Multiple Orthogonal Polynomials in several variables

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Abstract

TODO Write the abstract

1 Introduction

Multiple Orthogonality is a theory that extends standard orthogonality. In it, polynomials (defined on the real line in this introduction) satisfy orthogonality relations with respect to more than one measure. There are also two different types of multiple orthogonality, which will be explained shortly.

First, let's consider r different real measures $\mu_1, \dots, \mu_r$ such that $\Omega_j = \operatorname{supp}(\mu_j) \subseteq \mathbb{R}$ $(j = 1, \dots, r)$. We will use multi-indices $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, and denote $|\vec{n}| := n_1 + \dots + n_r$. These multi-indices determine the orthogonality relations with each measure. With these preliminaries we will present the first definition.

Definition 1.1 (Type II Multiple Orthogonal Polynomials)**.** *Let* $\vec{n} = (n_1, \dots, n_r)$*. A monic polynomial* $P_{\vec{n}}(x)$ *is a* ***type II multiple orthogonal polynomial*** *if* $\deg(P_{\vec{n}}) = |\vec{n}|$ *and*

$$\int_{\Omega_j} P_{\vec{n}}(x) x^k d\mu_j(x) = 0, \quad k = 0, \dots, n_j - 1, \quad j = 1, \dots, r \tag{1}$$

This means $P_{\vec{n}}$ is orthogonal to $1, x, x^2, \dots, x^{n_j - 1}$ with respect to each measure μ_j, $(j = 1, \dots, r)$. If we define the inner product

$$\langle f, g \rangle_j = \int_{\Omega_j} f(x) g(x) d\mu_j(x) \tag{2}$$

conditions (1) can also be written as

$$\left\langle P_{\vec{n}}, x^k \right\rangle_j = 0, \quad k = 0, \dots, n_j - 1, \quad j = 1, \dots, r \tag{3}$$

In addition to type II there exists another type of multiple orthogonality: type I multiple orthogonality.

Definition 1.2 (Type I Multiple Orthogonal Polynomials)**.** *Let* $\vec{n} = (n_1, \dots, n_r)$*. Type I Multiple Orthogonal Polynomials are presented in a vector* $(A_{\vec{n},1}(x), \dots, A_{\vec{n},r}(x))$*, where* $\deg(A_{\vec{n},j}) \leq n_j - 1$*,* $(j = 1, \dots, r)$ *and these polynomials satisfy*

$$\sum_{j=1}^{r} \int_{\Omega_j} A_{\vec{n},j}(x) x^k d\mu_j(x) = 0, \quad k = 0, \dots, |\vec{n}| - 2 \tag{4}$$

and the normalization condition

$$\sum_{j=1}^{r} \int_{\Omega_j} A_{\vec{n},j}(x) x^{|\vec{n}|-1} d\mu_j(x) = 1. \tag{5}$$

As we did previously with the type II MOP, the relations (4) and (5) can be written, using the inner products (2), as

$$\sum_{j=1}^{r} \left\langle A_{\vec{n},j}, x^k \right\rangle_j = \begin{cases} 0 & \text{if} \quad k = 0, \dots, |\vec{n}| - 2 \\ 1 & \text{if} \quad k = |\vec{n}| - 1 \end{cases} \tag{6}$$

Whenever the measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup_{i=1}^{r} \Omega_i$, *i.e.*, $d\mu_j = w_j(x) d\mu(x)$, $(j = 1, \dots, r)$, it is possible to define the *Type I function* as

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x). \tag{7}$$

Using the type I function, we can rewrite the orthogonality relations

$$\int_{\Omega} Q_{\vec{n}}(x) x^k d\mu(x) = \begin{cases} 0 & \text{if} \quad k = 0, \dots, |\vec{n}| - 2 \\ 1 & \text{if} \quad k = |\vec{n}| - 1 \end{cases} \tag{8}$$

Nevertheless, not every multi-index $\vec{n} \in \mathbb{N}^r$ provides a type I vector of polynomials or a type II polynomial. Let $\vec{n} = (n_1, \dots, n_r)$ be a multi-index and let $\mu_1, \dots, \mu_r$ be r positive measures. If $(A_{\vec{n},1}(x), \dots, A_{\vec{n},r}(x))$ is a vector of type I MOP, then, if we denote

$$\begin{aligned} A_{\vec{n},1}(x) &= a_{n_1-1,1} x^{n_1-1} + a_{n_1-2,1} x^{n_1-2} + \dots + a_{1,1} x + a_{0,1} \\ &\vdots \\ A_{\vec{n},r}(x) &= a_{n_r-1,r} x^{n_r-1} + a_{n_r-2,r} x^{n_r-2} + \dots + a_{1,r} x + a_{0,r} \end{aligned} \tag{9}$$

and apply the orthogonality conditions (4) and (5), then we get a linear system of $|\vec{n}|$ equations and $|\vec{n}|$ unknown coefficients. Thus, the type I MOP will exist and it will be unique if and only if the following matrix is regular:

$$A = \begin{pmatrix} M_{n_1}^{(1)} \\ \hline M_{n_2}^{(2)} \\ \hline \vdots \\ \hline M_{n_r}^{(r)} \end{pmatrix}, \quad \text{where} \quad M_{n_j}^{(j)} = \begin{pmatrix} m_0^{(j)} & m_1^{(j)} & \cdots & m_{n-1}^{(j)} \\ m_1^{(j)} & m_2^{(j)} & \cdots & m_n^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n_j-1}^{(j)} & m_{n_j}^{(j)} & \cdots & m_{n+n_j-2}^{(j)} \end{pmatrix}, \tag{10}$$

$j = 1, \dots, r$ and $m_k^{(j)} = \int_{\Omega_j} x^k d\mu_j(x)$ are the moments of the measures ($k \geq 0$).

On the other hand, if we consider the type II polynomial as:

$$P_{\vec{n}} = x^{|\vec{n}|} + a_{|\vec{n}|-1}x^{|\vec{n}|-1} + \cdots + a_1 x + a_0$$

and apply the conditions (1), we get another linear system with $|\vec{n}|$ equations and $|\vec{n}|$ unknown coefficients. In fact, the coefficient matrix of this linear system is A^t, the transpose matrix of the type I MOP. Thus, we have the following result.

Proposition 1.3. *Given a multi-index $\vec{n} \in \mathbb{N}^r$ and r positive measures, $\mu_1, \dots, \mu_r$, the following statements are equivalent:*

1. *There exist a unique vector $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$ of type I MOP.*
2. *There exist a unique type II multiple orthogonal polynomial $P_{\vec{n}}$.*
3. *The matrix A defined in* (10) *is regular.*

Following this proposition, we provide a new definition.

Definition 1.4. *A multi-index $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ is **normal** if it satisfies the conditions of Proposition 1.3. A system of r measures $\mu_1, \dots, \mu_r$ is **perfect** if every $\vec{n} \in \mathbb{N}^r$ is normal.*

There are some perfect systems, standing out the Angelesco systems and the AT-systems, see [4, Sections 23.1.1 and 23.1.2]

We will be mainly focused on type II multiple orthogonal polynomials. In next sections, a definition of type II multiple orthogonal polynomials on several variables will be provided, along with a few simple examples and a generalized version of the Jacobi-Piñeiro polynomials. REVIEW No poner esto en el poster que los piñeiro no ha dado la vida

2 Orthogonal Polynomials Systems in several variables

First of all, we introduce the notation that will be used. Let $\Pi^d = \mathbb{R}[x_1, \dots, x_d]$ be the space of polynomials in d variables. If $d = 2$, we use the variables x, y. For $n \in \mathbb{N}_0$, the space generated by all the degree n monomials is denoted by

$$\mathcal{P}_n^d = \left\langle x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} : k_1 + k_2 + \cdots + k_d = n, k_1, k_2, \dots, k_d \in \mathbb{N}_0 \right\rangle$$

It is possible to check by induction that the number of different monomials of degree n with d variables is $\binom{n+d-1}{n}$. So, this means

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{n}.$$

In order to work with multivariate polynomials, we will use the notation $\mathbb{P}_n$ as a column polynomial vector. In order to understand this notation, we denote the vector of degree j monomials as

$$\mathbb{X}_j = \left(x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} \right)_{k_1+k_2+\cdots+k_d=j}.$$

For example, if $d = 2$ and we use the "degree reverse lexicographic ordering" in $\mathbb{N}^2$, then

$$\mathbb{X}_0 = (1)\,, \mathbb{X}_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbb{X}_2 = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}, \ldots, \mathbb{X}_j = \begin{pmatrix} x^j \\ x^{j-1}y \\ \vdots \\ y^j \end{pmatrix}.$$

Thus, a column polynomial vector of degree n can be represented as

$$\mathbb{P}_n = G_{n,n}\mathbb{X}_n + G_{n,n-1}\mathbb{X}_{n-1} + \cdots G_{n,1}\mathbb{X}_1 + G_{n,0}\mathbb{X}_0,$$

where $G_{n,j}$ are matrices of size $\binom{n+d-1}{n} \times \binom{j+d-1}{j}$ and they are the polynomial's "coefficients". Hence, $\mathbb{P}_n$ is a vector of polynomials of size $\binom{n+d-1}{n}$. We denote

$$r_n^d = \dim \mathcal{P}_n^d = \binom{n+d-1}{n}.$$

Given a multi-dimensional measure $\mu(x_1, \ldots, x_d)$, with support $\Omega \subseteq \mathbb{R}^d$, we can extend the definition of a multivariate product $\langle f, g \rangle_\mu$ and its functional $\mathcal{L}_\mu[f \cdot g]$ to column vectors. If $F = (f_1, f_2, \ldots, f_n)^t$ and $G = (g_1, g_2, \ldots, g_m)^t$ are column vectors of functions of size n and m, respectively, then we define

$$\langle F, G \rangle := \mathcal{L}_\mu[F \cdot G^T] = \int_\Omega F \cdot G^T d\mu(x_1, \ldots, x_d) = \left(\int_\Omega f_i \cdot g_j d\mu(x_1, \ldots, x_d) \right)_{i,j=1}^{n,m}. \tag{11}$$

REVIEW como pongo esto para que salga en forma de matriz :(

In fact, we are applying the standard product $\langle f_i, g_j \rangle_\mu$ or the functional $\mathcal{L}_\mu[f_i \cdot g_j]$ to each pair i, j and placing the results in a matrix.

Let $\{\mathbb{P}_n\}_{n \geq 0}$ be a system of polynomial vectors such that

$$\langle \mathbb{P}_n, \mathbb{P}_k \rangle_\mu = \mathcal{L}_\mu[\mathbb{P}_n \mathbb{P}_k^t] = \begin{cases} 0_{r_n^d \times r_k^d} & \text{if} \quad k = 0, \ldots, n-1 \\ S_n & \text{if} \quad k = n \end{cases}$$

Where S_n is a regular squared matrix of size $r_n^d \times r_n^d$. Due to orthogonality, it is possible to give an equivalent condition:

$$\langle \mathbb{P}_n, \mathbb{X}_k \rangle_\mu = \mathcal{L}_\mu[\mathbb{P}_n \mathbb{X}_k^t] = \begin{cases} 0_{r_n^d \times r_k^d} & \text{if} \quad k = 0, \ldots, n-1 \\ S_n & \text{if} \quad k = n \end{cases} \tag{12}$$

Then, $\{\mathbb{P}_n\}_{n \geq 0}$ is called a system of orthogonal polynomials with respect to the measure μ or the functional $\mathcal{L}_\mu$. Further information about this topic is available in [2, Ch. III, Section 3.2].

3 Type II MOP in several variables

In order to give a definition of type II orthogonality, we will take as a reference the definition of type II orthogonality in the univariate case. given $r \in \mathbb{N}$, $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, r d-dimensional measures $\mu_1, \ldots, \mu_r$ and their respective matrix inner products defined in (11), we are going to define the type II multiple orthogonal polynomial vector as a monic polynomial vector $\mathbb{P}_{\vec{n}} = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k}\mathbb{X}_k$ which satisfies

$$\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_k \rangle_i = 0, \quad k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r \tag{13}$$

Note the similarity between conditions (3) and (13). However, when we work with one variable, it is known type II polynomial is a monic polynomial whose degree is exactly $|\vec{n}|$. This is because the number of coefficients of a degree $|\vec{n}|$ univariate monic polynomial ($|\vec{n}|$) is equal to both the number of orthogonality conditions and the size of matrices A defined in (10). Due to the differences between univariate and multivariate case, the degree n of a polynomial vector $\mathbb{P}_{\vec{n}}$ might not be equal to $|\vec{n}|$. For now, this degree n will be considered as unknown, getting to know it later.

As mencioned above, in the univariate multiple orthogonality, $\deg(P_{\vec{n}}) = |\vec{n}|$. The main reason why this happens is because the condition $\left\langle P_{\vec{n}}, x^k \right\rangle = 0$ is a linear equation (only one). Hence, we are looking for a polynomial with $|\vec{n}|$ unknown coefficients and we have $|\vec{n}|$ linear equations deduced from conditions (3). Then, it is possible to build a system of linear equations with coefficient matrix A^t given in (10).

If we work with d variables, attending to (12), $\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_k \rangle$ is a matrix of size $r_n^d \times r_k^d$. Then, equation $\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_k \rangle_i = 0$ gives us $r_n^d \times r_k^d$ linear equations.

Now, the question is: How many linear equations is it possible to get from a multi-index $\vec{n} = (n_1, \ldots, n_r)$? The answer to this problem is easy since, if we fix $j \in \{1, \ldots, r\}$:

- $\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_0 \rangle_j = 0_{r_n^d \times r_0^d}$, we get $r_n^d \times r_0^d$ equations.
- $\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_1 \rangle_j = 0_{r_n^d \times r_1^d}$, we get $r_n^d \times r_1^d$ equations.
- ...
- $\left\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_{n_j - 1} \right\rangle_j = 0_{r_n^d \times r_{n_i-1}^d}$, we get $r_n^d \times r_{n_j-1}^d$ equations.

If we collect all the equations, we obtain $r_n^d \cdot \sum_{k=0}^{n_j-1} r_k^d$ linear equations for each $j \in \{1, \ldots, r\}$. Collecting the number of equations of each j, we finally get that the number of equations is

$$r_n^d \sum_{j=1}^{r} \sum_{k=0}^{n_j-1} r_k^d. \tag{14}$$

On the other hand, since $\mathbb{P}_{\vec{n}} = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k}\mathbb{X}_k$, where $G_{n,k}$ are matrices of dimensions $r_n^d \times r_k^d$, if we consider $\mathbb{P}_{\vec{n}}$ a vector of degree n monic polynomials, then this matrices give us $\sum_{k=0}^{n-1} r_n^d r_k^d = r_n^d \sum_{k=0}^{n-1} r_k^d$ unknown coefficients.

We want this system to have only one unique solution for a multi-index $\vec{n}$. Then, we will say $\vec{n}$ is a **admissible** multi-index if there exist a number $n \in \mathbb{N}_0$ such that:

$$\sum_{k=0}^{n-1} r_k^d = \sum_{i=1}^{r} \sum_{k=0}^{n_i-1} r_k^d. \tag{15}$$

This number n is the degree of type II polynomials. In order to emphasise the degree of the polynomials, in the next sections we denote type II MOP as $\mathbb{P}_{\vec{n}}^n$.

REVIEW No me termina de hacer gracia el concepto de admisible porque al final está relacionado con un número n. A la hora de hablar de que un multi-indice es admisible lo suyo es decir después para qué n. He pensado que quizá estaría bien decir que es n-admisible?

4 The bidimensional case

From now on, we will assume $d = 2$ and $x := x_1, y := x_2$. Notice that, in the bidimensional case, $r_n^2 = \binom{n+1}{n} = n + 1$, which makes things much easier: $\mathbb{X}_j$ is a polynomial vector of size $j + 1$, $\mathbb{P}_n$ is a vector of size $n + 1$, and $G_{n,k}$ is a $(n+1)\times(k+1)$ matrix. Also, condition (15) gets simpler because some sums become sums of arithmetic progressions. When $d = 2$, a multi-index $\vec{n} = (n_1, \ldots, n_r)$ is admissible if there exists a number $n \in \mathbb{N} \cup \{0\}$ such that

$$n(n+1) = \sum_{j=1}^{r} n_j(n_j+1). \tag{16}$$

Observe some of the admissible multi-indices and their respective degree n in tables 1 and 2, and a visual representation of some of them when $r = 2$ and $r = 3$ in figure 1.

5 Type I MOP in two variables

Let us recall type I multiple orthogonal polynomials in Definition 1.2. Given $\vec{n} = (n_1, \ldots, n_r)$, it consists of a polynomial list $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$ where $\deg(A_{\vec{n},j}) \le n_j - 1$ $(j = 1, \ldots, r)$ and polynomials $A_{\vec{n},j}$ satisfy (4) and (5). These two conditions can be also expresed in a more compact way if we use the inner products, see (6).

In order to generalize this type of orthogonality, the first problem is the sum located in orthogonality condition. This forces us to define polynomials, or polynomial

n	admissible multi-indices when $r = 2$ and $d = 2$
0	$(0, 0)$
1	$(0, 1), (1, 0)$
2	$(0, 2), (2, 0)$
3	$(0, 3), (2, 2), (3, 0)$
4	$(0, 4), (4, 0)$
5	$(0, 5), (5, 0)$
6	$(0, 6), (3, 5), (5, 3), (6, 0)$
7	$(0, 7), (7, 0)$
8	$(0, 8), (5, 6), (6, 5), (8, 0)$
9	$(0, 9), (9, 0)$

Table 1: admissible multi-indices when using $r = 2$ measures

n	admissible multi-indices when $r = 3$ and $d = 2$
0	$(0, 0, 0)$
1	$(0, 0, 1), (0, 1, 0), (1, 0, 0)$
2	$(0, 0, 2), (0, 2, 0), (1, 1, 1), (2, 0, 0)$
3	$(0, 0, 3), (0, 2, 2), (0, 3, 0), (2, 0, 2), (2, 2, 0), (3, 0, 0)$
4	$(0, 0, 4), (0, 4, 0), (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), (4, 0, 0)$
5	$(0, 0, 5), (0, 5, 0), (2, 3, 3), (3, 2, 3), (3, 3, 2), (5, 0, 0)$
6	$(0, 0, 6), (0, 3, 5), (0, 5, 3), (0, 6, 0), (1, 4, 4), (2, 2, 5), (2, 5, 2), (3, 0, 5),$ $(3, 5, 0), (4, 1, 4), (4, 4, 1), (5, 0, 3), (5, 2, 2), (5, 3, 0), (6, 0, 0)$
7	$(0, 0, 7), (0, 7, 0), (1, 3, 6), (1, 6, 3), (2, 4, 5), (2, 5, 4), (3, 1, 6), (3, 6, 1),$ $(4, 2, 5), (4, 5, 2), (5, 2, 4), (5, 4, 2), (6, 1, 3), (6, 3, 1), (7, 0, 0)$
8	$(0, 0, 8), (0, 5, 6), (0, 6, 5), (0, 8, 0), (3, 5, 5), (5, 0, 6), (5, 3, 5), (5, 5, 3),$ $(5, 6, 0), (6, 0, 5), (6, 5, 0), (8, 0, 0)$
9	$(0, 0, 9), (0, 9, 0), (2, 3, 8), (2, 6, 6), (2, 8, 3), (3, 2, 8), (3, 8, 2), (5, 5, 5), (6, 2, 6),$ $(6, 6, 2), (8, 2, 3), (8, 3, 2), (9, 0, 0)$

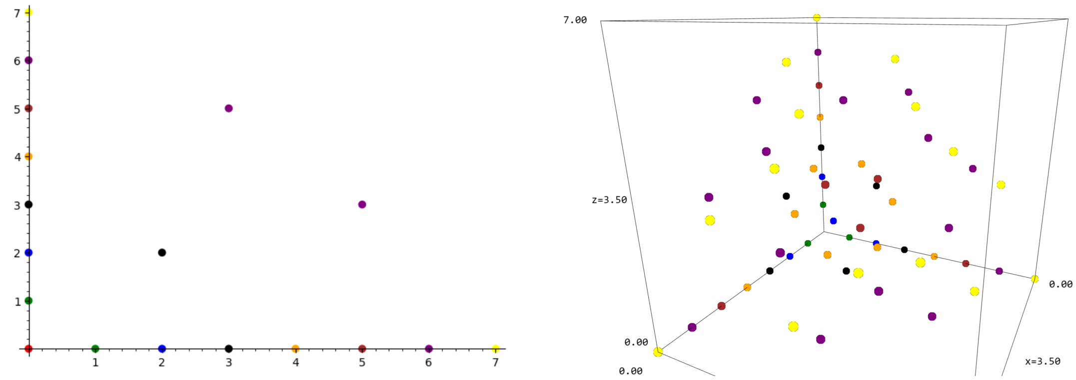
Table 2: admissible multi-indices when using $r = 3$ measures

vectors $\mathbb{A}_{\vec{n},j}$, of the same size. If these vectors had different size, it would not be possible to make the sum. This makes impossible to consider $\mathbb{A}_{\vec{n},j}$ a vector of degree $\leq n_j - 1$ polynomials, with size n_j. The solution to this problem will be explained below.

Let $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ such that the condition (16) holds for some $n \in \mathbb{N}$. Then, let $i \in \{1, \ldots, n\}$ and, for each j, the polynomial

$$A^{(i)}_{\vec{n},j}(x, y) = g^t_{n_j-1,n_j-1}\mathbb{X}_{n_j-1} + g^t_{n_j-1,n_j-2}\mathbb{X}_{n_j-2} + \cdots + g^t_{n_j-1,0}\mathbb{X}_0$$

with $g_{n_j-1,k} \in \mathbb{R}^{k+1}$ $(k = 0, \ldots, n_j - 1)$, is a bivariate polynomial such that $\deg A^{(i)}_{\vec{n},j} \leq n_j - 1$, $(j = 1, \ldots, r)$. Now, given r 2-dimensional measures $\mu_1, \ldots, \mu_r$, we will force the polynomials $A^{(i)}_{\vec{n},1}, A^{(i)}_{\vec{n},2}, \ldots, A^{(i)}_{\vec{n},r}$ to satisfy the first type I multiple orthogonality condition:



(a) $r = 2$ (b) $r = 3$

Figure 1: Graphic representation of admissible r-indices

$$\sum_{j=1}^{r} \left\langle \mathbb{X}_k, A_{\vec{n},j}^{(i)} \right\rangle_j = \begin{cases} 0_{(k+1)\times 1} & \text{if} \quad k = 0, \dots, n-2 \\ (e_i)_{n\times 1} = (0, \dots, 0, 1, 0, \dots, 0)^t & \text{if} \quad k = n-1 \end{cases} \tag{17}$$

It is easy to see that the number of unknown coefficients of the polynomials $A_{\vec{n},j}^{(i)}$ is the same as the number of linear equations given by conditions (17). If we repeat this for every possible $i \in \{1, \dots, n\}$, we get n lists of polynomials $A_{\vec{n},j}^{(i)}$, $j = 1, \dots, r, i = 1, \dots, n$. Now, let us denote

$$\mathbb{A}_{\vec{n},j} = \begin{pmatrix} A_{\vec{n},j}^{(1)} \\ \vdots \\ A_{\vec{n},j}^{(n)} \end{pmatrix}_{n\times 1},$$

a polynomial vector of size n whose components are bivariate polynomials of degree less than or equal to $n_j - 1$ ($j = 1, \dots, r$) and, fixed i, the polynomials $A_{\vec{n},1}^{(i)}, \dots, A_{\vec{n},r}^{(i)}$ satisfy (17). Thus, the polynomial vectors $\mathbb{A}_{\vec{n},j}$ will satisfy:

$$\sum_{j=1}^{r} \langle \mathbb{X}_k, \mathbb{A}_{\vec{n},j} \rangle_j = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \dots, n-2 \\ I_n & \text{if} \quad k = n-1 \end{cases} \tag{18}$$

Observe the similarity between equations (8) and (18).

Summarizing, we present the final definitions of the multiple orthogonal polynomials in two variables.

Definition 5.1 (Type II MOP in two variables)**.** *Let $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ such that the condition (16) holds for certain $n \in \mathbb{N}$. A vector of monic degree n polynomials $\mathbb{P}_{\vec{n}}^n = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k} \mathbb{X}_k$ is a **type II Multiple Orthogonal Polynomial Vector** if*

$$\langle \mathbb{X}_k, \mathbb{P}_{\vec{n}}^n \rangle_j = 0_{(k+1)\times(n+1)}, \quad k = 0, \dots, n_j - 1, \quad i = 1, \dots, r \tag{19}$$

The $n+1$ bivariate polynomials composing the polynomial vector $\mathbb{P}^n_{\vec{n}}$ will be denoted as

$$\mathbb{P}^n_{\vec{n}} = \begin{pmatrix} P^{n,1}_{\vec{n}}(x,y) \\ \vdots \\ P^{n,n+1}_{\vec{n}}(x,y) \end{pmatrix}_{(n+1)\times 1}.$$

Definition 5.2 (Type I MOP in two variables)**.** *Let $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ such that the condition* (16) *holds for certain $n \in \mathbb{N}$.* ***Type I Multiple Orthogonal Polynomials Vectors*** *are presented as $(\mathbb{A}_{\vec{n},1}, \ldots, \mathbb{A}_{\vec{n},r})$, where $\mathbb{A}_{\vec{n},j} = (A^{(1)}_{\vec{n},j}, \ldots, A^{(n)}_{\vec{n},j})^t$, $j = 1, \ldots, r$ and $A^{(i)}_{\vec{n},j}$ are polynomials of degree less than or equal to $n_j - 1$ for every $i = 1, \ldots, n$. In addition, these polynomial vectors satisfy conditions* (18)*.*

As in the univariate case, from Type I MOP, when the 2-dimensional measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup_{j=1}^{r} \Omega_j$ (with $\Omega_j = \mathrm{supp}(\mu_j)$), *i.e.*, $d\mu_j = w_j(x,y)d\mu(x,y)$, $(j = 1, \ldots, r)$, we can define the *Type I function*:

$$\mathbb{Q}_{\vec{n}} = \sum_{j=1}^{r} \mathbb{A}_{\vec{n},j} w_j(x,y). \tag{20}$$

Using this function, it is possible to rewrite (18) as

$$\langle \mathbb{X}_k, \mathbb{Q}_{\vec{n}} \rangle_\mu = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \ldots, n-2 \\ I_n & \text{if} \quad k = n-1. \end{cases} \tag{21}$$

We will often remove the sub-index μ from $\langle \cdot, \cdot \rangle_\mu$, assuming that the product uses the common measure.

Despite their differences, type I and type II MOP in two variables are equivalent.

Proposition 5.3. *Let $\vec{n} = (n_1, \ldots, n_r)$ be a multi-index satisfying* (16) *for $n \in \mathbb{N}_0$. Then the following statements are equivalent:*

1. *There exist unique polynomial vectors $(\mathbb{A}_{\vec{n},1}, \ldots, \mathbb{A}_{\vec{n},r})$ of bivariate type I multiple orthogonal polynomials.*
2. *There exist a unique bivariate type II multiple orthogonal polynomial $\mathbb{P}^n_{\vec{n}}$.*

As we can see, the bivariate case reduces the number of multi-indices $(n_1, \ldots, n_r)$ which it is possible to create multiple orthogonal polynomials with. Whereas in univariate case it is possible to question existence of MOP for every $\vec{n} \in \mathbb{N}^r$, in the bivariate case only some of the multi-indices can give us MOP, those such that there exist $n \in \mathbb{N}$ with $n(n+1) = \sum_{j=1}^{r} n_j(n_j+1)$. Given a admissible $\vec{n} \in \mathbb{N}^r$, we will say $\vec{n}$ is a **normal** multi-index if it satisfies the conditions of Proposition 5.3. A system $\mu_1, \ldots, \mu_r$ of bivariate measures is **perfect** if every admissible multi-index is normal.

6 First example

We are going to show you the first example of type II bivariate multiple orthogonal polynomials: Multiple product Laguerre polynomials. First, we will present univariate precedents. According to [3, Page 658, Section 3.6.1], type II multiple Laguerre polynomials of the first kind $L_{\vec{n}}^{\vec{\alpha}}(x)$ ($\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$) satisfy

$$\int_0^\infty x^k L_{\vec{n}}^{\vec{\alpha}}(x) x^{\alpha_j} e^{-x} dx = 0 \quad k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, r. \tag{22}$$

In order that all multi-indices to be normal we need to have $\alpha_j > -1$ and $\alpha_j - \alpha_i \notin \mathbb{Z}$ whenever $i \neq j$. On the other hand, bivariate product Laguerre polynomials [2, Ch. II, Section 2.2] are orthogonal with respect to the bidimensional weight function

$$w(x, y) = x^\alpha y^\beta e^{-x-y}, \quad \alpha, \beta > -1.$$

Following this two approaches, we present an example with $r = 3$. Let $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ such that $\alpha_j, \beta_j > -1$ and any possible pair α_i, β_j satisfies $\alpha_i - \beta_j \notin \mathbb{Z}$, and $\alpha_j - \alpha_i \notin \mathbb{Z}$, $\beta_j - \beta_i \notin \mathbb{Z}$ whenever $i \neq j$ (any difference between different parameters is not an integer). We consider the following system of 2-dimensional measures:

$$d\mu_1(x, y) = x^{\alpha_1} y^{\beta_1} e^{-x-y} d(x, y), \qquad d\mu_2(x, y) = x^{\alpha_2} y^{\beta_2} e^{-x-y} d(x, y), \tag{23}$$

$$d\mu_3(x, y) = x^{\alpha_3} y^{\beta_3} e^{-x-y} d(x, y).$$

Let $\vec{n} = (n_1, n_2, n_3) \in \mathbb{N}^3$ satisfying (16). Then, we have proven experimentally that many of the possible multi-indices are normal. For example, if we take

$$\alpha_1 = 0, \alpha_2 = 0.5, \alpha_3 = 1.3; \beta_1 = 0.8, \beta_2 = 0.4, \beta_3 = 2.1,$$

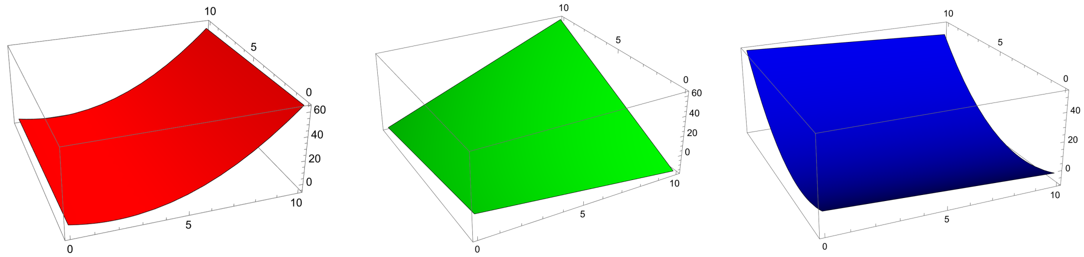
then we can calculate the polynomial vector $\mathbb{P}^2_{(1,1,1)}$:

$$\mathbb{P}^2_{(1,1,1)} = \begin{pmatrix} x^2 + -3.85556x - 0.444444y + 2.65556 \\ xy - 2.15556x - 1.94444y + 3.85556 \\ y^2 - 0.755556x - 5.14444y + 4.97556 \end{pmatrix}$$

In the following images it is possible to see a plot of the 3 polynomials of vector $\mathbb{P}^2_{(1,1,1)}$ in figure 2 and a picture representing all three polynomials together in figure 3.

7 Biorthogonality

Let $\mu_1, \ldots, \mu_r$ be a perfect system of r 2-dimensional measures. In this section, we will explain an orthogonality relation between type I and type II MOP. Analogous results for the univariate case are available in [4, Theorem 23.1.6].



(a) $(\mathbb{P}^2_{(1,1,1)})_1$ (b) $(\mathbb{P}^2_{(1,1,1)})_2$ (c) $(\mathbb{P}^2_{(1,1,1)})_3$

Figure 2: Graphic representation of $\mathbb{P}^2_{(1,1,1)}$

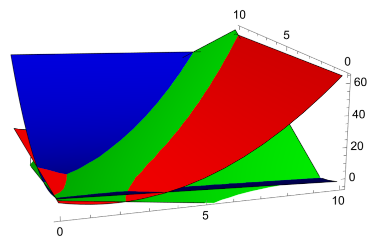


Figure 3: $\mathbb{P}^2_{(1,1,1)}$ polynomials together

Theorem 7.1. *Given $\vec{n}, \vec{m}$, both satisfying* (16) *for n and m, respectively, the following biorthogonality holds for type I and type II MOP:*

$$\langle \mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}} \rangle = \begin{cases} 0_{(n+1)\times m} & \textit{if} \quad \vec{m} \leq \vec{n} \\ 0_{(n+1)\times m} & \textit{if} \quad n \leq m-2 \\ I_{n+1} & \textit{if} \quad n = m-1 \end{cases} \tag{24}$$

Proof. We will prove each item. First, we have

$$\langle \mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}} \rangle = \sum_{j=1}^{r} \langle \mathbb{P}^n_{\vec{n}}, \mathbb{A}_{\vec{m},j} \rangle_j .$$

- Assume $\vec{m} \leq \vec{n}$, this means $m_j \leq n_j \;\; \forall j \in \{1, \ldots, r\}$. Then, remember $\mathbb{A}_{\vec{m},j}$ is a vector of polynomials of degree less than or equal to $m_j - 1 < n_j$ for every $j \in \{1, \ldots, r\}$ and $\langle \mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}} \rangle = 0$ follows from type II orthogonality conditions (19).

- If $n \leq m-2$, then each polynomial of $\mathbb{P}^n_{\vec{n}}$ is a linear combination of $\mathbb{X}_{n-2}, \ldots, \mathbb{X}_1, \mathbb{X}_0$. Then, type I orthogonality conditions (18) for $0 \leq k \leq n-2$ shows $\langle \mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}} \rangle = 0$.

- Finally, if $m = n+1$, then $\mathbb{P}^n_{\vec{n}}$ is a vector of degree $m-1$ polynomials. So, $\mathbb{P}^n_{\vec{n}} = \mathbb{X}_{m-1} + \sum_{k=0}^{m-2} G_k \mathbb{X}_k$. Due to the previous item, we have

$$\langle \mathbb{P}^{m-1}_{\vec{n}}, \mathbb{Q}_{\vec{m}} \rangle = \sum_{j=1}^{r} \langle \mathbb{P}^{m-1}_{\vec{n}}, \mathbb{A}_{\vec{m},j} \rangle_j = \sum_{j=1}^{r} \langle \mathbb{X}_{m-1}, \mathbb{A}_{\vec{m},j} \rangle_j = I_m = I_{n+1},$$

where in the second last equality we have used the normalization condition in (18).

$\square$

Observe that the first item emphasises the components of multi-indices $\vec{n}$ and $\vec{m}$, while the last two items are true for any two multi-indices with degree n and m whatever their components are.

This result, which is a two-dimensional extension of [4, Theorem 23.1.6], will be very useful in order to prove the next result.

8 Nearest Neighbor Recurrence Relation

The standard orthogonal polynomials always satisfy a Three-Term Recurrence Relation. That is, given an monic OPS $\{P_n\}$ with respect to a inner product $\langle \cdot, \cdot \rangle$ or its respective functional $\mathcal{L}$, there exist constants α_n, β_n such that

$$xP_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}.$$

The nearest Neighbor Recurrence Relation, in univariate multiple orthogonality, extends this result, allowing us to express a type II polynomial multiplied by x: $xP_{\vec{n}}$ as a linear combination of the polynomial itself, $P_{\vec{n}}$, a beyond neighbor, *i.e.* $P_{\vec{n}+e_k}$ and r above neighbors, see [4, Theorem 23.1.7]. We are going to give a proof of a first generalization to bivariate MOP of this result.

Theorem 8.1. *Let $\mu_1, \ldots, \mu_r$ be a perfect system of 2-dimensional measures. Let $\vec{n} \in \mathbb{N}^r$ be a multi-index satisfying (16) for $n \in \mathbb{N}$ and let us consider a path $\{\overrightarrow{m}_k : k = 0, \ldots, n+1\}$ where $\overrightarrow{m}_0 = \vec{0}, \overrightarrow{m}_n = \vec{n}$, each $\overrightarrow{m}_k$ satisfy (16) for k and $\overrightarrow{m}_k \leq \overrightarrow{m}_{k+1}$ for $k = 0, \ldots, n$. Then, there exist matrices $A_0, \ldots, A_r$ of sizes $(n+1) \times (n-j+1)$, $(j = 0, \ldots, r)$ such that*

$$x\mathbb{P}^n_{\vec{n}} = L_{n+1,1}\mathbb{P}^{n+1}_{\overrightarrow{m}_{n+1}} + A_0\mathbb{P}^n_{\vec{n}} + \sum_{j=1}^{r} A_j \mathbb{P}^{n-j}_{\overrightarrow{m}_{n-j}}, \tag{25}$$

where $L_{n+1,1} = (I_{n+1}|0_{(n+1)\times 1})$.

In addition, $A_j = \left\langle x\mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\overrightarrow{m}_{n-j+1}} \right\rangle$.

Proof. As we are in a perfect system, we can assume the existence of $\mathbb{P}^k_{\overrightarrow{m}_k}$ for every $\vec{m}_k, k = 0, \ldots, n+1$. As we know, $x\mathbb{P}^n_{\vec{n}}$ is a vector of degree $n+1$ polynomials and its size is also $n+1$. It is possible to write $x\mathbb{P}^n_{\vec{n}}$ as a linear combination of $\{\mathbb{P}^k_{\overrightarrow{m}_k}, k = 0, \ldots, n+1\}$ whose coefficients are matrices M_k:

$$x\mathbb{P}^n_{\overrightarrow{n}} = \sum_{k=0}^{n+1} M_k \mathbb{P}^k_{\overrightarrow{m}_k},$$

where M_k is a $(n+1) \times (k+1)$ matrix $(k = 0, \ldots, n+1)$.

Observing the leader term of $x\mathbb{P}^n_{\vec{n}}$, which is $x\mathbb{X}_n$, and comparing it with the leader term of $M_{n+1}\mathbb{P}^{n+1}_{\vec{m}_{n+1}}$, which is $M_{n+1}\mathbb{X}_{n+1}$ we can easily check $M_{n+1} = L_{(n+1),1}$. Now, we have

$$\mathbb{P}^n_{\vec{n}} = L_{n+1,1}\mathbb{P}^{n+1}_{\vec{m}_{n+1}} + \sum_{k=0}^{n} M_k \mathbb{P}^k_{\vec{m}_k}.$$

Now, we apply the inner product to both sides of the equation by $\mathbb{Q}_{\vec{m}_l}$ and observe:

- If $\vec{m}_l \leq \vec{m}_j$, which always happens if $l \leq j \leq n+1$, then, due to the first item of Theorem 7.1, $\left\langle \mathbb{P}^j_{\vec{m}_j}, \mathbb{Q}_{\vec{m}_l} \right\rangle = 0_{(j+1)\times l}$ whenever $l \leq j$.
- If $j \leq l-2$, then, from the second item of Theorem 7.1, $\left\langle \mathbb{P}^j_{\vec{m}_j}, \mathbb{Q}_{\vec{m}_l} \right\rangle = 0_{(j+1)\times l}$.
- Finally, if $j = l-1$, from the last item of Theorem 7.1 we deduce $\left\langle \mathbb{P}^j_{\vec{m}_j}, \mathbb{Q}_{\vec{m}_l} \right\rangle = I_{j+1}$.

Summarizing, we have

$$\left\langle x\mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}_l} \right\rangle = L_{n+1,1} \left\langle \mathbb{P}^{n+1}_{\vec{m}_{n+1}}, \mathbb{Q}_{\vec{m}_l} \right\rangle + \sum_{j=0}^{n} M_j \left\langle \mathbb{P}^j_{\vec{m}_j}, \mathbb{Q}_{\vec{m}_l} \right\rangle .$$

Applying the previous items deduced from Theorem 7.1, we have

$$\left\langle x\mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}_l} \right\rangle = M_{l-1} \qquad l = 1, \ldots, n+1, \tag{26}$$

Now, if $l \leq n-r$, then $\vec{m}_l \leq \vec{m}_{n-r}$. So, choosing $l \leq n-r$,

$$\left\langle x\mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}_l} \right\rangle = \sum_{j=1}^{r} \left\langle \mathbb{P}^n_{\vec{n}}, x\mathbb{A}_{\vec{m}_l,j} \right\rangle_j ,$$

but $x\mathbb{A}_{\vec{m}_l,j}$ are vectors of degree less than or equal to $(\vec{m}_l)_j \leq (\vec{m}_{n-r})_j < n_j$. Then, the product vanishes because of type II orthogonality (19). This means

$$\left\langle x\mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}_l} \right\rangle = M_{l-1} = 0 \quad l = 1, \ldots, n-r.$$

Thus, the final expression is

$$\mathbb{P}^n_{\vec{n}} = L_{n+1,1}\mathbb{P}^{n+1}_{\vec{m}_{n+1}} + \sum_{k=n-r}^{n} M_k \mathbb{P}^k_{\vec{m}_k}.$$

If we rename $A_{n-k} = M_k$ for $k = n-r, ..., n$, then we get the desired expression (25).

Finally, due to (26),

$$A_j = M_{n-j} = \left\langle x\mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}_{n-j+1}} \right\rangle .$$

$\square$

9 Conclusion

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