Calculus 1 Final Exam Solutions

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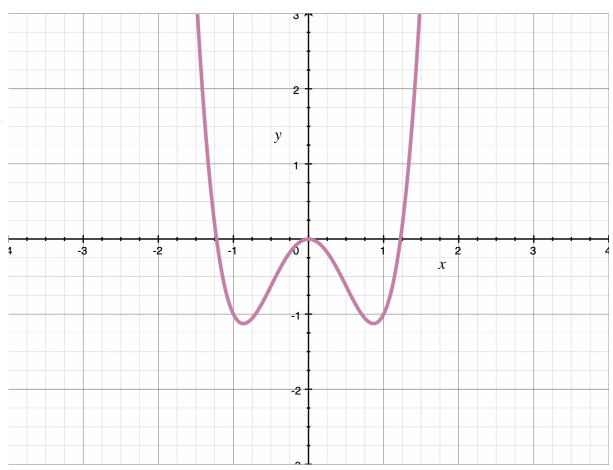
Calculus 1 Final Exam Answer Key

1. (5 pts)	■	B	C	D	E
2. (5 pts)	A	B	C	D	■
3. (5 pts)	A	B	■	D	E
4. (5 pts)	A	B	C	■	E
5. (5 pts)	A	B	■	D	E
6. (5 pts)	■	B	C	D	E
7. (5 pts)	A	B	C	D	■
8. (5 pts)	■	B	C	D	E

9. (15 pts)



10. (15 pts) $36x^2 + 8x$

11. (15 pts) -22.9 ft/s

12. (15 pts) $V = 6\sqrt{6}$ m^3

Calculus 1 Final Exam Solutions

1. A. Use the formula

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

where $y(t)$ is the position of the ball at time t, g is the gravitational constant, v_0 is the initial velocity, and y_0 is the initial position.

In this problem $g = 32$ ft/s^2, $v_0 = 32$ ft/s, and $y_0 = 0$ (the ball is thrown from the ground).

$$y(t) = -\frac{1}{2}(32)t^2 + 32t + 0$$

$$y(t) = -16t^2 + 32t$$

Velocity is the derivative of position so $v(t) = y'(t)$.

$$v(t) = -32t + 32$$

Set velocity equal to 0 to find time t at the ball's maximum height.

$$0 = -32t + 32$$

$$32t = 32$$

$$t = 1$$

Plug $t = 1$ into the equation $y(t)$ to find the position of the ball at its maximum height.

$$y(1) = -16(1)^2 + 32(1)$$

$$y(1) = -16 + 32$$

$$y(1) = 16$$

The ball's maximum height is 16 ft.

2. E. We need to make sure that the piecewise function is continuous through each domain and at each "break" point (the transition between functions).

The piece $x + 3$ is continuous for all of x.

The piecewise function is discontinuous at the transition $x = -3$.

The piece $1/x$ is discontinuous at $x = 0$, which is in the interval $-3 < x < 3$.

The piecewise function is discontinuous at $x = 3$.

The piece $\sqrt{x - 3}$ is continuous for all $x > 3$.

Therefore, the piecewise function is discontinuous at $x = -3, 0, 3$.

3. C. Find the limit at the endpoints by plugging the endpoints into $x^3 - 5x + 3$.

$$f(-5) = (-5)^3 - 5(-5) + 3 = -97$$

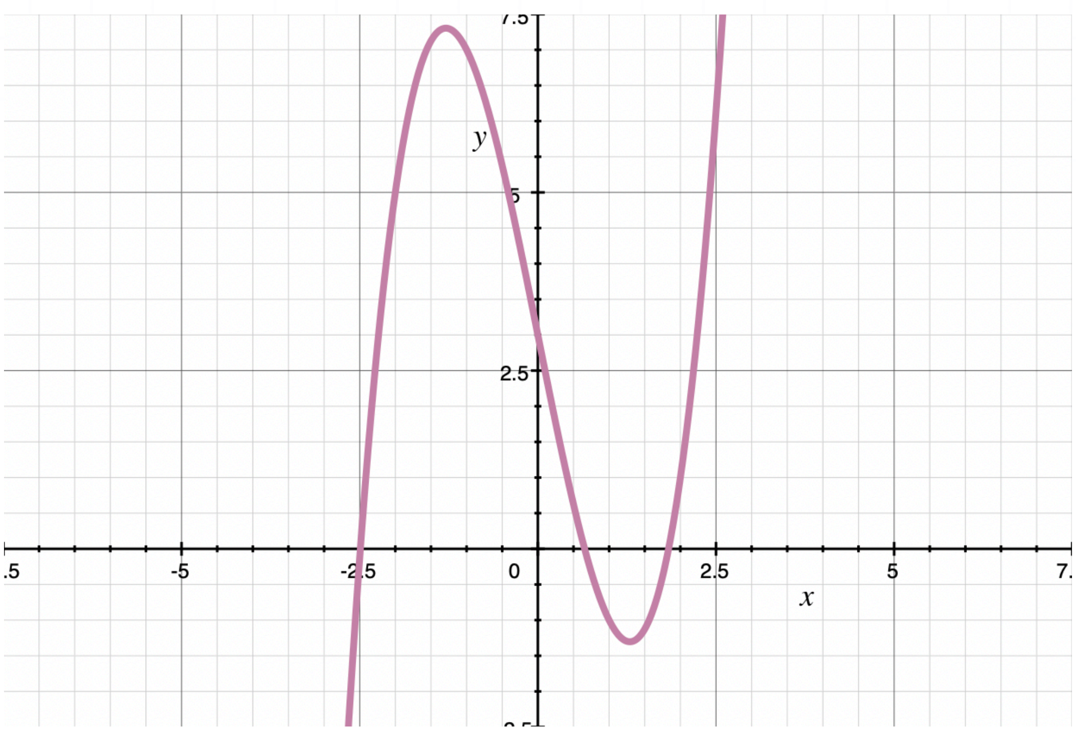
$$f(0) = (0)^3 - 5(0) + 3 = 3$$

$$f(1) = (1)^3 - 5(1) + 3 = -1$$

$$f(5) = 5^3 - 5(5) + 3 = 103$$

Since $f(-5) < 0$ and $f(0) > 0$, the Intermediate Value Theorem states that there must be some x-value in $[-5,0]$ where $f(x) = 0$. This means that there is at least one zero in the interval $[-5,0]$. Since $f(0) > 0$ and $f(1) < 0$, there is at least one zero in the interval $[0,1]$, and since $f(1) < 0$ and $f(5) > 0$, there is at least one zero in the interval $[1,5]$.

The graph of the function shows the zeros in each of these intervals.



4. D. Use substitution with $u = 5x^2 - 3$ and $u' = 10x$ to rewrite the function as $f(x) = 7u^4$. Then the derivative is

$$f'(x) = 28u^3u'$$

Back-substitute.

$$f'(x) = 28(5x^2 - 3)^3(10x)$$

$$f'(x) = 280x(5x^2 - 3)^3$$

5. C. To evaluate the limit,

$$\lim_{x \to 3} \frac{x + 6}{x}$$

plug the value that's being approached into the function, then simplify the answer.

$$\frac{3 + 6}{3}$$

$$\frac{9}{3}$$

$$3$$

6. A. For $f(x) = -6x^3 + 10x^2 + 3$, the second derivative is positive at $x = 0$,

$$f''(x) = -36(0) + 20 = 20 > 0$$

so the function is concave up at that critical point, which means there's a relative minimum there.

The second derivative is negative at $x = 10/9$,

$$f''(x) = -36\left(\frac{10}{9}\right) + 20 = -20 < 0$$

so the function is concave down at that critical point, which means there's a relative maximum there.

7. E. Find the limit.

$$\lim_{x \to 4} g(x)$$

$$\lim_{x \to 4} 2x - 3$$

$$8 - 3$$

$$5$$

Find $f(5)$.

$$f(5) = 5(5)^3$$

$$f(5) = 625$$

$$\lim_{x \to 4} f[g(x)] = 625$$

8. A. Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q = 4{,}000e^{-0.01p}$$

$$\frac{dq}{dt} = -40e^{-0.01p}\frac{dp}{dt}$$

From the question, we know that $p = 120$ and $dq/dt = -24$, so we'll plug those in.

$$-24 = -40e^{-0.01(120)}\frac{dp}{dt}$$

$$-24 = -40e^{-1.20}\frac{dp}{dt}$$

Solve for dp/dt, which is the rate we were asked to find.

$$\frac{dp}{dt} = \frac{-24}{-40e^{-1.20}}$$

$$\frac{dp}{dt} = \frac{3}{5e^{-1.20}}$$

$$\frac{dp}{dt} = \frac{3e^{1.20}}{5}$$

$$\frac{dp}{dt} \approx 1.99$$

9. Find the first derivative.

$$f(x) = 2x^4 - 3x^2$$

$$f'(x) = 8x^3 - 6x$$

Find the second derivative.

$$f''(x) = 24x^2 - 6$$

Find the zeros of the function.

$$2x^4 - 3x^2 = 0$$

$$x^2(2x^2 - 3) = 0$$

$$x = 0, \pm\sqrt{\frac{3}{2}}$$

$$x = 0, \pm\frac{\sqrt{6}}{2}$$

Find the zeros of the first derivative.

$$8x^3 - 6x = 0$$

$$2x(4x^2 - 3) = 0$$

$$x = 0, \ \ \pm\sqrt{\frac{3}{4}}$$

$$x = 0, \ \ \pm\frac{\sqrt{3}}{2}$$

Find the zeros of the second derivative.

$$24x^2 - 6 = 0$$

$$6(4x^2 - 1) = 0$$

$$x = \pm \sqrt{\frac{1}{4}}$$

$$x = \pm \frac{1}{2}$$

Organize the information into a sign chart.

f(x)	decreasing	critical pt	increasing	decreasing	critical pt	increasing
f'(x)	-	0	+	-	0	+
x	x<-√3/2	x=-√3/2	-√3/2<x<0	0<x<√3/2	x=√3/2	√3/2<x

f(x)	concave up	inf. point	concave dn.	inf. point	concave up
f''(x)	+	0	-	0	+
x	x<-1/2	x=-1/2	-1/2<x<1/2	x=1/2	1/2<x

To summarize, we can say

$f(x)$ is decreasing from $(-\infty, -0.866)$ and $(0, 0.866)$, and then increasing from $(-0.866, 0)$ and $(0.866, \infty)$.

$f(x)$ has two local minimums at $(-0.866, -1.125)$ and $(0.866, -1.125)$.

$f(x)$ is concave up from $(-\infty, -0.5)$ and $(0.5, \infty)$ and concave down from $(-0.5, 0.5)$.

$f(x)$ has inflection points at $(-0.5, -0.625)$ and $(0.5, -0.625)$.

10. Find the derivative of $f(x)$.

$$f(x) = 4x^3$$

$$f'(x) = 12x^2$$

Find the derivative of $g(x)$.

$$g(x) = \frac{1}{x} + 3$$

$$g'(x) = -x^{-2}$$

Plug these into the product rule formula.

$$f'(x)g(x) + f(x)g'(x)$$

$$12x^2\left(\frac{1}{x} + 3\right) + 4x^3(-x^{-2})$$

$$12x + 36x^2 - 4x$$

$$36x^2 + 8x$$

11. The ground, the wall, and the ladder form a right triangle, so we'll use the Pythagorean Theorem as the equation that relates the side lengths.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt}+2b\frac{db}{dt}=2c\frac{dc}{dt}$$

From the question, we know that the length of the ladder is $c=15$, and that the length of the ladder doesn't change, so $dc/dt=0$.

$$2a\frac{da}{dt}+2b\frac{db}{dt}=2(15)(0)$$

$$2a\frac{da}{dt}+2b\frac{db}{dt}=0$$

$$a\frac{da}{dt}+b\frac{db}{dt}=0$$

If we say that the vertical wall is side b, and that the horizontal ground is side a, then the question tells us that $b=6$ and that $da/dt=10$.

$$a(10)+6\frac{db}{dt}=0$$

$$10a+6\frac{db}{dt}=0$$

Find the value of a when $b=6$ and $c=15$.

$$a^2+b^2=c^2$$

$$a^2+6^2=15^2$$

$$a^2+36=225$$

$$a^2=189$$

$$a \approx 13.75$$

We're asked to solve for db/dt, so we'll plug in $a \approx 13.75$ and then solve the equation for db/dt.

$$10(13.75) + 6\frac{db}{dt} = 0$$

$$137.5 + 6\frac{db}{dt} = 0$$

$$6\frac{db}{dt} = -137.5$$

$$\frac{db}{dt} = -\frac{137.5}{6}$$

$$\frac{db}{dt} = -22.9$$

The top of the ladder is moving -22.9 ft/s.

12. The volume of a box is always given by $V = lwh$, but since we've been told that the box has a square base, we know that $l = w$, so we can simplify the volume equation to $V = l^2h$.

The surface area of a box is always given by $A = 2lw + 2lh + 2wh$. But since $l = w$, we can simplify this as

$$A = 2ll + 2lh + 2lh$$

$$A = 2l^2 + 4lh$$

We know that total surface area is 36, so

$$36 = 2l^2 + 4lh$$

$$18 = l^2 + 2lh$$

Solve this area equation for height h.

$$18 - l^2 = 2lh$$

$$h = \frac{18 - l^2}{2l}$$

Substitute this value into the volume equation.

$$V = l^2 h$$

$$V = l^2 \left(\frac{18 - l^2}{2l} \right)$$

$$V = \frac{18l - l^3}{2}$$

$$V = 9l - \frac{1}{2}l^3$$

Take the derivative,

$$V' = 9 - \frac{3}{2}l^2$$

then set it equal to 0 to find critical points.

$$9 - \frac{3}{2}l^2 = 0$$

$$\frac{3}{2}l^2 = 9$$

$$l^2 = \frac{18}{3}$$

$$l = \pm\sqrt{6}$$

It's nonsensical to have a negative length, so the only critical point is $l = \sqrt{6}$. Use the first derivative test with test values of $l = 2$ and $l = 3$ to confirm that the critical point represents a maximum. We get

$$V'(2) = 9 - \frac{3}{2}(2)^2$$

$$V'(2) = 9 - \frac{12}{2}$$

$$V'(2) = 3$$

and

$$V'(3) = 9 - \frac{3}{2}(3)^2$$

$$V'(3) = 9 - \frac{27}{2}$$

$$V'(3) = -\frac{9}{2}$$

Since we get a positive value to the left of the critical point and a negative value to the right of it, the function has a maximum at the critical point.

We found the length l associated with the critical point, but we were asked for the maximum volume, so now we just need to find the volume that corresponds with this length, which we'll do by plugging $l = \sqrt{6}$ into $V = 9l - (1/2)l^3$.

$$V = 9\sqrt{6} - \frac{1}{2}(\sqrt{6})^3$$

$$V = 9\sqrt{6} - \frac{6}{2}(\sqrt{6})$$

$$V = 9\sqrt{6} - 3\sqrt{6}$$

$$V = 6\sqrt{6}$$