Calculus 1 Workbook Solutions

Continuity

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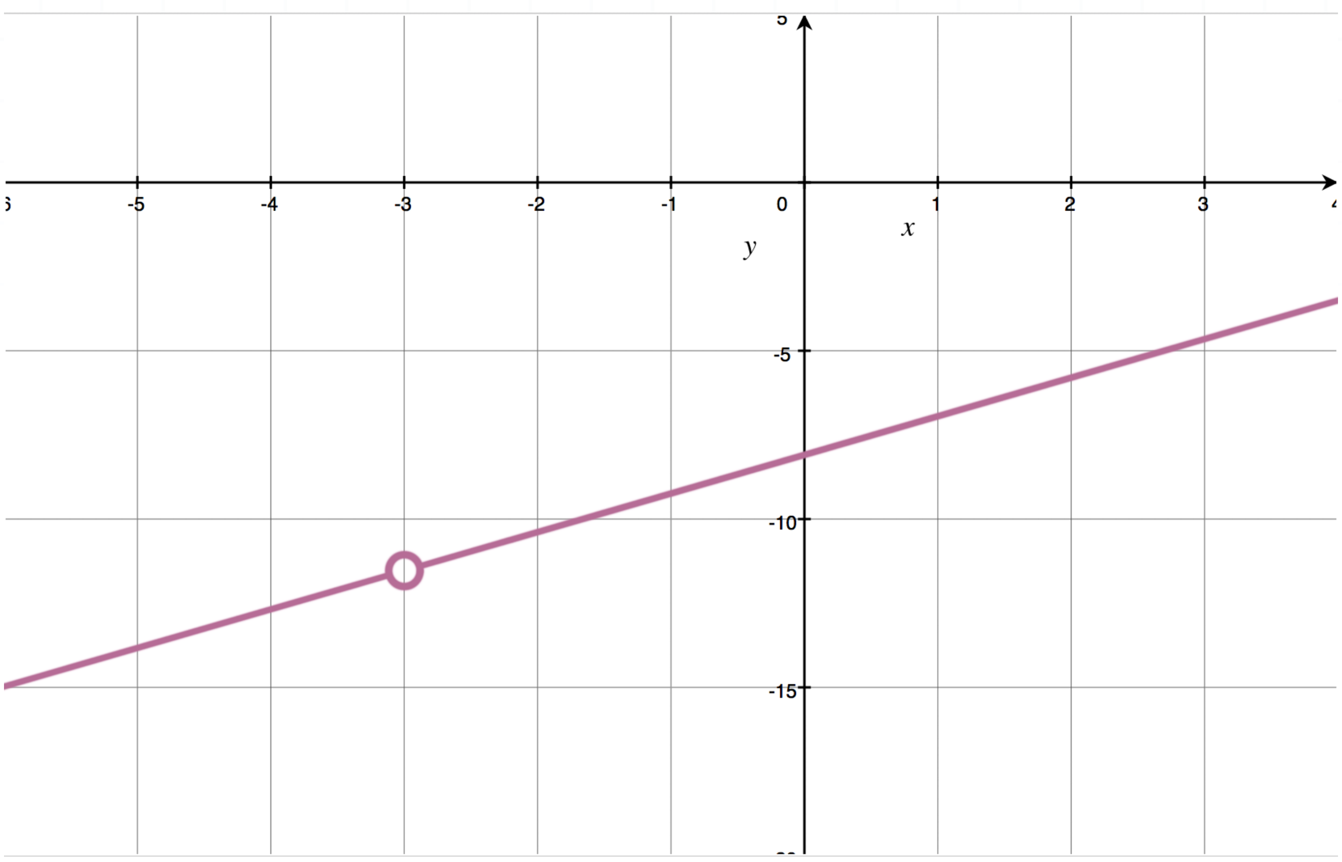
POINT DISCONTINUITIES

■ 1. Redefine the function as a continuous piecewise function.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

Solution:

The function is discontinuous at $x = -3$.



Factor and reduce to remove the discontinuity.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

$$f(x) = \frac{(x+3)(x-9)}{x+3}$$

$$f(x) = x - 9$$

Evaluate $f(x)$ at $x = -3$.

$$f(-3) = -3 - 9 = -12$$

Therefore, to make the function continuous, we have to redefine it as

$$f(x) = \begin{cases} \frac{x^2 - 6x - 27}{x+3} & x \neq -3 \\ -12 & x = -3 \end{cases}$$

We can see whether or not this function is continuous at $x = -3$ by looking at the limit as x approaches -3.

$$\lim_{x \to -3} \frac{x^2 - 6x - 27}{x+3} = \lim_{x \to -3} \frac{(x+3)(x-9)}{x+3}) = -12$$

Since -12 is also the value of the function at $x = -3$, we see that this function is continuous.

■ 2. Identify the non-removable discontinuities of the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

Solution:

Factor the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

$$k(x) = \frac{(x+5)(x-5)(x+3)}{(x+4)(x-3)}$$

No factors can be canceled. Which means the function has discontinuities at $x = -4$ and $x = 3$, both of which are non-removable.

■ 3. What is the set of removable discontinuities of the function?

$$j(\theta) = \frac{\cos^2\theta \cdot \sin^2\theta}{\tan^2\theta}$$

Solution:

We can rewrite the function as

$$j(\theta) = \frac{\cos^2\theta \cdot \sin^2\theta}{\tan^2\theta} = \frac{\cos^2\theta \cdot \sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta \cdot \sin^2\theta \cdot \cos^2\theta}{\sin^2\theta} = \cos^4\theta$$

The removable discontinuities are the values of θ that make the sine function equal to 0, which are all the multiples of π,

$$\theta = \pm 0, \ \pm \pi, \ \pm 2\pi, \ \pm 3\pi, \ \pm 4\pi, \ \dots$$

$\theta = n\pi$, where n is the set of all integers as well as the values that make the cosine function equal to 0, but where the original function itself has a finite limit, which are all the $(2n+1)/2$ multiples of π.

$$\theta = \pm\frac{1}{2}\pi, \ \pm\frac{3}{2}\pi, \ \pm\frac{5}{2}\pi, \ldots$$

$$\theta = \frac{2n+1}{2}\pi, \text{ where } n \text{ is the set of all integers}$$

Combining these two sets gives

$$\theta = \pm 0, \ \pm\frac{1}{2}\pi, \ \pm\pi, \ \pm\frac{3}{2}\pi, \ \pm 2\pi, \ \pm\frac{5}{2}\pi, \ \pm 3\pi, \ \ldots$$

$$\theta = \frac{n\pi}{2}, \text{ where } n \text{ is the set of all integers}$$

■ 4. Examine whether or not the function is continuous at $x = 0$.

$$g(x) = \begin{cases} 2 - x^2 & x \le 0 \\ x - 2 & x > 0 \end{cases}$$

Solution:

We can say that $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and $\lim_{x\to a} f(x) = f(a)$.

Evaluate $f(x)$ at $x = 0$.

$$f(0) = 2 - 0^2 = 2$$

The right-hand limit at $x = 0$ is

$$\lim_{x \to 0+} (x - 2) = 0 - 2 = -2$$

The right-hand limit at $x = 0$ isn't equivalent to the function's value at $x = 0$, so the function is not continuous there.

5. Where is the removable discontinuity in the graph of the function?

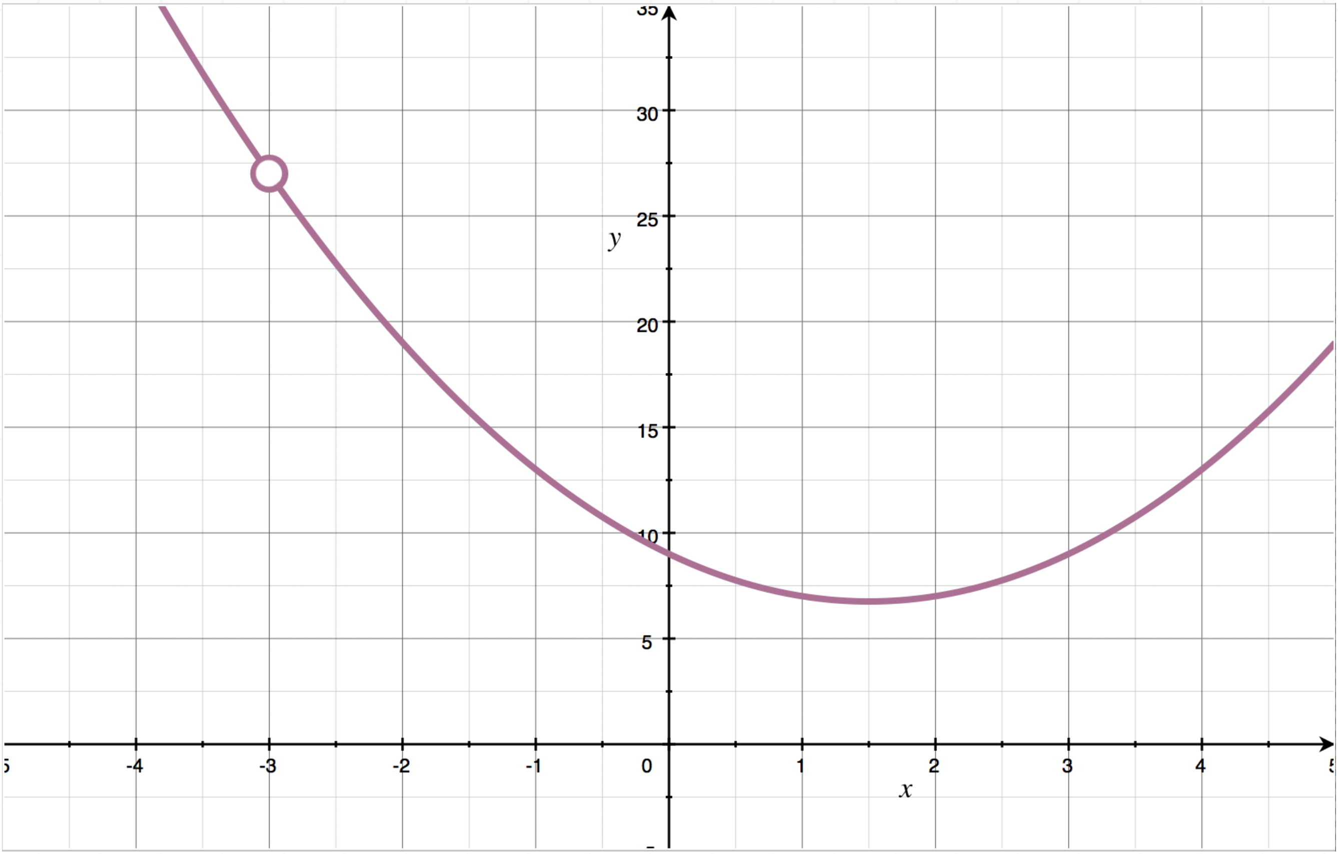
$$f(x) = \frac{x^3 + 27}{x + 3}$$

Solution:

If we factor the function, we can cancel a factor of $x + 3$. We could have also simplified the function by using polynomial long division to find the quotient.

$$f(x) = \frac{x^3 + 27}{x + 3} = \frac{(x + 3)(x^2 - 3x + 9)}{x + 3} = x^2 - 3x + 9$$

Because the factor of $x + 3$ cancels, the removable discontinuity is at $x + 3 = 0$, or $x = -3$.



■ 6. Identify the removable discontinuities in the function.

$$k(x) = \frac{x^4 - 2x^3 - 16x^2 + 2x + 15}{x^2 - 2x - 15}$$

Solution:

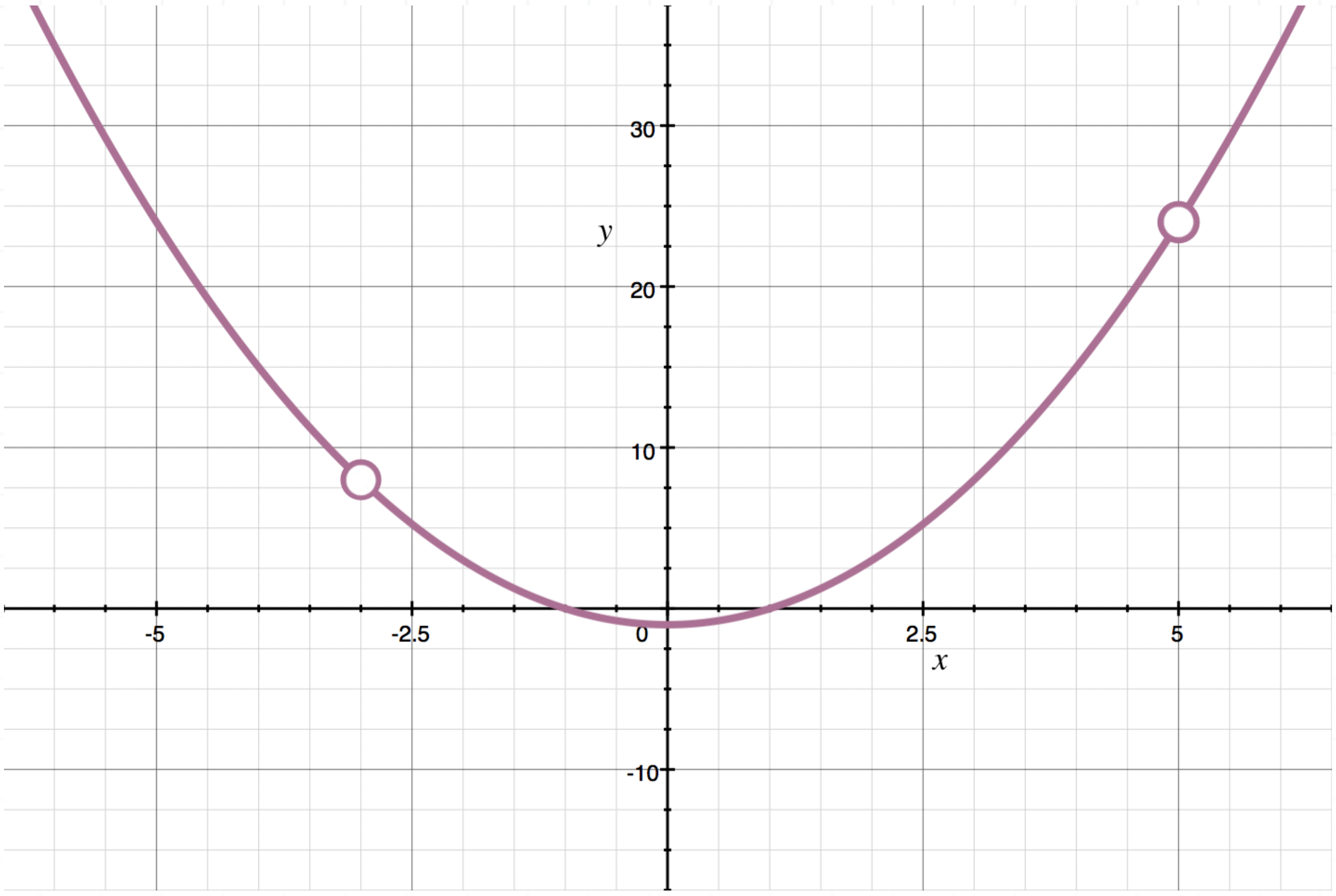
The function $k(x)$ has removable discontinuities at $x = -3$ and $x = 5$ because the function factors as

$$k(x) = \frac{(x+3)(x-5)(x+1)(x-1)}{(x+3)(x-5)}$$

and both factors from the denominator can be cancelled.

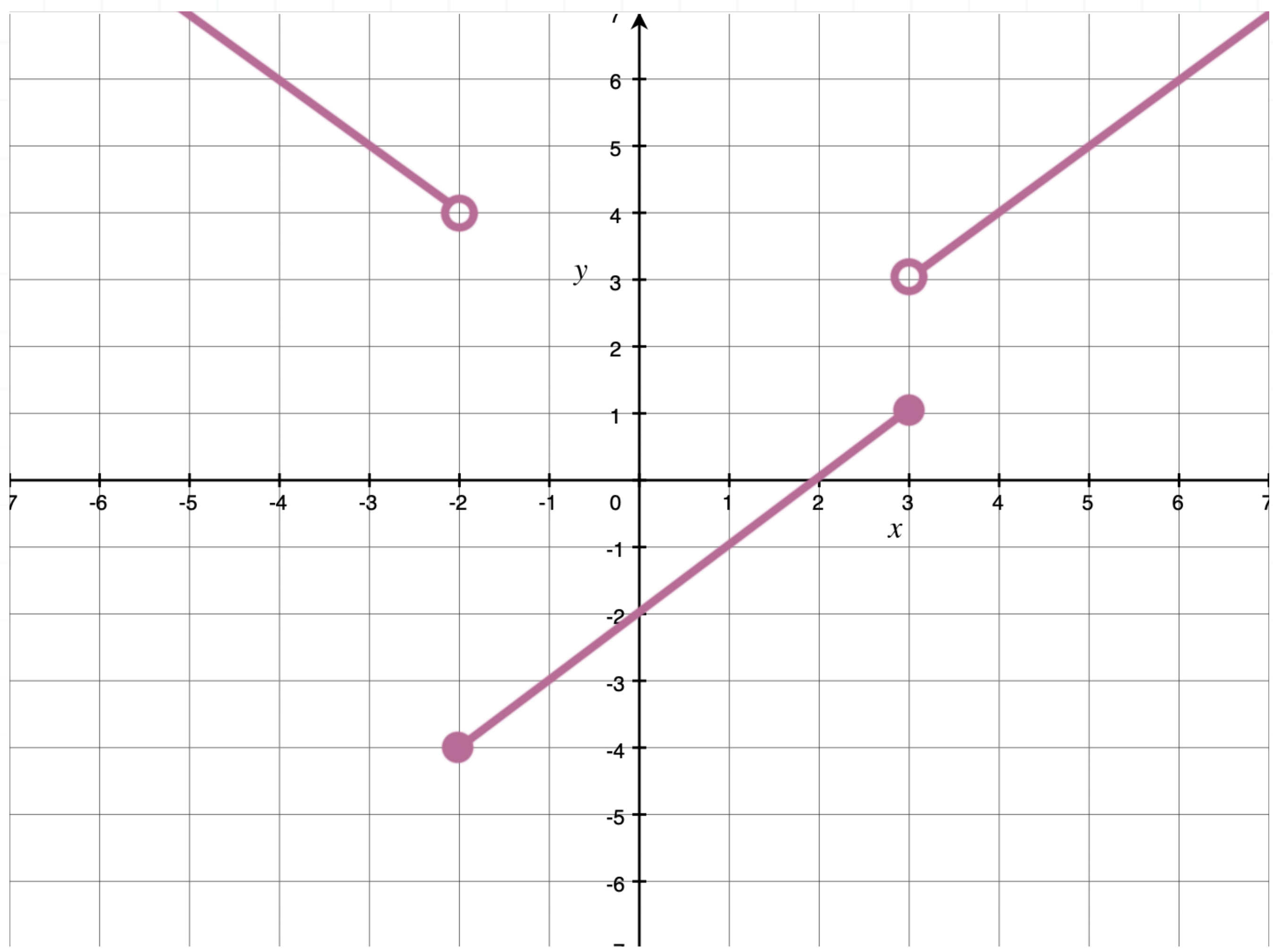
$$k(x) = (x+1)(x-1)$$

The graph is shown below.



JUMP DISCONTINUITIES

■ 1. What are the x-values where the graph of $f(x)$, shown below, has jump discontinuities?



Solution:

The function $f(x)$ has jump discontinuities at $x = -2$ and $x = 3$ because the left- and right-hand limits aren't equal at $x = -2$,

$$\lim_{x\to -2^-} f(x) = 4 \quad \neq \quad \lim_{x\to -2^+} f(x) = -4$$

and they aren't equal at $x = 3$.

$$\lim_{x \to 3^-} f(x) = 1 \quad \neq \quad \lim_{x \to 3^+} f(x) = 3$$

■ 2. Where are the jump discontinuities in the graph of the function?

$$h(x) = \begin{cases} -\frac{1}{3}x^2 + 2 & x < 0 \\ 3 & 0 \le x \le 1 \\ \frac{1}{3}x^2 + 4 & x > 1 \end{cases}$$

Solution:

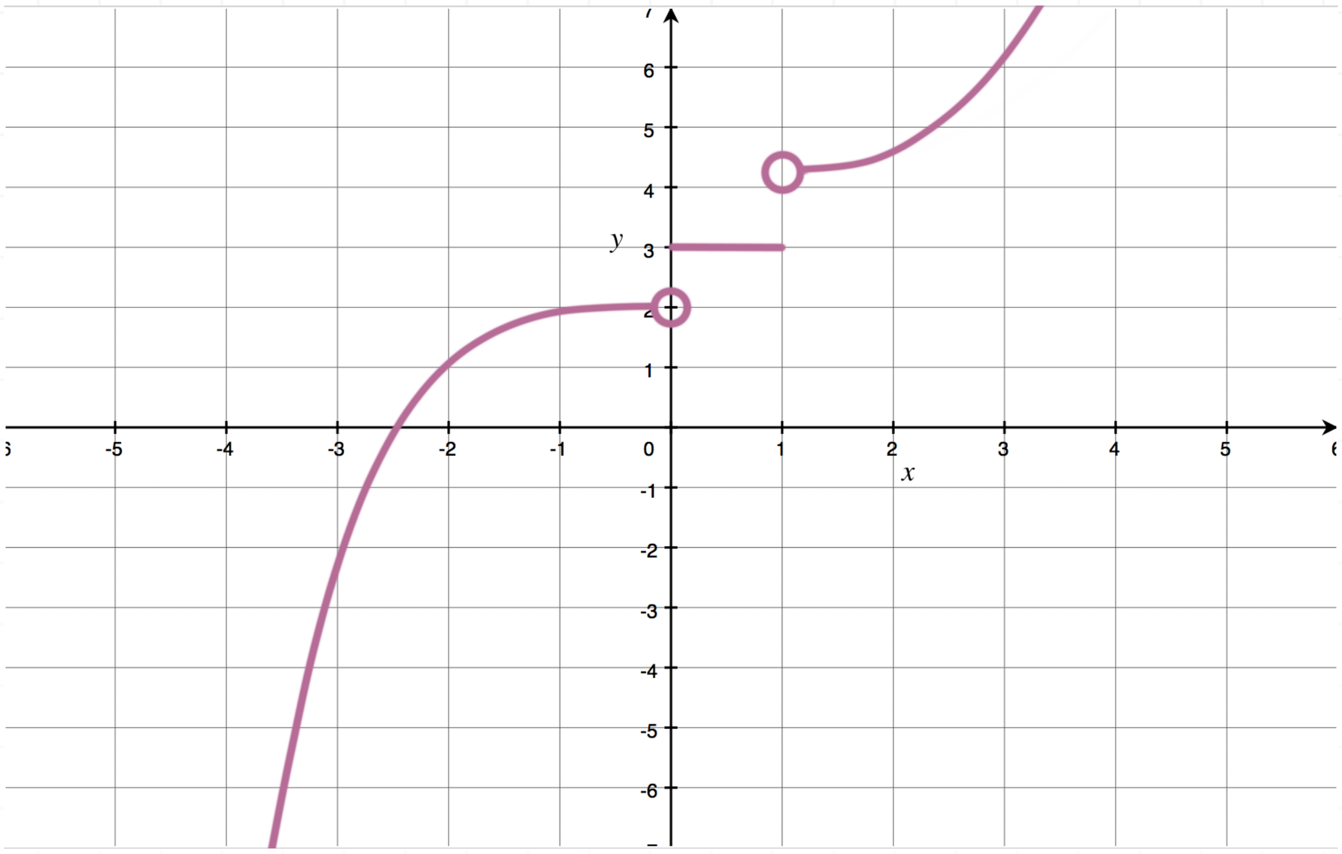
The function $h(x)$ has jump discontinuities at $x = 0$ and $x = 1$ because the left- and right-hand limits aren't equal at $x = 0$,

$$\lim_{x \to 0^-} f(x) = 2 \quad \neq \quad \lim_{x \to 0^+} f(x) = 3$$

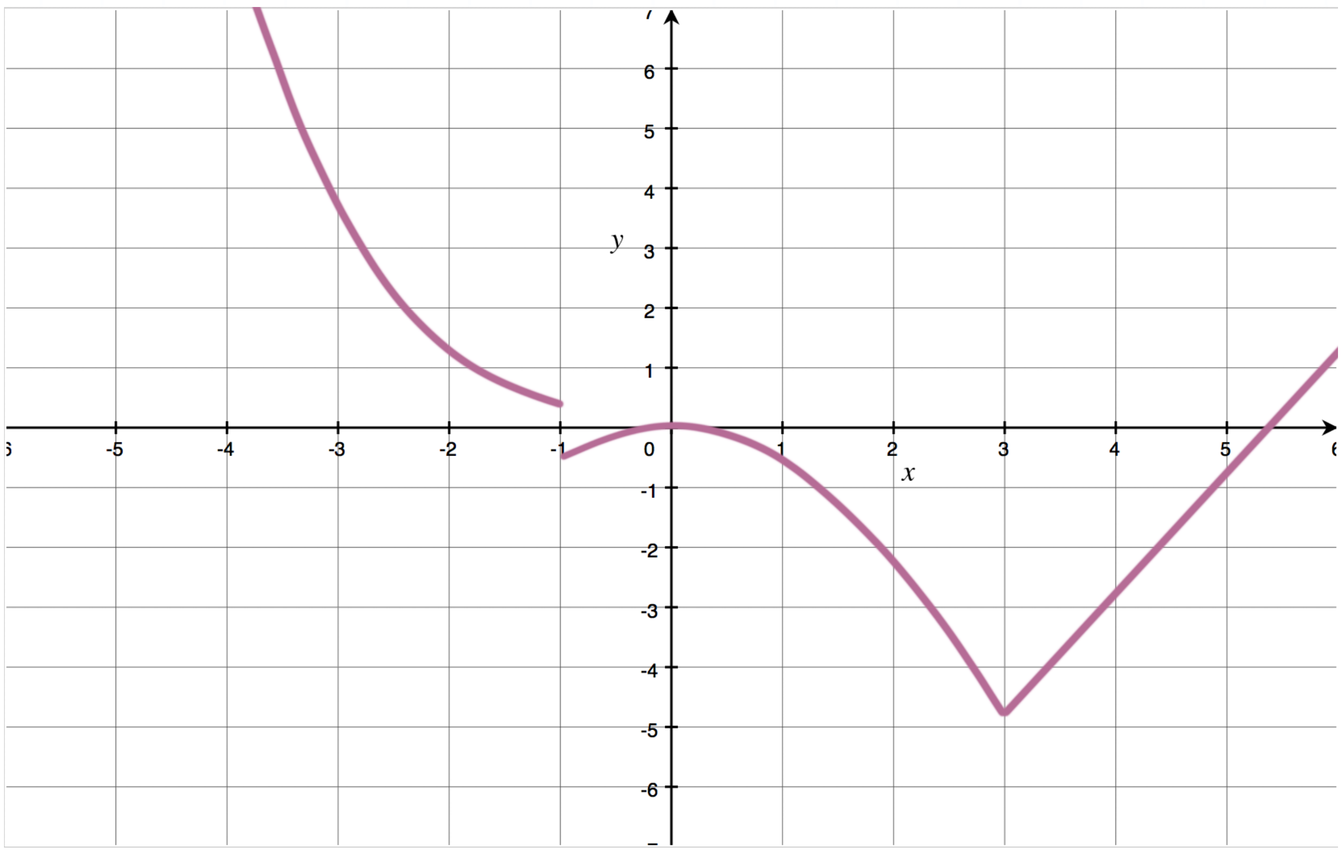
or at $x = 1$.

$$\lim_{x \to 1^-} f(x) = 3 \quad \neq \quad \lim_{x \to 1^+} f(x) = \frac{13}{3}$$

We can see the discontinuities in the function's graph, as well.



■ 3. What are the x-values where the graph of $g(x)$ has jump discontinuities?

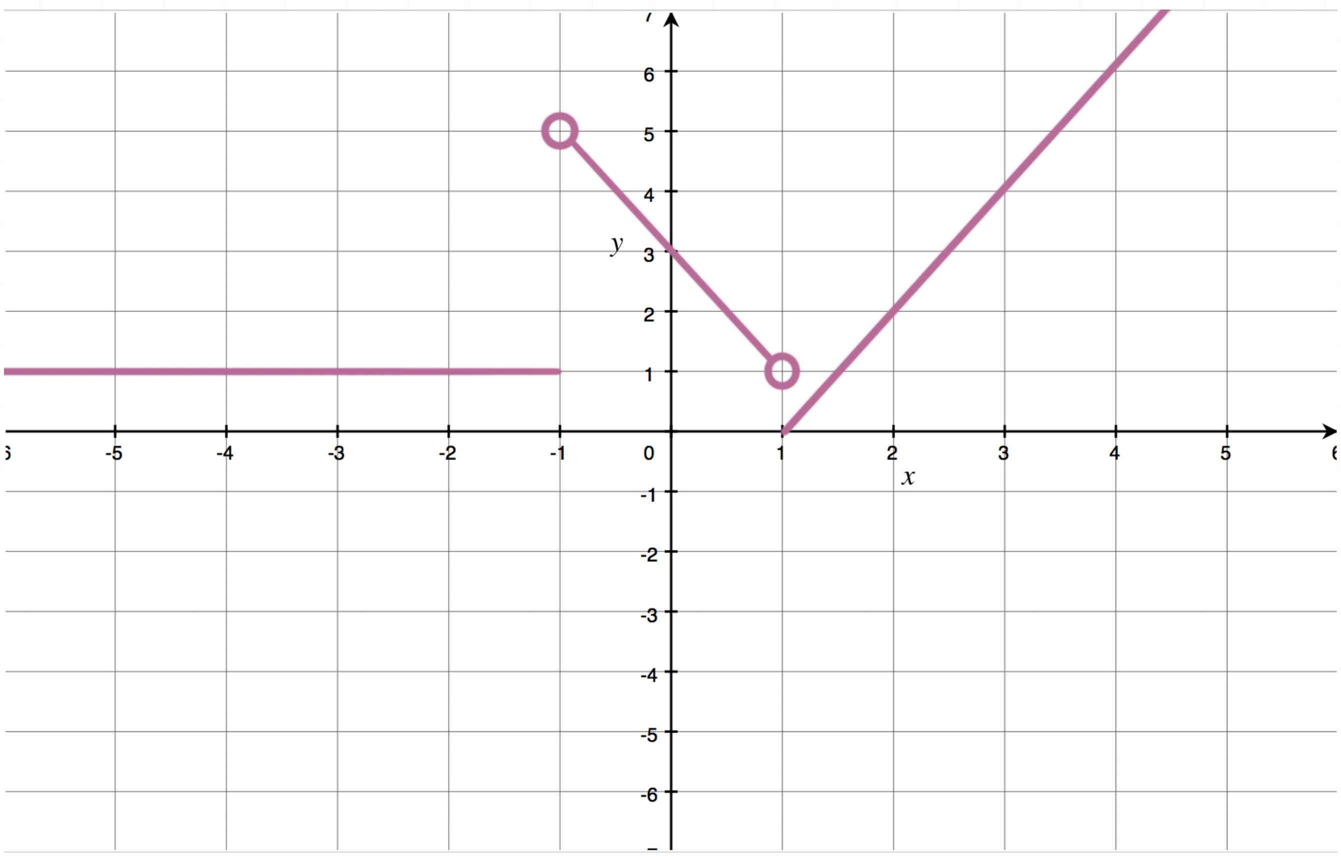


Solution:

The function $g(x)$ has a jump discontinuity at $x=-1$ because the left- and right-hand limits aren't equal there.

$$\lim_{x\to-1^-} f(x)=\frac{1}{3} \quad \neq \quad \lim_{x\to-1^+} f(x)=-\frac{1}{3}$$

■ 4. Show that $f(x)$ has jump discontinuity at $x=-1$ and $x=1$.



Solution:

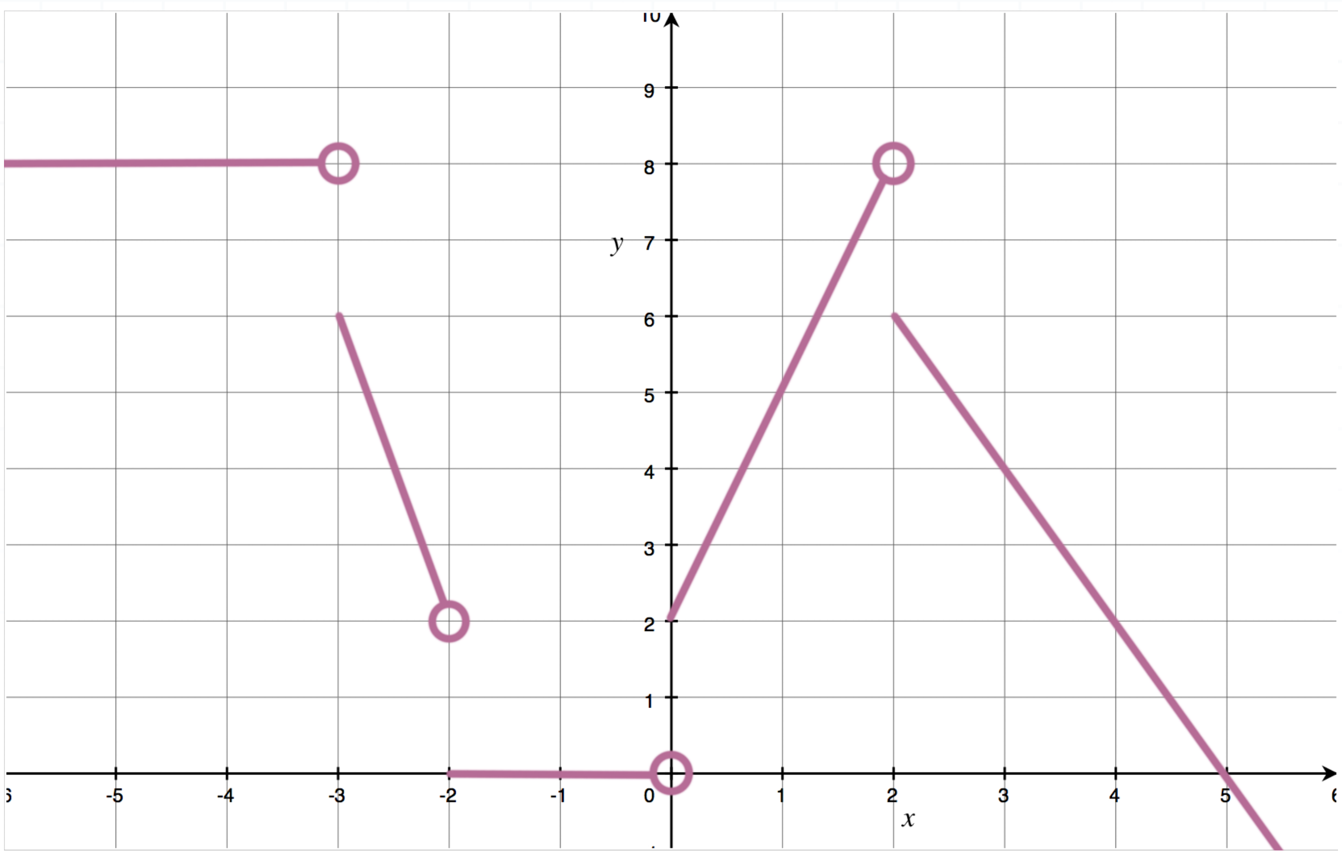
The function $f(x)$ has jump discontinuities at $x=-1$ and $x=1$ because the left- and right-hand limits aren't equal at $x=-1$

$$\lim_{x\to-1^-} f(x)=1 \quad \neq \quad \lim_{x\to-1^+} f(x)=5$$

or at $x = 1$.

$$\lim_{x \to 1^-} f(x) = 1 \quad \neq \quad \lim_{x \to 1^+} f(x) = 0$$

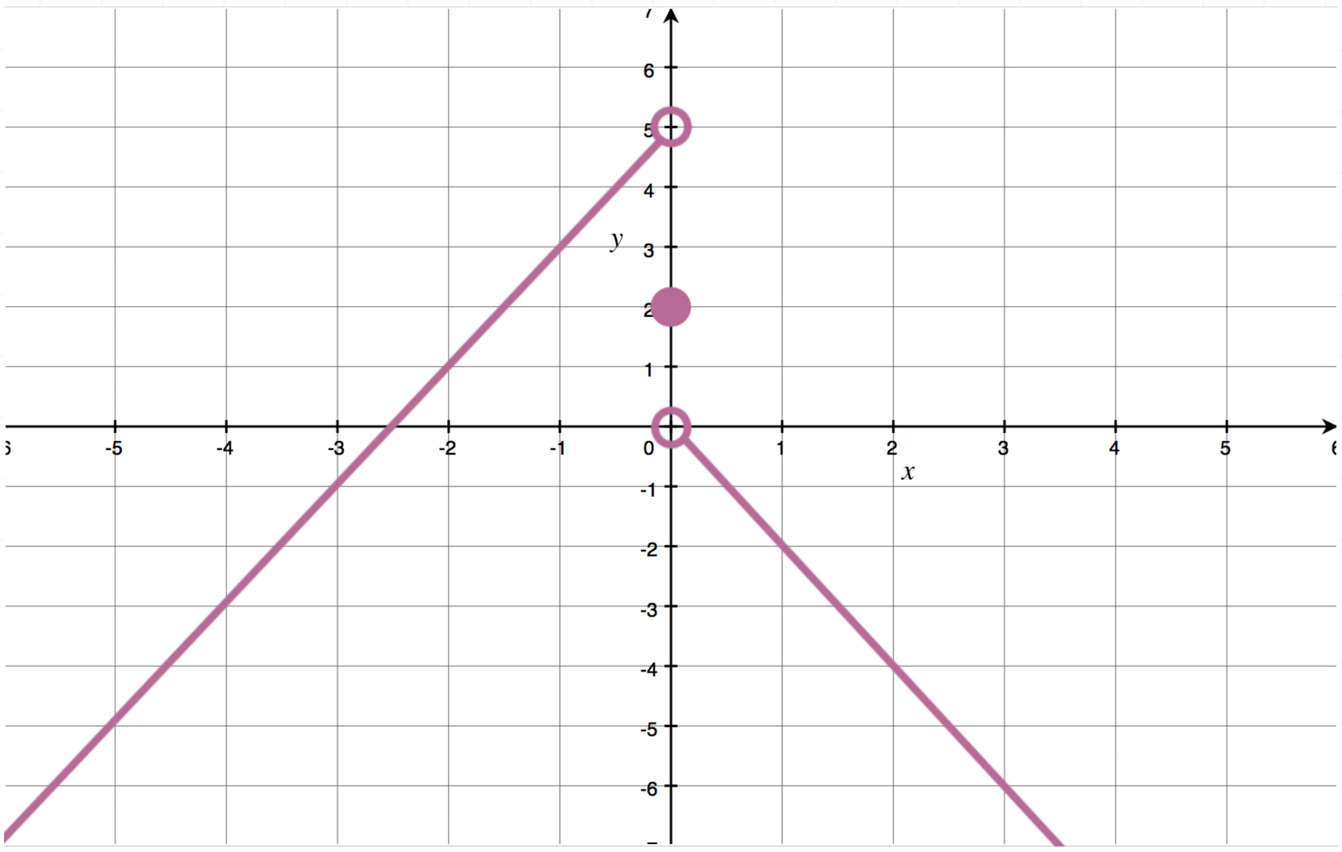
■ 5. Where are the jump discontinuities in the graph of the function shown below?



Solution:

The function has jump discontinuities at $x = -3$, $x = -2$, $x = 0$, and $x = 2$, because at each x-value, the left- and right-hand limits aren't equal.

■ 6. What are the x-values where the graph of $h(x)$, shown below, has jump discontinuities?



Solution:

The function $h(x)$ has a jump discontinuity at $x=0$ because the left- and right-hand limits aren't equal there.

$$\lim_{x\to 0^-} f(x) = 5 \quad \neq \quad \lim_{x\to 0^+} f(x) = 0$$

INFINITE DISCONTINUITIES

1. At what x-values does the function have infinite discontinuities?

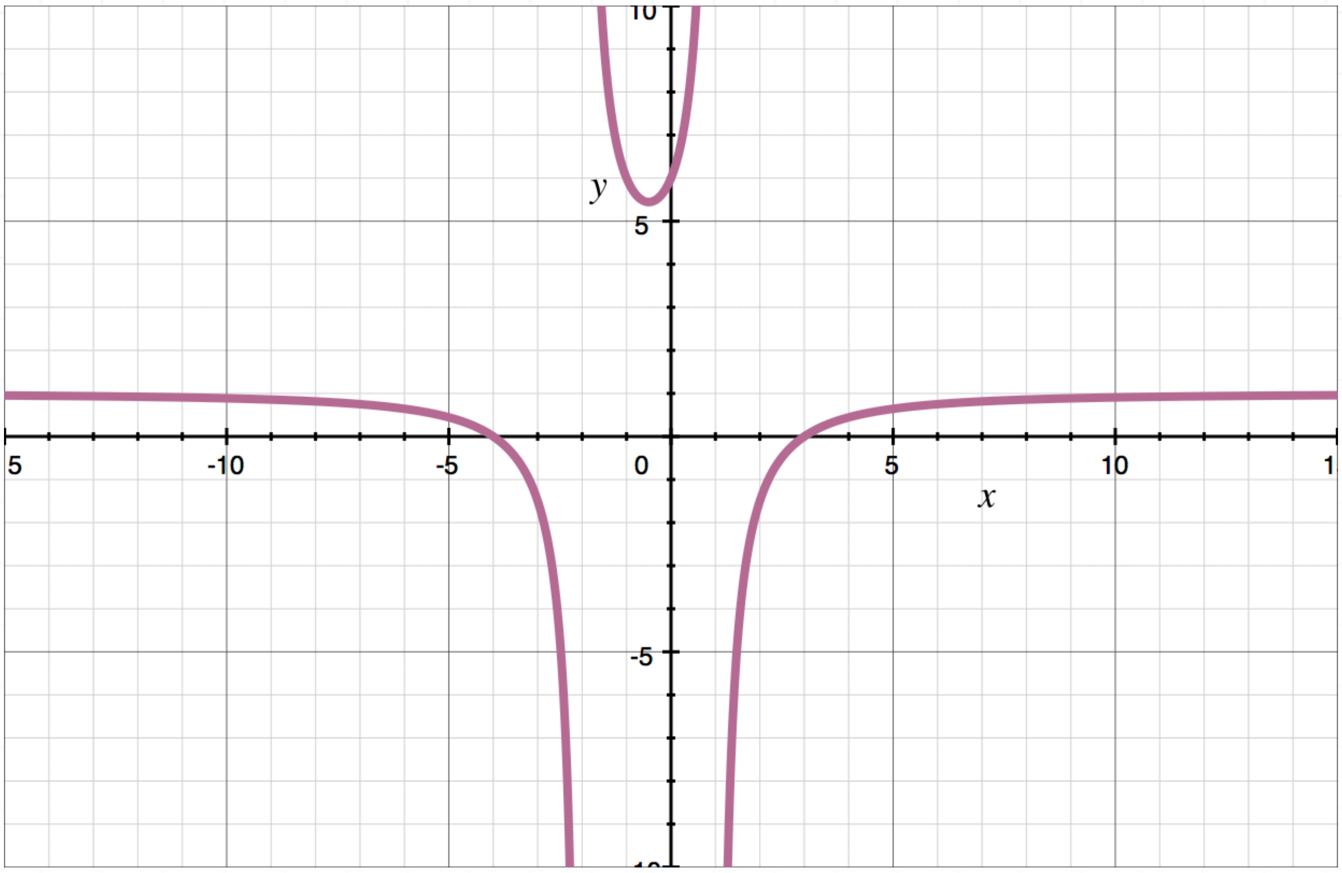
$$f(x) = \frac{x^2 + x - 12}{x^2 + x - 2}$$

Solution:

Factor the function.

$$f(x) = \frac{x^2 + x - 12}{x^2 + x - 2} = \frac{(x + 4)(x - 3)}{(x + 2)(x - 1)}$$

None of these factors cancel, which means that $x + 2 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -2$ and $x = 1$.



■ 2. Where are the infinite discontinuities of the function?

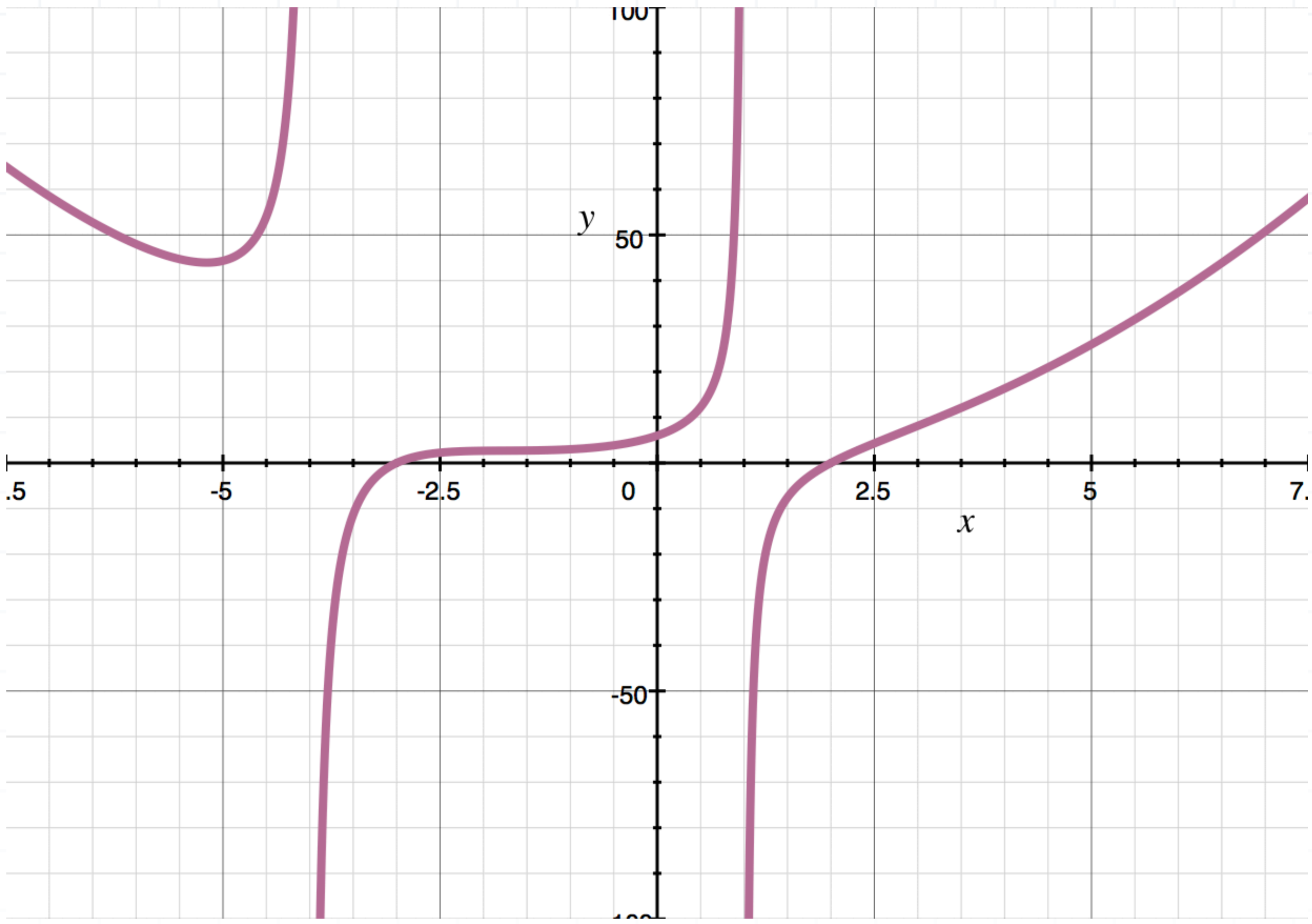
$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4}$$

Solution:

Factor the function.

$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4} = \frac{(x-2)(x^2+2x+4)(x+3)}{(x+4)(x-1)}$$

None of these factors cancel, which means that $x + 4 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -4$ and $x = 1$.



3. At what x-values does the function have infinite discontinuities?

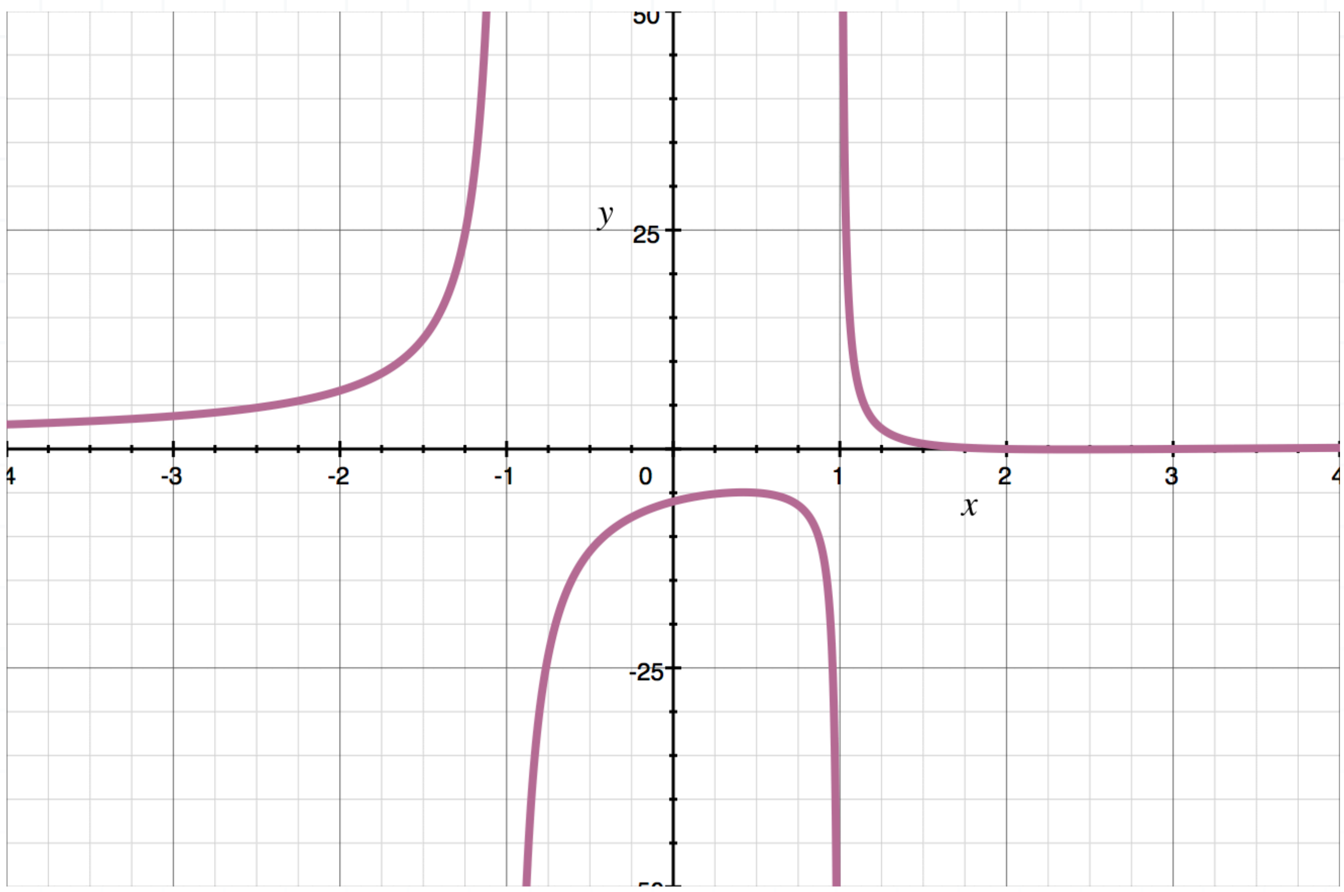
$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 1}$$

Solution:

Factor the function.

$$g(x) = \frac{x^2 - 5x + 6}{x^2 - 1} = \frac{(x-3)(x-2)}{(x+1)(x-1)}$$

None of these factors cancel, which means that $x + 1 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -1$ and $x = 1$.



4. Where are the infinite discontinuities of the function?

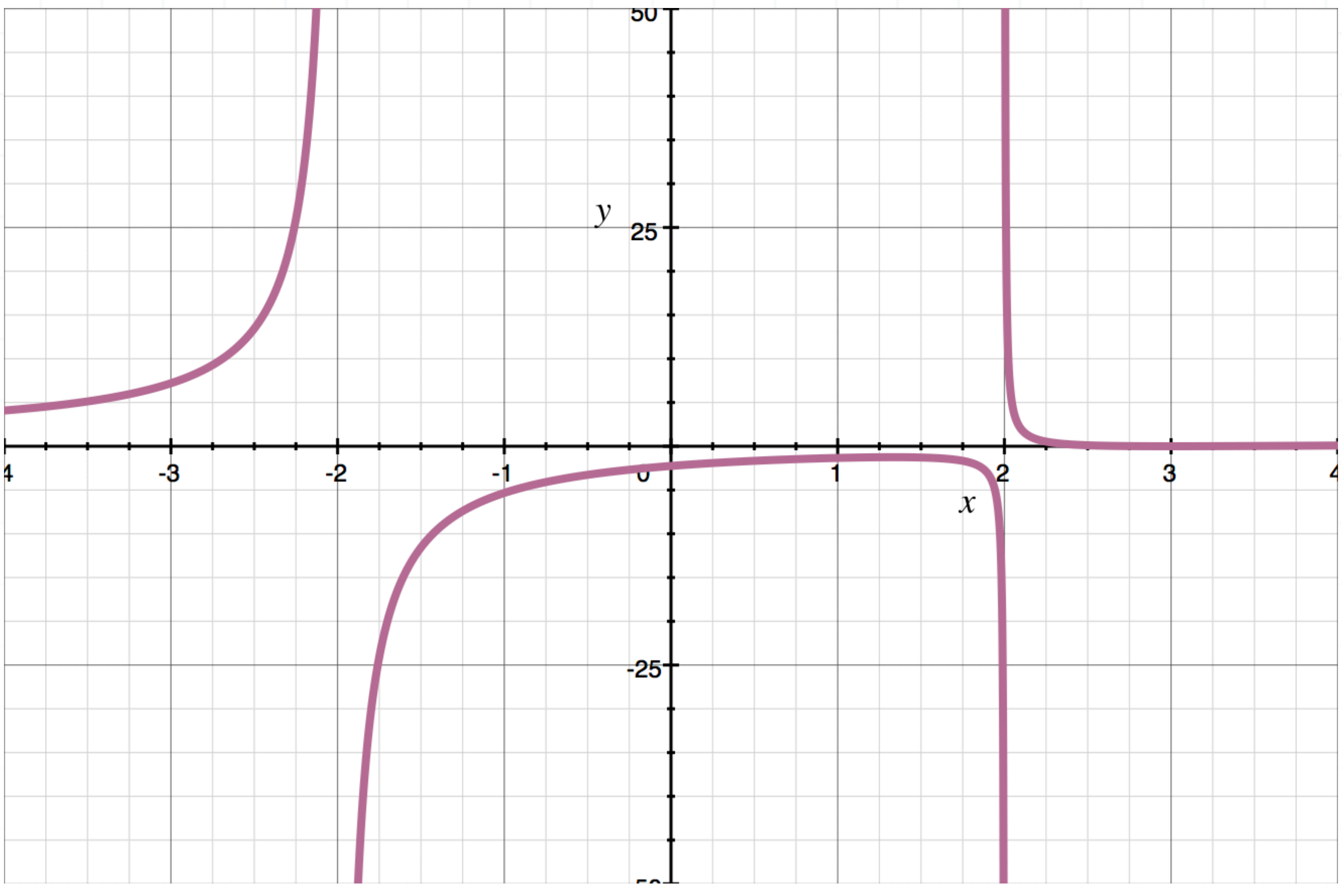
$$h(x) = \frac{x^2 - 6x + 9}{x^2 - 4}$$

Solution:

Factor the function.

$$h(x) = \frac{x^2 - 6x + 9}{x^2 - 4} = \frac{(x-3)^2}{(x+2)(x-2)}$$

None of these factors cancel, which means that $x+2=0$ and $x-2=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-2$ and $x=2$.



■ 5. At what x-values does the function have infinite discontinuities?

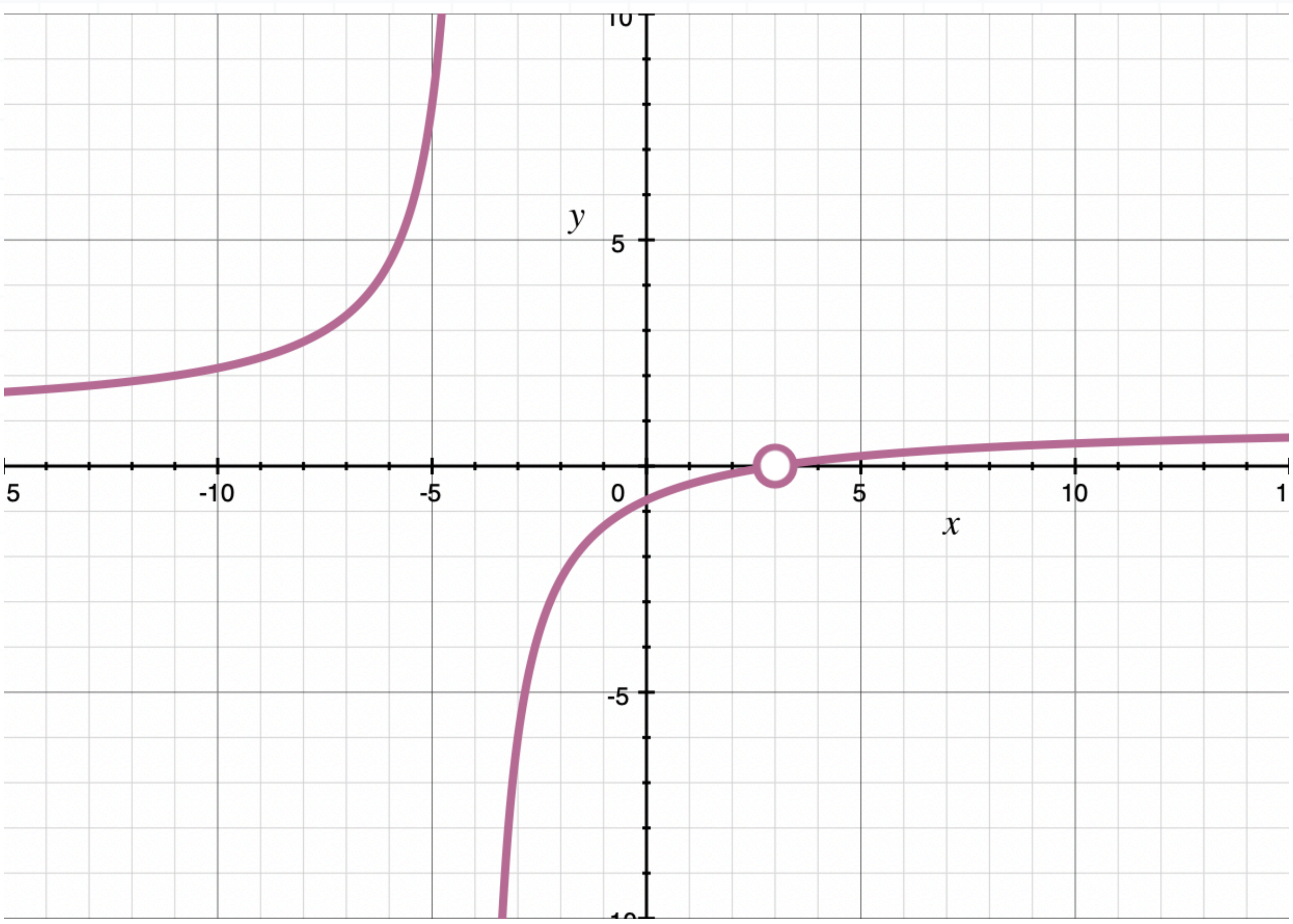
$$h(x)=\frac{x^2-6x+9}{x^2+x-12}$$

Solution:

Factor the function.

$$h(x)=\frac{x^2-6x+9}{x^2+x-12}=\frac{(x-3)^2}{(x+4)(x-3)}=\frac{x-3}{x+4}$$

In this form, we can see that the denominator is 0 at both $x = 3$ and $x = -4$. Because the $x - 3$ can be canceled, there's a point discontinuity at $x = 3$. Since the $x + 4$ can't be canceled, and, no matter how much we simplify the fraction, $x = -4$ will always make the denominator 0, that tells us there's a vertical asymptote at $x = -4$, and therefore an infinite discontinuity there.



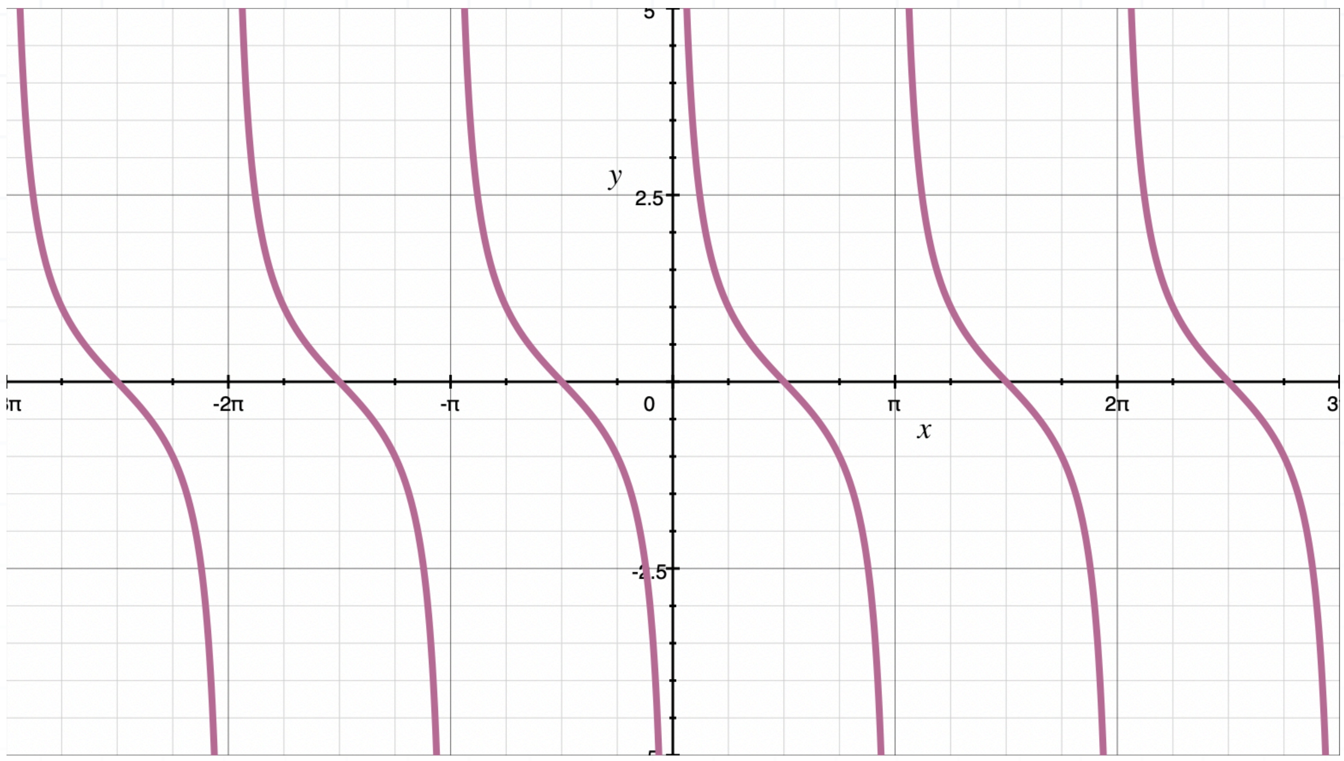
■ 6. Classify the discontinuities of $f(x) = \cot x$ on the interval $[0,2\pi]$.

Solution:

Factor the function.

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

None of these factors cancel, which means that the denominator will be 0 whenever $\sin x = 0$. The value of $\sin x$ is 0 at all integer multiples of π, so $f(x)$ has an infinite discontinuity at three points in the interval $[0,2\pi]$, which are at $x = 0$, $x = \pi$, and $x = 2\pi$.



ENDPOINT DISCONTINUITIES

■ 1. What is the value of the limit on the interval $[0,3]$?

$$\lim_{x \to 3} -\sqrt{x+5}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 3$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to 3^-} -\sqrt{x+5} = -2\sqrt{2} \quad \neq \quad \lim_{x \to 3^+} -\sqrt{x+5} = \text{DNE}$$

■ 2. What is the value of the limit on the interval $[\pi,2\pi]$?

$$\lim_{x \to \pi} \sin x$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = \pi$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to\pi^+} \sin x = 0 \quad \neq \quad \lim_{x\to\pi^-} \sin x = \text{DNE}$$

■ 3. What is the value of the limit on the interval $[4,\infty)$?

$$\lim_{x\to 4} -\frac{x+7}{x^2-6x+15}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x=4$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to 4^+} -\frac{x+7}{x^2-6x+15} = -\frac{11}{7} \quad \neq \quad \lim_{x\to 4^-} -\frac{x+7}{x^2-6x+15} = \text{DNE}$$

■ 4. What is the value of the limit on the interval $[-9/2,5/2]$?

$$\lim_{x\to\frac{5}{2}} \frac{x+3}{x^2+x+1}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 5/2$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to\frac{5}{2}^-} \frac{x+3}{x^2+x+1} = \frac{22}{39} \quad \neq \quad \lim_{x\to\frac{5}{2}^+} \frac{x+3}{x^2+x+1} = \text{DNE}$$

■ 5. What is the value of the limit on the interval $(-2,2]$?

$$\lim_{x\to-2} \sqrt{2x+4}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = -2$. The left-hand limit does not exist, which means that the one-sided limits are not equal.

$$\lim_{x\to-2^+} \sqrt{2x+4} = 0 \quad \neq \quad \lim_{x\to-2^-} \sqrt{2x+4} = \text{DNE}$$

■ 6. What is the value of the limit on the interval $[-\pi, \pi]$?

$$\lim_{x\to\pi} -\frac{5\cos x}{2}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = \pi$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to\pi^-} -\frac{5\cos x}{2} = \frac{5}{2} \quad \neq \quad \lim_{x\to\pi^+} -\frac{5\cos x}{2} = \text{DNE}$$

W W W . K R I S T A K I N G M A T H . C O M