Definition of the derivative

The derivative is the answer to the single most important question in Calculus 1, which is "How do we find the slope of a function at a particular point?"

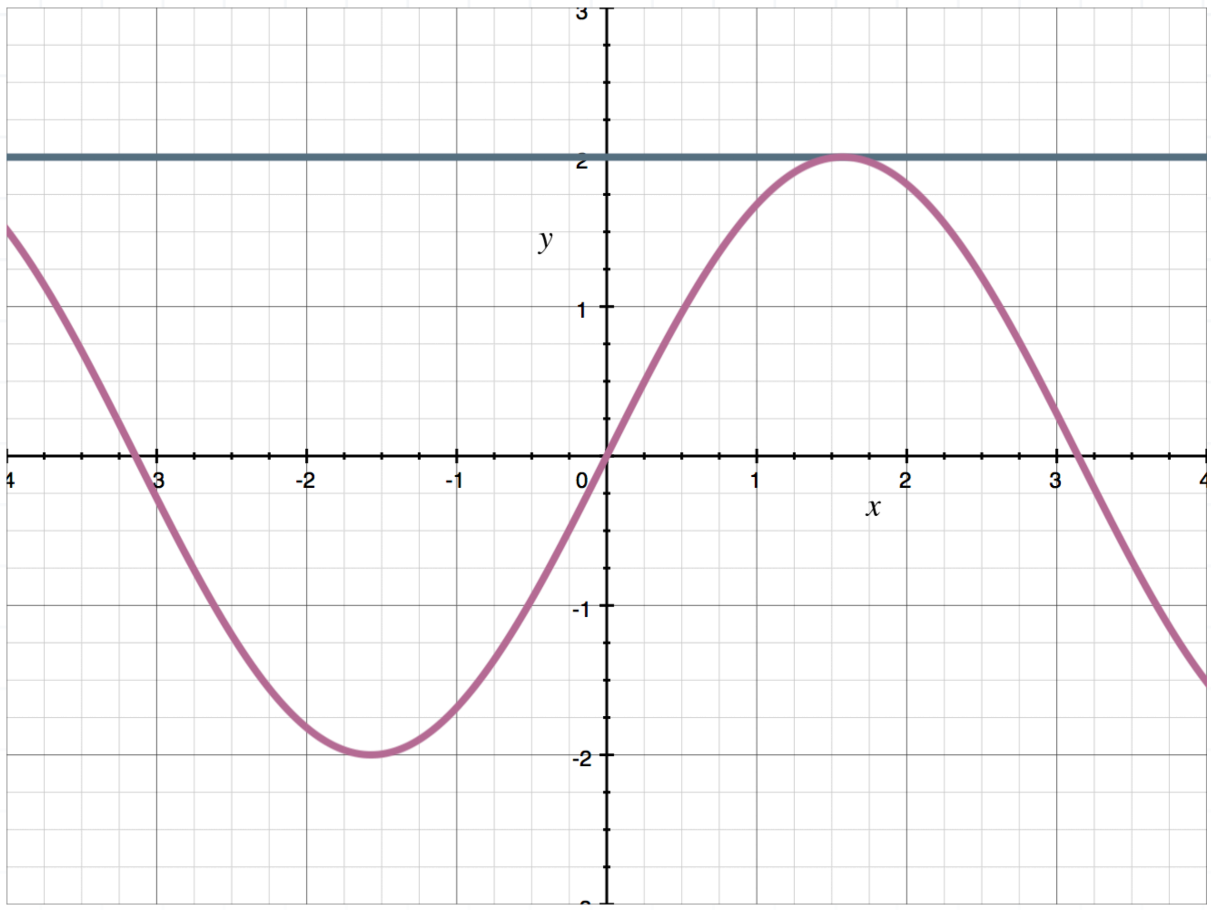
We already know, from Algebra, how to find the slope of a straight line. But in Calculus, we want to figure out how to find the slope of a function, even when it's curved.

To find that slope, we 1) calculate the derivative of the function in general, and then we 2) evaluate the derivative at the point we're interested in.

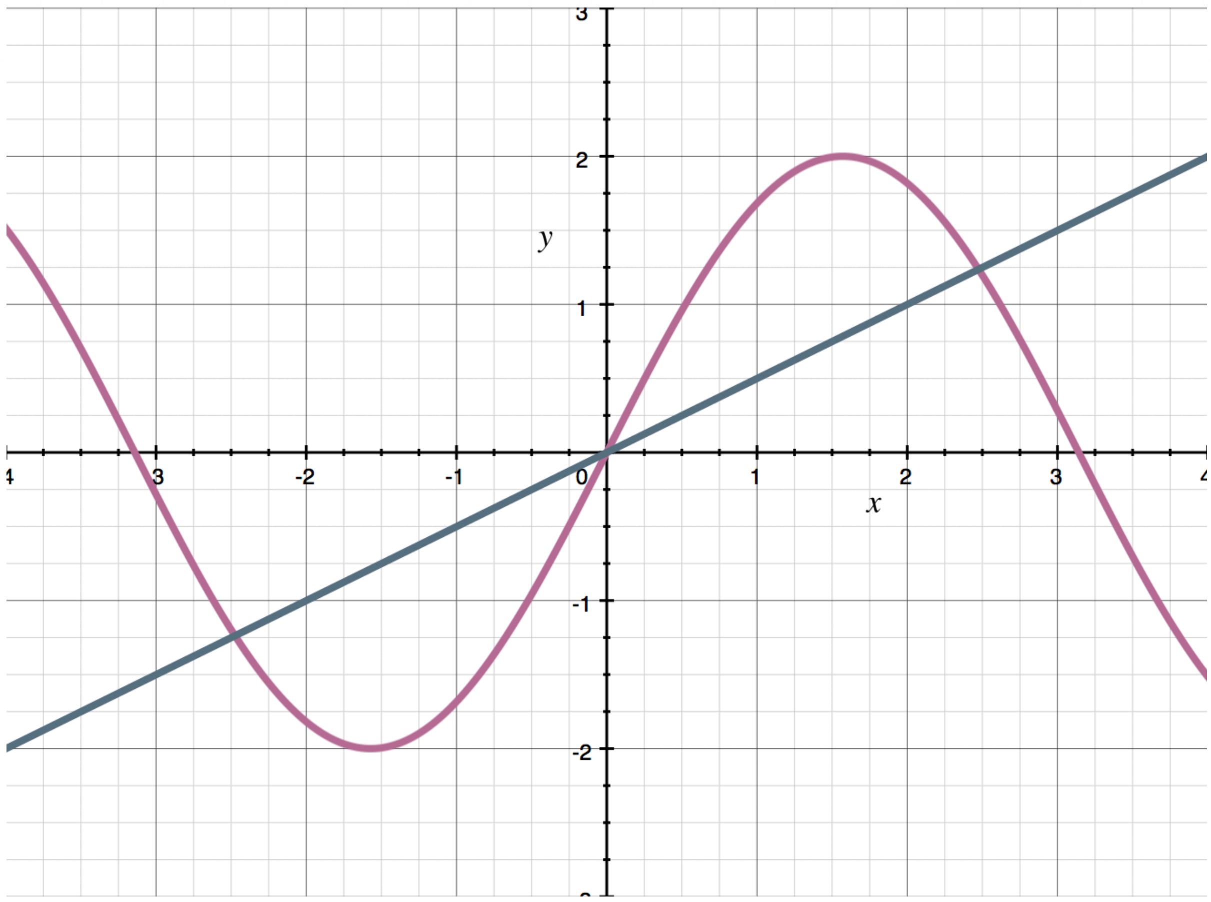
To better understand the idea of the derivative in general, we want to starting by thinking about the difference between a tangent line and a secant line.

Secant and tangent lines

A tangent line is a line that just barely touches the edge of a graph, intersecting it at exactly one specific point. The line doesn't cross the graph, it skims along the graph and stays along the same side of the graph.



A secant line, on the other hand, is a line that runs right through the graph, crossing it at a point.



In both graphs here, we're showing the same curve. Let's call it the function $f(x)$. Theoretically, we could use the secant line to approximate the rate of change of the curve between $x = 0$ and $x \approx 2.5$. Those are two of the points where the secant line intersects $f(x)$.

That would give us an idea about the **average rate of change** of the function in that interval, $[0,2.5]$. But as we can see from the graph, the slope of $f(x)$ is changing constantly throughout that interval. So, if we're actually interested in the slope of the function at $x \approx 1.5$, for example, the average rate of change over $[0,2.5]$ would give us somewhat of an estimate, but it wouldn't give us an exact rate of change at $x \approx 1.5$.

On the other hand, if we use the tangent line instead, we can see that it intersects the graph at one single point, about $x \approx 1.5$. So if we use the slope of the tangent instead of the slope of the secant line, we could get the **instantaneous rate of change** of the function at that exact point.

That's what the derivative allows us to do. Instead of using the secant line to settle for an average rate of change over an interval, it lets us use the tangent line to find the instantaneous (exact) rate of change at a specific point.

Building the derivative

We've said that slope of the secant line is the average rate of change over the interval between the points where the secant line intersects the graph, and that the slope of the tangent line instead indicates an instantaneous rate of change at the single point where it intersects the graph.

To express this mathematically, we start with a point $(c, f(c))$ on a graph, and then move a certain distance Δx to the right of that point, and call that new point $(c + \Delta x, f(c + \Delta x))$.

Connecting those points together gives us a secant line. Remember that the slope of any line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and if we plug the values from our two points $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$ into this slope equation, we get

$$m = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c}$$

$$m = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c}$$

$$m = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

This is the slope of any generic secant line that intersects the curve at two points $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$.

What we want to do now is turn this secant line into a tangent line, which we can do by moving the two intersection points closer and closer together. As we move the two points closer to each other, the secant line will start to look more and more like a tangent line.

Eventually, if we move the points so close together that we reduce the distance between them to 0, then the secant line will literally become the tangent line. Mathematically, this means that we're reducing the value of Δx to 0, since Δx represents the horizontal distance between the points.

Therefore, we can say that the **definition of the derivative** at $x = c$ is

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

We'll also see this same definition of the derivative formula written as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

and we can use either one to find the derivative of any function at a particular point.

Calculating the derivative

When it comes to actually applying the definition to calculate the derivative of a function at a particular point x, we'll

1. substitute $x + \Delta x$ for every x in the original function, then plug this result into the definition for $f(x + \Delta x)$, then

2. plug the original function $f(x)$ into the definition.

That fills out the definition of the derivative formula. From there, we'll simplify the **difference quotient**, which is the fraction that makes up the definition of the derivative, and then evaluate the limit.

Example

Use the definition of the derivative to differentiate the function.

$$f(x) = x^2 - 5x + 6$$

After replacing x with $(x + \Delta x)$ in $f(x)$,

$$f(x + \Delta x) = (x + \Delta x)^2 - 5(x + \Delta x) + 6$$

we'll substitute for $f(x + \Delta x)$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 5(x + \Delta x) + 6 - f(x)}{\Delta x}$$

Then plug $f(x)$ into the definition.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 5(x + \Delta x) + 6 - (x^2 - 5x + 6)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 5x - 5\Delta x + 6 - x^2 + 5x - 6}{\Delta x}$$

Collect like terms,

$$f'(x) = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 - 5x - 5\Delta x + 6 + 5x - 6}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 - 5\Delta x + 6 - 6}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x - 5\Delta x}{\Delta x}$$

then factor Δx out of the numerator and cancel out that common factor from the numerator and denominator.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(\Delta x + 2x - 5)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (\Delta x + 2x - 5)$$

Now we evaluate the limit using substitution, which means we'll substitute $\Delta x = 0$.

$$f'(x) = 0 + 2x - 5$$

$$f'(x) = 2x - 5$$

The answer we just got from this example is the derivative of the original function $f(x) = x^2 - 5x + 6$.

The amazing thing is that, once we have the derivative, we can find the slope of the function at any point we'd like!

For instance, if we want to know the slope of the function at $x = 1$, we plug $x = 1$ into the derivative.

$$f'(1) = 2(1) - 5$$

$$f'(1) = 2 - 5$$

$$f'(1) = -3$$

Then we can say that the slope of the function at $x = 1$ is -3.