27 Discutir y resolver el siguiente sistema para todos los valores reales a y b:

$$\left.\begin{aligned} ax + by &= 0 \\ 3x - 2y &= 2 \end{aligned}\right\}$$

Estudiamos $rg(A)$ y $rg(A^*)$ por Gauss:

$$A^* = \begin{array}{ccc} x & y & T.I \\ \end{array}$$
$$A^* = \left(\begin{array}{cc|c} a & b & 0 \\ 3 & -2 & 2 \end{array}\right) \underset{F_1 \leftrightarrow F_2}{\sim} \left(\begin{array}{cc|c} 3 & -2 & 2 \\ a & b & 0 \end{array}\right) \underset{F_2 \to F_2 - \frac{a}{3}F_1}{\sim} \left(\begin{array}{cc|c} 3 & -2 & 2 \\ 0 & b+\frac{2}{3}a & -\frac{2}{3}a \end{array}\right)$$

• Si $b + \frac{2}{3}a \neq 0 \longrightarrow b \neq -\frac{2}{3}a \longrightarrow rg(A) = 2 = rg(A^*) = n$ $\boxed{SCD}$

Última fila nunca es 0

$$\left.\begin{aligned} 3x - 2y &= 2 \\ \left(b+\tfrac{2}{3}a\right)y &= -\tfrac{2}{3}a \end{aligned}\right\} \to x = \frac{2+2y}{3} = \frac{2 - \frac{4a}{2a+3b}}{3} =$$

$$= \frac{\frac{\cancel{4a} + 6b - \cancel{4a}}{2a+3b}}{\cancel{3}} = \boxed{\frac{2b}{2a+3b}}$$

$\downarrow$

$$y = \frac{-\frac{2}{3}a}{b + \frac{2}{3}a} = \frac{\frac{-2a}{\cancel{3}}}{\frac{3b+2a}{\cancel{3}}} = \boxed{\frac{-2a}{2a+3b}}$$

• Si $b + \frac{2}{3}a = 0 \longrightarrow b = -\frac{2}{3}a \longrightarrow rg(A) = 1$

* si $-\frac{2}{3}a \neq 0 \longrightarrow a \neq 0 \longrightarrow rg(A^*) = 2 \neq rg(A)$ $\boxed{SI}$
* si $-\frac{2}{3}a = 0 \longrightarrow a = 0 \longrightarrow rg(A^*) = 1 = rg(A) < n$ $\boxed{SCI}$

Resolvemos cuando es SCI : $b = -\frac{2}{3}a$ y $a = 0$

$$\left(\begin{array}{cc|c} 3 & -2 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right) \rightarrow 3x - 2y = 2$$

$$3x = 2 + 2y$$

$$x = \frac{2+2y}{3}$$

2 inc - 1 ec = 1 par.
(α)

$$\boxed{\begin{array}{l} x = \frac{2+2\alpha}{3} \\ y = \alpha \\ (\alpha \in \mathbb{R}) \end{array}}$$

28 Discutir según los valores a, $b \in \mathbb{R}$ el sistema:

$$\left.\begin{aligned} x + ay + bz &= a \\ x + by + az &= 0 \\ 3y + 2z &= 1 \end{aligned}\right\}$$

Estudiamos rg(A) y rg(A*) por menores:

$$|A| = \begin{vmatrix} 1 & a & b \\ 1 & b & a \\ 0 & 3 & 2 \end{vmatrix} = 2b + 3b - (3a + 2a) = 5b - 5a$$

$$5b - 5a = 0 \longrightarrow b - a = 0 \longrightarrow a = b$$

- Si $a \neq b$: $rg(A) = 3 = rg(A^*) = n \longrightarrow \boxed{SCD}$

- Si $a = b$: $rg(A) < 3 \longrightarrow rg(A) = 2$

$$A = \begin{pmatrix} 1 & a & a \\ 1 & a & a \\ 0 & 3 & 2 \end{pmatrix} \longrightarrow \begin{array}{c|cc} & C_1 & C_2 \\ \hline F_2 & 1 & a \\ F_3 & 0 & 3 \end{array} \quad \begin{vmatrix} 1 & a \\ 0 & 3 \end{vmatrix} = 3 \neq 0$$

$$A^* = \left(\begin{array}{ccc|c} 1 & a & a & a \\ 1 & a & a & 0 \\ 0 & 3 & 2 & 1 \end{array}\right) \longrightarrow \begin{array}{c|ccc} & C_1 & C_2 & C_4 \\ \hline F_1 & 1 & a & a \\ F_2 & 1 & a & 0 \\ F_3 & 0 & 3 & 1 \end{array} \quad \begin{vmatrix} 1 & a & a \\ 1 & a & 0 \\ 0 & 3 & 1 \end{vmatrix} = a + 3a - a = 3a$$

$$3a = 0$$

$$a = 0$$

$$\begin{cases} * \text{ Si } a \neq 0 \rightarrow rg(A^*) = 3 \neq rg(A) \longrightarrow \boxed{SI} \\[2ex] * \text{ Si } a = 0 \rightarrow rg(A^*) = 2 = rg(A) < n \rightarrow \boxed{SCI} \end{cases}$$

29 Discutir y resolver el siguiente sistema según los parámetros $a,\ b \in \mathbb{R}$:

$$\left.\begin{array}{r} ax + y + z = 1 \\ ax + ay + z = b \\ ax + ay + az = b \\ y + (b+1)\,z = 1 \end{array}\right\}$$

Estudiamos $rg(A)$ y $rg(A^*)$ por Gauss:

$$A^* = \left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ a & a & 1 & b \\ a & a & a & b \\ 0 & 1 & b+1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 0 & a-1 & 0 & b-1 \\ 0 & a-1 & a-1 & b-1 \\ 0 & 1 & b+1 & 1 \end{array}\right)$$

$F_2 \to F_2 - F_1$

$F_3 \to F_3 - F_1$

$$\underset{F_3 \to F_3 - F_2}{\sim} \left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 0 & a-1 & 0 & b-1 \\ 0 & 0 & a-1 & 0 \\ 0 & 1 & b+1 & 1 \end{array}\right) \underset{F_2 \leftrightarrow F_4}{\sim} \left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 0 & 1 & b+1 & 1 \\ 0 & 0 & a-1 & 0 \\ 0 & a-1 & 0 & b-1 \end{array}\right)$$

$$\underset{F_4 \to F_4 - (a-1)F_2}{\sim} \left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 0 & 1 & b+1 & 1 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & -(a-1)\cdot(b+1) & b-a \end{array}\right) \sim \left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 0 & 1 & b+1 & 1 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & b-a \end{array}\right)$$

$F_4 \to F_4 + (b+1)F_3$

- Si $b - a \neq 0 \to a \neq b$: $rg(A) \neq rg(A^*)$ $\boxed{SI}$

• si $b - a = 0 \longrightarrow a = b$

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 0 & 1 & a+1 & 1 \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

* si $a - 1 = 0 \longrightarrow a = 1$:

$$\begin{array}{c} \begin{array}{cccc} x & y & z & T.I \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$rg(A) = rg(A^*) = 2 < n \rightarrow \boxed{SCI}$ 1 par (α)

$$\left.\begin{array}{r} x + y + z = 1 \\ y + 2z = 1 \end{array}\right\} \begin{array}{l} \rightarrow x = 1 - y - z = 1 - 1 + 2z - z = z \\ \rightarrow y = 1 - 2z \end{array}$$

$$\boxed{\begin{array}{l} x = \alpha \\ y = 1 - 2\alpha \qquad (\alpha \in \mathbb{R}) \\ z = \alpha \end{array}}$$

* si $a = 0$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right) \rightarrow rg(A) = rg(A^*) = 2 < n$$

$\boxed{SCI}$

$$\left.\begin{array}{r} y + z = 1 \\ -z = 0 \end{array}\right\} \begin{array}{l} \rightarrow y = 1 \\ \rightarrow z = 0 \end{array}$$

$x \in \mathbb{R}$

$\xrightarrow{}$ 1 par

$$\boxed{\begin{array}{l} x = \alpha \\ y = 1 \qquad (\alpha \in \mathbb{R}) \\ z = 0 \end{array}}$$

$*$ si $a \neq 0$ y $a \neq 1$:

$$\left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 0 & 1 & a+1 & 1 \\ 0 & 0 & a-1 & 0 \end{array}\right) \quad rg(A) = rg(A^*) = 3 = n$$

$$\boxed{SCD}$$

$$\left.\begin{array}{r} ax + y + z = 1 \\ y + (a+1)z = 1 \\ (a-1)z = 0 \end{array}\right\} \begin{array}{l} \to ax + 1 = 1 \to ax = 0 \to \boxed{x = 0} \\ \to \boxed{y = 1} \\ \to \boxed{z = 0} \end{array}$$

30 Estudiar según los valores de a y b el siguiente sistema:

$$\left.\begin{array}{r} ax+y+z+t=1 \\ x+ay+z+t=b \\ x+y+az+t=b^2 \\ x+y+z+t=b^3 \end{array}\right\}$$

Estudiamos rg(A) y rg(A*) por menores :

$$|A| = \begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} \underset{\uparrow}{=} \begin{vmatrix} a & 1-a & 1-a & 1-a \\ 1 & a-1 & 0 & 0 \\ 1 & 0 & a-1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} =$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 - C_1$

$C_4 \to C_4 - C_1$

$$= -1 \cdot \begin{vmatrix} 1-a & 1-a & 1-a \\ a-1 & 0 & 0 \\ 0 & a-1 & 0 \end{vmatrix} = -\left[(\overset{-(a-1)}{1-a}) \cdot (a-1)^2 \right] = (a-1)^3$$

$(a-1)^3 = 0 \longrightarrow a-1 = 0 \longrightarrow a = 1$

- Si $a \neq 1$: $rg(A) = 4 = rg(A^*) = n$ $(b \in \mathbb{R})$ $\boxed{SCD}$

  ↳ no puede ser > 4

- Si $a = 1$: $rg(A) = 1$ $(b \in \mathbb{R})$

$$A^* = \left(\underbrace{\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}}_{A} \left|\, \begin{matrix} 1 \\ b \\ b^2 \\ b^3 \end{matrix}\right.\right) \longrightarrow \begin{array}{c|cc|} & C_4 & C_5 \\ F_1 & 1 & 1 \\ F_2 & 1 & b \end{array} = b-1$$

$b-1 = 0 \longrightarrow b = 1$

* si $b \neq 1 \longrightarrow rg(A^*) = 2 \neq rg(A) = 1 \longrightarrow \boxed{SI}$

Todos los menores orden $3 = 0$

* si $b = 1 \longrightarrow rg(A^*) = 1 = rg(A) < n \longrightarrow \boxed{SCI}$

Todas las filas de A^* son iguales

31 Calcular a y b para que el sistema homogéneo tenga solución no trivial:

$$\left.\begin{aligned} x - ay + z &= 0 \\ x - y - z &= 0 \\ 2x - y - bz &= 0 \\ y + z &= 0 \end{aligned}\right\}$$

Un sistema homogéneo tendrá solución no trivial (SCI) si : $rg(A) < n$

$$A = \begin{pmatrix} 1 & -a & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -b \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \text{queremos que } rg(A) < 3$$

No tienen param.

$$\begin{array}{c|cc|} & C_1 & C_3 \\ F_2 & 1 & -1 \\ F_4 & 0 & 1 \end{array} = 1 \neq 0 \rightarrow rg(A) \geq 2 \text{ para todo } a, b \in \mathbb{R}.$$

Todos los menores de orden 3 construidos sobre el menor anterior deben dar 0.

Podemos añadir F_1 o F_3 y C_2 :

$$\begin{array}{c|ccc|} & C_1 & C_2 & C_3 \\ F_1 & 1 & -a & 1 \\ F_2 & 1 & -1 & -1 \\ F_4 & 0 & 1 & 1 \end{array} = \cancel{-1+1} - (-1-a) = 1 + a = 0 \rightarrow \boxed{a = -1}$$

$$\begin{array}{c|ccc|} & C_1 & C_2 & C_3 \\ F_2 & 1 & -1 & -1 \\ F_3 & 2 & -1 & -b \\ F_4 & 0 & 1 & 1 \end{array} = -1 - 2 - (-b - 2) = -3 + b + 2 = 0 \rightarrow \boxed{b = 1}$$

Si $a = -1$ y $b = 1$: $rg(A) = 2 < n$

- El sistema tendrá sol. no trivial (SCI) cuando $a = -1$ y $b = 1$.

32 Encontrar la factorización LU de las siguientes matrices:

a) $A = \begin{pmatrix} 1 & -3 & 1 \\ 3 & -8 & 5 \\ 1 & -5 & -2 \end{pmatrix}$ **b)** $A = \begin{pmatrix} 2 & 3 & 2 & 4 \\ 4 & 10 & -4 & 0 \\ -3 & -2 & -5 & -2 \\ -2 & 4 & 4 & -7 \end{pmatrix}$

a) $$A = L \cdot U = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}_{U}$$

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 3 & -8 & 5 \\ 1 & -5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = U$$

$F_2 \to F_2 - 3F_1$ $F_3 \to F_3 + 2F_2$

$F_3 \to F_3 - F_1$

b) $$A = L \cdot U = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -\frac{3}{2} & \frac{5}{8} & 1 & 0 \\ -1 & \frac{7}{4} & \frac{20}{3} & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 2 & 3 & 2 & 4 \\ 0 & 4 & -8 & -8 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & -49 \end{pmatrix}}_{U}$$

$$A = \begin{pmatrix} 2 & 3 & 2 & 4 \\ 4 & 10 & -4 & 0 \\ -3 & -2 & -5 & -2 \\ -2 & 4 & 4 & -7 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 2 & 4 \\ 0 & 4 & -8 & -8 \\ 0 & 5/2 & -2 & 4 \\ 0 & 7 & 6 & -3 \end{pmatrix}$$

$F_2 \to F_2 - 2F_1$

$F_3 \to F_3 + \frac{3}{2}F_1$

$F_4 \to F_4 + F_1$

$$\sim \begin{pmatrix} 2 & 3 & 2 & 4 \\ 0 & 4 & -8 & -8 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 20 & 11 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 2 & 4 \\ 0 & 4 & -8 & -8 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & -49 \end{pmatrix} = U$$

$F_3 \to F_3 - \frac{5}{8} F_2$

$F_4 \to F_4 - \frac{7}{4} F_2$

$F_4 \to F_4 - \frac{20}{3} F_3$