

# Calculus 1 Workbook Solutions

Chain rule



#### CHAIN RULE WITH POWER RULE

■ 1. Find h'(x) if  $h(x) = (3x^2 - 7)^4$ .

#### Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is  $3x^2 - 7$ , and that the derivative of that inside function is 6x. Therefore, the derivative is

$$h'(x) = 4(3x^2 - 7)^3(6x)$$

$$h'(x) = 24x(3x^2 - 7)^3$$

**2.** Find 
$$h'(x)$$
 if  $h(x) = \sqrt{2 - 4x^2}$ .

#### Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is  $2 - 4x^2$ , and that the derivative of that inside function is -8x. Therefore, the derivative is

$$h'(x) = \frac{1}{2}(2 - 4x^2)^{-\frac{1}{2}}(-8x)$$

$$h'(x) = -4x(2-4x^2)^{-\frac{1}{2}}$$



$$h'(x) = -\frac{4x}{\sqrt{2 - 4x^2}}$$

**3.** Find 
$$h'(x)$$
 if  $h(x) = (2x^2 - 6x + 5)^7$ .

To find the derivative, we have to apply chain rule. We'll say that the inside function is  $2x^2 - 6x + 5$ , and that the derivative of that inside function is 4x - 6. Therefore, the derivative is

$$h'(x) = 7(2x^2 - 6x + 5)^6(4x - 6)$$

$$h'(x) = 7(4x - 6)(2x^2 - 6x + 5)^6$$

**4.** Find 
$$h'(x)$$
 if  $h(x) = 2(x^3 + 4x^2 - 2x)^{-5}$ .

## Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is  $x^3 + 4x^2 - 2x$ , and that the derivative of that inside function is  $3x^2 + 8x - 2$ . Therefore, the derivative is

$$h'(x) = 2(-5)(x^3 + 4x^2 - 2x)^{-6}(3x^2 + 8x - 2)$$



$$h'(x) = -10(3x^2 + 8x - 2)(x^3 + 4x^2 - 2x)^{-6}$$

$$h'(x) = -\frac{10(3x^2 + 8x - 2)}{(x^3 + 4x^2 - 2x)^6}$$

**5.** Find 
$$f'(x)$$
 if  $f(x) = 3(5x^2 + \sin x)^4$ .

To find the derivative, we have to apply chain rule. We'll say that the inside function is  $5x^2 + \sin x$ , and that the derivative of that inside function is  $10x + \cos x$ . Therefore, the derivative is

$$f'(x) = 3(4)(5x^2 + \sin x)^3(10x + \cos x)$$

$$f'(x) = 12(10x + \cos x)(5x^2 + \sin x)^3$$

■ 6. Find 
$$g'(y)$$
 if  $g(y) = \sqrt{3y + (2y + y^2)^2}$ .

## Solution:

To find the derivative, we have to apply chain rule two times. We'll say that the inside function is  $3y + (2y + y^2)^2$ , and that the derivative of that inside function is  $3 + 2(2y + y^2)(2 + 2y)$  or  $3 + (4 + 4y)(2y + y^2)$ . Therefore, the derivative is



$$g'(y) = \frac{1}{2}(3y + (2y + y^2)^2)^{-\frac{1}{2}}(3 + (4 + 4y)(2y + y^2))$$

$$g'(y) = \frac{3 + (4 + 4y)(2y + y^2)}{2\sqrt{3y + (2y + y^2)^2}}$$



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## CHAIN RULE WITH TRIG, LOG, AND EXPONENTIAL FUNCTIONS

■ 1. Find f'(x).

$$f(x) = \ln(x^2 + 6x + 9)$$

#### Solution:

Take the derivative, remembering to apply chain rule.

$$f'(x) = \frac{1}{x^2 + 6x + 9} \cdot (2x + 6)$$

$$f'(x) = \frac{2x+6}{x^2+6x+9}$$

$$f'(x) = \frac{2(x+3)}{(x+3)(x+3)}$$

$$f'(x) = \frac{2}{x+3}$$

**2.** Find g'(x) if  $g(x) = 3\sin(4x^3) - 4\cos(6x) + 3\sec(2x^4)$ .

## Solution:

Differentiate one term at a time, remembering to apply chain rule.

$$g'(x) = 3\cos(4x^3)(12x^2) - 4(-\sin(6x))(6) + 3\sec(2x^4)\tan(2x^4)(8x^3)$$

$$g'(x) = 36x^2\cos(4x^3) + 24\sin(6x) + 24x^3\tan(2x^4)\sec(2x^4)$$

$$g'(x) = 12(2x^3 \tan(2x^4)\sec(2x^4) + 3x^2 \cos(4x^3) + 2\sin(6x))$$

**3.** Find 
$$h'(x)$$
 if  $h(x) = \cos(\sin x + 3x^3)$ .

To find the derivative, we have to apply chain rule. We'll say that the inside function is  $\sin x + 3x^3$ , and that the derivative of that inside function is  $\cos x + 9x^2$ . Therefore, the derivative is

$$h'(x) = -\sin(\sin x + 3x^3)(\cos x + 9x^2)$$

$$h'(x) = -(\cos x + 9x^2)\sin(\sin x + 3x^3)$$

**4.** Find 
$$f'(y)$$
 if  $f(y) = e^{y + \ln y} + 8^{\cos y}$ .

#### Solution:

To find the derivative, we have to apply chain rule two times. In the first term,  $e^{y+\ln y}$ , the inside function is  $y + \ln y$ , and the derivative of that inside

function is 1 + (1/y). In the second term,  $8^{\cos y}$ , the inside function is  $\cos y$ , and the derivative of that inside function is  $-\sin y$ . Therefore, the derivative is

$$f'(y) = e^{y+\ln y} \left(1 + \frac{1}{y}\right) + 8^{\cos y} \ln 8(-\sin y)$$

$$f'(y) = e^{y + \ln y} \left( 1 + \frac{1}{y} \right) - \ln 8 \sin(y) 8^{\cos y}$$

Use properties of exponential functions to simplify.

$$f'(y) = ye^y \left(1 + \frac{1}{y}\right) - \ln 8\sin(y)8^{\cos y}$$

$$f'(y) = ye^{y} \left(\frac{y+1}{y}\right) - \ln 8 \sin(y) 8^{\cos y}$$

$$f'(y) = e^{y}(y+1) - \ln 8 \sin(y) 8^{\cos y}$$

**5.** Find f'(x) if  $f(x) = \tan^5 x + \tan x^5$ .

## Solution:

To find the derivative, we have to apply chain rule two times. In the first term,  $\tan^5 x$ , the inside function is  $\tan x$ , and the derivative of that inside function is  $\sec^2 x$ . In the second term,  $\tan x^5$ , the inside function is  $x^5$ , and the derivative of that inside function is  $5x^4$ . Therefore, the derivative is

$$f'(x) = 5\tan^4 x(\sec^2 x) + \sec^2(x^5)(5x^4)$$



$$f'(x) = 5 \tan^4 x \sec^2 x + 5x^4 \sec^2(x^5)$$

■ 6. Find g'(x) if  $g(x) = \ln(e^{\sin x} - \sin^2 x)$ .

#### Solution:

To find the derivative, we have to apply chain rule three times. We'll say that the inside function is  $e^{\sin x} - \sin^2 x$ , and to find the derivative of that inside function we need to use chain rule for each term.

The derivative of that inside function is  $e^{\sin x}(\cos x) - 2\sin x(\cos x)$ , so the derivative g'(x) is

$$g'(x) = \frac{1}{e^{\sin x} - \sin^2 x} (e^{\sin x} (\cos x) - 2\sin x (\cos x))$$

$$g'(x) = \frac{e^{\sin x}(\cos x) - 2\sin x(\cos x)}{e^{\sin x} - \sin^2 x}$$

$$g'(x) = \frac{\cos x (e^{\sin x} - 2\sin x)}{e^{\sin x} - \sin^2 x}$$



#### CHAIN RULE WITH PRODUCT RULE

■ 1. Find y'(x) if  $y(x) = (3x - 2)(5x^3)^5$ .

#### Solution:

To find the derivative, we have to apply product rule.

$$y'(x) = \frac{d}{dx}(3x - 2) \cdot (5x^3)^5 + (3x - 2) \cdot \frac{d}{dx}(5x^3)^5$$

To find each derivative, we have to apply chain rule.

$$y'(x) = 3 \cdot (5x^3)^5 + (3x - 2) \cdot 5(5x^3)^4 (15x^2)$$

$$y'(x) = 3(5x^3)^5 + 75x^2(3x - 2)(5x^3)^4$$

**2.** Find h'(x) if  $h(x) = (x^2 - 5x)^2 (2x^3 - 3x^2)^5$ .

#### Solution:

To find the derivative, we have to apply product rule.

$$h'(x) = \frac{d}{dx}(x^2 - 5x)^2 \cdot (2x^3 - 3x^2)^5 + (x^2 - 5x)^2 \cdot \frac{d}{dx}(2x^3 - 3x^2)^5$$

To find each derivative, we have to apply chain rule.

$$h'(x) = 2(x^2 - 5x)(2x - 5) \cdot (2x^3 - 3x^2)^5 + (x^2 - 5x)^2 \cdot 5(2x^3 - 3x^2)^4 (6x^2 - 6x)$$

$$h'(x) = 2(2x - 5)(x^2 - 5x)(2x^3 - 3x^2)^5 + 5(6x^2 - 6x)(x^2 - 5x)^2(2x^3 - 3x^2)^4$$

■ 3. Find the derivative of the function.

$$y = (\sin(7x))(7e^{4x})(2x^6 + 1)$$

#### Solution:

The derivative of a function y = f(x)g(x)h(x) using the product rule is

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

To find each derivative, we have to apply chain rule. We'll let

$$f(x) = \sin(7x)$$

$$f'(x) = 7\cos(7x)$$

$$g(x) = 7e^{4x}$$

$$g'(x) = 28e^{4x}$$

$$h(x) = 2x^6 + 1$$

$$h'(x) = 12x^5$$

Then by product rule, the derivative is

$$y' = (7\cos(7x))(7e^{4x})(2x^6 + 1) + (\sin(7x))(28e^{4x})(2x^6 + 1) + (\sin(7x))(7e^{4x})(12x^5)$$

$$y' = 49e^{4x}(2x^6 + 1)\cos(7x) + 28e^{4x}(2x^6 + 1)\sin(7x) + 84x^5e^{4x}\sin(7x)$$

**4.** Find h'(x) if  $h(x) = \sin(4x)e^{3x^2+4}$ .

## Solution:

Use product rule to take the derivative.

$$h'(x) = \frac{d}{dx}\sin(4x) \cdot e^{3x^2+4} + \sin(4x) \cdot \frac{d}{dx}e^{3x^2+4}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \cos(4x)(4) \cdot e^{3x^2+4} + \sin(4x) \cdot e^{3x^2+4}(6x)$$

$$h'(x) = 4e^{3x^2+4}\cos(4x) + 6xe^{3x^2+4}\sin(4x)$$

Factor to simplify.

$$h'(x) = 2e^{3x^2+4}(3x\sin(4x) + 2\cos(4x))$$

■ 5. Find the derivative of the function.

$$y = \sin(x^2 e^{x^2})$$

## Solution:

If we use substitution with  $u = x^2 e^{x^2}$ , then we can rewrite the function as

$$y = \sin u$$

$$y' = \cos(u)u'$$

We need to plug in for u and u', so let's find u' using product rule and chain rule with

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$g(x) = e^{x^2}$$

$$g'(x) = e^{x^2}(2x) = 2xe^{x^2}$$

Then u' is

$$u' = (x^2)(2xe^{x^2}) + (2x)(e^{x^2})$$

$$u' = 2x^3e^{x^2} + 2xe^{x^2}$$

$$u' = 2xe^{x^2}(x^2 + 1)$$

Now we can back-substitute into the equation we found for y', and then simplify.

$$y' = \cos(x^2 e^{x^2}) \cdot 2x e^{x^2} (x^2 + 1)$$

$$y' = 2xe^{x^2}(x^2 + 1)\cos(x^2e^{x^2})$$

**6.** Find 
$$h'(x)$$
 if  $h(x) = \ln(x^3\sqrt{3x^4 - 2x^2 + 3})$ .

If we use substitution with  $u = x^3\sqrt{3x^4 - 2x^2 + 3}$ , then we can rewrite the function and its derivative as

$$y = \ln u$$

$$y' = \frac{1}{u} \cdot u'$$

We need to plug in for u and u', so let's find u' using product rule and chain rule with

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$g(x) = \sqrt{3x^4 - 2x^2 + 3}$$

$$g'(x) = \frac{1}{2}(3x^4 - 2x^2 + 3)^{-\frac{1}{2}}(12x^3 - 4x)$$

$$g'(x) = \frac{6x^3 - 2x}{\sqrt{3x^4 - 2x^2 + 3}}$$

Then

$$u' = (x^3) \left( \frac{6x^3 - 2x}{\sqrt{3x^4 - 2x^2 + 3}} \right) + (3x^2)(\sqrt{3x^4 - 2x^2 + 3})$$



$$u' = \frac{6x^6 - 2x^4}{\sqrt{3x^4 - 2x^2 + 3}} + 3x^2\sqrt{3x^4 - 2x^2 + 3}$$

Now we can back-substitute into the equation we found for y', and then simplify.

$$y' = \frac{1}{x^3 \sqrt{3x^4 - 2x^2 + 3}} \left( \frac{6x^6 - 2x^4}{\sqrt{3x^4 - 2x^2 + 3}} + 3x^2 \sqrt{3x^4 - 2x^2 + 3} \right)$$

$$y' = \frac{6x^3 - 2x}{3x^4 - 2x^2 + 3} + \frac{3}{x}$$



#### CHAIN RULE WITH QUOTIENT RULE

■ 1. Find h'(x).

$$h(x) = \frac{(2x+1)^3}{(3x-2)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(2x+1)^3 \cdot (3x-2)^2 - (2x+1)^3 \cdot \frac{d}{dx}(3x-2)^2}{((3x-2)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{3(2x+1)^2(2) \cdot (3x-2)^2 - (2x+1)^3 \cdot 2(3x-2)(3)}{((3x-2)^2)^2}$$

$$h'(x) = \frac{6(2x+1)^2(3x-2)^2 - 6(2x+1)^3(3x-2)}{(3x-2)^4}$$

$$h'(x) = \frac{6(2x+1)^2(3x-2) - 6(2x+1)^3}{(3x-2)^3}$$

Factor the numerator.

$$h'(x) = \frac{6(2x+1)^2(3x-2-(2x+1))}{(3x-2)^3}$$



$$h'(x) = \frac{6(2x+1)^2(3x-2-2x-1)}{(3x-2)^3}$$

$$h'(x) = \frac{6(2x+1)^2(x-3)}{(3x-2)^3}$$

 $\blacksquare$  2. Find h'(x).

$$h(x) = \frac{(4x+5)^5}{(x+3)^2}$$

### Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(4x+5)^5 \cdot (x+3)^2 - (4x+5)^5 \cdot \frac{d}{dx}(x+3)^2}{((x+3)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{5(4x+5)^4(4) \cdot (x+3)^2 - (4x+5)^5 \cdot 2(x+3)(1)}{(x+3)^4}$$

$$h'(x) = \frac{20(4x+5)^4(x+3)^2 - 2(4x+5)^5(x+3)}{(x+3)^4}$$

$$h'(x) = \frac{20(4x+5)^4(x+3) - 2(4x+5)^5}{(x+3)^3}$$



Factor the numerator.

$$h'(x) = \frac{2(4x+5)^4(10(x+3) - (4x+5))}{(x+3)^3}$$

$$h'(x) = \frac{2(4x+5)^4(10x+30-4x-5)}{(x+3)^3}$$

$$h'(x) = \frac{2(4x+5)^4(6x+25)}{(x+3)^3}$$

## $\blacksquare$ 3. Find h'(x).

$$h(x) = \ln\left(\frac{x^3}{x^2 + 3}\right)$$

### Solution:

If we use substitution with

$$u = \frac{x^3}{x^2 + 3}$$

then we can rewrite the function and take its derivative.

$$y = \ln u$$

$$y' = \frac{1}{u} \cdot u'$$



We need to plug in for u and u', so let's find u' using quotient rule, where f(x) is the numerator and g(x) is the denominator.

$$u' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$u' = \frac{(3x^2)(x^2 + 3) - (x^3)(2x)}{(x^2 + 3)^2}$$

Now we can back-substitute into the equation we found for y'.

$$h'(x) = \frac{1}{\frac{x^3}{x^2 + 3}} \cdot \frac{3x^2(x^2 + 3) - x^3(2x)}{(x^2 + 3)^2}$$

$$h'(x) = \frac{x^2 + 3}{x^3} \cdot \frac{3x^2(x^2 + 3) - x^3(2x)}{(x^2 + 3)^2}$$

$$h'(x) = \frac{1}{x} \cdot \frac{3(x^2 + 3) - x(2x)}{x^2 + 3}$$

$$h'(x) = \frac{3x^2 + 9 - 2x^2}{x(x^2 + 3)}$$

$$h'(x) = \frac{x^2 + 9}{x(x^2 + 3)}$$

## $\blacksquare$ 4. Find h'(x).

$$h(x) = \frac{\sec(2-x)}{2x + e^{-x}}$$



To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}\sec(2-x)\cdot(2x+e^{-x}) - \sec(2-x)\cdot\frac{d}{dx}(2x+e^{-x})}{(2x+e^{-x})^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{\sec(2-x)\tan(2-x)(-1)\cdot(2x+e^{-x}) - \sec(2-x)\cdot(2+e^{-x}(-1))}{(2x+e^{-x})^2}$$

$$h'(x) = \frac{-(2x + e^{-x})\sec(2 - x)\tan(2 - x) - (2 - e^{-x})\sec(2 - x)}{(2x + e^{-x})^2}$$

$$h'(x) = -\frac{(2x + e^{-x})\sec(2 - x)\tan(2 - x) + (2 - e^{-x})\sec(2 - x)}{(2x + e^{-x})^2}$$

## ■ 5. Find h'(x).

$$h(x) = \frac{2 + \ln(3x)}{x + \cot(2x)}$$

#### Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(2 + \ln(3x)) \cdot (x + \cot(2x)) - (2 + \ln(3x)) \cdot \frac{d}{dx}((x + \cot(2x)))}{(x + \cot(2x))^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{\frac{1}{3x}(3) \cdot (x + \cot(2x)) - (2 + \ln(3x)) \cdot (1 - \csc^2(2x)(2))}{(x + \cot(2x))^2}$$

$$h'(x) = \frac{\frac{x + \cot(2x)}{x} - (2 + \ln(3x))(1 - 2\csc^2(2x))}{(x + \cot(2x))^2}$$

$$h'(x) = \frac{\frac{x + \cot(2x)}{x}}{(x + \cot(2x))^2} - \frac{(2 + \ln(3x))(1 - 2\csc^2(2x))}{(x + \cot(2x))^2}$$

$$h'(x) = \frac{1}{x(x + \cot(2x))} - \frac{(2 + \ln(3x))(1 - 2\csc^2(2x))}{(x + \cot(2x))^2}$$

## $\blacksquare$ 6. Find h'(x).

$$h(x) = x^2 \sin\left(\frac{x^3 + 4x}{\sqrt{x^4 - 2}}\right)$$

#### Solution:

Use product rule with h'(x) = f(x)g'(x) + f'(x)g(x) and let

$$f(x) = x^2$$



$$f'(x) = 2x$$

and

$$g(x) = \sin\left(\frac{x^3 + 4x}{\sqrt{x^4 - 2}}\right)$$

To find the derivative g'(x), we have to apply chain rule. If we use substitution with

$$u = \frac{x^3 + 4x}{\sqrt{x^4 - 2}}$$

then we can rewrite g(x) and its derivative as

$$g(u) = \sin u$$

$$g'(u) = \cos u \cdot u'$$

We need to plug in for u and u', so let's find u' using quotient rule.

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2}) - (x^3 + 4x)\left(\frac{1}{2\sqrt{x^4 - 2}}\right)(4x^3)}{(\sqrt{x^4 - 2})^2}$$

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2}) - 2x^3(x^3 + 4x)\left(\frac{1}{\sqrt{x^4 - 2}}\right)}{x^4 - 2}$$

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2})}{x^4 - 2} - \frac{2x^3(x^3 + 4x)\left(\frac{1}{\sqrt{x^4 - 2}}\right)}{x^4 - 2}$$



$$u' = \frac{3x^2 + 4}{\sqrt{x^4 - 2}} - \frac{2x^3(x^3 + 4x)}{\sqrt{(x^4 - 2)^3}}$$

Now we can back-substitute into  $g'(u) = \cos u \cdot u'$ , and then simplify.

$$g'(x) = \cos\left(\frac{x^3 + 4x}{\sqrt{x^4 - 2}}\right) \cdot \left(\frac{3x^2 + 4}{\sqrt{x^4 - 2}} - \frac{2x^3(x^3 + 4x)}{\sqrt{(x^4 - 2)^3}}\right)$$

Now plug into the product rule formula.

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = x^2 \cos\left(\frac{x^3 + 4x}{\sqrt{x^4 - 2}}\right) \left(\frac{3x^2 + 4}{\sqrt{x^4 - 2}} - \frac{2x^3(x^3 + 4x)}{\sqrt{(x^4 - 2)^3}}\right) + 2x \sin\left(\frac{x^3 + 4x}{\sqrt{x^4 - 2}}\right)$$





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