Second derivatives with implicit differentiation

We use the same rules for finding the second derivative as we did to find the first derivative. That means that, when we're using implicit differentiation to take the derivative of an implicitly-defined function, we'll use implicit differentiation to find the first derivative, and then we'll apply implicit differentiation again to find the second derivative.

If the second derivative includes any value of y', we can substitute the value of the first derivative into the second derivative.

Let's work through an example so that we can see how to get to the second derivative of an implicitly-defined function.

Example

Use implicit differentiation to find the second derivative.

$$2y^2+6x^2=76$$

This is a function we could solve explicitly for y, but instead of taking those extra steps up front, let's instead use implicit differentiation to find the first and second derivatives. Using implicit differentiation, the first derivative is

$$4yy'+12x=0$$

$$4yy'=-12x$$

Solve for y'.

$$y' = \frac{-12x}{4y}$$

$$y' = -\frac{3x}{y}$$

To find the second derivative, we need to differentiate this first derivative we've just found. Since we have a quotient, we'll use quotient rule. We still need to follow our rules for implicit differentiation, multiplying by y' each time we take the derivative of y. The second derivative will be

$$y'' = -\frac{(3)(y) - (3x)(1)(y')}{(y)^2}$$

$$y'' = -\frac{3y - 3xy'}{y^2}$$

This is the second derivative, but it includes y', which we can substitute for using the equation we found earlier for the first derivative.

$$y'' = -\frac{3y - 3x\left(-\frac{3x}{y}\right)}{y^2}$$

$$y'' = -\frac{3y + \frac{9x^2}{y}}{y^2}$$

Find a common denominator within the numerator, then combine the fractions in the numerator into one fraction.

$$y'' = -\frac{\frac{3y^2}{y} + \frac{9x^2}{y}}{y^2}$$

$$y'' = -\frac{\frac{9x^2 + 3y^2}{y}}{y^2}$$

$$y'' = -\frac{9x^2 + 3y^2}{y}\left(\frac{1}{y^2}\right)$$

$$y'' = -\frac{9x^2 + 3y^2}{y^3}$$

For practice, let's try one more example. This time, we'll use the dy/dx notation for the derivative.

Example

Use implicit differentiation to find the second derivative.

$$xy + 24x = 6y^2$$

Using implicit differentiation (and product rule on the xy term), the first derivative is

$$\left[(1)(y) + (x)(1)\left(\frac{dy}{dx}\right)\right] + 24 = 12y\frac{dy}{dx}$$

$$y + x\frac{dy}{dx} + 24 = 12y\frac{dy}{dx}$$

Solve for dy/dx.

$$x\frac{dy}{dx} - 12y\frac{dy}{dx} = -y - 24$$

$$\frac{dy}{dx}(x - 12y) = -y - 24$$

$$\frac{dy}{dx} = \frac{-y - 24}{x - 12y}$$

To find the second derivative, we need to differentiate this first derivative we've just found. Since we have a quotient, we'll use quotient rule. We still need to follow our rules for implicit differentiation, multiplying by dy/dx each time we take the derivative of y. The second derivative will be

$$\frac{d^2y}{dx^2} = \frac{\left(-1\frac{dy}{dx}\right)(x - 12y) - (-y - 24)\left(1 - 12\frac{dy}{dx}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{dy}{dx}(x - 12y) + (y + 24)\left(1 - 12\frac{dy}{dx}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(-x\frac{dy}{dx} + 12y\frac{dy}{dx}\right) + \left(y - 12y\frac{dy}{dx} + 24 - 288\frac{dy}{dx}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + 12y\frac{dy}{dx} + y - 12y\frac{dy}{dx} + 24 - 288\frac{dy}{dx}}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + y + 24 - 288\frac{dy}{dx}}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y + 24 - (x + 288)\frac{dy}{dx}}{(x - 12y)^2}$$

This is the second derivative, but it includes dy/dx, which we can substitute for using the equation we found earlier for the first derivative.

$$\frac{d^2y}{dx^2} = \frac{y + 24 - (x + 288)\left(\frac{-y - 24}{x - 12y}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y + 24 + (x + 288)\left(\frac{y + 24}{x - 12y}\right)}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{y + 24 + \frac{(y + 24)(x + 288)}{x - 12y}}{(x - 12y)^2}$$

Find a common denominator within the numerator, then combine the fractions in the numerator into one fraction.

$$\frac{d^2y}{dx^2} = \frac{\frac{(y + 24)(x - 12y)}{x - 12y} + \frac{(y + 24)(x + 288)}{x - 12y}}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{(y + 24)(x - 12y) + (y + 24)(x + 288)}{x - 12y}}{(x - 12y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(y + 24)(x - 12y) + (y + 24)(x + 288)}{x - 12y}\left(\frac{1}{(x - 12y)^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{(y + 24)(x - 12y) + (y + 24)(x + 288)}{(x - 12y)^3}$$

We can factor a $y+24$ out to the front of the numerator in order to simplify further.

$$\frac{d^2y}{dx^2}=\frac{(y+24)\left[(x-12y)+(x+288)\right]}{(x-12y)^3}$$

$$\frac{d^2y}{dx^2}=\frac{(y+24)(x-12y+x+288)}{(x-12y)^3}$$

$$\frac{d^2y}{dx^2}=\frac{(y+24)(2x-12y+288)}{(x-12y)^3}$$

$$\frac{d^2y}{dx^2}=\frac{2(y+24)(x-6y+144)}{(x-12y)^3}$$