Ideas on presenting

- [ ] Knockout result: The audience clearly
  1. understands
     1.1. How SP relate to Ent. Zone
     1.2. How to Construct the Hier.
  2. To impress them
- [ ] High visual element
- [ ] "Applicable to wider audience"

20 min on part 1
5 min break
15 min on part 2
20 min break
30 min on part 3
5 min Conclusion

85 min

READING GROUP TALK: SIGNOMIAL OPTIMIZATION

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Part 1: AM-GM · Exponentials & Relative entropy

1. Definitions
2. Lemma
   ↳ Proof ☐
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   ↳ IMAGE ☑

Part 2: The SAGE-Cone

0. Idea
1. Definitions
2. Non-example of SAGE ☑
3. Lemma (No proof)
4. Pictures. ☐
   (Drawn by hand.)

Part 3: Signomial Optimization.

Unconstrained.

1. f_* to f_{SAGE}
2. Computation f_{SAGE}
3. Pólya type theorem ☐ Proof next
4. Observations ☐ is it complete?

Constrained.

1. f_*
2. f_{WD}
3. $f^{(p,q)}_{SAGE}$

☐ Proof next

☐ is it complete!

Conclusion

Definitions

Signomial: $g(x) := \sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)T} x)$

where $c_j \in \mathbb{R}$, $\alpha^{(j)} \in \mathbb{R}^n$ are parameters, $j \in [\ell]$

Signomial Optimization

$$\begin{aligned} \min \quad & g(x) \\ \text{s.t.} \quad & g_i(x) \geq 0 \quad \forall i \in [m] \\ & h_k(x) = 0 \quad \forall k \in [d] \end{aligned}$$

$g, g_1, \dots g_m, h_1, \dots, h_k \in$ Signomials

Special Cases

AM/GM-Exponentials

1. Signomial g
2. $g(x) \geq 0 \quad \forall x$
3. c_j's > 0 EXCEPT POSSIBLY for one Denoted

So $g(x) = b \exp(\alpha^T x) + \sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)T} x)$

$b \in \mathbb{R}, c \in \mathbb{R}^{\ell}_{++}; \alpha, \alpha^{(1)}, \dots, \alpha^{(\ell)} \in \mathbb{R}^n$

RELATIVE ENTROPY

$$D : \mathbb{R}^n \times \mathbb{R}^n \ni (v, \lambda) \longmapsto \sum_{j=1}^{n} v_j \log\left(\frac{v_j}{\lambda_j}\right) \in \mathbb{R} \cup \{\pm\infty\}$$

WEIGHTED AM/GM INEQUALITIES

$$\frac{w_1 x_1 + \cdots + w_n x_n}{w} \geq \sqrt[w]{x_1^{w_1} \cdots x_n^{w_n}}$$

$$\frac{1}{w} \sum_{j=1}^{n} w_j x_j \geq \sqrt[w]{\prod_{j=1}^{n} x_j^{w_j}}$$

Where $w_1, w_2, \ldots, w_n > 0$ and $w = \sum_{j=1}^{n} w_j$

$x_1, x_2, \ldots, x_n \geq 0$

Proof:

$$\ln\left(\frac{1}{w} \sum_{j=1}^{n} w_j x_j\right) \geq \sum_{j=1}^{n} \frac{w_j}{w} \ln x_j \quad \because \text{Jensen} \Rightarrow \ln \text{ is strictly concave}$$

$$= \ln\left(\sqrt[w]{\prod x_j^{w_j}}\right)$$

The result follows from apply. exp on both sides which is valid since ln is strictly inc.

$\square$

CONVEX CONJUGATE

ZR

: X a Topo. Space, X^* Topo. dual

$\langle \, , \, \rangle : X, X^* \longrightarrow \mathbb{R}$ duality pair

$$f^* : X^* \ni x^* \longmapsto \sup_{x \in X} \langle x^*, x \rangle - f$$

LEGENDRE TRANSFORM

ZR

. . .

Recall:

SLATER'S CONDITION: (Sufficient condition for Strong Duality in Convex optimization problem)

ZK

Let $A \in \mathbb{R}^{n \times k}$, $b \in \mathbb{R}^{k}$, $f_i(x)$ convex $\forall i \in [m] \cup \{0\}$

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t} \quad & f_i(x) \le 0 \quad \forall i \in [m] \\ & Ax = b \end{aligned}$$

Satisfies SLATER'S Condition if

$$\exists\, x^* \in \bigcap_{i=0}^{m} \mathrm{Dom}(f_i) \text{ s.t. } \begin{aligned} & f_i(x^*) < 0 \;\forall i \in [m] \\ & Ax^* = b \end{aligned}$$

Strictly feasible Pt

If SLATER'S Condition is satisfied then STRONG Duality holds, i.e,

Claim: Signomial programming is NP-Hard.

LEMMA 2

Let $Q \in \mathbb{R}^{n \times \ell}$, $c \in \mathbb{R}^{\ell}_{+}$, $b \in \mathbb{R}$, $\alpha \in \mathbb{R}^{n}$

① $g(x) := b \exp(\alpha^T x) + \sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)T} x) \geq 0$

$$\Longleftrightarrow$$

② $\exists\, \nu \in \mathbb{R}^{\ell}_{+}$ s.t. $D(\nu, ec) - b \leq 0$ (2.1) Cref's

$[\alpha^{(1)}, \dots, \alpha^{(\ell)}]\nu = \sum_{j=1}^{n} \nu_j \alpha^{(j)} = (\mathbb{1}^T \nu)\alpha$ (2.2) Expo's

Proof: (Via AM/GM)

"$\Longleftarrow$" Suppose you are given a ν satisfying ②

$$\frac{1}{1}\sum_{j=1}^{n} \left(\frac{\nu_j/\mathbb{1}^T\nu}{\nu_j/\mathbb{1}^T\nu}\right) c_j \exp\left(\alpha^{(j)T} x\right) \geq \left(\prod_{j=1}^{\ell} \left(\frac{c_j \exp(\alpha^{(j)T} x)}{\nu_j/\mathbb{1}^T\nu}\right)^{\nu_j/\mathbb{1}^T\nu}\right)^{\frac{1}{1}}$$

$\because$ Weighted AM/GM

$$= \prod_{j=1}^{\ell} \left(\frac{c_j \exp\left(\frac{\nu_j \alpha^{(j)T} x}{\mathbb{1}^T\nu}\right)}{\nu_j/\mathbb{1}^T\nu}\right) = \prod_{j=1}^{\ell} \left(\frac{c_j}{\nu_j/\mathbb{1}^T\nu}\right)^{\nu_j/\mathbb{1}^T\nu} \exp(\alpha^T x)$$

$\because \sum_j \nu_j \alpha^{(j)T} = \mathbb{1}^T \nu\, \alpha^T$ (2.2)

Where

$$\prod_{j=1}^{\ell} \left(\frac{c_j}{\nu_j/\mathbb{1}^T\nu}\right)^{\nu_j/\mathbb{1}^T\nu} \geq b$$

$\because$ Claim on next Page

$$\prod_{j=1}^{\ell}\left(\frac{c_j}{v_j/\mathbb{1}^T v}\right)^{v_j/\mathbb{1}^T v} = \exp\left\{-D\left(\frac{v}{\mathbb{1}^T v}, c\right)\right\}$$

$\because$ Def of Rel. Ent.

$$\exp\left\{-D\left(\frac{v}{\mathbb{1}^T v}, c\right)\right\} \geq -\xi\left[D\left(\frac{v}{\mathbb{1}^T v}, c\right) + \log(\xi) - 1\right] \quad \forall \xi > 0$$

$\because$ Definition of Convex Conjugate: ? (1)

$$f^*: X^* \ni x^* \mapsto \sup_{\xi \geq} \langle x^*, \xi \rangle - f(\xi)$$

Claim: $\exp(-x)$ is the Conv-Conj of $x \log x$

$$\exp(-p) = \sup_{\xi \geq 0} (1-p)\cdot\xi - \xi \log \xi$$

[Legendre Transform] [Recall]

$$= -\xi\left(D\left(\frac{v}{\mathbb{1}^T v}, c\right) + \log(\xi) - \log(e)\right)$$

$$= -\xi\left(\sum_{j=1}^{\ell} \frac{v_j}{\mathbb{1}^T v} \log\left(\frac{v_j}{c_j \mathbb{1}^T v}\right) + \log\left(\frac{\xi}{e}\right)\right)$$

$$= -\sum_{j=1}^{\ell} \xi \frac{v_j}{\mathbb{1}^T v} \log\left(\frac{\xi v_j}{\mathbb{1}^T v} \cdot \frac{1}{c_j e}\right)$$

$$= -D\left(\frac{\xi v}{\mathbb{1}^T v}, ec\right) \quad \because \text{Def. Rel. Ent.}$$

$$\geq -D(v, ec) \quad \because \text{choose } \xi := \mathbb{1}^T v$$

$$\geq -b \quad \because (2.1)$$

$\square$

"$\Rightarrow$"

$$g(x) \geq 0$$
$$\Updownarrow$$
$$g(x)\exp(-\alpha^T x) \geq 0$$
$$\Updownarrow \qquad \because \exp(-\alpha^T x) > 0$$
$$\sum_{j=1}^{\ell} c_j \exp\left(\left((\alpha^{(j)})^T x\right)_j - \alpha^T x\right) \geq -b$$

$\because$ Def of $g(x)$

Consider the problem

$$P := \inf_{x \in \mathbb{R}^n} c^T t$$
$$\text{s.t. } \exp(\alpha^{(j)T} x - \alpha^T x) = t_j \quad \forall j \in [\ell]$$

We want $P \geq -b$

$$P = \inf_{x \in \mathbb{R}^n,\, t \in \mathbb{R}^\ell} c^T t$$
$$\text{s.t. } \exp(\alpha^{(j)T} x - \alpha^T x) \leq t_j \quad \forall j \in [\ell]$$

$\because$ $\leq$ $\because$ Def of inf
$\geq$ $\because$ $c, t \geq 0$

$\because$ Definition of Lagrangian dual

$$\sup_{v \in \mathbb{R}_+^\ell} \inf_{\substack{x \in \mathbb{R}^n \\ t \in \mathbb{R}^\ell}} c^T t + \sum_{j=1}^{\ell} v_j \left(\exp(\alpha^{(j)T} x - \alpha^T x) - t_j\right)$$

Define $E(x) := \exp(\alpha^{(j)T} x - \alpha^T x) \leq t$ then

$$\sup_{v \in \mathbb{R}_+^\ell} \inf_{x \in \mathbb{R}^n} \inf_{t \in \mathbb{R}^\ell} c^T t + v^T E(x) - v^T t$$

Note:

$\inf_{x \in \mathbb{R}^n} v^T E(x)$ is attained when $\nabla_x v^T E(x) = 0$

$$\frac{\partial}{\partial x_i} v^T E(x) = \frac{\partial}{\partial x_j} \sum_{j=1}^{\ell} v_j \exp((\alpha^{(j)T} - \alpha) x)$$

$$= \sum_{j=1}^{\ell} v_j (\alpha^{(j)} - \alpha)_i \exp((\alpha^{(j)T} - \alpha) x)$$

so $\nabla_x (v^T E(x)) = \left(\sum_{j=1}^{\ell} (\alpha^{(j)} - \alpha)_i v_j \exp((\alpha^{(j)T} - \alpha) x) \right)_{i=1}^{n}$

$$\left((\alpha^{(j)} - \alpha)_i \right)_{\substack{i \in [n] \\ j \in [\ell]}} \left(v_j \exp((\alpha^{(j)T} - \alpha) x) \right)_{j \in [\ell]} = 0$$

Choose $v_j := v_j / \exp((\alpha^{(j)T} - \alpha) x))$ $\forall j$ then

$$\left((\alpha^{(j)} - \alpha)_i \right)_{\substack{i \in [n] \\ j \in [\ell]}} (v_j)_{j \in [\ell]} = 0 \Leftrightarrow \boxed{\sum v_j \alpha^{(j)} = \mathbb{1}^T v \alpha}$$

$$D = \sup_{\nu \in \mathbb{R}^\ell_+} \inf_{\epsilon \in \mathbb{R}^\ell_+} \inf_{x \in \mathbb{R}^n} c^T \epsilon + \nu^T (E(x) - \epsilon)$$

?

$$-\nu^T \begin{bmatrix} \log\left(\frac{\nu_1}{e c_1}\right) \\ \vdots \\ \log\left(\frac{\nu_\ell}{e c_\ell}\right) \end{bmatrix} = -\nu^T \left(\begin{bmatrix} \log(\nu_1) \\ \vdots \\ \log(\nu_\ell) \end{bmatrix} - \begin{bmatrix} \log(c_1) \\ \vdots \\ \log(c_\ell) \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right)$$

$$\sum_{j=1}^{\ell} \nu_j \log\left(\frac{\nu_j}{e c_j}\right) = \sum_{j=1}^{\ell} \log\left(\left(\frac{\nu_j}{e c_j}\right)^{\nu_j}\right) = \log\left(\prod_{j=1}^{n} \left(\frac{\nu_j}{e c_j}\right)^{\nu_j}\right)$$

$$\sup_{\nu \in \mathbb{R}^\ell_+} - \sum_{j=1}^{\ell} \nu_j \log\left(\frac{\nu_j}{e c_j}\right)$$
$$\text{s.t. } [\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell)}]\nu = (\mathbb{1}^T \nu)\alpha$$

$$\Longleftrightarrow$$

$$\sup_{\nu \in \mathbb{R}^\ell_+} -D(\nu, eC)$$
$$\text{s.t. } [\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell)}]\nu = (\mathbb{1}^T \nu)\alpha$$

Section 2: Characterizing the cone of Sums of AM/GM Exponentials

Exponents (fixed) — **Coefficients** ("variables")

$(\alpha, \alpha^{(2)}, \dots, \alpha^{(\ell)})$ $(b, c_2, \dots, c_\ell)$

$$g(x) := b\exp(\alpha^T x) + \sum_{j=1} c_j \exp(\alpha^{(j)T} x) \geq 0$$

$$\Leftrightarrow$$

$\exists\; \nu \in \mathbb{R}^\ell_+$

s.t. $A\nu := \begin{bmatrix} | & | & & | \\ \alpha^{(1)} & \alpha^{(2)} & \dots & \alpha^{(\ell)} \\ | & | & & | \end{bmatrix} \nu = (\mathbb{1}^T\nu)\alpha$ } linear constraint on ν

$D(\nu, ec) - b \leq 0$ } Entropy constr. on ν and coef c, b

What are we gonna do with this tool?

$$\max\ b$$
$$\text{s.t. } \underbrace{\sum_{j=1} c_j \exp(\alpha^{(j)T} x) - b}_{g(x) \text{ with }} \geq 0$$

$\alpha = 0$
$b \leq 0$

$$\Leftrightarrow$$

s.t. $\exists\; \nu \in \mathbb{R}^\ell_+$

s.t. $A\nu := \begin{bmatrix} | & | & & | \\ \alpha^{(1)} & \alpha^{(2)} & \dots & \alpha^{(\ell)} \\ | & | & & | \end{bmatrix} \nu = 0$

$D(\nu, ec) - b \leq 0$

Definition

For a <u>finite</u> set of vectors $M \subseteq \mathbb{R}^n$

$$SAGE(M) := \left\{ f = \sum_{i=1}^{l} f_i \;\middle|\; f_i \text{ is AM/GM Expo using Exponent. in } M \right\}$$

So if $M = \{\alpha^{(j)}\}_{j=1}^{l}$ then

$SAGE(\{\alpha^{(j)}\}_{j=1}^{l}) =$ the Paper gives a description when only l terms..

SAGE $\neq$ all Nonnegative Polynomial

Example

$$\left(\exp(x_1) - \exp(x_2) - \exp(x_3)\right)^2 \geq 0$$

is not SAGE because ... WHY

$$\sum_{i=1}^{3} \exp\left(2e_i^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + \exp([0,1,1]x)$$

$$- \exp([1,1,0]x) - \exp([0,1,1]x)$$

$$\text{"}Q\text{"} := \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \qquad C = (1,1,1,1)$$

$$b = -1 \qquad b' = -1$$

$$\alpha := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad QV = \mathbb{1}^T V \alpha$$

$$\alpha' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} V = (V_1 + V_2) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$V_1 \log\left(\frac{V_1}{e}\right) + V_2 \log\left(\frac{V_2}{e}\right) \leq -1 \qquad \begin{aligned} 2V_1 &: V_1 + V_2 \\ 2V_1 &: V_1 + V_1 \end{aligned} \Rightarrow \quad V_1 = V_2$$

Example: $\left(\exp(x_1) - \exp(x_2) - \exp(x_3)\right)^2 \geq 0$

$$= \sum_{i=1}^{3} \exp(2e_i^T x) + 2\exp([0,1,1]x)$$

$$-2\exp([1,1,0]x) - 2\exp([1,0,1]x)$$

is not in SAGE.

Exponents: $\alpha^{(j)}$'s $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$, α's $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

Coefficients: c_j's $(1,1,1,2)$ b's $-2, -2$

We are gonna use: the Lemma

↳ we need to choose 2 columns of A $\therefore$ linear indep.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} V = (V_1 + V_2)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow V_1 = V_2$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} W = (W_1 + W_2)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow W_2 = W_1 + W_2, \quad W_1 = 0$$

$$W_2 \log\left(\frac{W_2}{e}\right) \leq -2$$

$$\log(W_2) - \log(e) \leq -2/W_2$$

$$\log(W_2) \leq 1 - 2/W_2$$

Holds for no W_2

$$C_{SAGE}(\{\alpha^{(j)}\}_{j=1}^{\ell}) := \{c \in \mathbb{R}^{\ell} \mid \sum_{j=1}^{\ell} c_j \exp\{\alpha^{(j)^T} x\} \in SAGE(\{\alpha^{(j)}\})\}.$$

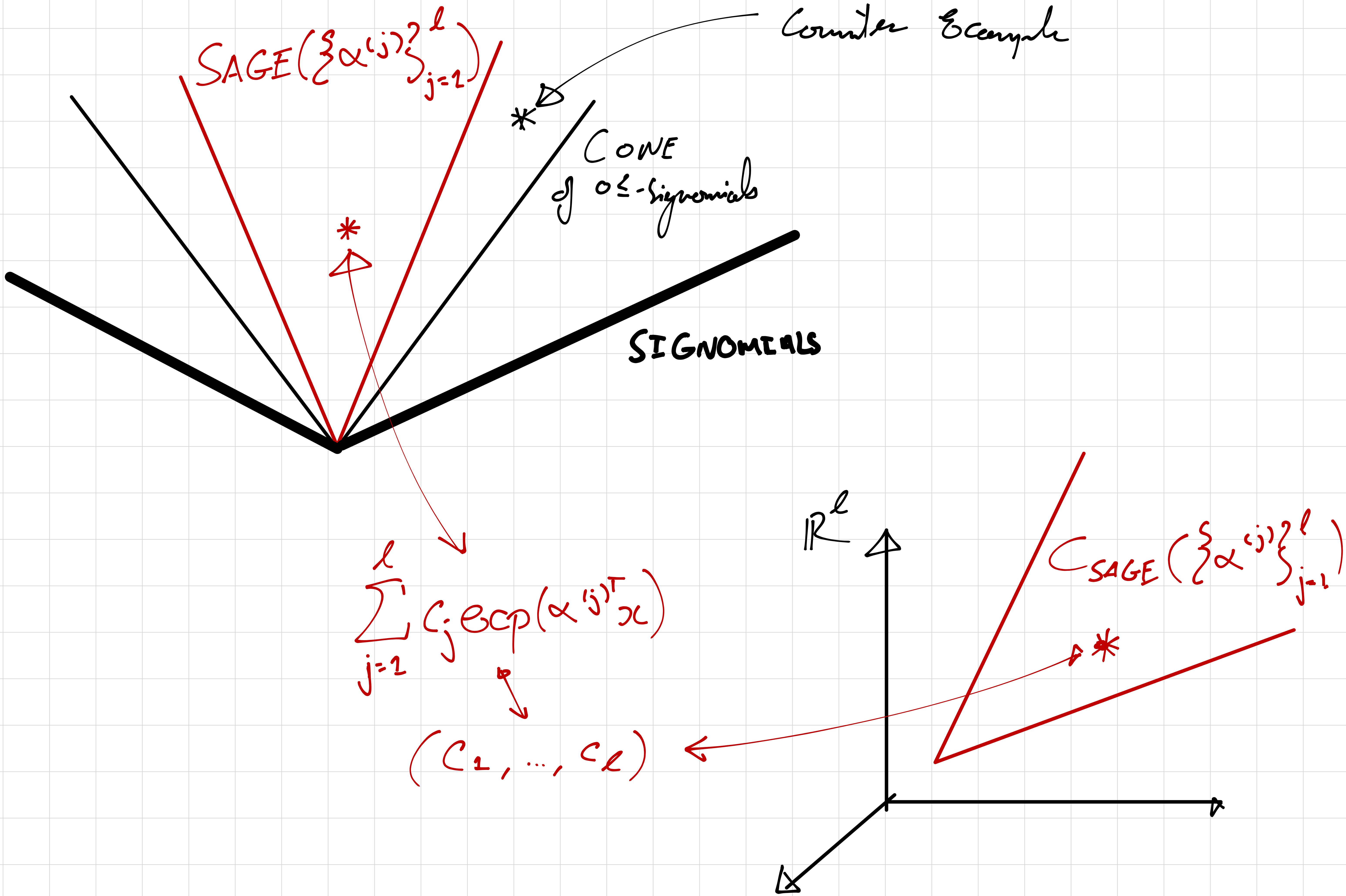
"The SET of all coefficients that correspond to AM/GM-Expo's in exponent $\{\alpha^{(j)}\}_{j=1}^{\ell}$"

<u>Proposition</u>

$$C_{SAGE}(\{\alpha^{(j)}\}_{j=1}^{\ell}) = \{c \in \mathbb{R}^{\ell} \mid \exists\, c^{(j)}, \nu^{(j)} \in \mathbb{R}^{\ell} \text{ s.t. } \sum_{j=1}^{\ell} c^{(j)} = c$$

$$\sum_{i=1}^{\ell} \alpha^{(i)} \nu_i^{(j)} = 0, \quad \nu_j^{(j)} = -\mathbb{1}^T \nu_{\setminus j}^{(j)}, \quad j \in [\ell]$$

$$D(\nu_{\setminus j}^{(j)}, e c_{\setminus j}^{(j)}) - c_j^{(j)} \leq 0, \quad \nu_{\setminus j}^{(j)} \in \mathbb{R}_+^{\ell-1} \}$$


$\mathrm{SAGE}(\{\alpha^{(j)}\}_{j=1}^{\ell})$
Counter Example
CONE
of $0\leq$-Signomials
SIGNOMIALS
$\sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)T} x)$
$(c_1, \dots, c_\ell)$
$\mathbb{R}^\ell$
$C_{\mathrm{SAGE}}(\{\alpha^{(j)}\}_{j=1}^{\ell})$

Signomial Optimization (Unconstrained)

Let f be a signomial in $x \in \mathbb{R}^n$ variables

Unconstrained SP

$$f_* = \inf_{x \in \mathbb{R}^n} f(x)$$

$$\Longleftrightarrow$$

$$f_* = \sup_{\gamma \in \mathbb{R}} \gamma \quad \dots \quad \text{(U-SP)}$$
$$\text{s.t. } f(x) - \gamma \geq 0$$

$\in$ "Nonnegative Signomial CONE"

Relax

Assume w.l.o.g $f(x) = \sum_{j=1}^{\ell} c_j \exp(\alpha^{(j)T} x)$ With $\alpha^{(1)} = 0$ Then

$$f_{SAGE} = \sup_{\gamma \in \mathbb{R}} \gamma$$
$$\text{s.t. } f(x) - \gamma \in SAGE(\{\alpha^{(j)}\}_{j=1}^{\ell})$$

Clearly $f_{SAGE} < f_*$ $\because$ $SAGE(\{\alpha^{(j)}\}_{j=1}^{\ell}) \subseteq$ "Nonnegative Signomial CONE"

$$\Longleftrightarrow$$

$$f_{SAGE} = \sup \gamma$$
$$\text{s.t. } (c_1 - \gamma, c_2, \dots, c_\ell) \in C_{SAGE}(0, \alpha^{(2)}, \dots, \alpha^{(\ell)})$$

This we can solve because the cone smaller

- CVXPY
- ECOS

Where is the Hierarchy?

Idea is to use a Pólya type argument

$$\boxed{\begin{aligned} f^{(p)}_{SAGE} &= \sup_{\gamma \in \mathbb{R}} \gamma \\ \text{s.t. } & \Big(\sum_{j=1}^{\ell} \exp\{\alpha^{(j)T} x\}\Big)^p (f(x)-\gamma) \in SAGE\Big(E_{p+1}(\{\alpha^{(j)}\}_{j=1}^{\ell})\Big) \\ \text{Where } & \\ E_p(\{\alpha^{(j)}\}_{j=1}^{\ell}) &= \Big\{ \sum_{j=1}^{\ell} \lambda_j \alpha^{(j)} \;\Big|\; \sum_{j=1}^{\ell} \lambda_j \le p, \ \lambda_j \in \mathbb{Z}_+ \Big\} \end{aligned}}$$

Recall Pólya (1975)

$p(x) \in \mathbb{R}[x]$ homogeneous, positive on $\mathbb{R}^n_+ \setminus \{0\}$ then for some $N \in \mathbb{N}$, all coefficients of

$$\Big(\sum_{i=1}^{n} x_i\Big)^N p(x) \text{ are positive}$$

" Just to recap the idea

- We check $\Big(\sum_{j=1}^{\ell} \exp\{\alpha^{(j)T} x\}\Big)^p (f(x)-\gamma) \in SAGE\Big(E_{p+1}(\{\alpha^{(j)}\}_{j=1}^{\ell})\Big)$ for higher and higher values of $p \in \mathbb{Z}_+$
- $\Big(\sum_{j=1}^{\ell} \exp\{\alpha^{(j)T} x\}\Big)^p (f(x)-\gamma) \ge 0 \Rightarrow (f(x)-\gamma) \ge 0$

  $\because \Big(\sum_{j=1}^{\ell} \exp\{\alpha^{(j)T} x\}\Big)^p > 0$

Observations:

I) $f^{(p)}_{SAGE} \leq f_* \quad \forall\, p \in \mathbb{Z}_+$ ($\because$ Previous Rem.)

II) $f^{(0)}_{SAGE} = f_{SAGE}$ ($\because$ By def)

III) $f^{(p)}_{SAGE} \leq f^{(p+1)}_{SAGE}$ (Lemma 3.4)

Quick Proof: $g_p(x) := \left(\sum_{j=1}^{l} \exp(\alpha^{(j)^T} x)\right)^p (f(x) - \gamma)$

$$g_{p+1}(x) := \left(\sum_{j=1}^{l} \exp(\alpha^{(j)^T} x)\right)^{p+1} (f(x) - \gamma)$$

Clearly: $g_{p+1}(x) = \sum_{j=1}^{l} \exp(\alpha^{(j)^T} x)\, g_p(x)$

$g_p(x) \in E_p(\{\alpha^{(j)}\}) \Rightarrow \exp(\alpha^{(j)^T} x)\, g_p(x) \in E_{p+1}(\{\alpha^{(j)}\})$

$\Rightarrow g_{p+1}(x) \in E_{p+1}(\{\alpha^{(j)}\})$ $\because$ Closed under $+$

~ Is there a guarantee of $\exists p$ s.t. ...? ~

IV) $f^{(p)}_{SAGE} \to f_*$ as $p \to \infty$

For certain families of Signomials

~ To be elaborated on later. ~

Signomial Optimization (Constrained)

Let $f(x), g_1(x), \ldots, g_m(x)$ be signomials in exponents $\{\alpha^{(j)}\}_{j=1}^{\ell}$

Let $C := \{ g_j(x) \mid j \in [m] \}$

$K_C := \{ x \in \mathbb{R}^n \mid g(x) \geq 0 \; \forall g \in C \}$

I need a name for these domain

Do we need a finite set of g's?

Where do we need K compactness?

Constrained SP

$$f_* = \sup_{\gamma \in \mathbb{R}} \gamma \qquad \text{(CSP)}$$
$$\text{s.t. } f(x) - \gamma \geq 0 \quad \forall x \in K$$

Weak duality

$$f_{WD} = \sup_{\gamma \in \mathbb{R}, \mu \in \mathbb{R}^m_+} \gamma$$
$$\text{s.t. } f(x) - \gamma - \sum_{i=1}^{m} \mu_j g_j(x) \geq 0$$

Two problems:

1. Problem is still intractable to check ≥ 0

2. ~~The problem is Non-convex~~
   ~~"$\Rightarrow$" No strong duality $\Rightarrow$ $f_{WD} \lneq f_*$~~

Solutions:

1. Relax Nonnegativity with SAGE memb.

2. Add "redundant" constraints (how does this Help?)

   ↳ Another trick used alot in POP.

Define: $R_q(C) := \left\{ \prod_{k=1}^{q} h_k \;\middle|\; h_k \in \{1\} \cup C \right\}$

$$f_{SAGE}^{(p,q)} = \sup_{\substack{\gamma \in \mathbb{R} \\ s_h(x) \in SAGE(E_p(\{\alpha^{(j)}\}_{j=1}^{\ell}))}}$$

$$\text{s.t. } f(x) - \gamma - \sum_{h(x) \in R_q(C)} s_h(x) h(x) \in SAGE\left(E_{p+q}\left(\{\alpha^{(j)}\}_{j=1}^{\ell}\right)\right)$$

?

Not tight.

~~Why is it convex now?~~

~~Why is it tractable now?~~

Lemma 3.2

Let $f(x), \{g_j(x)\}_{j=1}^{\ell} = C$ be signomials w.r.t. exponents $\{\alpha^{(j)}\}_{j=1}^{\ell} \subset \mathbb{R}^n$ with $\alpha^{(1)} = 0$

then $f_{SAGE}^{(p,q)} \leq f_{SAGE}^{(p',q')}$ $\quad p \leq p'$, $q \leq q'$ $\quad p, p', q, q' \in \mathbb{Z}_+$

Quick Proof:

$$SAGE(E_p) \subseteq SAGE(E_{p'})$$
$$R_q(C) \subseteq R_{q'}(C)$$
$$SAGE(E_{p+q}) \subseteq SAGE(E_{p'+q'})$$

$\square$

Conclusion: ...