T2: Computational learning

Fundamentos del Aprendizaje Automático

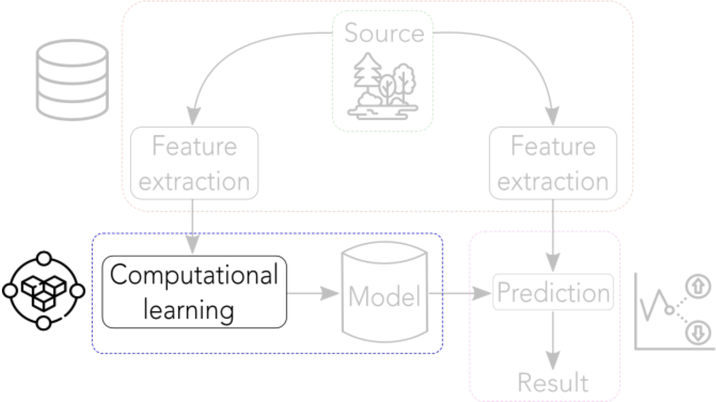
Curso 2025/2026

Structure

1. Introduction
   - Where are we?
   - Computational learning VS Decision theory
2. Bayesian decision theory
   - Two-class problem
   - General form
   - Risk
   - Discriminant functions
3. Statistical likelihood
   - Maximum likelihood estimation
4. Issues in computational learning
   - Bias-variance issues
   - Curse of dimensionality

Outline

Where are we?



Source
Feature extraction
Feature extraction
Computational learning
Model
Prediction
Result

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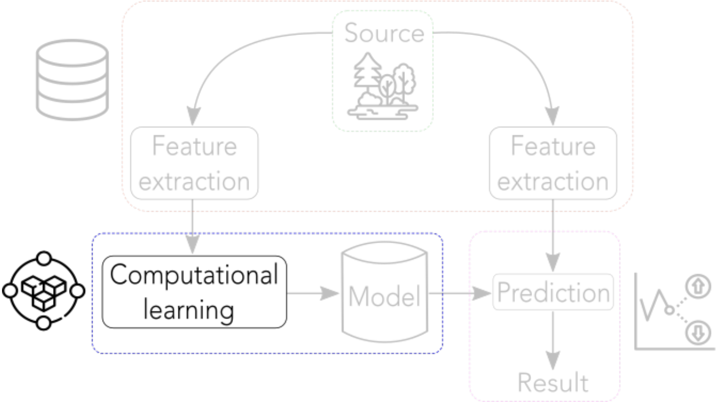

Computational learning:

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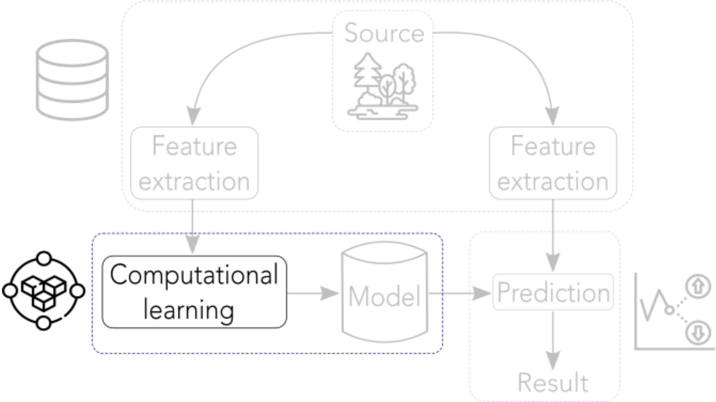

Computational learning:

- Infer knowledge from data

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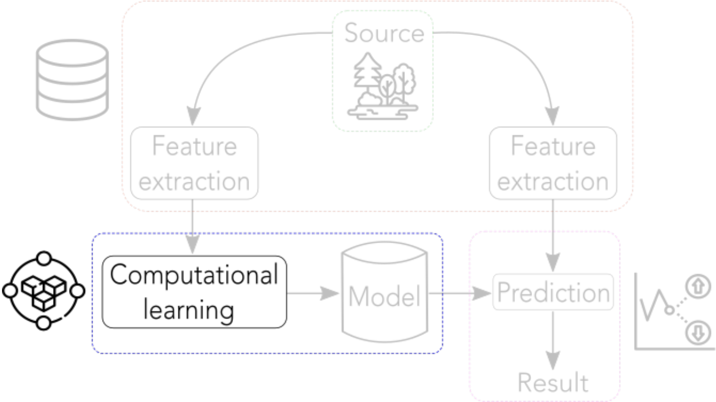

Computational learning:

- Infer knowledge from data
- Derive a model out of that knowledge

Computational learning VS Decision theory

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Two related, but different, concepts:

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- **Computational learning**: Studies the limits of learning from data

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  - Foundational mathematical framework for Machine Learning
  - Pattern classification is the most important subfield

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Bayesian decision theory (for classification):

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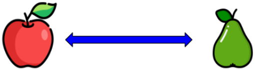
Task: 

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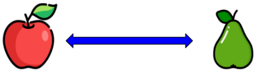

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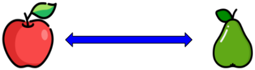
Task: 

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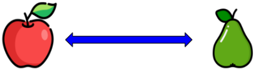
Task: 

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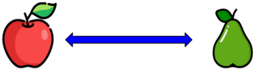
Task: 

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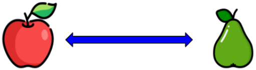
Task: 

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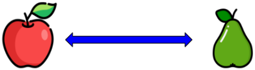
Task: 

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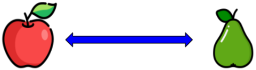
Task: 

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- Additional information may be incorporated!

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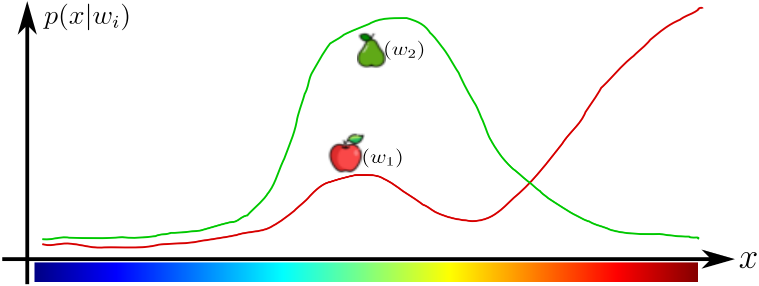

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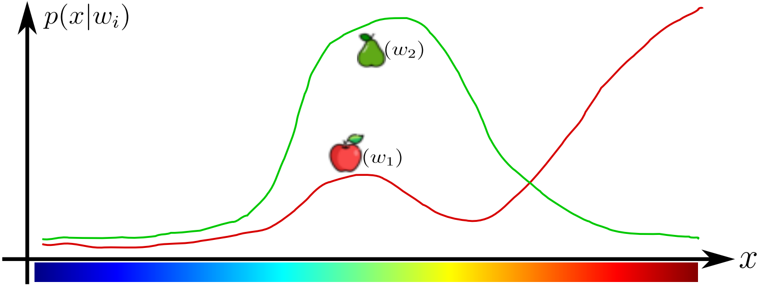

$$\int p(x|\omega_i)\, dx = 1 \quad \forall i \in \{1, 2\}$$

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$$\downarrow$$

$$p(\omega_j, x) = P(w_j|x) \cdot p(x) = p(x|\omega_j) \cdot P(\omega_j)$$

Bayes theorem

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$$P(\omega_j|x) = \frac{p(x|\omega_j)\cdot P(\omega_j)}{p(x)}$$

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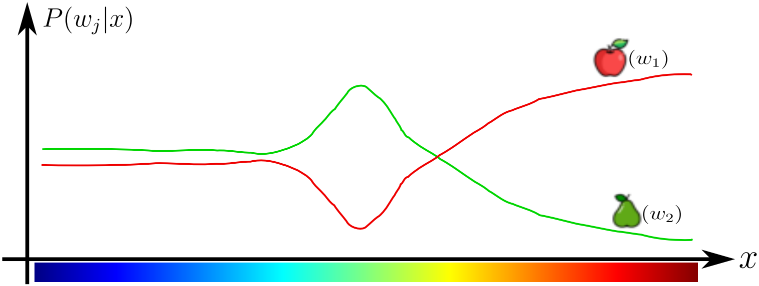
- **Evidence**: Scale factor $\Rightarrow p(x) = \sum_{\forall j} p(x|\omega_j) \cdot P(\omega_j)$

Posterior probability



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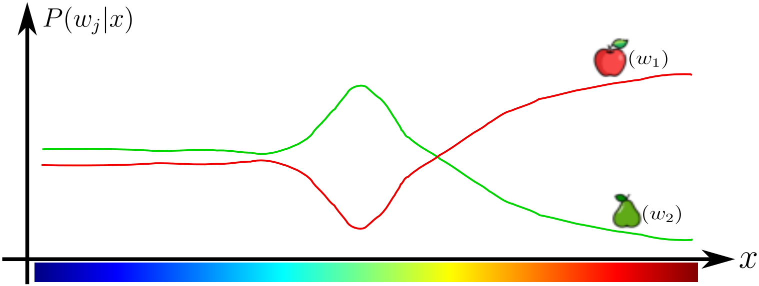

For any color $x = x_0 \longrightarrow P(\omega_1|x_0) + P(\omega_2|x_0) = 1$

Practical exercise

Estimate the probability of obtaining a particular fruit (state of nature) given *priors* and its color

1. Historically, the harvest of apples represents the 70% of the fruit, whereas that of pears is the 30%
2. The distribution of color for each fruit is:

Color ($[0, 100]$)	Blue (0)	Green (40)	Yellow (50)	Orange (70)	Red (100)
$p(x \mid w_1)$	0.05	0.25	0.15	0.1	0.45
$p(x \mid w_2)$	0.05	0.6	0.25	0.05	0.05

Q1. How likely is it to obtain a yellow apple? And a same-colored pear?

Q2. Obtain all posterior probabilities for all possible color and fruit combinations

Probability of error

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$$P(\text{error}|x_0) = \begin{cases} P(w_1|x_0) & \text{if we decide } w_2 \\ P(w_2|x_0) & \text{if we decide } w_1 \end{cases}$$

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- We can minimize this expression by selecting:
  - $\rightarrow$ w_1 when $P(w_1|x_0) > P(w_2|x_0)$
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- Will this rule minimize the average probability of error?

$$P(\text{error}) = \int_{-\infty}^{\infty} P(\text{error}, x)\, dx = \int_{-\infty}^{\infty} P(\text{error}|x) \cdot p(x)\, dx$$

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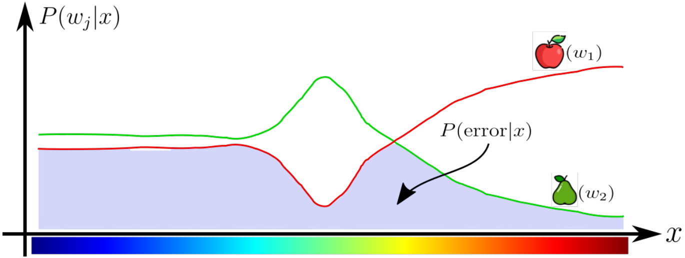

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Bayes decision rule for two-class classification

Decide w_1 if $p(x|w_1) \cdot P(w_1) > p(x|w_2) \cdot P(w_2)$;
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- The rule may be also written as:

$$w = \underset{w \in \{w_1, w_2\}}{\arg\max}\ P(w|x) = \underset{w \in \{w_1, w_2\}}{\arg\max}\ p(x|w) \cdot P(w)$$

Practical exercise

Considering the same exercise as before where

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2. The distribution of color for each fruit is:

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Q1. Which is the minimum error in each case considering the Bayes decision rule?

Q2. What would occur is priors were equiprobable? $P(w_1) = P(w_2) = 50\%$

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2) More than two states of nature
3) Actions beyond selecting labels
4) Loss function as a generalization of the probability of error

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  - $\rightarrow$ Cost of performing α_i when true state is $w_j \Rightarrow \lambda(\alpha_i | w_j)$

Definitions

- Feature space: use more than a single descriptor
  - $\rightarrow$ Fixed d-dimensional space: $\mathbf{x} \in \mathbb{R}^d$
- States of nature: allow more than two states
  - $\rightarrow$ Finite set of states: $\mathcal{W} = \left\{w_1, \ldots, w_{|\mathcal{W}|}\right\}$
- Actions: possibilitate other beyond classification
  - $\rightarrow$ Finite set of actions: $\mathcal{A} = \left\{\alpha_1, \ldots, \alpha_{|\mathcal{A}|}\right\}$ s.t. $\alpha_i : \mathbb{R}^d \rightarrow \mathcal{W}$
- Loss function: Generalization of probability of error $\Rightarrow$ errors are not equally important
  - $\rightarrow$ Cost of performing α_i when true state is $w_j \Rightarrow \lambda(\alpha_i|w_j)$

Bayes theorem (general form)

$$P(w_j|\mathbf{x}) = \frac{p(\mathbf{x}|w_j) \cdot P(w_j)}{p(\mathbf{x})} \quad \text{with} \quad p(\mathbf{x}) = \sum_{j=1}^{|\mathcal{W}|} p(\mathbf{x}|w_j) \cdot P(w_j)$$