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- Only requires the definition of a (dis)similarity measure:
  - ✓ Useful for feature-based and structural representations
- Does not derive a model out of the $\mathcal{T}$:
  - ✓ Adaptive
  - ✗ Inefficient as $|\mathcal{T}| \uparrow\uparrow$

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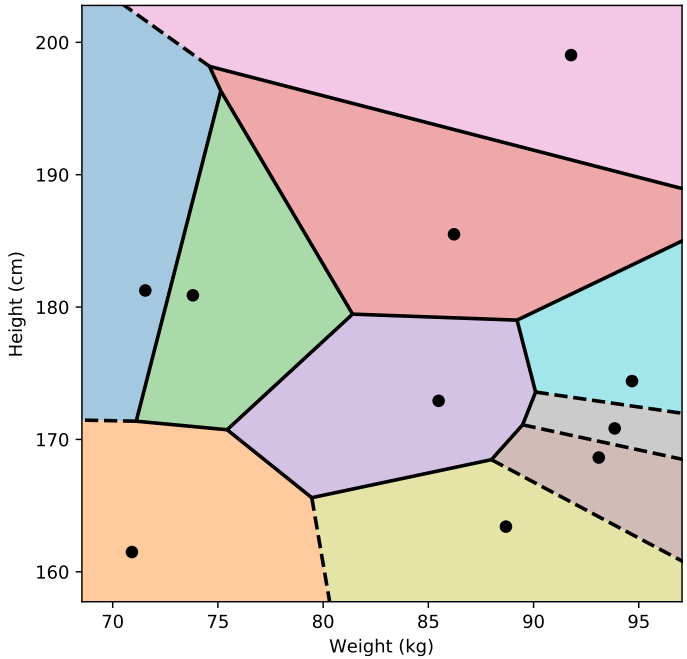

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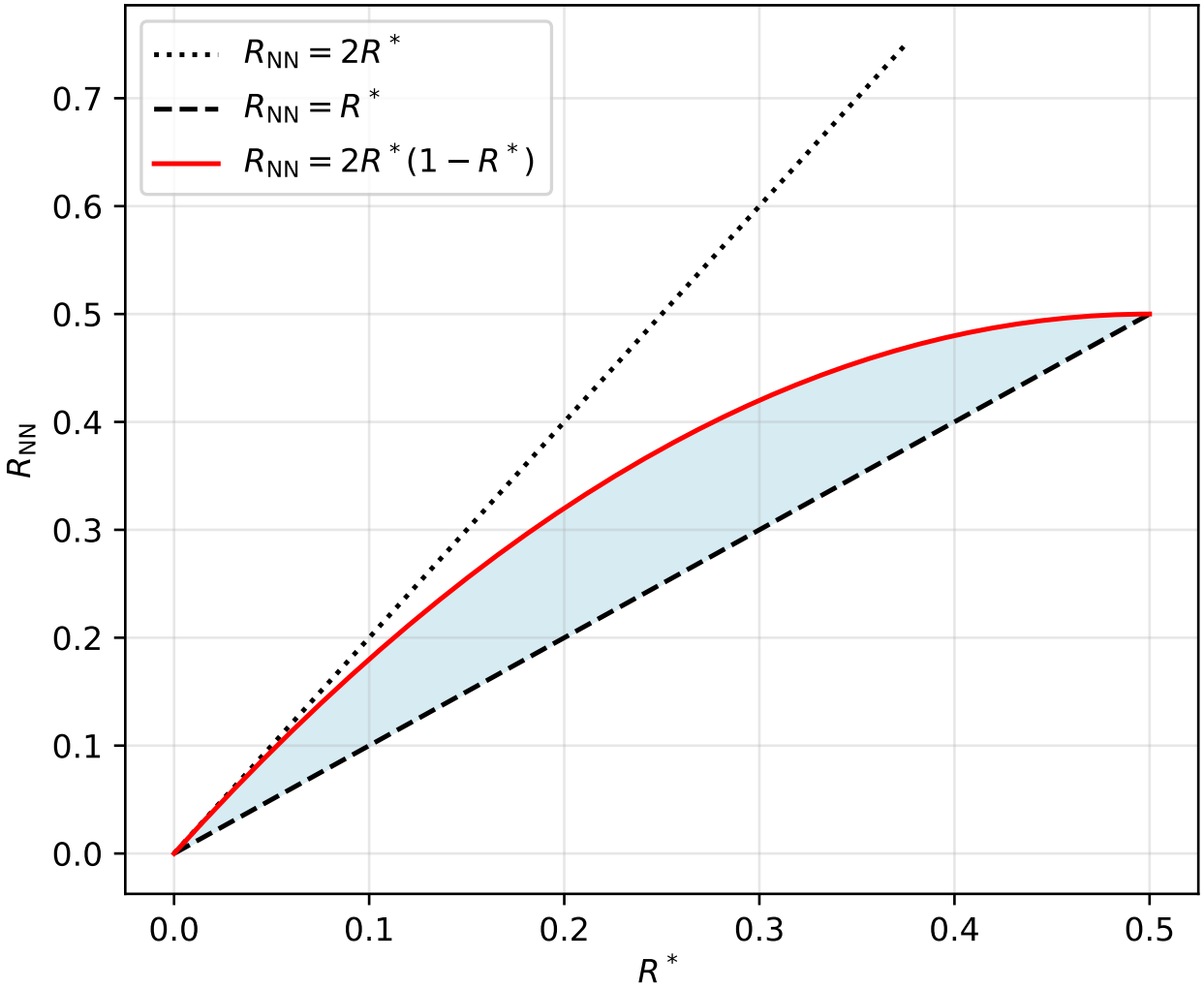

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- A metric must follow four properties:
  1. Nonnegativity: $D(\mathbf{a}, \mathbf{b}) \geq 0$
  2. Reflexivity: $D(\mathbf{a}, \mathbf{b}) = 0$ iif $\mathbf{a} = \mathbf{b}$
  3. Symmetry: $D(\mathbf{a}, \mathbf{b}) = D(\mathbf{b}, \mathbf{a})$
  4. Triangle inequality: $D(\mathbf{a}, \mathbf{b}) + D(\mathbf{b}, \mathbf{c}) \geq D(\mathbf{a}, \mathbf{c})$

Minkowski distance

Generalized metric that unifies most well-known distance measures:

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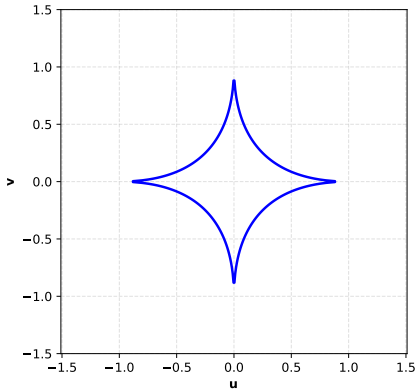
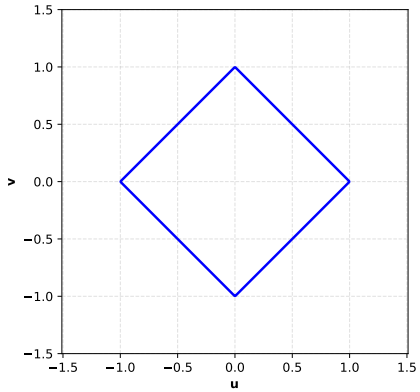
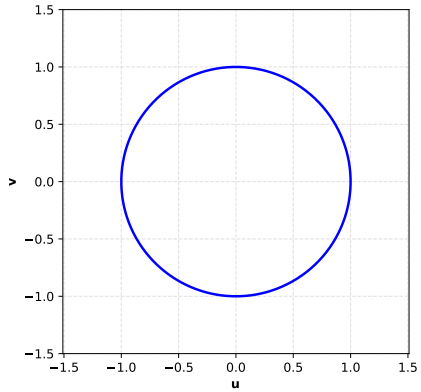


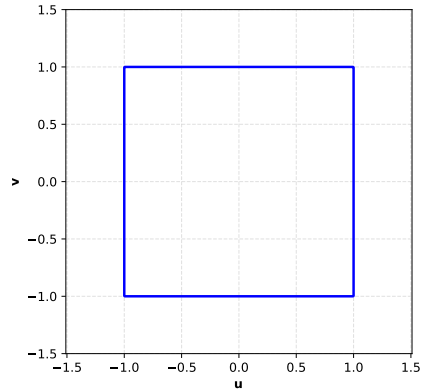
$p = 1$



$p = 2$



$p = \infty$



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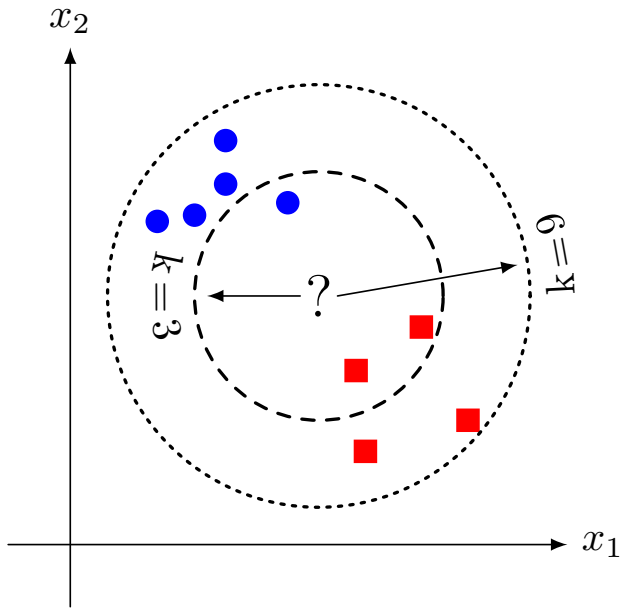

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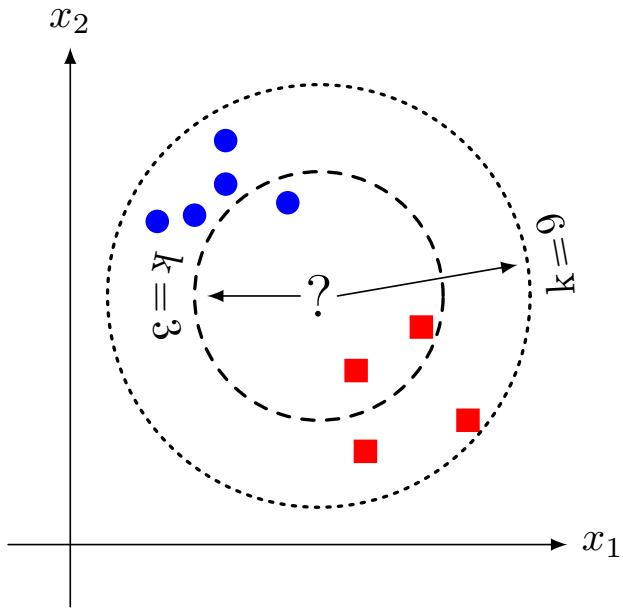

Hyperparameter k:

- Low k values: Sensitive to noise (high variance)
- High k values: Too smooth/general (high bias)

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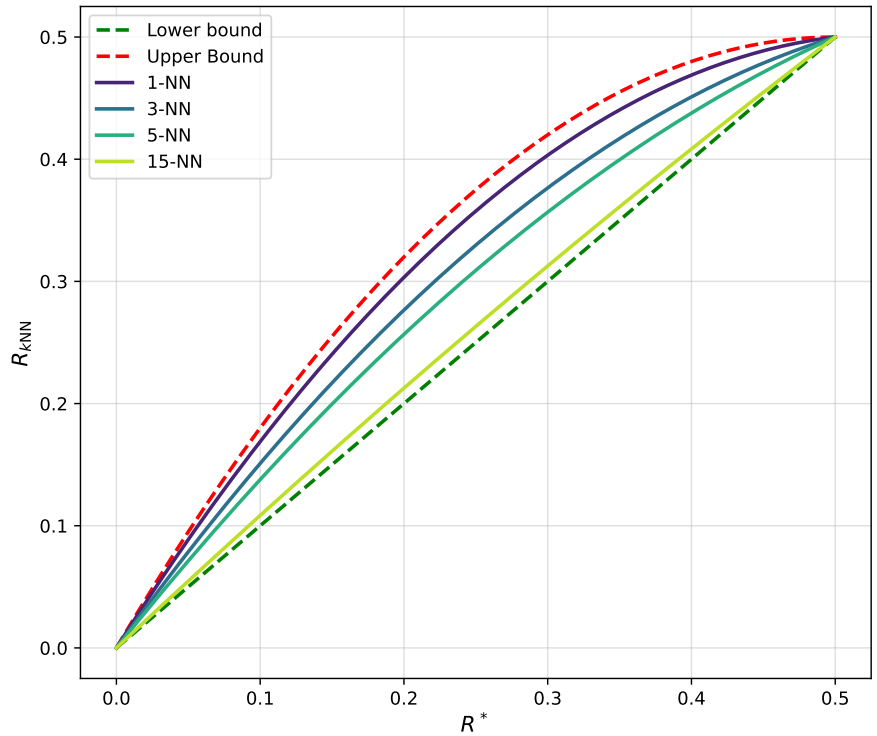
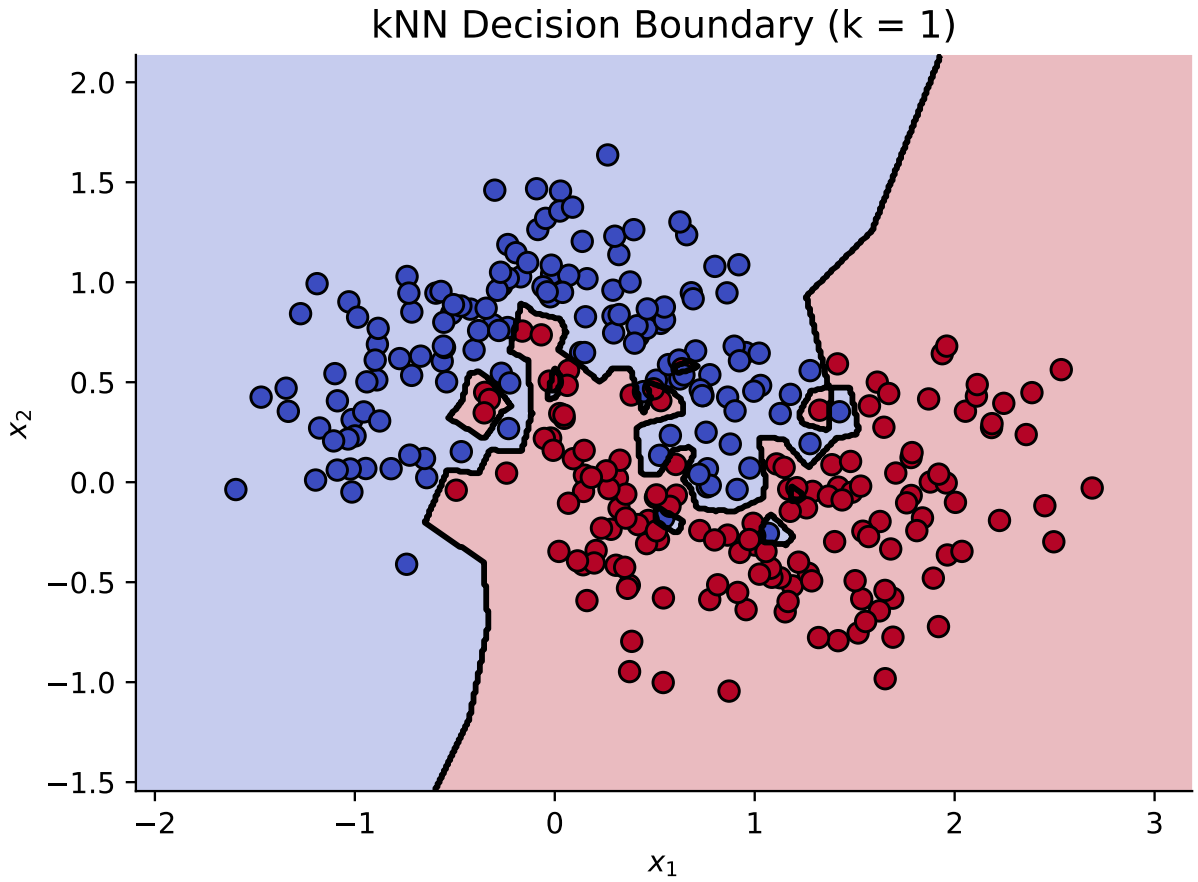


Decision boundaries



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- ✗ Inefficient classifier $\Rightarrow \mathcal{O}(n^2)$
  1. Fast similarity search
  2. Approximate search
  3. Data reduction

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  $\Rightarrow$ But there exist other types of models

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- Two examples of models:
  1. Decision tree
  2. Support Vector Machine

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- Tend to overfit:
  - ✗ Pruning strategies to improve generalization
  - ✓ Useful for outlier detection

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Example:

- Binary task: $\mathcal{W} = \{\text{male}, \text{female}\}$
- Feature space $\mathbb{R}^2$: weight (kg), height (cm)

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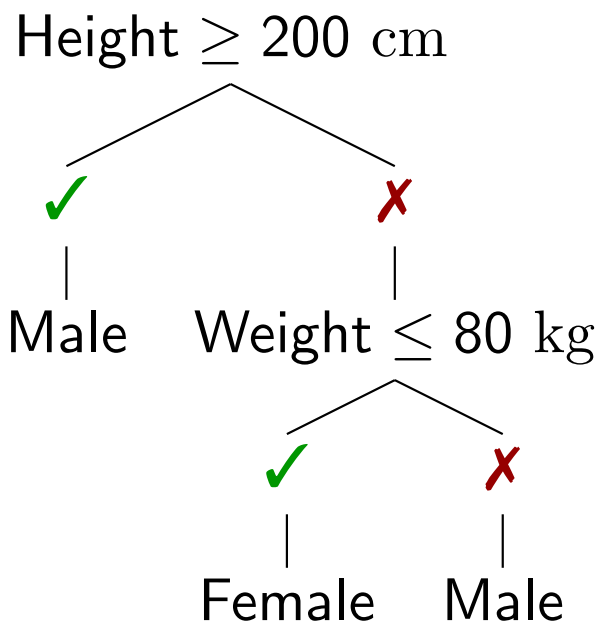

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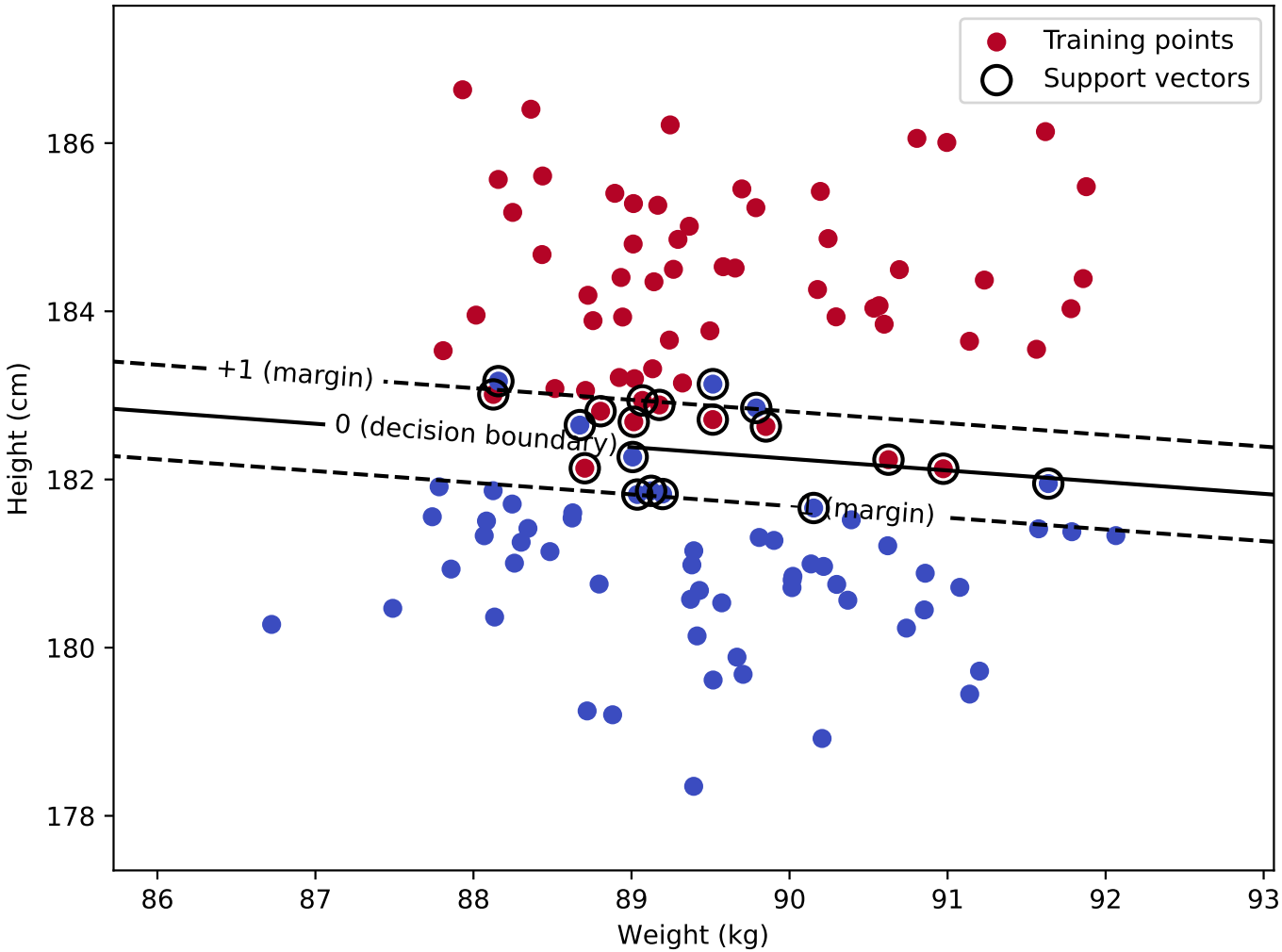
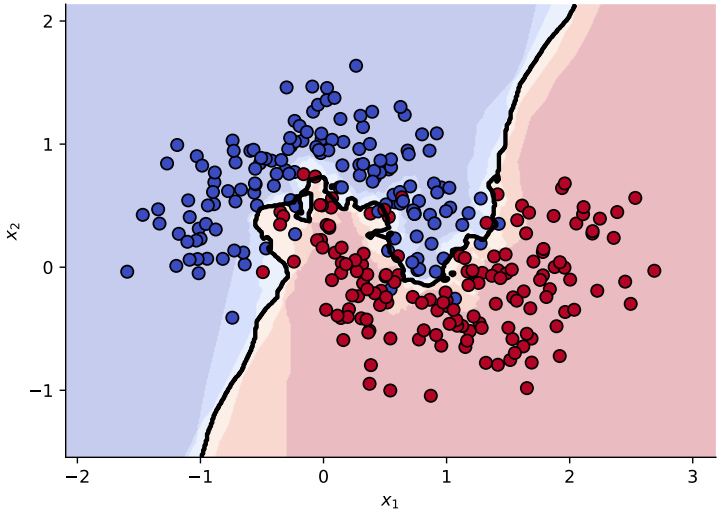
- Adaptations for multiclass scenarios
  → One-vs-all configuration
  → One-vs-one configuration

Support Vector Machine

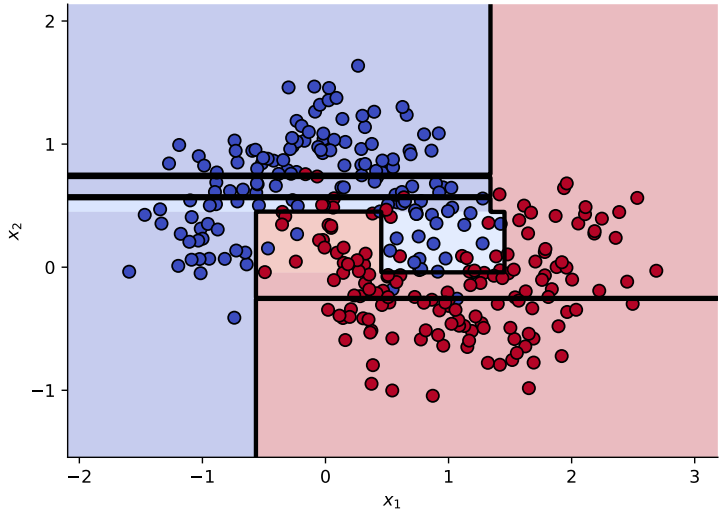


Decision boundaries (comparison)

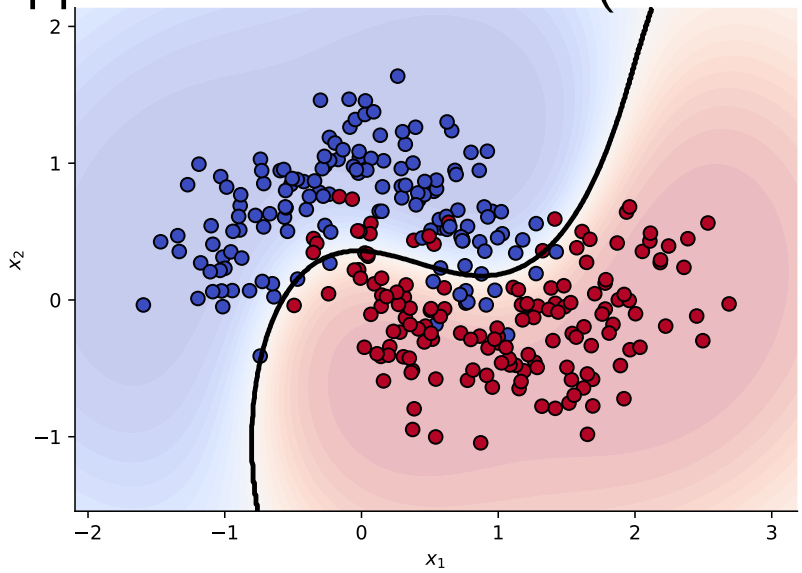
kNN with $k = 5$



Decision Tree with 5 levels



Support Vector Machine (RBF kernel)



T4: Nonparametric and distance-based learning

Fundamentos del Aprendizaje Automático

Curso 2025/2026