

Random Numbers

AI21BTECH11016

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Uncomment Part-1 of following c file

```
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/exrand.c
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/coeffs.h
```

and run the code using the following commands

```
gcc -o out exrands.c coeffs.h -lm
./out
```

This generates an output file "uni.dat".

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

Uncomment the Question-1.2 of following python file

```
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/cdf_plot.py
```

and run the code using the following command

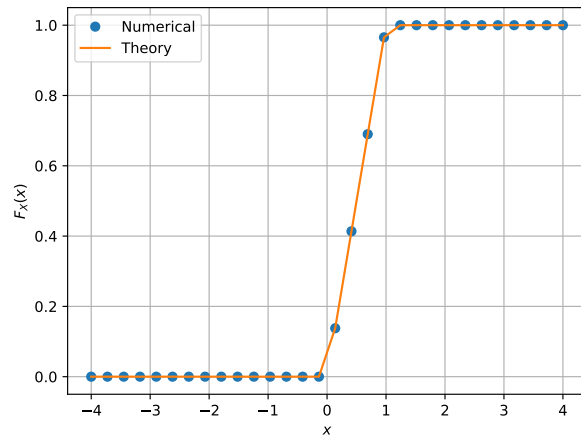


Fig. 1.2: The CDF of U

```
python3 cdf_plot.py
```

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: $F_U(x)$ is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(u) du \quad (1.2)$$

$$\text{PDF function : } p_U(u) = \begin{cases} 1, & u \in (0, 1) \\ 0, & \text{otherwise} \end{cases} \quad (1.3)$$

$$\text{if } x < 0 : \int_{-\infty}^x p_U(x) dx = 0 \quad (1.4)$$

$$\because p_U(x) = 0 \quad (1.5)$$

if $0 \leq x \leq 1$:

(1.6)

$$\int_{-\infty}^x p_U(x)dx = \int_{-\infty}^0 p_U(x)dx + \int_0^x p_U(x)dx \quad (1.7)$$

$$= \int_{-\infty}^0 0dx + \int_0^x 1dx \quad (1.8)$$

$$= 0 + x \quad (1.9)$$

$$= x \quad (1.10)$$

if $x > 1$:

(1.11)

$$\int_{-\infty}^x p_U(x)dx \quad (1.12)$$

$$= \int_{-\infty}^0 p_U(x)dx + \int_0^x p_U(x)dx + \int_1^x p_U(x)dx \quad (1.13)$$

$$= \int_{-\infty}^0 0dx + \int_0^1 1dx + \int_1^x 0dx \quad (1.14)$$

$$= 0 + 1 + 0 \quad (1.15)$$

$$= 1 \quad (1.16)$$

$$\Rightarrow F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (1.17)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.18)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.19)$$

Write a C program to find the mean and variance of U .

Solution: Uncomment the Q-1.4 portion of following c code

```
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/MV.c
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/coeffs.h
```

and run the following commands

```
gcc -o out MV.c coeffs.h -lm
./out
```

We get : Mean is 0.500031

Variance is 0.083247

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.20)$$

Solution: From 1.4 we have,

$$dF_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ dx, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.21)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.22)$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \quad (1.23)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.24)$$

$$E[U] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.25)$$

$$\text{variance} = E[U^2] - E[U]^2 \quad (1.26)$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.27)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Uncomment Part-2 of following c file,

```
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/exrand.c
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/coeffs.h
```

and run the code using the following commands

```
gcc -o out exrand.c coeffs.h -lm
./out
```

This generates an output file "gau.dat".

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What

properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2

Uncomment the Question-2.2 of following python file

```
https://github.com/JBA-12/Sim-Assignment1/blob/main/codes/cdf_plot.py
```

and run the code using the following command

```
python3 cdf_plot.py
```

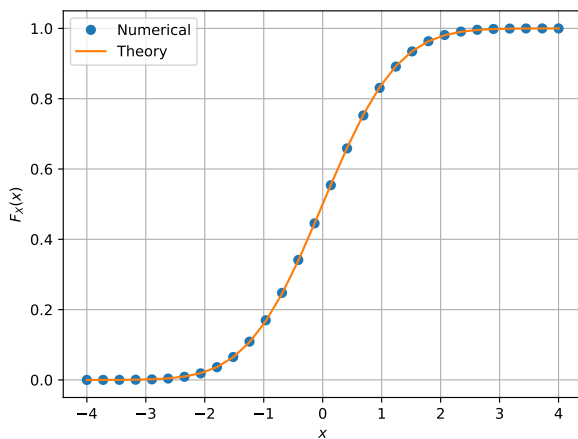


Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/raw/master/manual/codes/pdf_plot.py
```

and run the code using the following command

```
python3 pdf_plot.py
```

2.4 Find the mean and variance of X by writing a C program.

Solution: Uncomment the Q-2.4 portion of following c code

```
https://github.com/JBA-12/Sim-Assignment1/blob/main/codes/MV.c
```

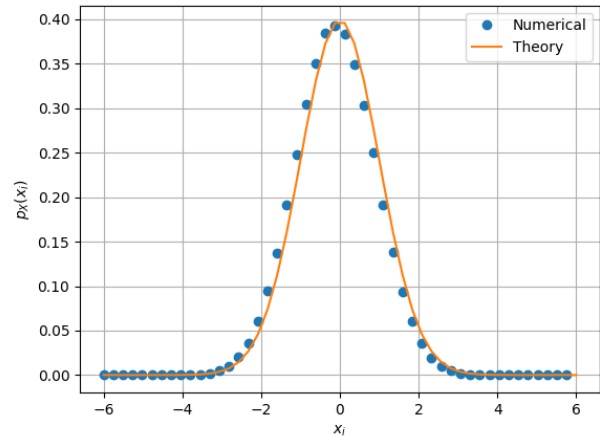


Fig. 2.3: The PDF of X

```
https://github.com/JBA-12/Sim-Assignment1/blob/main/codes/coeffs.h
```

and run the following commands

```
gcc -o out MV.c coeffs.h -lm
./out
```

We get : Mean is 0.000685
Variance is 1.000025

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution:

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (2.4)$$

$$E[U] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (2.5)$$

$$= \left[-\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \right]_{-\infty}^{\infty} \quad (2.6)$$

$$= 0 \quad (2.7)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (2.8)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (2.9)$$

$$= \sqrt{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2x^2 \exp(-x^2) \quad (2.10)$$

$$= 1 \quad (2.11)$$

$$\because \int_{-\infty}^{\infty} x^2 \exp -x^2 = \frac{\sqrt{\pi}}{2} \quad (2.12)$$

$$\text{variance} = E[U^2] - E[U]^2 \quad (2.13)$$

$$= 1 - 0 \quad (2.14)$$

$$= 1 \quad (2.15)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Uncomment the Part-3 portion of following c file

```
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/exrand.c
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/coeffs.h
```

and run the code using the following commands

```
gcc -o out exrands.c coeffs.h -lm
./out
```

This generates an output file "V.dat".

Plotting cdf for V:

The following code plots Fig. 1.2

Uncomment the Question-3.1 of following python file

```
https://github.com/JBA-12/Sim-Assignment1
/blob/main/codes/cdf_plot.py
```

and run the code using the following command

```
python3 cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(\ln(1 - U) \geq \frac{-x}{2}\right) \quad (3.4)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

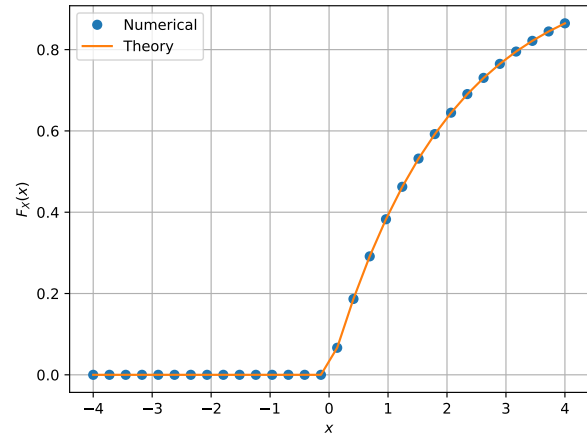


Fig. 3.1: The CDF of U

Here we have :

$$0 \leq 1 - \exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \geq 0 \quad (3.8)$$

$$1 - \exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \quad (3.9)$$

$$\therefore F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases} \quad (3.10)$$