Ejercicios

24. Utilizando la factorización LU del ejercicio 23b, obtener $|A|$ y resolver el sistema:

$$\left.\begin{aligned} x-3y+5z&=4\\ -4y+7z&=1\\ -x-2y+z&=0 \end{aligned}\right\}$$

23) b) $A=\begin{pmatrix}1&-3&5\\0&-4&7\\-1&-2&1\end{pmatrix}=L\cdot U=\underset{L}{\begin{pmatrix}1&0&0\\0&1&0\\-1&\frac{5}{4}&1\end{pmatrix}}\cdot\underset{U}{\begin{pmatrix}1&-3&5\\0&-4&7\\0&0&-\frac{11}{4}\end{pmatrix}}$

$$|A|=|U|\underset{\uparrow}{=}1\cdot(-4)\cdot\left(\frac{-11}{4}\right)=\boxed{11}$$

U triang.

$$AX=B\longrightarrow\underset{A}{\begin{pmatrix}1&-3&5\\0&-4&7\\-1&-2&1\end{pmatrix}}\cdot\underset{X}{\begin{pmatrix}x\\y\\z\end{pmatrix}}=\underset{B}{\begin{pmatrix}4\\1\\0\end{pmatrix}}$$

$$AX=B\longrightarrow L\underbrace{UX}_{Z}=B\longrightarrow LZ=B$$

$$\underset{L}{\begin{pmatrix}1&0&0\\0&1&0\\-1&\frac{5}{4}&1\end{pmatrix}}\cdot\underset{Z}{\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}}=\begin{pmatrix}4\\1\\0\end{pmatrix}\longrightarrow\left.\begin{aligned}&\boxed{z_1=4}\\&\boxed{z_2=1}\\&-z_1+\frac{5}{4}z_2+z_3=0\\&\qquad\downarrow\\&\boxed{z_3=\frac{11}{4}}\end{aligned}\right\}$$

$$\underset{\checkmark}{U}\,\underset{?}{X}=\underset{\checkmark}{Z}$$

$$\underbrace{\begin{pmatrix} 1 & -3 & 5 \\ 0 & -4 & 7 \\ 0 & 0 & -\frac{11}{4} \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \begin{pmatrix} 4 \\ 1 \\ \frac{11}{4} \end{pmatrix} \longrightarrow \left.\begin{aligned} x - 3y + 5z &= 4 \\ -4y + 7z &= 1 \\ -\tfrac{11}{4} z &= \tfrac{11}{4} \end{aligned}\right\}$$

$$\boxed{z = -1}$$

$$\boxed{y = -2}$$

$$\boxed{x = 3}$$

Ejercicios

25 Calcular la factorización de Cholesky de las matrices:

a) $\begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix}$ **b)** $\begin{pmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & -1 & -2 & 3 \\ 1 & -2 & 0 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$

a) A es simétrica ✓

$A_1 = 1 > 0 \quad A_2 = |A| = 4 > 0 \rightarrow$ A es definida positiva. ✓

Con fact. LU :

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} = U$$

$F_2 \rightarrow F_2 + F_1$

$$Q = L \cdot D^{\frac{1}{2}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}}_{D^{\frac{1}{2}}} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

$$A = Q \cdot Q^t = \underline{\underline{\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}}}$$

Con algoritmo: $A = \begin{pmatrix} 1 & -1 \\ -1 & 5 \end{pmatrix}$ $\quad Q = \begin{vmatrix} q_{11} & 0 \\ q_{21} & q_{22} \end{vmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

$q_{11} = \sqrt{a_{11}} = \sqrt{1} = 1$

$q_{21} = \frac{a_{21}}{q_{11}} = \frac{-1}{1} = -1 \qquad q_{22} = \sqrt{a_{22} - q_{21}^2} = \sqrt{5-(-1)^2} = \sqrt{4} = 2$

b) A es simétrica ✓

$A_1 = 4 > 0 \quad A_2 = 64 > 0 \quad A_3 = 1600 > 0 \rightarrow$ A es d.p ✓

Con fact. LU:

$$A = \begin{pmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 14 \\ 0 & 16 & -12 \\ 0 & -12 & 34 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 14 \\ 0 & 16 & -12 \\ 0 & 0 & 25 \end{pmatrix} = U$$

$F_2 \rightarrow F_2 - \frac{1}{2} F_1 \qquad F_3 \rightarrow F_3 + \frac{3}{4} F_2$

$\frac{14}{4} = \frac{7}{2} \qquad F_3 \rightarrow F_3 - \frac{7}{2} F_1$

$\frac{12}{16} = \frac{3}{4}$

$$Q = L \cdot D^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{7}{2} & -\frac{3}{4} & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix}$$

$$A = Q \cdot Q^t = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix}$$

• Con algoritmo: $A = \begin{pmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix}$ $Q = \begin{pmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$

$$q_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$$q_{21} = \frac{a_{21}}{q_{11}} = \frac{2}{2} = 1 \qquad q_{22} = \sqrt{a_{22} - q_{21}^2} = \sqrt{17 - 1^2} = 4$$

$$q_{31} = \frac{a_{31}}{q_{11}} = \frac{14}{2} = 7 \qquad q_{32} = \frac{a_{32} - q_{31} \cdot q_{21}}{q_{22}} = \frac{-5-7 \cdot 1}{4} = -3$$

$$q_{33} = \sqrt{a_{33} - q_{32}^2 - q_{31}^2} = \sqrt{83 - (-3)^2 - 7^2} = \sqrt{25} = 5$$

c) A es simétrica ✓

$A_1 = 2 > 0$ $A_2 = -2 < 0$ ✗ $\rightarrow$ A no es d.p

La fact. de Cholesky no existe.

Ejercicios

26 Utilizando la factorización de Cholesky del ejercicio 25b, obtener $|A|$ y resolver el sistema:

$$\left.\begin{array}{r} 4x + 2y + 14z = 14 \\ 2x + 17y - 5z = -101 \\ 14x - 5y + 83z = 155 \end{array}\right\}$$

25 **b)** $A = Q \cdot Q^t = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix}$

$$|A| = |Q|^2 = (2 \cdot 4 \cdot 5)^2 = (40)^2 = \boxed{1600}$$

$\uparrow$

Q triangular

$$AX = B \longrightarrow Q \cdot \underbrace{Q^t \cdot X}_{Z} = B \longrightarrow Q \cdot \underset{?}{Z} = B$$

$$\underset{Q}{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix}} \cdot \underset{Z}{\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}} = \begin{pmatrix} 14 \\ -101 \\ 155 \end{pmatrix} \longrightarrow \left.\begin{array}{l} 2z_1 = 14 \\ z_1 + 4z_2 = -101 \\ 7z_1 - 3z_2 + 5z_3 = 155 \end{array}\right\}$$

$z_1 = 7$

$z_2 = -27$

$z_3 = 5$

$$Q^t \cdot X = Z \longrightarrow \begin{pmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -27 \\ 5 \end{pmatrix}$$

$$\left.\begin{aligned} 2x + y + 7z &= 7 \\ 4y - 3z &= -27 \\ 5z &= 5 \end{aligned}\right\} \begin{aligned} &\longrightarrow \boxed{x = 3} \\ &\longrightarrow \boxed{y = -6} \\ &\longrightarrow \boxed{z = 1} \end{aligned}$$

Ejercicio ¿El conjunto de los números enteros $\underbrace{\mathbb{Z}}_{V}$ con las operaciones estándar es un e.v?

- Objeto o vectores de $\mathbb{Z}$: n^{os} enteros $(a, b, c, \dots)$.

① $a + b \in \mathbb{Z}$? ✓

⋮

⑤ ✓

⑥ $\alpha \cdot a \notin \mathbb{Z}$? $(\alpha \in \mathbb{R})$ ✗

$$\sqrt{2} \cdot 2 = 2\sqrt{2} \notin \mathbb{Z}$$

$\boxed{\text{No es e.v}}$

Ejercicio ¿El conjunto de polinomios de grado 2 con las operaciones usuales de polinomios es un e.v?

- Objetos o vectores V : pol. grado 2 $(P(x), Q(x), \dots)$

$$P(x) = ax^2 + bx + c \quad (a, b, c \in \mathbb{R})$$

① $\underbrace{P(x) + Q(x)}_{\text{pol. gr } 2} \in V$?

$$\left.\begin{array}{l} P(x) = x^2 \\ Q(x) = -x^2 + x + 1 \end{array}\right\} \quad P(x) + Q(x) = \cancel{x^2} - \cancel{x^2} + \underbrace{x + 1}_{\text{gr. } 1} \notin V$$

$\boxed{\text{No es e.v}}$

$\boxed{\underline{\text{Ejercicio}} : \text{Verificar que el conjunto } U = \{(x,y,z) \in \mathbb{R}^3 / x+y+z=0\} \text{ es un subespacio vectorial de } \mathbb{R}^3.}$

a) $\vec{0} = (\underset{x}{0}, \underset{y}{0}, \underset{z}{0}) \in U$? $\quad 0+0+0=0$ ✓ sí

b) Vectores de U : $\underbrace{z = -x-y}_{2 \text{ param.}} \longrightarrow (x, y, -x-y)$

$\vec{u} = (x_1, y_1, -x_1-y_1)$

$\vec{v} = (x_2, y_2, -x_2-y_2)$

$\vec{u}+\vec{v} = (x_1+x_2, y_1+y_2, -x_1-y_1-x_2-y_2) =$

$= (\underbrace{x_1+x_2}_{x}, \underbrace{y_1+y_2}_{y}, \underbrace{-(x_1+x_2)}_{-x} \underbrace{-(y_1+y_2)}_{-y}) \in U$ ✓

c) $\alpha \cdot \vec{u} \in U$

$\alpha \cdot \vec{u} = \alpha(x_1, y_1, -x_1-y_1) = (\underbrace{\alpha x_1}_{x}, \underbrace{\alpha y_1}_{y}, \underbrace{-\alpha x_1}_{-x} \underbrace{-\alpha y_1}_{-y}) \in U$ ✓

$\boxed{U \text{ es subespacio de } \mathbb{R}^3}$

Ejercicio: Considera el conjunto $U = \{(x,y) \in \mathbb{R}^2 / x \cdot y = 0\}$.

¿Es un subespacio de $\mathbb{R}^2$?

a) $\vec{0} = (\underset{x}{0}, \underset{y}{0}) \in U$? $\quad 0 \cdot 0 = 0$ ✓ Sí

b) Vectores de U: $x \cdot y = 0 \longrightarrow x = 0 \longrightarrow (0, y)$

$\qquad\qquad\qquad\qquad\quad \longrightarrow y = 0 \longrightarrow (x, 0)$

$$\vec{u} + \vec{v} = (0, y) + (x, 0) = (x, y) \notin U$$

$\boxed{U \text{ no es subespacio de } \mathbb{R}^2}$

$\underline{\text{Ejercicio}}$: ¿Es $U = \{(x,y) \in \mathbb{R}^2 \ / \ y = mx\}$ un subespacio de $\mathbb{R}^2$?

a) $\vec{0} = (\underset{x}{0}, \underset{y}{0}) \in U$? $\longrightarrow$ $0 = m \cdot 0$ ✓ Sí.

b) Vectores de U : $y = mx \longrightarrow (x, mx)$

$$\vec{u} + \vec{v} = (x_1, mx_1) + (x_2, mx_2) = (x_1 + x_2, mx_1 + mx_2) =$$

$$= (\underbrace{x_1 + x_2}_{x}, \underbrace{m(x_1 + x_2)}_{mx}) \in U \ ✓$$

c) $\alpha \cdot \vec{u} = \alpha \cdot (x_1, mx_1) = (\underbrace{\alpha x_1}_{x}, \underbrace{m \alpha x_1}_{mx}) \in U$ ✓

$\boxed{U \text{ es subespacio de } \mathbb{R}^2}$

$\underline{\text{Ejercicio}}$: ¿U es subespacio vectorial de $\mathbb{R}^3$?

$$U = \{(x,y,z) \in \mathbb{R}^3 \ / \ x = 2 + 4t, \ y = 1 - 2t, \ z = t, \ t \in \mathbb{R}\}$$