Random Numbers

AI21BTECH11016

CONTENTS

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Uncomment Part-1 of following c file

https://github.com/JBA−12/Sim−Assignment1
  /blob/main/codes/exrand.c
https://github.com/JBA−12/Sim−Assignment1
  /blob/main/codes/coeffs.h

and run the code using the following commands

gcc −o out exrands.c coeffs.h −lm
./out

This generates an output file "uni.dat".

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \qquad (1.1)$$

Solution: The following code plots Fig. **??**

Uncomment the Question-1.2 of following python file

https://github.com/JBA−12/Sim−Assignment1
  /blob/main/codes/cdf_plot.py

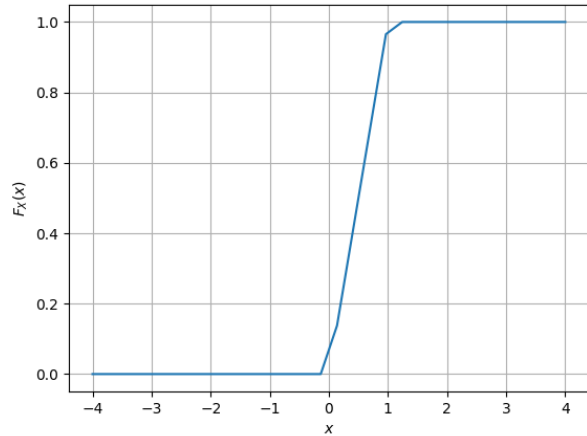and run the code using the following command

python3 cdf_plot.py

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: $F_U(x)$ is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(u)du$$
$$(1.2)$$

$$PDF\,function : p_U(u) = \begin{cases} 1, & u \in (0,1) \\ 0, & otherwise \end{cases}$$
$$(1.3)$$

$$if\,x < 0 : \int_{-\infty}^{x} p_U(x)dx = 0 \qquad (1.4)$$

$$\because p_U(x) = 0 \qquad (1.5)$$

$$if\,0 \le x \le 1 : \qquad (1.6)$$

$$\int_{-\infty}^{x} p_U(x)dx = \int_{-\infty}^{0} p_U(x)dx + \int_{0}^{x} p_U(x)dx$$
$$(1.7)$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{x} 1dx \qquad (1.8)$$

$$= 0 + x \qquad (1.9)$$

$$= x \qquad (1.10)$$

$$if\ x > 1:\qquad (1.11)$$

$$\int_{-\infty}^{x} p_U(x)dx \qquad (1.12)$$

$$= \int_{-\infty}^{0} p_U(x)dx + \int_{0}^{x} p_U(x)dx + \int_{1}^{x} p_U(x)dx \qquad (1.13)$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{x} 0dx \qquad (1.14)$$

$$= 0 + 1 + 0 \qquad (1.15)$$

$$= 1 \qquad (1.16)$$

$$\Rightarrow F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \qquad (1.17)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \qquad (1.18)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \qquad (1.19)$$

Write a C program to find the mean and variance of U.
Solution: Uncomment the Q-1.4 portion of following c code

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/MV.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

and run the following commands

```
gcc −o out MV.c coeffs.h −lm
./out
```

```
We get : Mean is 0.500031
        Variance is 0.083247
```

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \qquad (1.20)$$

Solution: From 1.4 we have,

$$dF_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ dx, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \qquad (1.21)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \qquad (1.22)$$

$$E[U] = \int_{0}^{1} x dx = \frac{1}{2} \qquad (1.23)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \qquad (1.24)$$

$$E[U] = \int_{0}^{1} x^2 dx = \frac{1}{3} \qquad (1.25)$$

$$\text{variance} = E[U^2] - E[U]^2 \qquad (1.26)$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \qquad (1.27)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \qquad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Uncomment Part-2 of following c file,

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/exrand.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

and run the code using the following commands

```
gcc −o out exrands.c coeffs.h −lm
./out
```

This generates an output file "gau.dat".

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. **??**

Uncomment the Question-2.2 of following python file

and run the code using the following command

python3 cdf_plot.py

Fig. 2.2: The CDF of X

Fig. 2.3: The PDF of X

We get : Mean is 0.000685
Variance is 1.000025
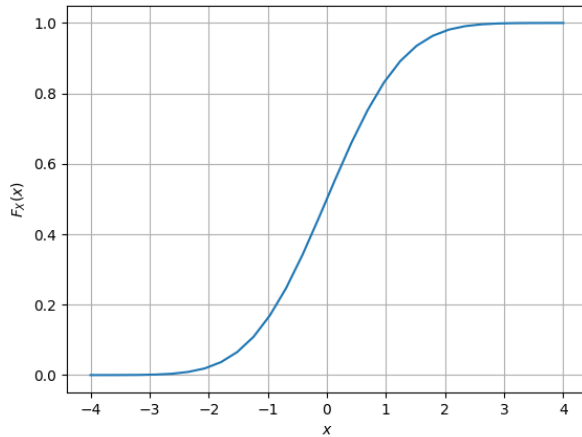
2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \qquad (2.2)$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. **??** using the code below

and run the code using the following command

python3 pdf_plot.py

2.4 Find the mean and variance of X by writing a C program.
Solution: Uncomment the Q-2.4 portion of following c code

and run the following commands

gcc −o out MV.c coeffs.h −lm
./out

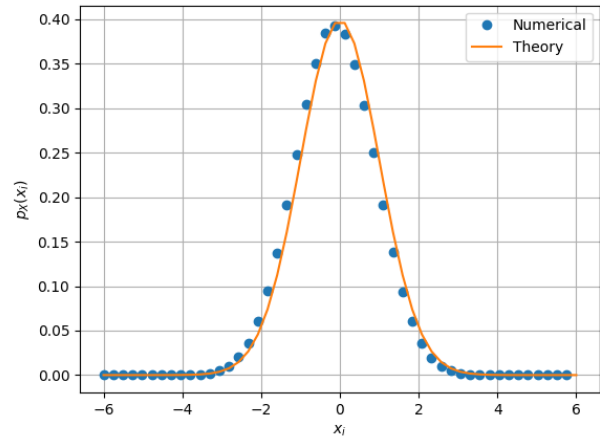2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.
Solution:

$$E\left[U\right] = \int_{-\infty}^{\infty} x \, dF_U(x) \qquad (2.4)$$

$$E\left[U\right] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \qquad (2.5)$$

$$= \left[-\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \qquad (2.6)$$

$$= 0 \qquad (2.7)$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 \, dF_U(x) \qquad (2.8)$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \qquad (2.9)$$

$$= \sqrt{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2x^2 \exp\left(-x^2\right) \qquad (2.10)$$

$$= 1 \qquad (2.11)$$

$$\because \int_{-\infty}^{\infty} x^2 \exp -x^2 = \frac{\sqrt{\pi}}{2} \quad (2.12)$$

$$variance = E\left[U^2\right] - E\left[U\right]^2 \quad (2.13)$$

$$= 1 - 0 \quad (2.14)$$

$$= 1 \quad (2.15)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Uncomment the Part-3 portion of following c file

https://github.com/JBA−12/Sim−Assignment1
/blob/main/codes/exrand.c
https://github.com/JBA−12/Sim−Assignment1
/blob/main/codes/coeffs.h

and run the code using the following commands

gcc −o out exrands.c coeffs.h −lm
./out

This generates an output file "V.dat".

Plotting cdf for V:
The following code plots Fig. **??**

Uncomment the Question-3.1 of following python file

https://github.com/JBA−12/Sim−Assignment1
/blob/main/codes/cdf_plot.py

and run the code using the following command

python3 cdf_plot.py

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = \Pr(V \le x) \quad (3.2)$$
$$= \Pr(-2ln(1-U) \le x) \quad (3.3)$$
$$= \Pr\left(ln(1-U) \ge \frac{-x}{2}\right) \quad (3.4)$$
$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$
$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$
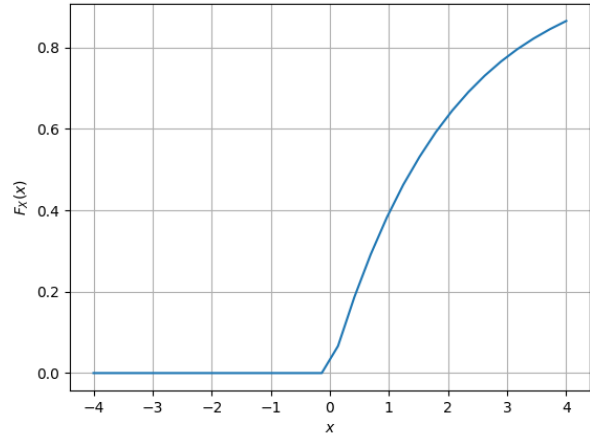$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

Fig. 3.1: The CDF of U

Here we have :

$$0 \le 1 - \exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \ge 0 \quad (3.8)$$
$$1 - \exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \quad (3.9)$$

$$\therefore F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \end{cases} \quad (3.10)$$