Ejercicio: Obtener una factorización SVD de la matriz:

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\underset{3\times 2}{A} = \underset{3\times 3}{U} \cdot \underset{3\times 2}{\Sigma} \cdot \underset{2\times 2}{V^t}$$

$$A^t \cdot A = \underset{2\times 3}{\begin{pmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{pmatrix}} \cdot \underset{3\times 2}{\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}}$$

$$|A^t \cdot A - \lambda I| = 0 \longrightarrow \begin{vmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{vmatrix} = 0$$

$$(9-\lambda)^2 - 81 = 0 \longrightarrow \cancel{81} - 18\lambda + \lambda^2 - \cancel{81} = 0$$

$$\lambda(\lambda - 18) = 0 \longrightarrow \lambda = 0$$

$$\downarrow$$

$$\lambda = 18$$

$$\longrightarrow \boxed{\begin{array}{l} \lambda_1 = 18 \longrightarrow \sigma_1 = \sqrt{\underset{9\cdot 2}{18}} = 3\sqrt{2} \\ \lambda_2 = 0 \ \times \end{array}}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

• Para $\lambda_1 = 18$: $(A^t \cdot A - 18 I) \cdot \vec{v} = \vec{0}$ $\quad \begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix}$

$$\begin{pmatrix} -9 & -9 \\ -9 & -9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{array}{l} -\cancel{9}x - \cancel{9}y = 0 \end{array}\right\} \begin{array}{l} y = -x \\ \underset{\alpha}{\overset{}{x}} \in \mathbb{R} \end{array}$$

$x = \alpha$

$$V_{\lambda_1} = \{ (\alpha, -\alpha) \in \mathbb{R}^2 / \alpha \in \mathbb{R} \} \rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad |\vec{v}_1| = \sqrt{2}$$

$$\vec{v}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

• Para $\lambda_2 = 0$: $(A^t \cdot A) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} 9 & -9 \\ -9 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{array}{l} 9x - 9y = 0 \end{array}\right\} \longrightarrow \begin{array}{l} x = y \\ y \in \mathbb{R} \end{array}$$

$$V_{\lambda_2} = \{ (\alpha, \alpha) \in \mathbb{R}^2 / \alpha \in \mathbb{R} \} \longrightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |\vec{v}_2| = \sqrt{2}$$

$$\vec{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$V = \begin{array}{cc} \vec{v}_1 & \vec{v}_2 \\ \end{array}$$
$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A \cdot \vec{v}_1 = \frac{1}{3\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} =$$

$$= \frac{1}{3\sqrt{2}} \begin{pmatrix} \frac{2}{\sqrt{2}} \\ -\frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

No tenemos σ_2 ni σ_3 para calcular $\vec{u}_2$ y $\vec{u}_3$.

Debemos construir una base ortonormal de 3 vectores: $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$

$$B = \{\vec{u}_1, \underbrace{(0,1,0)}_{\vec{e}_2}, \underbrace{(0,0,1)}_{\vec{e}_3}\}$$

Gram-Schmidt:

$$\vec{w}_1 = \vec{u}_1 = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}$$

$$\vec{w}_2 = \vec{e}_2 - \frac{\vec{e}_2 \cdot \vec{w}_1}{\underbrace{|\vec{w}_1|^2}_{=1}} \cdot \vec{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{9} \\ \frac{5}{9} \\ \frac{4}{9} \end{pmatrix}$$

$$\vec{w}_3 = \vec{e}_3 - \frac{\vec{e}_3 \bullet \vec{w}_1}{\underbrace{|\vec{w}_1|^2}_{=1}} \vec{w}_1 - \frac{\vec{e}_3 \bullet \vec{w}_2}{|\vec{w}_2|^2} \cdot \vec{w}_2 =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \cdot \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} - \frac{\frac{4}{\cancel{9}}}{\frac{5}{\cancel{9}}} \cdot \begin{pmatrix} \frac{2}{9} \\ \frac{5}{9} \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\underbrace{|\vec{w}_2|}_{=\frac{\sqrt{5}}{3}}} = \frac{3}{\sqrt{5}} \cdot \begin{pmatrix} \frac{2}{9} \\ \frac{5}{9} \\ \frac{4}{9} \end{pmatrix} = \begin{pmatrix} \frac{2}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \end{pmatrix}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\underbrace{|\vec{w}_3|}_{=\frac{1}{\sqrt{5}}}} = \sqrt{5} \cdot \begin{pmatrix} -\frac{2}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} -\frac{2\sqrt{5}}{5} \\ 0 \\ \frac{\sqrt{5}}{5} \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} \frac{1}{3} & \frac{2}{3\sqrt{5}} & -\frac{2\sqrt{5}}{5} \\ -\frac{2}{3} & \frac{5}{3\sqrt{5}} & 0 \\ \frac{2}{3} & \frac{4}{3\sqrt{5}} & \frac{\sqrt{5}}{5} \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \cdot \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{V^t}$$

$\underline{\underline{\text{Ejercicio}}}$: Calcular una factorización SVD de la matriz:

$$A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\underset{3\times 2}{A} = \underset{3\times 3}{U} \cdot \underset{3\times 2}{\Sigma} \cdot \underset{2\times 2}{V^t}$$

$$A^t \cdot A = \begin{pmatrix} 20 & -10 \\ -10 & 5 \end{pmatrix}$$

$$|A^t \cdot A - \lambda I| = 0 \longrightarrow \lambda(\lambda - 25) = 0 \quad \begin{cases} \lambda = 0 \\ \lambda = 25 \end{cases}$$

$$\boxed{\begin{array}{l} \lambda_1 = 25 \longrightarrow \sigma_1 = \sqrt{25} = 5 \\ \lambda_2 = 0 \quad \times \end{array}} \rightarrow \Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- $\underline{\text{Para } \lambda_1 = 25:}$

$$V_{\lambda_1} = \{(-2\alpha, \alpha) \in \mathbb{R}^2 / \alpha \in \mathbb{R}\} \xrightarrow[\alpha = -1]{} \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow \vec{v}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$$

- <u>Para $\lambda_2 = 0$:</u>

$$V_{\lambda_2} = \{ (\alpha, 2\alpha) \in \mathbb{R}^2 / \alpha \in \mathbb{R} \} \longrightarrow \vec{V}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \longrightarrow \vec{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A \cdot \vec{v}_1 = \frac{1}{5} \underset{3\times 2}{\begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}} \cdot \underset{2\times 1}{\begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}} =$$

$$= \frac{1}{5} \begin{pmatrix} \frac{10}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$$

$$B = \{ \vec{u}_1 , \vec{e}_2 , \vec{e}_3 \}$$

$$\vec{w}_1 = \vec{u}_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \qquad \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$$

$$\vec{w}_2 = \vec{e}_2 - \frac{\vec{e}_2 \cdot \vec{w}_1}{\underset{=1}{|\vec{w}_1|^2}} \cdot \vec{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}$$

$$\vec{w}_3 = \vec{e}_3 - \frac{\cancelto{0}{\vec{e}_3 \cdot \vec{w}_1}}{\underset{=1}{|\vec{w}_1|^2}} \vec{w}_1 - \frac{\cancelto{0}{\vec{e}_3 \cdot \vec{w}_2}}{\underset{=\frac{4}{5}}{|\vec{w}_2|^2}} \cdot \vec{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{|\vec{w}_2|} = \frac{\sqrt{5}}{2} \begin{pmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \\ 0 \end{pmatrix}$$

$$\vec{u}_3 = \frac{\vec{w}_3}{\underset{=1}{|\vec{w}_3|}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{\sqrt{5}}{5} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = \underset{U}{\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{\sqrt{5}}{5} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}} \cdot \underset{\Sigma}{\begin{pmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}} \cdot \underset{V^t}{\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}}$$

Ejercicio: Mediante Python, utilizar la factorización SVD para comprimir la imagen "cameraman.tif" (256×256 píxeles) según los valores de K: $1, 10, 20, 30, \ldots,$ inferiores a $K_{MÁX}$. Mostrar en cada caso el % de almacenamiento de la imagen comprimida vs la original.

```python
# -*- coding: utf-8 -*-

#Paquetes necesarios.
import matplotlib.pyplot as plt #Para mostrar imágenes.
import numpy as np #Para usar matrices y calcular SVD.
from PIL import Image #Para cargar imágenes.


#Mostramos la imagen original.
A = Image.open("C:/Users/Titan/Desktop/SVD/cameraman.tif")
plt.figure(figsize = (9, 6))
plt.title("Imagen original")
plt.axis("off")
plt.imshow(A, cmap = 'gray');

#Calculamos la SVD de la imagen.
U, sigma, V = np.linalg.svd(A)

#Valor máximo de k para que exista compresión.
m, n = np.shape(A)
k_max = np.floor(m * n / (m + n + 1))
k_max = np.int_(k_max)


#Mostramos la imagen comprimida para distintos valores de k.
for k in range(1, k_max - 1, 10):
    if k > 1: k = k - 1
    Acomp = U[:, :k] @ np.diag(sigma[:k]) @ V[:k, :]
    plt.figure(figsize=(9, 6))
    title = "Imagen comprimida: k = %s. " % k
    p = k*(m + n + 1)/(m * n)*100
    title += "%% almacenamiento: %.2f" % p
    plt.title(title)
    plt.axis("off")
    plt.imshow(Acomp, cmap='gray')


#Representamos los valores singulares (ordenados).
plt.figure(1)
plt.semilogy(sigma)
plt.title("Valores singulares")
plt.show()

#Representamos la suma acumulativa normalizada.
plt.figure(2)
plt.plot(np.cumsum(sigma)/np.sum(sigma))
plt.title("Valores singulares: suma acumulativa norm.")
plt.show()
```