Ejercicio: Verificar que P es una matriz ortogonal. Comprobar, además, que las columnas de P forman un conjunto ortonormal.

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{pmatrix}$$

$$P \cdot P^t = \begin{array}{c} \begin{array}{ccc} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \\ \downarrow & \downarrow & \downarrow \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{pmatrix} \end{array} \cdot \begin{pmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ✓ P es ortogonal.

$$\vec{p}_1 \cdot \vec{p}_2 = \frac{2}{9} - \frac{2}{5} + \frac{8}{45} = \frac{10 - 18 + 8}{45} = 0$$

$$\vec{p}_1 \cdot \vec{p}_3 = \frac{2}{9} + 0 - \frac{10}{45} = \frac{2}{9} - \frac{2}{9} = 0$$

$$\vec{p}_2 \cdot \vec{p}_3 = \frac{4}{9} + 0 - \frac{20}{45} = \frac{4}{9} - \frac{4}{9} = 0$$

$$|\vec{p}_1| = \sqrt{\frac{1}{9} + \frac{4}{5} + \frac{4}{45}} = \sqrt{\frac{5+36+4}{45}} = \sqrt{\frac{45}{45}} = \underline{\underline{1}}$$

$$|\vec{p}_2| = \sqrt{\frac{4}{9} + \frac{1}{5} + \frac{16}{45}} = \sqrt{\frac{20+9+16}{45}} = \sqrt{\frac{45}{45}} = \underline{\underline{1}}$$

$$|\vec{p}_3| = \sqrt{\frac{4}{9} + 0 + \frac{25}{45}} = \sqrt{\frac{20+25}{45}} = \sqrt{\frac{45}{45}} = \underline{\underline{1}}$$

das colunas de P formam um conjunto ortonormal ✓

Ejercicio: Determinar una matriz P que diagonalice ortogonalmente a $A = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$. Comprobar que $D = P^t \cdot A \cdot P$.

Como A es simétrica, será diagonalizable ortogonalmente.

$$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\cdot(1-\lambda) - 4 = 0 \rightarrow -2 + 2\lambda - \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0 \rightarrow \lambda = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-6)}}{2\cdot 1} = \frac{-1 \pm 5}{2} =$$

$$= \begin{cases} \lambda_1 = 2 & m_1 = 1 \\ \lambda_2 = -3 & m_2 = 1 \end{cases}$$

- Para $\lambda_1 = 2$: $(A - 2I)\cdot\vec{v} = \vec{0}$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{array}{r} \cancel{-4x + 2y = 0} \\ 2x - y = 0 \end{array}\right\} \begin{array}{l} \overset{\alpha}{\overset{\shortparallel}{x}} \in \mathbb{R} \\ \rightarrow y = 2x \end{array}$$

$$V_{\lambda_1} = \{ (\alpha, 2\alpha) \in \mathbb{R}^2 / \alpha \in \mathbb{R} \}$$

$$B_{\lambda_1} = \{ (1,2) \} \longrightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$|\vec{v}_1| = \sqrt{1^2 + 2^2} = \sqrt{5} \longrightarrow \vec{p}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

- <u>Para $\lambda_2 = -3$:</u> $(A + 3I) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{matrix} x + 2y = 0 \\ \cancel{2x + 4y = 0} \end{matrix}\right\} \rightarrow \begin{matrix} x = -2y \\ \underset{\alpha}{\overset{\shortparallel}{y}} \in \mathbb{R} \end{matrix}$$

$$V_{\lambda_2} = \{ (-2\alpha, \alpha) \in \mathbb{R}^2 / \alpha \in \mathbb{R} \}$$

$$B_{\lambda_2} = \{ (-2, 1) \} \longrightarrow \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$|\vec{v}_2| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \longrightarrow \vec{p}_2 = \begin{pmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} \vec{p}_1 \downarrow & \vec{p}_2 \downarrow \end{array} \\ P = & \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \end{array} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Comprobamos :

$$D = P^t \cdot A \cdot P = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \frac{1}{5} \cdot \begin{pmatrix} 2 & 4 \\ 6 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 10 & 0 \\ 0 & -15 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

<u>Ejercicio</u> : Diagonalizar ortogonalmente la matriz :

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$$

$$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & -1-\lambda & 4 \\ -2 & 4 & -1-\lambda \end{vmatrix} \underset{F_2 \to F_2 + F_3}{=} \begin{vmatrix} 2-\lambda & 2 & -2 \\ 0 & \underline{\underline{3-\lambda}} & \underline{\underline{3-\lambda}} \\ -2 & 4 & -1-\lambda \end{vmatrix} =$$

$$= (3-\lambda) \cdot \begin{vmatrix} 2-\lambda & 2 & -2 \\ 0 & 1 & 1 \\ -2 & 4 & -1-\lambda \end{vmatrix} \underset{C_2 \to C_2 - C_3}{=} (3-\lambda) \cdot \begin{vmatrix} 2-\lambda & 4 & -2 \\ 0 & 0 & 1 \ (-1) \\ -2 & 5+\lambda & -1-\lambda \end{vmatrix} =$$

$$= (3-\lambda)(-1) \cdot \Big((2-\lambda) \cdot (5+\lambda) + 8 \Big) = 0$$

$$= (\lambda - 3) \cdot \big(10 + 2\lambda - 5\lambda - \lambda^2 + 8 \big) = 0 \rightarrow \lambda = 3$$

$$\downarrow$$

$$-\lambda^2 - 3\lambda + 18 = 0 \quad \ldots \quad \begin{array}{l} \nearrow \lambda = 3 \\ \searrow \lambda = -6 \end{array}$$

$\lambda_1 = 3 \qquad m_1 = 2$

$\lambda_2 = -6 \qquad m_2 = 1$

• <u>Para $\lambda_1 = 3$:</u> $(A - 3I) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} -1 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \left. \begin{array}{l} -x + 2y - 2z = 0 \\ \\ \\ \end{array} \right\}$$

$$\left. \begin{array}{l} x = 2y - 2z \\ \underset{\alpha}{\overset{\shortparallel}{y}},\ \underset{\beta}{\overset{\shortparallel}{z}} \in \mathbb{R} \end{array} \right\} \qquad V_{\lambda_1} = \{ (2\alpha - 2\beta, \alpha, \beta) \in \mathbb{R}^3 / \alpha, \beta \in \mathbb{R} \}$$

$$B_{\lambda_1} = \{ (2,1,0), (-2,0,1) \} \qquad \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Son ortogonales $\vec{v}_1$ y $\vec{v}_2$? $\quad \vec{v}_1 \cdot \vec{v}_2 = -4 \neq 0 \rightarrow$ <u>NO</u>

$$\vec{w}_1 = \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{|\vec{w}_1|^2} \cdot \vec{w}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \frac{4}{5} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix}$$

$$|w_1|^2 = w_1 \cdot w_1 = 2^2 + 1^2 = 5 \qquad \vec{v}_2 \cdot \vec{w}_1 = \vec{v}_2 \cdot \vec{v}_1 = -4$$

$$\vec{p}_1 = \frac{\vec{w}_1}{\underset{\sqrt{5}}{\underset{\shortparallel}{|\vec{w}_1|}}} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix} \qquad \vec{p}_2 = \frac{\vec{w}_2}{|\vec{w}_2|} = \frac{5}{3\sqrt{5}} \begin{pmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-2}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{pmatrix}$$

$$\vec{w}_2 = \sqrt{\left(-\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + 1^2} = \sqrt{\frac{4}{25} + \frac{16}{25} + 1} = \sqrt{\frac{45}{25}} = \sqrt{\frac{9}{5}} =$$

$$= \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

- <u>Para $\lambda_2 = -6$</u> : $(A + 6I) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{aligned} 8x + 2y - 2z &= 0 \\ 2x + 5y + 4z &= 0 \\ -2x + 4y + 5z &= 0 \end{aligned}\right\}$$

$$+ \quad \cancel{} \quad 9y + 9z = 0$$

$$y = -z$$

① $\xrightarrow{:2}$ $4x + y - z = 0 \rightarrow 4x - z - z = 0$

$$4x - 2z = 0$$

$$4x = 2z \rightarrow x = \frac{1}{2} z$$

② $2 \cdot \frac{1}{2} z - 5z + 4z = 0 \rightarrow \cancel{0 = 0}$ $\qquad \underset{\alpha}{\underset{\shortparallel}{z}} \in \mathbb{R}$

$$V_{\lambda_2} = \{ (\tfrac{\alpha}{2}, -\alpha, \alpha) \in \mathbb{R}^3 / \alpha \in \mathbb{R} \} \xrightarrow{\alpha = 2} B_{\lambda_2} = \{ (1, -2, 2) \}$$

$$\vec{V_3} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \qquad |\vec{V_3}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$$

$$\vec{P_3} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$P = \begin{array}{c} \begin{array}{ccc} \vec{P_1} & \vec{P_2} & \vec{P_3} \end{array} \\ \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & -\frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & \frac{2}{3} \end{pmatrix} \end{array} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$$D = P^t \cdot A \cdot P \checkmark$$

$$\boxed{\begin{array}{l}\underline{\underline{\text{Ejercicio}}}: \text{ Obtener la factorización SVD de la matriz :} \\[2em] \qquad\qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{2\times 3}\end{array}}$$

$$\underset{m\times n}{A} = \underset{m\times m}{U} \cdot \underset{m\times n}{\Sigma} \cdot \underset{n\times n}{V^t} \longrightarrow A_{2\times 3} = \underset{2\times 2}{U} \cdot \underset{2\times 3}{\Sigma} \cdot \underset{3\times 3}{V^t}$$

$$A^t \cdot A = \underset{3\times 2}{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \cdot \underset{2\times 3}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} = \underset{3\times 3}{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$|A^t \cdot A - \lambda I| = 0 \longrightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^3 - (1-\lambda) = 0 \qquad (1-\lambda)\cdot\left((1-\lambda)^2 - 1\right) = 0 \rightarrow \boxed{\lambda = 1}$$

$$\downarrow$$

$$\cancel{1} - 2\underline{\underline{\lambda}} + \lambda^2 \cancel{-1} = 0$$

$$\lambda(\lambda - 2) = 0 \rightarrow \boxed{\lambda = 0}$$

$$\hookrightarrow \boxed{\lambda = 2}$$

$$\boxed{\begin{array}{l} \lambda_1 = 2 \rightarrow \sigma_1 = \sqrt{2} \\ \lambda_2 = 1 \rightarrow \sigma_2 = \sqrt{1} = 1 \\ \lambda_3 = 0 \quad \text{X} \end{array}}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}_{2\times 3} = \underline{\underline{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}}$$

- **Para $\lambda_1 = 2$** : $(A^t \cdot A - 2I) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{array}{l} \cancel{-x+y=0} \\ x-y=0 \\ -z=0 \end{array}\right\} \begin{array}{l} \underset{y}{\overset{\alpha}{\shortparallel}} \in \mathbb{R} \\ \rightarrow x=y \\ \rightarrow z=0 \end{array}$$

$$V_{\lambda_1} = \{ (\alpha, \alpha, 0) \in \mathbb{R}^3 / \alpha \in \mathbb{R} \} \rightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad |\vec{V}_1| = \sqrt{2}$$

$$\vec{v}_1 = \frac{\vec{V}_1}{|\vec{V}_1|} = \underline{\underline{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}}}$$

- **Para $\lambda_2 = 1$** : $(A^t \cdot A - I) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{array}{l} y=0 \\ x=0 \\ 0=0 \end{array}\right\} \; z \in \mathbb{R}$$

$$V_{\lambda_2} = \{ (0, 0, \alpha) \in \mathbb{R}^3 / \alpha \in \mathbb{R} \} \qquad \vec{V}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad |\vec{V}_2| = 1$$

$$\vec{v}_2 = \frac{\vec{V}_2}{|\vec{V}_2|} = \underline{\underline{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}}$$

• $\underline{\text{Para } \lambda_3 = 0}$: $(A^t \cdot A) \cdot \vec{v} = \vec{0}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{array}{r} \cancel{x + y = 0} \\ x + y = 0 \\ z = 0 \end{array}\right\} \longrightarrow x = -y$$

$$\underset{\alpha}{\overset{}{y}} \in \mathbb{R}$$

$$V_{\lambda_3} = \{(-\alpha, \alpha, 0) \in \mathbb{R}^3 / \alpha \in \mathbb{R}\}$$

$$\vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \longrightarrow |\vec{v}_3| = \sqrt{2} \longrightarrow \vec{v}_3 = \frac{\vec{v}_3}{|\vec{v}_3|} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\underline{\underline{V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}_{3 \times 3}}} \qquad \underline{\underline{U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2 \times 2}}}$$

$$\vec{u}_1 = \frac{1}{\sigma_1} A \cdot \vec{v}_1 = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A \cdot \vec{v}_2 = \frac{1}{1} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}}$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\Sigma} \cdot \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}}_{V^t}$$