Orthogonal Polynomials

Applications and Bivariate Multiple Orthogonality

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1. Introduction to Orthogonal Polynomials

Orthogonal Polynomials theory is a branch of approximation theory which could be useful in many other research topics as approximation of functions, differential equations, numerical analysis, stochastic processes and even in other sciences like optics. Everybody knows polynomials and orthogonality, however, what are "orthogonal polynomials"?

Let us consider $\mathbb{R}[x]$, the vector space of univariate polynomials. As a vector space, we can provide $\mathbb{R}[x]$ with an inner product $\langle\cdot,\cdot\rangle : \mathbb{R}[x]\times\mathbb{R}[x] \longrightarrow \mathbb{R}$, which it is often defined using integral functionals. This is, given a one-dimensional measure μ, we can define the inner product

$$\langle P, Q\rangle_\mu := \int P(x)Q(x)d\mu(x), \quad \forall P,Q \in \mathbb{R}[x]$$

Thus, using this inner product, two polynomials P, Q are orthogonal if $\langle P,Q\rangle_\mu = 0$. This motivates the first definition

Definition 1. *Let $\{P_n\}_{n\geq 0}$ be a sucession of polynomials. Then it is a **Orthogonal Polynomials Sucession** (OPS) if:*

1. $\deg(P_n) = 0, \;\; n \geq 0.$
2. $\langle P_n, P_k\rangle = 0, \;\; 0 \geq k < n, n \geq 0.$
3. $\langle P_n, P_n\rangle \neq 0, \;\; n \geq 0.$

Then, an OPS shapes a orthogonal basis of $\mathbb{R}[x]$. You can build the OPS from the measure using many different ways: applying Gram-Schmidt orthogonalization process to the usual basis $\{1, x, x^2, \dots\}$, solving linear systems of equations, or applying other proper methods like the Three Term Relation or Rodrigues Formula for the so called 'classic' orthogonal polynomials.

2. Multiple Orthogonality

Multiple Orthogonality is a theory that extends standard orthogonality. In it, polynomials (defined on the real line) satisfy orthogonality relations with respect to more than one measure. There are also two different types of multiple orthogonality.

First, let us consider r different real measures $\mu_1,\dots,\mu_r$ such that $\Omega_j = \operatorname{supp}(\mu_j) \subseteq \mathbb{R}$ and denote as $\langle\cdot,\cdot\rangle_j$ the respective integral inner product ($j = 1,\dots,r$). We will use multi-indices $\vec{n} = (n_1,\dots,n_r) \in \mathbb{N}^r$, and denote $|\vec{n}| := n_1 + \cdots + n_r$. These multi-indices determine the orthogonality relations with each measure.

Definition 2 (Type II Multiple Orthogonal Polynomials). *Let $\vec{n} = (n_1,\dots,n_r)$. A monic polynomial $P_{\vec{n}}(x)$ is a **type II multiple orthogonal polynomial** if $\deg(P_{\vec{n}}) = |\vec{n}|$ and*

$$\boxed{\left\langle P_{\vec{n}}, x^k\right\rangle_j = 0, \quad k = 0,\dots,n_j - 1, \;\; j = 1,\dots,r} \tag{1}$$

Definition 3 (Type I Multiple Orthogonal Polynomials). *Let $\vec{n} = (n_1,\dots,n_r)$. **Type I Multiple Orthogonal Polynomials** are presented in a vector $(A_{\vec{n},1}(x),\dots,A_{\vec{n},r}(x))$, where $\deg(A_{\vec{n},j}) \leq n_j - 1$, $(j = 1,\dots,r)$ and satisfy*

$$\boxed{\sum_{j=1}^{r}\left\langle A_{\vec{n},j}, x^k\right\rangle_j = \begin{cases} 0 & \text{if} \quad k = 0,\dots,|\vec{n}|-2 \\ 1 & \text{if} \quad k = |\vec{n}|-1. \end{cases}} \tag{2}$$

Multiple Orthogonal Polynomials theory has been hardly researched recently. It has applications to Hermite-Padé rational approximations, random matrices, non-intersecting random paths or integrable systems. However, this theory has only been studied in the univariate case. Mi recent research is mainly focused on finding generalisations of MOP to bivariate Orthogonal Polynomials.

3. First approach: Polynomial Vectors

The first option we studied was using the polynomial vector notation for bivariate orthogonal polynomials (see [1, Ch. III, Section 3.2]). In this sense, we denote $\mathbb{X}_k = (x^k, x^{k-1}y,\dots,y^k)^t$ and polynomial vectors as

$$\mathbb{P}_n = \sum_{k=0}^{n} G_{n,k}\mathbb{X}_k,$$

where $G_{n,k}$ is a $(n+1)\times(k+1)$ matrix $(k = 0,\dots,n)$. Given a bidimensional measure $\mu(x,y)$ and its respective integral inner product, it is possible to extend the latter to vectors by applying it to each pair of polynomials.

$$\langle\mathbb{P}_n, \mathbb{P}_m\rangle_\mu := \int \mathbb{P}_n\cdot\mathbb{P}_m^t d\mu(x,y) = \left(\int P_n^{(i)}(x,y)P_m^{(j)}(x,y)d\mu(x,y)\right)_{i,j=1}^{n,m}. \tag{3}$$

With these preliminaries, we can define the two types of orthogonality.

Definition 4 (Bivariate Type II multiple orthogonality). *given $r \in \mathbb{N}$, $\vec{n} = (n_1,\dots,n_r) \in \mathbb{N}^r$, r 2-dimensional measures $\mu_1,\dots,\mu_r$ and their respective matrix inner products defined in* (3)*, we define the **bivariate type II multiple orthogonal polynomial vector** as a monic polynomial vector*

$$\mathbb{P}_{\vec{n}} = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k}\mathbb{X}_k$$

which satisfies

$$\boxed{\left\langle\mathbb{P}_{\vec{n}}, \mathbb{X}_k\right\rangle_j = 0_{(n+1)\times(k+1)}, \quad k = 0,\dots,n_j-1, j = 1,\dots,r.} \tag{4}$$

Definition 5 (Bivariate Type I multiple orthogonality). *Let $\vec{n} = (n_1,\dots,n_r) \in \mathbb{N}^r$ and, for each j, we define the **bivariate type I multiple orthogonal polynomial vectors** as $\mathbb{A}_{\vec{n},j} = (A_{\vec{n},j}^{(1)},\dots,A_{\vec{n},j}^{(n)})^t$, a polynomial vector of size n whose components are bivariate polynomials of degree less than or equal to $n_j - 1$ $(j = 1,\dots,r)$ and which will satisfy:*

$$\boxed{\sum_{j=1}^{r}\left\langle\mathbb{X}_k, \mathbb{A}_{\vec{n},j}\right\rangle_j = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0,\dots,n-2 \\ I_n & \text{if} \quad k = n-1 \end{cases}} \tag{5}$$

With these two definitions, type II and type I bivariate MOP are equivalent, in the sense that, for a multi-index $\vec{n} \in \mathbb{N}^r$, the type II polynomial vector exists if, and only if, the type I polynomial vectors does. The main problem we found with this approach is that there exist several multi-indices $\vec{n} \in \mathbb{N}^r$ such that the type I and type II polynomials do not exist or they are not unique. In fact, the multi-indices which allow the existence of bivariate type I and II MOP are those such that there exist a number $n \in \mathbb{N}_0$ satisfying:

$$\boxed{n(n+1) = \sum_{j=1}^{r} n_j(n_j+1).} \tag{6}$$

This number n is the degree of type II polynomials and the size of the type I vectors. We call these multi-indices n-**admissible**.

4. Second Approach: Using the Koornwinder Basis

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5. Future research topics

Multiple Orthogonal Polynomials is a theory that is currently being widely researched. At this moment, we have generalised to the bivariate case some of the main univariate results. But there are still many results and applications to work on:

- Find a Nearest Neighbor Relation for more general paths and other types of Nearest Neighbor Relations.
- Christoffel-Darboux formula.
- Look for applications to Hermite-Padé rational approximations and random matrices.
- Generalise existing examples to the bivariate case: Jacobi-Piñeiro, Jacobi-Angelesco, OP defined in the Simplex, etc.
- ...

References and Acknowledgements

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