Chain rule with trig, log, and exponential functions

We've looked at derivatives of trigonometric, exponential, and logarithmic functions, but so far we've always kept the arguments equal to x. In other words, we've found the derivatives of $y = \sin x$, $y = e^x$, and $y = \ln x$.

But we want to be able to differentiate these kinds of functions, even when the argument is something other than x. For instance, $y = \sin(3x)$, $y = e^{-x^2}$, and $y = \ln(x^2 + 4x + 2)^3$.

A new argument

We'll differentiate all of these functions by applying chain rule. In fact, we've been using chain rule all along, even when the argument was simply x, but the effect of chain rule was invisible to us.

For instance, with a basic trig function like $y = \cos x$, the argument is x, so x acts like an "inside function." When we differentiate $y = \cos x$, we have take the derivative of the $\cos$ part to get $-\sin$, but then we have to multiply by the derivative of the inside function. The derivative of the inside function x is 1, so the derivative of $y = \cos x$ actually looks like this:

$$y = \cos x$$

$$y' = (-\sin x)(1)$$

$$y' = -\sin x$$

In this case, applying chain rule and multiplying by 1 doesn't change the value of the derivative, which is what we mean when we say that the effect of chain rule was invisible to us.

But when the argument is anything other than x, the derivative of the argument will be something other than 1, and therefore applying chain rule will of course have an actual effect on the value of the derivative. For instance, the derivative of $y=\sin(2x)$ will be $y'=2\cos(2x)$, and the derivative of $y=\sec(3x^2)$ will be $y'=6x\sec(3x^2)\tan(3x^2)$.

So if we modify our trig derivative rules to account for chain rule, they now look like this:

Trigonometric function	**Its derivative**
$y=\sin(g(x))$	$y'=g'(x)(\cos(g(x)))$
$y=\cos(g(x))$	$y'=g'(x)(-\sin(g(x)))$
$y=\tan(g(x))$	$y'=g'(x)(\sec^2(g(x)))$
$y=\cot(g(x))$	$y'=g'(x)(-\csc^2(g(x)))$
$y=\sec(g(x))$	$y'=g'(x)(\sec(g(x))\tan(g(x)))$
$y=\csc(g(x))$	$y'=g'(x)(-\csc(g(x))\cot(g(x)))$

And just like with trig functions, we need to apply chain rule every time we take the derivative of an exponential or logarithmic function. The "inside function" of a trig function is its argument; the "inside function" of an exponential function is its exponent, and the "inside function" of a log function is its argument.

If we want to show chain rule as part of the exponential derivative formulas, we get

Exponential function	**Its derivative**
$y = e^{g(x)}$	$y' = e^{g(x)}g'(x)$
$y = a^{g(x)}$	$y' = a^{g(x)}(\ln a)g'(x)$

And the logarithmic derivative formulas, after adding in chain rule, become

Log function	**Its derivative**
$y = \log_a g(x)$	$y' = \frac{1}{g(x)\ln a}g'(x)$
$y = \ln g(x)$	$y' = \frac{1}{g(x)}g'(x)$

Let's look at an example of a trig function in which the argument is something other than x.

Example

Use chain rule to find the derivative.

$$y = 8x^5 - 9\cot(7x^4)$$

Dealing with one term at a time, and remembering to use chain rule to handle the derivative of $-9\cot(7x^4)$, we get

$$y' = 8(5)x^{5-1} - 9(-\csc^2(7x^4))(7(4)x^{4-1})$$

$$y' = 40x^4 - 9(-\csc^2(7x^4))(28x^3)$$

$$y' = 40x^4 + 252x^3\csc^2(7x^4)$$

Now let's do one with an exponential function.

Example

Find the derivative of the exponential function.

$$y = 42^{6x}$$

In this function, $a = 42$ and the exponent is $6x$. We'll differentiate by applying the formula for exponential derivatives.

$$y' = a^{g(x)}(\ln a)g'(x)$$

$$y' = 42^{6x}(\ln(42))(6)$$

$$y' = 6(42)^{6x}\ln(42)$$

And let's do one more with a log function, just so that we can see the log derivative formulas in action.

Example

Find the derivative of the logarithmic function.

$$g(x) = \ln\sqrt{x^3 + x}$$

Take the derivative, remembering to apply chain rule.

$$g'(x) = \frac{1}{\sqrt{x^3 + x}} \cdot \frac{1}{2}(x^3 + x)^{-\frac{1}{2}} \cdot (3x^2 + 1)$$

$$g'(x) = \frac{(3x^2 + 1)(x^3 + x)^{-\frac{1}{2}}}{2\sqrt{x^3 + x}}$$

$$g'(x) = \frac{3x^2 + 1}{2\sqrt{x^3 + x}\sqrt{x^3 + x}}$$

$$g'(x) = \frac{3x^2 + 1}{2(x^3 + x)}$$

$$g'(x) = \frac{3x^2 + 1}{2x(x^2 + 1)}$$