Orthogonal Polynomials

Applications and Bivariate Multiple Orthogonality

J. Antonio Villegas, Lidia Fernández

(jantoniovr@ugr.es, lidiafr@ugr.es)

Departamento de Matemática Aplicada and Instituto de Matemáticas IMAG, Universidad de Granada.

1. Introduction to Orthogonal Polynomials

Orthogonal Polynomials theory is a branch of approximation theory which could be useful in many other research topics as approximation of functions, differential equations, numerical analysis, stochastic processes and even in other sciences like optics. Everybody knows polynomials and orthogonality, however, what are "orthogonal polynomials"?

Let us consider $\mathbb{R}[x]$, the vector space of univariate polynomials. As a vector space, we can provide $\mathbb{R}[x]$ with an inner product $\langle\cdot,\cdot\rangle : \mathbb{R}[x]\times\mathbb{R}[x] \longrightarrow \mathbb{R}$, which it is often defined using integral functionals. This is, given a one-dimensional measure μ, we can define the inner product

$$\langle P, Q\rangle_\mu := \int P(x)Q(x)d\mu(x), \quad \forall P, Q \in \mathbb{R}[x] \tag{1}$$

Thus, using this inner product, two polynomials P, Q are orthogonal if $\langle P, Q\rangle_\mu = 0$. This motivates the first definition

Definition 1. *Let $\{P_n\}_{n\geq 0}$ be a sucession of polynomials. Then it is a* ***Orthogonal Polynomials Sucession*** *(OPS) if:*

1. $\deg(P_n) = 0, \quad n \geq 0.$
2. $\langle P_n, P_k\rangle = 0, \quad 0 \geq k < n, n \geq 0.$
3. $\langle P_n, P_n\rangle \neq 0, \quad n \geq 0.$

Then, an OPS shapes a orthogonal basis of $\mathbb{R}[x]$. You can build the OPS from the measure using many different ways: applying Gram-Schmidt orthogonalization process to the usual basis $\{1, x, x^2, \dots\}$, solving linear systems of equations, or applying other proper methods like the Three Term Relation or Rodrigues Formula for the so called 'classic' orthogonal polynomials.

2. Multiple Orthogonality

Multiple Orthogonality is a theory that extends standard orthogonality. In it, polynomials (defined on the real line) satisfy orthogonality relations with respect to more than one measure. There are also two different types of multiple orthogonality.

First, let us consider r different real measures $\mu_1, \dots, \mu_r$ and denote as $\langle\cdot,\cdot\rangle_j$ the respective integral inner product given by (1) $(j = 1, \dots, r)$. We will use multi-indices $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, and denote $|\vec{n}| := n_1 + \cdots + n_r$.

Definition 2 (Type II Multiple Orthogonal Polynomials)**.** *Let $\vec{n} = (n_1, \dots, n_r)$. A monic polynomial $P_{\vec{n}}(x)$ is a* ***Type II Multiple Orthogonal Polynomial*** *if $\deg(P_{\vec{n}}) = |\vec{n}|$ and*

$$\boxed{\left\langle P_{\vec{n}}, x^k\right\rangle_j = 0, \quad 0 \leq k \leq n_j - 1, \quad j = 1, \dots, r} \tag{2}$$

Definition 3 (Type I Multiple Orthogonal Polynomials)**.** *Let $\vec{n} = (n_1, \dots, n_r)$.* ***Type I Multiple Orthogonal Polynomials*** *are presented in a vector $(A_{\vec{n},1}(x), \dots, A_{\vec{n},r}(x))$, where $\deg(A_{\vec{n},j}) \leq n_j - 1$, $(j = 1, \dots, r)$ and satisfy*

$$\boxed{\sum_{j=1}^{r}\left\langle A_{\vec{n},j}, x^k\right\rangle_j = \begin{cases} 0 & \text{if} \quad 0 \leq k \leq |\vec{n}| - 2 \\ 1 & \text{if} \quad k = |\vec{n}| - 1. \end{cases}} \tag{3}$$

Multiple Orthogonal Polynomials theory has been hardly researched recently. It has applications to Hermite-Padé rational approximations, random matrices, non-intersecting random paths or integrable systems. However, this theory has only been studied in the univariate case. Our recent research is mainly focused on finding generalisations of MOP to bivariate Orthogonal Polynomials. We are currently focusing in two different possible approaches.

3. First approach: Polynomial Vectors

The first option we studied was using the polynomial vector notation for bivariate orthogonal polynomials (see [2, Ch. III, Section 3.2]). In this sense, we denote $\mathbb{X}_k = (x^k, x^{k-1}y, \dots, y^k)^t$ and polynomial vectors as

$$\mathbb{P}_n = \sum_{k=0}^{n} G_{n,k}\mathbb{X}_k,$$

where $G_{n,k}$ is a $(n+1)\times(k+1)$ matrix $(k = 0, \dots, n)$. Given a bidimensional measure $\mu(x,y)$ and its respective integral inner product, it is possible to extend the latter to vectors by applying it to each pair of polynomials.

$$\langle \mathbb{P}_n, \mathbb{P}_m\rangle_\mu := \int \mathbb{P}_n \cdot \mathbb{P}_m^t d\mu(x,y) = \left(\int P_n^{(i)}(x,y)P_m^{(j)}(x,y)d\mu(x,y)\right)_{i,j=1}^{n,m}. \tag{4}$$

With these preliminaries, we can define the two types of orthogonality. Let $r \in \mathbb{N}$, r 2-dimensional measures $\mu_1, \dots, \mu_r$, their respective matrix inner products defined in (4) and $\vec{n} \in \mathbb{N}^r$ satisfying

$$\boxed{\exists\, n \in \mathbb{N} \text{ s.t. } n(n+1) = \sum_{j=1}^{r} n_j(n_j+1).} \tag{5}$$

Definition 4 (Bivariate Type II multiple orthogonality)**.** *We define the* ***bivariate type II multiple orthogonal polynomial vector*** *as a monic polynomial vector*

$$\mathbb{P}_{\vec{n}} = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k}\mathbb{X}_k$$

which satisfies

$$\boxed{\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_k\rangle_j = 0_{(n+1)\times(k+1)}, \quad k = 0, \dots, n_j - 1, j = 1, \dots, r.} \tag{6}$$

Definition 5 (Bivariate Type I multiple orthogonality)**.** *For each $j \in \{1, \dots, r\}$, we define the* ***bivariate type I multiple orthogonal polynomial vectors*** *as $\mathbb{A}_{\vec{n},j} = (A_{\vec{n},j}^{(1)}, \dots, A_{\vec{n},j}^{(n)})^t$, a polynomial vector of size n whose components are bivariate polynomials of degree less than or equal to $n_j - 1$ $(j = 1, \dots, r)$ and which will satisfy:*

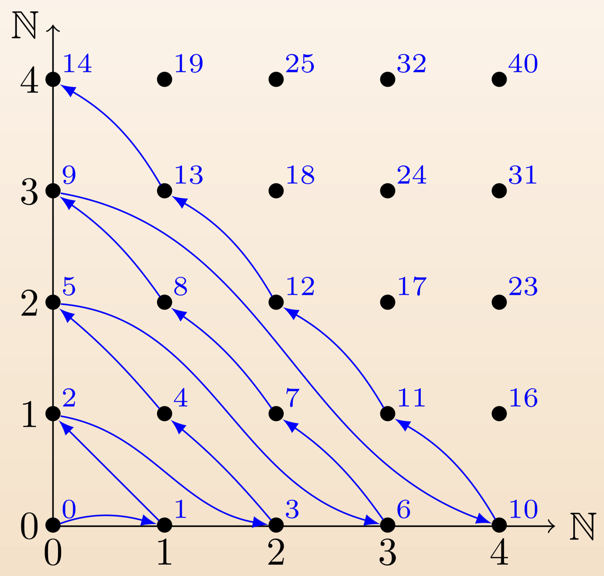
$$\boxed{\sum_{j=1}^{r}\langle \mathbb{X}_k, \mathbb{A}_{\vec{n},j}\rangle_j = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \dots, n-2 \\ I_n & \text{if} \quad k = n-1 \end{cases}} \tag{7}$$

With these two definitions, type II and type I bivariate MOP are equivalent, in the sense that, for a multi-index $\vec{n} \in \mathbb{N}^r$ satisfying (5), the type II polynomial vector exists if, and only if, the type I polynomial vectors does. The main problem we found with this approach is that there exist several multi-indices $\vec{n} \in \mathbb{N}^r$ not satisfying (5), so that type I and type II polynomials do not exist or they are not unique.

4. Second approach: Graded Reverse Lex. Order

The second way we tried was by using the basis of bivariate polynomials ordered by the graded reverse lexicographic order, this is

$$\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \dots\}$$



As we know $\mathbb{N}$ and $\mathbb{N}^2$ are isomorphic, we will use the "Cantor Pairing Function" π, which is a bijection between $\mathbb{N}$ and $\mathbb{N}^2$. Its expression is given by

$$\pi(k_1, k_2) = \frac{1}{2}(k_1 + k_2)(k_1 + k_2 + 1) + k_2;$$

Thus, we tried to do a generalization of univariate multiple orthogonality where $|\vec{n}| = \pi(\exp(P_{\vec{n}}))$, instead of the degree of type I polynomial, where the function "exp" represents the exponent of a polynomial (not the exponential function), *i.e.*, if

$$P(x,y) = c_k x^t y^s + c_{k-1}x^{t+1}y^{s-1} + \cdots c_2 y + c_1 x + c_0,$$

then $\exp(P) = (t,s)$ and $\pi(t,s) = k$.

Now, given $r \in \mathbb{N}$, $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ and a system of r 2-dimensional measures $\mu_1, \dots, \mu_r$, we are now presenting the definitions of type I and II multiple orthogonality following this approach.

Definition 6 (Type II Multiple Orthogonality)**.** *The* ***bivariate type II multiple orthogonal polynomial*** *$P_{\vec{n}}(x,y)$ is a monic polynomial such that $\exp(P_{\vec{n}}) = \pi^{-1}(|\vec{n}|)$, and satisfying*

$$\boxed{\left\langle P_{\vec{n}}, x^t y^s\right\rangle_j = 0, \quad 0 \leq \pi(t,s) \leq n_j - 1, j = 1, \dots, r.} \tag{8}$$

Definition 7. ***Bivariate type I multiple orthogonal polynomials*** *are often presented in a vector $(A_{\vec{n},1}(x,y), \dots, A_{\vec{n},r}(x,y))$ where $\exp(A_{\vec{n},j}) \leq \pi^{-1}(n_j - 1)$. This means, if $n_j - 1 = \pi(t_j, s_j)$, then $A_{\vec{n},j} = c_{n_j-1}x^{t_j}y^{s_j} + \cdots + c_0$. This polynomials satisfy*

$$\boxed{\sum_{j=1}^{r}\left\langle A_{\vec{n},j}, x^t y^s\right\rangle_j = \begin{cases} 0 & \text{if} \quad 0 \leq \pi(t,s) \leq |\vec{n}| - 2 \\ 1 & \text{if} \quad \pi(t,s) = |\vec{n}| - 1 \end{cases}}. \tag{9}$$

As occurs in the first approach, Type II and Type I polynomials existence is equivalent.

With the first approach only a few multi-indices allowed us to create MOP vectors, whereas in this second approach it is possible to calculate bivariate MOP with any $\vec{n} \in \mathbb{N}^r$. Apart from that, the degree of the type II polynomials and the number of orthogonality conditions in type I orthogonality is exactly $|\vec{n}|$, just like happens in univariate multiple orthogonality. Nevertheless, the conditions (8) and (9) are less intuitive than (6) and (5) with respect to the original conditions (2) and (3).

5. Future research topics

Multiple Orthogonal Polynomials is a theory that is currently being widely researched. At this moment, we have generalised to the bivariate case some of the main univariate results. But there are still many results and applications to work on:

- Study product of two univariate MOP $P_{\vec{n}}(x)P_{\vec{m}}(y)$ as a bivariate MOP.
- Find Nearest Neighbour Relation generalizations to the bivariate case for general paths in both approaches.
- Find a possible connection between the two definitions.
- Look for applications to Hermite-Padé rational approximations and random matrices.
- Generalise existing examples to the bivariate case: Jacobi-Piñeiro, Jacobi-Angelesco, OP defined in the Simplex, etc.
- ...

References and Acknowledgements

[1] T. S. Chiara, *An Introduction to Orthogonal Polynomials*, Dover Books on Mathematics, Dover Publications, 2011

[2] C. F. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, 2nd edition, Encyclopedia of Mathematics and its Applications, vol. 155, Cambridge Univ. Press, Cambridge, 2014.

[3] W. van Assche, *Orthogonal and multiple orthogonal polynomials, random matrices, and painlevé equations*, in "Orthogonal Polynomials" (M. Foupouagnigni, W. Koepf, eds), Tutorials, Schools and Workshops in the Mathematical Sciences, Springer Nature Switzerland (2020) 629–683.

[4] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Cambridge University Press, Encyclopedia of mathematics and its applications, 98, (2005).

[5] A. Martínez-Finkelshtein, W. van Assche, *What is ... a multiple orthogonal polynomial?*, Notices of the American Mathematical Society, 63, (2016) 1029–1031.

Acknowledgements Universidad de Granada; GOYA: "Grupo de Ortogonalidad Y Aplicaciones" and "Departamento de Matemática Aplicada".

Research supported by IMAG, María de Maeztu Grant "IMAG CEX2020–001105–M"