Random Numbers

AI21BTECH11016

Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Uncomment Part-1 of following c file

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/exrand.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

and run the code using the following commands

```
gcc −o out exrands.c coeffs.h −lm
./out
```

This generates an output file "uni.dat".

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

Uncomment the Question-1.2 of following python file

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/cdf_plot.py
```

and run the code using the following command

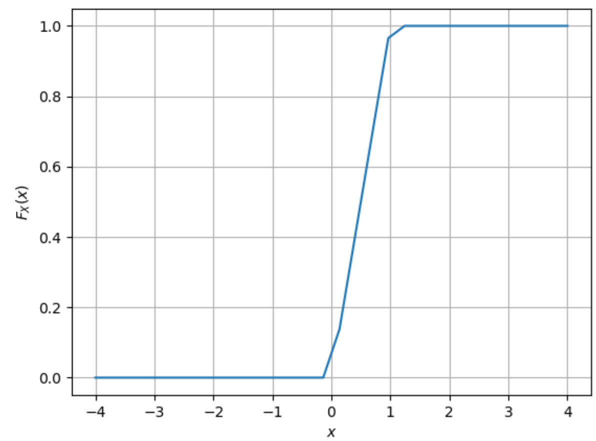

Fig. 1.2: The CDF of U

```
python3 cdf_plot.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution: $F_U(x)$ is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

$$PDF function : p_U(u) = \begin{cases} 1, & u \in (0,1) \\ 0, & otherwise \end{cases} \tag{1.3}$$

$$if x < 0 : \int_{-\infty}^{x} p_U(x)dx = 0 \tag{1.4}$$

$$\because p_U(x) = 0 \tag{1.5}$$

$$if 0 \le x \le 1 : \tag{1.6}$$

$$\int_{-\infty}^{x} p_U(x)dx = \int_{-\infty}^{0} p_U(x)dx + \int_{0}^{x} p_U(x)dx \tag{1.7}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{x} 1dx \tag{1.8}$$

$$= 0 + x \tag{1.9}$$

$$= x \tag{1.10}$$

$$if x > 1 : \tag{1.11}$$

$$\int_{-\infty}^{x} p_U(x)dx \tag{1.12}$$

$$= \int_{-\infty}^{0} p_U(x)dx + \int_{0}^{x} p_U(x)dx + \int_{1}^{x} p_U(x)dx \tag{1.13}$$

$$= \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{x} 0dx \tag{1.14}$$

$$= 0 + 1 + 0 \tag{1.15}$$

$$= 1 \tag{1.16}$$

$$\Rightarrow F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.17}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.18}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.19}$$

Write a C program to find the mean and variance of U.
Solution: Uncomment the Q-1.4 portion of following c code

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/MV.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

and run the following commands

```
gcc −o out MV.c coeffs.h −lm
./out
```

```
We get : Mean is 0.500031
Variance is 0.083247
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.20}$$

Solution: From 1.4 we have,

$$dF_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ dx, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.21}$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.22}$$

$$E[U] = \int_{0}^{1} x dx = \frac{1}{2} \tag{1.23}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.24}$$

$$E[U] = \int_{0}^{1} x^2 dx = \frac{1}{3} \tag{1.25}$$

$$variance = E\left[U^2\right] - E[U]^2 \tag{1.26}$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.27}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Uncomment Part-2 of following c file,

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/exrand.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

and run the code using the following commands

```
gcc −o out exrands.c coeffs.h −lm
./out
```

This generates an output file ”gau.dat”.

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What

properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2

Uncomment the Question-2.2 of following python file

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/cdf_plot.py
```

and run the code using the following command

```
python3 cdf_plot.py
```

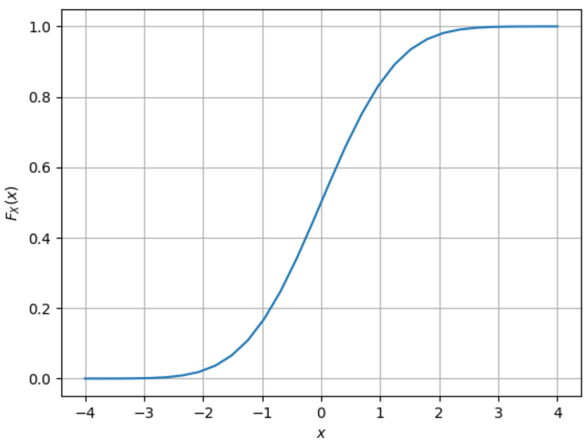

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/gadepall/probability/
    raw/master/manual/codes/pdf_plot.py
```

and run the code using the following command

```
python3 pdf_plot.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution: Uncomment the Q-2.4 portion of following c code

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/MV.c
```

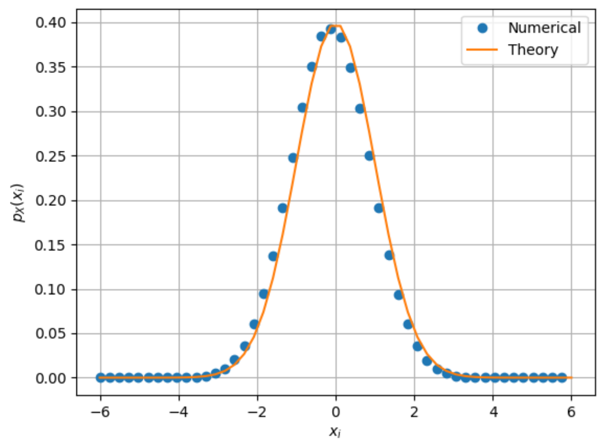

Fig. 2.3: The PDF of X

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

and run the following commands

```
gcc −o out MV.c coeffs.h −lm
./out
```

```
We get : Mean is 0.000685
         Variance is 1.000025
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution:

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{2.4}$$

$$E[U] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.5}$$

$$= \left[-\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right]_{-\infty}^{\infty} \tag{2.6}$$

$$= 0 \tag{2.7}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{2.8}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.9}$$

$$= \sqrt{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2x^2 \exp\left(-x^2\right) \tag{2.10}$$

$$= 1 \tag{2.11}$$

$$\because \int_{-\infty}^{\infty} x^2 \exp -x^2 = \frac{\sqrt{\pi}}{2} \tag{2.12}$$

$$variance = E\left[U^2\right] - E\left[U\right]^2 \tag{2.13}$$

$$= 1 - 0 \tag{2.14}$$

$$= 1 \tag{2.15}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{3.1}$$

and plot its CDF.
Solution: Uncomment the Part-3 portion of following c file

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/exrand.c
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/coeffs.h
```

and run the code using the following commands

```
gcc −o out exrands.c coeffs.h −lm
./out
```

This generates an output file ”V.dat”.

Plotting cdf for V:
The following code plots Fig. 1.2

Uncomment the Question-3.1 of following python file

```
https://github.com/JBA−12/Sim−Assignment1
    /blob/main/codes/cdf_plot.py
```

and run the code using the following command

```
python3 cdf_plot.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = \Pr\left(V \leq x\right) \tag{3.2}$$

$$= \Pr\left(-2ln(1-U) \leq x\right) \tag{3.3}$$

$$= \Pr\left(ln(1-U) \geq \frac{-x}{2}\right) \tag{3.4}$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

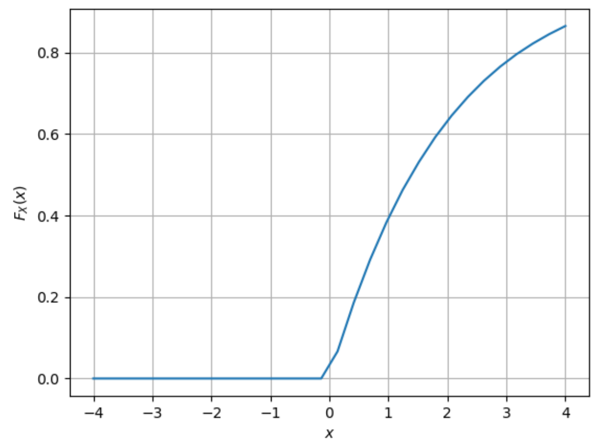

Fig. 3.1: The CDF of U

Here we have :

$$0 \leq 1 - \exp\left(-\frac{x}{2}\right) < 1 \quad \text{if } x \geq 0 \tag{3.8}$$

$$1 - \exp\left(-\frac{x}{2}\right) < 0 \quad \text{if } x < 0 \tag{3.9}$$

$$\therefore F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases} \tag{3.10}$$