Higher-order derivatives

Up to now, we've only dealt with first derivatives. The **first derivative** is exactly the value we've been finding all along; it's what you get when you take the derivative.

In this lesson, we want to turn to **higher-order derivatives**, which are the second derivative, third derivative, fourth derivative, etc.

The **second derivative** is the derivative of the derivative; it's the derivative of the first derivative. The **third derivative** is the derivative of the second derivative, the **fourth derivative** is the derivative of the third derivative, etc.

As long as each derivatives continues to be differentiable, theoretically, there's no limit to the number of derivatives we can find. Each derivative models the slope of the derivative before it. So, in the same way that the derivative models the slope of the original function, the second derivative models the slope of the first derivative, the third derivative models the slope of the second derivative, etc.

Function	**What it models**
$f(x)$	
$f'(x)$	Models the slope of $f(x)$
$f''(x)$	Models the slope of $f'(x)$
$f'''(x)$	Models the slope of $f''(x)$

$f^{(4)}(x)$	Models the slope of $f'''(x)$
...	...

Rules and notation

We use all the same rules for finding higher-order derivatives as we did to find the first derivative. In other words, the power, product, quotient, and chain rules still apply in the same way. And taking derivatives of trigonometric, exponential, logarithmic functions, etc., will be the same in higher-order derivative functions as it was in the original function.

It's nice to keep our derivative notation consistent.

Function	**1st derivative**	**2nd derivative**	**3rd derivative**
y	y'	y''	y'''
	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$
$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$

Applications

Higher-order derivatives are important in calculus and in real-life applications. Later we'll look in depth at how to use derivatives to sketch the graph of a function. The first derivative gives us information about where the function is increasing and decreasing, and the second derivative gives us information about where the function is concave up and down.

We'll also look at position functions. When an equation models the position of an object, its first derivative models the object's velocity, and its second derivative models the object's acceleration.

Let's work through an example so that we can see how to get to the second derivative of a function.

Example

Find the second derivative of the function.

$$y = x^3 + 3x^2 - 4x + 5$$

Start by using power rule to find the first derivative.

$$y = x^3 + 3x^2 - 4x + 5$$

$$y' = 3x^2 + 6x - 4$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (3x^2 + 6x - 4)'$$

$$y'' = 6x + 6$$

For practice, let's try one more example. This time, we'll use the dy/dx notation for all of our derivatives.

Example

Find the function's second and the third derivatives.

$$y = \cos^2 x$$

Use chain rule to find the first derivative.

$$y = \cos^2 x$$

$$\frac{dy}{dx} = 2\cos x(-\sin x)$$

$$\frac{dy}{dx} = -2\sin x \cos x$$

Now we can use product rule to find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-2\sin x \cos x)$$

$$\frac{d^2y}{dx^2} = -2\cos x \cos x + (-2\sin x(-\sin x))$$

$$\frac{d^2y}{dx^2} = -2\cos^2 x + 2\sin^2 x$$

$$\frac{d^2y}{dx^2} = 2\sin^2 x - 2\cos^2 x$$

Find the third derivative by taking the derivative of the second derivative.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(2\sin^2 x - 2\cos^2 x)$$

$$\frac{d^3y}{dx^3} = 2(2\sin x)(\cos x) - 2(2\cos x)(-\sin x)$$

$$\frac{d^3y}{dx^3} = 4\sin x\cos x + 4\sin x\cos x$$

$$\frac{d^3y}{dx^3} = 8\sin x\cos x$$