15 Calcular las inversas de las siguientes matrices utilizando determinantes:

$$A = \begin{pmatrix} -3 & -2 \\ 5 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -3 & 2 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} -3 & -2 \\ 5 & 1 \end{vmatrix} = -3 + 10 = 7 \neq 0$$

$$A^{-1} = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ -5 & -3 \end{pmatrix}$$

$$|B| = \begin{vmatrix} -3 & 2 & 0 \\ 2 & -3 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -9 + 2 - (-3 - 4) = 0$$

B^{-1} no existe (B es singular).

$$|C| = \begin{vmatrix} 1 & 4 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 3 - (1 + 8) = -6 \neq 0$$

$$\operatorname{adj}(C) = \begin{pmatrix} \overset{+}{C_{11}} & \overset{-}{C_{12}} & \overset{+}{C_{13}} \\ \overset{-}{C_{21}} & \overset{+}{C_{22}} & \overset{-}{C_{23}} \\ \overset{+}{C_{31}} & \overset{-}{C_{32}} & \overset{+}{C_{33}} \end{pmatrix} = \begin{pmatrix} 2 & -2 & 2 \\ -4 & 1 & -1 \\ 4 & -1 & -5 \end{pmatrix}$$

$$C_{11} = + \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2 \qquad C_{12} = - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2 \qquad C_{13} = + \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} = -4 \qquad C_{22} = +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad C_{23} = -\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{31} = +\begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} = 4 \qquad C_{32} = -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1 \qquad C_{33} = +\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = -5$$

$$C^{-1} = \frac{-1}{6}\begin{pmatrix} 2 & -4 & 4 \\ -2 & 1 & -1 \\ 2 & -1 & -5 \end{pmatrix}$$

16 Decidir si la siguiente matriz tiene inversa para algún $a \in \mathbb{R}$:

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 \\ 1 & 4 & 1 & 3 \\ 0 & 2 & 3 & a \\ 2 & 4 & 1 & 1 \end{pmatrix}$$

Existirá A^{-1} si $|A| \neq 0$ (o $rg(A) = 4$):

$$|A| = \begin{vmatrix} 1 & 3 & -2 & 0 \\ 1 & 4 & 1 & 3 \\ 0 & 2 & 3 & a \\ 2 & 4 & 1 & 1 \end{vmatrix} \underset{\substack{F_2 \to F_2 - F_1 \\ F_4 \to F_4 - 2F_1}}{=} \begin{vmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 3 & a \\ 0 & -2 & 5 & 1 \end{vmatrix} =$$

$$+1 \cdot \begin{vmatrix} 1 & 3 & 3 \\ 2 & 3 & a \\ -2 & 5 & 1 \end{vmatrix} = 3 - 6a + 30 - (-18 + 5a + 6) =$$

$$= 33 - 6a + 12 - 5a = 45 - 11a = 0$$

$$11a = 45 \rightarrow a = \frac{45}{11}$$

- A^{-1} existirá cuando $a \neq \frac{45}{11}$.

18 Calcular A^{-1}, donde:

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a & 1 & 0 & \cdots & 0 \\ a^2 & a & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a^n & a^{n-1} & a^{n-2} & \cdots & 1 \end{pmatrix}$$

Aplicamos Gauss - Jordan : $(A|I) \sim (I|A^{-1})$

$$(A|I) = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ a & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ a^2 & a & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & & & & \cdots & & & \\ a^n & a^{n-1} & a^{n-2} & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 \end{array}\right) \sim$$

$(n+1) \times (n+1)$

$F_2 \to F_2 - aF_1$
$F_3 \to F_3 - a^2 F_1$
$\vdots$
$F_{n+1} \to F_{n+1} - a^n F_1$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -a & 1 & 0 & \cdots & 0 \\ 0 & a & 1 & \cdots & 0 & -a^2 & 0 & 1 & \cdots & 0 \\ & \cdots & & & & & \cdots & & & \\ 0 & a^{n-1} & a^{n-2} & \cdots & 1 & -a^n & 0 & 0 & \cdots & 1 \end{array}\right) \sim$$

$F_3 \to F_3 - aF_2$
$\vdots$
$F_{n+1} \to F_{n+1} - a^{n-1} F_2$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -a & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & -a & 1 & \cdots & 0 \\ & \cdots & & & & & \cdots & & & \\ 0 & 0 & a^{n-2} & \cdots & 1 & 0 & -a^{n-1} & 0 & \cdots & 1 \end{array}\right) \sim$$

$\vdots$
$F_{n+1} \to F_{n+1} - a^{n-2} \cdot F_3$

$$\left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & -a & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & -a & 1 & \cdots & 0 \\ & & \cdots & & & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & -a^{n-2} & \cdots & 1 \end{array}\right)$$

$$\sim \cdots \sim \left(\underbrace{\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ -a & 1 & 0 & \cdots & 0 \\ 0 & -a & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \end{array}}_{A^{-1}}\right)$$

20 Obtener la solución de la ecuación matricial:

$$5AX - \frac{1}{2}\,\mathrm{tr}(C)B = B^t C$$

donde:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 \\ -2 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$$

$$5AX = B^t \cdot C + \frac{1}{2}\,tr(C)\cdot B \xrightarrow{\cdot 2} 10AX = 2B^t \cdot C + tr(C)\cdot B$$

$$\cancel{(10A)^{-1} \cdot 10A}X = (10A)^{-1} \cdot \left(2B^t \cdot C + tr(C)\cdot B\right) \qquad (KA)^{-1} = \frac{1}{K}A^{-1}$$

$$\boxed{X = \frac{1}{10}A^{-1} \cdot \left(2B^t \cdot C + tr(C)\cdot B\right)}$$

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$2B^t \cdot C + tr(C)\cdot B = 2 \cdot \begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} -1 & 0 \\ -2 & 3 \end{pmatrix} =$$

$$= 2 \cdot \begin{pmatrix} -9 & 3 \\ 12 & -6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} -17 & 6 \\ 26 & -15 \end{pmatrix}$$

$$X = \frac{1}{10} \cdot \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} -17 & 6 \\ 26 & -15 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 26 & -15 \\ -\frac{43}{2} & \frac{21}{2} \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{13}{5} & -\frac{3}{2} \\ -\frac{43}{20} & \frac{21}{20} \end{pmatrix}}}$$

21 Resolver la siguiente ecuación matricial:

$$\left[4\cdot\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix} - 3X^{-1}\right]^t = \left(2X^t\right)^{-1}$$

Trasponemos toda la ecuación: $(A^t)^t = A$

$$4\cdot\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix} - 3X^{-1} = \left[(2X^t)^{-1}\right]^t \qquad (A^t)^{-1} = (A^{-1})^t$$

$$4\cdot\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix} - 3X^{-1} = \left[(2X^t)^t\right]^{-1} \qquad (KA)^t = KA^t$$

$$4\cdot\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix} - 3X^{-1} = \left[2X\right]^{-1} \qquad (KA)^{-1} = \frac{1}{K}A^{-1}$$

$$4\cdot\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix} - 3X^{-1} = \frac{1}{2}X^{-1}$$

$$\frac{1}{2}X^{-1} + 3X^{-1} = 4\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix}$$

$$\frac{7}{2}X^{-1} = 4\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix} \longrightarrow X^{-1} = \frac{8}{7}\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix}$$

Invertimos toda la ecuación: $(A^{-1})^{-1} = A$

$$X = \left(\frac{8}{7}\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix}\right)^{-1} = \frac{7}{8}\begin{pmatrix}1 & 0\\ -1 & 3\end{pmatrix}^{-1} = \frac{7}{8}\cdot\frac{1}{3}\begin{pmatrix}3 & 0\\ 1 & 1\end{pmatrix} =$$

$$= \frac{7}{8} \begin{pmatrix} 1 & 0 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{7}{8} & 0 \\ \frac{7}{24} & \frac{7}{24} \end{pmatrix}}}$$

24 Resolver el siguiente sistema matricial:

$$\left.\begin{aligned} X + AY &= B \\ X^t + Y^t C &= D \end{aligned}\right\}$$

sabiendo que:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 6 & 7 \\ 9 & 10 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} X + AY &= B \\ X^t + Y^t \cdot C &= D \end{aligned}\right\} \xrightarrow{(\ )^t} \left.\begin{aligned} X + AY &= B \\ X + (Y^t \cdot C)^t &= D^t \end{aligned}\right\} \qquad (A \cdot B)^t = B^t \cdot A^t$$

② ①

$$\left.\begin{aligned} X + AY &= B \\ \ominus \quad X + C^t \cdot Y &= D^t \end{aligned}\right\}$$

$$\boxed{Y = (A - C^t)^{-1} \cdot (B - D^t)}$$

$$\cancel{\ } \quad A\underline{Y} - C^t \cdot \underline{Y} = B - D^t \longrightarrow (A - C^t) \cdot Y = B - D^t$$

dcha!

$$Y = \left[\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ 7 & 10 \end{pmatrix}\right]^{-1} \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] =$$

$$= \begin{pmatrix} -4 & -6 \\ -3 & -9 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{18} \cdot \begin{pmatrix} -9 & 6 \\ 3 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} =$$

$$= \frac{1}{18} \begin{pmatrix} -15 & 15 \\ 7 & -7 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -\frac{5}{6} & \frac{5}{6} \\ \frac{7}{18} & -\frac{7}{18} \end{pmatrix}}}$$

$$X = B - AY = B - A\cdot(A - C^t)^{-1}\cdot(B - D^t)$$

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \cdot \frac{1}{18} \cdot \begin{pmatrix} -15 & 15 \\ 7 & -7 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{18}\begin{pmatrix} -9 & 9 \\ -53 & 53 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{53}{18} & -\frac{53}{18} \end{pmatrix} =$$

$$= \underline{\underline{\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{53}{18} & -\frac{35}{18} \end{pmatrix}}}$$

17 Aplicando el método de Gauss-Jordan, hallar la inversa de la matriz:

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ -2 & 3 & -4 & 1 \\ 5 & -8 & 11 & -4 \\ -2 & 3 & -4 & 2 \end{pmatrix}$$

$$(A \mid I) = \left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ -2 & 3 & -4 & 1 & 0 & 1 & 0 & 0 \\ 5 & -8 & 11 & -4 & 0 & 0 & 1 & 0 \\ -2 & 3 & -4 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$F_2 \to F_2 + 2F_1$

$F_3 \to F_3 - 5F_1$

$F_4 \to F_4 + 2F_1$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & -9 & -5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \sim$$

$F_1 \to F_1 + F_2$

$F_3 \to F_3 + 3F_2$

$F_4 \to F_4 - F_2$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \sim$$

$F_1 \to F_1 - 2F_3$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & 1 & -5 & -2 & 0 \\ 0 & 1 & 0 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \sim \begin{array}{l} F_1 \to F_1 - 4F_4 \\ F_2 \to F_2 - 3F_4 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & -2 & -4 \\ 0 & 1 & 0 & 0 & 2 & 4 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) = (I \mid A^{-1})$$

$$\underbrace{}_{I} \qquad \underbrace{}_{A^{-1}}$$