Digital Signal Processing
EE3900

Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:

https://github.com/JBA−12/EE3900/blob/main/fourier/codes/1.1.py

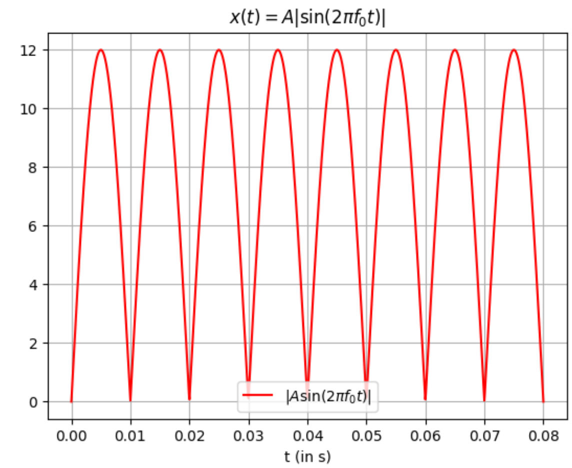

Fig. 0

1.2 Show that $x(t)$ is periodic and find its period.
Solution: A signal $x(t)$ is said to be periodic with fundamental period T if

$$x(t + nT) = x(t) \forall n \in \mathbb{Z} \tag{1.2}$$

Let T be fundamental period of $x(t)$. Comparing (1.2) and (1.1), we get

$$A_0 \left|\sin\left(2\pi f_0 t\right)\right| = A_0 \left|\sin\left(2\pi f_0 (t + T)\right)\right| \tag{1.3}$$

$$\left|\sin\left(2\pi f_0 t\right)\right| = \left|\sin\left(2\pi f_0 (t + T)\right)\right| \tag{1.4}$$

$$\left|\sin\left(2\pi f_0 t\right)\right| = \left|\sin\left(2\pi f_0 t + 2\pi f_0 T\right)\right| \tag{1.5}$$

As $|sin\theta|$ is periodic with fundamental period $F = \pi$, Hence,

$$\left|\sin\left(t\right)\right| = \left|\sin\left(t + F\right)\right| \tag{1.6}$$

Hence,$2\pi f_0 T = \pi$, therefore, fundamental period(T) is

$$T = \frac{\pi}{2\pi f_0} = \frac{1}{2 f_0} \tag{1.7}$$

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.3}$$

Mulitply $e^{-j2\pi l f_0 t}$ on both sides of (2.3), we get,

$$x(t) e^{-j2\pi l f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} e^{-j2\pi l f_0 t} \tag{2.4}$$

Integrating (2.4) w.r.t. t from $-T$ to T, and $T = \frac{1}{f_0}$, we get,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi (k-l) f_0 t}\, dt \tag{2.5}$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi (k-l) f_0 t}\, dt \tag{2.6}$$

Consider the following cases.
case-1:$k = l$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi (k-l) f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{0}\, dt \tag{2.7}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.8}$$

case-2: $k \neq l$
Let $n = f_0(k-l)$, here $n \in \mathbb{Z}$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi (k-l) f_0 t}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{2n\pi}\, dt \tag{2.9}$$

Here, $2n\pi T = 2f_0(k-l)T\pi$, and $2n\pi T = (k-l)\pi$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.10}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \cos(2n\pi) + j\sin(2n\pi)\, dt \tag{2.11}$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.12}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.13}$$

$$= -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.14}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.15}$$

$$= -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.16}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.17}$$

$$\tag{2.18}$$

$$= -\sin((k-l)\pi) + \sin(-(k-l)\pi) \tag{2.19}$$

$$+ j\cos((k-l)\pi) - j\cos(-(k-l)\pi) \tag{2.20}$$

$$\tag{2.21}$$

Since $k - l \in \mathbb{Z}$,$\sin((k-l)\pi) = 0$ and $\sin(-(k-l)\pi) = 0$, simillarly, as $\cos(\theta) = \cos(-\theta)$, we get $\cos((k-l)\pi) - \cos(-(k-l)\pi) = 0$
From (2.18),

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\, dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.22}$$

$$= 0 + j0 = 0 \tag{2.23}$$

Hence, we have,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi (k-l) f_0 t}\, dt = \begin{cases} 0 & k \neq l \\ \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt & k = l \end{cases} \tag{2.24}$$

From (2.5),

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi (k-l) f_0 t}\, dt \tag{2.25}$$

$$= c_k \times \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\, dt \tag{2.26}$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\, dt \tag{2.27}$$

$$\therefore c_k = \frac{2}{T} \int_{-\frac{1}{T}}^{\frac{1}{T}} x(t)e^{-j2\pi k f_0 t}\, dt \tag{2.28}$$

2.2 Find c_k for (1.1)
Solution: We know that,

$$c_k = 2f_0 \int_{0}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\, dt \tag{2.29}$$

when $t \in \left(0, \frac{1}{2f_0}\right)$, $x(t) = A_0 \sin(2\pi f_0 t)$

$$c_k = 2f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \left(\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\right) e^{-j2\pi k f_0 t}\, dt \tag{2.30}$$

$$= A_0 f_0 \int_{0}^{\frac{1}{2f_0}} \left(\frac{e^{j2\pi(1-k) f_0 t} - e^{j2\pi(-1-k) f_0 t}}{j}\right) dt \tag{2.31}$$

$$= A_0 f_0 \Bigg(\frac{e^{j2\pi(1-k) f_0 t}}{-2\pi(1-k) f_0} \Bigg|_{0}^{\frac{1}{2f_0}} \tag{2.32}$$

$$- \frac{e^{j2\pi(-1-k) f_0 t}}{-2\pi(-1-k) f_0} \Bigg|_{0}^{\frac{1}{2f_0}} \Bigg) \tag{2.33}$$

$$= A_0 \left[\frac{e^{j\pi(1-k)} - 1}{2\pi(k-1)} - \frac{e^{-j\pi(1+k)} - 1}{2\pi(k+1)}\right] \tag{2.34}$$

Hence,

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k = even \\ 0 & k = odd \end{cases} \tag{2.35}$$

2.3 Verify (1.1) using python.
Solution:

https://github.com/JBA−12/EE3900/blob/main /fourier/codes/2.3.py

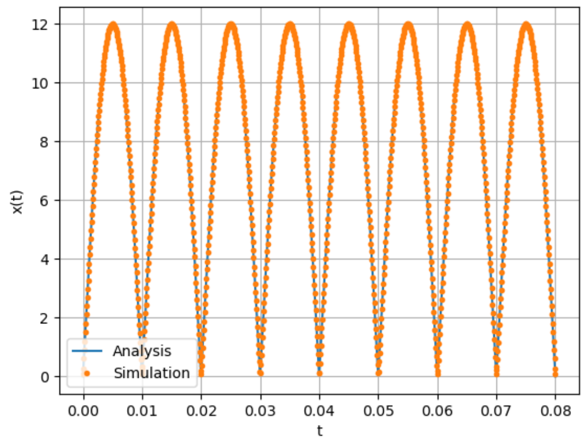

Fig. 0

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.36}$$

and obtain the formulae for a_k and b_k.
Solution: Using (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.37}$$

As,

$$e^{j2\pi k f_0 t} = \cos(2\pi k f_0 t) + j\sin(2\pi k f_0 t) \tag{2.38}$$

From (2.1), we have,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \left[\cos(2\pi k f_0 t) + j\sin(2\pi k f_0 t)\right] \tag{2.39}$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t) \tag{2.40}$$

$$= \sum_{k=-\infty}^{-1} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] \tag{2.41}$$

$$+ c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] \tag{2.42}$$

$$= \sum_{k=1}^{\infty} \left[c_{-k} \cos(2\pi k f_0 t) - jc_{-k} \sin(2\pi k f_0 t)\right] \tag{2.43}$$

$$+ c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] \tag{2.44}$$

$$= c_0 + \sum_{k=1}^{\infty} \Big((c_k + c_{-k}) \cos(2\pi k f_0 t) \tag{2.45}$$

$$+ j(c_k - c_{-k}) \sin(2\pi k f_0 t) \Big) \tag{2.46}$$

Substituting $a_k = c_k + c_{-k}$ and $b_k = j(c_k - c_{-k})$, we get,

$$x(t) = c_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.47}$$

$$= \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.48}$$

$$\therefore a_k = \begin{cases} c_k + c_{-k} & k \neq 0 \\ c_0 & k = 0 \end{cases} \tag{2.49}$$

$$b_k = j(c_k - c_{-k}) \tag{2.50}$$

Using (2.2),

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.51}$$

$$c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi k f_0 t}\, dt \tag{2.52}$$

$$a_k = c_k + c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \left[e^{-j2\pi k f_0 t} + e^{j2\pi k f_0 t} \right] dt \tag{2.53}$$

$$= 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos(2\pi k f_0 t)\, dt \tag{2.54}$$

Similarly, for b_k, we get,

$$b_k = -j \left\{ 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin\{2\pi k f_0 t\}\, dt \right\} \tag{2.55}$$

2.5 Find a_k and b_k for (1.1)
Solution: Using (2.49) and (2.50) with (2.35),

$$a_k = c_k + c_{-k} = \begin{cases} \frac{4A_0}{\pi(1-k^2)} & k = even \\ \frac{2A_0}{\pi} & k = 0 \\ 0 & k = odd \end{cases} \tag{2.56}$$

$$b_k = j(c_k - c_{-k}) = 0 \tag{2.57}$$

2.6 Verify (2.36) using python.
Solution:

https://github.com/JBA−12/EE3900/blob/main
/fourier/codes/2.6.py

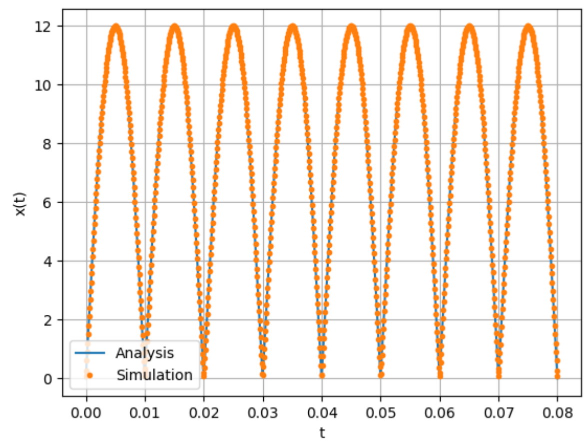

Fig. 0

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

Solution: Let us consider $x = t - t_0$. Fourier transform of $g(t - t_0)$ is given as

$$g(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f t}\, dt \tag{3.5}$$

$$= \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f((t-t_0)+t_0)}\, du \tag{3.6}$$

$$= \int_{-\infty}^{\infty} g(x) e^{-j2\pi f(x+t_0)}\, dt \tag{3.7}$$

$$= \int_{-\infty}^{\infty} g(x) e^{-j2\pi f(x+t_0)}\, d(x - t_0) \tag{3.8}$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi f t_0} g(x) e^{-j2\pi f x}\, d(x) \tag{3.9}$$

$$= e^{-j2\pi f t_0} \left\{ \int_{-\infty}^{\infty} g(x) e^{-j2\pi f x}\, d(x) \right\} \tag{3.10}$$

Using (3.3) in equation (3.10), we get,

$$g(t-t_0) \xleftrightarrow{\mathcal{F}} G(f)e^{-\jmath 2\pi f t_0} \tag{3.11}$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.12}$$

Solution: Let $g(t) \xleftrightarrow{\mathcal{F}} G(f)$, then

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{\jmath 2\pi f t}\, df \tag{3.13}$$

Consider $g(-k)$,

$$g(-k) = \int_{-\infty}^{\infty} G(f)e^{\jmath 2\pi f k}\, df \tag{3.14}$$

Let $f = t$,then,

$$g(-k) = \int_{-\infty}^{\infty} G(t)e^{\jmath 2\pi t k}\, dt \tag{3.15}$$

Substituting $k = f$ and in the (3.15), we get,

$$g(-f) = \int_{-\infty}^{\infty} G(t)e^{\jmath 2\pi f t}\, dt \tag{3.16}$$

Comparing (3.16) with (3.3), we can say that,

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.17}$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{F}} ?$

Solution: From (3.3), fourier transform of $\delta(t)$ is,

$$\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t)e^{-\jmath 2\pi f t}\, dt \tag{3.18}$$

$$= \int_{-\infty}^{\infty} \delta(0)e^{-\jmath 2\pi f 0}\, dt \tag{3.19}$$

$$= \int_{-\infty}^{\infty} \delta(0)\, dt \tag{3.20}$$

$$= 1 \tag{3.21}$$

Hence, $\delta(t) \xleftrightarrow{\mathcal{F}} 1$

3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$

Solution: Suppose $g(t) \xleftrightarrow{\mathcal{F}} G(f)$. Hence,

$$g(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t} \tag{3.22}$$

$$g(t)e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t}e^{-j2\pi f_0 t} \tag{3.23}$$

$$g(t)e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} G(f)e^{-j2\pi f_0 f} \tag{3.24}$$

$$\tag{3.25}$$

From (3.16),

$$g(t-f_0) \xleftrightarrow{\mathcal{F}} G(f)e^{-\jmath 2\pi f t f_0} \tag{3.26}$$

$$g(t)e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} G(f)e^{-j2\pi f_0 f} \tag{3.27}$$

$$\tag{3.28}$$

From (3.13),

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1 \tag{3.29}$$

$$1 \xleftrightarrow{\mathcal{F}} \delta(-f) = \delta(f) \tag{3.30}$$

Hence,

$$g(t-f_0) \xleftrightarrow{\mathcal{F}} \delta((f+f_0)) \tag{3.31}$$

$$g(t)e^{\jmath 2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi (f-f_0)t}\, dt \tag{3.32}$$

$$= G(f-f_0) \tag{3.33}$$

Hence,

$$e^{-\jmath 2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \delta(-(f+f_0)) = \delta(f+f_0) \tag{3.34}$$

3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$

Solution: We know that

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right) \tag{3.35}$$

$$\tag{3.36}$$

Hence,

$$\mathcal{F}(\cos(2\pi f_0 t)) = \mathcal{F}(\frac{1}{2}\left(e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}\right)) \tag{3.37}$$

$$\mathcal{F}(\cos(2\pi f_0 t)) = \frac{1}{2}\mathcal{F}(\left(e^{\jmath 2\pi f_0 t}\right)) + \frac{1}{2}\mathcal{F}(e^{-\jmath 2\pi f_0 t}) \tag{3.38}$$

$$= \frac{1}{2}\mathcal{F}(\left(e^{\jmath 2\pi f_0 t}\right)) + \frac{1}{2}\mathcal{F}(e^{-\jmath 2\pi f_0 t}) \tag{3.39}$$

$$= \frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right) \tag{3.40}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution: As obtained earlier, from equation

(2.35),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{3.41}$$

$$e^{j2\pi k f_0 t} \xleftrightarrow{\mathcal{F}} = \delta f - k f_0 \tag{3.42}$$

Hence, from the value of c_k,

$$x(t) \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \delta\left(f + k f_0\right) c_k \tag{3.43}$$

$$\implies x(t) \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \frac{2A_0}{\pi} \frac{\delta\left(f + 2k f_0\right)}{1 - 4k^2} \tag{3.44}$$

Fourier transform of $x(t)$ is verified in the

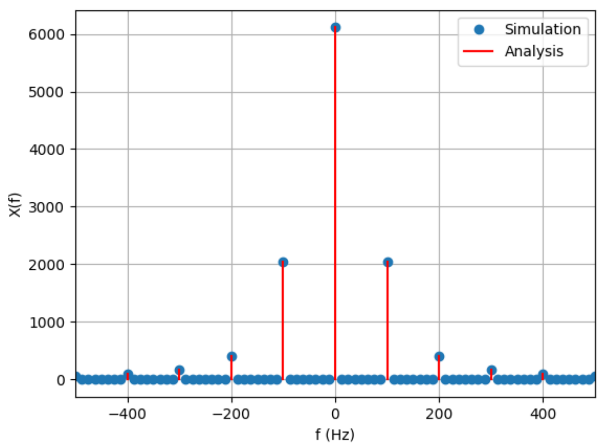

Fig. 0: Fourier Transform of $x(t)$.

following figure. The figure is plotted using the below python code.

```
https://github.com/JBA−12/EE3900/blob/main
/fourier/codes/3.8.py
```

3.9 Show that

$$\operatorname{rect} t \xleftrightarrow{\mathcal{F}} \operatorname{sinc} f \tag{3.45}$$

Verify using python.
Solution: We know that,

$$rect(t) = \begin{cases} 0 & t < \frac{-1}{2} \\ 1 & \frac{-1}{2} < t < \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases} \tag{3.46}$$

$$\operatorname{sinc}(f) = \frac{\sin \pi f}{\pi f} \tag{3.47}$$

Applying fourier transform we get,

$$\operatorname{rect} t \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \operatorname{rect} t e^{-\jmath 2\pi f t} \, dt \tag{3.48}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\jmath 2\pi f t} \, dt \tag{3.49}$$

$$= \frac{e^{\jmath \pi f} - e^{-\jmath \pi f}}{\jmath 2\pi f} = \frac{\sin \pi f}{\pi f} = \operatorname{sinc} f \tag{3.50}$$

The below python code plots the figure 0

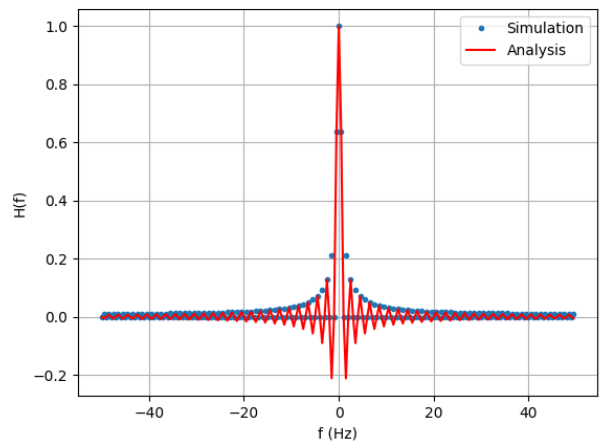

Fig. 0: Fourier Transform of rect t).

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https://github.com/JBA−12/EE3900/blob/main
/fourier/codes/3.9.py
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3.10 $\operatorname{sinc} t \xleftrightarrow{\mathcal{F}}$? Verify using python.
Solution: From (3.10), we have

$$\operatorname{sinc} t \xleftrightarrow{\mathcal{F}} \operatorname{rect}(-f) = \operatorname{rect} f \tag{3.51}$$

Since rect f is an even function. The below python code plots the figure 0

```
https://github.com/JBA−12/EE3900/blob/main
/fourier/codes/3.10.py
```

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: The function $H(f)$ is a low pass filter which filters out even harmonics and leaves the zero frequency component behind. The rectangular function represents an ideal

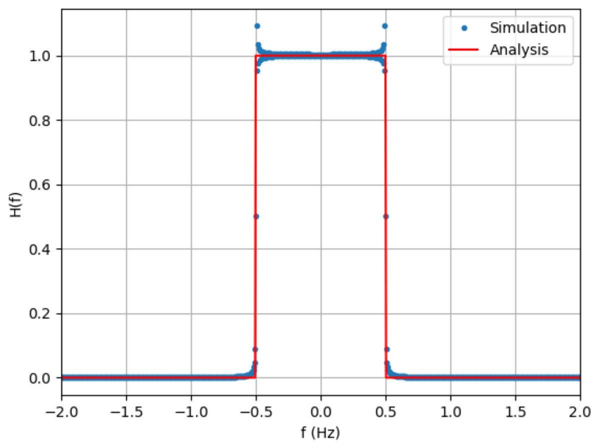

Fig. 0: Fourier Transform of rect t).

low pass filter. Suppose the cutoff frequency is $f_c = 50$ Hz, then

$$H(f) = \text{rect}\left(\frac{f}{2f_c}\right) = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.2}$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.

Solution: Suppose $g(t) \xleftrightarrow{\mathcal{F}} G(f)$. Then, for some nonzero $a \in \mathbb{R}$, let $u = at$,

$$g(at) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(at)e^{-\jmath 2\pi ft}\, dt \tag{4.3}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} g(u)e^{\left(-\jmath 2\pi \frac{f}{a}t\right)}\, dt \tag{4.4}$$

$$= \frac{1}{a}G\left(\frac{f}{a}\right) \tag{4.5}$$

Using (4.5), from (4.2),

$$h(t) \xleftrightarrow{\mathcal{F}} \frac{\pi V_0}{2A_0}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.6}$$

$$h(t) \xleftrightarrow{\mathcal{F}} \frac{\pi V_0 2f_c}{2A_0}\frac{1}{2f_c}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.7}$$

$$h(t) \xleftrightarrow{\mathcal{F}} \frac{2\pi f_c V_0}{A_0}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.8}$$

$$h(t) = \frac{2\pi V_0}{A_0}f_c\,\text{sinc}\left(2f_ct\right) \tag{4.9}$$

4.3 Verify your result using convolution.

Solution: Fourier transform of $x(t)$ and $h(t)$ respectively is

$$X(f) = \sum_{k=-\infty}^{\infty}\frac{2A_0}{\pi}\frac{\delta\left(f + 2kf_0\right)}{1 - 4k^2} \tag{4.10}$$

$$H(f) = \frac{\pi V_0}{2A_0}\text{rect}\left(\frac{f}{2f_c}\right) \tag{4.11}$$

$$X(f) \times H(f) = \sum_{k=-\infty}^{\infty} V_0\frac{\delta\left(f + 2kf_0\right)}{1 - 4k^2} \times \text{rect}\left(\frac{f}{2f_c}\right) \tag{4.12}$$

$$X(f) \times H(f) = \sum_{k=0}^{0} V_0\frac{\delta\left(f + 2kf_0\right)}{1 - 4k^2} \tag{4.13}$$

Hence,

$$X(f) \times H(f) = V_0\frac{\delta\left(f\right)}{1 - 4 \times 0} \tag{4.14}$$

$$X(f) \times H(f) = V_0\delta\left(f\right) \tag{4.15}$$

Since $1 \xleftrightarrow{\mathcal{F}} \delta\left(0\right)$, Hence,

$$V_0\delta\left(t\right) \xleftrightarrow{\mathcal{F}^{-1}} V_0 \times 1 \tag{4.16}$$

$$\implies H(t) \circledast x(t) = V_0 \tag{4.17}$$

Hence verified. The following python code plots the figure ??

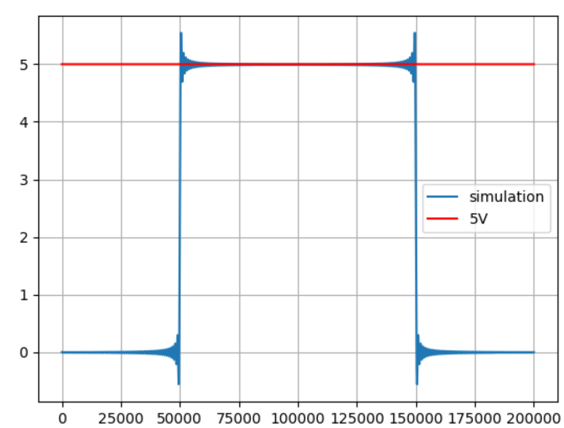

Fig. 0: Convolution of the $x(t)$ and $h(t)$.

https://github.com/JBA−12/EE3900/blob/main/fourier/codes/4.3.py

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.

Solution: The Butterworth filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \left(\frac{f}{f_c}\right)^{2n}\right)} \tag{5.1}$$

where n is the order of the filter and f_c is the cutoff frequency. The attenuation at frequency f is given by

$$A = -10 \log_{10} |H(f)|^2 \tag{5.2}$$
$$= -20 \log_{10} |H(f)| \tag{5.3}$$

We consider the following design parameters for our lowpass analog Butterworth filter:

a) Passband edge, $f_p = 50$ Hz
b) Stopband edge, $f_s = 100$ Hz
c) Passband attenuation, $A_p = -1$ dB
d) Stopband attenuation, $A_s = -20$ dB

We are required to find a desriable order n and cutoff frequency f_c for the filter. From (5.3),

$$A_p = -10 \log_{10} \left[1 + \left(\frac{f_p}{f_c}\right)^{2n}\right] \tag{5.4}$$

$$A_s = -10 \log_{10} \left[1 + \left(\frac{f_s}{f_c}\right)^{2n}\right] \tag{5.5}$$

Thus,

$$\left(\frac{f_p}{f_c}\right)^{2n} = 10^{-\frac{A_p}{10}} - 1 \tag{5.6}$$

$$\left(\frac{f_s}{f_c}\right)^{2n} = 10^{-\frac{A_s}{10}} - 1 \tag{5.7}$$

Therefore, on dividing the above equations and solving for n,

$$n = \frac{\log\left(10^{-\frac{A_s}{10}} - 1\right) - \log\left(10^{-\frac{A_p}{10}} - 1\right)}{2\left(\log f_s - \log f_p\right)} \tag{5.8}$$

In this case, making appropriate susbstitutions gives $n = 4.29$. Hence, we take $n = 5$. Solving for f_c in (5.6) and (5.7),

$$f_{c1} = f_p \left[10^{-\frac{A_p}{10}} - 1\right]^{-\frac{1}{2n}} = 57.23\,\text{Hz} \tag{5.9}$$

$$f_{c2} = f_s \left[10^{-\frac{A_s}{10}} - 1\right]^{-\frac{1}{2n}} = 63.16\,\text{Hz} \tag{5.10}$$

Hence, we take $f_c = \sqrt{f_{c1} f_{c2}} = 60\,\text{Hz}$ approximately.

5.2 Design a Chebyshev filter for $H(f)$.bjjk,bv,kgg
Solution: The Chebyshev filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \epsilon^2 C_n^2 \left(\frac{f}{f_c}\right)\right)} \tag{5.11}$$

where

a) n is the order of the filter
b) ϵ is the ripple
c) f_c is the cutoff frequency
d) $C_n = \cosh^{-1}(n \cosh x)$ denotes the n^th order Chebyshev polynomial, given by

$$c_n(x) = \begin{cases} \cos\left(n \cos^{-1} x\right) & |x| \le 1 \\ \cosh\left(n \cosh^{-1} x\right) & \text{otherwise} \end{cases} \tag{5.12}$$

We are given the following specifications:

a) Passband edge (which is equal to cutoff frequency), $f_p = f_c$
b) Stopband edge, f_s
c) Attenuation at stopband edge, A_s
d) Peak-to-peak ripple δ in the passband. It is given in dB and is related to ϵ as

$$\delta = 10 \log_{10}\left(1 + \epsilon^2\right) \tag{5.13}$$

and we must find a suitable n and ϵ. From (5.13),

$$\epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \tag{5.14}$$

At $f_s > f_p = f_c$, using (5.12), A_s is given by

$$A_s = -10 \log_{10} \left[1 + \epsilon^2 c_n^2 \left(\frac{f_s}{f_p}\right)\right] \tag{5.15}$$

$$\implies c_n\left(\frac{f_s}{f_p}\right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \tag{5.16}$$

$$\implies n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_p}\right)} \tag{5.17}$$

We consider the following specifications:

a) Passband edge/cutoff frequency, $f_p = f_c = 60\,\text{Hz}$.
b) Stopband edge, $f_s = 100\,\text{Hz}$.
c) Passband ripple, $\delta = 0.5\,\text{dB}$
d) Stopband attenuation, $A_s = -20\,\text{dB}$

$\epsilon = 0.35$ and $n = 3.68$. Hence, we take $n = 4$ as the order of the Chebyshev filter.

5.3 Design a circuit for your Butterworth filter.
Solution: Looking at the table of normalized

element values L_k, C_k, of the Butterworth filter for order 5, and noting that de-normalized values L'_k and C'_k are given by

$$C'_k = \frac{C_k}{\omega_c} \qquad L'_k = \frac{L_k}{\omega_c} \tag{5.18}$$

De-normalizing these values, taking $f_c = 60$ Hz,

$$C'_1 = C'_5 = 1.64\,\text{mF} \tag{5.19}$$
$$L'_2 = L'_4 = 4.29\,\text{mH} \tag{5.20}$$
$$C'_3 = 5.31\,\text{mF} \tag{5.21}$$

The L-C network is shown in Fig. 0.

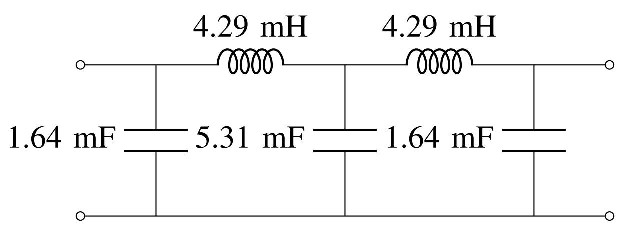

Fig. 0: L-C Butterworth Filter

Below python code plot the figure 0

```
https://github.com/JBA−12/EE3900/blob/main
    /fourier/codes/5.3.py
```

5.4 Design a circuit for your Chebyshev filter.
Solution: Looking at the table of normalized element values of the Chebyshev filter for order 3 and 0.5 dB ripple, and de-nomrmalizing those values, taking $f_c = 50\,\text{Hz}$,

$$C'_1 = 4.43\,\text{mF} \tag{5.22}$$
$$L'_2 = 3.16\,\text{mH} \tag{5.23}$$
$$C'_3 = 6.28\,\text{mF} \tag{5.24}$$
$$L'_4 = 2.23\,\text{mH} \tag{5.25}$$

The L-C network is shown in Fig. 0. Below

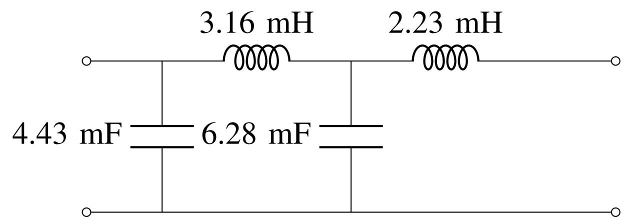

Fig. 0: L-C Chebyshev Filter

python code plot the figure 0

```
https://github.com/JBA−12/EE3900/blob/main
    /fourier/codes/5.4.py
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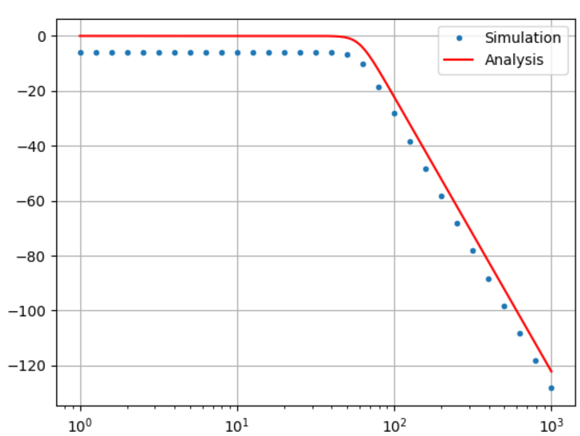

Fig. 0: Simulation of Chebyshev filter.

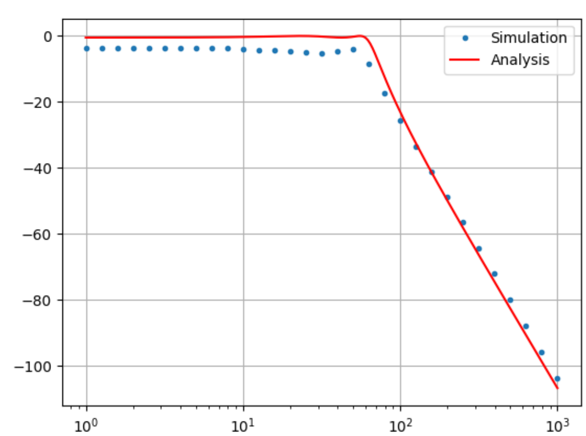

Fig. 0: Simulation of Chebyshev filter.