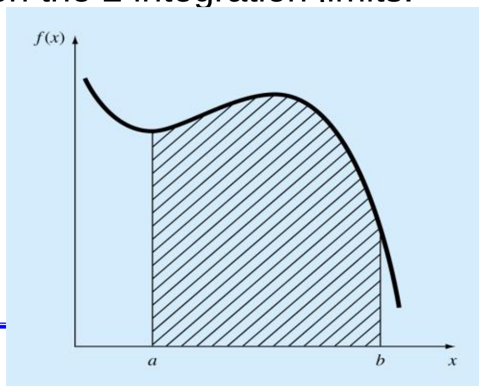
- The definite integral of function f on the interval [a,b] and represented symbolically as $\int_a^b f(x)dx$

  where a is called the lower limit and b is called the upper limit of given integral.
- Numerical integration is the process of computing the value of a definite integral from a set of numerical values of the function referred to as integrand.
- It is important because there are many integrals whose value cannot be obtained in closed form.
- The integrals have to be computed numerically.

Introduction

- When we applied to the integration of a function of a single variable, the process is sometimes called numerical integration or numerical quadrature.
- Graphically integration is simply to find the area under a certain curve between the 2 integration limits.

$$I = \int_a^b f(x).dx = A$$



Numerical Integration

- In practice, given set of values for a function f(x) the following table of data.

X	a	x_1	x_2	…	b
f(x)	f(a)	$f(x_1)$	$f(x_2)$	…	f(b)

is used to compute the value of the integral $\int_a^b f(x)dx$

- By dividing the interval (a,b) into a finite number of equal intervals, finding the areas of those subintervals and summing all such area of the required integration is possible

Quadrature Formula

$$\int_a^b f(x)dx = h[ny_0 + \frac{n^2}{2}\Delta y_0 + (\frac{n^3}{3} - \frac{n^2}{2})\frac{\Delta^2 y_0}{2} +$$

$$(\frac{n^4}{4} - n^3 + n^2)\frac{\Delta^3 y_0}{6} \;......]$$

where $x_1 - x_0 = x_2 - x_1 = \ldots\ldots x_n - x_{n-1} = h$

$\Delta y_0 = y_1 - y_0$

$\Delta y_1 = y_2 - y_1$

Trapezoidal Rule

- Take n = 1 in the quadrature formula we get trapezoidal rule.
- As n = 1 the values of $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ etc is zero

$$\int_a^b f(x)dx = h[y_0 + \frac{1}{2}\Delta y_0]$$

$$= \frac{h}{2}[2 * y_0 + \Delta y_0]$$

$$= \frac{h}{2}[2 * y_0 + y_1 - y_0]$$

$$= \frac{h}{2}[y_0 + y_1]$$

$$\int_{x_0}^{x_0 + h} f(x)dx = = \frac{h}{2}[y_0 + y_1] \quad ....(1)$$

Trapezoidal Rule

- Similarly

$$\int_{x_0+h}^{x_0+2h} f(x)dx = \frac{h}{2}[y_1 + y_2] \quad ....(2)$$

$$\int_{x_0+2h}^{x_0+3h} f(x)dx = \frac{h}{2}[y_2 + y_3] \quad ....(3)$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}[y_{n-1} + y_n] \quad ....(n)$$

Trapezoidal Rule

- Add (1), (2), (3) , ……, (n)

$$\int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_0+2h} f(x)dx + \int_{x_0+2h}^{x_0+3h} f(x)dx + ... + \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx$$

$$= \frac{h}{2}[y_0 + y_1 + y_1 + y_2 + ... + y_{n-1} + y_n]$$

Trapezoidal Rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[y_0 + 2*(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

- This formula is called trapezoidal rule.

Trapezoidal Rule

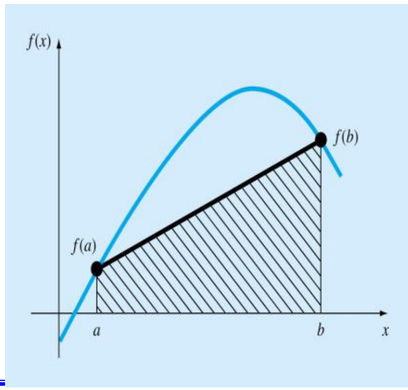
- The trapezoidal rule uses a polynomial of the first degree to replace the function to be integrated.

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$

$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

$$\boxed{I = (b-a)\frac{f(a) + f(b)}{2}}$$



Trapezoidal Rule (Example)

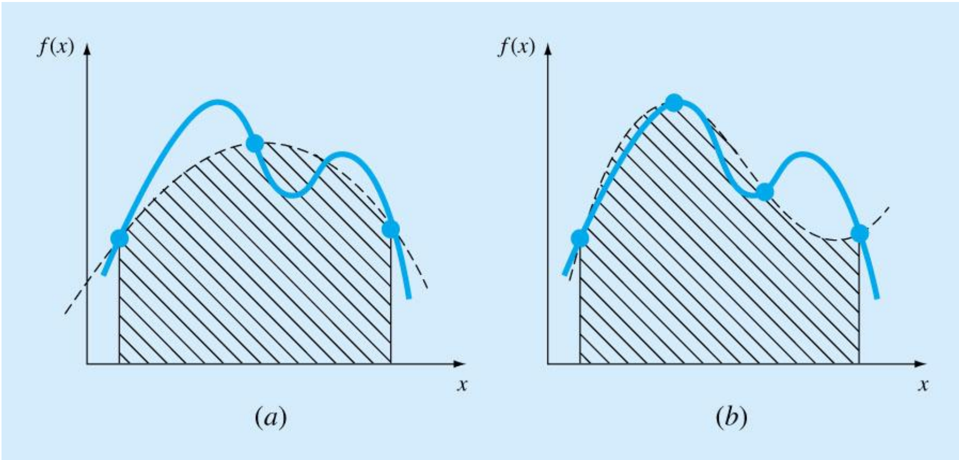
- Evaluate $$\int_0^1 \frac{dx}{1+x^2}$$
- By the trapezoidal rule where the interval (0,1) is sub-divided into 6 equal parts.
- The general trapezoidal formula is

$$= \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{n-1} + y_n]$$

Simpson's Rule

- One drawback of the trapezoidal rule is that the error is related to the second derivative of the function.
- More accurate estimate of an integral is obtained if a high-order polynomial is used to connect the points.
- The formulas that result from taking the integrals under such polynomials are called ***Simpson's Rules***.
- **Simpson's 1/3 Rule :** Results when a second-order interpolating polynomial is used.
- **Simpson's 3/8 Rule :** Results when a third-order (cubic) interpolating polynomial is used.

Simpson's Rules



Simpson's 1/3 Rule

Simpson's 3/8 Rule

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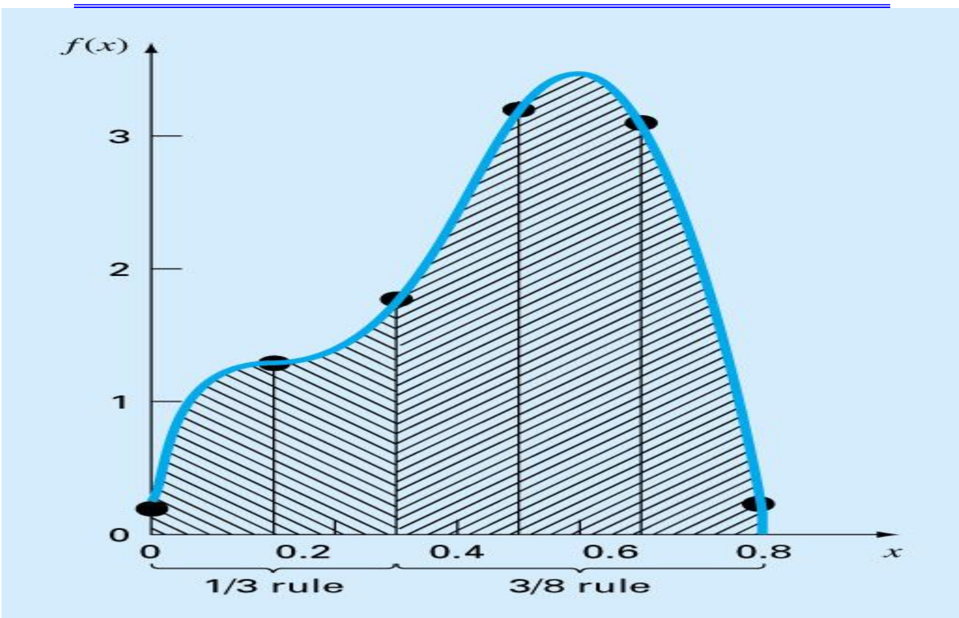
Simpson's Rules



Simpson's 1/3 Rule

- By putting n = 2 in the quadrature formula we get Simpson's 1/3 rule.
- If n = 2 then the value of $\Delta^3 y_0$, $\Delta^4 y_{0,\ldots}$ etc is zero

$$\int_a^b f(x)dx = h[2y_0 + 2\Delta y_0 + (\tfrac{8}{3} - 2)\frac{\Delta^2 y_0}{2}]$$

$$= h\,[2*y_0 + 2(y_1 - y_0) + \tfrac{2}{3}\frac{\Delta^2 y_0}{2}]$$
$$= h\,[2*y_0 + 2*y_1 - 2*y_0 + \tfrac{1}{3}\Delta^2 y_0]$$
$$= h\,[2*y_0 + 2*y_1 - 2*y_0) + \tfrac{1}{3}(y_2 - 2*y_1 + y_0)]$$

Simpson's 1/3 Rule

$$\int_a^b f(x)dx = h[2y_0 + 2\Delta y_0 + (\tfrac{8}{3} - 2)\frac{\Delta^2 y_0}{2}]$$

$$= h\,[2*y_0 + 2(y_1 - y_0) + \tfrac{2}{3}\frac{\Delta^2 y_0}{2}]$$
$$= h\,[2*y_0 + 2*y_1 - 2*y_0 + \tfrac{1}{3}\Delta^2 y_0]$$
$$= h\,[(2*y_0 + 2*y_1 - 2*y_0) + \tfrac{1}{3}(y_2 - 2*y_1 + y_0)]$$
$$= \tfrac{h}{3}\,[6*y_1 + y_2 - 2*y_1 + y_0]$$
$$= \tfrac{h}{3}\,[y_2 + 4*y_1 + y_0]$$

Simpson's 1/3 Rule

$$\int_{x_0}^{x_0+2h} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2] \ldots\ldots..(1)$$

$$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}[y_2 + 4y_3 + y_4] \ldots\ldots..(2)$$

$$\int_{x_0+4h}^{x_0+6h} f(x)dx = \frac{h}{3}[y_4 + 4y_5 + y_6] \ldots\ldots..(3)$$

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$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n] \ldots\ldots..(n)$$

Take sum of integration

$$\int_{x_0}^{x_0+2h} f(x)dx + \int_{x_0+2h}^{x_0+4h} f(x)dx + \int_{x_0+4h}^{x_0+6h} f(x)dx + \ldots.+$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) +$$

$$(y_4 + 4y_5 + y_6) + \ldots.+(y_{n-2} + 4y_{n-1} + y_n)]$$

Simpson's 1/3 Rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2}) + y_n]$$

- Note : It should be noted that this rule requires the divisions of the whole range into an even number of subintervals of width h.

Simpson's 3/8 Rule

- By putting n = 3 in the quadrature formula we get Simpson's 3/8 rule.
- If n = 3 then the value of $\Delta^4 y_0$, $\Delta^5 y_{0,....}$ etc is zero

$$\int_a^b f(x)dx = h[ny_0 + \frac{n^2}{2}\Delta y_0 + (\frac{n^3}{3} - \frac{n^2}{2})\frac{\Delta^2 y_0}{2} + (\frac{n^4}{4} - n^3 + n^2)\frac{\Delta^3 y_0}{6}]$$

$$\int_a^b f(x)dx = h[3y_0 + \frac{9}{2}\Delta y_0 + (9 - \frac{9}{2})\frac{\Delta^2 y_0}{2} + (\frac{81}{4} - 27 + 9)\frac{\Delta^3 y_0}{6}]$$

Simpson’s 3/8 Rule

$$\int_a^b f(x)dx = h[3y_0 + \frac{9}{2}(y_1 - y_0) + (9 - \frac{9}{2})(\frac{y_2 - 2y_1 + y_0}{2})$$
$$+ (\frac{81}{4} - 27 + 9)(\frac{y_3 - 3y_2 + 3y_1 - y_0}{6})]$$

$$\int_a^b f(x)dx = \frac{3h}{8}[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0)$$
$$+ (y_3 - 3y_2 + 3y_1 - y_0)]$$

Simpson’s 3/8 Rule

$$\int_{x_0}^{x_0+3h} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2) + y_3] \text{........}(1)$$

$$\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}[y_3 + 3(y_4 + y_5) + y_6] \text{........}(2)$$

$$\int_{x_0+6h}^{x_0+9h} f(x)dx = \frac{3h}{8}[y_6 + 3(y_7 + y_8) + y_9] \text{........}(3)$$

Simpson's 3/8 Rule

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$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}[y_{n-3} + 3(y_{n-2}+y_{n-1}) + y_n] \ldots\ldots..(n)$$

Add all integration

$$\int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + \ldots.. +$$

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1+y_2) + y_3 +$$

$$y_3 + 3(y_4+y_5) + y_6 +$$

$$\ldots.. + y_{n-3} + 3(y_{n-2}+y_{n-1}) + y_n]$$

Simpson's 3/8 Rule

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3}) + y_n]$$

- Note : It should be noted that the entire range must be divided into an even number of subintervals of width h Or Total number of points must be odd.