Multiple Orthogonal Polynomials in Two Variables

Lidia Fernández and Juan Antonio Villegas-Recio
(lidiafr@ugr.es, jantoniovr@ugr.es)
Departamento de Matemática Aplicada and Instituto de Matemáticas IMAG, Universidad de Granada, Spain.

1. Introduction

Multiple Orthogonality is a theory that extends standard orthogonality. In it, polynomials (defined on the real line) satisfy orthogonality relations with respect to more than one measure. There are also two different types of multiple orthogonality.

First, let us consider r different real measures $\mu_1, \dots, \mu_r$ such that $\Omega_j = \operatorname{supp}(\mu_j) \subseteq \mathbb{R}$ and denote as $\langle \cdot, \cdot \rangle_j$ the respective integral inner product ($j = 1, \dots, r$). We will use multi-indices $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, and denote $|\vec{n}| := n_1 + \cdots + n_r$. These multi-indices determine the orthogonality relations with each measure.

Definition 0.1 (Type II Multiple Orthogonal Polynomials). *Let $\vec{n} = (n_1, \dots, n_r)$. A monic polynomial $P_{\vec{n}}(x)$ is a* ***type II multiple orthogonal polynomial*** *if $\deg(P_{\vec{n}}) = |\vec{n}|$ and*

$$\boxed{\langle P_{\vec{n}}, x^k \rangle_j = 0, \quad k = 0, \dots, n_j - 1, \quad j = 1, \dots, r} \tag{1}$$

Definition 0.2 (Type I Multiple Orthogonal Polynomials). *Let $\vec{n} = (n_1, \dots, n_r)$. Type I Multiple Orthogonal Polynomials are presented in a vector $(A_{\vec{n},1}(x), \dots, A_{\vec{n},r}(x))$, where $\deg(A_{\vec{n},j}) \le n_j - 1$, ($j = 1, \dots, r$) and these polynomials satisfy*

$$\boxed{\sum_{j=1}^{r} \langle A_{\vec{n},j}, x^k \rangle_j = \begin{cases} 0 & \text{if} \quad k = 0, \dots, |\vec{n}| - 2 \\ 1 & \text{if} \quad k = |\vec{n}| - 1 \end{cases}} \tag{2}$$

Whenever the measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup_{i=1}^{r} \Omega_i$, *i.e.*, $d\mu_j = w_j(x) d\mu(x)$, ($j = 1, \dots, r$), it is possible to define the *Type I function* as

$$Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x) w_j(x). \tag{3}$$

Using the type I function, we can rewrite the orthogonality relations

$$\langle Q_{\vec{n}}, x^k \rangle_\mu = \begin{cases} 0 & \text{if} \quad k = 0, \dots, |\vec{n}| - 2 \\ 1 & \text{if} \quad k = |\vec{n}| - 1 \end{cases} \tag{4}$$

2. Bivariate OP

In order to work with bivariate polynomials, we will use the notation $\mathbb{P}_n$ as a column polynomial vector. Let us denote the vector of degree j monomials as

$$\mathbb{X}_j = (x^j, x^{j-1}y, \dots, y^j)^t.$$

Thus, a column polynomial vector of degree n can be represented as

$$\mathbb{P}_n = G_{n,n}\mathbb{X}_n + G_{n,n-1}\mathbb{X}_{n-1} + \cdots G_{n,1}\mathbb{X}_1 + G_{n,0}\mathbb{X}_0,$$

where $G_{n,j}$ are matrices of size $(n+1) \times (j+1)$.

Given a bidimensional measure $\mu(x, y)$, with support $\Omega \subseteq \mathbb{R}^2$, we can extend the definition of inner product $\langle f, g \rangle_\mu$ to column vectors. If $F = (f_1, f_2, \dots, f_n)^t$ and $G = (g_1, g_2, \dots, g_m)^t$ are column vectors of functions, then we define

$$\langle F, G \rangle := \mathcal{L}_\mu[F \cdot G^T] = \int_\Omega F \cdot G^T d\mu = \left(\int_\Omega f_i \cdot g_j d\mu \right)_{i,j=1}^{n,m}. \tag{5}$$

In fact, we are applying the standard product $\langle f_i, g_j \rangle_\mu$ or the functional $\mathcal{L}_\mu[f_i \cdot g_j]$ to each pair i, j and placing the results in a matrix.

Let $\{\mathbb{P}_n\}_{n \ge 0}$ be a system of polynomial vectors such that

$$\langle \mathbb{P}_n, \mathbb{P}_k \rangle_\mu = \mathcal{L}_\mu[\mathbb{P}_n \mathbb{P}_k^t] = \begin{cases} 0_{(n+1)\times(k+1)} & \text{if} \quad k = 0, \dots, n-1 \\ S_n & \text{if} \quad k = n. \end{cases}$$

Where S_n is a regular squared matrix of size $(n+1) \times (n+1)$. Due to orthogonality, it is possible to give an equivalent condition:

$$\langle \mathbb{P}_n, \mathbb{X}_k \rangle_\mu = \mathcal{L}_\mu[\mathbb{P}_n \mathbb{X}_k^t] = \begin{cases} 0_{(n+1)\times(k+1)} & \text{if} \quad k = 0, \dots, n-1 \\ S_n & \text{if} \quad k = n. \end{cases} \tag{6}$$

Valid multi-indices for $r = 2, r = 3$

Observe some of the valid multi-indices and their respective degree n in the following table.

n	$r = 2$ indices	$r = 3$ indices
1	$(0,1), (1,0)$	$(0,0,0)$
2	$(0,2), (2,0)$	$(0,0,1), (0,1,0), (1,0,0)$
0	$(0,0)$	$(0,0,2), (0,2,0), (1,1,1), (2,0,0)$
3	$(0,3), (2,2), (3,0)$	$(0,0,3), (0,2,2), (0,3,0), (2,0,2), (2,2,0), (3,0,0)$
4	$(0,4), (4,0)$	$(0,0,4), (0,4,0), (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (4,0,0)$
5	$(0,5), (5,0)$	$(0,0,5), (0,5,0), (2,3,3), (3,2,3), (3,3,2), (5,0,0)$
6	$(0,6), (3,5), (5,3), (6,0)$	$(0,0,6), (0,3,5), (0,5,3), (0,6,0), (1,4,4), (2,2,5), (2,5,2), (3,0,5), (3,5,0), (4,1,4), (4,4,1), (5,0,3), (5,2,2), (5,3,0), (6,0,0)$
7	$(0,7), (7,0)$	$(0,0,7), (0,7,0), (1,3,6), (1,6,3), (2,4,5), (2,5,4), (3,1,6), (3,6,1), (4,2,5), (4,5,2), (5,2,4), (5,4,2), (6,1,3), (6,3,1), (7,0,0)$
8	$(0,8), (5,6), (6,5), (8,0)$	$(0,0,8), (0,5,6), (0,6,5), (0,8,0), (3,5,5), (5,0,6), (5,3,5), (5,5,3), (5,6,0), (6,0,5), (6,5,0), (8,0,0)$
9	$(0,9), (9,0)$	$(0,0,9), (0,9,0), (2,3,8), (2,6,6), (2,8,3), (3,2,8), (3,8,2), (5,5,5), (6,2,6), (6,6,2), (8,2,3), (8,3,2), (9,0,0)$

3. Bivariate Type II MOP

Given $r \in \mathbb{N}$, $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, r 2-dimensional measures $\mu_1, \dots, \mu_r$ and their respective matrix inner products defined in (5), we define the type II multiple orthogonal polynomial vector as a monic polynomial vector

$$\mathbb{P}_{\vec{n}} = \mathbb{X}_n + \sum_{k=0}^{n-1} G_{n,k}\mathbb{X}_k$$

which satisfies

$$\langle \mathbb{P}_{\vec{n}}, \mathbb{X}_k \rangle_j = 0, \quad k = 0, \dots, n_i - 1, \quad j = 1, \dots, r. \tag{7}$$

Since $G_{n,k}$ are matrices whose dimensions are $(n+1) \times (k+1)$, if we consider $\mathbb{P}_{\vec{n}}$ a vector of degree n monic polynomials, then this matrices give us $\frac{1}{2}n(n+1)^2$ unknown coefficients. We want the system to have only one solution for a multi-index $\vec{n}$. Then, $\vec{n}$ is a valid multi-index if there exist a number $n \in \mathbb{N}_0$ such that:

$$n(n+1) = \sum_{j=1}^{r} n_j(n_j + 1). \tag{8}$$

This number n is the degree of type II polynomials. In order to emphasise the degree of the polynomials, in the next sections we denote type II MOP as $\mathbb{P}^n_{\vec{n}}$.

4. Bivariate Type I MOP

Let $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ such that the condition (8) holds for some $n \in \mathbb{N}$. Then, let $i \in \{1, \dots, n\}$ and, for each j, we define $A^{(i)}_{\vec{n},j}(x, y)$, a bivariate polynomial of degree $\le n_j - 1$, ($j = 1, \dots, r$). Now, given r 2-dimensional measures $\mu_1, \dots, \mu_r$, the polynomials $A^{(i)}_{\vec{n},1}, \dots, A^{(i)}_{\vec{n},r}$ satisfy

$$\sum_{j=1}^{r} \left\langle \mathbb{X}_k, A^{(i)}_{\vec{n},j} \right\rangle_j = \begin{cases} 0_{(k+1)\times 1} & \text{if} \quad k = 0, \dots, n-2 \\ (e_i)_{n \times 1} & \text{if} \quad k = n-1 \end{cases} \tag{9}$$

If we repeat this for every possible $i \in \{1, \dots, n\}$, we get n lists of polynomials $A^{(i)}_{\vec{n},j}$, $j = 1, \dots, r$, $i = 1, \dots, n$. Now, let us denote

$$\mathbb{A}_{\vec{n},j} = (A^{(1)}_{\vec{n},j}, \dots, A^{(n)}_{\vec{n},j})^t,$$

a polynomial vector of size n whose components are bivariate polynomials of degree less than or equal to $n_j - 1$ ($j = 1, \dots, r$). Thus, the polynomial vectors $\mathbb{A}_{\vec{n},j}$ will satisfy:

$$\sum_{j=1}^{r} \langle \mathbb{X}_k, \mathbb{A}_{\vec{n},j} \rangle_j = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \dots, n-2 \\ I_n & \text{if} \quad k = n-1 \end{cases} \tag{10}$$

When the 2-dimensional measures are all absolutely continuous with respect to a common positive measure μ defined in $\Omega = \bigcup_{j=1}^{r} \Omega_j$ (with $\Omega_j = \operatorname{supp}(\mu_j)$), *i.e.*, $d\mu_j = w_j(x, y) d\mu(x, y)$, ($j = 1, \dots, r$), we can define the *Type I function*:

$$\mathbb{Q}_{\vec{n}} = \sum_{j=1}^{r} \mathbb{A}_{\vec{n},j} w_j(x, y). \tag{11}$$

Using this function, it is possible to rewrite (10) as

$$\langle \mathbb{X}_k, \mathbb{Q}_{\vec{n}} \rangle_\mu = \begin{cases} 0_{(k+1)\times n} & \text{if} \quad k = 0, \dots, n-2 \\ I_n & \text{if} \quad k = n-1. \end{cases} \tag{12}$$

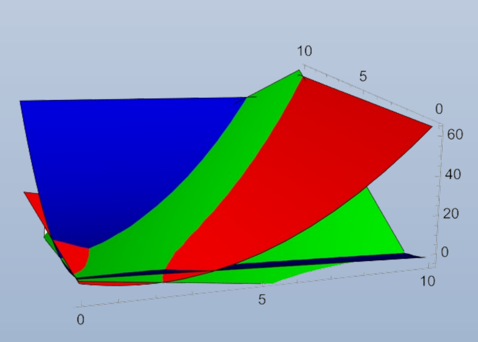
5. Numerical example

Starting from the well-known Laguerre product polynomials and the generalisation to multiple orthogonality of classical Laguerre polynomials, we have implemented $\mathbb{P}^2_{(1,1,1)}$, a multiple orthogonal polynomial vector with respect to the measures given by $d\mu_j = x^{\alpha_j} y^{\beta_j} e^{-x-y} d(x, y)$ ($i = 1, 2, 3$) and the multi-index $(1, 1, 1)$. We have choosen the values

$$\alpha_1 = 0, \alpha_2 = 0.5, \alpha_3 = 1.3; \beta_1 = 0.8, \beta_2 = 0.4, \beta_3 = 2.1,$$

and got the polynomial vector

$$\mathbb{P}^2_{(1,1,1)} = \begin{pmatrix} x^2 + -3.85556x - 0.444444y + 2.65556 \\ xy - 2.15556x - 1.94444y + 3.85556 \\ y^2 - 0.755556x - 5.14444y + 4.97556 \end{pmatrix}$$



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