FRAMING THE COLLATZ CONJECTURE USING MATRIX SERIES AND DIOPHANTINE EQUATIONS

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ABSTRACT

Stacking in columns all even numbers as powers of 2 of odd numbers and observing the length of the displacements between odd numbers, I found a rich set of periodic patterns. I explicit the function $\tilde{f}$ that describes the image of each odd number towards each odd number (skipping the even numbers). This function defines a passage matrix between odd numbers, which is sparse and highly ordered by submatrices defined by linear functions, that I call series. For Collatz transition matrix of dimensions $[I_2, I_1]$, the n^{th} order passage matrix is built by matricial product of passage matrices of adequate dimensions-it is not an endomorphism.

I explicit the rules of the series multiplication and show that they obey to linear Diophantine equations. I characterize several aspects of the n^{th} order passage matrix, such as the rank evolution as function of the input dimension I_1, the nilpotence of its series -defined as linear functions- and their slope change. Finally, I explicit the convergence mechanism, ie the apparition of submatrices with zero ordinate at a given abscissa, and which has two phases: $\forall x \in \mathbb{N}$ at the order $n = \left\lceil \frac{ln(I_1)}{ln(2)} \right\rceil$ there is a series with abcissa x, and ordinate $y \in \mathbb{N}$. The second phase of the convergence happens at constant x, and involves only one series.

The Collatz conjecture can then be framed as showing the convergence towards zero of the ordinate of a finite number of integers defining the initial passage matrix.

Keywords Collatz conjecture, Syracuse conjecture, 3x+1 problem, Czech conjecture, Ulam conjecture, Kakutani problem, Diophantine equation, Matrix series.

1 Introduction

The Collatz conjecture is named after the mathematician Lothar Collatz, who formulated it in 1937 [Wikipedia contributors, 2023], [**?**]. Let u be an application from $\mathbb{N}$ to $\mathbb{N}$.

$$u_{(n+1)} = \begin{cases} \frac{u_{(n)}}{2} & \text{if } u_{(n)} \text{ is even} \\ 3u_{(n)} + 1 & \text{if } u_{(n)} \text{ is odd} \end{cases}$$

One starts with any natural number $u_{(n)}$ and, after some iterations, obtains the cycle $1 \to 4 \to 2$. To prove the conjecture one must show this happens for all natural numbers.

2^9	512	1536	2560	3584	4608	5632	6656	7680	8704	9728	10752	11776	12800	13824	14848	15872	16896	17920	18944	19968
2^8	256	768	1280	1792	2304	2816	3328	3840	4352	4864	5376	5888	6400	6912	7424	7936	8448	8960	9472	9984
2^7	128	384	640	896	1152	1408	1664	1920	2176	2432	2688	2944	3200	3456	3712	3968	4224	4480	4736	4992
2^6	64	192	320	448	576	704	832	960	1088	1216	1344	1472	1600	1728	1856	1984	2112	2240	2368	2496
2^5	32	96	160	224	288	352	416	480	544	608	672	736	800	864	928	992	1056	1120	1184	1248
2^4	16	48	80	112	144	176	208	240	272	304	336	368	400	432	464	496	528	560	592	624
2^3	8	24	40	56	72	88	104	120	136	152	168	184	200	216	232	248	264	280	296	312
2^2	4	12	20	28	36	44	52	60	68	76	84	92	100	108	116	124	132	140	148	156
2^1	2	6	10	14	18	22	26	30	34	38	42	46	50	54	58	62	66	70	74	78
2^0	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39
d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Δd	0	1	-2	2	-1	3	-4	4	-2	5	-10	6	-3	7	-9	8	-4	9	-15	10
sign	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1
Δd	0	1	-2	2	-1	3	-4	4	-2	5	-10	6	-3	7	-9	8	-4	9	-15	10
Δd	0	1	-2	2	-1	3	-4	4	-2	5	-10	6	-3	7	-9	8	-4	9	-15	10
Δd	0	1	-2	2	-1	3	-4	4	-2	5	-10	6	-3	7	-9	8	-4	9	-15	10
Δd	0	1	-2	2	-1	3	-4	4	-2	5	-10	6	-3	7	-9	8	-4	9	-15	10
Δd	0	1	-2	2	-1	3	-4	4	-2	5	-10	6	-3	7	-9	8	-4	9	-15	10

Table 1: Even numbers reordered in columns of powers of two, on top of odd numbers.

2 Reordering $\mathbb{N}$ in even and odd numbers, reveals a highly ordered pattern

Let's put the odd numbers in the central row (2^0, dark gray) comprising all odd numbers 1,3,5,... Now make columns of even numbers multiplying odd number with all powers of two (light gray rows). With this procedure, we cover the totality of the natural numbers.

Start with an odd number, say 1 (in light green), multiply it by 3 and add 1, we obtain 4. 4 is situated in the same column as 1. We divide 4 by 2 twice and obtain 1 again. Let's pick now 5: 5*3+1 = 16, 16 is two columns at left from 5. $16/2^4 = 1$. With this arrangement the Collatz suite can also be seen as a suite between odd numbers. 1 goes back to 1 and 5 goes to 1.

Let's see what is the logic between each jump, from an even number to an even number. For instance:

$$5 \times 3 + 1 = 16 = 5 \times 2^4 - \mathbf{2} \times 2^5 = (5 - \mathbf{2} \times 2) \times 2^4$$

If we only consider how many steps of two were made to go from an odd number to an odd number, we see that in the case of 5, we took two to the left ($\Delta d = -2$). Considering the sign of each Δd, we observe an alternation of signs: +1, -1, +1, -1, ... (yellow and orange cells).

Now if we consider only the positive steps (row 2^1, $\Delta d > 0$ in red), we observe a very salient pattern: they become increasingly bigger, one at a time: $3 \rightarrow 5 : 1, 7 \rightarrow 11 : 2, 11 \rightarrow 34 : 3....$ This is simply explained by the fact that the first row (2^1) preserves the order of the zero order row (2^0), since they are proportional, and the position difference increases linearly.

To understand the logic of the negative shifts, we can proceed backwards and examine the row 2^2. In this row, the elements that result from 3 times plus one an odd number, are highlighted in light green. When observing the corresponding antecedent elements (the odd numbers), we see that the light green elements (which are a subset of all the negative shifts) also follow a linear negative progression (0, -1, -2, ...).

Overall, determining two consecutive elements within a row which are themselves image of an odd number, allows to compute the periodicity of the associated series (for instance 64 and 448 for the 2^6 row).

Following the logic from example 1, we find the series of $\Delta d_{(n)}$:

$$\Delta d_{(n)}^{\,n\ odd} = \begin{cases} +\frac{n+1}{4} & \text{if} \quad n+1 \equiv 0[4] & m=1 \\ -\frac{n-1}{8} & \text{if} \quad n-1 \equiv 0[8] & m=2 \\ \frac{3n+1-n.2^{2m-1}}{2^{2m}} & \text{if} \quad n - \frac{5.2^{2m-1}-1}{3} \equiv 0[2^{2m}] & m \in \mathbb{N}\backslash\{0\} \\ \frac{3n+1-n.2^{2m}}{2^{2m+1}} & \text{if} \quad n - \frac{2^{2m}-1}{3} \equiv 0[2^{2m+1}] & m \in \mathbb{N}\backslash\{0\} \end{cases}$$

Therefore the application f from an odd number to an odd number that describes the Collatz series can be written as:

$$\begin{aligned} f\colon 2\mathbb{N}+1 &\longrightarrow 2\mathbb{N}+1 \\ n &\longmapsto n + 2\Delta d_{(n)} \end{aligned}$$

3 Transition matrix J and n^{th} order transition matrices

We can now consider the series of odd to odd numbers as a non-square passage matrix J. J_{ij} the connection between input odd number with index j (and value 2j+1) and odd number with index i (and value 2i+1). For instance 5 and 1 are connected asymmetrically (5 projects to 1).

$$J_{ij} = \begin{cases} 1 & \text{if } j \to i \\ 0 & \text{otherwise} \end{cases}$$

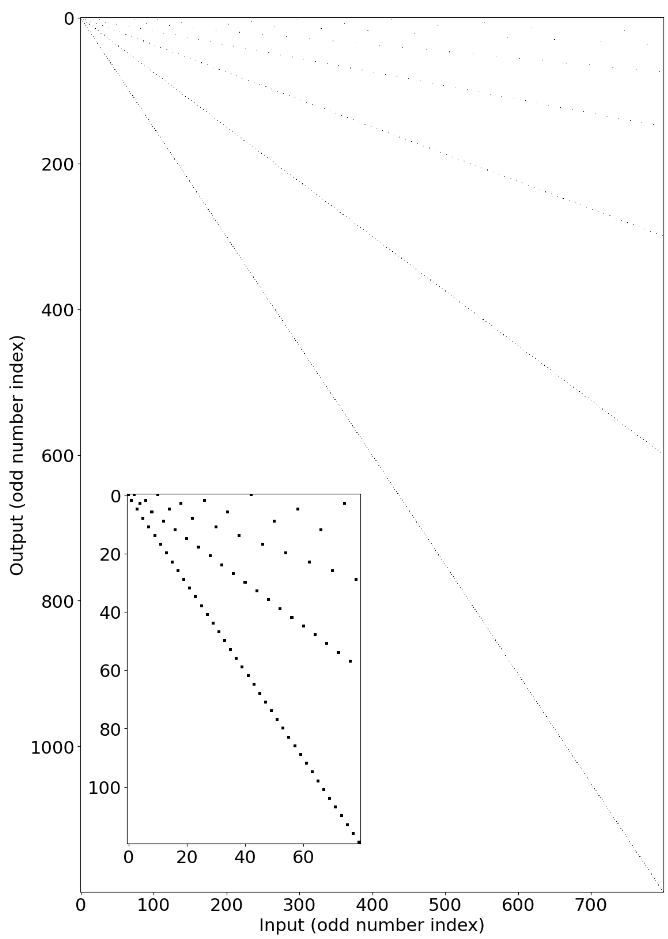

Figure 1: Transition matrix for odd numbers 1, 3,...,1601 (indexes from 0 to 800). Inset: zoom for the first 81 indexes.

In the connectivity matrix, we see that the positive connections are the ones which are below the identity line J_{ii}, whereas all the negative connections are above it. The negative connections form several lines of lower angle with respect to the horizontal. As the dimension of the transition matrix increases, we see more and more lines corresponding to higher powers of two.

Given an initial odd number of corresponding in index j. In vector notation this number is $X_{(t=1)} = \delta_{ij}$, with a 1 on the index j, the next odd number (with non null vector component) will be given by doing the matrix multiplication:

$$X_{(t=2)} = J_1.X_{(t=1)} \qquad \text{therefore} \qquad X_{(n)} = J_n J_{n-1}...J_1.X(1)$$

$J_{k,k\in\mathbb{N}}$ is a matrix with different dimensions than J_1 (of dimensions $[I_2, I_1]$), whose submatrix $J_k|_{(i=1,...I_2,j=1,...,I_1)}$ is equal to J_1.

To prove the conjecture, one has to prove that, for any matrix J_1 of input dimension I_1 (number of columns), for large m, the matrix $C^{(n)} = J_n J_{n-1}...J_1$ converges towards an empty matrix except for the first row that is equal to one.

In figure 2, we see the result of the matrix product of two matrices $C^{(2)} = J_2.J_1$. The notation $C^{(n)}$ designates a matrix which is the product of n matrices J_w, $w \in [1, n]$. $C^{(n)} = J_n.J_{n-1}...J_1$. Each matrix J_k has dimensions $[I_{k+1}, I_k]$.

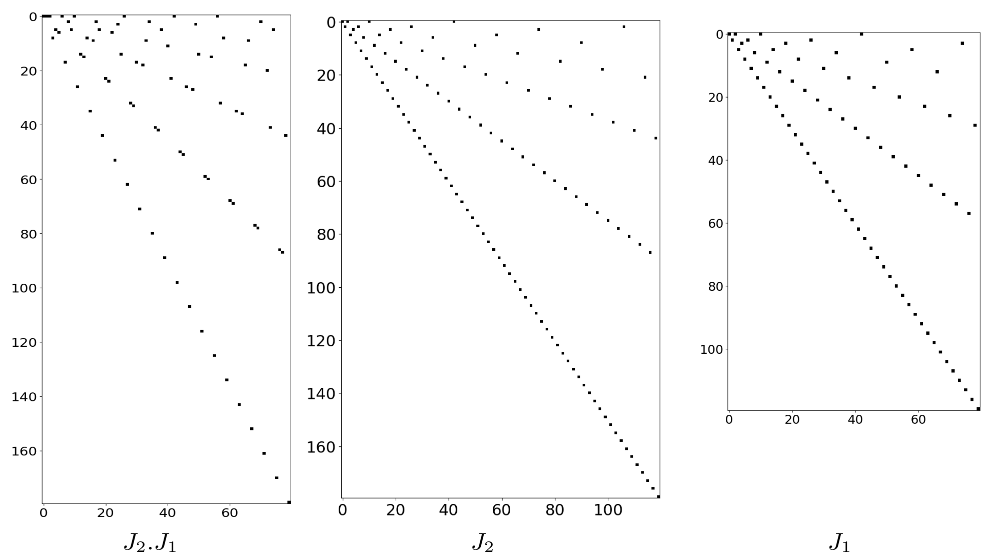

Figure 2: $C^{(2)} = J_2.J_1$

The transition matrices capture the dynamics of the Collatz series. We can redefine the function f that transforms an odd number into an odd number into a function $\tilde{f}$ that transforms a natural number into a natural number. Same for $\tilde{\Delta}d_{(d)} = \tilde{\Delta}d_{(2n+1)}$. The transformation is trivial and deals with the fact that is easier to manipulate the indexes of the odd numbers 0,1,2,3 for respectively 1,3,5,7 following the convention $2d + 1 = n$, n being an odd number and d a natural number.

Lemma 3.1. *Transition matrix between odd numbers:*

$$\begin{aligned}\tilde{f} : \mathbb{N} &\longrightarrow \mathbb{N}\\ d &\longmapsto d + \tilde{\Delta}d_{(d)}\end{aligned}$$

$\tilde{f}$ can be written as:

$$\tilde{f}_{(d)} = \begin{cases} d + \frac{6d+4-(2d+1).2^{2m-1}}{2^{2m}} & \text{if} \quad 2d+1-\frac{5.2^{2m-1}-1}{3} \equiv 0[2^{2m}] \qquad m \in \mathbb{N}\backslash\{0\} \\ d + \frac{6d+4-(2d+1).2^{2m}}{2^{2m+1}} & \text{if} \quad 2d+1-\frac{2^{2m}-1}{3} \equiv 0[2^{2m+1}] \qquad m \in \mathbb{N}\backslash\{0\} \end{cases}$$

In the formula of $\tilde{f}_{(d)}$, for a given power of 2 (parametrized by m), we can rewrite the lines observed in the transition matrix as series $\{J_{ap+c,p}\}_{p\in\mathbb{N}}$. For instance, the condition $2d+1-\frac{5.2^{2m-1}-1}{3} \equiv 0[2^{2m}]$ can be rewritten as $2d+1-\frac{5.2^{2m-1}-1}{3} = p.2^{2m} \quad (p,m)\in(\mathbb{N}^*)^2$. Substituting d in the formula of $\tilde{f}_{(d)}$ gives the following series:

$$\begin{cases} \{J_{3p+2,\ (2^{2m}p+\frac{5.2^{2m-1}-1}{3}-1)/2}\} & (p,m)\in(\mathbb{N}^*)^2 \\ \{J_{3p,\ (2^{2m+1}p+\frac{2^{2m}-1}{3}-1)/2}\} & (p,m)\in(\mathbb{N}^*)^2 \end{cases}$$

The next figure shows the matrix $C^{(n)} = J_n.J_{n-1}...J_1$, for an input dimension $I_1 = 12$. All odd numbers between 1 and 25 (of indexes $d \in [0,12]$) converge effectively to index 0, ie to the odd number 1.

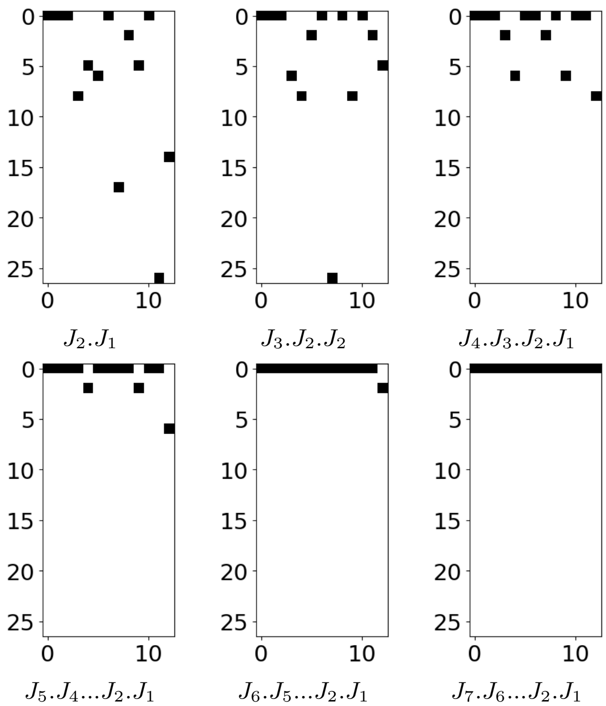

Figure 3: The dimensions of the 6 matrices are (27, 13), (39, 13), (57, 13), (84, 13), (126, 13), (189, 13). For visualization purposes only the first he zoom on the rows to rows are shown ($d \in [0,26]$).

In Appendix 1, we compute these series up to $m = 10$.

4 Matrix multiplication of submatrices defined as linear functions

If we consider the line defining the positive weights on the matrix J_1, the abcissas are defined as even occurrences of d ($d+1 \equiv 0[2]$) and the ordinates are defined by the image of these abscissas following the function $d \mapsto d + \frac{d+1}{2}$, therefore as a set of points $\{J_{3p+2,\ 2p+1}\}_{p\in\mathbb{N}}$

The components of $J^{(2)}$ that result from the product of J_1 with J_2 (taking into account the change of dimension), and that come from the product of the positive weights with themselves is given by the matricial product of the two series:

$$\begin{aligned}\{J_{3p+2,\ 2p+1}\}_{p\in\mathbb{N}}\cdot\{J_{3p+2,\ 2p+1}\}_{p\in\mathbb{N}} &= \{J_{3(3p+2)+2,\ 2(3p+2)+1}\}_{p\in\mathbb{N}}\cdot\{J_{3(2p+1)+2,\ 2(2p+1)+1}\}_{p\in\mathbb{N}}\\ &= \{J_{9p+8,\ 6p+5}\}_{p\in\mathbb{N}}\cdot\{J_{6p+5,\ 4p+3}\}_{p\in\mathbb{N}}\\ &= \{J_{9p+8,\ 4p+3}\}_{p\in\mathbb{N}}\end{aligned}$$

$$\{J_{3p,\ 4p}\}^2_{p\in\mathbb{N}} = \{J_{9p,\ 16p}\}_{p\in\mathbb{N}}$$

$$\{J_{3p+2,\ 8p+6}\}^2_{p\in\mathbb{N}} = \{J_{9p+8,\ 64p+54}\}_{p\in\mathbb{N}}$$

$$\begin{aligned}\{J_{3p+2,\ 2p+1}\}_{p\in\mathbb{N}}\cdot\{J_{3p,\ 4p+1}\}_{p\in\mathbb{N}} &= \{J_{3(3p+1)+2,\ 2(3p+1)+1}\}_{p\in\mathbb{N}}\cdot\{J_{3(2p+1),\ 4(2p+1)+1}\}_{p\in\mathbb{N}}\\ &= \{J_{9p+5,\ 6p+3}\}_{p\in\mathbb{N}}\cdot\{J_{6p+3,\ 8p+5}\}_{p\in\mathbb{N}}\\ &= \{J_{9p+5,\ 8p+5}\}_{p\in\mathbb{N}}\end{aligned}$$

$$\begin{aligned}\{J_{5p,\ 3p+1}\}_{p\in\mathbb{N}}\cdot\{J_{2p,\ 5p+1}\}_{p\in\mathbb{N}} &= \{J_{5(2p+1),\ 3(2p+1)+1}\}_{p\in\mathbb{N}}\cdot\{J_{2(3p+2),\ 5(3p+2)+1}\}_{p\in\mathbb{N}}\\ &= \{J_{10p+5,\ 6p+4}\}_{p\in\mathbb{N}}\cdot\{J_{6p+4,\ 15p+11}\}_{p\in\mathbb{N}}\\ &= \{J_{10p+5,\ 15p+11}\}_{p\in\mathbb{N}}\end{aligned}$$

Theorem 4.1. *The matricial multiplication of sparse submatrices defined following the series $\{J_{a_1p+b_1,\ a_2p+b_2}\}_{p\in\mathbb{N}}\cdot\{J_{a_3p+b_3,\ a_4p+b_4}\}_{p\in\mathbb{N}}$, -$(a_i, b_i)\in\mathbb{N}^2$- is either nul or it is a series $\{J_{a_lp+b_l,\ a_rp+b_r}\}_{p\in\mathbb{N}}$. To determine it, we must find must find two series $s_2^*(p)$ and $s_3^*(p)$ of respectively $s_2(p) = a_2p+b_2$ and $s_3(p) = a_3p+b_3$ such that $(s_2\circ s_2^*)(p) = (s_3\circ s_3^*)(p)$. The resulting matricial product will be $\{J_{s_1\circ s_2^*(p), s_4\circ s_3^*(p)}\}_{p\in\mathbb{N}}$*

$\exists(s_2^*(p), s_3^*(p))\in(\mathbb{N}\to\mathbb{N})^2 : (s_2\circ s_2^*)_{(p)} = (s_3\circ s_3^*)_{(p)} \iff \exists(c_2,c_3)\in\mathbb{Z}^2 : a_2c_2 - a_3c_3 = b_3 - b_2$

In case of existence:

$s_2^*(p) = \dfrac{a_3}{GCD(a_2,a_3)}p + c_2$ *and* $s_3^*(p) = \dfrac{a_2}{GCD(a_2,a_3)}p + c_3,$ *GCD: greatest common divisor*

Proof.

$$\begin{aligned}(s_2\circ s_2^*)_{(p)} = (s_3\circ s_3^*)_{(p)} &\iff a_2(\frac{a_3}{GCD(a_2,a_3)}p + c_2) + a_3(\frac{a_2}{GCD(a_2,a_3)}p + c_3) = b_3 - b_2\\ &\iff \exists(c_2,c_3)\in\mathbb{Z}^2 : a_2c_2 - a_3c_3 = b_3 - b_2\end{aligned}$$

This is a linear Diophantine equation. If this equation has a solution $(c_2^\bullet, c_3^\bullet)\in\mathbb{N}^2$, the ensemble of solutions are $(c_2,c_3) = \{(c_2^\bullet + ka_3,\ c_3^\bullet + ka_2)\}_{k\in\mathbb{Z}}$

As $(a_2,a_3)\in\mathbb{N}^2$, we look for the smallest positive solution in $\mathbb{N}^2$ such that :

$$\begin{cases} c_2^\bullet + ka_3 \geq 0 \\ c_3^\bullet + ka_2 \geq 0 \end{cases}$$

We then choose k such that $k = max(\lceil\frac{-c_2^\bullet}{a_3}\rceil, \lceil\frac{-c_3^\bullet}{a_2}\rceil)$.

Bachet-Bézout identity doesn't garantee the existence of solutions to the equation:

$$a_2c_2 - a_3c_3 = b_3 - b_2$$

when the equation:

$$a_2x - a_3y = GCD(a_2, a_3)$$

has no solution.

$\square$

Corollary 4.2. $s_2^*(p) = s_3(p)$ *and* $s_3^*(p) = s_2(p) \iff GCD(a_2, a_3) = 1$ *and* $a_2b_3 - a_3b_2 = b_3 - b_2$

Corollary 4.3. $\{J_{3p+2,\ 2p+1}\}^{(n)}_{p\in\mathbb{N}} = \{J_{3^np+3^n-1,\ 2^np+2^n-1}\}_{p\in\mathbb{N}}$

Proof.

• $h(x) = ax + b$. The n^{th} convolution of h with itself is given by: $a^n.x + b.\frac{a^n-1}{a-1}$.

• $GCD(a, b) = 1 \implies GCD(a, b^n) = 1$

$$\begin{aligned} GCD(a,b) = 1 &\iff \exists x, y \in \mathbb{Z}^2 : ax + by = 1 \quad \text{(Bézout)} \\ &\implies (ax+by)^n = a.(\sum_{i=1}^{n} a^{i-1}x^iy^{n-i}b^{n_i}) + b^ny^n = 1 \\ &\implies (ax+by)^n = a.(\sum_{i=1}^{n} a^{i-1}x^iy^{n-i}b^{n_i}) + b^ny^n = 1 \\ &\exists \tilde{x}, \tilde{y} \in \mathbb{Z}^2 : a\tilde{x} + b^n\tilde{y} = 1 \end{aligned}$$

• Consider by recurrence the hypothesis $H(n) : \{J_{3p+2,\ 2p+1}\}^{(n)}_{p\in\mathbb{N}} = \{J_{3^np+3^n-1,\ 2^np+2^n-1}\}_{p\in\mathbb{N}}$

• For n = 2:

$s_2(p) = 2p + 1$ and $s_3(p) = 3p + 2$.

$GCD(2, 3) = 1$ and $\exists c_2, c_3 \in \mathbb{Z}^2 : 2c_2 - 3c_3 = 2 - 1 = 1, \qquad (c_2, c_3) = (2, 1)$

Using precedent corollary: $s_2^*(p) = s_3(p)$ and $s_3^*(p) = s_2(p)$.

$\{J_{3p+2,\ 2p+1}\}^{(2)} = \{J_{s_3\circ s_3,\ s_2\circ s_3}\}\{J_{s_3\circ s_2,\ s_2\circ s_2}\} = \{J_{3^2p+2\frac{3^2-1}{3-1},\ 2^2p+\frac{2^2-1}{2-1}}\}$

• For n = n:

$\{J_{3p+2,\ 2p+1}\}.\{J_{3^np+3^n-1,\ 2^np+2^n-1}\}$:

$s_2(p) = 2p + 1$ and $s_3(p) = 3^np + 3^n - 1$.

$GCD(2, 3^n) = 1$ and $\exists c_2, c_3 \in \mathbb{Z}^2 : 2c_2 - 3^nc_3 = 3^n - 2, \qquad (c_2, c_3) = (3^n - 1, 1)$

Then:

$$\begin{aligned} \{J_{3p+2,\ 2p+1}\}.\{J_{3^np+3^n-1,\ 2^np+2^n-1}\} &= \{J_{(3p+2)\circ(3^np+3^n-1),\ (2p+1)\circ(3^np+3^n-1)}\}.\{J_{(3^np+3^n-1)\circ(2p+1),\ (2^np+2^n-1)\circ(2p+1)}\} \\ &= \{J_{3^{n+1}p+3^{n+1}-1, 2^{n+1}p+2^{n+1}-1}\} \end{aligned}$$

$\square$

Corollary 4.4. *On the existence of solutions to the Diophantine equation*
$C^{(n+1)} = J_{n+1}C^{(n)}$. *As* $J_{n+1} = \sum_{i=1}^{M}\{J_{a_1^ip+b_1^i,\ a_2^ip+b_2^i}\}_{p\in\mathbb{N}}$ *and* $C^{(n)} = \sum_{i=1}^{N}\{J_{a_1^ip+b_1^i,\ a_2^ip+b_2^i}\}_{p\in\mathbb{N}}$.

Bachet-Bézout identity doesn't garantee the existence of solutions to the Diophantine equation:

$$a_2c_2 - a_3c_3 = b_3 - b_2$$

when the equation:

$$a_2x - a_3y = GCD(a_2, a_3)$$

has no solution.

5 Evolution of the rank as function of the input dimension and of the iterations $rank(C^{(m)}_{(I_1)})$

Now, we observe how does the rank evolves as the input dimension I_1 ($I_1 = d + 1$) of the transition matrix increases: it appears that the evolution of the rank of J_1 is a monotonically continuous increasing linear-plateau-linear function. The slope lower than one is due to the fact that half of even numbers have negative increases Δd. The monotonic increase is partly due to the positive increases every two even numbers.

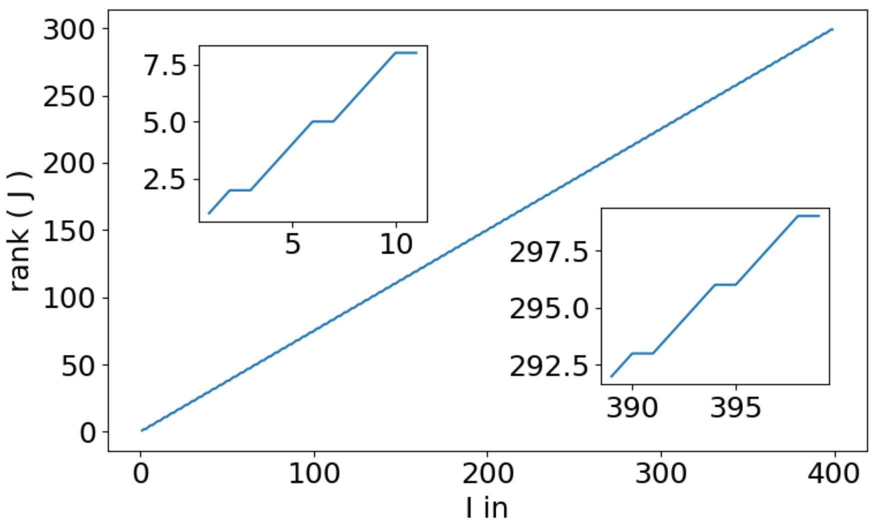

Figure 4: $Rank(J_1) = f(I_1)$. J_1 is a matrix of dimension $[I_2, I_1]$

Proposition 5.1. *Let $\{J_{cp+e,ap+b}\}_{p\in\mathbb{N}}$ be a submatrix of J_1. The numbers of existing integers ($i : i = ap + b, p \in \mathbb{N}$) up to the number d is equal to $1 + \lfloor \frac{d-b}{a} \rfloor \ = 1 + \lfloor \frac{I_1-b-1}{a} \rfloor$.*

Proposition 5.2. *The rank of the transition matrix $C^{(1)} = J_1$ of dimensions $[I_2, I_1]$ is equal to:*

$$rank(C^{(1)})_{(I_1)} = \lfloor \frac{I_1 - 2}{2} \rfloor + \lfloor \frac{I_1 - 1}{4} \rfloor + 2$$

Proof. The formula giving the series of sparse submatrices of the transition matrix J_1, for every $(p, m) \in (\mathbb{N}^*)^2$ shows that the non-nil index are always equal either to $3p+2$ or to $3p$. The column number, or abscissa, is a function of (p, m), but the ordinate (non-nil row value) is only dependent on p. Therefore, only the two first components of $\tilde{f}(d)$ (conditions $d + 1 \equiv 0[2]$ and $d \equiv 0[4]$) contribute to the rank (see also figure 1 and Appendix 2). The other components contribute with repeated columns.

$$rank(C^{(1)}) = \sum_{\substack{j=0 \\ j=2p+1, p\in\mathbb{N}}}^{d} 1 + \sum_{\substack{j=0 \\ j=4p, p\in\mathbb{N}}}^{d} 1 = (\lfloor \frac{d-1}{2} \rfloor + 1) + (\lfloor \frac{d}{4} \rfloor + 1) = \lfloor \frac{I_1 - 2}{2} \rfloor + \lfloor \frac{I_1 - 1}{4} \rfloor + 2$$

□

Proposition 5.3. *The rank of the second order transition matrix $C^{(2)}$ of dimensions $[I_3, I_1]$ is equal to:*

$$rank(C^{(2)})_{(I_1)} = \lfloor\frac{I_1-4}{4}\rfloor + \lfloor\frac{I_1-5}{8}\rfloor + \lfloor\frac{I_1-6}{8}\rfloor + \lfloor\frac{I_1-1}{16}\rfloor + \lfloor\frac{I_1-9}{32}\rfloor + \lfloor\frac{I_1-25}{64}\rfloor + 6$$

Proof. Following the same argument as before, when computing the matricial product of the first ten series (see Appendix 2), we realise that the rank is determined by series with repeated rows and different column numbers. The series which have non repeated column numbers, and most frequent columns are: $\{J^{(1)}_{9p+8,\ 4p+3}\}_{p\in\mathbb{N}}$, $\{J^{(1)}_{9p+5,\ 8p+4}\}_{p\in\mathbb{N}}$, $\{J^{(1)}_{9p+6,\ 8p+5}\}_{p\in\mathbb{N}}$, $\{J^{(1)}_{9p,\ 16p}\}_{p\in\mathbb{N}}$, $\{J^{(1)}_{9p+2,\ 32p+8}\}_{m\in\mathbb{N}}$, , $\{J^{(1)}_{9p+3,\ 64p+24}\}_{p\in\mathbb{N}}$

□

Proposition 5.4. *The rank of the transition matrix $C^{(n)}$ of dimensions $[I_n, I_1]$ has a lower bound equal to :*

$$rank(C^{(n)})_{(I_1)} \geq \lfloor\frac{I_1-2^n}{2^n}\rfloor + \lfloor\frac{I_1-1}{4^n}\rfloor + 2$$

Proof. Only the series $\{J_{3p+2,\ 2p+1}\}_{p\in\mathbb{N}}$ and $\{J_{3p,4p}\}^{(n)}_{p\in\mathbb{N}}$ contribute to the rank of J_1. The power of these series can be computed easily. We showed that $\{J_{3p+2,\ 2p+1}\}^{(n)}_{p\in\mathbb{N}} = \{J_{3^n p+3^n-1,\ 2^n p+2^n-1}\}_{p\in\mathbb{N}}$. This series contributes to the rank of the the transition matrix $C^{(n)}$ with the term: $1 + \lfloor\frac{I_1-1-(2^n-1)}{2^n}\rfloor$.

Same argument with the series: $\{J_{3p,4p}\}^{(n)}_{p\in\mathbb{N}} = \{J_{3^n p,4^n p}\}^{(n)}_{p\in\mathbb{N}}$. □

Proposition 5.5. *The rank of the third order transition matrix $C^{(3)}$ of dimensions $[I_4, I_1]$ is equal to:* $rank(C^{(3)})_{(I_1)} = 18 + \lfloor\frac{I_1-8}{8}\rfloor + \lfloor\frac{I_1-5}{16}\rfloor + \lfloor\frac{I_1-14}{16}\rfloor + \lfloor\frac{I_1-4}{16}\rfloor + \lfloor\frac{I_1-29}{32}\rfloor + \lfloor\frac{I_1-17}{32}\rfloor + \lfloor\frac{I_1-22}{32}\rfloor + \lfloor\frac{I_1-6}{64}\rfloor + \lfloor\frac{I_1-1}{64}\rfloor + \lfloor\frac{I_1-41}{64}\rfloor + \lfloor\frac{I_1-73}{128}\rfloor + \lfloor\frac{I_1-102}{128}\rfloor + \lfloor\frac{I_1-25}{128}\rfloor + \lfloor\frac{I_1-97}{128}\rfloor + \lfloor\frac{I_1-33}{256}\rfloor + \lfloor\frac{I_1-89}{256}\rfloor + \lfloor\frac{I_1-217}{512}\rfloor + \lfloor\frac{I_1-473}{1024}\rfloor$

Proposition 5.6. *The rank of the fourth order transition matrix $C^{(4)}$ of dimensions $[I_5, I_1]$ is equal to:* $rank(C^{(4)})_{(I_1)} = 54 + \lfloor\frac{I_1-16}{16}\rfloor + \lfloor\frac{I_1-24}{32}\rfloor + \lfloor\frac{I_1-14}{32}\rfloor + \lfloor\frac{I_1-21}{32}\rfloor + \lfloor\frac{I_1-20}{32}\rfloor + \lfloor\frac{I_1-36}{64}\rfloor + \lfloor\frac{I_1-49}{64}\rfloor + \lfloor\frac{I_1-54}{64}\rfloor + \lfloor\frac{I_1-62}{64}\rfloor + \lfloor\frac{I_1-61}{64}\rfloor + \lfloor\frac{I_1-4}{128}\rfloor + \lfloor\frac{I_1-81}{128}\rfloor + \lfloor\frac{I_1-86}{128}\rfloor + \lfloor\frac{I_1-65}{128}\rfloor + \lfloor\frac{I_1-70}{128}\rfloor + \lfloor\frac{I_1-41}{128}\rfloor + \lfloor\frac{I_1-29}{128}\rfloor + \lfloor\frac{I_1-68}{256}\rfloor + \lfloor\frac{I_1-105}{256}\rfloor + \lfloor\frac{I_1-134}{256}\rfloor + \lfloor\frac{I_1-150}{256}\rfloor + \lfloor\frac{I_1-225}{256}\rfloor + \lfloor\frac{I_1-25}{256}\rfloor + \lfloor\frac{I_1-73}{256}\rfloor + \lfloor\frac{I_1-93}{256}\rfloor + \lfloor\frac{I_1-102}{256}\rfloor + \lfloor\frac{I_1-1}{256}\rfloor + \lfloor\frac{I_1-5}{64}\rfloor + \lfloor\frac{I_1-33}{512}\rfloor + \lfloor\frac{I_1-22}{512}\rfloor + \lfloor\frac{I_1-153}{512}\rfloor + \lfloor\frac{I_1-230}{512}\rfloor + \lfloor\frac{I_1-457}{512}\rfloor + \lfloor\frac{I_1-477}{512}\rfloor + \lfloor\frac{I_1-89}{512}\rfloor + \lfloor\frac{I_1-129}{512}\rfloor + \lfloor\frac{I_1-97}{512}\rfloor + \lfloor\frac{I_1-486}{1024}\rfloor + \lfloor\frac{I_1-713}{1024}\rfloor + \lfloor\frac{I_1-801}{1024}\rfloor + \lfloor\frac{I_1-217}{1024}\rfloor + \lfloor\frac{I_1-385}{1024}\rfloor + \lfloor\frac{I_1-345}{1024}\rfloor + \lfloor\frac{I_1-289}{2048}\rfloor + \lfloor\frac{I_1-1753}{2048}\rfloor + \lfloor\frac{I_1-1497}{2048}\rfloor + \lfloor\frac{I_1-1881}{2048}\rfloor + \lfloor\frac{I_1-998}{2048}\rfloor + \lfloor\frac{I_1-1225}{2048}\rfloor + \lfloor\frac{I_1-3361}{4096}\rfloor + \lfloor\frac{I_1-2905}{4096}\rfloor + \lfloor\frac{I_1-473}{4096}\rfloor + \lfloor\frac{I_1-6617}{8192}\rfloor + \lfloor\frac{I_1-2521}{16384}\rfloor$

The rank of the 5th order passage Collatz matrix has 162 terms. The approach of computing analytically the rank of the transition matrix of order n seems unfortunately to be a dead-end due to the lack of apparent computational tractability or recurrence. However, it has the merit of showing that as the input dimension increases, more iterations are needed to converge. In a sense we will pursue the same strategy, in order to try to proove the conjecture: fix an input dimension I_1 and show that after certain number of iterations, all the integers below this number do converge.

Finally, just observing elementary properties, from the passage matrix, we can deduce a weak convergence inequality, stating that the rank decreases (non-strictly) with the iterations.

Theorem 5.7. $\forall n \in \mathbb{N},\ rank(C^{(n)}) \leq rank(C^{(n-1)}) \leq ... \leq rank(C^{(2)}) \leq rank(C^{(1)})$

Proof. For all matrices $(A, B) \in (M_{m,n}(\mathbb{R}), M_{n,p}(\mathbb{R}))$: $rank(A.B) \leq rank(A)$ and $rank(A.B) \leq rank(B)$. Then: $rank(C^{(n)}) = rank(J_n.C^{(n-1)}) \leq rank(C^{(n-1)})$ □

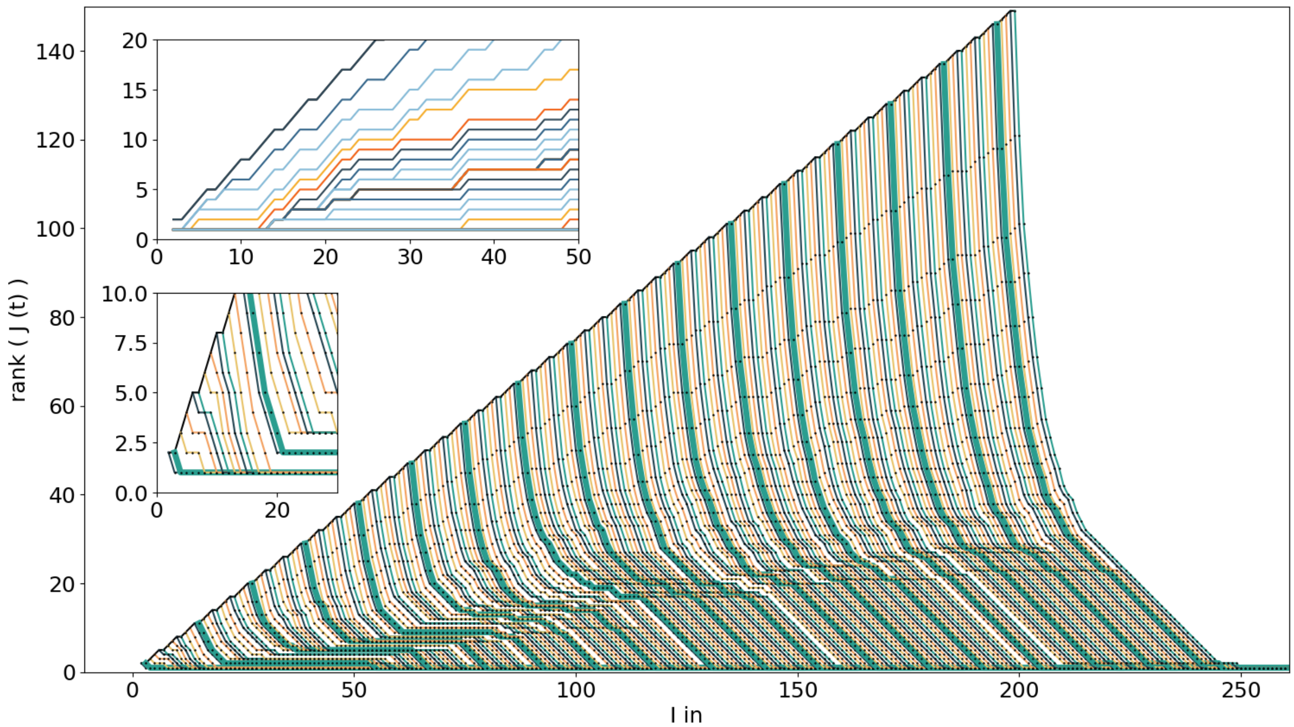

Figure 5: $Rank(C^{(m)}) = f(I_1)$. J_1 is a matrix of dimension $[I_2, I_1]$. Each colored line represents the rank of $C^{(m)}$ as m the iteration increases. Top left inset: rank of the Collatz matrix at fixed order m for different input dimensions I_1. Middle left inset: zoom. Remark the long plateaux in the bottom.

6 Series nilpotence

Example 6.1. $\{J_{3p+2,\ 8p+6}\}^{(2)}_{p\in\mathbb{N}} = \{J_{9p+8,\ 64p+54}\}_{p\in\mathbb{N}}$. *The submatrix of* $C^{(2)}$ *of dimensions* $[I_3, I_1]$ *has a nil contribution from the series* $\{J_{3p+2,\ 8p+6}\}^{(2)}_{p\in\mathbb{N}}$*, for* $I_1 - 1 < 54$.

Example 6.2. $\{J_{3p+2,\ 8p+6}\}^{(m)}_{p\in\mathbb{N}} = \{J_{3^m p+3^m-1,\ 8^m p+6\frac{8^m-1}{8-1}}\}_{p\in\mathbb{N}} = \{J_{3^m p+3^m-1,\ 8^m p+6\sum_{i=1}^{m} 8^{m-i}}\}_{p\in\mathbb{N}}$

$\forall I_1 \in \mathbb{N}^*, \exists m_o \in \mathbb{N}^*, \quad (m_0 : I_1 < 6\sum_{i=1}^{m_o} 8^{m_o-i} + 1) \quad : \forall m \geq m_o$*: the submatrix* $C^{(m)}$ *has a nil contribution from the series* $\{J_{3p+2,\ 8p+6}\}^{(m)}_{p\in\mathbb{N}}$.

Example 6.3. $\{J_{3p,\ 16p+2}\}^{(m)}_{p\in\mathbb{N}} = \{3^m p,\ 16^m p + 2\frac{16^m-1}{16-1}\}_{p\in\mathbb{N}} = \{3^m p,\ 16^m p + 2\sum_{i=1}^{m} 16^{m-i}\}_{p\in\mathbb{N}}$

$\forall I_1 \in \mathbb{N}^*, I_1$ *big enough*, $\exists m_o \in \mathbb{N}^* : \forall m \geq m_o$*: the submatrix* $C^{(m)}$ *has a unique contribution from the series* $\{J_{3p,\ 16p+2}\}^{(m)}_{p\in\mathbb{N}}$*, corresponding to the term in* $p = 0$ *of nil abscissa* $J_{0,\ 2\sum_{i=1}^{m_o} 16^{m_o-i}}$.

Choose m_0: $2\sum_{i=1}^{m_o} 16^{m_o-i} + 1 < I_1$ *and* $I_1 < 16^{m_o} + 2\sum_{i=1}^{m_o} 16^{m_o-i} + 1$

Definition 6.4. *Pure series:* $\{J_{a_1 p+b_1, a_2 p+b_2}\}$ *is a series (a submatrix of* J_1*). We call pure series the series which results from the* n^{th} *power of that series.*

Definition 6.5. *Composite series:* $\prod_{i=1}^{n}\{J_{a_1^i p+b_1^i, a_2^i p+b_2^i}\}$ *is a series, which is the product of n series of* J_1*.*

Proposition 6.6. *Nilpotence of the positive series:* $\forall I_1 \in \mathbb{N}$ *(input dimension of* J_1*)* $\exists m \in \mathbb{N} : \forall n \geq m :$ $\{J_{3p+2,\ 2p+1}\}^{(n)}_{p\in\mathbb{N}} = 0$

Proof. $\{J_{3p+2,\ 2p+1}\}^{(n)}_{p\in\mathbb{N}} = \{J_{3^n p+3^n-1,\ 2^n p+2^n-1}\}_{p\in\mathbb{N}}$.

As the column number $2^n p + 2^n - 1$ increases with n, take, for $p = 0$, $2^n - 1 > I_1 - 1$ and therefore choose $n > \left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor$. □

Lemma 6.7. *Every composite series of order n, with $n > \left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor$, has at most one non nil element for an input matrix of input dimension I_1.*

Proof. $\prod_{i=1}^{n}\{J_{a_1^i p+b_1^i, a_2^i p+b_2^i}\} = \{J_{3^n p+b_{1,n}, 2^m p+b_{2,n}}\}$.

Case a) $b_{2,n} > I_1 - 1$: then there are zero elements from this series in the matrix $C^{(n)}$ of dimensions $[I_n, I_1]$.
Case b) $b_{2,n} \in [0, I_1 - 1]$: as $2^n > 2^{\left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor}$, $\forall p > 0 : 2^n p + b_{2,n} > I_1 - 1$. Then there is only one element from this complex series in $C^{(n)}$ (which can be a zero ($b_{1,n} = 0$) or not). Not all series of a matrix $C_*^{(n)}$ of dimensions $[I_n, I_1*]$ with $I_1* >> I_1$ are present in the matrix $C^{(n)}$ (of dimensions $[I_n, I_1]$), but the ones which are present have only one element. Since $C^{(n)}$ is a transition matrix of order n, there is no need to proove that every integer $d \in [0, I_1 - 1]$ has an image. Then each pair $(d, C^{(n)}(d))$, belongs to one series, and this series has only one point. □

The first order matrix $J_1 = C^{(1)}$ has a certain number of series whose first element has a column equal to zero (we will call them zeros hereon), for instance $\{J_{3p,4p}\}_{p\in\mathbb{N}}$, $\{J_{3p,16p+2}\}_{p\in\mathbb{N}}$ and $\{J_{3p,64p+10}\}_{p\in\mathbb{N}}$, d = 0, 2, 10, for p = 0. They correspond to the natural odd numbers 1, 5, 21. These numbers are antecedents of respectively 2^2, 2^4 and 2^6 by the application 3n+1 (See Table 1).

The second order matrix $C^{(2)}$ has more zeros (in the sense of density, since they are infinitely already in $C^{(1)}$), and contains the precedent ones, for instance at = 0, 1, 2, 6, 10, 26. See Appendix 2.

7 Slope change

Definition 7.1. Δ*, the* ***slope*** *of a Collatz series of order n, $\{J_{a_{1,n}p+b_{1,n}, a_{2,n}p+b_{2,n}}\} = \{J_{3^n p+b_{1,n}, 2^m p+b_{2,n}}\}$ is the ratio $\frac{a_{1,n}}{a_{2,n}} = \frac{3^n}{2^m}$*

Proposition 7.2. *The slope change after multiplying by a first order series at left $\{J_{3p+b_1, 2^w p+b_2}\}_{p\in\mathbb{N}}\{J_{3^n p+b_{1,n}, 2^m p+b_{2,n}}\} = \{J_{3^{n+1}p+b_5, 2^{m+w}p+b_6}\}_{p\in\mathbb{N}}$ is equal to: $\frac{3^{n+1}}{2^{m+w}} - \frac{3^n}{2^m}$.*

Proposition 7.3. *With respect to the slope, the series product is commutative: $\Delta(\{J_{3^{q_1}p+b_1, 2^{r_1}p+b_2}\}\{J_{3^{q_2}p+b_3, 2^{r_2}p+b_4}\}) = \Delta(\{J_{3^{q_2}p+b_3, 2^{r_2}p+b_4}\}\{J_{3^{q_1}p+b_1, 2^{r_1}p+b_2}\})$. Moreover, the slope of the product is equal to $\frac{3^{q_1+q_2}}{2^{r_1+r_2}}$.*

Proof. See theorem 4.1. □

Proposition 7.4. *Let a Collatz matrix of order 1 be composed of m series: $C^{(1)} = \{J_{3p+2,2p+1}\}_{p\in\mathbb{N}} + \{J_{3p,2^2p}\}_{p\in\mathbb{N}} + ... + \{J_{3p+b_1, 2^m p+b_2}\}_{p\in\mathbb{N}}$. If we consider that the Collatz matrix of order 2, 3, ..., n have also m series (which is inexact), then the mean slope tends to 0 as n increases.*

Proof. Lets consider only the denominator of the slope. At order 1, the mean of the denominator is $\sum_{i=1}^{m} 2^i = 2.(2^m - 1)$ therefore the mean of the n^{th} power is $\frac{1}{m}(\sum_{i=1}^{m} 2^i)^n = 2^n.\frac{(2^m-1)^n}{m^n}$. Consider now the numerator of any collatz series of order n: it is always equal to 3^n. Therefore the mean slope is equal to: $\frac{3^n.m^n}{2^n.(2^m-1)^n} \sim \frac{(3m)^n}{(2^{m+1})^n} \to 0 \quad as \quad n \to \infty$. This argument explains why the majority of slopes become smaller as the order of the Collatz matrix increases. Even though there are series, as we will see next, that on the contrary become larger. □

Example 7.5. $\Delta(\{J_{3p+2,2p+1}\}^n) = (\frac{3}{2})^n$*. Nonetheless, the positive series at the n^{th} power becomes quickly nilpotent.*

Example 7.6. *Let a composite series κ composed of w products of order 1 series $\{J_{3p+_b 2,2^r p+1}\}$, $r \geq 2$ and $n-w$ products of the positive series $\{J_{3p+2,2p+1}\}$. We can write $\Delta(\kappa) = \frac{1}{K}(\frac{3}{2})^n, K \in \mathbb{N}$. Therefore, $\exists n \in \mathbb{N} : \Delta(\kappa) > 1$.*

As we will see next, even if it explains visually what we see in the Collatz matrices as the order increases, this slope change is not what causes convergence of the series towards zero. As we multiply Collatz series with themselves, we first generate new series with higher abcissa (b_2) and, by multiplying iteratively by new series, we find a series with that abcissa (b_2) and zero ordinate ($b_1 = 0$). We will see in the section transient increase and decrease, that in the regime $n > \left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor$, we precisely have composite series which precisely alternate positive increases with negative increases in the abcissa and in the slope change, and which remain at a fixed abcissa for several iterations.

8 Convergence

Proposition 8.1. *If the product $\{J_{a_1 p+b_1,a_2 p+b_2}\}_{p\in\mathbb{N}}\{J_{a_3 p+b_3,a_4 p+b_4}\}_{p\in\mathbb{N}}$ is non nil and can be written as $\{J_{a_5 p+b_5,a_6 p+b_6}\}_{p\in\mathbb{N}}$, then $b_6 \geq b_4$, and b_5 can be higher, equal or lower than b_3.*

Proof. $\{J_{a_1 p+b_1,\ a_2 p+b_2}\}_{p\in\mathbb{N}}.\{J_{a_3 p+b_3,\ a_4 p+b_4}\}_{p\in\mathbb{N}} = \{J_{\frac{a_1.a_3}{GCD(a_2,a_3)}p+a_1 c_2+b_1,\ \frac{a_4.a_2}{GCD(a_2,a_3)}p+a_4.c_3+b_4}\}_{p\in\mathbb{N}}$

$b_5 = a_1 c_2 + b_1$, c_2 depends only on $(a_2, a_3, b_3 - b_2)$ and not directly on b_3.

$b_6 = a_4.c_3 + b_4 \geq b_4$.

□

Definition 8.2. *A **zero** of a series is defined as the first element corresponding to $p = 0$ for series $\{J_{a_1 p+b_1,\ a_2 p+b_2}\}_{p\in\mathbb{N}}$ in which there is no row offset ($b_1 = 0$).*

Proposition 8.3. *$C^{(n+1)} = J_{n+1}C^{(n)}$. As $J_{n+1} = \sum_{i=1}^{M}\{J_{a_1^i p+b_1^i,\ a_2^i p+b_2^i}\}_{p\in\mathbb{N}}$ and $C^{(n)} = \sum_{i=1}^{N}\{J_{a_1^i p+b_1^i,\ a_2^i p+b_2^i}\}_{p\in\mathbb{N}}$,a necessary condition for the submatrix $\{J_{a_1^i p+b_1^i,\ a_2^i p+b_2^i}\}\{J_{a_1^j p+b_1^j,\ a_2^j p+b_2^j}\}_{p\in\mathbb{N}}$ to have a zero is that $b_1^i = 0$.*

Proof. One can rename $\{J_{a_1^i p+b_1^i,\ a_2^i p+b_2^i}\}\{J_{a_1^j p+b_1^j,\ a_2^j p+b_2^j}\}_{p\in\mathbb{N}}$ as $\{J_{a_1 p+b_1,\ a_2 p+b_2}\}\{J_{a_3 p+b_3,\ a_4 p+b_4}\}_{p\in\mathbb{N}}$. As seen in the previous proposition, $b_5 = a_1 c_2 + b_1 \geq b_1$. As $b_1 \geq 0$, $b_1 = 0$ is a necessary condition for having $b_5 = 0$. □

Proposition 8.4. *When $b_1 = 0$, the matricial multiplication of sparse submatrices defined following the series $\{J_{a_1 p,\ a_2 p+b_2}\}_{p\in\mathbb{N}}.\{J_{a_3 p+b_3,\ a_4 p+b_4}\}_{p\in\mathbb{N}} = \{J_{a_1(\frac{a_3}{GCD(a_2,a_3)}p+c_2),\ a_4(\frac{a_2}{GCD(a_2,a_3)}p+c_3)+b_4}\}_{p\in\mathbb{N}}$ has no offset on the columns when $c_2 = 0$ ie $a_3|b_2 - b_3$ and $c_3 = \frac{b_2-b_3}{a_3}$.*

The resulting zero is equal to $a_4.c_3 + b_4$.

When $c_3 = 0$ we also have creation of a zero at fixed abcissa.

Proof. Straightforward, looking at the Diophantine equation $a_2 c_2 - a_3 c_3 = b_3 - b_2$. □

Proposition 8.5. *For a Collatz matrix of order n, $n \leq \left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor$ we can obtain a zero both decreasing the line number (ordinate) at a fixed abcissa and by varying the line number and the abscissa.*

Example 8.6. *Consider a J_1 matrix of dimension $I_1 - 1 = 113$. $\{J_{3p,\ 1024p+170}\}_{p\in\mathbb{N}}.\{J_{3p+2,\ 2p+1}\}_{p\in\mathbb{N}} = \{J_{9p,\ 2048p+113}\}_{p\in\mathbb{N}}$. The abscissa from the series $\{J_{3p+2,\ 2p+1}\}_{p\in\mathbb{N}}$ went from 1 to 113, and the ordinate from 2 to 0.*

Proposition 8.7. *Roots of $C^{(q+1)}$. The series $\{J_{3p,\ a_2p+b_2}\}_{p\in\mathbb{N}}.\{J_{3^qp+b_3,\ a_4p+b_4}\}_{p\in\mathbb{N}}$, with $a_2 = 2^{2m}$ and $b_2 = (\frac{2^{2m}-1}{3} - 1)\frac{1}{2}$, $\quad m \in \mathbb{N}$, is going to have a zero if: $\quad 3^q|\frac{1}{2}(\frac{2^{2m}-1}{3} - 1) - b_3$. The resulting zero is equal to $a_4.\frac{\frac{1}{2}(\frac{2^{2m}-1}{3}-1)-b_3}{3^q} + b_4$.*

Proposition 8.8. *Stability of roots of $C^{(q)}_{q\geq q^0}$. If there is a integer q^0 such that the submatrix $\{J_{3^{q^0}p,\ a_2p+b_2}\}_{p\in\mathbb{N}}$ is a zero of $C^{(q^0)}$ (at b_2), then $\forall q \geq q^0$, b_2 is also going to be a zero.*

Proof. It suffices to remark that $C^{(q)} = J_qJ_{q-1}...J_{q^0+1}C^{(q^0)}$. From each submatrix $J_i, i \in [q, q^0+1]$, consider the submatrix $\{J_{3p,4p}\}_{p\in\mathbb{N}}$. As $\{J_{3p,4p}\}^{(q-q^0)}_{p\in\mathbb{N}} = \{J_{3^{q-q^0}p,4^{q-q^0}p}\}_{p\in\mathbb{N}}$.

Then $\{J_{3^{q-q^0}p,4^{q-q^0}p}\}_{p\in\mathbb{N}}\{J_{3^{q^0}p,\ a_2p+b_2}\}_{p\in\mathbb{N}} = \{J_{3^q,a_2.4^{q-q^0}p+b_2}\}_{p\in\mathbb{N}}$ □

Proposition 8.9. *If $GCD(a_2, a_3) = 1$:*

$\{J_{a_1p+b_1,a_2p+b_2}\}_{p\in\mathbb{N}}\{J_{a_3p+b_2,a_4p+b_4}\}_{p\in\mathbb{N}} = \{J_{a_1a_3p+b_1,a_4a_2p+b_4}\}$.

If $b_1 > b_2$ (respectively $b_1 < b_2$) there will be an increase (respectively decrease) in the ordinate, at the same constant abscissa (b_4).

Proof. In the presented case $c_2 = c_3 = 0$. □

Example 8.10. *We multiply at left the series $\{J_{27p+6,16p+3}\}_{p\in\mathbb{N}}$ from $C^{(3)}$ by the series $\{J_{3p+2,8p+6}\}_{p\in\mathbb{N}}$:*

$$\{J_{3p+2,8p+6}\}_{p\in\mathbb{N}}\{J_{27p+6,16p+3}\}_{p\in\mathbb{N}} = \{J_{81p+2,128p+3}\}_{p\in\mathbb{N}}$$

The abscissa 3 stays the same in the result, and the ordinate 6 decreases to 2.

Proposition 8.11. *Let a series from $C^{(n)}$ be multiplied at left with a series from J_{n+1} :*

$\{J_{3p+b_1,\ 2^{2m}p+b_2}\}_{p\in\mathbb{N}}\{J_{3^np+b_3,a_4p+b_4}\}_{p\in\mathbb{N}}, \quad b_1 \in \{0, 2\}, m \in \mathbb{N}.$

As $GCD(2^{2m}, 3^n) = 1$, given the precedent proposition, we will have an increase or a reduction in the line number when $b_2 = b_3$, and the resulting series will be: $\{J_{3^{n+1}p+b_1,a_4.2^{2m}p+b_4}\}_{p\in\mathbb{N}}$

Example 8.12. *In the following figure, I show both the passage matrices $J_i, i \in [1, 5]$ as well as the i^{th} project transition / product matrices, as they converge.*

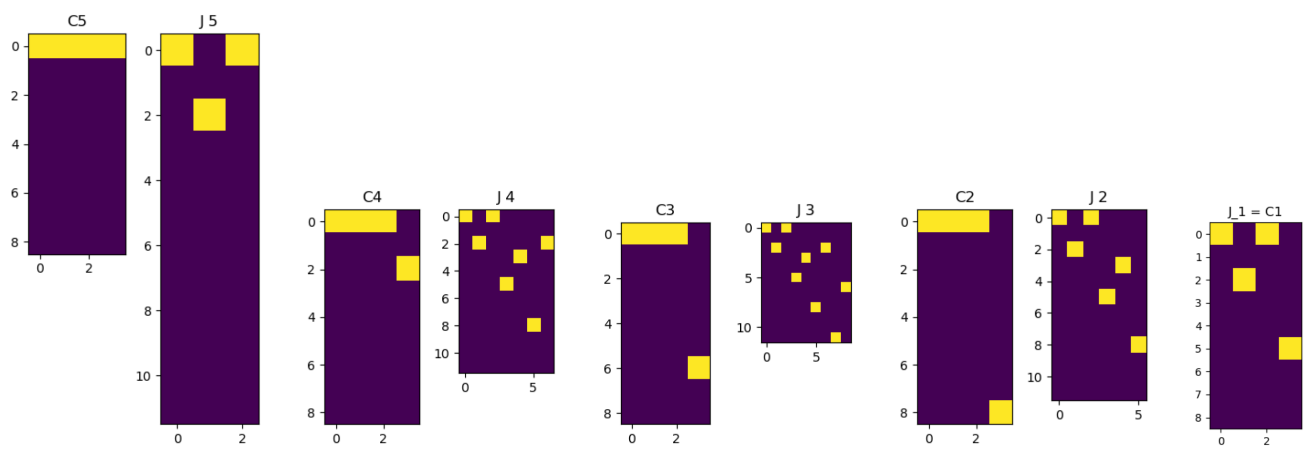

Figure 6: J_i and $C^{(i)}$ for $i \in [1, 3]$, $C^{(5)}$ converges to zero (d=0, corresponding to 1).

I show the convergence of the Collatz matrix, $C^{(i)}$ for $d \in [0, 3]$ for $i \geq 5$.

$J_1 = \{J_{3p,4p\leq 3}\} + \{J_{3p+2,2p+1\leq 3}\} + \{J_{3p,16p+2\leq 3}\}$

For $d \leq 3$, J_1 has two zeros, $d = 0$ and $d = 2$, corresponding to $p = 0$, for the submatrices $\{J_{3p,4p}\}$ and $\{J_{3p,16p+2}\}$.

$J_2 = \{J_{3p,4p\leq 5}\} + \{J_{3p+2,2p+1\leq 5}\} + \{J_{3p,16p+2\leq 5}\}$

$$\begin{aligned} J_2.J_1 = &\{J_{3p,4p\leq 5}\}\{J_{3p,4p\leq 3}\} + \{J_{3p,4p\leq 5}\}\{J_{3p+2,2p+1\leq 3}\} + \{J_{3p,4p\leq 5}\}\{J_{3p,16p+2\leq 3}\} \\ &+ \{J_{3p+2,2p+1\leq 5}\}\{J_{3p,4p\leq 3}\} + \{J_{3p+2,2p+1\leq 5}\}\{J_{3p+2,2p+1\leq 3}\} + \{J_{3p+2,2p+1\leq 5}\}\{J_{3p,16p+2\leq 3}\} \\ &+ \{J_{3p,16p+2\leq 5}\}\{J_{3p,4p\leq 3}\} + \{J_{3p,16p+2\leq 5}\}\{J_{3p+2,2p+1\leq 3}\} + \{J_{3p,16p+2\leq 5}\}\{J_{3p,16p+2\leq 3}\} \end{aligned}$$

This sum of submatrices has three zeros, the two precedent $\{J_{3p,4p\leq 5}\}\{J_{3p,4p\leq 3}\} = \{J_{9p,16p\leq 3}\}$ and $\{J_{3p,4p\leq 5}\}\{J_{3p,16p+2\leq 3}\} = \{J_{9p,64p+2\leq 3}\}$ plus a new zero at d = 1, $\{J_{3p,16p+2\leq 5}\}\{J_{3p+2,2p+1\leq 3}\} = \{J_{9p,\ 32p+1\leq 3}\}$, as we can see it on the $C^{(2)}$ matrix.

From these summands one can discard the ones whose zeros are higher than 5, as we multiply at left the $i^{(th)}$ order passage matrix by the corresponding first order passage matrix J_{i+1}, the zeros only grow, they do not decrease.

Therefore we disregard the terms:

$\{J_{3p,4p\leq 5}\}\{J_{3p+2,2p+1\leq 3}\} = \{J_{9p+6,8p+5\leq 3}\}$

$\{J_{3p+2,2p+1\leq 5}\}\{J_{3p,16p+2\leq 3}\} = \{J_{9p+5,32p+18\leq 3}\}$

$\{J_{3p,16p+2\leq 5}\}\{J_{3p,4p\leq 3}\} = \{J_{9p+3,64p+24\leq 3}\}$

$\{J_{3p+2,2p+1\leq 5}\}\{J_{3p,4p\leq 3}\} = \{J_{9p+5,8p+4\leq 3}\}$

$\{J_{3p,4p+5\leq 5}\}\{J_{3p,4p\leq 3}\} = \{J_{9p+3,16p+12\leq 3}\}$

The only remaining term is:

$\{J_{3p+2,2p+1\leq 5}\}\{J_{3p+2,2p+1\leq 3}\} = \{J_{9p+8,4p+3\leq 3}\}$

$J_3 = \{J_{3p,4p\leq 8}\} + \{J_{3p+2,2p+1\leq 8}\} + \{J_{3p,16p+2\leq 8}\} + \{J_{3p+2,8p+6\leq 8}\}$

The only non zero value in the range $d \in [0, 3]$ in the product $J3.J2.J1$ is the summand in which the $c_2 = 0$:

$\{J_{3p,4p\leq 8}\}\{J_{9p+8,4p+3\leq 3}\} = \{J_{27p+6,16p+3\leq 3}\}$

$J_4 = \{J_{3p,4p\leq 6}\} + \{J_{3p+2,2p+1\leq 6}\} + \{J_{3p,16p+2\leq 6}\} + \{J_{3p+2,8p+6\leq 6}\}$

The only non zero value in the range $d \in [0, 3]$ in the product $J4.J3.J2.J1$ is the summand in which the $c_2 = 0$:

$\{J_{3p+2,8p+6\leq 6}\}\{J_{27p+6,16p+3\leq 3}\} = \{J_{81p+2,128p+3\leq 3}\}$

$J_5 = \{J_{3p,4p\leq 2}\} + \{J_{3p+2,2p+1\leq 2}\} + \{J_{3p,16p+2\leq 2}\}$

$\{J_{3p,16p+2\leq 2}\}\{J_{81p+2,128p+3\leq 3}\} = \{J_{243p,2048p+3\leq 3}\}$

At fifth order ($243 = 3^5$), we found how $d = 3$ becomes a zero.

Although by now we understood that the basic operations of product between series is explained by solving a Diophantine equation, we are now going to try to explicit a bit further how does the series interact between them to converge. First in a mixed scenario with Collatz passage matrices of order $q \leq \left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor$, in which we will see changes both in abcissa and ordinate of the series, and then at $q > \left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor$, where we will only have changes in the ordinate with a fixed abscissa.

8.1 Convergence at varying abscissa and ordinate

Example 8.13. *Let's consider the case $d = 7$, which is slighly more interesting than the case $d = 3$ that we just saw, and start backwards.*

$$\begin{array}{l}\{J_{243p,\ 4096p+7}\}\\ \{J_{3p,16p+2}\}\{J_{81p+2,256p+7}\} \quad (c_2, c_3) = (0, 0)\\ \{J_{3p,16p+2}\}\{J_{3p+2,32p+26}\}\{J_{27p+26,8p+7}\} \quad (c_2, c_3) = (0, 0)\\ \{J_{3p,16p+2}\}\{J_{3p+2,32p+26}\}\{J_{3p+2,2p+1}\}\{J_{9p+8,4p+3}\}, \quad (c_2, c_3) = (8, 1)\end{array}$$

To converge to the point $d = 7$ (b_4) of ordinate 0, remark that we start from the point of abcissa $d = 3$ (b_4) (in the series $\{J_{9p+8,4p+3}\}$ and after multiplying at left by $\{J_{3p+2,2p+1}\}$, the solution of the Diophantine equation imposes that $c_3 = 1$ which means that $b_4^{'} = 3 + 4.1 = 7$.

The multiplying the series $\{J_{9p+8,4p+3}\}$ at left by $\{J_{3p+2,2p+1}\}$ results in a series in which both the abscissa $(3 \to 7)$ and the ordinate change $(8 \to 26)$. 26 is precisely the abscissa of the series $\{J_{3p+2,32p+26}\}$, which means that in the Diophantine equation $b3 - b_2 = 0$, and that there will be a change in ordinate at fixed abscissa, in this case a drastic reduction towards 2. The last step $(2 \to 0)$ is similar.

Proposition 8.14. *Let the composite series:*

$$\{J_{a_{1,k}p+b_{1,k},\ a_{2,k}p+b_{2,k}}\}_k . \{J_{a_{1,k-1}p+b_{1,k-1},\ a_{2,k-1}p+b_{2,k-1}}\}_{k-1} \cdots \\ \{J_{a_{1,2}p+b_{1,2},\ a_{2,2}p+b_{2,2}}\}_2 \{J_{a_3^{(1)}p+b_3^{(1)},\ a_4^{(1)}p+b_4^{(1)}}\}_1$$

and call $\{J_{a_{1,2}p+b_{1,2},\ a_{2,2}p+b_{2,2}}\}_2 \{J_{a_3^{(1)}p+b_3^{(1)},\ a_4^{(1)}p+b_4^{(1)}}\}_1 = \{J_{a_3^{(2)}p+b_3^{(2)},\ a_4^{(2)}p+b_4^{(2)}}\}$

Let $\{c_2^{(i)}, c_3^{(i)}\}_{i \in [1,k-1]}$*, the solutions of the Diophantine equation associated with the series product* $\{J\}_{i+1}.(\prod_{i=1}^{m}\{J\}_i)$*. The algorithm describing the evolution of the abcissa* $b_4^{(i)}$ *of the composite series* $\prod_{i=1}^{m}\{J\}_i$ *is given by:*

$$\begin{cases} b_4^{(i+1)} & \leftarrow b_4^{(i)} + c_3^{(i)}.a_4^{(i)} \\ a_4^{(i+1)} & \leftarrow a_4^{(i)} a_{2,i} \end{cases}$$

8.2 Convergence at varying abscissa and fixed ordinate

Proposition 8.15. $\{J_{3^q p,\ 2^m p+b_4}\}$ *is a series with a zero in* b_4*. Multiplying at left this series by* $\{J_{3p,\ 2^m p+b2}\}$ *results in a new series with a new zero, if the equation* $2^m.c_2 - 3^q.c_3 = -b_2$ *has a solution such that* $c_2 = 0$ *and* $c_3 = \frac{b_2}{3^q}$.

Example 8.16. $\{J_{3p,\ 256p+42}\}\{J_{3p,\ 4p}\} = \{J_{9p,\ 1024p+56}\},\ c_2 = 0,\ c_3 = 14.$

8.3 Convergence at fixed abscissa

Proposition 8.17. *For a Collatz passage matrix of order q,* $q > \left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor$*, the changes in the ordinate at a fixed abcissa follow the following algorithm:*

Let $\{J_{3^q p+b_3^{(q)}, a_4 p+b_4}\}$ *be a submatrix of* $C^{(q)}$*. The term* $b_3^{(q)}$ *designates the fact that it is the coefficient from the* q^{th} *Collatz matrix (it is not a power).*

Let $\{J_{3p+b_1,a_2p+b_2}\}, b_1 \in \{0, 2\}$ *be a submatrix of* J_{q+1}.

To find the value of b_3 at the next iterations $b_3^{(q+1)}$:

- Find (b_2, a_2, b_1) such that $c_2^{(q+1)} = \frac{b_3^{(q)} - b_2}{a_2} \in \mathbb{N}$ and $c_3^{(q+1)} = 0$

- The next ordinate (at the same abcissa) will be : $b_3^{(q+1)} = 3c_2^{(q+1)} + b_1$

Proof. As a reminder, for a Collatz matrix of order q and input dimension I_1, every column, or abscissa g, has a series associated to it and the series has the form $\{J_{3^q p+b_3, a_4 p+b_4}\}, b_4 = g$. Apply then the multiplication by a first order series of J_{q+1}, $\{J_{3p+b_1, a_2 p+b_2}\}, b_1 \in \{0, 2\}$. Solve the Diophantine equation: $a_2 c_2 - a_3 c_3 = b_3 - b_2$, with $c_3 = 0$. Therefore $c_2 = \frac{b_3 - b_2}{a_2}$ and $b_3^{(q+1)} = 3c_2^{(q+1)} + b_1$.

□

Example 8.18.

$b_3^{(q)} = 5, \quad (b_2, a_2, b_1) = (1, 2, 2), \quad c_2^{(q+1)} = \frac{5-1}{2} = 2$

$b_3^{(q+1)} = 3.2 + 2 = 8, \quad (b_2, a_2, b_1) = (0, 4, 0), \quad c_2^{(q+1)} = \frac{8-0}{4} = 2$

$b_3^{(q+2)} = 3.2 = 6, \quad (b_2, a_2, b_1) = (6, 8, 2), \quad c_2^{(q+2)} = \frac{6-6}{8} = 0$

$b_3^{(q+3)} = 0 + 2 = 2, \quad (b_2, a_2, b_1) = (2, 2, 0), \quad c_2^{(q+3)} = \frac{2-2}{2} = 0$

$b_3^{(q+4)} = 0$

We just did the cycle $5 \to 8 \to 6 \to 2 \to 0$. which corresponds to the Collatz cycle $11 \to 17 \to 13 \to 5 \to 1$, when disregarding the even numbers.

8.4 Regularities of the zeros

As we can see in Appendix 2 in the red boxes, there are very salient regularities in $C^{(2)}$. These can be observed in the space of the series (rather than in $\mathbb{N}$), in which the zeros are not evenly spaced (0, 2, 10, 42, 170) corresponding to $\{J_{3p,4p}\}$ times (respectively): $\{J_{3p,4p}\}_{p\in\mathbb{N}}$, $\{J_{3p,\ 16p+2}\}_{p\in\mathbb{N}}$, $\{J_{3p,\ 64p+10}\}_{p\in\mathbb{N}}$, $\{J_{3p,\ 256p+42}\}_{p\in\mathbb{N}}$, $\{J_{3p,\ 1024p+170}\}_{p\in\mathbb{N}}$. This regularity is trivial, since it involves the same Diophantine equation $4c_2 - 3c_3 = 0$.

Nonetheless, there is a regularity that seems less trivial, and that we observe along the columns. It corresponds to the coefficient b_1 of the resulting product,that as we can see over columns is periodic : 8, 6, 5, 0, 2, 3. The periodicity of this coefficient is itself due to the periodicity of the solution c_2 variable in the Diophantine equations 2, 2, 1, 0, 0, 1. The zeros themselves result to be periodic also.

Proposition 8.19. *Zeros of $C^{(2)}$. If m: $\{J_{3p,2^{2m+1}p} + \frac{1}{2}(\frac{2^{2m}-1}{3} - 1)\}_{p\in\mathbb{N}}.\{J_{3p,2^{2n+1}p} + \frac{1}{2}(\frac{2^{2n}-1}{3} - 1)\}_{p\in\mathbb{N}}$ or $\{J_{3p,2^{2m+1}p} + \frac{1}{2}(\frac{2^{2m}-1}{3} - 1)\}_{p\in\mathbb{N}}.\{J_{3p+2,2^{2n}p} + \frac{1}{2}(\frac{5.2^{2n-1}-1}{3} - 1)\}_{p\in\mathbb{N}}$ is a zero, then m+3 is also a zero.*

Proof. TBD.

□

Example 8.20. *Periodicity of zeros in $C^{(2)}$, $C^{(3)}$ in the series space.*

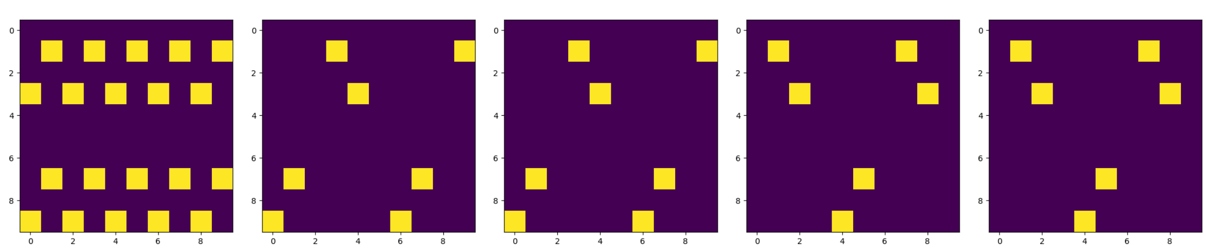

Figure 7: Zeros in the series space: a) $C^{(2)}$: similar to Appendix 2.b) and c) First and third line of a, multiplied at left by the basis series of J_1 d and e) Second and fourth line of a, multiplied at left by the basis series of J_1.

9 Large transients

As we will see, the transient increase and decrease is governed by the same mechanism as the convergence.

Example 9.1. *Consider a Collatz matrix of dimension $d + 1 = I_1 = 19$.*

As a remainder consider the power of the positive series:

$\{J_{3p+2,\ 2p+1}\}^{(n)}_{p\in\mathbb{N}} = \{J_{3^n p+3^n-1,\ 2^n p+2^n-1}\}_{p\in\mathbb{N}} = \{J_{a_3 p+b_3,\ a_4 p+b_4}\}_{p\in\mathbb{N}}$

The abscissa of the b_4 parameter is, for the fourth order Collatz matrix of equal to $2^4 - 1 = 15$ and the ordinate b_3 is equal to $3^n - 1 = 80$. $\left\lfloor \frac{ln(18)}{ln(2)} \right\rfloor = 4$.

Let's follow what happens to the input number of abscissa d = 15 from now on.

$b_3^{(4)} = 80,\quad (b_2, a_2, b_1) = (0, 4, 0),\quad c_2^{(5)} = \frac{80-0}{4} = 20$

$b_3^{(5)} = 3.20 + 0 = 60,\quad (b_2, a_2, b_1) = (0, 4, 0),\quad c_2^{(6)} = \frac{60-0}{4} = 15$

$b_3^{(6)} = 3.15 + 0 = 45,\quad (b_2, a_2, b_1) = (1, 2, 2),\quad c_2^{(7)} = \frac{45-1}{2} = 22$

$b_3^{(7)} = 3.22 + 2 = 68,\quad (b_2, a_2, b_1) = (0, 4, 0),\quad c_2^{(8)} = \frac{68-0}{4} = 17$

$b_3^{(8)} = 3.17 + 0 = 51,\quad (b_2, a_2, b_1) = (1, 2, 2),\quad c_2^{(9)} = \frac{51-1}{2} = 25$

$b_3^{(9)} = 25.3 + 2 = 77$

Summarizing until now, the abcissa (d = 15) and the ordinates (80 , 60, 45, 68, 51, 77) are described by the series (at p=0)

$\{J_{3p+2,\ 2p+1}\}\{J_{3p,\ 4p}\}\{J_{3p+2,\ 2p+1}\}\{J_{3p,\ 4p}\}\{J_{3p,\ 4p}\}\{J_{3p+2,\ 2p+1}\}^{(4)}$

As we see the downs come with the negative series $\{J_{3p,\ 4p}\}$ and the ups with the positive series $\{J_{3p+2,\ 2p+1}\}$.

As shown in Appendix 1, the series, the congruence conditions and the expression of $\tilde{f}_{(d)}$, the function giving the image of the indexes of each odd number in the Collatz series are equivalent expressions. The positive series has the particularity that $\tilde{f}_{(d)} - d > 0$ and the negative series that $\tilde{f}_{(d)} - d < 0$. In particular, all the positive jumps correspond to indexes of odd indexes of odd numbers that are divisible by two.

10 Conclusion

Reordering the natural numbers in even numbers that are multiples of two of odd numbers, I could find a very clear order in the way the Collatz series alternates between odd numbers. I described the function giving the next odd number as a function of each odd number and show that this function has periodic positive and negative components. These regularities allow to define a transition matrix, which has also simple structure its rank increases quasi-linearly with the input dimension. We measured how the rank if the powers of this matrix converge, first using numerical simulations and then using a semi-analytical approach for the first iterations of the transition matrix. With this approach, in order to prove the Collatz conjecture, one has to fix a arbitrary natural number I_1 (input dimension of the transition matrix), and show that as the number of iterations (multiplications) of this matrix increases, the first row converges towards $d = 0$, which corresponds to 1, in the usual Collatz series. In particular, I prove a weak convergence criterion, stating that the rank decreases (non-strictly) with the iterations.

As the transition matrix is composed of a sum of submatrices defined by affine functions (both series in x and y) , I explicited here the multiplication rules of these submatrices, that I call series, and which allow to understand the dynamics of these series. Notably, the link between solving a Diophantine equation and iterating over the Collatz series is made.

Notably, I found that for a transition matrix of input dimension I_1, one needs $n > \left\lfloor \frac{ln(I_1)}{ln(2)} \right\rfloor$ iterations to have a transition matrix in which there is only one series associated to it, which means that the ordinates of all the points will evolve independently. In particular, for the large transients we came to understand, using the series framework, the particularities of each number in the sense that it explicits the non trivial divisibility conditions using the congruences with powers of two. Solving the Collatz conjecture in this framework imply using the series constraints to understand how the x axis is filled iteratively towards the right with summands that are multiples of powers of two.

11 Appendix 1

In the next table we compute $\tilde{f}_{(d)}$ up to $m = 10$.

$\tilde{f}_{(d)}$	Condition	Series	Sign
$d + \frac{d+1}{2}$	$d + 1 \equiv 0[2]$	$\{J_{3p+2,\ 2p+1}\}_{p\in\mathbb{N}}$	+
$d - \frac{d}{4}$	$d \equiv 0[4]$	$\{J_{3p,\ 4p}\}_{p\in\mathbb{N}}$	-
$d - \frac{5d+2}{8}$	$d - 6 \equiv 0[8]$	$\{J_{3p+2,\ 8p+6}\}_{p\in\mathbb{N}}$	-
$d - \frac{13d+6}{16}$	$d - 2 \equiv 0[16]$	$\{J_{3p,\ 16p+2}\}_{p\in\mathbb{N}}$	-
$d - \frac{29d+14}{32}$	$d - 26 \equiv 0[32]$	$\{J_{3p+2,\ 32p+26}\}_{p\in\mathbb{N}}$	-
$d - \frac{61d+30}{64}$	$d - 10 \equiv 0[64]$	$\{J_{3p,\ 64p+10}\}_{p\in\mathbb{N}}$	-
$d - \frac{125d+62}{128}$	$d - 106 \equiv 0[128]$	$\{J_{3p+2,\ 128p+106}\}_{p\in\mathbb{N}}$	-
$d - \frac{253d+126}{256}$	$d - 42 \equiv 0[256]$	$\{J_{3p,\ 256p+42}\}_{p\in\mathbb{N}}$	-
$d - \frac{509d+254}{512}$	$d - 426 \equiv 0[512]$	$\{J_{3p+2,\ 512p+426}\}_{p\in\mathbb{N}}$	-
$d - \frac{1021d+510}{1024}$	$d - 170 \equiv 0[1024]$	$\{J_{3p,\ 1024p+170}\}_{p\in\mathbb{N}}$	-

12 Appendix 2: $C^{(2)}_{(I_1)}$

In the next table, I adopt the following notation for describing the matricial product:

$$J_{ap+b,cp+d}.J_{ep+f,gp+h} = \begin{bmatrix} ap+b \\ cp+d \end{bmatrix} \bigotimes \begin{bmatrix} ep+f \\ gp+h \end{bmatrix}$$

$\otimes$	$\begin{bmatrix}3p+2\\2p+1\end{bmatrix}$	$\begin{bmatrix}3p\\4p\end{bmatrix}$	$\begin{bmatrix}3p+2\\8p+6\end{bmatrix}$	$\begin{bmatrix}3p\\16p+2\end{bmatrix}$	$\begin{bmatrix}3p+2\\32p+26\end{bmatrix}$	$\begin{bmatrix}3p\\64p+10\end{bmatrix}$	$\begin{bmatrix}3p+2\\128p+106\end{bmatrix}$	$\begin{bmatrix}3p\\256p+42\end{bmatrix}$	$\begin{bmatrix}3p+2\\512p+426\end{bmatrix}$	$\begin{bmatrix}3p\\1024p+170\end{bmatrix}$
$\begin{bmatrix}3p+2\\2p+1\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+8}\\4p+3\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+5}\\8p+4\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+8}\\16p+14\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+5}\\32p+18\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+8}\\64p+58\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+5}\\128p+74\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+8}\\256p+234\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+5}\\512p+298\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+8}\\1024p+938\end{bmatrix}$	$\begin{bmatrix}\mathbf{9p+5}\\2048p+1194\end{bmatrix}$
$\begin{bmatrix}3p\\4p\end{bmatrix}$	$\begin{bmatrix}9p+6\\8p+5\end{bmatrix}$	$\begin{bmatrix}9p\\16p\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{6}\\32p+22\end{bmatrix}$	$\begin{bmatrix}9p\\64p+2\end{bmatrix}$	$\begin{bmatrix}9p+6\\128p+90\end{bmatrix}$	$\begin{bmatrix}9p\\256p+10\end{bmatrix}$	$\begin{bmatrix}9p+6\\512p+362\end{bmatrix}$	$\begin{bmatrix}9p\\1024p+42\end{bmatrix}$	$\begin{bmatrix}9p+6\\2048p+1450\end{bmatrix}$	$\begin{bmatrix}9p\\4096p+170\end{bmatrix}$
$\begin{bmatrix}3p+2\\8p+6\end{bmatrix}$	$\begin{bmatrix}9p+5\\16p+9\end{bmatrix}$	$\begin{bmatrix}9p+2\\32p+8\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{5}\\64p+38\end{bmatrix}$	$\begin{bmatrix}9p+2\\128p+34\end{bmatrix}$	$\begin{bmatrix}9p+5\\256p+154\end{bmatrix}$	$\begin{bmatrix}9p+2\\512p+138\end{bmatrix}$	$\begin{bmatrix}9p+5\\1024p+618\end{bmatrix}$	$\begin{bmatrix}9p+2\\2048p+554\end{bmatrix}$	$\begin{bmatrix}9p+5\\4096p+2474\end{bmatrix}$	$\begin{bmatrix}9p+2\\8192p+2218\end{bmatrix}$
$\begin{bmatrix}3p\\16p+2\end{bmatrix}$	$\begin{bmatrix}9p\\32p+1\end{bmatrix}$	$\begin{bmatrix}9p+3\\64p+24\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{0}\\128p+6\end{bmatrix}$	$\begin{bmatrix}9p+3\\256p+98\end{bmatrix}$	$\begin{bmatrix}9p\\512p+26\end{bmatrix}$	$\begin{bmatrix}9p+3\\1024p+394\end{bmatrix}$	$\begin{bmatrix}9p\\2048p+106\end{bmatrix}$	$\begin{bmatrix}9p+3\\4096p+1578\end{bmatrix}$	$\begin{bmatrix}9p\\8192p+426\end{bmatrix}$	$\begin{bmatrix}9p+3\\16384p+6314\end{bmatrix}$
$\begin{bmatrix}3p+2\\32p+26\end{bmatrix}$	$\begin{bmatrix}9p+2\\64p+17\end{bmatrix}$	$\begin{bmatrix}9p+8\\128p+120\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{2}\\256p+70\end{bmatrix}$	$\begin{bmatrix}9p+8\\512p+482\end{bmatrix}$	$\begin{bmatrix}9p+2\\1024p+282\end{bmatrix}$	$\begin{bmatrix}9p+8\\2048p+1930\end{bmatrix}$	$\begin{bmatrix}9p+2\\4096+1130\end{bmatrix}$	$\begin{bmatrix}9p+8\\8192p+7722\end{bmatrix}$	$\begin{bmatrix}9p+2\\16384p+4522\end{bmatrix}$	$\begin{bmatrix}9p+8\\32768p+30890\end{bmatrix}$
$\begin{bmatrix}3p\\64p+10\end{bmatrix}$	$\begin{bmatrix}9p+3\\128p+49\end{bmatrix}$	$\begin{bmatrix}9p+6\\256p+184\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{3}\\512p+198\end{bmatrix}$	$\begin{bmatrix}9p+6\\1024p+738\end{bmatrix}$	$\begin{bmatrix}9p+3\\2048p+794\end{bmatrix}$	$\begin{bmatrix}9p+6\\4096p+2954\end{bmatrix}$	$\begin{bmatrix}9p+3\\8192p+3178\end{bmatrix}$	$\begin{bmatrix}9p+6\\16384p+11818\end{bmatrix}$	$\begin{bmatrix}9p+3\\32768p+12714\end{bmatrix}$	$\begin{bmatrix}9p+6\\65536p+47274\end{bmatrix}$
$\begin{bmatrix}3p+2\\128p+106\end{bmatrix}$	$\begin{bmatrix}9p+8\\256p+241\end{bmatrix}$	$\begin{bmatrix}9p+5\\512p+312\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{8}\\1024p+966\end{bmatrix}$	$\begin{bmatrix}9p+5\\2048p+1250\end{bmatrix}$	$\begin{bmatrix}9p+8\\4096p+3866\end{bmatrix}$	$\begin{bmatrix}9p+5\\8192p+5002\end{bmatrix}$	$\begin{bmatrix}9p+8\\16384p+15466\end{bmatrix}$	$\begin{bmatrix}9p+5\\32768p+20010\end{bmatrix}$	$\begin{bmatrix}9p+8\\65536p+61866\end{bmatrix}$	$\begin{bmatrix}9p+5\\131072p+80042\end{bmatrix}$
$\begin{bmatrix}3p\\256p+42\end{bmatrix}$	$\begin{bmatrix}9p+6\\512p+369\end{bmatrix}$	$\begin{bmatrix}9p\\1024p+56\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{6}\\2048p+147\end{bmatrix}$	$\begin{bmatrix}9p\\4096p+226\end{bmatrix}$	$\begin{bmatrix}9p+6\\8192p+5914\end{bmatrix}$	$\begin{bmatrix}9p\\16384p+906\end{bmatrix}$	$\begin{bmatrix}9p+6\\32768p+23658\end{bmatrix}$	$\begin{bmatrix}9p\\65536p+3626\end{bmatrix}$	$\begin{bmatrix}9p+6\\131072p+94634\end{bmatrix}$	$\begin{bmatrix}9p\\262144p+14506\end{bmatrix}$
$\begin{bmatrix}3p+2\\512p+426\end{bmatrix}$	$\begin{bmatrix}9p+5\\1024p+625\end{bmatrix}$	$\begin{bmatrix}9p+2\\2048p+568\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{5}\\4096p+2502\end{bmatrix}$	$\begin{bmatrix}9p+2\\8192p+2274\end{bmatrix}$	$\begin{bmatrix}9p+5\\16384p+10010\end{bmatrix}$	$\begin{bmatrix}9p+2\\32768p+9098\end{bmatrix}$	$\begin{bmatrix}9p+5\\65536p+40042\end{bmatrix}$	$\begin{bmatrix}9p+2\\131072p+36394\end{bmatrix}$	$\begin{bmatrix}9p+5\\262144p+160170\end{bmatrix}$	$\begin{bmatrix}9p+2\\524288p+145578\end{bmatrix}$
$\begin{bmatrix}3p\\1024p+170\end{bmatrix}$	$\begin{bmatrix}9p\\2048p+113\end{bmatrix}$	$\begin{bmatrix}9p+3\\4096p+159\end{bmatrix}$	$\begin{bmatrix}9p+\mathbf{0}\\8192p+454\end{bmatrix}$	$\begin{bmatrix}9p+3\\16384p+63\end{bmatrix}$	$\begin{bmatrix}9p\\32768p+1818\end{bmatrix}$	$\begin{bmatrix}9p+3\\65536p+25482\end{bmatrix}$	$\begin{bmatrix}9p\\131072p+7274\end{bmatrix}$	$\begin{bmatrix}9p+3\\262144p+10193\end{bmatrix}$	$\begin{bmatrix}9p\\544288p+29098\end{bmatrix}$	$\begin{bmatrix}9p+3\\1048576p+407722\end{bmatrix}$

In blue are the components that contribute to the rank, in red the components that have a nul row component (for p=0), which are the ones responsible for the convergence towards 0 (corresponding to the number

1 in the usual formulation). Of course, these components include the components that have a nul row component of $C^{(1)}_{(I_1)}$ (for instance (0,0), (2,0), (10,0)) which correspond to the first element of the series $\{J_{3p,4p}\}$, $\{J_{3p,16p+2}\}$ and $\{J_{3p,64p+10}\}$, are included in the series $\{J_{9p,16p}\}$, $\{J_{9p,64p+2}\}$ and $\{J_{9p,256p+10}\}$. Aslo remark that we observe two types of periodicity. The first one, correspond to the alternance in the first component of the result vector along lines (see second row for instance). This is trivially explained by the fact that they correspond to the solution of the same Diophantine equation. The second component is a periodicity in the coefficient b of the first component along the columns. This component is equal to (in order): 8, 6, 5, 0, 2, 3 (and then 8, 6...). See fourth column for instance. Now, I show the solutions (c_2, c_3) of the Diophantine equation $a_2c_2 - a_3c_3 = b_3 - b_2$

$\otimes$	$\begin{bmatrix}3p+2\\2p+1\end{bmatrix}$	$\begin{bmatrix}3p\\4p\end{bmatrix}$	$\begin{bmatrix}3p+2\\8p+6\end{bmatrix}$	$\begin{bmatrix}3p\\16p+2\end{bmatrix}$	$\begin{bmatrix}3p+2\\32p+26\end{bmatrix}$	$\begin{bmatrix}3p\\64p+10\end{bmatrix}$	$\begin{bmatrix}3p+2\\128p+106\end{bmatrix}$	$\begin{bmatrix}3p\\256p+42\end{bmatrix}$	$\begin{bmatrix}3p+2\\512p+426\end{bmatrix}$	$\begin{bmatrix}3p\\1024p+170\end{bmatrix}$
$\begin{bmatrix}3p+2\\2p+1\end{bmatrix}$	$(2,1)$	$(1,1)$	$(2,1)$	$(1,1)$	$(2,1)$	$(1,1)$	$(2,1)$	$(1,1)$	$(2,1)$	$(1,1)$
$\begin{bmatrix}3p\\4p\end{bmatrix}$	$(2,2)$	$(0,0)$	$(2,2)$	$(0,0)$	$(2,2)$	$(0,0)$	$(2,2)$	$(0,0)$	$(2,2)$	$(0,0)$
$\begin{bmatrix}3p+2\\8p+6\end{bmatrix}$	$(1,4)$	$(0,2)$	$(1,4)$	$(0,2)$	$(1,4)$	$(0,2)$	$(1,4)$	$(0,2)$	$(1,4)$	$(0,2)$
$\begin{bmatrix}3p\\16p+2\end{bmatrix}$	$(0,0)$	$(1,6)$	$(0,0)$	$(1,6)$	$(0,0)$	$(1,6)$	$(0,0)$	$(1,6)$	$(0,0)$	$(1,6)$
$\begin{bmatrix}3p+2\\32p+26\end{bmatrix}$	$(0,8)$	$(2,30)$	$(0,8)$	$(2,30)$	$(0,8)$	$(2,30)$	$(0,8)$	$(2,30)$	$(0,8)$	$(2,30)$
$\begin{bmatrix}3p\\64p+10\end{bmatrix}$	$(1,24)$	$(2,46)$	$(1,24)$	$(2,46)$	$(1,24)$	$(2,46)$	$(1,24)$	$(2,46)$	$(1,24)$	$(2,46)$
$\begin{bmatrix}3p+2\\128p+106\end{bmatrix}$	$(2,120)$	$(1,78)$	$(2,120)$	$(1,78)$	$(2,120)$	$(1,78)$	$(2,120)$	$(1,78)$	$(2,120)$	$(1,78)$
$\begin{bmatrix}3p\\256p+42\end{bmatrix}$	$(2,184)$	$(0,14)$	$(2,184)$	$(0,14)$	$(2,184)$	$(0,14)$	$(2,184)$	$(0,14)$	$(2,184)$	$(0,14)$
$\begin{bmatrix}3p+2\\512p+426\end{bmatrix}$	$(1,312)$	$(0,142)$	$(1,312)$	$(0,142)$	$(1,312)$	$(0,142)$	$(1,312)$	$(0,142)$	$(1,312)$	$(0,142)$
$\begin{bmatrix}3p\\1024p+170\end{bmatrix}$	$(0,56)$	$(1,398)$	$(0,56)$	$(1,398)$	$(0,56)$	$(1,398)$	$(0,56)$	$(1,398)$	$(0,56)$	$(1,398)$

References

Wikipedia contributors. Collatz conjecture, 2023. URL `en.wikipedia.org/wiki/Collatz_conjecture`.