Minkowski Distance Objective Function

A way around this is to choose as objective the distance from an idealized solution. To begin consider the individual optimization problems whose solutions are sometimes called the *aspirational* solution \cite{}.

$$\begin{aligned} \psi_{\pm}^{(1)} = \max \setminus \min \;\; & w^T \mu \\ s.t. \;\; & w \in \Delta^N \end{aligned} \tag{1}$$

$$\begin{aligned} \psi_{\pm}^{(2)} = \max \setminus \min \;\; & w^T \Phi^{(2)} w \\ s.t. \;\; & w \in \Delta^N \end{aligned} \tag{2}$$

$$\begin{aligned} \psi_{\pm}^{(3)} = \max \setminus \min \;\; & w^T \Phi^{(3)} (w \otimes w) \\ s.t. \;\; & w \in \Delta^N \end{aligned} \tag{3}$$

$$\begin{aligned} \psi_{\pm}^{(4)} = \max \setminus \min \;\; & w^T \Phi^{(4)} (w \otimes w \otimes w) \\ s.t. \;\; & w \in \Delta^N \end{aligned} \tag{4}$$

For a fixed choice (made by the investor) of $\lambda \in \Delta^4$ define the following optimization problem:

$$\begin{aligned} \min \;\; \not{f}(w) := & \Big(\frac{\psi_{+}^{(1)} - w^T \Phi^{(1)}}{\psi_{+}^{(1)}} \Big)^{\lambda_1} + \Big(\frac{w^T \Phi^{(2)} w - \psi_{-}^{(2)}}{\psi_{-}^{(2)}} \Big)^{\lambda_2} \\ & + \Big(\frac{\psi_{+}^{(3)} - w^T \Phi^{(3)} (w \otimes w)}{\psi_{+}^{(3)}} \Big)^{\lambda_3} + \Big(\frac{w^T \Phi^{(4)} (w \otimes w) - \psi_{-}^{(4)}}{\psi_{-}^{(4)}} \Big)^{\lambda_4} \\ s.t. \;\; & w \in \Delta^N \end{aligned} \tag{5}$$

Recasting the objective as a signomial

The problems in **(5)** can be recast into the form of a *signomial* which is the following form:

$$f(\mathbf{x}) = \sum_{j=1}^{\ell} c_j e^{\alpha^{(j)} x}$$

we do this as follows: Define

- $z_1 := \log(1 - \frac{1}{\psi_{+}^{(1)}} w^T \Phi^{(1)})$
- $z_2 := \log(\frac{1}{\psi_{-}^{(2)}} w^T \Phi^{(2)} w - 1)$
- $z_3 := \log(1 - \frac{1}{\psi_{+}^{(3)}} w^T \Phi^{(3)} (w \otimes w))$
- $z_4 := \log(\frac{1}{\psi_{-}^{(4)}} w^T \Phi^{(4)} (w \otimes w \otimes w) - 1)$

Since $w \in \Delta^N$ we have that

$$\begin{aligned} z_3 \in \; & [\log(1 - \max_{w \in \Delta^N} \frac{1}{\psi_{+}^{(3)}} w^T \Phi^{(3)} (w \otimes w)), \log(1 - \min_{w \in \Delta^N} \frac{1}{\psi_{+}^{(3)}} w^T \Phi^{(3)} (w \otimes w))] \\ = \; & [\log(0), \log(1 - \frac{\psi_{-}^{(3)}}{\psi_{+}^{(3)}}] \end{aligned}$$

Then equation **(5)** becomes

$$
\begin{aligned}
\min \ & \not{f}(z) := \sum_{j=1}^{4} e^{\lambda_i z_i} \\
s.t. \ & z_1 \in (-\infty, \log(1 - \frac{\psi_-^{(1)}}{\psi_+^{(1)}})] \\
& z_2 \in (-\infty, \log(\frac{\psi_-^{(2)}}{\psi_+^{(2)}} - 1)] \\
& z_3 \in (-\infty, \log(1 - \frac{\psi_-^{(3)}}{\psi_+^{(3)}})] \\
& z_4 \in (-\infty, \log(\frac{\psi_-^{(4)}}{\psi_+^{(4)}} - 1)]
\end{aligned}
\tag{6}
$$

[Signomial Optimization]()