Multiple Orthogonal Polynomials in several variables

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Abstract

TODO Write the abstract

1 Introduction

Multiple Orthogonality is an extension of the standard orthogonality. It consists of Polynomials (defined on the real line in this introduction) that satisfy orthogonality relations with respect to more than one measure. There are also two different types of multiple orthogonality, which will be explained shortly.

First, let's consider r different real measures $\mu_1, \ldots, \mu_r$ such that $\Omega_i = \text{supp}(\mu_i) \subseteq \mathbb{R}$ $(i = 1, \ldots, r)$. We will use multi-indexes $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, and denote $n := |\vec{n}| = n_1 + \cdots + n_r$. These multi-indexes determine the orthogonality relations with each measure. With these preliminaries we will present the first definition.

Definition 1.1 (Type II Multiple Orthogonal Polynomials). *Let* $\vec{n} = (n_1, \ldots, n_r)$. *A monic polynomial* $P_{\vec{n}}(x)$ *is a type II multiple orthogonal polynomial if* $\deg(P_{\vec{n}}) \leq n$ *and*

$$\int_{\Omega_i} P_{\vec{n}}(x) x^k d\mu_i(x) = 0, \quad k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r \tag{1}$$

This means $P_{\vec{n}}$ is orthogonal to $1, x, x^2, \ldots, n^{n_i-1}$ with respect to each measure μ_i, $(i = 1, \ldots, r)$. If we define the product $\langle f, g \rangle_i = \int_{\Omega_i} f(x) g(x) d\mu(x)$, conditions (1) can also be written as

$$\left\langle P_{\vec{n}}, x^k \right\rangle_i = 0, \quad k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r \tag{2}$$

There is another type of multiple orthogonality: The type I MOP.

Definition 1.2 (Type I Multiple Orthogonal Polynomials). *Let $\vec{n} = (n_1, \ldots, n_r)$. Type I Multiple Orthogonal Polynomials are presented in a vector $(A_{\vec{n},1}(x), \ldots, A_{\vec{n},r}(x))$, where $\deg(A_{\vec{n},i}) \leq n_i - 1$, $(i = 1, \ldots, r)$ and these polynomials satisfy the relation*

$$\sum_{i=1}^{r} \int_{\Omega_i} A_{\vec{n},i}(x) x^k d\mu_i(x) = 0, \quad k = 0, \ldots, n-2 \tag{3}$$

and the normalization condition

$$\sum_{i=1}^{r} \int_{\Omega_i} A_{\vec{n},i}(x) x^{n-1} d\mu_i(x) = 1. \tag{4}$$

As we did previously with the type II MOP, the relations (3) and (4) can be written, using the dot products, as

$$\sum_{i=1}^{r} \langle A_{\vec{n},i}, x^k \rangle_i = \begin{cases} 0 & \text{if} \quad k = 0, \ldots, n-2 \\ 1 & \text{if} \quad k = n-1 \end{cases} \tag{5}$$

Whenever the measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup_{i=1}^{r} \Omega_i$, i.e., $d\mu_i = w_i(x) d\mu(x)$, $(i = 1, \ldots, r)$, it is possible to define the *Type I function* as

$$Q_{\vec{n}}(x) = \sum_{i=1}^{r} A_{\vec{n},i}(x) w_i(x). \tag{6}$$

Using the type I function, we can rewrite the orthogonality relations

$$\int_{\Omega} Q_{\vec{n}}(x) x^k d\mu(x) = \begin{cases} 0 & \text{if} \quad k = 0, \ldots, n-2 \\ 1 & \text{if} \quad k = n-1 \end{cases} \tag{7}$$

Nevertheless, not every multi-index $\vec{n} \in \mathbb{N}^r$ provides a type I vector of polynomials or a type II polynomial. Let $\vec{n} = (n_1, \ldots, n_r)$ be a multi-index and let $\mu_1, \ldots, \mu_r$ be r positive measures. If $(A_{\vec{n},1}(x), \ldots, A_{\vec{n},r}(x))$ is the vector of type I MOP, then, if we denote

$$A_{\vec{n},1}(x) = a_{n_1-1,1} x^{n_1-1} + a_{n_1-2,1} x^{n_1-2} + \cdots + a_{1,1} x + a_{0,1}$$
$$\vdots \tag{8}$$
$$A_{\vec{n},r}(x) = a_{n_r-1,r} x^{n_r-1} + a_{n_r-2,r} x^{n_r-2} + \cdots + a_{1,r} x + a_{0,r}$$

and apply the orthogonality conditions (3) and (4), then we get a linear system of n equations and n unknown coefficients. Thus, the type I MOP will exist if and only if the following matrix is regular:

$$A = \begin{pmatrix} M_{n_1}^{(1)} \\ \hline M_{n_2}^{(2)} \\ \hline \vdots \\ \hline M_{n_r}^{(r)} \end{pmatrix}, \quad \text{where} \quad M_{n_j}^{(j)} = \begin{pmatrix} m_0^{(j)} & m_1^{(j)} & \cdots & m_{n-1}^{(j)} \\ m_1^{(j)} & m_2^{(j)} & \cdots & m_n^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n_j-1}^{(j)} & m_{n_j}^{(j)} & \cdots & m_{n+n_j-2}^{(j)} \end{pmatrix}, \tag{9}$$

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$j = 1, \ldots, r$ and $m_k^{(j)} = \int_{\Omega_j} x^k d\mu_j(x)$ are the moments of the measures $(k \geq 0)$.

On the other hand, if we consider the type II polynomial as:

$$P_{\vec{n}} = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a0$$

and apply the conditions (1), we get another linear system with n equations and n unknown coefficients. In fact, the coefficients matrix of this linear system is A^t, the transpose matrix of the type I MOP. Thus, we have the following result.

Proposition 1.3. *Given a multi-index $\vec{n} \in \mathbb{N}^r$ and r positive measures, $\mu_1, \ldots, \mu_r$, the following statements are equivalent:*

1. *There exist an unique vector $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$ of type I MOP.*

2. *There exist an unique type II multiple orthogonal polynomial $P_{\vec{n}}$.*

3. *The matrix A defined in (9) is regular.*

Following this proposition, we provide a new definition.

Definition 1.4. *A multi-index $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ is **normal** if it satisfies the conditions of the proposition 1.3. A system of r measures $\mu_1, \ldots, \mu_r$ is **perfect** if every $\vec{n} \in \mathbb{N}^r$ is normal.*

There are some perfect systems, standing out the Angelesco systems and the AT-systems, see [3, Sections 23.1.1 and 23.1.2]

We will be mainly focused on the type II multiple orthogonal polynomials. In the next sections, a definition of type II multiple orthogonal polynomials on several variables will be provided, and also some easy examples and a generalized version of the Jacobi-Piñeiro polynomials.

2 Orthogonal Polynomials Systems in several variables

First of all, we will introduce the notation that will be used. Let $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$ be the space of polynomials in d variables. If $d = 2$, we will use the variables x, y. For $n \in \mathbb{N}_0$, the space generated by all the degree n monomials is denoted by

$$\mathcal{P}_n^d = \left\langle x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} : k_1 + k_2 + \cdots + k_d = n, k_1, k_2, \ldots, k_d \in \mathbb{N}_0 \right\rangle$$

It is possible to check by induction that the number of different monomials of degree n with d variables is $\binom{n+d-1}{n}$. So, this means

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{n}.$$

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In order to work with multivariate polynomials, we will use the notation $\mathbb{P}_n$ as a column polynomial vector. In order to understand this notation, we denote the vector of degree j monomials as

$$\mathbb{X}_j = \left(x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d} \right)_{k_1+k_2+\cdots+k_d=j}.$$

For example, if $d = 2$ and we use the "degree reverse lexicographic ordering" in $\mathbb{N}^2$, then

$$\mathbb{X}_0 = (1), \mathbb{X}_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \mathbb{X}_2 = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}, \ldots, \mathbb{X}_j = \begin{pmatrix} x^j \\ x^{j-1}y \\ \vdots \\ y^j \end{pmatrix}.$$

Thus, a column polynomial vector of degree n can be represented as

$$\mathbb{P}_n = G_{n,n}\mathbb{X}_n + G_{n,n-1}\mathbb{X}_{n-1} + \cdots G_{n,1}\mathbb{X}_1 + G_{n,0}\mathbb{X}_0,$$

where $G_{n,j}$ are matrices of size $\binom{n+d-1}{n} \times \binom{j+d-1}{j}$ and they are the polynomial's "coefficients". Hence, $\mathbb{P}_n$ is a vector of polynomials whose size is $\binom{n+d-1}{n}$. We will denote

$$r_n^d = \dim \mathcal{P}_n^d = \binom{n+d-1}{n}.$$

Given a multi-dimensional measure $\mu(x_1, \ldots, x_d)$, whose support is $\Omega \subseteq \mathbb{R}^d$, we can extend the definition of a multivariate product $\langle f, g \rangle_\mu$ and its functional $\mathcal{L}_\mu[f \cdot g]$ to column vectors. If $F = (f_1, f_2, \ldots, f_n)^t$ and $G = (g_1, g_2, \ldots, g_m)^t$ are column vectors of functions of size n and m, respectively, then we define

$$\langle F, G \rangle := \mathcal{L}_\mu[F \cdot G^T] = \int_\Omega F \cdot G^T d\mu(x_1, \ldots, x_d) = \left(\int_\Omega f_i \cdot g_j d\mu(x_1, \ldots, x_d) \right)_{i=1,\ldots,n; \ j=1,\ldots,m}.$$
$$(10)$$

In fact, we are applying the standard product $\langle f_i, g_j \rangle_\mu$ or the functional $\mathcal{L}_\mu[f_i \cdot g_j]$ to each pair i, j and placing the results in a matrix.

Let's consider it is possible to build a system of polynomials $\{\mathbb{P}_n\}_{n \geq 0}$ where

$$\langle \mathbb{P}_n, \mathbb{P}_k \rangle_\mu = \mathcal{L}_\mu[\mathbb{P}_n \mathbb{P}_k^t] = \begin{cases} 0_{r_n^d \times r_k^d} & \text{if} \quad k = 0, \ldots, n-1 \\ S_n & \text{if} \quad k = n \end{cases}$$

Where S_n is a regular squared matrix of size $r_n^d \times r_n^d$. Due to orthogonality, it is possible to give a equivalent condition:

$$\langle \mathbb{P}_n, \mathbb{X}_k \rangle_\mu = \mathcal{L}_\mu[\mathbb{P}_n \mathbb{X}_k^t] = \begin{cases} 0_{r_n^d \times r_k^d} & \text{if} \quad k = 0, \ldots, n-1 \\ S_n & \text{if} \quad k = n \end{cases} \quad (11)$$

Then, $\{\mathbb{P}_n\}_{n \geq 0}$ is called a system of orthogonal polynomials with respect to the measure μ or the functional $\mathcal{L}_\mu$. You can check more information about the existence and uniqueness of this system in [1, Ch. III, Section 3.2].

3   Type II MOP in several variables

In order to respect the original Type II MOP in one variable, given $r \in \mathbb{N}$, $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, r d-dimensional measures $\mu_1, \ldots, \mu_r$ and their respective dot products defined in (10), we are going to define the type II multiple orthogonality as a monic polynomial vector $\mathbb{P}_n = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k} \mathbb{X}_k$ which satisfies

$$\langle \mathbb{P}_n, \mathbb{X}_k \rangle_i = 0, \quad k = 0, \ldots, n_i - 1, \quad i = 1, \ldots, r \tag{12}$$

You can check the similarity between conditions (2) and (12). Nevertheless, when we work with one variable, it is known the type II polynomial is a monic polynomial whose degree is exactly $|\vec{n}|$. That is because the number of coefficients of a degree $|\vec{n}|$ univariate monic polynomial ($|\vec{n}|$) is equal to the number of orthogonality conditions and the size of matrices A defined in (9). Due to the differences between the univariate and the multivariate case, the degree n of the polynomial vector $\mathbb{P}_n$ might not be equal to $|\vec{n}|$. For now, this degree n will be considered as unknown, getting to know it later.

As mencioned previously, in the univariate multiple orthogonality, $\deg(P_{\vec{n}}) = |\vec{n}|$. The main reason this happens is because the condition $\langle P_{\vec{n}}, x^k \rangle = 0$ is a linear equation (only one). Hence, we are looking for a polynomial with $|\vec{n}|$ unknown coefficients and we have $|\vec{n}|$ linear equations deducted from the conditions (2). Then, it is possible to build a system of linear equations whose coefficient matrix is A^t given in (9).

If we work with d variables, attending to (11), remember the result of $\langle \mathbb{P}_n, \mathbb{X}_k \rangle$ is a matrix of size $r_n^d \times r_k^d$. Thus, it is possible to get $r_n^d \times r_k^d$ linear equations from each orthogonality condition.

Now, the question is: How many linear equations is it possible to get from a multi-index $\vec{n} = (n_1, \ldots, n_r)$? The answer to this problem is easy since, if we fix $i \in \{1, \ldots, r\}$:

- $\langle \mathbb{P}_n, \mathbb{X}_0 \rangle_i = 0_{r_n^d \times r_0^d}$, we get $r_n^d \times r_0^d$ equations.

- $\langle \mathbb{P}_n, \mathbb{X}_1 \rangle_i = 0_{r_n^d \times r_1^d}$, we get $r_n^d \times r_1^d$ equations.

- $\ldots$

- $\langle \mathbb{P}_n, \mathbb{X}_{n_i-1} \rangle_i = 0_{r_n^d \times r_{n_i-1}^d}$, we get $r_n^d \times r_{n_i-1}^d$ equations.

If we collect all the equations, we obtain $r_n^d \cdot \sum_{k=0}^{n_i-1} r_k^d$ linear equations for each $i \in \{1, \ldots, r\}$. Collecting the number of equations of each i, we finally get the number of equations is

$$r_n^d \sum_{i=1}^{r} \sum_{k=0}^{n_i-1} r_k^d. \tag{13}$$

On the other hand, since $\mathbb{P}_n = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k} \mathbb{X}_k$, where $G_{n,k}$ are matrices whose dimensions are $r_n^d \times r_k^d$, if we consider $\mathbb{P}_n$ a vector of degree n monic polynomials, then this matrices give us $\sum_{k=0}^{n-1} r_n^d r_k^d = r_n^d \sum_{k=0}^{n-1} r_k^d$ unknown coefficients.

If we want this system to have only one solution for a multi-index $\vec{n}$. Then, $\vec{n}$ is a valid multi-index if there exist a number $n \in \mathbb{N} \cup \{0\}$ such that:

$$\sum_{k=0}^{n-1} r_k^d = \sum_{i=1}^{r} \sum_{k=0}^{n_i-1} r_k^d. \tag{14}$$

4   The bidimensional case

Since this moment, we will assume $d = 2$ and $x := x_1, y := x_2$. Notice that, in the bidimensional case, $r_n^2 = \binom{n+1}{n} = n + 1$, which makes things much easier: $\mathbb{X}_j$ is a polynomial vector of size $j + 1$, $\mathbb{P}_n$ is a vector of size $n + 1$, and $G_{n,k}$ is a $(n + 1) \times (k + 1)$ matrix. Also, the condition (14) gets simpler because some sums become sums of an arithmetic progression. When $d = 2$, a multi-index $\vec{n} = (n_1, \ldots, n_r)$ is valid if there exists a number $n \in \mathbb{N} \cup \{0\}$ such that

$$n(n+1) = \sum_{i=1}^{r} n_i(n_i + 1). \tag{15}$$

Observe some of the valid multi-indexes and their respective degree n.

n	Valid multi-indexes when $r = 2$ and $d = 2$
0	$(0,0)$
1	$(0,1),(1,0)$
2	$(0,2),(2,0)$
3	$(0,3),(2,2),(3,0)$
4	$(0,4),(4,0)$
5	$(0,5),(5,0)$
6	$(0,6),(3,5),(5,3),(6,0)$
7	$(0,7),(7,0)$
8	$(0,8),(5,6),(6,5),(8,0)$
9	$(0,9),(9,0)$

Table 1: Valid multi-indexes when using $r = 2$ measures

Check this visual representation of the valid multi-indexes when $r = 2$ and $r = 3$.

n	Valid multi-indexes when $r = 3$ and $d = 2$
0	$(0,0,0)$
1	$(0,0,1), (0,1,0), (1,0,0)$
2	$(0,0,2), (0,2,0), (1,1,1), (2,0,0)$
3	$(0,0,3), (0,2,2), (0,3,0), (2,0,2), (2,2,0), (3,0,0)$
4	$(0,0,4), (0,4,0), (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (4,0,0)$
5	$(0,0,5), (0,5,0), (2,3,3), (3,2,3), (3,3,2), (5,0,0)$
6	$(0,0,6), (0,3,5), (0,5,3), (0,6,0), (1,4,4), (2,2,5), (2,5,2), (3,0,5),$ $(3,5,0), (4,1,4), (4,4,1), (5,0,3), (5,2,2), (5,3,0), (6,0,0)$
7	$(0,0,7), (0,7,0), (1,3,6), (1,6,3), (2,4,5), (2,5,4), (3,1,6), (3,6,1),$ $(4,2,5), (4,5,2), (5,2,4), (5,4,2), (6,1,3), (6,3,1), (7,0,0)$
8	$(0,0,8), (0,5,6), (0,6,5), (0,8,0), (3,5,5), (5,0,6), (5,3,5), (5,5,3),$ $(5,6,0), (6,0,5), (6,5,0), (8,0,0)$
9	$(0,0,9), (0,9,0), (2,3,8), (2,6,6), (2,8,3), (3,2,8), (3,8,2), (5,5,5), (6,2,6),$ $(6,6,2), (8,2,3), (8,3,2), (9,0,0)$

Table 2: Valid multi-indexes when using $r = 3$ measures
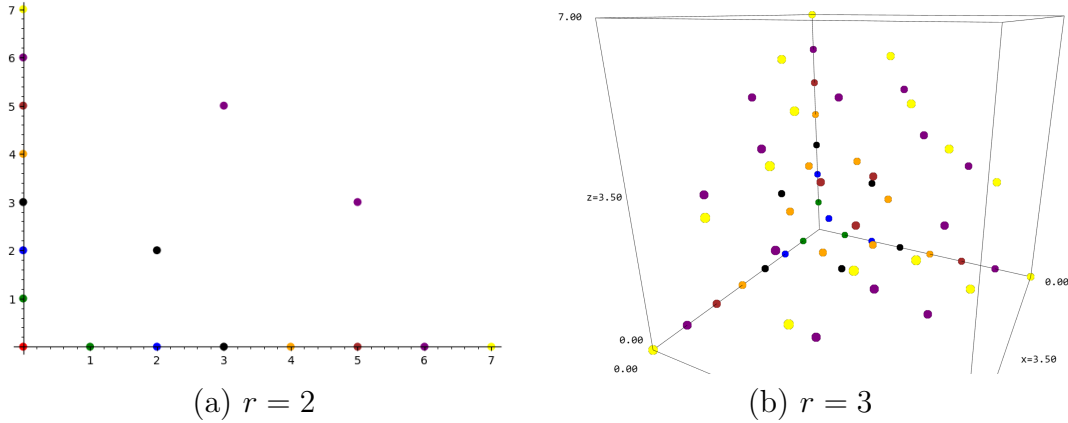

(a) $r = 2$ (b) $r = 3$

Figure 1: Graphic representation of the valid r-indexes

5   Type I MOP in two variables

Remember the type I multiple orthogonality in definition 1.2. Given $\vec{n} = (n_1, \ldots, n_r)$, it consists of a polynomial vector $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$ where $\deg(A_{\vec{n},j}) \le n_j - 1$ ($j = 1, \ldots, r$) and the polynomials satisfy (3) and (4). This two conditions can be also expresed in a more compact way if we use the dot products, see (7).

In order to generalize this type of orthogonality, the first problem is the sum located in the orthogonality condition. This forces us to define polynomials, or polynomial vectors $\mathbb{A}_{\vec{n},j}$, of the same size. If these vectors had different size, it would not be possible to make the sum. This makes impossible to considerate $\mathbb{A}_{\vec{n},j}$ a vector of degree $\le n_j - 1$ polynomials, with size n_j. We will tell you how this problem was solved and the definition we give.

Let $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ such that the condition (15) holds for some $n \in \mathbb{N}$.

Then, let $i \in \{1, \ldots, n\}$ and, for each j, the polynomial

$$A_{\vec{n},j}^{(i)}(x,y) = g_{n_j-1,n_j-1}^t \mathbb{X}_{n_j-1} + g_{n_j-1,n_j-2}^t \mathbb{X}_{n_j-2} + \cdots + g_{n_j-1,0}^t \mathbb{X}_0,$$

where $g_{n_j-1,k} \in \mathbb{R}^{k+1}$, $(k = 0, \ldots, n_j - 1)$ is a bivariate polynomial such that $\deg A_{\vec{n},j}^{(i)} \leq n_j - 1$, $(j = 1, \ldots, r)$. Now, given r 2-dimensional measures $\mu_1, \ldots, \mu_r$, we will force the polynomials $A_{\vec{n},1}^{(i)}, A_{\vec{n},2}^{(i)}, \ldots, A_{\vec{n},r}^{(i)}$ to satisfy the first type I multiple orthogonality condition:

$$\sum_{j=1}^{r} \left\langle \mathbb{X}_k, A_{\vec{n},j}^{(i)} \right\rangle_j = \begin{cases} 0_{(k+1) \times 1} & \text{if} \quad k = 0, \ldots, n-2 \\ (e_i)_{n \times 1} = (0, \ldots, 0, 1, 0, \ldots, 0)^t & \text{if} \quad k = n-1 \end{cases} \tag{16}$$

You can easily check the number of unknown coefficients of the polynomials $A_{\vec{n},j}^{(i)}$ is the same as the number of linear equations given by conditions (16). If we repeat this for every possible $i \in \{1, \ldots, n\}$, we get n lists of polynomials $A_{\vec{n},j}^{(i)}$, $j = 1, \ldots, r, i = 1, \ldots, n$. Now, let's denote

$$\mathbb{A}_{\vec{n},j} = \begin{pmatrix} A_{\vec{n},j}^{(1)} \\ \vdots \\ A_{\vec{n},j}^{(n)} \end{pmatrix}_{n \times 1},$$

a polynomial vector of size n whose components are bivariate polynomials of degree less than or equal to $n_j - 1$ $(j = 1, \ldots, r)$ and, fixed i, the polynomials $A_{\vec{n},1}^{(i)}, \ldots, A_{\vec{n},r}^{(i)}$ satisfy (16). Thus, the polynomial vectors $\mathbb{A}_{\vec{n},j}$ will satisfy:

$$\sum_{j=1}^{r} \langle \mathbb{X}_k, \mathbb{A}_{\vec{n},j} \rangle_j = \begin{cases} 0_{(k+1) \times n} & \text{if} \quad k = 0, \ldots, n-2 \\ I_n & \text{if} \quad k = n-1 \end{cases} \tag{17}$$

Observe the similarity between equations (7) and (17).

Summarizing, we present the final definitions of the multiple orthogonal polynomials in two variables.

Definition 5.1 (Type II MOP in two variables). *Let $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ such that the condition (15) holds for certain $n \in \mathbb{N}$. A vector of monic degree n polynomials $\mathbb{P}_{\vec{n}}^n = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k} \mathbb{X}_k$ is a **type II Multiple Orthogonal Polynomial Vector** if*

$$\langle \mathbb{X}_k, \mathbb{P}_{\vec{n}}^n \rangle_j = 0_{(k+1) \times (n+1)}, \quad k = 0, \ldots, n_j - 1, \quad i = 1, \ldots, r \tag{18}$$

Definition 5.2 (Type I MOP in two variables). *Let $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ such that the condition (15) holds for certain $n \in \mathbb{N}$. **Type I Multiple Orthogonal Polynomials Vectors** are presented as $(\mathbb{A}_{\vec{n},1}, \ldots, \mathbb{A}_{\vec{n},r})$, where $\mathbb{A}_{\vec{n},j} = (A_{\vec{n},j}^{(1)}, \ldots, A_{\vec{n},j}^{(n)})^t$, $j = 1, \ldots, r$ and $A_{\vec{n},j}^{(i)}$ are polynomials of degree less than or equal to $n_j - 1$ for every $i = 1, \ldots, n$. In addition, these polynomials vectors satisfy the conditions (17).*

As the univariate case, from the Type I MOP, in case the 2-dimensional measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup\limits_{j=1}^{r} \Omega_j$, i.e., $d\mu_j = w_j(x,y)d\mu(x,y)$, $(j = 1, \ldots, r)$, we can define the *Type I function*:

$$\mathbb{Q}_{\vec{n}} = \sum_{j=1}^{r} \mathbb{A}_{\vec{n},j} w_j(x,y). \tag{19}$$

Using this function, it is possible to rewrite (17) as

$$\langle \mathbb{X}_k, \mathbb{Q}_{\vec{n}} \rangle_\mu = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \ldots, n-2 \\ I_n & \text{if} \quad k = n-1 \end{cases} \tag{20}$$

As we can see, the bivariate case reduces the number of multi-indexes $(n_1, \ldots, n_r)$ which it is possible to create multiple orthogonal polynomials with. Whereas in the univariate it is possible to question the existence of MOP for every $\vec{n} \in \mathbb{N}^r$, in the bivariate case only some of the multi-indexes can give us MOP, those such that there exist $n \in \mathbb{N}$ with $n(n+1) = \sum\limits_{j=1}^{r} n_j(n_j+1)$. Given a valid $\vec{n} \in \mathbb{N}^r$, we will say it is **normal** if it is possible to calculate its type II polynomial $\mathbb{P}_{\vec{n}}^n$ and its type I polynomials $\mathbb{A}_{\vec{n},j}$ $(j = 1, \ldots, r)$. The system of 2-dimensional measures $\mu_1, \ldots, \mu_r$ will be called **perfect** if every valid multi-index is normal.

REVIEW Revisa esto Lidia porque normal y perfecto son conceptos que en una variable tienen sentido porque los tipo I y los tipo II son equivalentes, aquí de momento como poco tan seguros no estamos.

TODO Cambiar notación de las primeras definiciones de $P_{\vec{n}}^n$ y mejorar la explicación de qué es n en este caso.

TODO Quitar de la introducción el $n = |\vec{n}|$ para no confundir con las siguientes secciones

6   Biorthogonality

Let $\mu_1, \ldots, \mu_r$ be a perfect system of r 2-dimensional measures. In this section, we will explain an orthogonality relation between the type I and type II MOP. Check the preceding univariate result in [3, Theorem 23.1.6].

Theorem 6.1. *Given $\vec{n}, \vec{m}$, both satisfying* (15) *for n and m, respectively, the following biorthogonality holds for type I and type II MOP:*

$$\langle \mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\vec{m}} \rangle = \begin{cases} 0_{(n+1)\times m} & \text{if} \quad \vec{m} \leq \vec{n} \\ 0_{(n+1)\times m} & \text{if} \quad n \leq m-2 \\ I_{n+1} & \text{if} \quad n = m-1 \end{cases} \tag{21}$$

Proof. We will prove each item. First, we have

$$\langle \mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\vec{m}} \rangle = \sum_{j=1}^{r} \langle \mathbb{P}_{\vec{n}}^n, \mathbb{A}_{\vec{m},j} \rangle_j .$$

- Assume $\vec{m} \leq \vec{n}$, this means $m_j \leq n_j \;\; \forall j \in \{1, \ldots, r\}$. Then, remember $\mathbb{A}_{\vec{m},j}$ is a vector of polynomials of degree less than or equal to $m_j - 1 < n_j$ for every $j \in \{1, \ldots, r\}$ and $\langle \mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\vec{m}} \rangle = 0$ follows from type II orthogonality conditions (18).

- If $n \leq m-2$, then each polynomial of $\mathbb{P}_{\vec{n}}^n$ is a linear combination of $\mathbb{X}_{n-2}, \ldots, \mathbb{X}_1, \mathbb{X}_0$. Then, the type I orthogonality conditions (17) for $0 \leq k \leq n - 2$ shows $\langle \mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\vec{m}} \rangle = 0$.

- Finally, if $m = n + 1$, then $\mathbb{P}_{\vec{n}}^n$ is a vector of degree $m - 1$ polynomials. So, $\mathbb{P}_{\vec{n}}^n = \mathbb{X}_{m-1} + \sum_{k=0}^{m-2} G_k \mathbb{X}_k$. Because of the previous item, we have

$$\left\langle \mathbb{P}_{\vec{n}}^{m-1}, \mathbb{Q}_{\vec{m}} \right\rangle = \sum_{j=1}^{r} \left\langle \mathbb{P}_{\vec{n}}^{m-1}, \mathbb{A}_{\vec{m},j} \right\rangle_j = \sum_{j=1}^{r} \left\langle \mathbb{X}_{m-1}, \mathbb{A}_{\vec{m},j} \right\rangle_j = I_m = I_{n+1},$$

where in the second last equality we have used the normalization condition in (17).

$\square$

Observe the first item emphasises the components of the multi-indexes $\vec{n}$ and $\vec{m}$, while the last two items are true for any two multi-indexes with degree n and m whatever their components be.

This result, which is a two-dimensional extension of [3, Theorem 23.1.6], will be very useful in order to prove the next result.

7   Nearest Neighbor Recurrence Relation

The standard orthogonal polynomials always satisfy a Three-Term Recurrence Relation. That is, given an MOPS $\{P_n\}$ with respect to a dot product $\langle \cdot, \cdot \rangle$ or its respective functional $\mathcal{L}$, there exist constants α_n, β_n such that

$$x P_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}.$$

The nearest Neighbor Recurrence Relation, in univariate multiple orthogonality, extends this result, allowing us to express a type II polynomial multiplied by x: $x P_{\vec{n}}$ as a linear combination of the polynomial itself, $P_{\vec{n}}$, a beyond neighbor, *i.e.* $P_{\vec{n}+e_k}$ and r above neighbors, see [3, Theorem 23.1.7]. We are going to give a proof of a first generalization to bivariate MOP of this result.

Theorem 7.1. *Let $\mu_1, \ldots, \mu_r$ be a perfect system of 2-dimensional measures. Let $\vec{n} \in \mathbb{N}^r$ be a multi-index satisfying (15) for $n \in \mathbb{N}$ and let us consider a path $\{\overrightarrow{m_k} : k = 0, \ldots, n+1\}$ where $\overrightarrow{m_0} = \vec{0}, \overrightarrow{m_n} = \vec{n}$, each $\overrightarrow{m_k}$ satisfy (15) for k and $\overrightarrow{m_k} \leq \overrightarrow{m_{k+1}}$ for $k = 0, \ldots, n+1$. Then, there exist matrices $A_0, \ldots, A_r$ of sizes $(n+1) \times (n-j+1)$, $(j = 0, \ldots, r)$ such that*

$$x\mathbb{P}_{\vec{n}}^n = L_{n+1,1}\mathbb{P}_{\overrightarrow{m_{n+1}}}^{n+1} + A_0\mathbb{P}_{\vec{n}}^n + \sum_{j=1}^r A_j \mathbb{P}_{\overrightarrow{m_{n-j}}}^{n-j}, \tag{22}$$

where $L_{n+1,1} = (I_{n+1}|0_{(n+1)\times 1})$.

In addition, $A_j = \left\langle x\mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\overrightarrow{m_{n-j+1}}}\right\rangle$.

Proof. As we are in a perfect system, we can assume the existence of $\mathbb{P}_{\overrightarrow{m_k}}^k$ for every $k = 0, \ldots, n+1$. As we know, $x\mathbb{P}_{\vec{n}}^n$ is a vector of degree $n+1$ polynomials and its size is also $n+1$. It is possible to write $x\mathbb{P}_{\vec{n}}^n$ as a linear combination of $\{\mathbb{P}_{\overrightarrow{m_k}}^k, k = 0, \ldots, n+1\}$ whose coefficients are matrices M_k:

$$x\mathbb{P}_{\vec{n}}^n = \sum_{k=0}^{n+1} M_k \mathbb{P}_{\overrightarrow{m_k}}^k,$$

where M_k is a $(n+1) \times (k+1)$ matrix $(k = 0, \ldots, n+1)$.

Observing the leader term of $x\mathbb{P}_{\vec{n}}^n$, which is $x\mathbb{X}_n$, and equalizing this to the leader term of $M_{n+1}\mathbb{P}_{\overrightarrow{m_{n+1}}}^{n+1}$, which is $M_{n+1}\mathbb{X}_{n+1}$ we can easily check $M_{n+1} = L_{(n+1),1}$. Now, we have

$$\mathbb{P}_{\vec{n}}^n = L_{n+1,1}\mathbb{P}_{\overrightarrow{m_{n+1}}}^{n+1} + \sum_{k=0}^n M_k \mathbb{P}_{\overrightarrow{m_k}}^k.$$

Now, apply the dot product to both sides of the equation by $\mathbb{Q}_{\overrightarrow{m_l}}$ and observe:

- If $\overrightarrow{m_l} \leq \overrightarrow{m_j}$, which always happens if $l \leq j \leq n+1$, then, due to the first item of theorem 6.1, $\left\langle \mathbb{P}_{\overrightarrow{m_j}}^j, \mathbb{Q}_{\overrightarrow{m_l}}\right\rangle = 0_{(j+1)\times l}$ whenever $l \leq j$.

- If $j \leq l-2$, then, from the second item of theorem 6.1, $\left\langle \mathbb{P}_{\overrightarrow{m_j}}^j, \mathbb{Q}_{\overrightarrow{m_l}}\right\rangle = 0_{(j+1)\times l}$.

- Finally, if $j = l-1$, from the last item of theorem 6.1 we deduce $\left\langle \mathbb{P}_{\overrightarrow{m_j}}^j, \mathbb{Q}_{\overrightarrow{m_l}}\right\rangle = I_{j+1}$.

Summarizing, we have

$$\left\langle x\mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\overrightarrow{m_l}}\right\rangle = L_{n+1,1}\left\langle \mathbb{P}_{\overrightarrow{m_{n+1}}}^{n+1}, \mathbb{Q}_{\overrightarrow{m_l}}\right\rangle + \sum_{j=0}^n M_j \left\langle \mathbb{P}_{\overrightarrow{m_j}}^j, \mathbb{Q}_{\overrightarrow{m_l}}\right\rangle.$$

Applying the previous items deducted from theorem 6.1, we have

$$\left\langle x\mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\overrightarrow{m_l}}\right\rangle = M_{l-1} \quad l = 1, \ldots, n+1, \tag{23}$$

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Now, if $l \leq n - r$, then $\overrightarrow{m_l} \leq \overrightarrow{m_{n-r}}$. So, choosing $l \leq n - r$:

$$\left\langle x\mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\overrightarrow{m_l}} \right\rangle = \sum_{j=1}^r \left\langle \mathbb{P}_{\vec{n}}^n, x\mathbb{A}_{\overrightarrow{m_l},j} \right\rangle_j.$$

But $x\mathbb{A}_{\overrightarrow{m_l},j}$ are vectors of degree less than or equal to $(\overrightarrow{m_l})_j \leq (\overrightarrow{m_{n-r}})_j < n_j$. Then, the product vanishes because of type II orthogonality (18). This means

$$\left\langle x\mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\overrightarrow{m_l}} \right\rangle = M_{l-1} = 0 \quad l = 1, \ldots, n - r.$$

Thus, the final expression is

$$\mathbb{P}_{\vec{n}}^n = L_{n+1,1}\mathbb{P}_{\overrightarrow{m_{n+1}}}^{n+1} + \sum_{k=n-r}^n M_k \mathbb{P}_{\overrightarrow{m_k}}^k.$$

If we rename $A_k = M_{n-k}$ for $k = n - r, ..., n$, then we get the desired expression (22).

Finally, due to (23),
$$A_j = M_{n-j} = \left\langle x\mathbb{P}_{\vec{n}}^n, \mathbb{Q}_{\overrightarrow{m_{n-j+1}}} \right\rangle.$$

$\square$

REVIEW Espero haberlo hecho bien

8 Conclusion

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enfatizar en las coordenadas

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