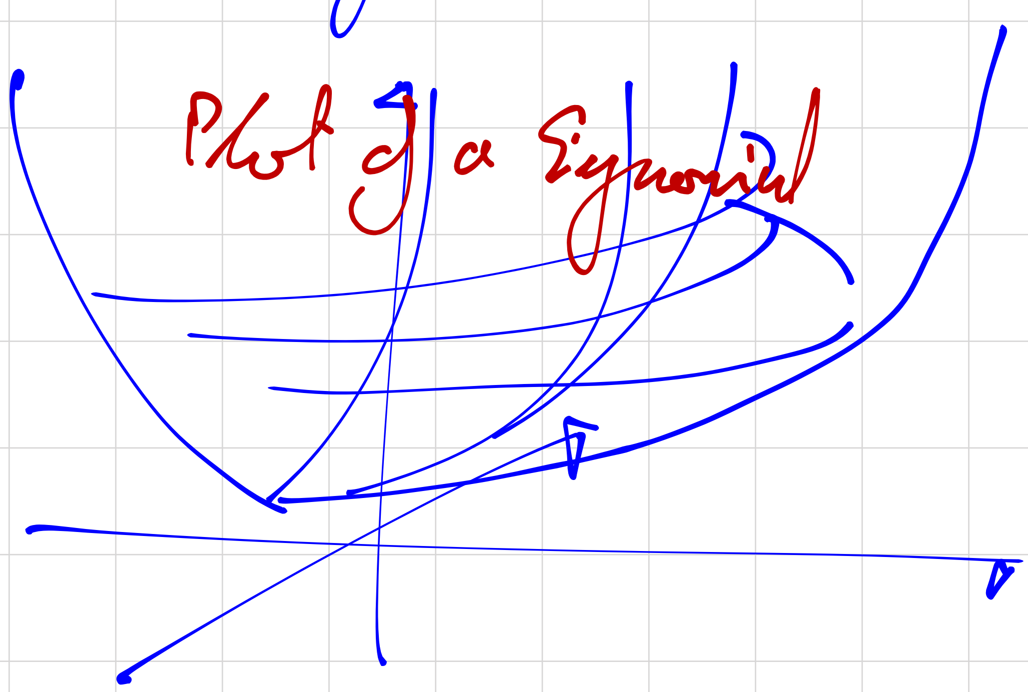
Thoughts on presentation:

- * Revisit the book on Bose-Hubble proof □
- * Ed.. on Overview before begin! □
- * Look into L.'s fancy slide technique □
- * Do the proof in recorded + written □
  + SPEED up
- * Images □



Something on the Coefs

[Need an image!]

Example 1 : Plot

- * IPM idea
- * Do the presentation in two parts
  2 > Out
  3 New

1 Hour 45 min

20 min – 5 min intervals
20 min – 5 min ..
20 min
– 15 min break

10 min last Exercises

Last 10 is a Recap.

- * Talk should be

READING GROUP TALK: SIGNOMIAL OPTIMIZATION

TABLE OF CONTENTS / OVERVIEW

Definitions

Signomial: $g(x) := \sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)^T} x)$

where $c_j \in \mathbb{R}$, $\alpha^{(j)} \in \mathbb{R}^n$ are parameters, $j \in [\ell]$

Signomial Optimization

$$
\begin{aligned}
\min \quad & g(x) \\
\text{s.t.} \quad & g_i(x) \geq 0 \quad \forall\, i \in [m] \\
& h_k(x) = 0 \quad \forall\, k \in [d]
\end{aligned}
$$

$g, g_1, \dots g_m, h_1, \dots, h_k \in$ Signomials

Special Cases

AM/GM-Exponentials

1. Signomial g
2. $g(x) \geq 0 \quad \forall\, x$
3. c_j's > 0 EXCEPT POSSIBLY for one Denoted

So $g(x) = b\exp(\alpha^T x) + \sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)^T} x)$

$b \in \mathbb{R}, c \in \mathbb{R}^{\ell}_{++}$; $\alpha, \alpha^{(1)}, \dots, \alpha^{(\ell)} \in \mathbb{R}^n$

Relative Entropy

$$D : \mathbb{R}^n \times \mathbb{R}^n \ni (v, \lambda) \longmapsto \sum_{j=1}^{n} v_j \log\left(\frac{v_j}{\lambda_j}\right) \in \mathbb{R} \cup \{\pm\infty\}$$

Weighted AM/GM Inequalities

$$\frac{w_1 x_1 + \cdots + w_n x_n}{w} \geq \sqrt[w]{x_1^{w_1} \cdots x_n^{w_n}}$$

$$\frac{1}{w} \sum_{j=1}^{n} w_j x_j \geq \sqrt[w]{\prod_{j=1}^{n} x_j^{w_j}}$$

Where $w_1, w_2, \ldots, w_n > 0$ and $w = \sum_{j=1}^{n} w_j$

$x_1, x_2, \ldots, x_n \geq 0$

Proof:

$$\ln\left(\frac{1}{w} \sum_{j=1}^{n} w_j x_j\right) \geq \sum_{j=1}^{n} \frac{w_j}{w} \ln x_j$$ $\because$ Jensen $\Rightarrow$ ln is strictly Concave

$$= \ln\left(\sqrt[w]{\prod x_j^{w_j}}\right)$$

The result follows from apply. exp on both sides which is valid since ln is strictly inc.

$\square$

Convex Conjugate

ZN

: X a Topo. Space, X^* Topo. dual
$\langle , \rangle : X, X^* \to \mathbb{R}$ duality Pair

$$f^* : X^* \ni x^* \mapsto \sup_{x \in X} \langle x^*, x \rangle - f$$

Recall:

SLATER'S CONDITION: ZK (Sufficient condition for Strong Duality in Convex optimization problem)

Let $A \in \mathbb{R}^{n \times k}$, $b \in \mathbb{R}^{k}$, $f_i(x)$ convex $\forall i \in [m] \cup \{0\}$

$$\min \; f_0(x)$$
$$\text{s.t.} \quad f_i(x) \leq 0 \quad \forall i \in [m]$$
$$Ax = b$$

Satisfies SLATER'S condition if

$$\exists \, x^* \in \bigcap_{i=0}^{m} \mathcal{D}om(f_i) \text{ s.t. } f_i(x^*) < 0 \; \forall i \in [m]$$
$$Ax^* = b$$

Strictly feasible Pt

If SLATER'S condition is satisfied then STRONG Duality holds, i.e, LEMMA 2

Let $Q \in \mathbb{R}^{n \times \ell}$, $c \in \mathbb{R}^{\ell}_{+}$, $b \in \mathbb{R}$, $\alpha \in \mathbb{R}^{n}$

① $g(x) := b \exp(\alpha^T x) + \sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)^T} x) \geq 0$

$\Longleftrightarrow$

② $\exists$ $v \in \mathbb{R}^{\ell}_{+}$ s.t. $D(v, ec) - b \leq 0$ (2.1) Coef's

$[\alpha^{(1)}, \dots, \alpha^{(\ell)}] v = \sum_{j=1}^{n} v_j \alpha^{(j)} = (\mathbb{1}^T v) \alpha$ (2.2) Expo's

Proof: (Via AM/GM)

"$\Longleftarrow$" Suppose you are given a v satisfying ②

$$\frac{1}{1} \sum_{j=1}^{n} \left(\frac{v_j/\mathbb{1}^T v}{v_j/\mathbb{1}^T v}\right) c_j \exp\left(\alpha^{(j)^T} x\right) \geq \left(\prod_{j=1}^{\ell} \left(\frac{c_j \exp(\alpha^{(j)^T} x)}{v_j/\mathbb{1}^T v}\right)^{v_j/\mathbb{1}^T v}\right)^{\frac{1}{1}}$$

$\because$ Weighted AM/GM [Recall]

$$= \prod_{j=1}^{\ell} \left(\frac{c_j \exp\left(\frac{v_j \alpha^{(j)^T} x}{\mathbb{1}^T v}\right)}{v_j/\mathbb{1}^T v}\right) = \prod_{j=1}^{\ell} \left(\frac{c_j}{v_j/\mathbb{1}^T v}\right)^{v_j/\mathbb{1}^T v} \exp(\alpha^T x)$$

$\because \sum_j v_j \alpha^{(j)^T} = \mathbb{1}^T v \alpha^T$ (2.2)

Where

$$\prod_{j=1}^{\ell} \left(\frac{c_j}{v_j/\mathbb{1}^T v}\right)^{v_j/\mathbb{1}^T v} \geq b$$

$$\prod_{j=1}^{\ell} \left(\frac{c_j}{v_j/\mathbb{1}^T v}\right)^{v_j/\mathbb{1}^T v} = \exp\left\{-D\left(\frac{v}{\mathbb{1}^T v}, c\right)\right\}$$

$\because$ Def of Rel. Ent.

[Recall]

$$\exp\left\{-D\left(\frac{v}{\mathbb{1}^T v}, c\right)\right\} \geq -\xi\left[D\left(\frac{v}{\mathbb{1}^T v}, c\right) + \log(\xi) - 1\right] \quad \forall \xi > 0$$

$\because$ Definition of Convex Conjugate: ?

$$f^*: X^* \ni x^* \mapsto \sup_{\xi \geq} \langle x^*, \xi \rangle - f(\xi) \qquad (1)$$

<u>Claim:</u> $\exp(-x)$ is the Conv-Conj of $x \log x$

$$-\exp(-p) = \sup_{\xi \geq 0} (1-p)\cdot\xi - \xi \log \xi$$

[Recall]

$$= -\xi\left(D\left(\frac{v}{\mathbb{1}^T v}, c\right) + \log(\xi) - \log(e)\right)$$

$$= -\xi\left(\sum_{j=1}^{n} \frac{v_j}{\mathbb{1}^T v} \log\left(\frac{v_j}{c_j \mathbb{1}^T v}\right) + \log\left(\frac{\xi}{e}\right)\right)$$

$$= -\sum_{j=1}^{n} \xi \frac{v_j}{\mathbb{1}^T v} \log\left(\frac{\xi v_j}{\mathbb{1}^T v} \cdot \frac{1}{c_j e}\right)$$

$$= -D\left(\frac{\xi v}{\mathbb{1}^T v}, ec\right) \quad \because \text{Def. Rel. Ent.}$$

$$\geq -D(v, ec) \quad \because \text{choose } \xi := \mathbb{1}^T v$$

$$\geq -b \quad \because (2.1)$$

$\square$

"$\Rightarrow$"

$$g(x) \geq 0$$

$$\Longleftrightarrow$$

$$g(x)\exp(-\alpha^T x) \geq 0 \qquad \because \exp(-\alpha^T x) > 0$$

$$\Longleftrightarrow$$

$$\sum_{j=1}^{\ell} c_j \exp\left(\left(\alpha^{(j)T} x\right)_j - \alpha^T x\right) \geq -b \qquad \because \text{Def of } g(x)$$

Consider the problem

$$P := \inf_{x \in \mathbb{R}^n} c^T t$$
$$\text{s.t. } \exp\left(\alpha^{(j)T} x - \alpha^T x\right) = t_j \quad \forall j \in [\ell]$$

We want $P \geq -b$

$$P = \inf_{x \in \mathbb{R}^n,\, t \in \mathbb{R}^\ell} c^T t$$
$$\text{s.t. } \exp\left(\alpha^{(j)T} x - \alpha^T x\right) \leq t_j \quad \forall j \in [\ell]$$

$\because$ $\leq$: Def of inf
$\geq$: $c, t \geq 0$

$$\sup_{\lambda \in \mathbb{R}^\ell_+} \inf_{x \in \mathbb{R}^n,\, t \in \mathbb{R}^\ell} (c - \lambda)^T \left(t_j - \exp(\dots)\right)_{j=1}^{n}$$

?

$$\sup_{\nu \in \mathbb{R}^\ell_+} \; -\sum_{j=1}^{\ell} \nu_j \log\left(\frac{\nu_j}{e c_j}\right)$$

$$\text{s.t.} \quad [\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell)}]\,\nu = (\mathbb{1}^T \nu)\,\alpha$$

$\Leftrightarrow$

$$\sup_{\nu \in \mathbb{R}^\ell_+} \; -D(\nu, eC)$$

$$\text{s.t.} \quad [\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell)}]\,\nu = (\mathbb{1}^T \nu)\,\alpha$$

Invoke SLATER'S CONDITION

[[RECALL]]

Section 2: Characterizing the cone of Sums of AM/GM Exponentials

Exponents (fixed) **Coefficients** ("Variables")

$(\alpha, \alpha^{(2)}, \dots, \alpha^{(\ell)})$ $(b, c_2, \dots, c_\ell)$

$$g(x) := b\exp(\alpha^T x) + \sum_{j=1} c_j \exp(\alpha^{(j)T} x) \geq 0$$

$$\Longleftrightarrow$$

$\exists\ \nu \in \mathbb{R}^\ell_+$

s.t. $A\nu := \begin{bmatrix} | & | & & | \\ \alpha^{(1)} & \alpha^{(2)} & \dots & \alpha^{(\ell)} \\ | & | & & | \end{bmatrix} \nu = (\mathbb{1}^T\nu)\alpha$ } linear constraint on ν

$D(\nu, ec) - b \leq 0$ } Entropy const. on ν and coef c, b

What are we gonna do with this tool?

$$\max\ b$$
$$\text{s.t.}\ \underbrace{\sum_{j=1} c_j \exp(\alpha^{(j)T} x) - b}_{g(x) \text{ with }} \geq 0$$

$\alpha = 0$

$b \leq 0$

$$\Longleftrightarrow$$

s.t. $\exists\ \nu \in \mathbb{R}^\ell_+$

s.t. $A\nu := \begin{bmatrix} | & | & & | \\ \alpha^{(1)} & \alpha^{(2)} & \dots & \alpha^{(\ell)} \\ | & | & & | \end{bmatrix} \nu = 0$

$D(\nu, ec) - b \leq 0$

Definition

For a finite set of vectors $M \subseteq \mathbb{R}^n$

$$SAGE(M) := \left\{ f = \sum_{i=1}^{\ell} f_i \;\middle|\; f_i \text{ is AM/GM Expo using Exponents in } M \right\}$$

So if $M = \{\alpha^{(j)}\}_{j=1}^{\ell}$ then

$SAGE(\{\alpha^{(j)}\}_{j=1}^{\ell}) =$ the paper gives a description when only ℓ terms..

SAGE $\subsetneq$ all Nonnegative Signomial

Example

$$(\exp(x_1) - \exp(x_2) - \exp(x_3))^2 \geq 0$$

is not SAGE because ... WHY

$$\sum_{i=1}^{3} \exp\left(2e_i^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + \exp([0,1,1]x)$$

$$- \exp([1,1,0]x) - \exp([0,1,1]x)$$

$$C_{SAGE}(\{\alpha^{(j)}\}_{j=1}^{\ell}) := \{c \in \mathbb{R}^{\ell} \mid \sum_{j=1}^{\ell} c_j \exp\{\alpha^{(j)T} x\} \in SAGE(\{\alpha^{(j)}\})\},$$

"The SET of all coefficients that correspond to AM/GM-Expo's in exponent $\{\alpha^{(j)}\}_{j=1}^{\ell}$"

Proposition 6

$$\begin{aligned} C_{SAGE}(\{\alpha^{(j)}\}_{j=1}^{\ell}) &:= \text{Def} \\ &= \text{Characterization 1} \\ &= \text{Characterization 2} \end{aligned} \Bigg\} \text{Ugly.}$$

$$C^*_{SAGE}(\{\alpha^{(j)}\}_{j=1}^{\ell}) = \{ \ldots \text{Relative Entropy} \ldots \}$$

Part 3

Unconstrained Signomial Programming

$$f_* = \inf_{x \in \mathbb{R}^n} f(x)$$

NON-Negativity Characterizations

$$\Longleftrightarrow$$

$$f_* = \sup_{\gamma \in \mathbb{R}} \gamma$$

$$\text{s.t. } f(x) - \gamma \in \text{"Nonnegative Signomials"}$$

Assume w.l.o.g $f(x) = \sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)T} x)$

with $c_1 \in \mathbb{R}, \quad \ldots ?$

Relax:

$$f_{SAGE} = \sup_{\gamma \in \mathbb{R}} \gamma$$

$$\text{s.t. } f(x) - \gamma \in SAGE(\{\alpha^{(j)}\}_{j=1}^{\ell})$$

Prop 6.

$$f_{SAGE} = \sup_{\gamma \in \mathbb{R}} \gamma$$

$$\text{s.t. } (c_1 - \gamma, c_2, \ldots, c_\ell) \in C_{SAGE}(\{\alpha^{(j)}\}_{j=1}^{\ell})$$

(Make steps explicit)

EXTEND

$$f_{SAGE}^{(p)} = \sup_{\gamma \in \mathbb{R}} \gamma$$

$$\text{s.t. } \left(\sum_{j=1}^{\ell} \exp\{\alpha^{(j)T} x\}\right)^p [f(x) - \gamma] \in \left(E_{p+1}(\{\alpha^{(j)}\}_{j=1}^{\ell})\right)$$

Question:

Is Signomial optimization NP-hard?

↳ what is the proof.

Notes: Make detours in a different Column..