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EXCELENCIA MARÍA DE MAEZTU

MULTIPLE ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

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MULTIPLE ORTHOGONALITY

Polynomials satisfy orthogonality relations with respect to r measures: $\mu_1, \ldots, \mu_r$. Let $\langle \cdot, \cdot \rangle_j$ be the respective integral inner product ($j = 1, \ldots, r$) and $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$, $|\vec{n}| = n_1 + \cdots + n_r$.

Type II Multiple Orthogonal Polynomials

$P_{\vec{n}}(x)$ is a monic polynomial with $\deg(P_{\vec{n}}) = |\vec{n}|$ and

$$\left\langle P_{\vec{n}}, x^k \right\rangle_j = 0, \quad k = 0, \ldots, n_j - 1, \;\; j = 1, \ldots, r \tag{1}$$

Type I Multiple Orthogonal Polynomials

$(A_{\vec{n},1}(x), \ldots, A_{\vec{n},r}(x))$, vector of polynomials, $\deg(A_{\vec{n},j}) \leq n_j - 1$, ($j = 1, \ldots, r$) and

$$\sum_{j=1}^{r} \left\langle A_{\vec{n},j}, x^k \right\rangle_j = \begin{cases} 0 & \text{if} \quad k = 0, \ldots, |\vec{n}| - 2 \\ 1 & \text{if} \quad k = |\vec{n}| - 1. \end{cases} \tag{2}$$

BIVARIATE OP

$$\mathbb{X}_j = (x^j, x^{j-1}y, \ldots, y^j)^t.$$

A column polynomial vector of degree n can be represented as

$$\mathbb{P}_n = G_{n,n}\mathbb{X}_n + G_{n,n-1}\mathbb{X}_{n-1} + \cdots + G_{n,1}\mathbb{X}_1 + G_{n,0}\mathbb{X}_0,$$

where $G_{n,j}$ are matrices of size $(n+1) \times (j+1)$.

Given a bidimensional measure $\mu(x, y)$, $\mathrm{supp}(\mu) = \Omega \subseteq \mathbb{R}^2$, we can extend the definition of inner product $\langle f, g \rangle_\mu$ to column vectors.

$F = (f_1, f_2, \ldots, f_n)^t$, $G = (g_1, g_2, \ldots, g_m)^t$ column vectors of functions. Then we define

$$\langle F, G \rangle_\mu := \mathcal{L}_\mu[F \cdot G^T] = \int_\Omega F \cdot G^T d\mu = \left(\int_\Omega f_i \cdot g_j d\mu \right)_{i,j=1}^{n,m}. \tag{3}$$

BIVARIATE MULTIPLE ORTHOGONAL POLYNOMIALS

Type II Multiple Orthogonal Polynomials

$$\mathbb{P}_{\vec{n}}^{n} = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k}\mathbb{X}_k$$

which satisfies

$$\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_k \rangle_j = 0_{(n+1)\times(k+1)}, \quad k = 0, \ldots, n_j - 1, j = 1, \ldots, r. \tag{4}$$

Type I Multiple Orthogonal Polynomials

$$\mathbb{A}_{\vec{n},j} = (A_{\vec{n},j}^{(1)}(x,y), \ldots, A_{\vec{n},j}^{(n)}(x,y))^t \ \ (\deg A_{\vec{n},j}^{(i)} \leq n_j - 1) \ \ (j = 1, \ldots, r)$$

satisfying

$$\sum_{j=1}^{r} \langle \mathbb{X}_k, \mathbb{A}_{\vec{n},j} \rangle_j = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \ldots, n-2 \\ I_n & \text{if} \quad k = n-1 \end{cases} \tag{5}$$

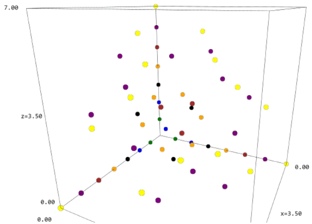
Type I and Type II MOP are equivalent: $\exists\, \mathbb{P}_{\vec{n}}^{n} \Leftrightarrow \exists\, \mathbb{A}_{\vec{n},j} \ \ (j = 1, \ldots, r)$.

ADMISSIBLE MULTI-INDICES

A multi-index $\vec{n}$ is **admissible** if there exist a number $n \in \mathbb{N}_0$ such that:

$$\boxed{n(n+1) = \sum_{j=1}^{r} n_j(n_j+1).} \tag{6}$$

This number n is the degree of type II polynomials $\mathbb{P}_{\vec{n}}^{n}$ and the size of type I polynomial vectors $\mathbb{A}_{\vec{n},j}$.



BIORTHOGONALITY

There is a relation between type I and type II MOP:

Theorem 1

Let $\mu_1, \ldots, \mu_r$ be a perfect system of r 2-dimensional measures. Given $\vec{n}$, $\vec{m}$ multi-indices satisfying (6) for n and m, respectively, the following biorthogonality holds for type I and type II MOP:

$$\boxed{\langle \mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}} \rangle = \begin{cases} 0_{(n+1)\times m} & \text{if} \quad \vec{m} \leq \vec{n} \\ 0_{(n+1)\times m} & \text{if} \quad n \leq m-2 \\ I_{n+1} & \text{if} \quad n = m-1. \end{cases}} \tag{7}$$

NEAREST NEIGHBOR RELATION

Generalisation of the TTRR satisfied by any OPS.

Theorem 2

Let $\mu_1, \ldots, \mu_r$ be a perfect system of 2-dimensional measures. Let $\vec{n} \in \mathbb{N}^r$ be a multi-index satisfying (6) for $n \in \mathbb{N}$ and let us consider a path $\{\vec{m}_k : k = 0, \ldots, n+1\}$ where $\vec{m}_0 = \vec{0}$, $\vec{m}_n = \vec{n}$, each $\vec{m}_k$ satisfies (6) for k and $\vec{m}_k \leq \vec{m}_{k+1}$ for $k = 0, \ldots, n$. Then, there exist matrices $A_0, \ldots, A_r$ of sizes $(n+1) \times (n-j+1)$, $(j = 0, \ldots, r)$ such that

$$x\mathbb{P}^n_{\vec{n}} = L_{n+1,1}\mathbb{P}^{n+1}_{\vec{m}_{n+1}} + A_0\mathbb{P}^n_{\vec{n}} + \sum_{j=1}^{r} A_j\mathbb{P}^{n-j}_{\vec{m}_{n-j}}, \tag{8}$$

where $L_{n+1,1} = (I_{n+1}|0_{(n+1)\times 1})$. In addition, $A_j = \left\langle x\mathbb{P}^n_{\vec{n}}, \mathbb{Q}_{\vec{m}_{n-j+1}} \right\rangle$.

FUTURE RESEARCH TOPICS

- Find a Nearest Neighbor Relation for more general paths and other types of Nearest Neighbor Relations.
- Christoffel-Darboux formula.
- Look for applications to Hermite-Padé rational approximation and random matrices.
- Generalise existing examples to the bivariate case: Jacobi-Piñeiro, Jacobi-Angelesco, OP defined in the Simplex, etc.
- ...

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