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Support Constraint on Variables in Polynomial Optimization (Not Complete)

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Note type standard
Tags

Given a Polynomial Optimization Problem with the added requirement that at most $k < m$ of the variables can be nonzero, i.e. at most k variables are supported,

$$\begin{aligned} f_* := \min\ & f(\mathbf{x}) \\ s.t.\ & \mathbf{x} \in \mathbb{R}^n \\ & g_j(\mathbf{x}) \geq 0 \;\forall\, j \in [m] \\ & h_k(\mathbf{x}) = 0 \;\forall\, k \in [d] \\ & \mathbf{x}^\alpha = 0 \;\forall\, \alpha \in \mathbb{N}^n \; s.t. \; |\mathrm{supp}(\alpha)| \geq k+1 \\ & = \\ \min\ & L(f) \\ s.t.\ & L : \mathbb{R}\widetilde{[\mathbf{x}]} \\ & L(1) = 1 \\ & M(g_0 L) := L(\widetilde{[\mathbf{x}]}\widetilde{[\mathbf{x}]} \\ & M(g_j L) := L(g_j(\mathbf{x})\widetilde{[\mathbf{x}]}\widetilde{[\mathbf{x}]} \\ & M(h_k L) := L(h_k\widetilde{[\mathbf{x}]}\widetilde{[\mathbf{x}]} \\ & L(\mathbf{x}^\alpha \mathbf{x}^\beta) = 0 \;\forall\, \alpha, \beta \in \mathbb{N}^n_{t-1} \; s.t. \; |\mathrm{supp}(\alpha + \beta)| \geq k+1 \end{aligned}$$

Where $\widetilde{[\mathbf{x}]}$ is the vector of all Definition Monomials that do not contain more than k distinct variables.
Sums of (polynomial) Squares

Proof
Recall Correspondence between Measures and Moments.
Suppose that μ satisfies the first problem and let L be the corresponding linear functional noting that all the moments with support larger than $k+1$ are set to zero.

$$L(\mathbf{x}^\alpha) = \begin{cases} 0 & |\mathrm{supp}(\alpha)| \geq k+1 \\ \int \mathbf{x}^\alpha d\mu & else \end{cases}$$

Hence, we may aswell ignore them. This results in the reformulation with $\widetilde{[\mathbf{x}]}$ in stead of $[\mathbf{x}]$. The rest of the proof follows the same pattern as in Result Polynomial Optimization is Equivalent to Optimization over Probability Measures Result Optimization over Probability Measures is equivalent to Optimization over Moments.
□

How big are the moment matrices

Recall that

$$|[\mathbf{x}]_t| = \binom{n+t}{t}$$

Similarly we have

$$|[\widetilde{\mathbf{x}}]_t| = \sum_{j=k+1}^{t} \binom{j+k-1}{k-1}$$

The reasoning is as follows :

1. All monomials of degree k or less necessarily have support less than $k+1$.
2. Among all monomials of degree $k+1$ the number of them that have support $k+1$ is $n \cdot (n-1) \cdot (n-2) \cdots (n-k) = \frac{n!}{(n-k-1)!}$. The number that have support k or less is given by $\binom{k+1+(k-1)}{k-1}$. You can think of this as having $k+1$ objects to assign to k bins, Stars and Bars.

How does this affect the Sums of Squares Relaxations?

$$\begin{aligned} &\sup_{\lambda \in \mathbb{R}} \ \lambda \\ &s.t. \ \ f(x) - \lambda \in \widetilde{\mathcal{M}}_{2t}(g_1, g_2, \ldots, g_m) + \widetilde{\mathcal{I}}_{2t}(h_1, h_2, \ldots, h_d) \end{aligned} \tag{?}$$

Where

$$\widetilde{\mathcal{M}}(g_1, g_2, \ldots, g_m) := \{\sigma_0 + \sum_{j \in [m]} g_j \sigma_j \ : \ \sigma_0, \sigma_1, \ldots, \sigma_m \in \Sigma[\widetilde{\mathbf{x}}]\}$$

and

$$\widetilde{\mathcal{I}}(h_1, h_2, \ldots, h_d) := \{\sum_{k \in [d]} h_k p_k \ : \ p_1, p_2, \ldots, p_d \in \mathbb{R}[\widetilde{\mathbf{x}}]\}$$

Consequences (TO DO)

- Find more literature on this topic.
- Convergence to a sparse solution
- If there is flatness does that mean we can extend flatly and keep the sparsity?
- Is there a bound on the difference between the full solution and the sparse one?
- What does this cardinality constraint do in the case of signomial optimization?
- Does it synergize with term/corr. sparsity?
- What other problems can be phrased like this?
  - Selection type problems.

Another way of looking at the constraint

$$L(\mathbf{x}^\alpha \mathbf{x}^\beta) = 0 \ \forall \ \alpha, \beta \in \mathbb{N}^n_{t-1} \ s.t. \ |\mathrm{supp}(\alpha + \beta)| \geq k+1$$

is to look at the Hamming Distance of the supports of all pairs.....(Need to think on this.)