Equation of the tangent line with implicit differentiation

In the same way we learned how to find the equation of the tangent line previously, we can find the equation of a tangent line for a function, even when we have to use implicit differentiation to find the function's derivative.

For instance, in the last lesson, we used implicit differentiation to find that the derivative of $x^3 + y^3 = 9xy$ was

$$y' = \frac{3y - x^2}{y^2 - 3x}$$

If we wanted to find the equation of the tangent line to $x^3 + y^3 = 9xy$ at the point $(2,4)$, we'd find the slope of the tangent line by substituting the point into the derivative, y'.

$$y'(2,4) = \frac{3(4) - 2^2}{4^2 - 3(2)}$$

$$y'(2,4) = \frac{12 - 4}{16 - 6}$$

$$y'(2,4) = \frac{8}{10}$$

$$y'(2,4) = \frac{4}{5}$$

Once we have the slope at the point of tangency, we plug the slope $m = 4/5$ and the point $(2,4)$ into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = \frac{4}{5}(x - 2)$$

$$y - 4 = \frac{4}{5}x - \frac{8}{5}$$

$$y = \frac{4}{5}x - \frac{8}{5} + \frac{20}{5}$$

$$y = \frac{4}{5}x + \frac{12}{5}$$

Finding the tangent line equation

From this, we can generalize the set up steps we can use every time in order to find the equation of the tangent line to an implicitly-defined function at a particular point.

1. Find the derivative using implicit differentiation.

2. Evaluate the derivative at the given point to find the slope of the tangent line.

3. Plug the slope of the tangent line and the given point into the point-slope formula for the equation of a line, $y - y_1 = m(x - x_1)$.

4. Simplify the tangent line equation.

Let's do another full example, starting from the beginning with a new function.

Example

Find the equation of the tangent line at $(1,2)$.

$$3y^2 - 2x^5 = 10$$

We could actually solve this equation for y, and then differentiate using traditional techniques, but it's a little tedious to do it that way. It's probably faster for us if we just leave the equation the way it's written, and use implicit differentiation.

Remember that whenever we use implicit differentiation to take the derivative of a term involving y, we have to multiply the result by the derivative of y, which we can write as dy/dx or as y'. The derivative is

$$6y\frac{dy}{dx} - 10x^4 = 0$$

Now we'll simplify and solve for dy/dx.

$$6y\frac{dy}{dx} = 10x^4$$

$$\frac{dy}{dx} = \frac{10x^4}{6y}$$

$$\frac{dy}{dx} = \frac{5x^4}{3y}$$

Keep in mind that, when we're finding the equation of the tangent line, we actually don't need to solve explicitly for dy/dx. Once we use implicit differentiation, we can move directly to plugging in the point of tangency, without first solving for dy/dx. Either way, whether we solve for dy/dx before or after plugging in the point of tangency, our next step is to plug in $(1,2)$.

$$\frac{dy}{dx}(1,2) = \frac{5(1)^4}{3(2)}$$

$$\frac{dy}{dx}(1,2) = \frac{5}{6}$$

Now that we have the slope of the tangent line $m = 5/6$, and the point of tangency, we can plug directly into the point-slope formula for the equation of the tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{5}{6}(x - 1)$$

$$y - 2 = \frac{5}{6}x - \frac{5}{6}$$

$$y = \frac{5}{6}x - \frac{5}{6} + \frac{12}{6}$$

$$y = \frac{5}{6}x + \frac{7}{6}$$