1 Estudiar si son lineales las siguientes aplicaciones:

a) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (z, x + y, -z)$

1) $f(\vec{0}) = \vec{0}$ $\quad f(\underset{x}{0}, \underset{y}{0}, \underset{z}{0}) = (0, 0+0, -0) = (0, 0, 0)$ ✓

2) $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$ ✓

$\vec{u} = (x_1, y_1, z_1) \qquad \vec{v} = (x_2, y_2, z_2)$

$f(\vec{u} + \vec{v}) = f\big((x_1, y_1, z_1) + (x_2, y_2, z_2)\big) =$

$= f(\underbrace{x_1 + x_2}_{x}, \underbrace{y_1 + y_2}_{y}, \underbrace{z_1 + z_2}_{z}) =$

$= (z_1 + z_2, x_1 + x_2 + y_1 + y_2, -z_1 - z_2) \leftarrow$

$f(\vec{u}) + f(\vec{v}) = f(x_1, y_1, z_1) + f(x_2, y_2, z_2) =$

$= (z_1, x_1 + y_1, -z_1) + (z_2, x_2 + y_2, -z_2) =$

$= (z_1 + z_2, x_1 + x_2 + y_1 + y_2, -z_1 - z_2) \leftarrow$

3) $f(\alpha\vec{u}) = \alpha f(\vec{u})$

$$f(\alpha\vec{u}) = f(\underbrace{\alpha x_1}_{x}, \underbrace{\alpha y_1}_{y}, \underbrace{\alpha z_1}_{z}) = (\alpha z_1, \alpha x_1 + \alpha y_1, -\alpha z_1)$$

$$\alpha f(\vec{u}) = \alpha \cdot (z_1, x_1 + y_1, -z_1) = (\alpha z_1, \alpha x_1 + \alpha y_1, -\alpha z_1)$$

$\boxed{f \text{ es aplicación lineal}}$

3 Sea $f : \mathbb{R}^4 \to \mathbb{R}^3$ la aplicación lineal definida por:

$$f(x, y, z, t) = (x - y + z + t, 2x - 2y + 3z + 4t, 3x - 3y + 4z + 5t)$$

a) Hallar una base y la dimensión de $\operatorname{Im} f$.

b) Hallar una base y la dimensión de $\operatorname{Ker} f$.

a) Imagen de f :

$$f(\underset{x}{1}, \underset{y}{0}, \underset{z}{0}, \underset{t}{0}) = (1, 2, 3)$$

$$f(0, \underset{y}{1}, 0, 0) = (-1, -2, -3)$$

$$f(0, 0, \underset{z}{1}, 0) = (1, 3, 4)$$

$$f(0, 0, 0, \underset{t}{1}) = (1, 4, 5)$$

$$\operatorname{Im} f = L\{(1,2,3), (-1,-2,-3), (1,3,4), (1,4,5)\}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$F_2 \to F_2 + F_1$

$F_3 \to F_3 - F_1$

$F_4 \to F_4 - F_1$

$F_4 \to F_4 - 2F_3$

$$\boxed{\mathcal{B}_{\operatorname{Im} f} = \{(1,2,3), (0,1,1)\} \qquad \dim(\operatorname{Im} f) = 2}$$

b) Núcleo de f:

$$\left.\begin{array}{l} x-y+z+t=0 \\ 2x-2y+3z+4t=0 \\ 3x-3y+4z+5t=0 \end{array}\right\} \quad \begin{array}{cccc} x & y & z & t \\ \end{array}$$
$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 3 & 4 \\ 3 & -3 & 4 & 5 \end{pmatrix} \sim \begin{array}{l} F_2 \to F_2 - 2F_1 \\ F_3 \to F_3 - 3F_1 \end{array}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{2} \end{pmatrix} \quad \left.\begin{array}{r} x-y+z+t=0 \\ z+2t=0 \end{array}\right\} \longrightarrow z=-2t$$

S.C.I → 4inc - 2ec = 2 par

$$x-y-2t+t=0 \to x=y+t$$

$$\left\{\begin{array}{l} x=\alpha+\beta \\ y=\alpha \\ z=-2\beta \\ t=\beta \end{array}\right. \quad (\alpha,\beta\in\mathbb{R}) \to (x,y,z,t)=\alpha(1,1,0,0)+\beta(1,0,-2,1)$$

$$\operatorname{Ker} f = L\{(1,1,0,0),(1,0,-2,1)\}$$

$$\boxed{B_{\operatorname{Ker} f}=\{(1,1,0,0),(1,0,-2,1)\} \qquad \dim(\operatorname{Ker} f)=2}$$

$$\dim(\operatorname{Ker} f)=\dim(\underset{\mathbb{R}^4}{\underset{\shortparallel}{V}})-\dim(\operatorname{Im} f)=4-2=2 \checkmark$$

8 Dada la aplicación lineal $f : \mathbb{R}^3 \to \mathbb{R}^4$ tal que:

$$
\begin{aligned}
f(1,1,0) &= (1,0,0,1)\\
f(2,1,0) &= (0,1,0,1)\\
f(0,-1,1) &= (0,0,0,0)
\end{aligned}
$$

Determinar la imagen del vector $\vec{v} = \left(-\frac{5}{3}, -2, 1\right)$.

Comprobamos si los vectores $(1,1,0)$, $(2,1,0)$ y $(0,-1,1)$ forman una base de $\mathbb{R}^3$:

$$\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = -1 \neq 0 \longrightarrow f \text{ existe y es única}$$

son base de $\mathbb{R}^3$ $\qquad B = \{(1,1,0),(2,1,0),(0,-1,1)\}$

Expresamos $\vec{v}$ como C. L de B:

$$\left(-\frac{5}{3}, -2, 1\right) = \alpha(1,1,0) + \beta(2,1,0) + \gamma(0,-1,1)$$

$$
\begin{aligned}
-\frac{5}{3} &= \alpha + 2\beta \\
-2 &= \alpha + \beta - \gamma \\
1 &= \gamma
\end{aligned}
\quad
\left\{
\begin{aligned}
&\alpha + 2\beta = -\frac{5}{3} \\
\to\ &\alpha + \beta = -1 \\
\to\ &\underline{\gamma = 1}
\end{aligned}
\right.
\quad \ominus \quad \beta = -\frac{5}{3} + 1 = \underline{\frac{-2}{3}}
$$

$$\alpha = -1 - \beta = -1 + \frac{2}{3} = \underline{\frac{-1}{3}}$$

$$\left(-\frac{5}{3}, -2, 1\right) = -\frac{1}{3}(1,1,0) - \frac{2}{3}(2,1,0) + 1\cdot(0,-1,1)$$

$$f(\vec{v}) = f\left(-\frac{5}{3}, -2, 1\right) = -\frac{1}{3} f(1,1,0) - \frac{2}{3} f(2,1,0) + f(0,-1,1)$$

$$= -\frac{1}{3}\cdot(1,0,0,1) - \frac{2}{3}(0,1,0,1) + (0,0,0,0) =$$

$$= \boxed{\left(-\frac{1}{3}, -\frac{2}{3}, 0, -1\right)}$$

10 Dada la aplicación lineal $f : \mathbb{R}^3 \to \mathbb{R}^2$ definida por:

$$f(x, y, z) = (2x + y + z, x + y + z)$$

a) Calcular unas bases de los subespacios núcleo e imagen.

b) Determinar si es inyectiva, suprayectiva o biyectiva.

c) Determinar los subespacios $f(U)$ y $f(W)$, siendo:

$$U = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + y = 0\}$$
$$W = \{(x, y, z) \in \mathbb{R}^3 \,/\, x + y + z = 0\}$$

d) Determinar la imagen inversa $f^{-1}(S)$, siendo $S = L\{(2, 1)\}$.

e) Calcular la matriz asociada a f respecto de las bases canónicas:

$$C_3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \text{ y } C_2 = \{(1, 0), (0, 1)\}$$

f) Calcular la imagen del vector $\vec{v} = (0, 3, -3)$ utilizando la matriz del apartado anterior.

a)
$$\left.\begin{aligned} 2x + y + z &= 0 \\ \ominus \quad x + y + z &= 0 \end{aligned}\right\} \to y + z = 0 \to \boxed{y = -z}$$

$$x \quad \cancel{} \quad = 0 \to \boxed{x = 0}$$

S.C.I : 1 par.

$$\left.\begin{aligned} x &= 0 \\ y &= -\alpha \\ z &= \alpha \end{aligned}\right\} \to (x, y, z) = \alpha\,\underset{L.I}{(0, -1, 1)}$$

$$\boxed{B_{\ker f} = \{(0, -1, 1)\}} \qquad \dim(\ker f) = 1$$

$$f(\underset{x}{1, 0, 0}) = (2, 1) \qquad f(\underset{y}{0, 1, 0}) = (1, 1) \qquad f(\underset{z}{0, 0, 1}) = (1, 1)$$

$$\operatorname{Im} f = L\{\underbrace{(2,1),(1,1)}_{L.I},\cancel{(1,1)}\}$$

$$\boxed{B_{\operatorname{Im} f} = \{(2,1),(1,1)\}} \qquad \dim(\operatorname{Im} f) = 2$$

b) Como $\dim(\operatorname{Ker} f) = 1 \neq 0 \rightarrow$ No inyectiva ni biyectiva.

Como $\dim(\operatorname{Im} f) = 2 \overset{?}{=} \dim(\underset{\mathbb{R}^2}{\overset{\shortparallel}{V'}}) = 2$

$\boxed{f \text{ es suprayectiva}}$

c) Calculamos las bases de U y W:

- $x + y = 0 \longrightarrow y = -x \longrightarrow \begin{cases} x = \alpha \\ y = -\alpha \\ z = \beta \end{cases} \quad (\alpha, \beta \in \mathbb{R})$

$z \in \mathbb{R}$

Nº par: 3 inc - 1 ec

$= 2$ par

$$(x,y,z) = \alpha(1,-1,0) + \beta(0,0,1)$$

L.I s. gen de U

$$B_U = \{(1,-1,0),(0,0,1)\}$$

$\cdot\ x+y+z=0 \longrightarrow x=-y-z$

$n^{\underline{o}}$ par : 3 inc − 1 ec = 2 par

$$\left.\begin{aligned} x &= -\alpha-\beta \\ y &= \alpha \\ z &= \beta \end{aligned}\right\}$$

$$(x,y,z)=\alpha(-1,1,0)+\beta(-1,0,1)$$

$$B_W=\{(-1,1,0),(-1,0,1)\}$$

$$f(x,y,z)=(2x+y+z,\ x+y+z)$$

$$f(U)=L\{f(\underset{x}{1},\underset{y}{-1},\underset{z}{0}),f(0,0,1)\}=\boxed{L\{(1,0),(1,1)\}}$$

imágenes de Bu

$$f(W)=L\{f(\underset{x}{-1},\underset{y}{1},\underset{z}{0}),f(-1,0,1)\}=L\{(-1,0),\cancel{(-1,0)}\}=$$

imágenes de Bw

$$=\boxed{L\{(-1,0)\}}$$

d) $\overset{-1}{f}(S)=\{(x,y,z)\in\mathbb{R}^3 / f(x,y,z)\in S\}$ $\quad S=L\{(2,1)\}$

$$(2x+y+z,\ x+y+z)=\alpha(2,1)$$

$$\left.\begin{array}{l} 2x+y+z = 2\underline{\underline{\alpha}} \\ x+y+z = \alpha \end{array}\right\} \rightarrow \cancel{2x}+y+z = \cancel{2x}+2y+2z$$

$$y+z=0$$

ec. imp. de $f^{-1}(S)$.

$$\boxed{f^{-1}(S) = \{(x,y,z) \in \mathbb{R}^3 \,/\, y+z=0\}}$$

e) $$A = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{2\times 3} \end{array}$$

$\mathbb{R}^2 \leftarrow \mathbb{R}^3$

f) $$A \cdot \vec{v}^{\,t} = f(\vec{v})^t \rightarrow \underset{2\times \underline{\underline{3}}}{\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \cdot \underset{\underline{\underline{3}}\times 1}{\begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

en columnas

$$\vec{v} = (0,3,-3)$$

$$\boxed{f(\vec{v}) = f(0,3,-3) = (0,0)}$$

12 Sea $f : \mathbb{R}^3 \to \mathbb{R}^2$ la aplicación lineal cuya matriz asociada en bases canónicas, C_3 y C_2, es:

$$A = \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}$$

Calcular las matrices asociadas a f en las bases:

a) C_3 canónica de $\mathbb{R}^3$ y $B_2 = \{(2,1), (1,1)\}$ de $\mathbb{R}^2$.

b) $B_3 = \{(1,1,1), (1,1,0), (1,0,0)\}$ de $\mathbb{R}^3$ y C_2 canónica de $\mathbb{R}^2$.

c) B_3 y B_2.

a) $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$

$$\begin{tikzcd}
\vec{v} & f(\vec{v}) \\
C_3 \arrow[r, "A"] & C_2 \\
B \arrow[u, "P"] \arrow[r, "A'"] & B' \arrow[u, "\varphi"'] \\
\vec{v}_B & f(\vec{v}_B)_{B'}
\end{tikzcd}
\qquad
\begin{tikzcd}
C_3 \arrow[r, "A"] \arrow[dr, "A'"'] & C_2 \arrow[d, dotted, bend right, "\varphi^{-1}"'] \\
& B_2 \arrow[u, "\varphi"']
\end{tikzcd}$$

$$A' = Q^{-1} \cdot A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix} =$$

multiplicamos en orden inverso.

$\varphi \equiv$ vectores de B_2 en col.

$$= \frac{1}{1} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 9 & -10 \\ 0 & -13 & 17 \end{pmatrix}$$

$\to$ mismo tamaño que A.

2×3

b)

$$\begin{tikzcd} C_3 \arrow[r, "A"] & C_2 \\ B_3 \arrow[u, "P"] \arrow[ur, "A'"'] & \end{tikzcd}$$

$$A' = A \cdot P =$$

al revés !

$$= \underset{2\times 3}{\begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}} \cdot \underset{3\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} = \underline{\underline{\begin{pmatrix} 4 & 7 & 2 \\ 4 & -3 & 1 \end{pmatrix}_{2\times 3}}}$$

$P \equiv$ vectores

de B_3 en col.

c)

$$\begin{tikzcd} C_3 \arrow[r, "A"] & C_2 \arrow[d, dashed, shift right, "Q^{-1}"'] \\ B_3 \arrow[u, "P"] \arrow[r, "A'"] & B_2 \arrow[u, shift right, "Q"'] \end{tikzcd}$$

$$A' = Q^{-1} \cdot A \cdot P =$$

$$= \underset{2\times 2}{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}} \cdot \underset{2\times 3}{\begin{pmatrix} 2 & 5 & -3 \\ 1 & -4 & 7 \end{pmatrix}} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} =$$

$$= \underset{2\times 3}{\begin{pmatrix} 1 & 9 & -10 \\ 0 & -13 & 17 \end{pmatrix}} \cdot \underset{3\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 0 & 10 & 1 \\ 4 & -13 & 0 \end{pmatrix}}$$