Ejercicio: Sea la matriz $A = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$.

Determinar sus valores y vectores propios.

restar λ a la diag. ppal.

- Polinomio característico: $P(\lambda) = |A - \lambda I|$

$$P(\lambda) = \left| \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| =$$

$$= \left| \begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| =$$

$$= \begin{vmatrix} 4-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} = (4-\lambda)\cdot(2-\lambda)^2$$

- Ecuación característica: $|A - \lambda I| = 0 \rightarrow (4-\lambda)\cdot(2-\lambda)^2 = 0$

$4-\lambda = 0 \rightarrow$ $\lambda_1 = 4 \rightarrow m_1 = 1$

$2-\lambda = 0 \rightarrow$ $\lambda_2 = 2 \rightarrow m_2 = 2$

• Para $\lambda_1 = 4$: $(A-\lambda I)\cdot\vec{v} = \vec{0} \rightarrow (A-4I)\cdot\vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 3 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{array}{l} \cancel{0 = 0} \\ x - 2y = 0 \\ 3x - 2z = 0 \end{array}\right\}$$

S.C.I $\rightarrow$ 1 par

$$\begin{cases} x = 2y \\ 6y - 2z = 0 \longrightarrow 2z = 6y \rightarrow z = 3y \quad (z \in \mathbb{R}) \end{cases}$$

sub. propio asociado a $\lambda_1 = 4$

$$\begin{cases} x = 2\alpha \\ y = \alpha \quad (\alpha \in \mathbb{R}) \\ z = 3\alpha \end{cases} \longrightarrow V_{\lambda_1} = \{ (2\alpha, \alpha, 3\alpha) \in \mathbb{R}^3 \,/\, \alpha \in \mathbb{R} \}$$

$$(x,y,z) = \alpha(2,1,3)$$

L.I

$B_{\lambda_1} = \{(2,1,3)\}$ $\quad \dim(V_{\lambda_1}) = 1$ $\quad \boxed{\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}}$

• <u>Para $\lambda_2 = 2$</u> : $(A - 2I) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{array}{l} \cancel{2x = 0} \\ x = 0 \\ \cancel{3x = 0} \end{array}\right\}$$

3 inc − 1 ec = 2 par

$$\left\{\begin{array}{l} x = 0 \\ y = \alpha \quad (\alpha, \beta \in \mathbb{R}) \\ z = \beta \end{array}\right.$$

$$V_{\lambda_2} = \{ (0, \alpha, \beta) \in \mathbb{R}^3 / \alpha, \beta \in \mathbb{R} \}$$

$$(x, y, z) = \alpha (0, 1, 0) + \beta (0, 0, 1)$$

L.I

$$B_{\lambda_2} = \{ (0,1,0), (0,0,1) \} \qquad \dim (V_{\lambda_2}) = 2$$

$$\boxed{\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

Ejercicio : Siendo la matriz $A = \begin{pmatrix} 1 & 2 & 10 \\ 2 & 1 & 10 \\ -1 & -1 & -6 \end{pmatrix}$.

a) Calcular sus autovalores.

b) Calcular sus autovectores.

c) En caso de ser diagonalizable la matriz A, hallar la matriz diagonal D y la matriz de paso P.

a) $|A - \lambda I| = 0 \longrightarrow \begin{vmatrix} 1-\lambda & 2 & 10 \\ 2 & 1-\lambda & 10 \\ -1 & -1 & -6-\lambda \end{vmatrix} =$

$$= (1-\lambda)^2 \cdot (-6-\lambda) - 20 - 20 - \left(-10(1-\lambda) - 10(1-\lambda) + 4(-6-\lambda)\right) =$$

$$= (1 - 2\lambda + \lambda^2) \cdot (-6-\lambda) - 40 + 20(1-\lambda) - 4(-6-\lambda) =$$

$$= -6 + 12\lambda - 6\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 40 + 20 - 20\lambda + 24 + 4\lambda =$$

$= -\lambda^3 - 4\lambda^2 - 5\lambda - 2 = 0 \rightarrow$ Ruffini.

$$\begin{array}{r|rrrr} & -1 & -4 & -5 & -2 \\ -1 & \downarrow & 1 & 3 & 2 \\ \hline & -1 & -3 & -2 & \boxed{0} \end{array}$$

Divisores del T.I : $\pm 1, \pm 2$

$\lambda = -1$

$-\lambda^2 - 3\lambda - 2 = 0 \longrightarrow \lambda = \dfrac{3 \pm \sqrt{9 - 4\cdot(-1)\cdot(-2)}}{2\cdot(-1)} = \ldots$

$= \begin{cases} \nearrow \lambda = -2 \\ \searrow \lambda = -1 \end{cases}$

$$\boxed{\begin{array}{ll} \lambda_1 = -1 & m_1 = 2 \\ \lambda_2 = -2 & m_2 = 1 \end{array}}$$

b)

- Para $\lambda_1 = -1$: $(A + I)\cdot\vec{v} = \vec{0}$

$$\begin{pmatrix} 2 & 2 & 10 \\ 2 & 2 & 10 \\ -1 & -1 & -5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{array}{l} \cancel{2x + 2y + 10z = 0} \\ \cancel{2x + 2y + 10z = 0} \\ -x - y - 5z = 0 \end{array}\right\}$$

$x = -y - 5z$

$$\begin{cases} x = -\alpha - 5\beta \\ y = \alpha \\ z = \beta \end{cases} \quad (\alpha, \beta \in \mathbb{R})$$

$V_{\lambda_1} = \{ (-\alpha - 5\beta, \alpha, \beta) \in \mathbb{R}^3 / \alpha, \beta \in \mathbb{R} \}$

$(x, y, z) = \alpha(-1, 1, 0) + \beta(-5, 0, 1)$

L.I $\rightarrow$ base

$$\boxed{\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}$$

• Para $\lambda_2 = -2$: $(A+2I)\cdot\vec{v} = \vec{0}$

GAUSS

$$\begin{pmatrix} 3 & 2 & 10 \\ 2 & 3 & 10 \\ -1 & -1 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{aligned} 3x + 2y + 10z &= 0 \\ 2x + 3y + 10z &= 0 \\ -x - y - 4z &= 0 \end{aligned}\right\}$$

$$\begin{pmatrix} 3 & 2 & 10 \\ 2 & 3 & 10 \\ -1 & -1 & -4 \end{pmatrix} \underset{F_1 \leftrightarrow F_3}{\sim} \begin{pmatrix} -1 & -1 & -4 \\ 2 & 3 & 10 \\ 3 & 2 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ \cancel{0} & \cancel{-1} & \cancel{-2} \end{pmatrix}$$

$F_1 \to -F_1$

$F_2 \to F_2 + 2F_1$

$F_3 \to F_3 + 3F_1$

$$\left.\begin{aligned} x + y + 4z &= 0 \\ y + 2z &= 0 \end{aligned}\right\} \begin{aligned} &\longrightarrow x - 2z + 4z = 0 \longrightarrow x = -2z \\ &\rightarrow y = -2z \qquad\qquad z \in \mathbb{R} \end{aligned}$$

$$\left\{\begin{aligned} x &= -2\alpha \\ y &= -2\alpha \quad (\alpha \in \mathbb{R}) \\ z &= \alpha \end{aligned}\right. \qquad V_{\lambda_2} = \{ (-2\alpha, -2\alpha, \alpha) \in \mathbb{R}^3 \,/\, \alpha \in \mathbb{R} \}$$

$(x, y, z) = \alpha(-2, -2, 1)$

L.I $\rightarrow$ base

$$\boxed{\vec{v}_3 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}}$$

c) ¿ A es diagonalizable ? $\rightarrow$ $\boxed{\text{sí}}$

n.º autovec.

① $m_1 + m_2 = n$?

$2 + 1 = 3$ ✓

② $m_1 = \dim(V_{\lambda_1})$? $\quad 2 = 2$ ✓

$m_2 = \dim(V_{\lambda_2})$? $\quad 1 = 1$ ✓

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad P = \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{pmatrix} -1 & -5 & -2 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

Se verificará que $A \cdot P = P \cdot D$:

$$AP = \begin{pmatrix} 1 & 5 & 4 \\ -1 & 0 & 4 \\ 0 & -1 & 2 \end{pmatrix} = P \cdot D \quad ✓$$

Ejercicio: Diagonalizar la matriz $A = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

A es triangular $\Longrightarrow$ los λ están en la diag. ppal.

$\lambda_1 = 4 \qquad m_1 = 2$

$\lambda_2 = 5 \qquad m_2 = 1$

• Para $\lambda_1 = 4$: $(A - 4I)\cdot\vec{v} = \vec{0}$ $\longrightarrow$ $\boxed{A \text{ no es diagonalizable}}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{array}{l} \cancel{0 = 0} \\ x = 0 \\ z = 0 \end{array}\right\} y \in \mathbb{R}$$

$$\left\{\begin{array}{l} x = 0 \\ y = \alpha \\ z = 0 \end{array}\right. \quad (\alpha \in \mathbb{R}) \qquad \boxed{\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

ec. param.

$V_{\lambda_1} = \{ (0, \alpha, 0) \in \mathbb{R}^3 \,/\, \alpha \in \mathbb{R} \}$ $\qquad \dim(V_{\lambda_1}) = 1 \neq m_1 = 2$

$(x, y, z) = \alpha(0, 1, 0)$ $\longrightarrow$ $B_{\lambda_1} = \{(0,1,0)\}$

L. I → base

Ejercicio: Analizar si la matriz $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{pmatrix}$ es diagonalizable, y en caso afirmativo, calcular la matriz diagonal D y la matriz de paso P.

A es triangular $\Longrightarrow$ los λ están en la diag. ppal.

$$\boxed{\begin{matrix} \lambda_1 = 1 \\ m_1 = 2 \end{matrix}} \qquad \boxed{\begin{matrix} \lambda_2 = -1 \\ m_2 = 1 \end{matrix}}$$

- Para $\lambda_1 = 1$:

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{matrix} y + 2z = 0 \\ -4z = 0 \\ -2z = 0 \end{matrix}\right\} \begin{matrix} \rightarrow y = 0 \\ x \in \mathbb{R} \\ \rightarrow z = 0 \end{matrix}$$

$$\left\{\begin{matrix} x = \alpha \\ y = 0 \\ z = 0 \end{matrix}\right. \quad (\alpha \in \mathbb{R}) \longrightarrow V_{\lambda_1} = \{ (\alpha, 0, 0) \in \mathbb{R}^3 / \alpha \in \mathbb{R} \}$$

$$(x, y, z) = \alpha (1, 0, 0)$$

L. I $\rightarrow$ base

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$B_{\lambda_1} = \{(1,0,0)\} \qquad \dim(V_{\lambda_1}) = 1$$

¡ solo 1 autovector, pero necesitamos 2 L.I !

$$\boxed{\text{La matriz } A \text{ no es diagonalizable, pues } m_1 = 2 \neq \dim(V_{\lambda_1}) = 1 .}$$

Ejercicio: Sabiendo que la matriz $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ es diagonalizable y que la matriz diagonal y la matriz de paso son:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Calcular A^{200}.

$$A^n = P \cdot D^n \cdot P^{-1} \longrightarrow A^{200} = P \cdot D^{200} \cdot P^{-1}$$

$$A^{200} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{200} \cdot \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = (*)$$

$$P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}^t = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$|P| = 1 - (-1-1) = 3$$

$$(*) = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2^{200} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} 2^{200} & -1 & -1 \\ 2^{200} & 1 & 0 \\ 2^{200} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} =$$

$$= \boxed{\frac{1}{3} \begin{pmatrix} 2^{200}+2 & 2^{200}-1 & 2^{200}-1 \\ 2^{200}-1 & 2^{200}+2 & 2^{200}-1 \\ 2^{200}-1 & 2^{200}-1 & 2^{200}+2 \end{pmatrix}}$$