Digital Signal Processing

AI21BTECH11016
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Contents

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: The following code removes the out of band noise

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/2.3.py
```

and execute the code using the following command

```
python3 2.3.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/3.2.py
```

and run the code using the following command

```
python3 3.2.py
```

3.3 Repeat the above exercise using a C code.
Solution: The following C code yields y(n).dat file

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/3.3.c
```

and run the code using the following commands

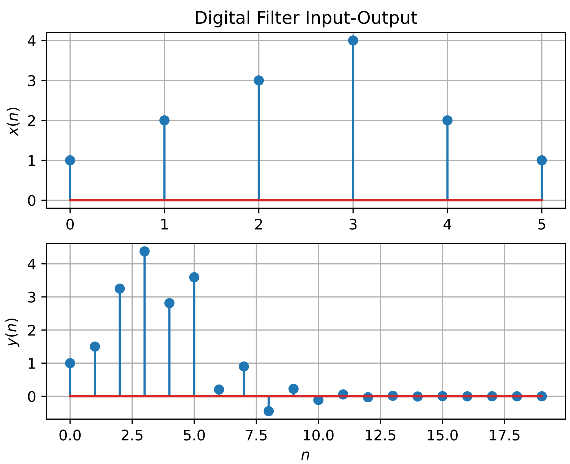

Fig. 3.2

```
gcc 3.3.c
./a.out
```

The following python code inputs y(n) produced using C code and yields Fig. 3.3.

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/3.3.py
```

and run the code using the following command

```
python3 3.3.py
```

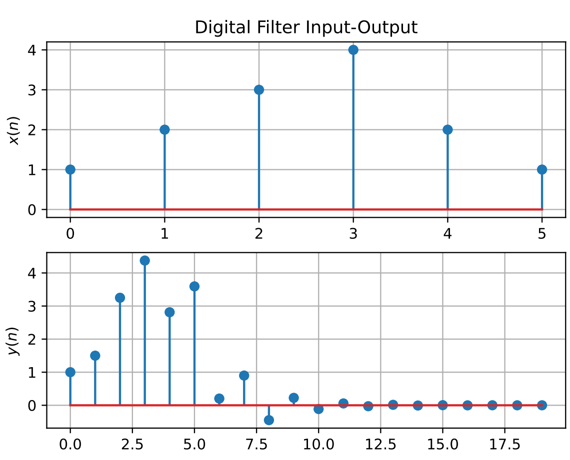

Fig. 3.3

4 $\mathcal{Z}$-transform

4.1 The $\mathcal{Z}$-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \tag{4.4}$$

substitute n - k = t

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n-k} \tag{4.5}$$

$$= z^{-k}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.6}$$

From (4.2), we get

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.7}$$

Substitute n = 1, we get

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.8}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution:

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.9}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.10}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.11}$$

from (3.2) assuming that the $\mathcal{Z}$-transform is a linear operation.
Solution: Applying (4.8) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.12}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.13}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.16}$$

Solution: Consider the Z-transform of δ

$$\mathcal{Z}\{\delta(n)\} = \delta(0) + 0 = 1 \tag{4.17}$$

and from (4.15),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.18}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.19}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.20}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.21}$$

$$= \sum_{n=-0}^{\infty} \left(az^{-1}\right)^n \tag{4.22}$$

$$= \frac{1}{1 - az^{-1}} \quad \left|az^{-1}\right| < |1| \tag{4.23}$$

$$= \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.24}$$

$$\therefore a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.25}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.26}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution: The following code plots Fig. 4.6.

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/4.6.py
```

and run the code using the following command

```
python3 4.6.py
```

Using (4.13), we observe that $\left|H\left(e^{\jmath\omega}\right)\right|$ is given by

$$\left|H\left(e^{\jmath\omega}\right)\right| = \left|\frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}}\right| \tag{4.27}$$

$$= \left|\frac{1 + \cos 2\omega - \jmath \sin 2\omega}{1 + \frac{1}{2}\cos\omega - \frac{1}{2}\jmath\sin\omega}\right| \tag{4.28}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} \tag{4.29}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.30}$$

$$= \sqrt{\frac{8\,(2\cos\omega)^2}{5 + 4\cos\omega}} \tag{4.31}$$

$$\left|H\left(e^{\jmath\omega}\right)\right| = \frac{4\,|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.32}$$

Using (4.32) and the plot we can conclude that,

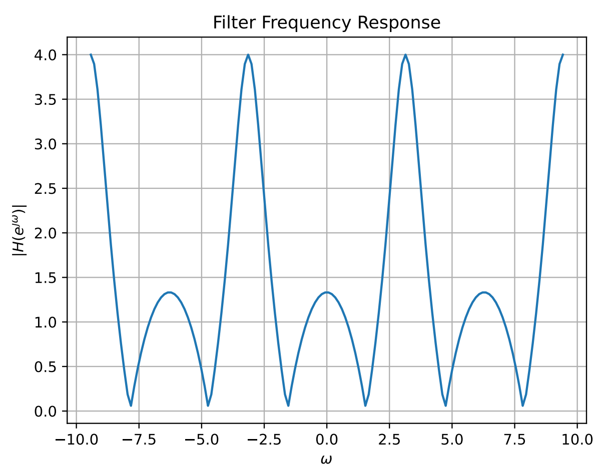

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

a) The Plot $\left|H\left(e^{\jmath\omega}\right)\right|$ is Symmetric about $\omega = 0$
b) The maximum and minimum values of the plot are 4 and 0 respectively
c) Consider (4.32),
   The period of numerator is π and the period of denominator is 2π

$\therefore$ The period of $|H(e^{\jmath\omega})|$ is $LCM(\pi, 2\pi) = 2\pi$ i.e.,

$$\left|H\left(e^{\jmath(\omega+2\pi)}\right)\right| = \frac{4\left|\cos(\omega+2\pi)\right|}{\sqrt{5+4\cos(\omega+2\pi)}} \tag{4.33}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5+4\cos\omega}} \tag{4.34}$$

$$\left|H\left(e^{\jmath(\omega+2\pi)}\right)\right| = \left|H(e^{\jmath\omega})\right| \tag{4.35}$$

$\Rightarrow$ it is Periodic with a period of 2π

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.
Solution:

$$H(e^{\jmath\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-\jmath\omega n} \tag{4.36}$$

Multiply both sides with $e^{\jmath\omega k}$ and integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})\, e^{\jmath\omega k} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-\jmath\omega n} e^{\jmath\omega k} d\omega \tag{4.37}$$

$$= \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-\jmath\omega(n-k)} d\omega \tag{4.38}$$

$$= h(k)2\pi \tag{4.39}$$

Since,

$$\int_{-\pi}^{\pi} e^{-\jmath\omega(n-k)} d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.40}$$

$$\therefore h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})\, e^{\jmath\omega n} d\omega \tag{4.41}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.13).
Solution: From (4.13), we have

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$ to perform long division

$$\begin{array}{r|l} & \;\; 2x - 4 \\ \hline \frac{1}{2}x + 1 & \;\; x^2 \qquad + 1 \\ & -x^2 - 2x \\ \hline & \qquad -2x + 1 \\ & \qquad\; 2x + 4 \\ \hline & \qquad\qquad 5 \end{array}$$

From above division we can write,

$$1 + z^{-2} = (1 + \frac{1}{2}z^{-1})(2z^{-1} - 4) + 5 \tag{5.3}$$

$$\frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} = 2z^{-1} - 4 + \frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

From (4.13), we can write

$$H(z) = -4 + 2z^{-1} + \frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.5}$$

$$= -4 + 2z^{-1} + 5\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.6}$$

$$= 1 - \frac{1}{2}z^{-1} + 5\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.7}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} + 4\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.8}$$

$$= \sum_{n=-\infty}^{\infty} u(n)\left(-\frac{1}{2}\right)^n z^{-n} + \sum_{n=-\infty}^{\infty} u(n-2)\left(-\frac{1}{2}\right)^{n-2} z^{-n} \tag{5.9}$$

Therefore, from (4.1),

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.10}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.11}$$

and there is a one to one relationship between $h(n)$ and $H(z)$.
$h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.13),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.12}$$

From (4.20),

$$\frac{1}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) \quad |z| > \frac{1}{2} \tag{5.13}$$

$$\frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \tag{5.14}$$

$$\Rightarrow H(z) \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \tag{5.15}$$

$$\tag{5.16}$$

(Since Z-transform is a linear operator)

$$\therefore h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

From (5.12),Consider the first part:

$$\frac{1}{1+\frac{1}{2}z^{-1}} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^n \tag{5.18}$$

This sum converges when $|z| > \frac{1}{2}$
$\Rightarrow$ ROC is $|z| > \frac{1}{2}$

$$\frac{1}{1+\frac{1}{2}z^{-1}} = \frac{2z}{1+2z} \tag{5.19}$$

$$= \sum_{n=-\infty}^{-1} (2z)^{-n} \tag{5.20}$$

This sum converges when $|z| < \frac{1}{2}$
$\Rightarrow$ ROC is $|z| < \frac{1}{2}$
Therefore, ROC of $H(z)$ will be

$$|z| \neq \frac{1}{2} \tag{5.21}$$

5.3 Sketch $h(n)$. Is it bounded? Convergent? **Solution:** The following code plots Fig. 5.3.

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/5.3.py
```

and run the code using the following command

```
python3 5.3.py
```

From the plot, we can conclude that it is convergent to 0
$\Rightarrow$ It is bounded as well.

5.4 Convergent? Justify using the ratio test.

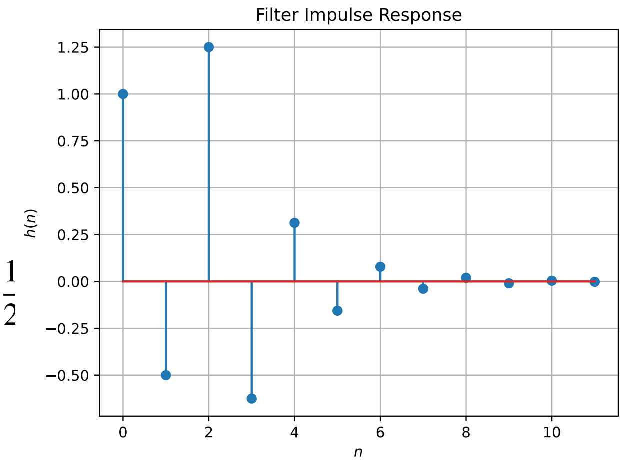

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

Solution: Using the ratio test for convergence

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(1+\frac{1}{4}\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(1+\frac{1}{4}\right)}\right| \tag{5.22}$$

$$= \lim_{n\to\infty}\left|-\frac{1}{2}\right| \tag{5.23}$$

$$= \frac{1}{2} < 1 \tag{5.24}$$

$\therefore$ h(n) is Convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.25}$$

Is the system defined by (3.2) stable for the impulse response in (5.11)?
Solution: From 5.2,

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.26}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.27}$$

$$= \frac{1}{1-\left(\frac{-1}{2}\right)} + \frac{1}{1-\left(\frac{-1}{2}\right)} \tag{5.28}$$

$$= \frac{4}{3} < \infty \tag{5.29}$$

using the fomula for the sum of an infinite geometric progression

∴ The system is stable.

5.6 Verify the above result using a python code.
Solution: The following code verifies whether the given system is stable or not

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/5.6.py
```

run the code using the following command

```
python3 5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.30)$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/5.7.py
```

run the code using the following command

```
python3 5.7.py
```

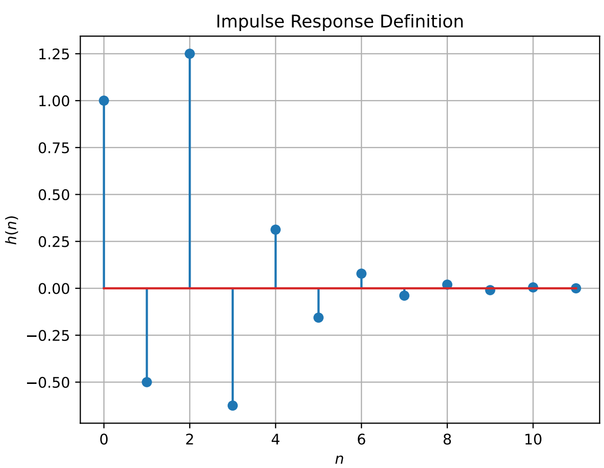

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.31)$$

Comment. The operation in (5.31) is known as *convolution*.

Solution:

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.32)$$

$$= \sum_{k=0}^{5} x(k)h(n-k) \quad (5.33)$$

The following code plots Fig. 5.9.

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/5.8.py
```

run the code using the following command

```
python3 5.8.py
```

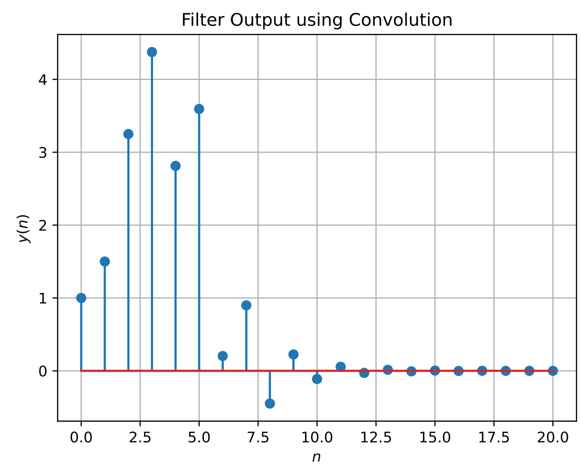

Fig. 5.8: $y(n)$ from the definition of convolution

This plot is same as y(n) in Fig. 3.2

Therefore,

$$y(n) = x(n) * h(n) \quad (5.34)$$

5.9 Express the above convolution using a Teoplitz matrix.
Solution: From (3.1),we can write

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \quad (5.35)$$

Their convolution is given by the product of the following Toeplitz matrix $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix} \tag{5.36}$$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} = \mathbf{T}\mathbf{x} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \tag{5.37}$$

The following python code computes the convolution using Teoplitz matrix.

```
https://github.com/JBA−12/EE3900/blob/main
    /A1/codes/5.9.py
```

run the code using the following command

```
python3 5.9.py
```

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.38}$$

Solution: From (5.31) we know that,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.39}$$

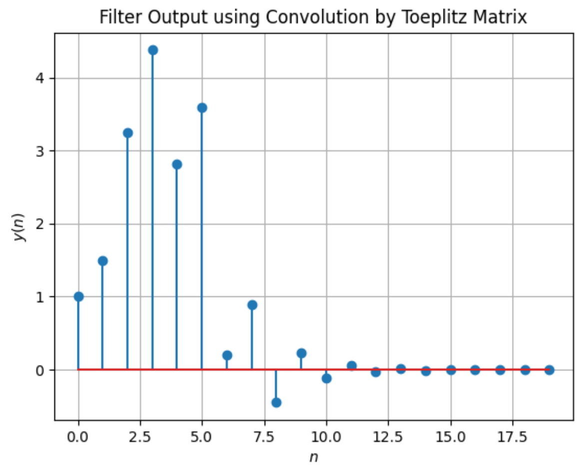

Fig. 5.9: $y(n)$ from the definition of convolution using Teoplitz matrix

Substitute k = n-i

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \tag{5.40}$$

$$= \sum_{i=\infty}^{-\infty} x(n-i)h(i) \tag{5.41}$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \tag{5.42}$$

Since, the order of limit doesn't matter in case of summation.Therefore, now we have

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.43}$$

from (5.31)

$$\therefore y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.44}$$