6 Obtener unas ecuaciones paramétricas e implícitas del subespacio de $\mathbb{R}^4$:

$$U = L\{(2,1,0,-3),(0,-1,-2,0),(3,0,1,-1)\}$$

Obtenemos una base de U:

$$\begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & -1 & -2 & 0 \\ 3 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & -1 & -2 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{7}{2} \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 4 & \frac{7}{2} \end{pmatrix}$$

$F_3 \to F_3 - \frac{3}{2}F_1$ $\qquad$ $F_3 \to F_3 - \frac{3}{2}F_2$

$B_U = \{(2,1,0,-3),(0,-1,-2,0),(3,0,1,-1)\} \to \dim(U) = 3$

$(x,y,z,t) = \alpha(2,1,0,-3) + \beta(0,-1,-2,0) + \gamma(3,0,1,-1)$

$$\boxed{\begin{array}{l} x = 2\alpha + 3\gamma \\ y = \alpha - \beta \\ z = -2\beta + \gamma \\ t = -3\alpha - \gamma \\ (\alpha,\beta,\gamma \in \mathbb{R}) \end{array}}$$

$\xrightarrow{\text{II}} x = 2\alpha - 9\alpha - 3t = -7\alpha - 3t$

$\xrightarrow{\text{III}} \alpha = y + \beta$

$\xrightarrow{\text{II}} z = -2\beta - 3\alpha - t$

$\xrightarrow{\text{I}} \gamma = -3\alpha - t$

ec. param. de U

Nº ec. imp $= \underset{\mathbb{R}^4}{\dim(V)} - \dim(U) = 4 - 3 = 1$ ec. imp.

$$x = -7y - 7\beta - 3t \longrightarrow x = -7y + \frac{7(3y+t+z)}{5} - 3t \quad (*)$$

$$z = -2\beta - 3y - 3\beta - t \longrightarrow z = -5\beta - 3y - t$$

$$\downarrow$$

$$5\beta = -3y - t - z$$

$$\beta = \frac{-3y - t - z}{5} = -\frac{(3y+t+z)}{5}$$

$$(*) \xrightarrow{\cdot 5} 5x = -3y + 21y + 7t + 7z - 3t$$

$$\boxed{5x + 14y - 7z + 8t = 0}$$

9 Dado el siguiente subespacio vectorial:

$$W = \{(x, y, z, t) \in \mathbb{R}^4 \,/\, \overbrace{x + y + z + t = 0, y - 2z - t = 0}^{\text{ec. imp.}}\}$$

Hallar una base de W y la dimensión de W.

$$\text{Ec. imp} \xrightarrow{\text{Resolver}} \text{Ec. par} \xrightarrow{\text{C.L}} \text{sist. gen} \xrightarrow{\text{Gauss}} \text{Base}$$

$$\begin{array}{cccc} x & y & z & t \\ \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -1 \end{pmatrix} \longrightarrow \left.\begin{array}{l} x + 2z + t + z + t = 0 \rightarrow x = -3z - 2t \\ y = 2z + t \qquad\qquad (z, t \in \mathbb{R}) \end{array}\right\}$$

S.C.I

$N^{\underline{o}}$ par = 4 inc − 2 ec = 2 par

$$\left.\begin{array}{l} x = -3\alpha - 2\beta \\ y = 2\alpha + \beta \\ z = \alpha \\ t = \beta \end{array}\right\} \quad (x, y, z, t) = (-3\alpha - 2\beta, 2\alpha + \beta, \alpha, \beta) =$$

$$= \alpha\,\underline{(-3, 2, 1, 0)} + \beta\,\underline{(-2, 1, 0, 1)}$$

ec. par.

sist. gen de W $\xrightarrow[\text{L.T}]{}$ base

$$\boxed{B_W = \{(-3, 2, 1, 0), (-2, 1, 0, 1)\} \qquad \dim(W) = 2}$$

x y z t

11 Dado el espacio vectorial $\mathbb{R}^4$, consideramos los subespacios:

$$U_1 = L\{(1,2,0,1)\}$$
$$U_2 = \{(x,y,z,t)\,/\,x-y+z+t=0, y-z=0\}$$
$$U_3 = \{(\alpha,\alpha+\beta,\gamma,\beta)\,/\alpha,\beta,\gamma\in\mathbb{R}\}$$

¿Pertenece el vector $\vec{u} = (2,4,0,2)$ a U_1, U_2 o U_3? En caso afirmativo, calcular sus coordenadas en unas bases elegidas previamente.

- $\vec{u} \in U_1$? $\rightarrow \boxed{\text{SÍ}}$

$\vec{u} = (2,4,0,2) = \alpha\cdot(1,2,0,1) \rightarrow \alpha = 2$ ✓

coordenadas de $\vec{u}$ respecto a la base de B_{U_1}

$B_{U_1} = \{(1,2,0,1)\}$

$$\boxed{\vec{u} = (\underset{x}{2},\underset{y}{4},\underset{z}{0},\underset{t}{2}) = (2)_{B_{U_1}}}$$

- $\vec{u} \in U_2$? $\boxed{\text{NO}}$

$\vec{u}$ debe cumplir las ec. implícitas de U_2.

$x - y + z + t = 0$? $\quad 1 - 2 + 0 + 1 = 0$ ✓

$y - z = 0$? $\quad 2 - 0 \neq 0$ ✗

- $\vec{u} \in U_3$? $\rightarrow \boxed{\text{SÍ}}$

$\vec{u} = (2,4,0,2) = (\alpha, \alpha+\beta, \gamma, \beta) \rightarrow$

$$\left.\begin{array}{l} 2 = \alpha \\ 4 = \alpha+\beta \\ 0 = \gamma \\ 2 = \beta \end{array}\right\} \begin{array}{l} \rightarrow \alpha = 2 \\ \rightarrow 4 = 2+2 ✓ \\ \rightarrow \gamma = 0 \\ \rightarrow \beta = 2 \end{array}$$

$$\boxed{\vec{u} = (2,4,0,2) = (2,2,0)_{B_{U_3}}}$$

coord. de $\vec{u}$ con respecto a B_{U_3}

$$(\alpha, \alpha+\beta, \gamma, \beta) = \alpha(1, \underline{\underline{1, 0, 0}}) + \beta(0, \underline{\underline{1, 0}}, 1) + \gamma(0, \underline{\underline{0, 1, 0}})$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{L.I}$$

$$B_{U_3} = \{(1,1,0,0), (0,1,0,1), (0,0,1,0)\}$$

12 Encontrar una base para el subespacio U de $\mathbb{R}^4$ cuyas ecuaciones paramétricas son:

$$x_1 = \lambda + \alpha + \beta, \quad x_2 = \lambda - \alpha + 3\beta, \quad x_3 = \lambda + 2\alpha, \quad x_4 = 2\lambda + 3\alpha + \beta$$

$$(x_1, x_2, x_3, x_4) = (\lambda + \alpha + \beta,\ \lambda - \alpha + 3\beta,\ \lambda + 2\alpha,\ 2\lambda + 3\alpha + \beta)$$

$$= \lambda (1,1,1,2) + \alpha (1,-1,2,3) + \beta (1,3,0,1)$$

⇝ s. gen de U

$$U = L\{(1,1,1,2), (1,-1,2,3), (1,3,0,1)\}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 3 \\ 1 & 3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & 1 \\ \cancel{0} & \cancel{2} & \cancel{-1} & \cancel{-1} \end{pmatrix}$$

$F_2 \to F_2 - F_1$

$F_3 \to F_3 - F_1$

$$\boxed{B_U = \{(1,1,1,2), (0,-2,1,1)\}}$$

13 Determinar una base para la suma y la intersección de los subespacios U_1 y U_2, engendrados por $\{(1,2,1,0),(-1,1,1,1)\}$ y $\{(2,-1,0,1),(1,-1,3,7)\}$, respectivamente.

L. I   L. I

$\{B_{U_1}, B_{U_2}\} \rightarrow$ s. gen de $U_1 + U_2$

$B_{U_1} = \{(1,2,1,0),(-1,1,1,1)\}$

$B_{U_2} = \{(2,-1,0,1),(1,-1,3,7)\}$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & -5 & -2 & 1 \\ 0 & -3 & 2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 4/3 & \frac{8}{3} \\ 0 & 0 & 4 & 8 \end{pmatrix}$$

$F_2 \rightarrow F_2 + F_1$

$F_3 \rightarrow F_3 - 2F_1$

$F_4 \rightarrow F_4 - F_1$

$F_3 \rightarrow F_3 + \frac{5}{3} F_2$

$F_4 \rightarrow F_4 + F_2$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ \cancel{0} & \cancel{0} & \cancel{4} & \cancel{8} \end{pmatrix}$$

$F_3 \rightarrow \frac{3}{4} F_3$

$$\boxed{B_{U_1+U_2} = \{(1,2,1,0),(0,3,2,1),(0,0,1,2)\}}$$

$\dim(U_1+U_2) = 3$

C.L de B_U = C.L de B_W

$$\alpha(1,2,1,0)+\beta(-1,1,1,1)=\gamma(2,-1,0,1)+\delta(1,-1,3,7)$$

$$\left.\begin{array}{l}\alpha-\beta=2\gamma+\delta\\ 2\alpha+\beta=-\gamma-\delta\\ \alpha+\beta=3\delta\\ \beta=\gamma+7\delta\end{array}\right\}\quad \begin{array}{l}\alpha-\beta-2\gamma-\delta=0\\ 2\alpha+\beta+\gamma+\delta=0\\ \alpha+\beta-3\delta=0\\ \beta-\gamma-7\delta=0\end{array}$$

$$\begin{array}{cccc}\alpha & \beta & \gamma & \delta\end{array}$$

$$\begin{pmatrix}1&-1&-2&-1\\2&1&1&1\\1&1&0&-3\\0&1&-1&-7\end{pmatrix}\underset{F_2\leftrightarrow F_4}{\sim}\begin{pmatrix}1&-1&-2&-1\\0&1&-1&-7\\1&1&0&-3\\2&1&1&1\end{pmatrix}\sim$$

$F_3\to F_3-F_1$

$F_4\to F_4-2F_1$

$$\begin{pmatrix}1&-1&-2&-1\\0&1&-1&-7\\0&2&2&-2\\0&3&5&3\end{pmatrix}\sim\begin{array}{c}\begin{array}{cccc}\alpha&\beta&\gamma&\delta\end{array}\\ \begin{pmatrix}1&-1&-2&-1\\0&1&-1&-7\\0&0&4&12\\\cancel{0}&\cancel{0}&\cancel{8}&\cancel{24}\end{pmatrix}\end{array}\begin{array}{l}\\ \\ \xrightarrow{:4}\\ \text{SCI}\end{array}$$

$F_3\to F_3-2F_2$

$F_4\to F_4-3F_2$

$$\left.\begin{array}{r} \alpha - \beta - 2\gamma - \delta = 0 \\ \beta - \gamma - 7\delta = 0 \\ \gamma + 3\delta = 0 \end{array}\right\} \begin{array}{l} \to \alpha - 4\delta + 6\delta - \delta = 0 \to \alpha = -\delta \\ \to \beta + 3\delta - 7\delta = 0 \to \beta = 4\delta \\ \to \gamma = -3\delta \end{array}$$

4 inc − 3 ec = 1 par $\qquad (\delta \in \mathbb{R})$

$$\boxed{\begin{array}{ll} \alpha = -t & \\ \beta = 4t & (t \in \mathbb{R}) \\ \gamma = -3t & \\ \delta = t & \end{array}}$$

• si $t = 1$: $\gamma = -3$, $\delta = 1$: $-3(2,-1,0,1) + 1\cdot(1,-1,3,7)$

$$\boxed{B_{U_1 \cap U_2} = \{(-5,2,3,4)\}}$$

$$\dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) =$$

$$= 2 + 2 - 3 = 1 \checkmark$$

15 En el espacio vectorial $\mathbb{R}^3$ se consideran los subespacios vectoriales:

$$U = \{(x, y, z) \,/\, x + y + z = 0\}$$
$$W = \{(t, 2t, 3t) \,/\, t \in \mathbb{R}\}$$

Verificar que $\mathbb{R}^3$ es suma directa de U y W, es decir: $\mathbb{R}^3 = U \oplus W$.

1 $U \cap W = \{\vec{0}\} = \{(0,0,0)\}$?

Obtenemos las ec. implícitas de W :

$(x,y,z) = t\underbrace{(1,2,3)}_{\text{sist. gen de } W} \longrightarrow B_W = \{(1,2,3)\} \quad \dim(W) = 1$

$$\left.\begin{array}{l} x = t \\ y = 2t \quad (t \in \mathbb{R}) \\ z = 3t \end{array}\right\} \begin{array}{l} \rightarrow t = x \\ \rightarrow y = 2x \\ \rightarrow z = 3x \end{array} \Big\}$$

ec. par. de W

ec. imp. de W

$N^{\underline{o}}$ ec. imp $= \dim(\overset{\mathbb{R}^3}{\overset{\shortparallel}{V}}) - \dim(W) = 3 - 1 = 2$ ec. imp.

ec. imp. de $U \cap W$:

$$\left.\begin{array}{l} x + \underset{=}{y} + \underset{=}{z} = 0 \\ y = 2x \\ z = 3x \end{array}\right\} \begin{array}{l} \rightarrow x + 2x + 3x = 0 \rightarrow 6x = 0 \rightarrow \boxed{x = 0} \\ \rightarrow y = \boxed{0} \\ \rightarrow z = \boxed{0} \end{array}$$

$U \cap W = \{(0,0,0)\}$ ✓

② $\dim(U) + \dim(W) = \dim(V)$ ✓ $\Rightarrow$ $\boxed{U \oplus W = \mathbb{R}^3}$

$$\underset{2}{\overset{\shortparallel}{\dim(U)}} + \underset{1}{\overset{\shortparallel}{\dim(W)}} = \underset{3}{\dim(\underset{\mathbb{R}^3}{\overset{\shortparallel}{V}})}$$