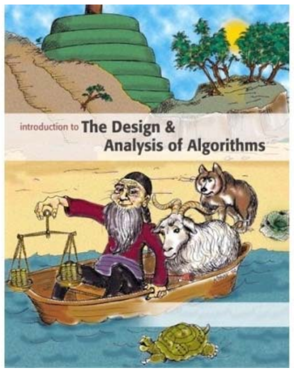
Introduction to

Algorithm Design and Analysis

[3] Recursion



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In the Last Class …

- **Asymptotic growth rate**
  - O, Ω, Θ
  - o, ω
- **Brute force algorithms**
  - By iteration
  - By recursion

Recursion

- **Recursion in algorithm design**
  - The divide and conquer strategy
  - Proving the correctness of recursive procedures
- **Solving recurrence equations**
  - Some elementary techniques
  - Master theorem

Recursion in Algorithm Design

- **Computing n! with Fac(n)**
  - if n=1 then return 1 else return Fac(n-1)*n

M(1)=0 and M(n)=M(n-1)+1 for n>0
(critical operation: multiplication)

- **Hanoi Tower**
  - if n=1 then move d(1) to peg3 else
    Hanoi(n-1, peg1, peg2); move d(n) to peg3; Hanoi(n-1, peg2, peg3)

M(1)=1 and M(n)=2M(n-1)+1 for n>1
(critical operation: move)

Recursion in Algorithm Design

- **Counting the Number of Bits**
  - Input: a positive decimal integer *n*
  - Output: the number of binary digits in *n*'s binary representation

Int BitCounting (int n)

```
1.  If(n==1) return 1;
2.  Else
3.    return BitCounting(n div 2) +1;
```

$$T(n)=\begin{cases}0 & n=1\\ T\left(\lfloor n/2\rfloor\right)+1 & n>1\end{cases}$$

Divide and Conquer

- **Divide**
  - Divide the "big" problem to smaller ones
- **Conquer**
  - Solve the "small" problems by recursion
- **Combine**
  - Combine results of small problems, and solve the original problem

Divide and Conquer

The general pattern

solve(*I*)

  ***n*=size(*I*);**

  if (*n*≤smallSize)

    solution=directlySolve(*I*)

  else

    divide *I* into I_1,… I_k;

    for each *i*∈{1,…,k}

      S_i=solve(I_i);

    solution=combine(S_1 ,… ,S_k);

  return solution

$T(n)=B(n)$ for $n \le smallSize$

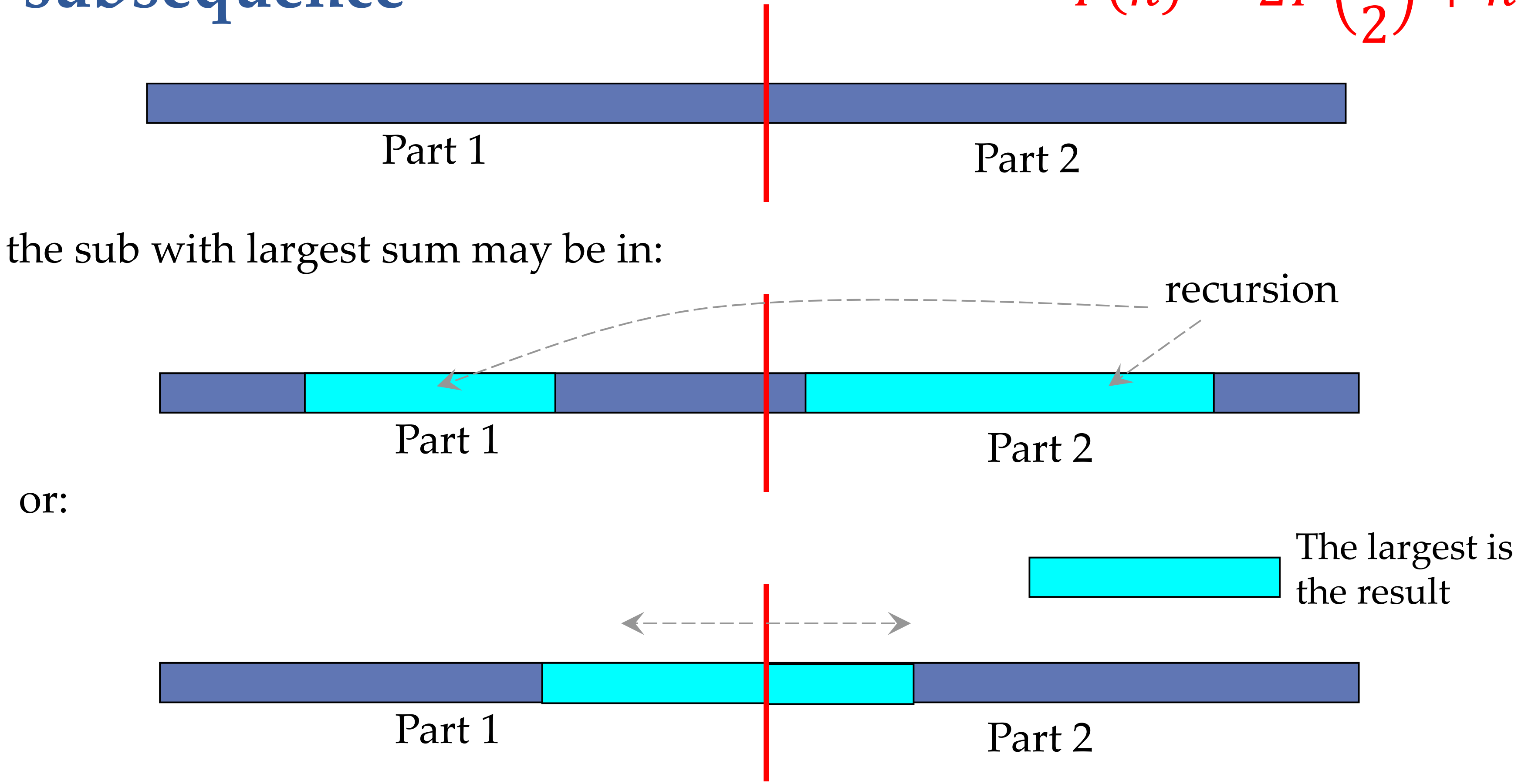
$$T(n)=D(n)+\sum_{i=1}^{k} T(size(I_i))+C(n)$$

for n>smallSize

Examples

Max sum subsequence

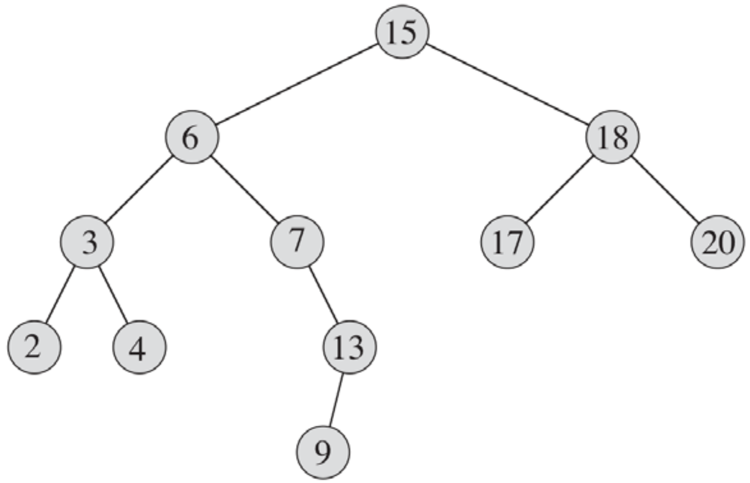
$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



Examples

- **Arrays**

- **Trees**

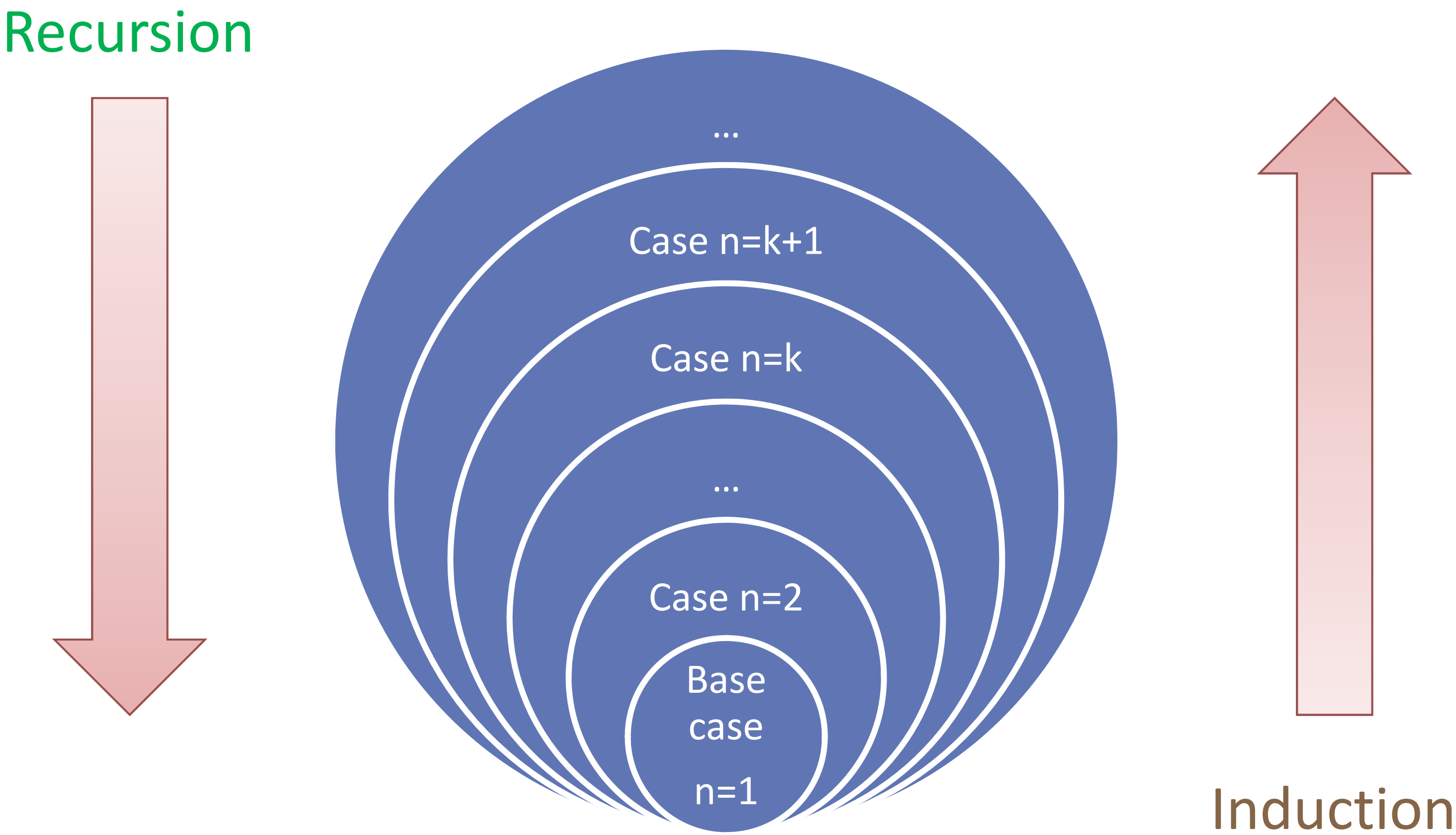


Workhorse

- "Hard division, easy combination"
- "Easy division, hard combination"

Usually, the "real work" is in one part.

Correctness of Recursion



Analysis of Recursion

- **Solving recurrence equations**

- **E.g., Bit counting**
    - Critical operation: add
    - The recurrence relation

$$T(n) = \begin{cases} 0 & n = 1 \\ T(\lfloor n/2 \rfloor) + 1 & n > 1 \end{cases}$$

Analysis of Recursion

- **Backward substitutions**

By the recursion equation : $T(n) = T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$

For simplicity , let $n = 2^k$ (k is a nonnegativ e integer),

that is, $k = \log n$

$$T(n) = T\left(\frac{n}{2}\right) + 1 = T\left(\frac{n}{4}\right) + 1 + 1 = T\left(\frac{n}{8}\right) + 1 + 1 + 1 = \ldots\ldots$$

$$T(n) = T\left(\frac{n}{2^k}\right) + \log n = \log n \quad (T(1)=0)$$

Smooth Functions

- $f(n)$
  - Nonnegative eventually non-decreasing function defined on the set of natural numbers
- **$f(n)$ is called smooth**
  - If $f(2n) \in \Theta(f(n))$.

- **Examples of smooth functions**
  - $\log n$, n, $n\log n$ and n^{α} ($\alpha \geq 0$)
  - E.g., $2n\log 2n = 2n(\log n + \log 2) \in \Theta(n\log n)$

Even Smoother

- **Let $f(n)$ be a smooth function, then, for any fixed integer $b \geq 2$, $f(bn) \in \Theta(f(n))$.**
  - That is, there exist positive constants c_b and d_b and a nonnegative integer n_0 such that

$$d_b f(n) \leq f(bn) \leq c_b f(n) \quad \text{for } n \geq n_0.$$

It is easy to prove that the result holds for $b = 2^k$,
for the second inequality:
$f(2^k n) \leqslant c_2^k f(n)$ *for* $k = 1,2,3\ldots$ *and* $n \geqslant n_0$.
For an arbitrary integer $b \geqslant 2$, $2^{k-1} \leqslant b \leqslant 2^k$
Then, $f(bn) \leqslant f(2^k n) \leqslant c_2^k f(n)$, *we can use* c_2^k *as* c_b.

Smoothness Rule

- **Let $T(n)$ be an eventually non-decreasing function and $f(n)$ be a smooth function.**
  - If $T(n) \in \Theta(f(n))$ for values of n that are powers of $b (b \geq 2)$, then $T(n) \in \Theta(f(n))$.

$$\begin{aligned}
&\textit{Just proving the big-Oh part:}\\
&\textit{By the hypothsis: } T(b^k) \leqslant cf(b^k) \textit{ for } b^k \geqslant n_0.\\
&\textit{By the prior result: } f(bn) \leqslant c_b f(n) \textit{ for } n \geqslant n_0.\\
&\textit{Let } n_0 \leqslant b^k \leqslant n \leqslant b^{k+1},\\
&T(n) \leqslant T(b^{k+1}) \leqslant cf(b^{k+1}) = cf(bb^k) \leqslant cc_b f(b^k) \leqslant cc_b f(n)
\end{aligned}$$

Non-decreasing hypothesis Prior result Non-decreasing

Computing the Fibonacci Number

$f_1=0$

$f_2=1$

$f_n=f_{n-1}+f_{n-2}$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ……

$$a_n = r_1 \quad a_{n\ 1} + r_2 a_{n\ 2} + \cdots + r_m a_{n\ k}$$

is called linear homogeneous relation of degree k.

For the special case of Fibonacci: $a_n=a_{n-1}+a_{n-2}$, $r_1=r_2=1$

Characteristic Equation

- For a linear homogeneous recurrence relation of degree k

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k}$$

the polynomial of degree k

$$x^k = r_1 x^{k-1} + r_2 x^{k-2} + \cdots + r_k$$

is called its characteristic equation.

- The characteristic equation of linear homogeneous recurrence relation of degree 2 is:

$$\boxed{x^2 - r_1 x - r_2 = 0}$$

Solution of Recurrence Relation

- If the characteristic equation $x^2 - r_1 x - r_2 = 0$ of the recurrence relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots s_1 and s_2, then

$$a_n = us_1^n + vs_2^n$$

where u and v depend on the initial conditions, is the explicit formula for the sequence.

- If the equation has a single root s, then, both s_1 and s_2 in the formula above are replaced by s

Proof of the Solution

$$Remember \quad equation \;:\; x^2 - r_1 x - r_2 = 0$$

$$We\ need\ to\ prove\ that\ : us_1^n + vs_2^n = r_1 a_{n\ 1} + r_2 a_{n\ 2}$$

$$us_1^n + vs_2^n = us_1^{n\ 2} s_1^2 + vs_2^{n\ 2} s_2^2$$

$$= us_1^{n\ 2}(r_1 s_1 + r_2) + vs_2^{n\ 2}(r_1 s_2 + r_2)$$

$$= r_1 us_1^{n\ 1} + r_2 us_1^{n\ 2} + r_1 vs_2^{n\ 1} + r_2 vs_2^{n\ 2}$$

$$= r_1(us_1^{n\ 1} + vs_2^{n\ 1}) + r_2(us_1^{n\ 2} + vs_2^{n\ 2})$$

$$= r_1 a_{n\ 1} + r_2 a_{n\ 2}$$

Back to Fibonacci Sequence

$f_0=0$

$f_1=1$

$f_n=f_{n-1}+f_{n-2}$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ……

Explicit formula for Fibonacci Sequence

The characteristic equation is x^2-x-1=0, which has roots:

$$s_1=\frac{1+\sqrt{5}}{2} \quad and \quad s_2=\frac{1-\sqrt{5}}{2}$$

Note: (by initial conditions) $f_1=us_1+vs_2=1 \;\; and \;\; f_2=us_1^{\,2}+vs_2^{\,2}=1$

which results:

$$f_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Guess and Prove

- **Example: T(n)=2T($\lfloor$n/2$\rfloor$) +n**
- **Guess**
  - $T(n) \in O(n)$?
    - $T(n) \le cn$, to be pr
  - $T(n) \in O(n^2)$?
    - $T(n) \le cn^2$, to be prove
  - **Or maybe**, $T(n) \in O(n\log$
    - $T(n) \le cn\log n$, to be prove
- **Prove**
  - by substitution

Try to prove $T(n) \le cn$:

However:

$T($

$$
\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor)+n \\
&\le 2(c\lfloor n/2 \rfloor \log (\lfloor n/2 \rfloor))+n \\
&\le cn\log (n/2)+n \\
&= cn \log n - cn \log 2 + n \\
&= cn \log n - cn + n \\
&\le cn \log n \quad \text{for } c \ge 1
\end{aligned}
$$

Divide and Conquer Recursions

- **Divide and conquer**
  - Divide the "big" problem to smaller ones
  - Solve the "small" problems by recursion
  - Combine results of small problems, and solve the original problem
- **Divide and conquer recursion**

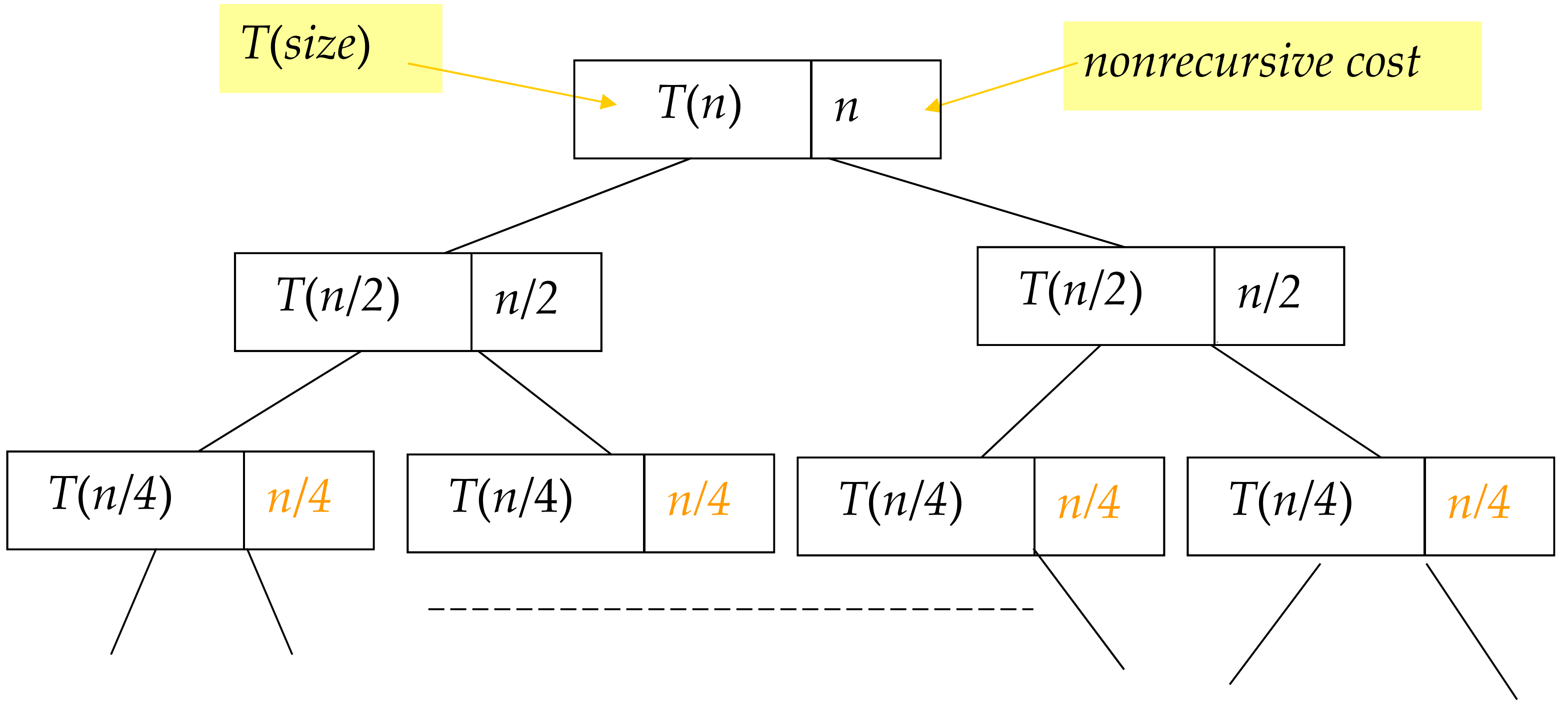
$$T(n) = b\ T(n/c) + f(n)$$

divide

combine

conquer

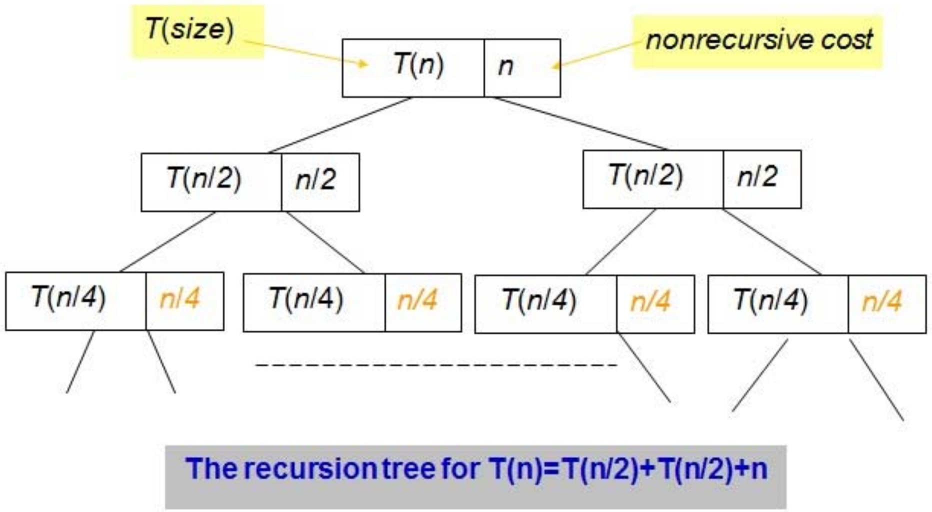
Recursion Tree



The recursion tree for $T(n) = 2T(n/2) + n$

Recursion Tree

- **Node**
  - Non-leaf
    - Non-recursive cost
    - Recursive cost
  - Leaf
    - Base case
- **Edge**
  - Recursion



Recursion Tree

Recursive cost — Non-recursive cost

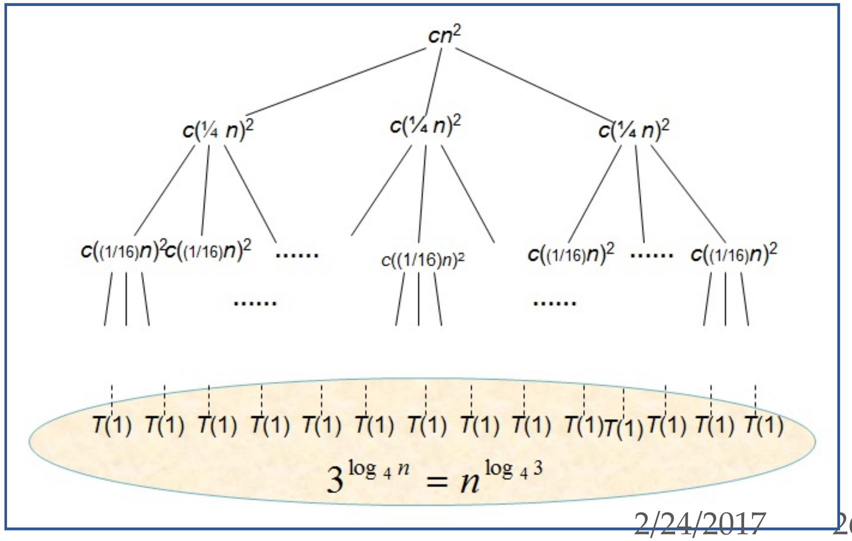
- $T(n) = \underbrace{3T(n/4)}_{\text{Recursive cost}} + \underbrace{\Theta(n^2)}_{\text{Non-recursive cost}}$

\# of sub-problems — size of sub-problems

- **Total cost**

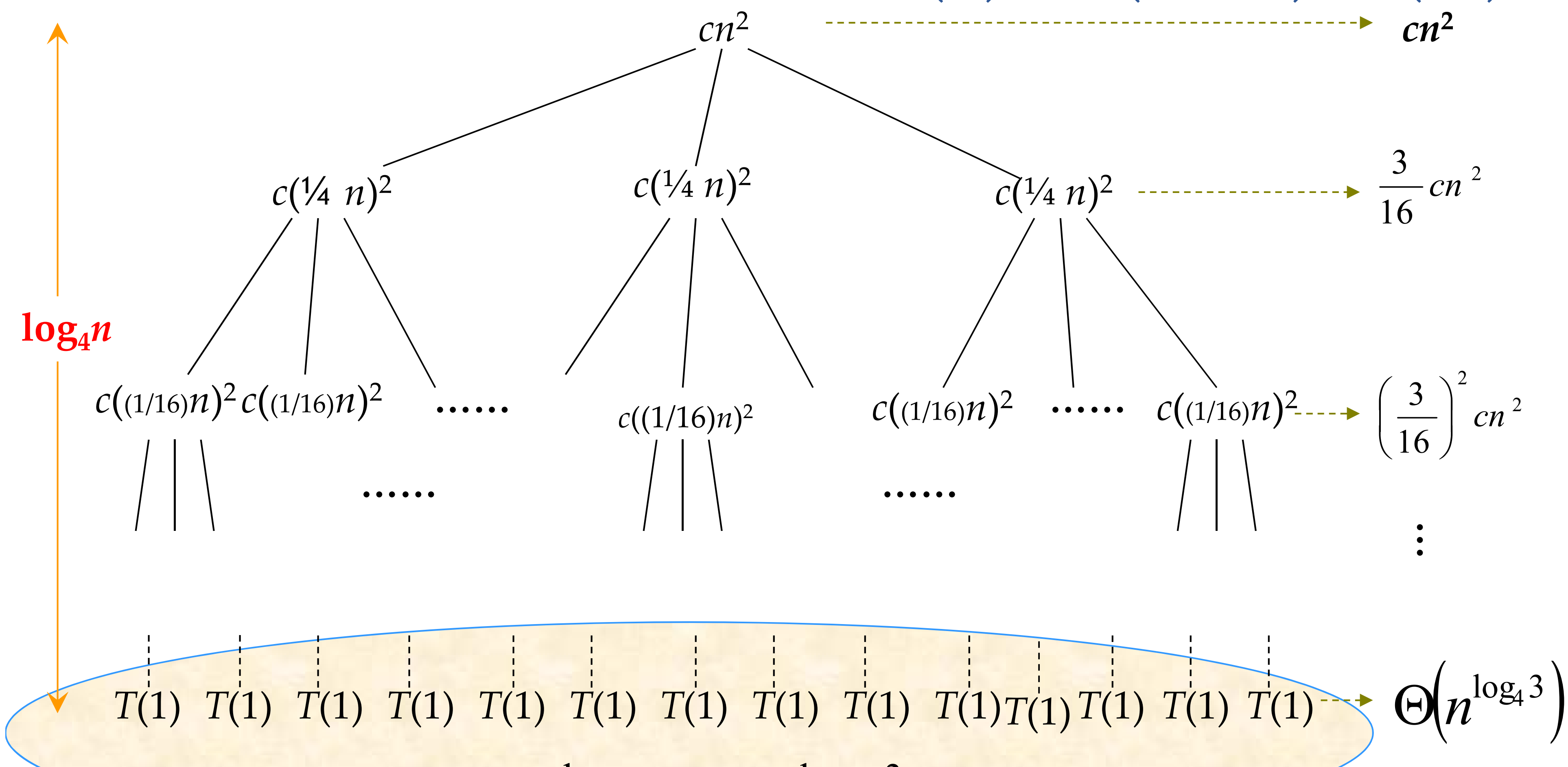
Σ

Sum of row sums



Sum of Row-sums

$$T(n)=3T(\lfloor n/4 \rfloor)+\Theta(n^2)$$



Note: $3^{\log_4 n} = n^{\log_4 3}$

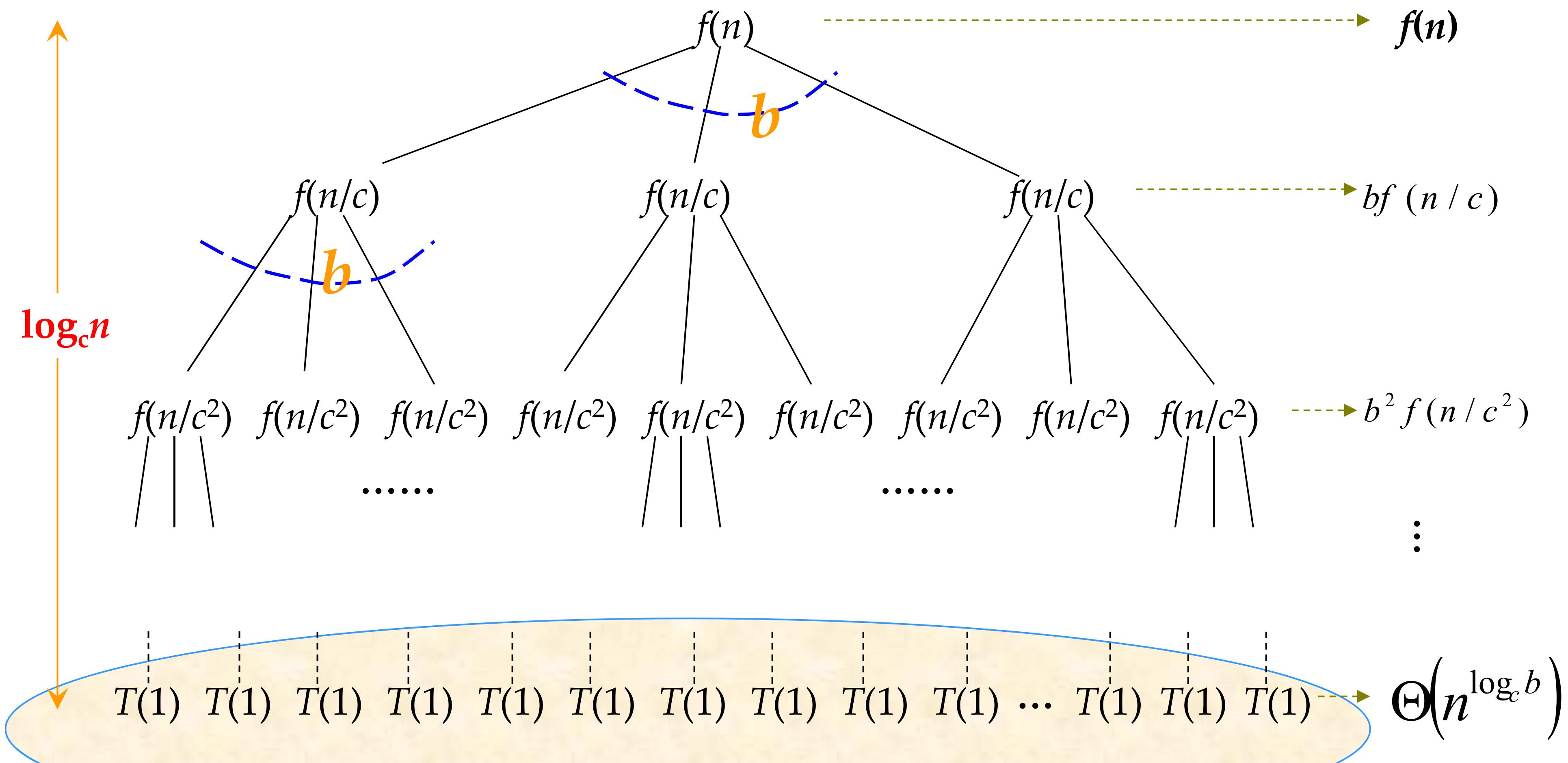
Total: $\Theta(n^2)$

Solving the Divide-and-Conquer Recurrence

- **The recursion equation for divide-and-conquer, the general case: $T(n)=bT(n/c)+f(n)$**
- **Observations:**
  - Let base-cases occur at depth D(leaf), then $n/c^D=1$, that is $D=\log(n)/\log(c)$
  - Let the number of leaves of the tree be L, then $L=b^D$, that is $L=b^{(\log(n)/\log(c))}$.
  - By a little algebra: $L=n^E$, where $E=\log(b)/\log(c)$, called *critical exponent*.

Recursion Tree for

$T(n)=bT(n/c)+f(n)$



Note: $b^{\log_c n} = n^{\log_c b}$

Total ?

Divide-and-Conquer - the Solution

- **The solution of divide-and-conquer equation is the non-recursive costs of all nodes in the tree, which is the sum of the row-sums**
  - The recursion tree has depth $D = \log(n) / \log(c)$, so there are about that many row-sums.

- **The 0^{th} row-sum**
  - is $f(n)$, the nonrecursive cost of the root.
- **The D^{th} row-sum**
  - is n^E, assuming base cases cost 1, or $\Theta(n^E)$ in any event.

Solution by Row-sums

- **[Little Master Theorem] Row-sums decide the solution of the equation for divide-and-conquer:**
  - Increasing geometric series: $T(n) \in \Theta(n^E)$
  - Constant: $T(n) \in \Theta(f(n) \log n)$
  - Decreasing geometric series: $T(n) \in \Theta(f(n))$

This can be generalized to get a result not using explicitly row-sums.

Master Theorem

- **Loosening the restrictions on $f(n)$**
  - Case 1: $f(n) \in O(n^{E-\varepsilon})$, $(\varepsilon > 0)$, then:
  $$T(n) \in \Theta(n^E)$$
  - Case 2: $f(n) \in \Theta(n^E)$, as all node depth contribute about equally:
  $$T(n) \in \Theta(f(n)\log(n))$$
  - case 3: $f(n) \in \Omega(n^{E+\varepsilon})$, $(\varepsilon > 0)$, and if $bf(n/c) \le \theta f(n)$ for some constant $\theta < 1$ and all sufficiently large n, then:
  $$T(n) \in \Theta(f(n))$$

The positive ε is critical, resulting gaps between cases as well

Using Master Theorem

- Example 1: $T(n) = 9T(\frac{n}{3}) + n$
  $b = 9, c = 3, E = 2, f(n) = n = O(n^{E-1})$
  Case 1 applies: $T(n) = \Theta(n^2)$
- Example 2: $T(n) = T(\frac{2}{3}n) + 1$
  $b = 1, c = \frac{3}{2}, E = 0, f(n) = 1 = \Theta(n^E)$
  Case 2 applies: $T(n) = \Theta(\log n)$
- Example 3: $T(n) = 3T(\frac{n}{4}) + n \log n$
  $b = 3, c = 4, E = \log_4 3, f(n) = \Omega(n^{E+\epsilon})$
  $bf(\frac{n}{4}) = \frac{3}{4} n \log n - \frac{3}{2} n$
  Case 3 applies: $T(n) = \Theta(n \log n)$

Looking at the Gap

- **Often, none of the 3 cases in the Master Theorem apply.**

Your task:
Design such a recursion

Thank you!

Q & A

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