Random Numbers

AI21BTECH11016

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1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/1.1.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

Run the code using the following commands

```
gcc −o out 1.1.c coeffs.h −lm
./out
```

This generates an output file "uni.dat".

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/1.2.py
```

and run the code using the following command

```
python3 1.2.py
```

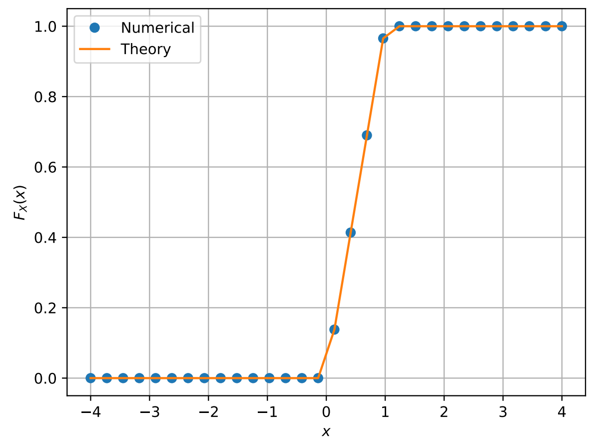

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: $F_U(x)$ is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

$$PDF function: p_U(u) = \begin{cases} 1, & u \in (0,1) \\ 0, & otherwise \end{cases} \tag{1.3}$$

$$if x < 0 : \int_{-\infty}^{x} p_U(x)dx = 0 \tag{1.4}$$

$$\because p_U(x) = 0 \tag{1.5}$$

$$if 0 \le x \le 1 : \tag{1.6}$$

$$\int_{-\infty}^{x} p_U(x)dx = \int_{-\infty}^{0} p_U(x)dx + \int_{0}^{x} p_U(x)dx \tag{1.7}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{x} 1dx \tag{1.8}$$

$$= 0 + x \tag{1.9}$$

$$= x \tag{1.10}$$

$$ifx > 1 : \quad (1.11)$$

$$\int_{-\infty}^{x} p_U(x)dx \quad (1.12)$$

$$= \int_{-\infty}^{0} p_U(x)dx + \int_{0}^{x} p_U(x)dx + \int_{1}^{x} p_U(x)dx \quad (1.13)$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{x} 0dx \quad (1.14)$$

$$= 0 + 1 + 0 \quad (1.15)$$

$$= 1 \quad (1.16)$$

$$\Rightarrow F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (1.17)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \quad (1.18)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.19)$$

Write a C program to find the mean and variance of U.
Solution:

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/1.4.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

Run the code using the following commands

```
gcc −o out 1.4.c coeffs.h −lm
./out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.20)$$

Solution: From 1.4 we have,

$$dF_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ dx, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.21)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.22)$$

$$E[U] = \int_{0}^{1} x dx = \frac{1}{2} \quad (1.23)$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.24)$$

$$E[U] = \int_{0}^{1} x^2 dx = \frac{1}{3} \quad (1.25)$$

$$variance = E\left[U^2\right] - E[U]^2 \quad (1.26)$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.27)$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/2.1.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

Run the code using the following commands

```
gcc −o out 2.1.c coeffs.h −lm
./out
```

This generates an output file ”gau.dat”.

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2

Uncomment the Question-2.2 of following python file

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/2.2.py
```

and run the code using the following command

```
python3 2.2.py
```

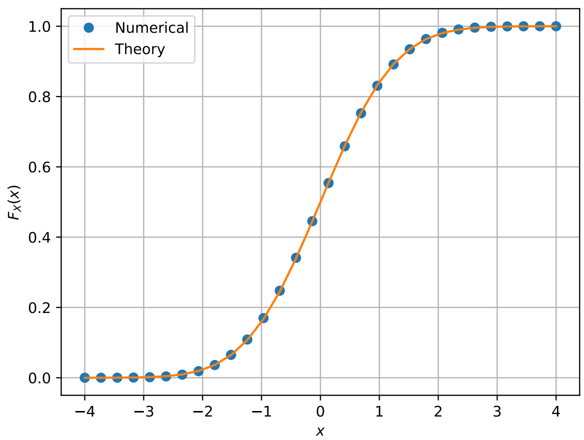

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/2.3.py
```

and run the code using the following command

```
python3 2.3.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution:

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/2.4.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

and run the following commands

```
gcc −o out 2.4.c coeffs.h −lm
./out
```

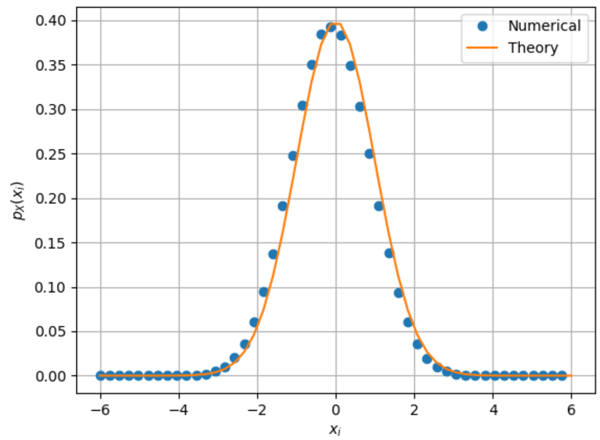

Fig. 2.3: The PDF of X

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution:

$$E[X] = \int_{-\infty}^{\infty} x dF_U(x) \tag{2.4}$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.5}$$

$$= \left[-\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \tag{2.6}$$

$$= 0 \tag{2.7}$$

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{2.8}$$

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.9}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2x^2 \exp\left(-x^2\right) \sqrt{2} dx \tag{2.10}$$

$$= 1 \tag{2.11}$$

$$\because \int_{-\infty}^{\infty} x^2 \exp -x^2 dx = \frac{\sqrt{\pi}}{2} \tag{2.12}$$

$$variance = E\left[X^2\right] - E[X]^2 \tag{2.13}$$

$$= 1 - 0 \tag{2.14}$$

$$= 1 \tag{2.15}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution:

```
https://github.com/JBA-12/Sim-Assignment1
    /blob/main/codes/3.1.c
https://github.com/JBA-12/Sim-Assignment1
    /blob/main/codes/coeffs.h
```

and run the code using the following commands

```
gcc -o out 3.1.c coeffs.h -lm
./out
```

This generates an output file ”V.dat”.

Plotting cdf for V:
The following code plots Fig. 1.2

```
https://github.com/JBA-12/Sim-Assignment1
    /blob/main/codes/3.1.py
```

and run the code using the following command

```
python3 3.1.py
```

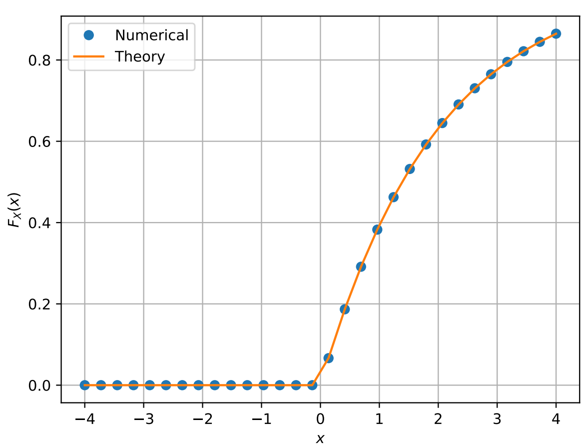

Fig. 3.1: The CDF of U

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$\begin{aligned} F_V(x) &= \Pr(V \le x) && (3.2)\\ &= \Pr(-2ln(1-U) \le x) && (3.3)\\ &= \Pr\left(ln(1-U) \ge \frac{-x}{2}\right) && (3.4)\\ &= \Pr\left(1-U \ge \exp\left(-\frac{x}{2}\right)\right) && (3.5)\\ &= \Pr\left(U \le 1-\exp\left(-\frac{x}{2}\right)\right) && (3.6)\\ &= F_U\left(1-\exp\left(-\frac{x}{2}\right)\right) && (3.7) \end{aligned}$$

Here we have :

$$0 \le 1-\exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \ge 0 \tag{3.8}$$

$$1-\exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \tag{3.9}$$

$$\therefore F_V(x) = \begin{cases} 0 & x < 0 \\ 1-\exp\left(-\frac{x}{2}\right) & x \ge 0 \end{cases} \tag{3.10}$$