Sketching f(x) from f'(x)

We talked earlier about graph sketching, and looked at how to pull together all of the optimization, intercept, and asymptote information together in order to get a picture of what the graph looks like.

Now we want to apply that graph sketching process to the relationship between the graphs of a function, its derivative, and its second derivative. The goal here is to be able to start with either $f(x)$ or $f'(x)$ or $f''(x)$, and use information about only one of those functions in order to sketch graphs of the other two.

This process can get pretty confusing, but we can organize what we know about the relationships between these functions in a chart, which will help us navigate as we go. Let's start building that chart now.

The chart

Let's make a chart with the three functions, $f(x)$, $f'(x)$, and $f''(x)$, as three columns. When we talked about critical points and the first derivative test, we learned that when the derivative is equal to 0, the original function has a critical point.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	

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When a function is equal to 0, it has an x-intercept, because it's crossing the x-axis.

And we learned that when the derivative is positive, the original function is increasing, but when the derivative is negative, the original function is decreasing.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	
Increasing	Positive (above the x-axis)	
Decreasing	Negative (below the x-axis)	

If the value of a function is positive, the graph of the function is of course above the x-axis, and if the value of a function is negative, its graph is below the x-axis at that point.

When we talked about inflection points and the second derivative test, we learned that the original function has an inflection point when its second derivative is 0. And we learned that the original function is concave up when the second derivative is positive, and that the original function is concave down when the second derivative is negative.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	
Increasing	Positive (above the x-axis)	
Decreasing	Negative (below the x-axis)	

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Inflection point		0 (x-intercept)
Concave up		Positive (above the x-axis)
Concave down		Negative (below the x-axis)

At this point, we want to realize that the first row of our chart tells us that $f(x)$ has a critical point when $f'(x)=0$. This relationship exists because $f'(x)$ is the derivative of $f(x)$.

But the same relationship exists between $f'(x)$ and $f''(x)$: the second derivative $f''(x)$ is the derivative of the first derivative $f'(x)$. So the same relationship will exist. Which means we can use the information from the first three rows of our chart, to fill in the empty second column in the last three rows of the chart.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	
Increasing	Positive (above the x-axis)	
Decreasing	Negative (below the x-axis)	
Inflection point	Critical point	0 (x-intercept)
Concave up	Increasing	Positive (above the x-axis)
Concave down	Decreasing	Negative (below the x-axis)

Now let's imagine for a moment that we're done with the first column of this chart, and we're looking just at the second and third columns. Again, we have the same relationship between $f'(x)$ and $f''(x)$ as we do between

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$f(x)$ and $f'(x)$. So we could, for instance, say that the first derivative $f'(x)$ will have an inflection point whenever the second derivative $f''(x)$ has a critical point.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	0 (x-intercept)	
Increasing	Positive (above the x-axis)	
Decreasing	Negative (below the x-axis)	
Inflection point	Critical point	0 (x-intercept)
Concave up	Increasing	Positive (above the x-axis)
Concave down	Decreasing	Negative (below the x-axis)
	Inflection point	Critical point

And, in the same way, we know that the first derivative will be concave up when the second derivative is increasing, and concave down when the second derivative is decreasing. So we get a final version of this chart.

$f(x)$	$f'(x)$	$f''(x)$
Critical point	**0 (x-intercept)**	
Increasing	**Positive (above the x-axis)**	
Decreasing	**Negative (below the x-axis)**	
Inflection point	**Critical point**	**0 (x-intercept)**

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Concave up	Increasing	Positive (above the x-axis)
Concave down	Decreasing	Negative (below the x-axis)
	Inflection point	Critical point
	Concave up	Increasing
	Concave down	Decreasing

By color-coding the chart, we can really see the patterns. Furthermore, we realize that we only need to remember the relationships in the middle three rows, because they capture all the information from the first three rows and the last three rows.

So we could really simplify the chart, and say that for any three functions, where you have an original function, its derivative, and its second derivative, the relationships between them look like this:

$f(x)$	$f'(x)$	$f''(x)$
Inflection point	Critical point	0 (x-intercept)
Concave up	Increasing	Positive (above the x-axis)
Concave down	Decreasing	Negative (below the x-axis)

It may be easier for some people to use the expanded version of the chart, but easier for others to use this condensed version of the chart, so feel free to work with whichever one works better for you.

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A "possible" graph

Oftentimes when we're doing these kinds of sketching problems, we'll be asked to sketch a "possible" graph. What that means is that we're just supposed to sketch a graph that doesn't contradict any of the information we've been given.

For instance, we might be given some information about $f''(x)$ and asked to sketch a possible $f'(x)$. From the graph of $f''(x)$, we may not have enough information to sketch $f'(x)$ exactly as it actually is, point for point. But we can make a rough sketch of it that at least doesn't contradict any of the relationships that we outlined in the chart we made.

So that'll be our goal, and now that we everything in place, let's put this into practice with an example.

Example

Given the graph of $f'(x)$, sketch a possible $f(x)$ and a possible $f''(x)$.

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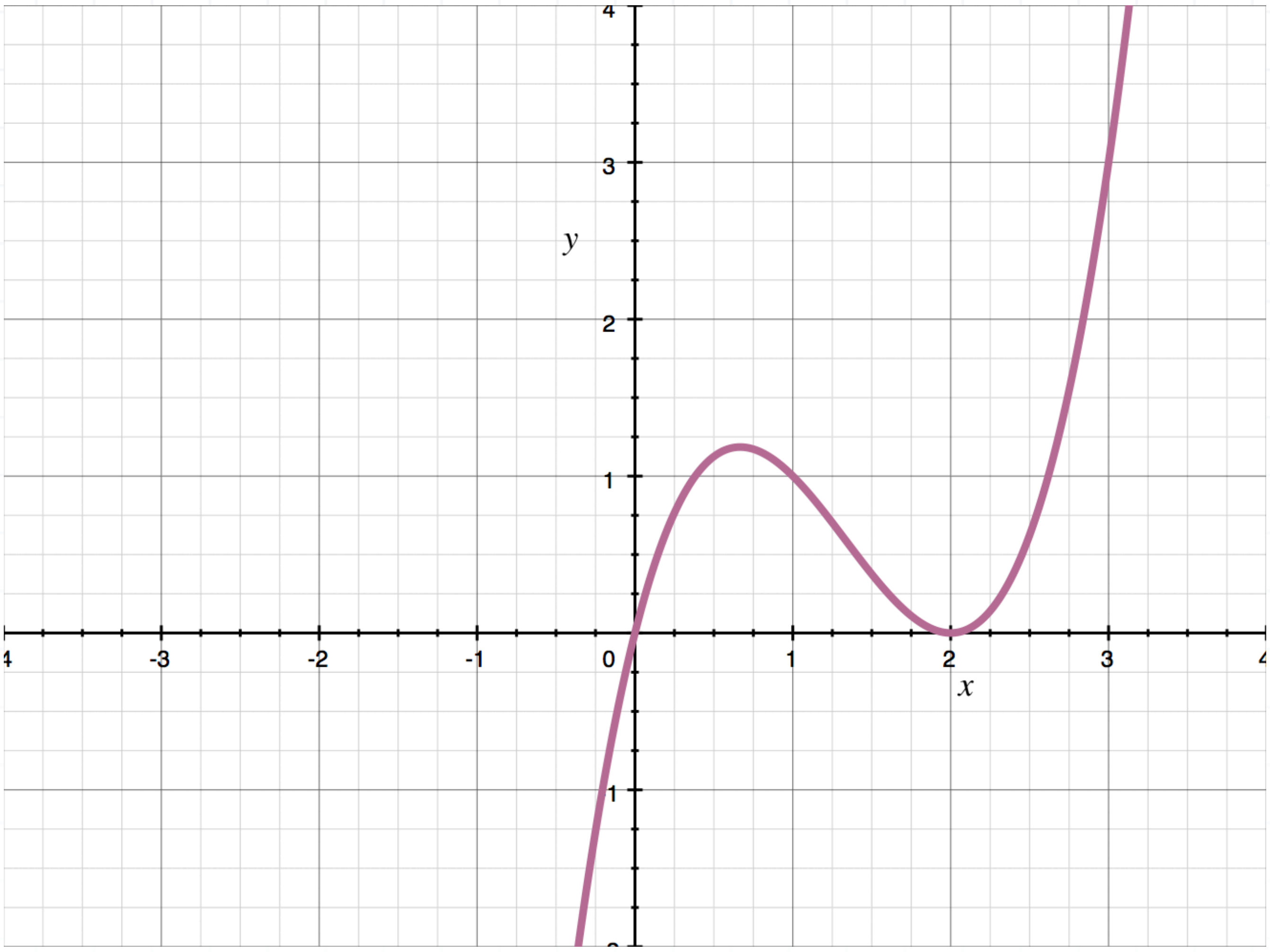

First, let's work on collecting information from this graph of $f'(x)$. We'll work right down the $f'(x)$ column in the center of the chart, listing in order the pieces of information there. From the graph we have of $f'(x)$, it may be hard to tell the exact value of some of these pieces, but an estimate is all we need.

$f'(x)$	
0	$x = 0$ and $x = 2$
Positive	$0 < x < \infty$
Negative	$-\infty < x < 0$
Critical point	$x = 2/3$ and $x = 2$

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Increasing	$-\infty < x < 2/3$ and $2 < x < \infty$
Decreasing	$2/3 < x < 2$
Inflection point	$x = 4/3$
Concave up	$4/3 < x < \infty$
Concave down	$-\infty < x < 4/3$

To translate this information about the graph of $f'(x)$ into a graph of $f(x)$, all we need to do is swap out the labels from the second column of the chart for their corresponding labels from the first column of the chart:

$f(x)$

Critical point	$x = 0$ and $x = 2$
Increasing	$0 < x < \infty$
Decreasing	$-\infty < x < 0$
Inflection point	$x = 2/3$ and $x = 2$
Concave up	$-\infty < x < 2/3$ and $2 < x < \infty$
Concave down	$2/3 < x < 2$
	$x = 4/3$
	$4/3 < x < \infty$
	$-\infty < x < 4/3$

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So when it comes to sketching $f(x)$, we can ignore the last three pieces of information, and simplify the table.

$f(x)$	
Critical point	$x = 0$ and $x = 2$
Increasing	$0 < x < \infty$
Decreasing	$-\infty < x < 0$
Inflection point	$x = 2/3$ and $x = 2$
Concave up	$-\infty < x < 2/3$ and $2 < x < \infty$
Concave down	$2/3 < x < 2$

If we put all of this information together, one possible graph of $f(x)$ might look like this:

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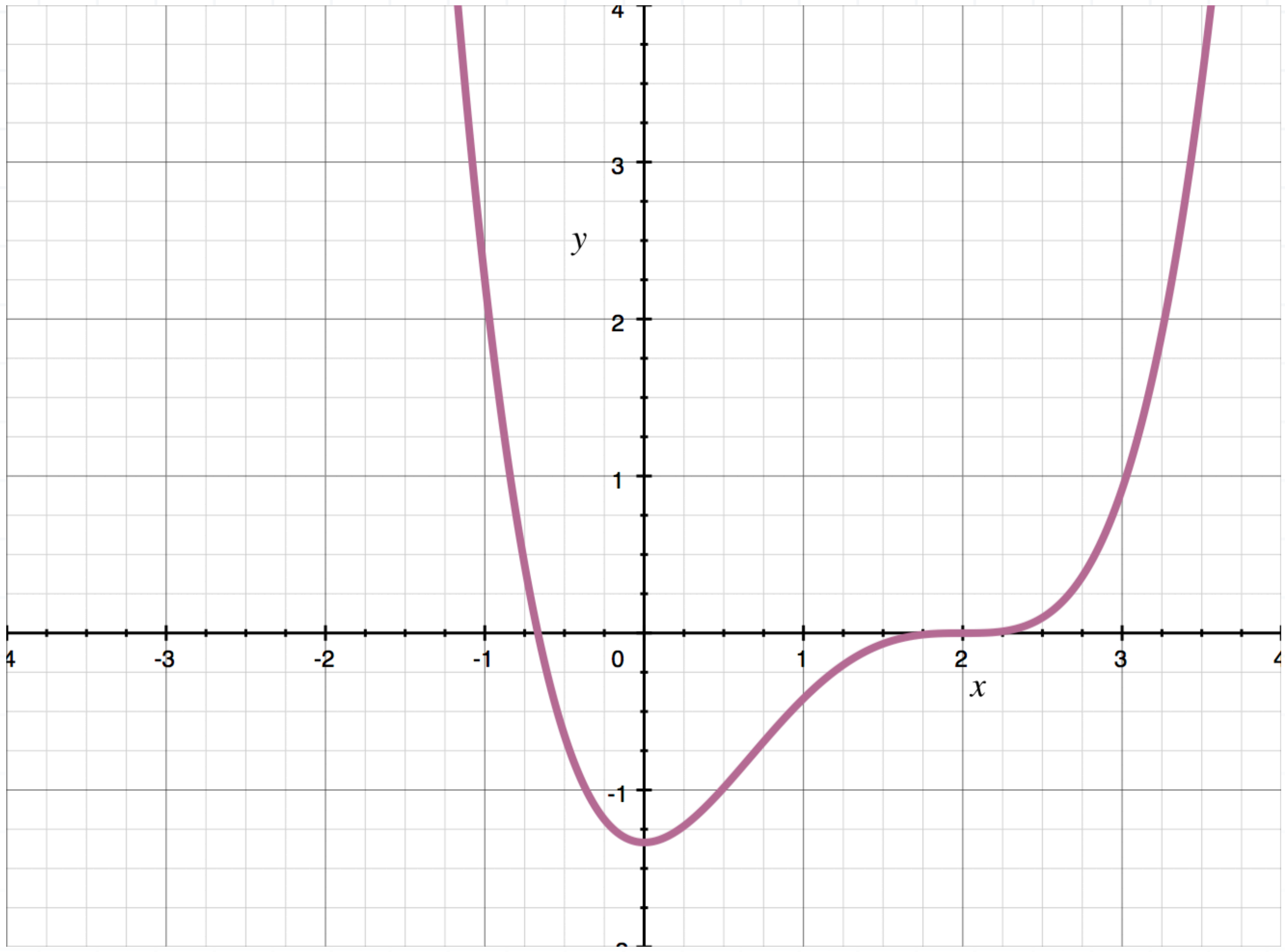

Now we'll go back to the original information we collect about $f'(x)$, but this time swap out the labels from the second column of the chart for their corresponding labels from the first column of the chart:

$f''(x)$	
	$x=0$ and $x=2$
	$0<x<\infty$
	$-\infty<x<0$
0	$x=2/3$ and $x=2$
Positive	$-\infty<x<2/3$ and $2<x<\infty$
Negative	$2/3<x<2$

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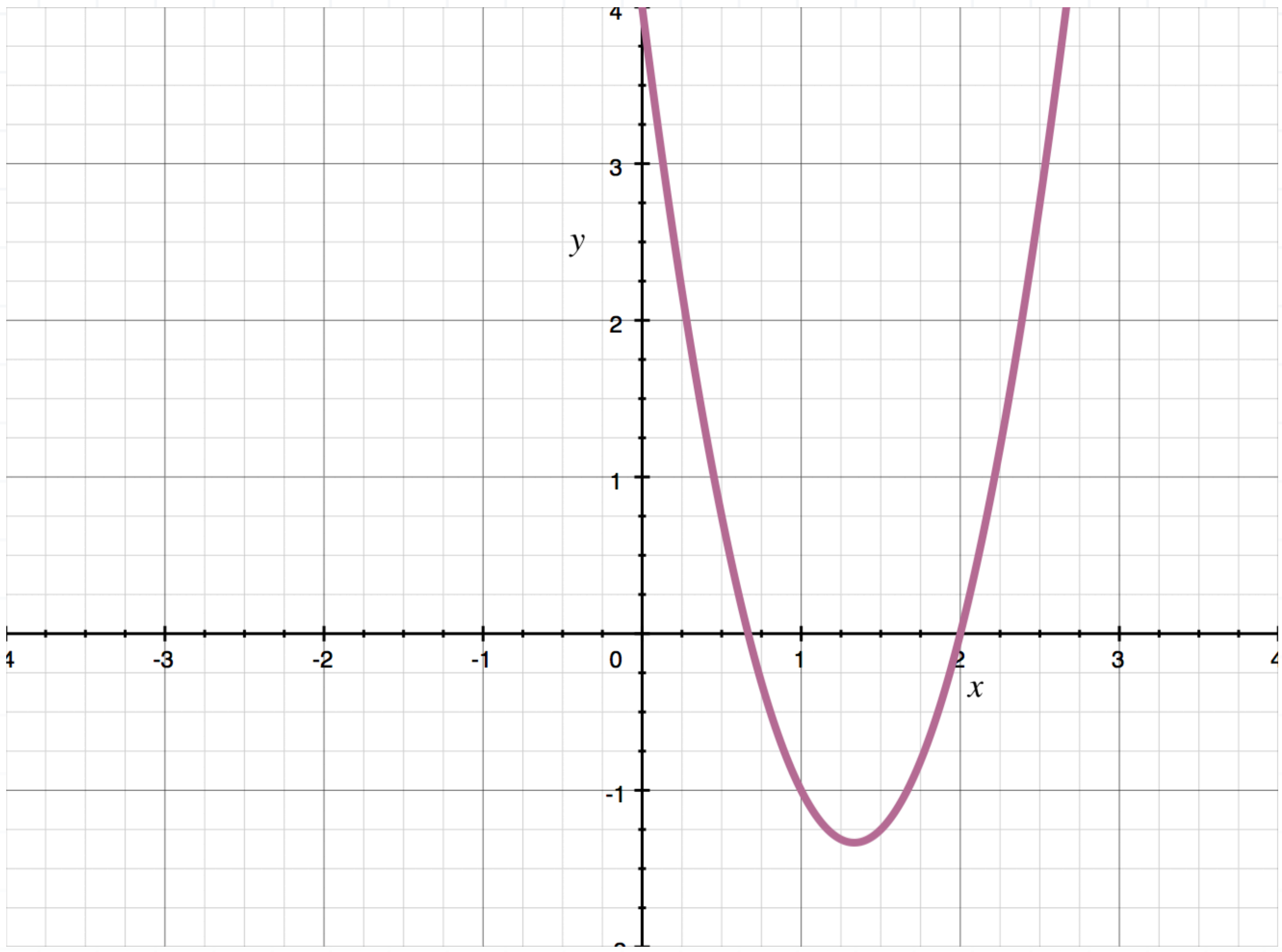
Critical point	$x = 4/3$
Increasing	$4/3 < x < \infty$
Decreasing	$-\infty < x < 4/3$

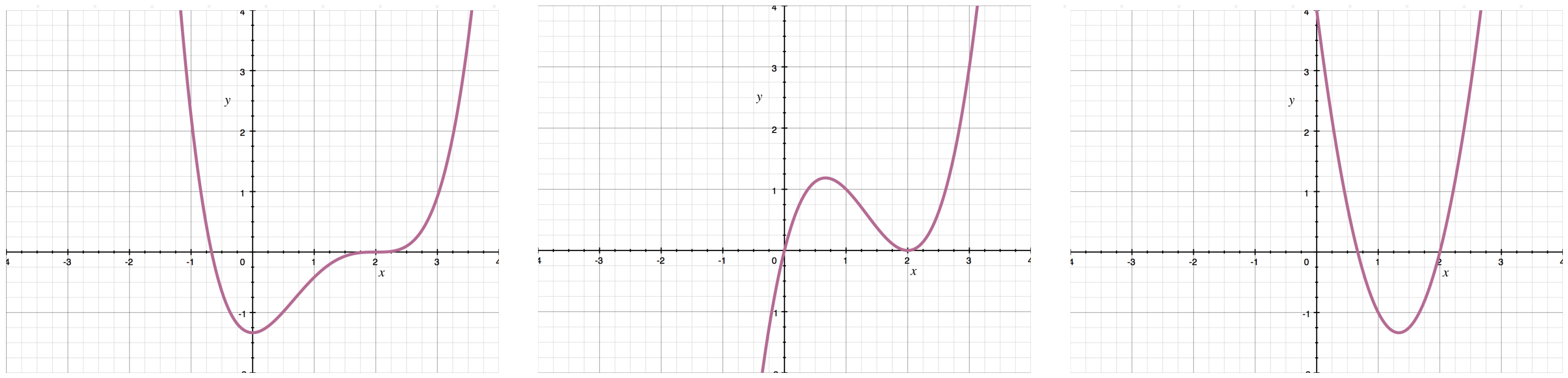
So when it comes to sketching $f''(x)$, we can ignore the first three pieces of information, and simplify the table.

$f''(x)$	
0	$x = 2/3$ and $x = 2$
Positive	$-\infty < x < 2/3$ and $2 < x < \infty$
Negative	$2/3 < x < 2$
Critical point	$x = 4/3$
Increasing	$4/3 < x < \infty$
Decreasing	$-\infty < x < 4/3$

If we put all of this information together, one possible graph of $f''(x)$ might look like this:



There’s one last thing we should notice about this last example. The graphs of $f(x)$, $f'(x)$, and $f''(x)$, in that order, are



Here’s what we see:

- The original function $f(x)$ is quadric (a fourth-degree function)

- The derivative $f'(x)$ is cubic (a third-degree function)

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- The second derivative $f''(x)$ is parabolic (a second-degree function)

This should make sense! If the original function is quadric, then it'll include an x^4 term. When we differentiate that term, we'll get an x^3 term in the derivative function. And when we differentiate that term, we'll get an x^2 term in the second derivative function.

Understanding how $f(x) \to f'(x) \to f''(x)$ can follow a pattern like $x^4 \to x^3 \to x^2$, and how $f''(x) \to f'(x) \to f(x)$ can follow the opposite $x^2 \to x^3 \to x^4$ pattern, can really help us to know ahead of time what our graph will generally look like, since we know the approximate shape of parabolic, cubic, and quadric functions.

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