Ejercicio : Determinar unas ecuaciones paramétricas, una base y la dimensión del subespacio de $\mathbb{R}^4$

$$U = \begin{cases} x + y - z - t = 0 \\ 2x - y + z = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & -1 & 1 & 0 \end{pmatrix} \sim \begin{array}{cccc} x & y & z & t \end{array}$$
$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -3 & 3 & 2 \end{pmatrix}$$

$F_2 \to F_2 - 2F_1$

$$\left.\begin{array}{l} x + y - z - t = 0 \\ -3y + 3z + 2t = 0 \end{array}\right\} \to$$

4 inc − 2 ec = 2 par

$$\boxed{\begin{array}{l} x = \cancel{\alpha} + \beta \cancel{-\alpha} - \frac{2}{3}\beta = \frac{1}{3}\beta \\ y = \frac{3\alpha + 2\beta}{3} = \alpha + \frac{2}{3}\beta \\ z = \alpha \\ t = \beta \qquad (\alpha, \beta \in \mathbb{R}) \end{array}}$$

ec. par. de U

$$(x, y, z, t) = \left(\frac{1}{3}\beta, \alpha + \frac{2}{3}\beta, \alpha, \beta\right) =$$

$$= \alpha\,\underline{(0,1,1,0)} + \beta\,\underline{\left(\frac{1}{3}, \frac{2}{3}, 0, 1\right)}$$

↳ sist. gen. de U → base

l. I

$$\boxed{B_U = \left\{(0,1,1,0), \left(\frac{1}{3}, \frac{2}{3}, 0, 1\right)\right\} \qquad \dim(U) = 2}$$

x y z t

Ejercicio : En $\mathbb{R}^4$ se consideran los subespacios vectoriales :

$$U = L\{(1,0,0,1),(1,1,1,1),(0,2,2,0)\}$$

$$W \equiv \begin{cases} 3x + y - z - 3t = 0 \\ y - t = 0 \end{cases}$$

Determinar una base del subespacio $U+W$ y una base del subespacio $U \cap W$.

• **Base de U** : $B_U = \{(1,0,0,1),(0,1,1,0)\} \rightarrow \dim(U) = 2$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix}$$

$F_2 \rightarrow F_2 - F_1$

• **Base de W** :

$$\begin{cases} 3x + y - z - 3t = 0 \rightarrow z = 3x - 2t \\ y - t = 0 \rightarrow y = t \end{cases} \rightarrow \begin{aligned} x &= \alpha \\ y &= \beta \\ z &= 3\alpha - 2\beta \\ t &= \beta \end{aligned} \quad (\alpha, \beta \in \mathbb{R})$$

$N^{\underline{o}}$ par $= \underset{\text{inc}}{4} - \underset{\text{finales}}{2\ \text{ec}} = 2$ par

$$(x,y,z,t) = \alpha\,(\underline{1,0,3,0}) + \beta\,(\underline{0,1,-2,1})$$

← s. gen → L.I → base

$$B_W = \{(1,0,3,0),(0,1,-2,1)\} \to \dim(W) = 2$$

- **U + W** : $\{B_U, B_W\}$ → sist. gen de $U+W$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 \\ \cancel{0} & \cancel{0} & \cancel{-3} & \cancel{1} \end{pmatrix}$$

↑ sist. gen $\qquad F_3 \to F_3 - F_1 \qquad F_4 \to F_4 - F_2$

$$\boxed{B_{U+W} = \{(1,0,0,1),(0,1,1,0),(0,0,3,-1)\}} \quad \dim(U+W) = 3$$

- **U ∩ W** : C.L de B_U = C.L de B_W

$$\alpha\,(1,0,0,1) + \beta\,(0,1,1,0) = \gamma\,(1,0,3,0) + \delta\,(0,1,-2,1)$$

$$\left.\begin{aligned} \alpha &= \gamma \\ \beta &= \delta \\ \cancel{\beta} &\cancel{= 3\gamma - 2\delta} \\ \alpha &= \delta \end{aligned}\right\} \begin{aligned} &\to \gamma = \alpha \\ &\to \beta = \alpha \\ &\to \cancel{\beta = 3\alpha - 2\alpha = \alpha} \\ &\to \delta = \alpha \quad \text{SCI} \\ &\quad (\alpha \in \mathbb{R}) \end{aligned}$$

4 inc − 3 ec. fin = 1 par.

$$\left.\begin{aligned} \alpha &= t \\ \beta &= t \\ \gamma &= t \\ \delta &= t \end{aligned}\right\} \quad (t \in \mathbb{R}) \quad \xrightarrow{t = 1} \quad \gamma = \delta = 1$$

$$\boxed{B_{U \cap W} = \{(1, 1, 1, 1)\}}$$

$$\dim(U \cap W) = 1$$

$$\begin{array}{ccccccc} \dim(U + W) & = & \dim(U) & + & \dim(W) & - & \dim(U \cap W) \\ \| & & \| & & \| & & \| \\ 3 & & 2 & + & 2 & - & 1 \end{array}$$

Ejercicio: Se consideran los subespacios de $\mathbb{R}^3$:

$$U_1 = L\{(1,0,1)\} \rightarrow L.I \rightarrow \text{base}$$

$$U_2 = L\{(1,0,0),(0,1,1)\} \rightarrow L.I \rightarrow \text{base}$$

$$U_3 = L\{(1,0,0),(0,0,1)\} \rightarrow L.I \rightarrow \text{base}$$

Estudiar si son subespacios suplementarios en $\mathbb{R}^3$:

a) U_1 y U_2. b) U_1 y U_3. c) U_2 y U_3.

a) U_1 y U_2 son suplementarios en $\mathbb{R}^3$ si:

$$\underset{\substack{\parallel \\ 1}}{\dim(U_1)} + \underset{\substack{\parallel \\ 2}}{\dim(U_2)} = \underset{\substack{\parallel \\ 3}}{\dim(U_1+U_2)} = \underset{\substack{\parallel \\ 3}}{\dim}(\overset{\substack{\mathbb{R}^3 \\ \parallel}}{V})$$

$$1 + 2$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$F_3 \rightarrow F_3 - F_1$

$\dim(U_1 \oplus U_2) = 3$

U_1 y U_2 son suma directa:

$U_1 \oplus U_2$

$$\boxed{U_1 \text{ y } U_2 \text{ son suplementarios en } \mathbb{R}^3: \ U_1 \oplus U_2 = \mathbb{R}^3}$$

b) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \dim(U_1 + U_3) = 2$

$F_3 \rightarrow F_3 - F_1$

U_1 y U_3 no son suma directa

$$\underbrace{\dim(U_1)}_{1} + \underbrace{\dim(U_3)}_{2} \neq \underbrace{\dim(U_1 + U_3)}_{2}$$

$\boxed{U_1 \text{ y } U_3 \text{ no son suplementarios en } \mathbb{R}^3}$

c) U_2 y U_3 tienen en común el vector $(1,0,0) \Rightarrow U_2 \cap U_3 \neq \{\vec{0}\}$

$\boxed{U_2 \text{ y } U_3 \text{ no son sup. en } \mathbb{R}^3}$

Ejercicio : Se considera el subespacio vectorial U de $\mathbb{R}^5$, engendrado por los vectores $(1,2,-1,1,0)$, $(1,3,0,-1,1)$ y $(0,1,1,-2,1)$. Hallar unas ecuaciones paramétricas de un subespacio suplementario W.

Hallamos una base de U : $B_U = \{(1,2,-1,1,0),(0,1,1,-2,1)\}$

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 1 & 3 & 0 & -1 & 1 \\ 0 & 1 & 1 & -2 & 1 \end{pmatrix} \underset{F_2 \to F_2 - F_1}{\sim} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 & 1 \end{pmatrix} \to \dim(U) = 2$$

Para que W sea suplementario, la B_W deberá tener:

$\dim(W) = \dim(\mathbb{R}^5) - \dim(U) = 5 - 2 =$ 3 vectores L.I entre sí y con los de B_U

$$B_W = \{(0,0,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$$

$$(x,y,z,s,t) = \alpha(0,0,1,0,0) + \beta(0,0,0,1,0) + \gamma(0,0,0,0,1)$$

Ec. par. de W.

$$\boxed{\begin{aligned} x &= 0 \\ y &= 0 \\ z &= \alpha \\ s &= \beta \\ t &= \gamma \end{aligned}}$$

$(\alpha, \beta, \gamma \in \mathbb{R})$

Nº ec imp. $= \dim(\mathbb{R}^5) - \dim(W) =$

$= 5 - 3 = 2$ ec. imp.

$$\boxed{\begin{aligned} x &= 0 \\ y &= 0 \end{aligned}}$$ Ec. imp. de W

$\underline{\underline{\text{Ejercicio}}}$: Sea $f: \mathbb{R}^3 \to \mathbb{R}^2$ la aplicación definida por:

$$f(x,y,z) = (3x - 2z,\ x - y + z)$$

a) Calcular $f(\vec{x})$ y $f(\vec{y})$, siendo $\vec{x} = (2,-1,1)$ e $\vec{y} = (1,2,-5)$.

b) Comprobar que f es una aplicación lineal.

a) $f(\vec{x}) = f(\underset{x}{2}, \underset{y}{-1}, \underset{z}{1}) = (3\cdot 2 - 2\cdot 1,\ 2+1+1) = \boxed{(4,4)}$

$f(\vec{y}) = f(1,2,-5) = (3\cdot 1 - 2\cdot(-5),\ 1-2-5) = \boxed{(13,-6)}$

b) $f(\alpha\vec{u} + \beta\vec{v}) = \alpha f(\vec{u}) + \beta f(\vec{v})$

$\vec{u} = (x_1, y_1, z_1)$

$\vec{v} = (x_2, y_2, z_2)$

$$f(\underbrace{\alpha x_1 + \beta x_2}_{x},\ \underbrace{\alpha y_1 + \beta y_2}_{y},\ \underbrace{\alpha z_1 + \beta z_2}_{z}) =$$

✓

$$= (3\alpha x_1 + 3\beta x_2 - 2\alpha z_1 - 2\beta z_2,\ \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2 + \alpha z_1 + \beta z_2)$$

$$\alpha f(\vec{u}) + \beta f(\vec{v}) =$$

$$= \alpha \cdot (3x_1 - 2z_1, x_1 - y_1 + z_1) + \beta(+ \beta(3x_2 - 2z_2, x_2 - y_2 + z_2) =$$

$$= (3\alpha x_1 + 3\beta x_2 - 2\alpha z_1 - 2\beta z_2, \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2 + \alpha z_1 + \beta z_2)$$

$\boxed{f \text{ es aplicación lineal}}$

Ejercicio: Determinar los subespacios núcleo e imagen de la aplicación lineal $f : \mathbb{R}^3 \to \mathbb{R}^2$, definida por:

$$f(x,y,z) = (x+2y+3z,\ 2x+4y+6z)$$

- Núcleo de f:

Ec. imp. de Ker f.

$$\left.\begin{array}{l} x+2y+3z=0 \\ \cancel{2x+4y+6z=0} \end{array}\right\} \to x = -2y-3z$$

N° par $= 3$ inc $- 1$ ec final $= 2$ par

ec. par. del Ker f

$$\boxed{\begin{array}{ll} x = -2\alpha - 3\beta & \\ y = \alpha & (\alpha \in \beta) \\ z = \beta & \end{array}}$$

$$\boxed{\operatorname{Ker} f = \{(x,y,z) \in \mathbb{R}^3 \,/\, x+2y+3z=0\}}$$

$$N^{\circ}\text{ec. imp.} = \dim(\underset{\mathbb{R}^3}{\overset{\shortparallel}{V}}) - \dim(\operatorname{Ker} f)$$

$$\dim(\operatorname{Ker} f) = \dim(\mathbb{R}^3) - N^{\circ}\text{ec imp} =$$

$$= 3 - 1 = \boxed{2}$$

• <u>Imagen de f :</u>

$$f(\underset{x}{1},\underset{y}{0},\underset{z}{0}) = (1,2)$$

$$f(0,\underset{y}{1},0) = (2,4)$$

$$f(0,0,\underset{z}{1}) = (3,6)$$

$$\operatorname{Im} f = L\{(1,2), \cancel{(2,4), (3,6)}\}$$

$$\boxed{\operatorname{Im} f = L\{(1,2)\}} \qquad \dim(\operatorname{Im} f) = 1$$

$$\underset{\underset{3}{\parallel}}{\dim(V)} = \underset{\underset{2}{\parallel}}{\dim(\operatorname{Ker} f)} + \underset{\underset{1}{\parallel}}{\dim(\operatorname{Im} f)}$$

$$3 = 2 + 1$$