Multiple Orthogonal Polynomials in several variables

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December 6, 2023

Abstract

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1 Introduction

Multiple Orthogonality is an extension of the standard orthogonality. It consists of Polynomials (defined on the real line in this introduction) that satisfy orthogonality relations with respect to more than one measure. There are also two different types of multiple orthogonality, which will be explained shortly.

First, let's consider r different real measures $\mu_1, \dots, \mu_r$ such that $\Omega_i \mathrm{supp}(\mu_i) \subseteq \mathbb{R}$ $(i = 1, \dots, r)$. We will use multi-indexes $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, and denote $n := |\vec{n}| = n_1 + \cdots + n_r$. These multi-indexes determine the orthogonality relations with each measure. With these preliminaries we will present the first definition.

Definition 1.1 (Type II Multiple Orthogonal Polynomials)**.** *Let* $\vec{n} = (n_1, \dots, n_r)$*. A monic polynomial* $P_{\vec{n}}(x)$ *is a type II multiple orthogonal polynomial if* $\deg(P_{\vec{n}}) \leq n$ *and*

$$\int_{\Omega_i} P_{\vec{n}}(x) x^k d\mu_i(x) = 0, \quad k = 0, \dots, n_i - 1, \quad i = 1, \dots, r \tag{1}$$

This means $P_{\vec{n}}$ is orthogonal to $1, x, x^2, \dots, n^{n_i - 1}$ with respect to each measure μ_i, $(i = 1, \dots, r)$. If we define the product $\langle f, g \rangle_i = \int_{\Omega_i} f(x) g(x) d\mu(x)$, conditions (1) can also be written as

$$\left\langle P_{\vec{n}}, x^k \right\rangle_i = 0, \quad k = 0, \dots, n_i - 1, \quad i = 1, \dots, r \tag{2}$$

There is another type of multiple orthogonality: The type I MOP.

Definition 1.2 (Type I Multiple Orthogonal Polynomials)**.** *Let* $\vec{n} = (n_1, \ldots, n_r)$*. Type I Multiple Orthogonal Polynomials are presented in a vector* $(A_{\vec{n},1}(x), \ldots, A_{\vec{n},r}(x))$*, where* $\deg(A_{\vec{n},i}) \leq n_i - 1$*,* $(i = 1, \ldots, r)$ *and these polynomials satisfy the relation*

$$\sum_{i=1}^{r} \int_{\Omega_i} A_{\vec{n},i}(x) x^k d\mu_i(x) = 0, \quad k = 0, \ldots, n-2 \tag{3}$$

and the normalization condition

$$\sum_{i=1}^{r} \int_{\Omega_i} A_{\vec{n},i}(x) x^{n-1} d\mu_i(x) = 1. \tag{4}$$

As we did previously with the type II MOP, the relations (3) and (4) can be written, using the dot products, as

$$\sum_{i=1}^{r} \left\langle A_{\vec{n},i}, x^k \right\rangle_i = \begin{cases} 0 & \text{if} \quad k = 0, \ldots, n-2 \\ 1 & \text{if} \quad k = n-1 \end{cases} \tag{5}$$

Whenever the measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup_{i=1}^{r} \Omega_i$, i.e., $d\mu_i = w_i(x) d\mu(x)$, $(i = 1, \ldots, r)$, it is possible to define the *Type I function* as

$$Q_{\vec{n}}(x) = \sum_{i=1}^{r} A_{\vec{n},i}(x) w_i(x). \tag{6}$$

Using the type I function, we can rewrite the orthogonality relations

$$\int_{\Omega} Q_{\vec{n}}(x) x^k d\mu(x) = \begin{cases} 0 & \text{if} \quad k = 0, \ldots, n-2 \\ 1 & \text{if} \quad k = n-1 \end{cases} \tag{7}$$

Nevertheless, not every multi-index $\vec{n} \in \mathbb{N}^r$ provides a type I vector of polynomials or a type II polynomial. Let $\vec{n} = (n_1, \ldots, n_r)$ be a multi-index and let $\mu_1, \ldots, \mu_r$ be r positive measures. If $(A_{\vec{n},1}(x), \ldots, A_{\vec{n},r}(x))$ is the vector of type I MOP, then, if we denote

$$\begin{aligned} A_{\vec{n},1}(x) &= a_{n_1-1,1} x^{n_1-1} + a_{n_1-2,1} x^{n_1-2} + \cdots + a_{1,1} x + a_{0,1} \\ &\vdots \\ A_{\vec{n},r}(x) &= a_{n_r-1,r} x^{n_r-1} + a_{n_r-2,r} x^{n_r-2} + \cdots + a_{1,r} x + a_{0,r} \end{aligned} \tag{8}$$

and apply the orthogonality conditions (3) and (4), then we get a linear system of n equations and n unknown coefficients. Thus, the type I MOP will exist if and only if the following matrix is regular:

$$A = \begin{pmatrix} M_{n_1}^{(1)} \\ \hline M_{n_2}^{(2)} \\ \hline \vdots \\ \hline M_{n_r}^{(r)} \end{pmatrix}, \quad \text{where} \quad M_{n_j}^{(j)} = \begin{pmatrix} m_0^{(j)} & m_1^{(j)} & \cdots & m_{n-1}^{(j)} \\ m_1^{(j)} & m_2^{(j)} & \cdots & m_n^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n_j-1}^{(j)} & m_{n_j}^{(j)} & \cdots & m_{n+n_j-2}^{(j)} \end{pmatrix}, \tag{9}$$

$j = 1, \dots, r$ and $m_k^{(j)} = \displaystyle\int_{\Omega_j} x^k d\mu_j(x)$ are the moments of the measures ($k \geq 0$).

On the other hand, if we consider the type II polynomial as:

$$P_{\vec{n}} = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a0$$

and apply the conditions (1), we get another linear system with n equations and n unknown coefficients. In fact, the coefficients matrix of this linear system is A^t, the transpose matrix of the type I MOP. Thus, we have the following result.

Proposition 1.3. *Given a multi-index $\vec{n} \in \mathbb{N}^r$ and r positive measures, $\mu_1, \dots, \mu_r$, the following statements are equivalent:*

1. *There exist an unique vector $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$ of type I MOP.*
2. *There exist an unique type II multiple orthogonal polynomial $P_{\vec{n}}$.*
3. *The matrix A defined in (9) is regular.*

Following this proposition, we provide a new definition.

Definition 1.4. *A multi-index $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ is **normal** if it satisfies the conditions of the proposition 1.3. A system of r measures $\mu_1, \dots, \mu_r$ is **perfect** if every $\vec{n} \in \mathbb{N}^r$ is normal.*

There are some perfect systems, standing out the Angelesco systems and the AT-systems, see [3, Sections 23.1.1 and 23.1.2]

We will be mainly focused on the type II multiple orthogonal polynomials. In the next sections, a definition of type II multiple orthogonal polynomials on several variables will be provided, and also some easy examples and a generalized version of the Jacobi-Piñeiro polynomials.

2 Orthogonal Polynomials Systems in several variables

First of all, we will introduce the notation that will be used. Let $\Pi^d = \mathbb{R}[x_1, \dots, x_d]$ be the space of polynomials in d variables. If $d = 2$, we will use the variables x, y. For $n \in \mathbb{N}_0$, the space generated by all the degree n monomials is denoted by

$$\mathcal{P}_n^d = \left\langle x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} : k_1 + k_2 + \cdots + k_d = n, k_1, k_2, \dots, k_d \in \mathbb{N}_0 \right\rangle$$

It is possible to check by induction that the number of different monomials of degree n with d variables is $\binom{n+d-1}{n}$. So, this means

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{n}.$$

In order to work with multivariate polynomials, we will use the notation $\mathbb{P}_n$ as a column polynomial vector. In order to understand this notation, we denote the vector of degree j monomials as

$$\mathbb{X}_j = \left(x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}\right)_{k_1+k_2+\cdots+k_d=j}.$$

For example, if $d = 2$ and we use the "degree reverse lexicographic ordering" in $\mathbb{N}^2$, then

$$\mathbb{X}_0 = (1), \mathbb{X}_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbb{X}_2 = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}, \ldots, \mathbb{X}_j = \begin{pmatrix} x^j \\ x^{j-1}y \\ \vdots \\ y^j \end{pmatrix}.$$

Thus, a column polynomial vector of degree n can be represented as

$$\mathbb{P}_n = G_{n,n}\mathbb{X}_n + G_{n,n-1}\mathbb{X}_{n-1} + \cdots G_{n,1}\mathbb{X}_1 + G_{n,0}\mathbb{X}_0,$$

where $G_{n,j}$ are matrices of size $\binom{n+d-1}{n} \times \binom{j+d-1}{j}$ and they are the polynomial's "coefficients". Hence, $\mathbb{P}_n$ is a vector of polynomials whose size is $\binom{n+d-1}{n}$. We will denote

$$r_n^d = \dim \mathcal{P}_n^d = \binom{n+d-1}{n}.$$

Given a multi-dimensional measure $\mu(x_1, \ldots, x_d)$, whose support is $\Omega \subseteq \mathbb{R}^d$, we can extend the definition of a multivariate product $\langle f, g \rangle_\mu$ and its functional $\mathcal{L}_\mu[f \cdot g]$ to column vectors. If $F = (f_1, f_2, \ldots, f_n)^t$ and $G = (g_1, g_2, \ldots, g_m)^t$ are column vectors of functions of size n and m, respectively, then we define

$$\langle F, G \rangle := \mathcal{L}_\mu[F \cdot G^T] = \int_\Omega F \cdot G^T d\mu(x_1, \ldots, x_d) = \left(\int_\Omega f_i \cdot g_j d\mu(x_1, \ldots, x_d)\right)_{i=1,\ldots,n;\ \ j=1,\ldots,m}. \tag{10}$$

In fact, we are applying the standard product $\langle f_i, g_j \rangle_\mu$ or the functional $\mathcal{L}_\mu[f_i \cdot g_j]$ to each pair i, j and placing the results in a matrix.

Let's consider it is possible to build a system of polynomials $\{\mathbb{P}_n\}_{n \geq 0}$ where

$$\langle \mathbb{P}_n, \mathbb{P}_k \rangle_\mu = \mathcal{L}_\mu[\mathbb{P}_n \mathbb{P}_k^t] = \begin{cases} 0_{r_n^d \times r_k^d} & \text{if} \quad k = 0, \ldots, n-1 \\ S_n & \text{if} \quad k = n \end{cases}$$

Where S_n is a regular squared matrix of size $r_n^d \times r_n^d$. Due to orthogonality, it is possible to give a equivalent condition:

$$\langle \mathbb{P}_n, \mathbb{X}_k \rangle_\mu = \mathcal{L}_\mu[\mathbb{P}_n \mathbb{X}_k^t] = \begin{cases} 0_{r_n^d \times r_k^d} & \text{if} \quad k = 0, \ldots, n-1 \\ S_n & \text{if} \quad k = n \end{cases} \tag{11}$$

Then, $\{\mathbb{P}_n\}_{n \geq 0}$ is called a system of orthogonal polynomials with respect to the measure μ or the functional $\mathcal{L}_\mu$. You can check more information about the existence and uniqueness of this system in [1, Ch. III, Section 3.2].

3 Type II MOP in several variables

In order to respect the original Type II MOP in one variable, given $r \in \mathbb{N}$, $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, r d-dimensional measures $\mu_1, \ldots, \mu_r$ and their respective dot products defined in (10), we are going to define the type II multiple orthogonality as a monic polynomial vector $\mathbb{P}_n = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k} \mathbb{X}_k$ which satisfies

$$\langle \mathbb{P}_n, \mathbb{X}_k \rangle_i = 0, \quad k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r \tag{12}$$

You can check the similarity between conditions (2) and (12). Nevertheless, when we work with one variable, it is known the type II polynomial is a monic polynomial whose degree is exactly $|\vec{n}|$. That is because the number of coefficients of a degree $|\vec{n}|$ univariate monic polynomial ($|\vec{n}|$) is equal to the number of orthogonality conditions and the size of matrices A defined in (9). Due to the differences between the univariate and the multivariate case, the degree n of the polynomial vector $\mathbb{P}_n$ might not be equal to $|\vec{n}|$. For now, this degree n will be considered as unknown, getting to know it later.

As mencioned previously, in the univariate multiple orthogonality, $\deg(P_{\vec{n}}) = |\vec{n}|$. The main reason this happens is because the condition $\left\langle P_{\vec{n}}, x^k \right\rangle = 0$ is a linear equation (only one). Hence, we are looking for a polynomial with $|\vec{n}|$ unknown coefficients and we have $|\vec{n}|$ linear equations deducted from the conditions (2). Then, it is possible to build a system of linear equations whose coefficient matrix is A^t given in (9).

If we work with d variables, attending to (11), remember the result of $\langle \mathbb{P}_n, \mathbb{X}_k \rangle$ is a matrix of size $r_n^d \times r_k^d$. Thus, it is possible to get $r_n^d \times r_k^d$ linear equations from each orthogonality condition.

Now, the question is: How many linear equations is it possible to get from a multi-index $\vec{n} = (n_1, \ldots, n_r)$? The answer to this problem is easy since, if we fix $i \in \{1, \ldots, r\}$:

- $\langle \mathbb{P}_n, \mathbb{X}_0 \rangle_i = 0_{r_n^d \times r_0^d}$, we get $r_n^d \times r_0^d$ equations.
- $\langle \mathbb{P}_n, \mathbb{X}_1 \rangle_i = 0_{r_n^d \times r_1^d}$, we get $r_n^d \times r_1^d$ equations.
- $\ldots$
- $\langle \mathbb{P}_n, \mathbb{X}_{n_i-1} \rangle_i = 0_{r_n^d \times r_{n_i-1}^d}$, we get $r_n^d \times r_{n_i-1}^d$ equations.

If we collect all the equations, we obtain $r_n^d \cdot \sum_{k=0}^{n_i-1} r_k^d$ linear equations for each $i \in \{1, \ldots, r\}$. Collecting the number of equations of each i, we finally get the number of equations is

$$r_n^d \sum_{i=1}^{r} \sum_{k=0}^{n_i-1} r_k^d. \tag{13}$$

On the other hand, since $\mathbb{P}_n = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k}\mathbb{X}_k$, where $G_{n,k}$ are matrices whose dimensions are $r_n^d \times r_k^d$, if we consider $\mathbb{P}_n$ a vector of degree n monic polynomials, then this matrices give us $\sum_{k=0}^{n-1} r_n^d r_k^d = r_n^d \sum_{k=0}^{n-1} r_k^d$ unknown coefficients.

If we want this system to have only one solution for a multi-index $\vec{n}$. Then, $\vec{n}$ is a valid multi-index if there exist a number $n \in \mathbb{N} \cup \{0\}$ such that:

$$\sum_{k=0}^{n-1} r_k^d = \sum_{i=1}^{r} \sum_{k=0}^{n_i-1} r_k^d. \tag{14}$$

4 The bidimensional case

Since this moment, we will assume $d = 2$ and $x := x_1, y := x_2$. Notice that, in the bidimensional case, $r_n^2 = \binom{n+1}{n} = n + 1$, which makes things much easier: $\mathbb{X}_j$ is a polynomial vector of size $j + 1$, $\mathbb{P}_n$ is a vector of size $n + 1$, and $G_{n,k}$ is a $(n + 1) \times (k + 1)$ matrix. Also, the condition (14) gets simpler because some sums become sums of an arithmetic progression. When $d = 2$, a multi-index $\vec{n} = (n_1, \dots, n_r)$ is valid if there exists a number $n \in \mathbb{N} \cup \{0\}$ such that

$$n(n + 1) = \sum_{i=1}^{r} n_i(n_i + 1). \tag{15}$$

Observe some of the valid multi-indexes and their respective degree n.

n	Valid multi-indexes when $r = 2$ and $d = 2$
0	$(0, 0)$
1	$(0, 1), (1, 0)$
2	$(0, 2), (2, 0)$
3	$(0, 3), (2, 2), (3, 0)$
4	$(0, 4), (4, 0)$
5	$(0, 5), (5, 0)$
6	$(0, 6), (3, 5), (5, 3), (6, 0)$
7	$(0, 7), (7, 0)$
8	$(0, 8), (5, 6), (6, 5), (8, 0)$
9	$(0, 9), (9, 0)$

Table 1: Valid multi-indexes when using $r = 2$ measures

Check this visual representation of the valid multi-indexes when $r = 2$ and $r = 3$.

5 Conclusion

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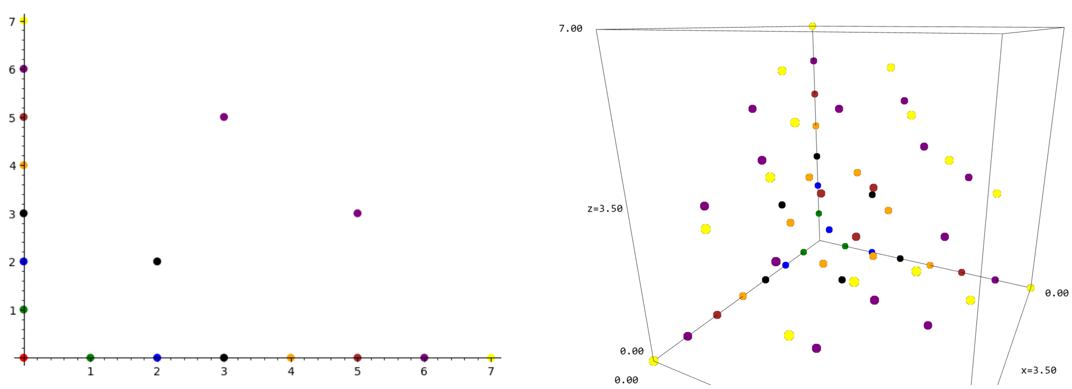
n	Valid multi-indexes when $r = 3$ and $d = 2$
0	$(0,0,0)$
1	$(0,0,1), (0,1,0), (1,0,0)$
2	$(0,0,2), (0,2,0), (1,1,1), (2,0,0)$
3	$(0,0,3), (0,2,2), (0,3,0), (2,0,2), (2,2,0), (3,0,0)$
4	$(0,0,4), (0,4,0), (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (4,0,0)$
5	$(0,0,5), (0,5,0), (2,3,3), (3,2,3), (3,3,2), (5,0,0)$
6	$(0,0,6), (0,3,5), (0,5,3), (0,6,0), (1,4,4), (2,2,5), (2,5,2), (3,0,5), (3,5,0), (4,1,4), (4,4,1), (5,0,3), (5,2,2), (5,3,0), (6,0,0)$
7	$(0,0,7), (0,7,0), (1,3,6), (1,6,3), (2,4,5), (2,5,4), (3,1,6), (3,6,1), (4,2,5), (4,5,2), (5,2,4), (5,4,2), (6,1,3), (6,3,1), (7,0,0)$
8	$(0,0,8), (0,5,6), (0,6,5), (0,8,0), (3,5,5), (5,0,6), (5,3,5), (5,5,3), (5,6,0), (6,0,5), (6,5,0), (8,0,0)$
9	$(0,0,9), (0,9,0), (2,3,8), (2,6,6), (2,8,3), (3,2,8), (3,8,2), (5,5,5), (6,2,6), (6,6,2), (8,2,3), (8,3,2), (9,0,0)$

Table 2: Valid multi-indexes when using $r = 3$ measures



(a) $r = 2$ (b) $r = 3$

Figure 1: Graphic representation of the valid r-indexes

References

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