Multiple Orthogonal Polynomials in Two Variables

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1. Introduction

Multiple Orthogonality is a theory that extends standard orthogonality. In it, polynomials (defined on the real line) satisfy orthogonality relations with respect to more than one measure. There are also two different types of multiple orthogonality.

First, let us consider r different real measures $\mu_1, \dots, \mu_r$ such that $\Omega_j = \operatorname{supp}(\mu_j) \subseteq \mathbb{R}$ and denote as $\langle \cdot, \cdot \rangle_j$ the respective integral inner product $(j = 1, \dots, r)$. We will use multi-indices $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, and denote $|\vec{n}| := n_1 + \cdots + n_r$. These multi-indices determine the orthogonality relations with each measure.

Definition 1 (Type II Multiple Orthogonal Polynomials). *Let $\vec{n} = (n_1, \dots, n_r)$. A monic polynomial $P_{\vec{n}}(x)$ is a **type II multiple orthogonal polynomial** if $\deg(P_{\vec{n}}) = |\vec{n}|$ and*

$$\boxed{\left\langle P_{\vec{n}}, x^k \right\rangle_j = 0, \quad k = 0, \dots, n_j - 1, \ j = 1, \dots, r} \tag{1}$$

Definition 2 (Type I Multiple Orthogonal Polynomials). *Let $\vec{n} = (n_1, \dots, n_r)$. **Type I Multiple Orthogonal Polynomials** are presented in a vector $(A_{\vec{n},1}(x), \dots, A_{\vec{n},r}(x))$, where $\deg(A_{\vec{n},j}) \leq n_j - 1$, $(j = 1, \dots, r)$ and satisfy*

$$\boxed{\sum_{j=1}^{r} \left\langle A_{\vec{n},j}, x^k \right\rangle_j = \begin{cases} 0 & \text{if} \quad k = 0, \dots, |\vec{n}| - 2 \\ 1 & \text{if} \quad k = |\vec{n}| - 1. \end{cases}} \tag{2}$$

Whenever the measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup_{i=1}^{r} \Omega_i$, *i.e.*, $d\mu_j = w_j(x) d\mu(x)$, $(j = 1, \dots, r)$, it is possible to define the *Type I function* as

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x). \tag{3}$$

Using the type I function, we can rewrite (2) as

$$\left\langle Q_{\vec{n}}, x^k \right\rangle_\mu = \begin{cases} 0 & \text{if} \quad k = 0, \dots, |\vec{n}| - 2 \\ 1 & \text{if} \quad k = |\vec{n}| - 1 \end{cases} \tag{4}$$

2. Bivariate OP

In order to work with bivariate polynomials, we will use the notation $\mathbb{P}_n$ as a column polynomial vector. Let us denote the vector of degree j monomials as

$$\mathbb{X}_j = (x^j, x^{j-1}y, \dots, y^j)^t.$$

A column polynomial vector of degree n can be represented as

$$\mathbb{P}_n = G_{n,n}\mathbb{X}_n + G_{n,n-1}\mathbb{X}_{n-1} + \cdots + G_{n,1}\mathbb{X}_1 + G_{n,0}\mathbb{X}_0,$$

where $G_{n,j}$ are matrices of size $(n+1) \times (j+1)$.

Given a bidimensional measure $\mu(x, y)$, with support $\Omega \subseteq \mathbb{R}^2$, we can extend the definition of inner product $\langle f, g \rangle_\mu$ to column vectors. If $F = (f_1, f_2, \dots, f_n)^t$ and $G = (g_1, g_2, \dots, g_m)^t$ are column vectors of functions, then we define

$$\langle F, G \rangle_\mu := \mathcal{L}_\mu[F \cdot G^T] = \int_\Omega F \cdot G^T d\mu = \left(\int_\Omega f_i \cdot g_j d\mu \right)_{i,j=1}^{n,m}. \tag{5}$$

3. Bivariate Type II MOP

Given $r \in \mathbb{N}$, $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, r 2-dimensional measures $\mu_1, \dots, \mu_r$ and their respective matrix inner products defined in (5), we define the type II multiple orthogonal polynomial vector as a monic polynomial vector

$$\mathbb{P}_{\vec{n}} = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k}\mathbb{X}_k$$

which satisfies

$$\boxed{\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_k \rangle_j = 0_{(n+1)\times(k+1)}, \quad k = 0, \dots, n_j - 1, j = 1, \dots, r.} \tag{6}$$

Since $G_{n,k}$ are matrices whose dimensions are $(n+1) \times (k+1)$, if we consider $\mathbb{P}_{\vec{n}}$ a vector of degree n monic polynomials, then this matrices give us $\frac{1}{2}n(n+1)^2$ unknown coefficients. We want the system to have only one solution for a multi-index $\vec{n}$. Then, $\vec{n}$ is a **valid** multi-index if there exist a number $n \in \mathbb{N}_0$ such that:

$$\boxed{n(n+1) = \sum_{j=1}^{r} n_j(n_j + 1).} \tag{7}$$

This number n is the degree of type II polynomials. In order to emphasise the degree of the polynomials, we denote type II MOP as $\mathbb{P}_{\vec{n}}^n$.

4. Bivariate Type I MOP

Let $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ such that condition (7) holds for some $n \in \mathbb{N}_0$. Then, let $i \in \{1, \dots, n\}$ and, for each j, we define $A_{\vec{n},j}^{(i)}(x, y)$, a bivariate polynomial of degree $\leq n_j - 1$, $(j = 1, \dots, r)$. Now, given r 2-dimensional measures $\mu_1, \dots, \mu_r$, the polynomials $A_{\vec{n},1}^{(i)}, \dots, A_{\vec{n},r}^{(i)}$ satisfy

$$\sum_{j=1}^{r} \left\langle \mathbb{X}_k, A_{\vec{n},j}^{(i)} \right\rangle_j = \begin{cases} 0_{(k+1)\times 1} & \text{if} \quad k = 0, \dots, n-2 \\ (e_i)_{n \times 1} & \text{if} \quad k = n-1 \end{cases} \tag{8}$$

If we repeat this for every possible $i \in \{1, \dots, n\}$, we get n lists of polynomials $A_{\vec{n},j}^{(i)}$, $j = 1, \dots, r, i = 1, \dots, n$. Now, let us denote

$$\mathbb{A}_{\vec{n},j} = (A_{\vec{n},j}^{(1)}, \dots, A_{\vec{n},j}^{(n)})^t,$$

a polynomial vector of size n whose components are bivariate polynomials of degree less than or equal to $n_j - 1$ $(j = 1, \dots, r)$. Thus, the polynomial vectors $\mathbb{A}_{\vec{n},j}$ will satisfy:

$$\boxed{\sum_{j=1}^{r} \langle \mathbb{X}_k, \mathbb{A}_{\vec{n},j} \rangle_j = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \dots, n-2 \\ I_n & \text{if} \quad k = n-1 \end{cases}} \tag{9}$$

When the 2-dimensional measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup_{j=1}^{r} \Omega_j$ (with $\Omega_j = \operatorname{supp}(\mu_j)$), *i.e.*, $d\mu_j = w_j(x, y) d\mu(x, y)$, $(j = 1, \dots, r)$, we can define the *Type I function*:

$$\mathbb{Q}_{\vec{n}} = \sum_{j=1}^{r} \mathbb{A}_{\vec{n},j} w_j(x, y). \tag{10}$$

Using this function, it is possible to rewrite (9) as

$$\langle \mathbb{X}_k, \mathbb{Q}_{\vec{n}} \rangle_\mu = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \dots, n-2 \\ I_n & \text{if} \quad k = n-1. \end{cases} \tag{11}$$

Despite their differences, type I and type II MOP in two variables are equivalent.

Proposition 1. *Let $\vec{n} = (n_1, \dots, n_r)$ be a multi-index satisfying (7) for $n \in \mathbb{N}_0$. Then the following statements are equivalent:*

1. *There exist a unique polynomials $(\mathbb{A}_{\vec{n},1}, \dots, \mathbb{A}_{\vec{n},r})$ of bivariate type I MOP.*
2. *There exist a unique bivariate type II multiple orthogonal polynomial $\mathbb{P}_{\vec{n}}^n$.*

We will say $\vec{n}$ is a **normal** multi-index if it satisfies the conditions of Proposition 1. A system $\mu_1, \dots, \mu_r$ of bivariate measures is **perfect** if every multi-index satisfying (7) is normal.

Valid multi-indices for $r = 2, r = 3$

Observe some of the valid multi-indices which satisfy (7) and their respective degree n in the following table.

n	$r = 2$ indices	$r = 3$ indices
1	$(0,1), (1,0)$	$(0,0,0)$
2	$(0,2), (2,0)$	$(0,0,1), (0,1,0), (1,0,0)$
0	$(0,0)$	$(0,0,2), (0,2,0), (1,1,1), (2,0,0)$
3	$(0,3), (2,2), (3,0)$	$(0,0,3), (0,2,2), (0,3,0), (2,0,2), (2,2,0), (3,0,0)$
4	$(0,4), (4,0)$	$(0,0,4), (0,4,0), (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (4,0,0)$
5	$(0,5), (5,0)$	$(0,0,5), (0,5,0), (2,3,3), (3,2,3), (3,3,2), (5,0,0)$
6	$(0,6), (3,5), (5,3), (6,0)$	$(0,0,6), (0,3,5), (0,5,3), (0,6,0), (1,4,4), (2,2,5), (2,5,2), (3,0,5), (3,5,0), (4,1,4), (4,4,1), (5,0,3), (5,2,2), (5,3,0), (6,0,0)$
7	$(0,7), (7,0)$	$(0,0,7), (0,7,0), (1,3,6), (1,6,3), (2,4,5), (2,5,4), (3,1,6), (3,6,1), (4,2,5), (4,5,2), (5,2,4), (5,4,2), (6,1,3), (6,3,1), (7,0,0)$

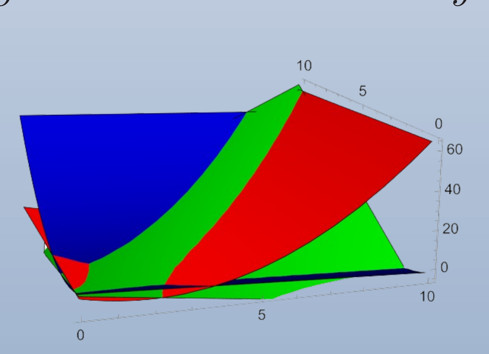
5. Numerical Example

Starting from the well-known Laguerre product polynomials in [1] and the generalisation to multiple orthogonality of classical Laguerre polynomials in [2], we have implemented $\mathbb{P}_{(1,1,1)}^2$, a multiple orthogonal polynomial vector with respect to the measures given by $d\mu_j = x^{\alpha_j} y^{\beta_j} e^{-x-y} d(x, y)$ $(i = 1, 2, 3)$ and the multi-index $(1, 1, 1)$. We have choosen the values

$$\alpha_1 = 0, \alpha_2 = 0.5, \alpha_3 = 1.3; \beta_1 = 0.8, \beta_2 = 0.4, \beta_3 = 2.1,$$

and got the polynomial vector

$$\mathbb{P}_{(1,1,1)}^2 = \begin{pmatrix} x^2 + -3.85556x - 0.444444y + 2.65556 \\ xy - 2.15556x - 1.94444y + 3.85556 \\ y^2 - 0.755556x - 5.14444y + 4.97556 \end{pmatrix}$$



6. Biorthogonality

There is a relation between type I and type II MOP:

Theorem 1. *Let $\mu_1, \dots, \mu_r$ be a perfect system of r 2-dimensional measures. Given $\vec{n}, \vec{m}$, both satisfying (7) for n and m, respectively, the following biorthogonality holds for type I and type II MOP:*

$$\boxed{\langle \mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\vec{m}} \rangle = \begin{cases} 0_{(n+1)\times m} & \text{if} \quad \vec{m} \leq \vec{n} \\ 0_{(n+1)\times m} & \text{if} \quad n \leq m-2 \\ I_{n+1} & \text{if} \quad n = m-1. \end{cases}} \tag{12}$$

7. Nearest Neighbor Relation

In univariate MOP, the nearest neighbor relation is a generalisation of the TTRR satisfied by any OPS. This is a extension to bivariate MOP.

Theorem 2. *Let $\mu_1, \dots, \mu_r$ be a perfect system of 2-dimensional measures. Let $\vec{n} \in \mathbb{N}^r$ be a multi-index satisfying (7) for $n \in \mathbb{N}$ and let us consider a path $\{\vec{m}_k : k = 0, \dots, n+1\}$ where $\vec{m}_0 = \vec{0}, \vec{m}_n = \vec{n}$, each $\vec{m}_k$ satisfy (7) for k and $\vec{m}_k \leq \vec{m}_{k+1}$ for $k = 0, \dots, n$. Then, there exist matrices $A_0, \dots, A_r$ of sizes $(n+1) \times (n-j+1)$, $(j = 0, \dots, r)$ such that*

$$\boxed{x\mathbb{P}_{\vec{n}}^n = L_{n+1,1}\mathbb{P}_{\vec{m}_{n+1}}^{n+1} + A_0\mathbb{P}_{\vec{n}}^n + \sum_{j=1}^{r} A_j \mathbb{P}_{\vec{m}_{n-j}}^{n-j},} \tag{13}$$

where $L_{n+1,1} = (I_{n+1} | 0_{(n+1)\times 1})$.

In addition, $A_j = \left\langle x\mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\vec{m}_{n-j+1}} \right\rangle$.

8. Future research topics

Multiple Orthogonal Polynomials is a theory that is currently being widely researched. At this moment, we have generalised to the bivariate case some of the main univariate results. But there are still many results and applications to work on:

- Find a Nearest Neighbor Relation for more general paths and other types of Nearest Neighbor Relations.
- Christoffel-Darboux formula.
- Look for applications to Hermite-Padé approximants and random matrices.
- Generalise existing examples to the bivariate case: Jacobi-Piñeiro, Jacobi-Angelesco, OP defined in the Simplex, etc.
- ...

References

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