Chain rule with product rule

We can tell by now that these derivative rules are very often used together. We've seen power rule used together with both product rule and quotient rule, and we've seen chain rule used with power rule.

In this lesson, we want to focus on using chain rule with product rule. But these chain rule/product rule problems are going to require power rule, too.

Let's look at an example of how we might see the chain rule and product rule applied together to differentiate the same function.

Example

Find the derivative of the function.

$$y = 8((6x)(3x^2))^{-4}$$

If we use substitution with $u = (6x)(3x^2)$, then we can rewrite the function as

$$y = 8u^{-4}$$

We'll differentiate using power rule. We also have to apply chain rule, and multiply by the derivative of the inside function.

$$y' = 8(-4)u^{-4-1}u'$$

$$y' = -32u^{-5}u'$$

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We need to plug in for u and u', so let's find u' using product rule.

$$u' = f(x)g'(x) + f'(x)g(x)$$

$$u' = (6x)(6x) + (6)(3x^2)$$

Now we can back-substitute into $y' = -32u^{-5}u'$, and then simplify.

$$y' = -32((6x)(3x^2))^{-5}((6)(3x^2) + (6x)(6x))$$

$$y' = -32(18x^3)^{-5}(18x^2 + 36x^2)$$

$$y' = -32(18x^3)^{-5}(54x^2)$$

We could leave the derivative like this, or we could rewrite it to make all the exponents positive.

$$y' = -\frac{32(54x^2)}{(18x^3)^5}$$

$$y' = -\frac{54x^2}{9^5x^{15}}$$

$$y' = -\frac{2}{2{,}187x^{13}}$$

In this last example, the product was nested inside the power function. Let's do another example, but this time where the power functions are nested inside the product.

Example

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Find the derivative of the function.

$$y = (x^2 + 1)^7(9x^4)$$

We have the product of $(x^2 + 1)^7$ and $9x^4$, so we need to use product rule. Let's start by listing out both of these functions, and their derivatives.

$$f(x) = (x^2 + 1)^7$$

$$f'(x) = 14x(x^2 + 1)^6$$

and

$$g(x) = 9x^4$$

$$g'(x) = 36x^3$$

We only needed power rule to find $g'(x)$, but we had to use power rule with chain rule in order to find $f'(x)$. If we'd found that derivative using substitution, we would have set $u = x^2 + 1$, and then found $u' = 2x + 0$, or $u' = 2x$. So $f'(x)$ was found to be

$$f'(x) = 7(u)^{7-1}u'$$

$$f'(x) = 7(x^2 + 1)^6(2x)$$

$$f'(x) = 14x(x^2 + 1)^6$$

Now we can plug everything we've found into the product rule formula.

$$y' = f(x)g'(x) + f'(x)g(x)$$

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$$y' = ((x^2 + 1)^7)(36x^3) + (14x(x^2 + 1)^6)(9x^4)$$

$$y' = 36x^3(x^2 + 1)^7 + 126x^5(x^2 + 1)^6$$

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