

# Digital Signal Processing

AI21BTECH11016

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## 1 SOFTWARE INSTALLATION

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
-sciipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

## 2 DIGITAL FILTER

2.1 Download the sound file from

```
https://github.com/JBA-12/EE3900/blob/main
/A1/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at <https://academo.org/demos/spectrum-analyzer>. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

**Solution:** There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.

**Solution:** The following code removes the out of band noise

```
https://github.com/JBA-12/EE3900/blob/main
/A1/codes/2.3.py
```

and execute the code using the following command

```
python3 2.3.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound\_With\_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

**Solution:** The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

## 3 DIFFERENCE EQUATION

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (3.1)$$

Sketch  $x(n)$ .

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$

$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch  $y(n)$ .

**Solution:** The following code yields Fig. 3.2.

```
https://github.com/JBA-12/EE3900/blob/main
/A1/codes/3.2.py
```

and run the code using the following command

```
python3 3.2.py
```

3.3 Repeat the above exercise using a C code.  
**Solution:** The following C code yields y(n).dat file

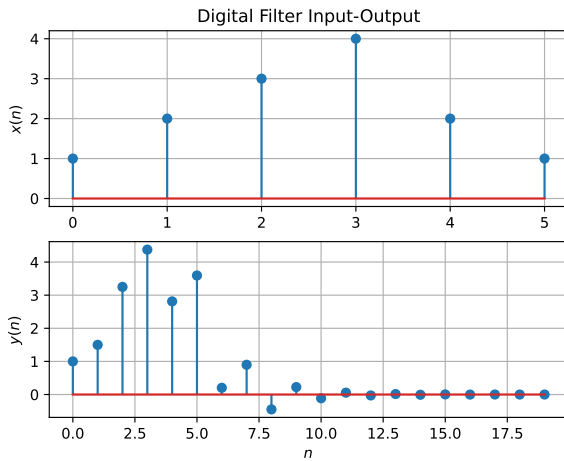


Fig. 3.2

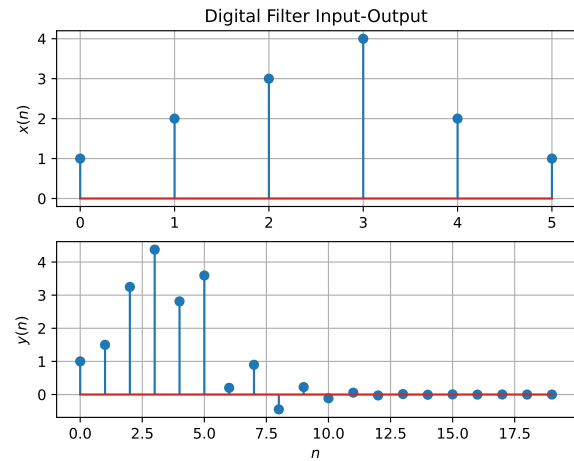


Fig. 3.3

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/3.3.c>

and run the code using the following commands

```
gcc 3.3.c
./a.out
```

The following python code inputs  $y(n)$  produced using C code and yields Fig. 3.3.

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/3.3.py>

and run the code using the following command

```
python3 3.3.py
```

#### 4 Z-TRANSFORM

4.1 The Z-transform of  $x(n)$  is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

**Solution:** From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \quad (4.4)$$

substitute  $n - k = t$

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n-k} \quad (4.5)$$

$$= z^{-k} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.6)$$

From (4.2), we get

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (4.7)$$

Substitute  $n = 1$ , we get

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.8)$$

4.2 Obtain  $X(z)$  for  $x(n)$  defined in problem 3.1.

**Solution:**

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.9)$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \quad (4.10)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.11)$$

from (3.2) assuming that the Z-transform is a linear operation.

**Solution:** Applying (4.8) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.12)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.13)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.14)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.15)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.16)$$

**Solution:** Consider the Z-transform of  $\delta$

$$\mathcal{Z}\{\delta(n)\} = \delta(0) + 0 = 1 \quad (4.17)$$

and from (4.15),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.18)$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.19)$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.20)$$

**Solution:**

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \quad (4.21)$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n \quad (4.22)$$

$$= \frac{1}{1 - az^{-1}} \quad |az^{-1}| < 1 \quad (4.23)$$

$$= \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.24)$$

$$\therefore a^n u(n) \stackrel{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.25)$$

4.6 Let

$$H(e^{j\omega}) = H(z = e^{j\omega}). \quad (4.26)$$

Plot  $|H(e^{j\omega})|$ . Comment.  $H(e^{j\omega})$  is known as the *Discret Time Fourier Transform* (DTFT) of  $x(n)$ .

**Solution:** The following code plots Fig. 4.6.

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/4.6.py>

and run the code using the following command

python3 4.6.py

Using (4.13), we observe that  $|H(e^{j\omega})|$  is given by

$$|H(e^{j\omega})| = \left| \frac{1 + e^{-2j\omega}}{1 + \frac{1}{2}e^{-j\omega}} \right| \quad (4.27)$$

$$= \left| \frac{1 + \cos 2\omega - j \sin 2\omega}{1 + \frac{1}{2} \cos \omega - \frac{1}{2}j \sin \omega} \right| \quad (4.28)$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2} \cos \omega\right)^2 + \left(\frac{1}{2} \sin \omega\right)^2}} \quad (4.29)$$

$$= \sqrt{\frac{2 + 2 \cos 2\omega}{\frac{5}{4} + \cos \omega}} \quad (4.30)$$

$$= \sqrt{\frac{8(2 \cos \omega)^2}{5 + 4 \cos \omega}} \quad (4.31)$$

$$|H(e^{j\omega})| = \frac{4 |\cos \omega|}{\sqrt{5 + 4 \cos \omega}} \quad (4.32)$$

Using (4.32) and the plot we can conclude that,

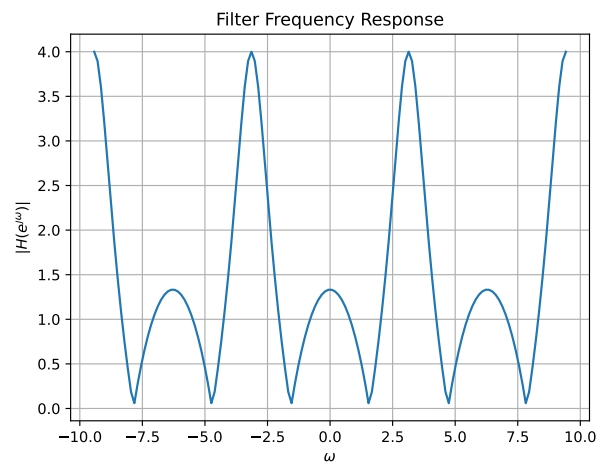


Fig. 4.6:  $|H(e^{j\omega})|$

- The Plot  $|H(e^{j\omega})|$  is Symmetric about  $\omega = 0$
- The maximum and minimum values of the plot are 4 and 0 respectively
- Consider (4.32),  
The period of numerator is  $\pi$  and the period of denominator is  $2\pi$

$\therefore$  The period of  $|H(e^{j\omega})|$  is  $LCM(\pi, 2\pi) = 2\pi$  i.e.,

$$\left| H(e^{j(\omega+2\pi)}) \right| = \frac{4|\cos(\omega+2\pi)|}{\sqrt{5+4\cos(\omega+2\pi)}} \quad (4.33)$$

$$= \frac{4|\cos \omega|}{\sqrt{5+4\cos \omega}} \quad (4.34)$$

$$\left| H(e^{j(\omega+2\pi)}) \right| = |H(e^{j\omega})| \quad (4.35)$$

$\Rightarrow$  it is Periodic with a period of  $2\pi$

4.7 Express  $h(n)$  in terms of  $H(e^{j\omega})$ .

**Solution:**

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \quad (4.36)$$

Multiply both sides with  $e^{j\omega k}$  and integrate from  $-\pi$  to  $\pi$

$$\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-j\omega n} e^{j\omega k} d\omega \quad (4.37)$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-j\omega(n-k)} d\omega \quad (4.38)$$

$$= h(k)2\pi \quad (4.39)$$

Since,

$$\int_{-\pi}^{\pi} e^{-j\omega(n-k)} d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \quad (4.40)$$

$$\therefore h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad (4.41)$$

## 5 IMPULSE RESPONSE

5.1 Using long division, find

$$h(n), \quad n < 5 \quad (5.1)$$

for  $H(z)$  in (4.13).

**Solution:** From (4.13), we have

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \quad (5.2)$$

Substitute  $z^{-1} = x$  to perform long division

$$\begin{array}{r} 2x-4 \\ \frac{1}{2}x+1 \overline{) x^2 + 1} \\ \underline{-x^2 - 2x} \phantom{+ 1} \\ -2x+1 \\ \underline{2x+4} \\ 5 \end{array}$$

From above division we can write,

$$1+z^{-2} = (1+\frac{1}{2}z^{-1})(2z^{-1}-4) + 5 \quad (5.3)$$

$$\frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} = 2z^{-1}-4 + \frac{5}{1+\frac{1}{2}z^{-1}} \quad (5.4)$$

From (4.13), we can write

$$H(z) = -4 + 2z^{-1} + \frac{5}{1+\frac{1}{2}z^{-1}} \quad (5.5)$$

$$= -4 + 2z^{-1} + 5 \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} \quad (5.6)$$

$$= 1 - \frac{1}{2}z^{-1} + 5 \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} \quad (5.7)$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} + 4 \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} \quad (5.8)$$

$$= \sum_{n=-\infty}^{\infty} u(n) \left(-\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{\infty} u(n-2) \left(-\frac{1}{2}\right)^{n-2} z^{-n} \quad (5.9)$$

Therefore, from (4.1),

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.10)$$

5.2 Find an expression for  $h(n)$  using  $H(z)$ , given that

$$h(n) \stackrel{Z}{\Leftrightarrow} H(z) \quad (5.11)$$

and there is a one to one relationship between  $h(n)$  and  $H(z)$ .

$h(n)$  is known as the *impulse response* of the system defined by (3.2).

**Solution:** From (4.13),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \quad (5.12)$$

From (4.20),

$$\frac{1}{1 + \frac{1}{2}z^{-1}} \stackrel{Z}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) \quad |z| > \frac{1}{2} \quad (5.13)$$

$$\frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \stackrel{Z}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \quad (5.14)$$

$$\Rightarrow H(z) \stackrel{Z}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \quad (5.15)$$

$$(5.16)$$

(Since Z-transform is a linear operator)

$$\therefore h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.17)$$

From (5.12), Consider the first part:

$$\frac{1}{1 + \frac{1}{2}z^{-1}} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n z^n \quad (5.18)$$

This sum converges when  $|z| > \frac{1}{2}$   
 $\Rightarrow$  ROC is  $|z| > \frac{1}{2}$

$$\frac{1}{1 + \frac{1}{2}z^{-1}} = \frac{2z}{1 + 2z} \quad (5.19)$$

$$= \sum_{n=-\infty}^{-1} (2z)^{-n} \quad (5.20)$$

This sum converges when  $|z| < \frac{1}{2}$   
 $\Rightarrow$  ROC is  $|z| < \frac{1}{2}$

Therefore, ROC of  $H(z)$  will be

$$|z| \neq \frac{1}{2} \quad (5.21)$$

5.3 Sketch  $h(n)$ . Is it bounded? Convergent? **Solution:** The following code plots Fig. 5.3.

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/5.3.py>

and run the code using the following command

python3 5.3.py

From the plot, we can conclude that it is convergent to 0

$\Rightarrow$  It is bounded as well.

5.4 Convergent? Justify using the ratio test.

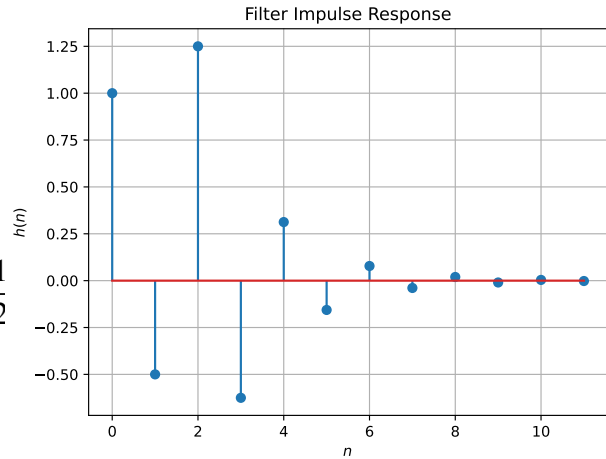


Fig. 5.3:  $h(n)$  as the inverse of  $H(z)$

**Solution:** Using the ratio test for convergence

$$\lim_{n \rightarrow \infty} \left| \frac{h(n+1)}{h(n)} \right| = \lim_{n \rightarrow \infty} \left| \frac{\left(-\frac{1}{2}\right)^{n+1} \left(1 + \frac{1}{4}\right)}{\left(-\frac{1}{2}\right)^{n-2} \left(1 + \frac{1}{4}\right)} \right| \quad (5.22)$$

$$= \lim_{n \rightarrow \infty} \left| -\frac{1}{2} \right| \quad (5.23)$$

$$= \frac{1}{2} < 1 \quad (5.24)$$

$\therefore h(n)$  is Convergent.

5.5 The system with  $h(n)$  is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.25)$$

Is the system defined by (3.2) stable for the impulse response in (5.11)?

**Solution:** From 5.2,

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.26)$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.27)$$

$$= \frac{1}{1 - \left(-\frac{1}{2}\right)} + \frac{1}{1 - \left(-\frac{1}{2}\right)} \quad (5.28)$$

$$= \frac{4}{3} < \infty \quad (5.29)$$

using the formula for the sum of an infinite geometric progression

∴ The system is stable.

5.6 Verify the above result using a python code.

**Solution:** The following code verifies whether the given system is stable or not

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/5.6.py>

run the code using the following command

python3 5.6.py

5.7 Compute and sketch  $h(n)$  using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.30)$$

This is the definition of  $h(n)$ .

**Solution:** The following code plots Fig. 5.7.

Note that this is the same as Fig. 5.3.

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/5.7.py>

run the code using the following command

python3 5.7.py

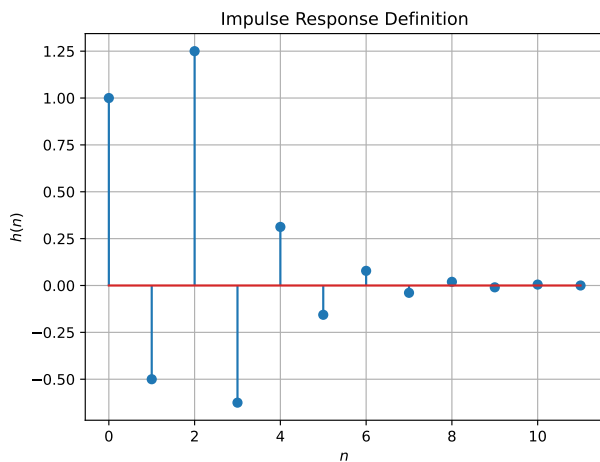


Fig. 5.7:  $h(n)$  from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.31)$$

Comment. The operation in (5.31) is known as *convolution*.

**Solution:**

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.32)$$

$$= \sum_{k=0}^5 x(k)h(n-k) \quad (5.33)$$

The following code plots Fig. 5.9.

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/5.8.py>

run the code using the following command

python3 5.8.py

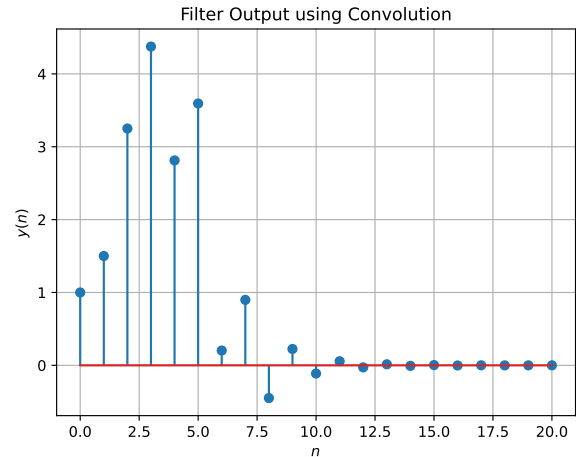


Fig. 5.8:  $y(n)$  from the definition of convolution

This plot is same as  $y(n)$  in Fig. 3.2

Therefore,

$$y(n) = x(n) * h(n) \quad (5.34)$$

5.9 Express the above convolution using a Teoplitz matrix.

**Solution:** From (3.1), we can write

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \quad (5.35)$$

Their convolution is given by the product of

the following Toeplitz matrix  $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix} \quad (5.36)$$

$$\mathbf{y} = \mathbf{x} \otimes \mathbf{h} = \mathbf{T}\mathbf{x} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \quad (5.37)$$

The following python code computes the convolution using Toeplitz matrix.

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/5.9.py>

run the code using the following command

python3 5.9.py

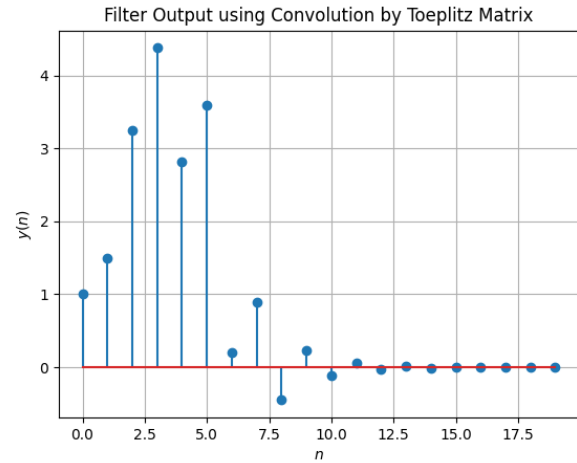


Fig. 5.9:  $y(n)$  from the definition of convolution using Toeplitz matrix

Substitute  $k = n-i$

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \quad (5.40)$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \quad (5.41)$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \quad (5.42)$$

Since, the order of limit doesn't matter in case of summation. Therefore, now we have

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.43)$$

from (5.31)

$$\therefore y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.44)$$

## 6 DFT

### 6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and  $H(k)$  using  $h(n)$ .

**Solution:** The following code plots Fig. 6.1.

<https://github.com/JBA-12/EE3900/blob/main/A1/codes/6.1.py>

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.38)$$

**Solution:** From (5.31) we know that,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.39)$$

and run the python code using the following command

```
python3 6.1.py
```

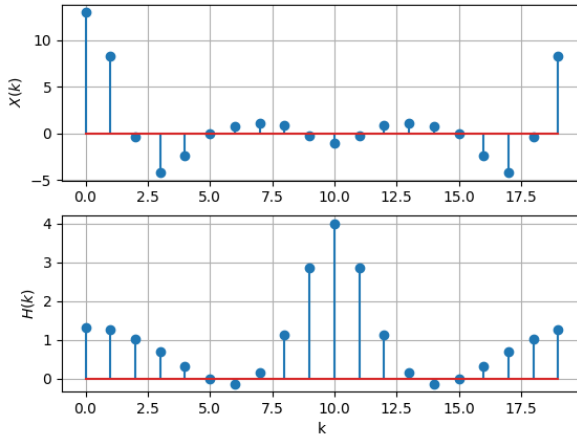


Fig. 6.1: Plot of real part of Discrete Fourier Transforms of  $x(n)$  and  $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

**Solution:** The following code plots Fig. 6.2.

```
https://github.com/JBA-12/EE3900/blob/main/A1/codes6.2.py
```

and run the code using the following command

```
python3 6.2.py
```

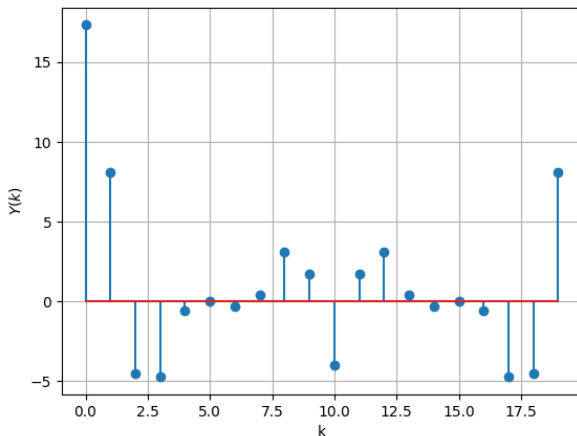


Fig. 6.2:  $Y(k)$  as product of  $X(k)$  and  $H(k)$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (6.3)$$

**Solution:** The following code plots Fig. 6.3.

```
https://github.com/JBA-12/EE3900/blob/main/A1/codes/6.3.py
```

and run the code using the following command

```
python3 6.3.py
```

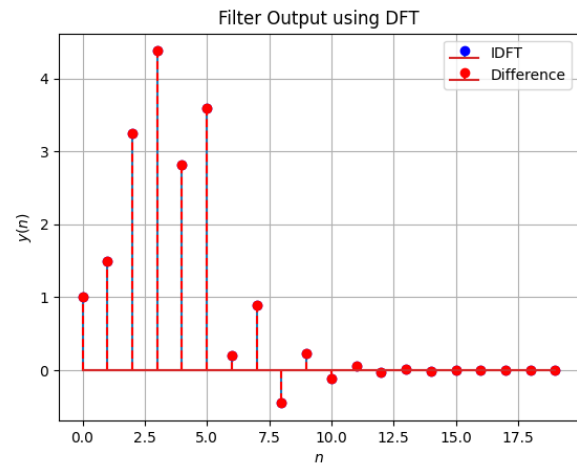


Fig. 6.3:  $y(n)$  using IDFT and Difference Equation

6.4 Repeat the previous exercise by computing  $X(k)$ ,  $H(k)$  and  $y(n)$  through FFT and IFFT.

**Solution:** The following code plots Fig. 6.4.

```
https://github.com/JBA-12/EE3900/blob/main/A1/codes/6.4.py
```

and run the code using the following command

```
python3 6.4.py
```

6.5 Wherever possible, express all the above equa-



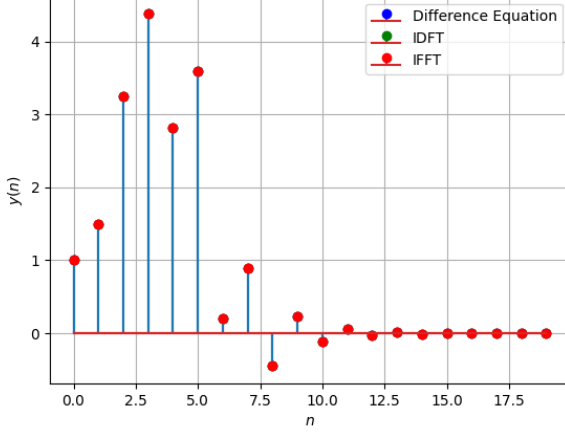


Fig. 6.4: The plot of  $y(n)$  using IFFT

tions as matrix equations. **Solution:**

$$\mathbf{x} = (x_0 \ x_1 \ \cdots \ x_{N-1})^T \quad (6.4)$$

$$\mathbf{h} = (h_0 \ h_1 \ \cdots \ h_{N-1})^T \quad (6.5)$$

$$\mathbf{y} = \mathbf{x} \otimes \mathbf{h} \quad (6.6)$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2N-1} \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 \\ 0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ 0 & 0 & h_{N-1} & \cdots & h_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{N-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \quad (6.7)$$

The convolution can be written using a Toeplitz matrix. Consider the DFT matrix

$$\mathbf{W} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \quad (6.8)$$

where  $\omega = e^{-j2\pi/N}$  is the  $N^{\text{th}}$  root of unity. Then the discrete Fourier transforms of  $\mathbf{x}$  and  $\mathbf{h}$  are

given by

$$\mathbf{X} = \mathbf{W}\mathbf{x} \quad (6.9)$$

$$\mathbf{H} = \mathbf{W}\mathbf{h} \quad (6.10)$$

$\mathbf{Y}$  is then given by

$$\mathbf{Y} = \mathbf{X} \circ \mathbf{H} \quad (6.11)$$

where  $\circ$  denotes the Hadamard product (element-wise multiplication)

But  $\mathbf{Y}$  is the discrete Fourier transform of the filter output  $\mathbf{y}$

$$\mathbf{Y} = \mathbf{W}\mathbf{y} \quad (6.12)$$

Thus,

$$\mathbf{W}\mathbf{y} = \mathbf{X} \circ \mathbf{H} \quad (6.13)$$

$$\Rightarrow \mathbf{y} = \mathbf{W}^{-1}(\mathbf{X} \circ \mathbf{H}) \quad (6.14)$$

$$= \mathbf{W}^{-1}(\mathbf{W}\mathbf{x} \circ \mathbf{W}\mathbf{h}) \quad (6.15)$$

This is the inverse discrete Fourier transform of  $\mathbf{Y}$

6.6 Verify the above equations by generating the DFT matrix in python.

## 7 FFT

1. The DFT of  $x(n)$  is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (7.1)$$

2. Let

$$\mathbf{W}_N = e^{-j2\pi/N} \quad (7.2)$$

Then the  $N$ -point DFT matrix is defined as

$$\mathbf{F}_N = [\mathbf{W}_N^{mn}], \quad 0 \leq m, n \leq N-1 \quad (7.3)$$

where  $\mathbf{W}_N^{mn}$  are the elements of  $\mathbf{F}_N$ .

3. Let

$$\mathbf{I}_4 = (\mathbf{e}_4^1 \ \mathbf{e}_4^2 \ \mathbf{e}_4^3 \ \mathbf{e}_4^4) \quad (7.4)$$

be the  $4 \times 4$  identity matrix. Then the 4 point DFT permutation matrix is defined as

$$\mathbf{P}_4 = (\mathbf{e}_4^1 \ \mathbf{e}_4^3 \ \mathbf{e}_4^2 \ \mathbf{e}_4^4) \quad (7.5)$$

4. The 4 point DFT diagonal matrix is defined as

$$\mathbf{D}_4 = \text{diag}(\mathbf{W}_8^0 \ \mathbf{W}_8^1 \ \mathbf{W}_8^2 \ \mathbf{W}_8^3) \quad (7.6)$$

5. Show that

$$W_N^2 = W_{N/2} \quad (7.7)$$

**Solution:**

$$W_N^2 = e^{-j2\pi/N} e^{-j2\pi/N} \quad (7.8)$$

$$= e^{2(-j2\pi/N)} \quad (7.9)$$

$$= e^{-j2\pi/\frac{N}{2}} \quad (7.10)$$

$$= W_{N/2} \quad (7.11)$$

6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \quad (7.12)$$

**Solution:**

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \quad (7.13)$$

$$= \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2 \mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2 \mathbf{F}_2 \end{bmatrix} \quad (7.14)$$

$$= \begin{bmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -j \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -\begin{pmatrix} 1 & 0 \\ 0 & -j \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{bmatrix} \quad (7.15)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \quad (7.16)$$

Now

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \quad (7.17)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.18)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad (7.19)$$

$$= \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \quad (7.20)$$

$$= \mathbf{F}_4 \quad (7.21)$$

since,

$$W_4^0 = 1 \quad (7.22)$$

$$W_4^1 = e^{-j\frac{\pi}{2}} = -j \quad (7.23)$$

$$W_4^2 = e^{-j\pi} = -1 \quad (7.24)$$

$$W_4^3 = e^{-j\frac{3\pi}{2}} = j \quad (7.25)$$

$$W_4^n = W_4^{n-4} \quad \forall n \geq 4 \quad (7.26)$$

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \quad (7.27)$$

**Solution:**

$$\begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \quad (7.28)$$

$$= \begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2} \mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2} \mathbf{F}_{N/2} \end{bmatrix} \quad (7.29)$$

Now

$$\mathbf{D}_{N/2} \mathbf{F}_{N/2} \quad (7.30)$$

$$= \begin{bmatrix} W_N^0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_N^{N/2-1} \end{bmatrix} \begin{bmatrix} W_{N/2}^0 & \cdots & W_{N/2}^0 \\ \vdots & \ddots & \vdots \\ W_{N/2}^0 & \cdots & W_{N/2}^{(N/2-1)^2} \end{bmatrix} \quad (7.31)$$

$$= \begin{bmatrix} W_N^0 W_{N/2}^0 & \cdots & W_N^0 W_{N/2}^0 \\ \vdots & \ddots & \vdots \\ W_N^{N/2-1} W_{N/2}^0 & \cdots & W_N^{N/2-1} W_{N/2}^{(N/2-1)^2} \end{bmatrix} \quad (7.32)$$

Thus

$$(\mathbf{D}_{N/2} \mathbf{F}_{N/2})_{ij} = W_N^i W_{N/2}^{ij} \quad (7.33)$$

$$= W_N^i W_N^{2ij} \quad (7.34)$$

$$= W_N^{i(2j+1)} \quad (7.35)$$

where  $i, j = 0, \dots, N/2-1$  Therefore,  $\mathbf{D}_{N/2} \mathbf{F}_{N/2}$  forms the first  $N/2$  rows of the odd-indexed

columns of  $\mathbf{F}_N$

$$W_N^{(i+N/2)(2j+1)} = \exp\left(-j\frac{2\pi}{N}(2j+1)\left(i + \frac{N}{2}\right)\right) \quad (7.36)$$

$$= \exp\left(-j\left(\frac{2\pi}{N}(2j+1)i + (2j+1)\pi\right)\right) \quad (7.37)$$

$$= -\exp\left(-j\frac{2\pi}{N}(2j+1)i\right) \quad (7.38)$$

$$= -W_N^{i(2j+1)} \quad (7.39)$$

Thus, the remaining  $N/2$  rows will be the negatives of the first  $N/2$  rows

$$(\mathbf{F}_{N/2})_{ij} = W_{N/2}^{ij} \quad (7.40)$$

$$= W_N^{i(2j)} \quad (7.41)$$

where  $i, j = 0, \dots, N/2 - 1$  Therefore,  $\mathbf{F}_{N/2}$  forms the first  $N/2$  rows of the even-indexed columns of  $\mathbf{F}_N$

$$W_N^{(i+N/2)(2j)} = \exp\left(-j\frac{2\pi}{N}(2j)\left(i + \frac{N}{2}\right)\right) \quad (7.42)$$

$$= \exp\left(-j\left(\frac{2\pi}{N}(2j)i + (2j)\pi\right)\right) \quad (7.43)$$

$$= \exp\left(-j\frac{2\pi}{N}(2j)i\right) \quad (7.44)$$

$$= W_N^{i(2j)} \quad (7.45)$$

Thus, the remaining  $N/2$  rows will be the same as the first  $N/2$  rows Therefore

$$\begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} = \mathbf{F}_N \mathbf{P}_N \quad (7.46)$$

where

$$\mathbf{P}_N = \begin{pmatrix} \mathbf{e}_N^1 & \mathbf{e}_N^3 & \dots & \mathbf{e}_N^{N-1} & \mathbf{e}_N^2 & \mathbf{e}_N^4 & \dots & \mathbf{e}_N^N \end{pmatrix} \quad (7.47)$$

Hence

$$\begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N = \mathbf{F}_N \mathbf{P}_N^2 = \mathbf{F}_N \quad (7.48)$$

$$\therefore \mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \quad (7.49)$$

for even  $N$

8. Find

$$\mathbf{P}_4 \mathbf{x} \quad (7.50)$$

**Solution:** Let  $\mathbf{x} = (x(0) \ x(1) \ x(2) \ x(3))^T$

$$\mathbf{P}_4 \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \quad (7.51)$$

$$= \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix} \quad (7.52)$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N \mathbf{x} \quad (7.53)$$

where  $\mathbf{x}, \mathbf{X}$  are the vector representations of  $x(n), X(k)$  respectively. **Solution:**

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (7.54)$$

$$\Rightarrow \mathbf{X} = \begin{bmatrix} \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(0)/N} \\ \vdots \\ \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-1)/N} \end{bmatrix} \quad (7.55)$$

$$= \begin{bmatrix} x(0) + \dots + x(N-1) \\ \vdots \\ x(0) + \dots + x(N-1) e^{-j2\pi(N-1)^2/N} \end{bmatrix} \quad (7.56)$$

$$\mathbf{X} = x(0) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \dots + x(N-1) \begin{bmatrix} 1 \\ \vdots \\ e^{-j2\pi(N-1)^2/N} \end{bmatrix} \quad (7.57)$$

$$= \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & e^{-j2\pi(N-1)^2/N} \end{bmatrix} \begin{bmatrix} x(0) \\ \vdots \\ x(N-1) \end{bmatrix} \quad (7.58)$$

$$= \mathbf{F}_N \mathbf{x} \quad (7.59)$$

10. Derive the following Step-by-step visualisation

of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.60)$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.61)$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (7.62)$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (7.63)$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.64)$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.65)$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \quad (7.66)$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \quad (7.67)$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \quad (7.68)$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \quad (7.69)$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \quad (7.70)$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \quad (7.71)$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \quad (7.72)$$

**Solution:**

$$X(k) = \sum_{n=0}^7 x(n) e^{-j2\pi kn/8}, \quad k = 0, \dots, 7 \quad (7.73)$$

$$= \sum_{n=0}^7 x(n) W_8^{kn} \quad (7.74)$$

$$= \sum_{n \text{ is even}} x(n) W_8^{kn} + \sum_{n \text{ is odd}} x(n) W_8^{kn} \quad (7.75)$$

$$= \sum_{m=0}^3 x(2m) W_8^{2km} + \sum_{m=0}^3 x(2m+1) W_8^{2km+k} \quad (7.76)$$

Now substitute  $W_8^2 = W_4$

$$X(k) = \sum_{m=0}^3 x(2m) W_4^{km} + W_8^k \sum_{m=0}^3 x(2m+1) W_4^{km} \quad (7.77)$$

Consider

$$x_1(n) = \{x(0), x(2), x(4), x(6)\} \quad (7.78)$$

$$x_2(n) = \{x(1), x(3), x(5), x(7)\} \quad (7.79)$$

Thus

$$X(k) = X_1(k) + W_8^k X_2(k) \quad k = 0, \dots, 7 \quad (7.80)$$

Now,  $X_1(k)$  and  $X_2(k)$  are 4-point DFTs which means they are periodic with period 4

$$X(k+4) = X_1(k+4) + W_8^{k+4} X_2(k+4) \quad (7.81)$$

$$= X_1(k) + e^{-j2\pi(k+4)/8} X_2(k) \quad (7.82)$$

$$= X_1(k) + e^{-j(2\pi k/8 + \pi)} X_2(k) \quad (7.83)$$

$$= X_1(k) - e^{-j2\pi k/8} X_2(k) \quad (7.84)$$

$$= X_1(k) - W_8^k X_2(k) \quad (7.85)$$

Therefore, for  $k = 0, 1, 2, 3$

$$X(k) = X_1(k) + W_8^k X_2(k) \quad (7.86)$$

$$X(k+4) = X_1(k) - W_8^k X_2(k) \quad (7.87)$$

which is the same as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.88)$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.89)$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (7.90)$$

compute the DFT using (7.53)

12. Repeat the above exercise using the FFT after zero padding  $\mathbf{x}$ .
13. Write a C program to compute the 8-point FFT.

## 8 EXERCISES

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
                                input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^M a(m) y(n-m) = \sum_{k=0}^N b(k) x(n-k) \quad (8.1)$$

where the input signal is  $x(n)$  and the output signal is  $y(n)$  with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

- 8.2 Repeat all the exercises in the previous sections for the above  $a$  and  $b$ .
- 8.3 What is the sampling frequency of the input signal?

**Solution:** Sampling frequency(fs)=44.1kHz.

- 8.4 What is type, order and cutoff-frequency of the above butterworth filter

**Solution:** The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

- 8.5 Modifying the code with different input parameters and to get the best possible output.