Sketching graphs

At this point, we have plenty of information about the graph of the function, and we can pull that information together in order to create a sketch.

Throughout this section, we've been working with the same function.

$$f(x) = x + \frac{4}{x}$$

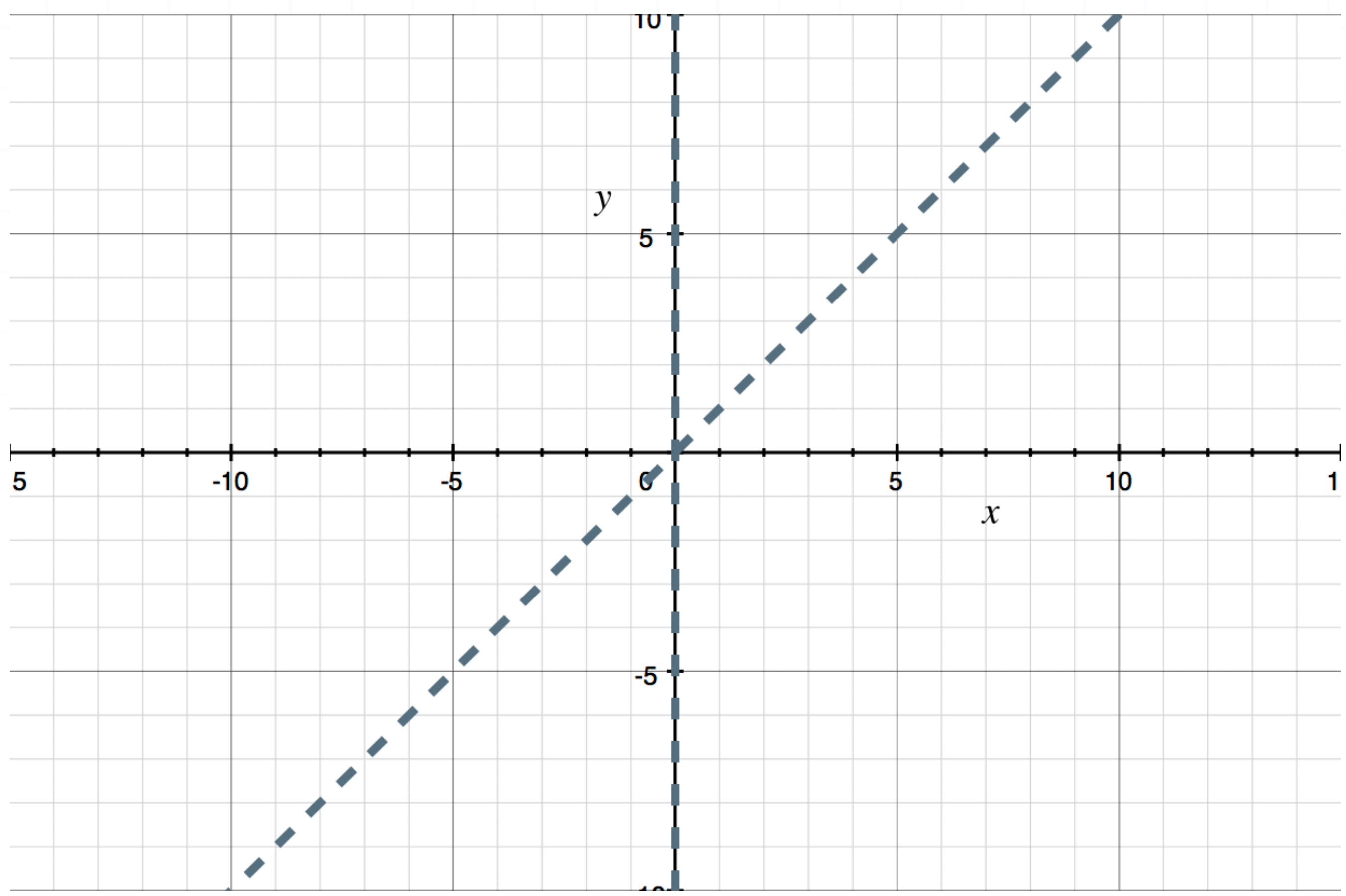
And we've calculated the following values for it:

- The critical points are $x = \pm 2$.
- The function is increasing on $-\infty < x < -2$, decreasing on $-2 < x < 2$, and increasing again on $2 < x < \infty$.
- The function has a local maximum at $x = -2$ and a local minimum at $x = 2$.
- The function has an inflection point at $x = 0$.
- The function is concave down on $-\infty < x < 0$ and concave up on $0 < x < \infty$.
- The function has no intercepts, so it doesn't cross the x- or y-axis.
- The function has a vertical asymptote at $x = 0$, no horizontal asymptote, and a slant asymptote at $y = x$.

From this information, we can get a pretty good sketch of the graph. We can also use information about the domain and range of the function and/or its symmetry.

Sketching the graph

A great starting point is to sketch in any asymptotes, because we know their exact equations, and they'll serve as guiding lines when we trace out the graph. Draw in the lines $x=0$ and $y=x$.



The function has a local maximum at $x=-2$ and a local minimum at $x=2$, so let's plug those values into the function in order to find those particular points along the graph. We get

$$f(-2)=-2+\frac{4}{-2}$$

$$f(-2) = -2 - 2$$

$$f(-2) = -4$$

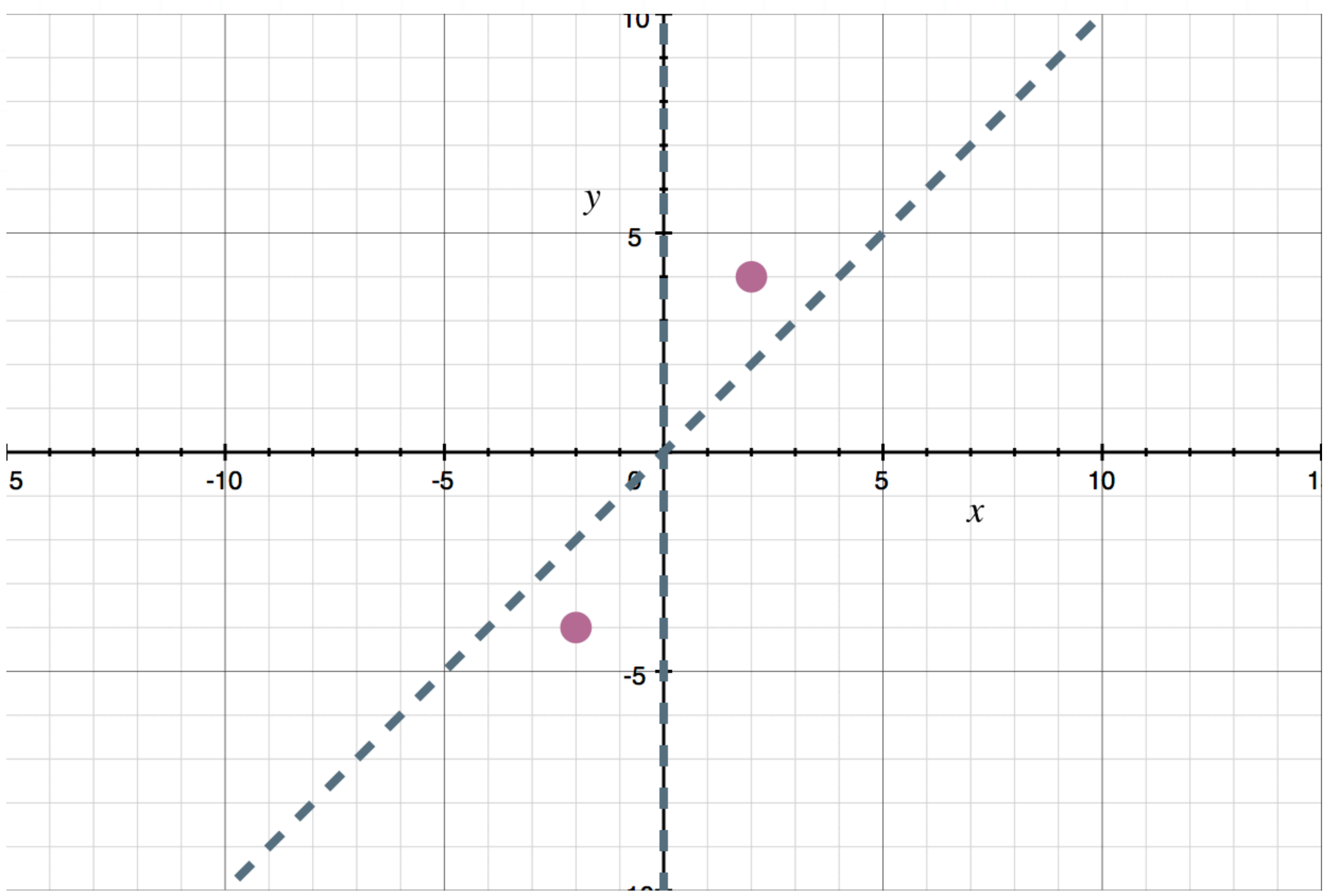
and

$$f(2) = 2 + \frac{4}{2}$$

$$f(2) = 2 + 2$$

$$f(2) = 4$$

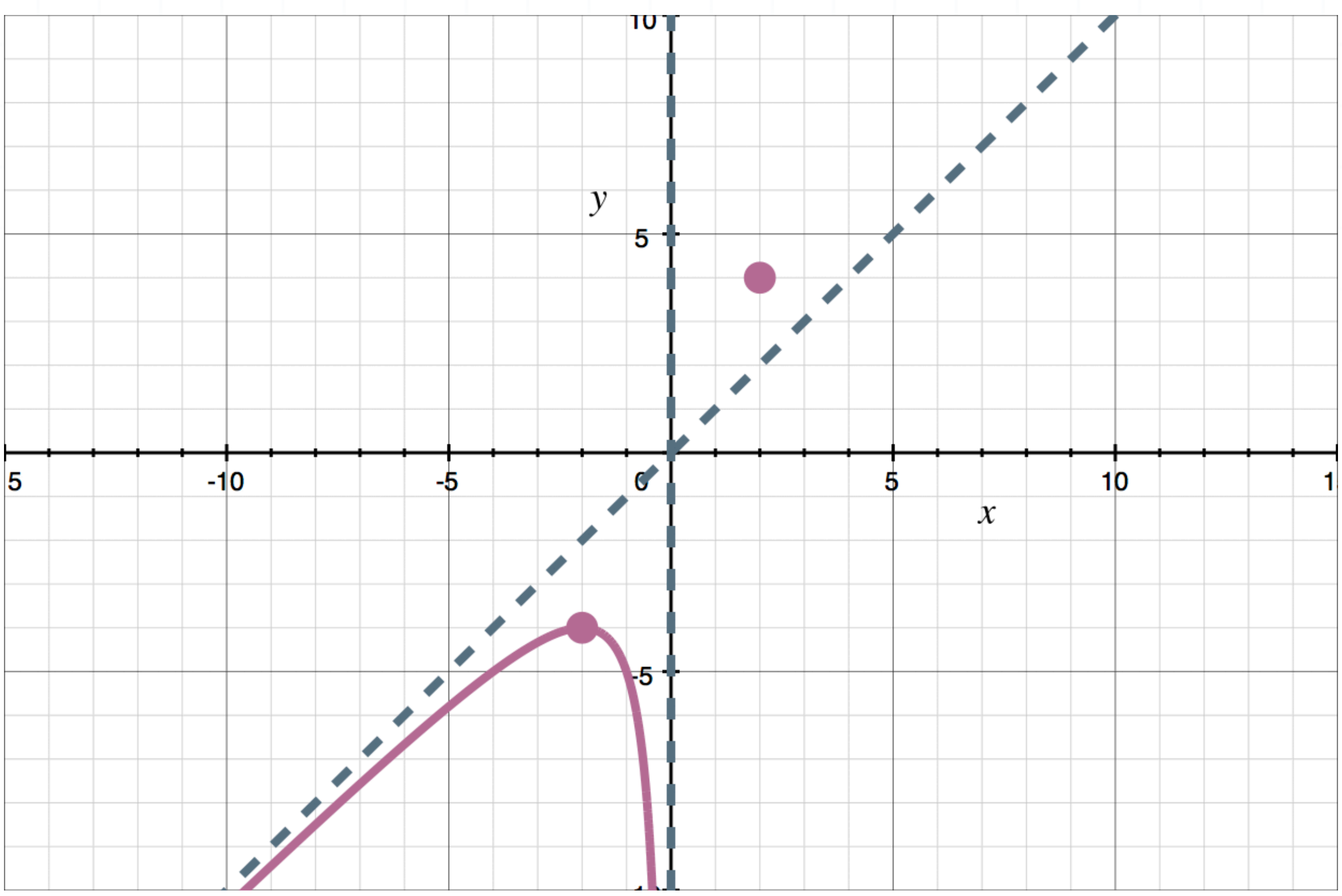
Add those critical points, $(-2, -4)$ and $(2,4)$, to the graph.



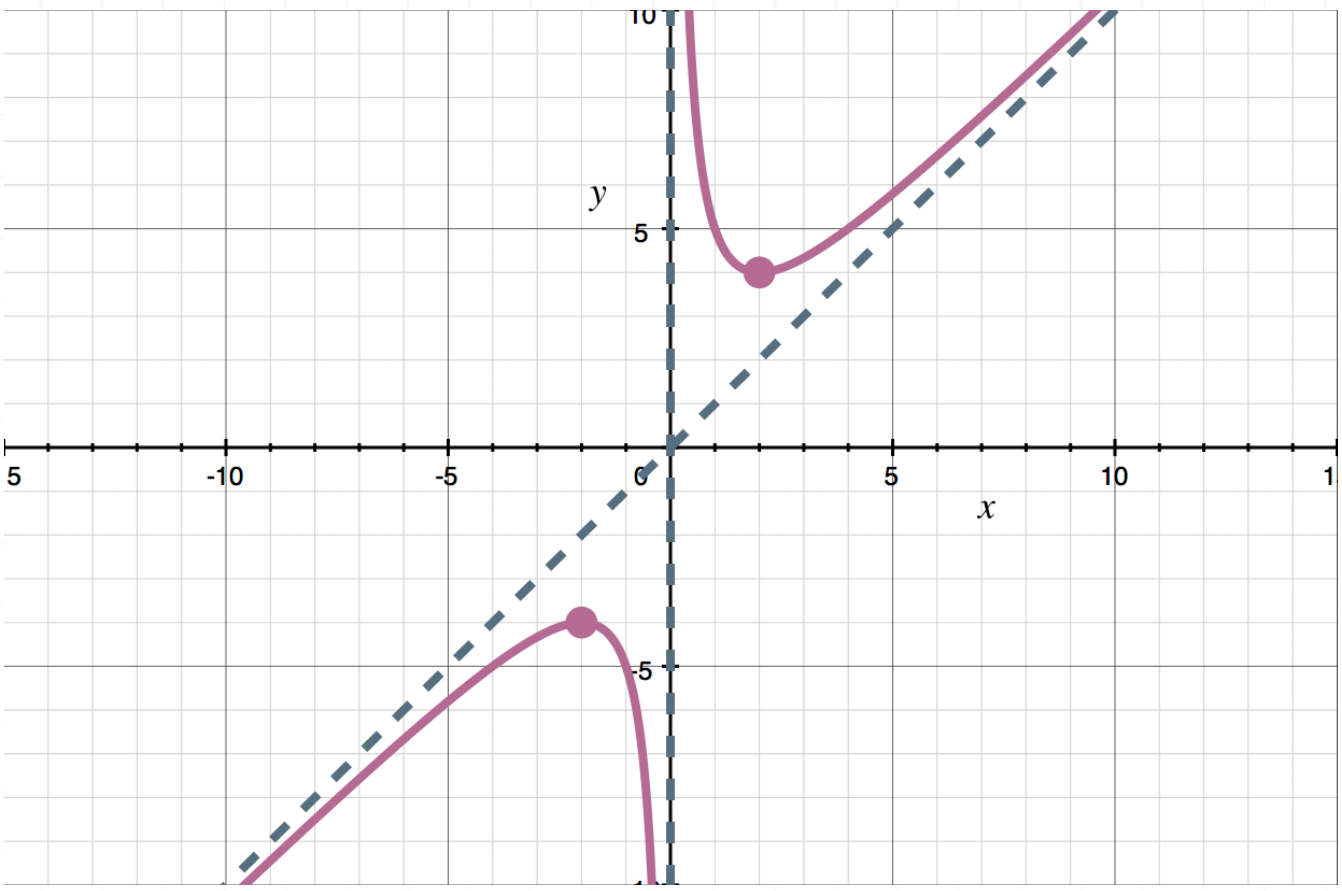
Now we need to hold a few things in our heads at once to sketch the rest of the graph.

First, we know the graph doesn't cross the y-axis, but we have points that are on the graph on either side of that line, which means the graph will have two separate, unattached pieces.

Starting with the piece to the left of the y-axis, we know that the curve can't cross $y = x$ or $x = 0$, so we'll have to stay inside that lower-left triangle that's created by those lines. We know the function is increasing to the left of that critical point on $-\infty < x < -2$, and decreasing to the right of it. And we know that $x = -2$ represents a local maximum. We also know that the function is concave down on $-\infty < x < 0$. If we put all that together, we can say that the left piece of the graph looks something like this:



For the piece to the right of the y-axis, we know that the curve can't cross $y = x$ or $x = 0$, so we'll have to stay inside that upper-right triangle that's created by those lines. We know the function is decreasing to the left of that critical point, and increasing to the right of it on $2 < x < \infty$. And we know that $x = 2$ represents a local minimum. We also know that the function is concave up on $0 < x < \infty$. If we put all that together, we can add the right piece of the graph:



This is the graph of the function.

Keep in mind, this kind of graph sketching takes lots of practice. First, we have to work through the entire optimization process, then find all the intercepts and asymptotes. Then, we have to put all of this information together to sketch the graph.

To make this graph sketching as manageable as possible, we want to make sure we put everything that's easy onto the graph. This would include asymptotes, coordinate points that represent the intercepts, and the coordinate points that represent the critical and inflection points.

This way, hopefully, all we have left to really "hold in our head" at one time is the increasing/decreasing and concave down/concave up behavior. But we'll have the asymptotes and some points already there to guide us.

All that being said, sketching graphs this way takes practice, so don't be discouraged if it doesn't come easily when you first start.

Here is a summary of steps we can use to sketch the graph of a function:

- Calculate the first derivative, find critical points, and use the first derivative test to determine where the function is increasing and decreasing, then classify those critical points as maxima or minima.

- Calculate the second derivative, find inflection points, and determine where the function is concave up and concave down.

- Find any vertical, horizontal, and slant asymptotes, and any x- and y- intercepts.

- Consider the domain and range of the function and determine any points of discontinuity.