Ejercicios

❽ Calcular el rango de la matriz a partir de sus menores:

$$A = \begin{pmatrix} -1 & 0 & 0 & 2 & -2 \\ 3 & 1 & -1 & 4 & 0 \\ 2 & 1 & -1 & 6 & -2 \\ 6 & 2 & -2 & 1 & 0 \end{pmatrix}$$

- Como A no es matriz nula $\rightarrow rg(A) > 0$.

- Menores de orden 1 : $-1 \neq 0 \rightarrow rg(A) \geq 1$.

- Menores de orden 2 : $\begin{array}{c c} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} F_1 \\ F_2 \end{array} & \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} \end{array} = -1 \cdot 1 - 3 \cdot 0 = -1 \neq 0 \rightarrow rg(A) \geq 2$

- Menores de orden 3 : a partir del menor de orden $2 \neq 0$.

$$\begin{array}{c c} & \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{array}{c} F_1 \\ F_2 \\ F_3 \end{array} & \begin{vmatrix} -1 & 0 & 0 \\ 3 & 1 & -1 \\ 2 & 1 & -1 \end{vmatrix} \end{array} = 0$$

$C_3 = -C_2$

$$\begin{array}{c c} & \begin{array}{ccc} & & C_4 \end{array} \\ \begin{array}{c} \\ \\ F_3 \end{array} & \begin{vmatrix} -1 & 0 & 2 \\ 3 & 1 & 4 \\ 2 & 1 & 6 \end{vmatrix} \end{array} = 0$$

$F_3 = F_1 + F_2$

$$\begin{array}{c c} & \begin{array}{ccc} & & C_5 \end{array} \\ \begin{array}{c} \\ \\ F_3 \end{array} & \begin{vmatrix} -1 & 0 & -2 \\ 3 & 1 & 0 \\ 2 & 1 & -2 \end{vmatrix} \end{array} = 0$$

$F_3 = F_1 + F_2$

$$\begin{array}{c c} & \begin{array}{ccc} & & C_3 \end{array} \\ \begin{array}{c} \\ \\ F_4 \end{array} & \begin{vmatrix} -1 & 0 & 0 \\ 3 & 1 & -1 \\ 6 & 2 & -2 \end{vmatrix} \end{array} = 0$$

$C_3 = -C_2$

$$\begin{array}{c c} & \begin{array}{ccc} C_1 & C_2 & C_4 \end{array} \\ \begin{array}{c} F_1 \\ F_2 \\ F_4 \end{array} & \begin{vmatrix} -1 & 0 & 2 \\ 3 & 1 & 4 \\ 6 & 2 & 1 \end{vmatrix} \end{array} = -1 + 0 + 12 - (12 - 8 + 0) = -1 + \cancel{12} - \cancel{12} + 8$$

$$= 7 \neq 0$$

$rg(A) \geq 3$

• Menores orden 4 : añadimos F_3 y C_3 / C_5 .

$$\begin{array}{cc} & \begin{array}{cccc} & & C_3 & \end{array} \\ F_3 & \begin{vmatrix} -1 & 0 & 0 & 2 \\ 3 & 1 & -1 & 4 \\ 2 & 1 & -1 & 6 \\ 6 & 2 & -2 & 1 \end{vmatrix} \end{array} \underset{\substack{\uparrow \\ C_3 = -C_2}}{=} 0 \qquad \begin{array}{cc} & \begin{array}{cccc} & & & C_5 \end{array} \\ F_3 & \begin{vmatrix} -1 & 0 & 2 & -2 \\ 3 & 1 & 4 & 0 \\ 2 & 1 & 6 & -2 \\ 6 & 2 & 1 & 0 \end{vmatrix} \end{array} \underset{\substack{\uparrow \\ F_2 = F_1 + F_2}}{=} 0$$

Como todos los menores de orden 4 son 0 $\rightarrow$ $\boxed{rg(A) = 3}$

❾ Resolver el ejercicio 7 utilizando menores.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ \lambda & -1 & 10 \\ -1 & \lambda & -6 \\ 2 & 5 & 1 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} C_2 & C_3 \end{array} \\ \begin{array}{c} F_1 \\ F_2 \end{array} & \begin{vmatrix} 2 & 1 \\ -1 & 10 \end{vmatrix} \end{array} = 20 - (-1) \cdot 1 = 21 \neq 0$$

$rg(A) \geq 2$

$$\begin{array}{cc} & \begin{array}{ccc} C_1 & & \end{array} \\ \begin{array}{c} \\ \\ F_3 \end{array} & \begin{vmatrix} 1 & 2 & 1 \\ \lambda & -1 & 10 \\ -1 & \lambda & -6 \end{vmatrix} \end{array} = 6 - 20 + \lambda^2 - (1 + 10\lambda - 12\lambda) =$$

$$= -14 + \lambda^2 - 1 + 2\lambda = \lambda^2 + 2\lambda - 15 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-15)}}{2 \cdot 1} = \frac{-2 \pm 8}{2} = \begin{cases} \boxed{\lambda = 3} \\ \lambda = -5 \end{cases}$$

$$\begin{array}{cc} & \begin{array}{ccc} C_1 & & \end{array} \\ \begin{array}{c} \\ \\ F_4 \end{array} & \begin{vmatrix} 1 & 2 & 1 \\ \lambda & -1 & 10 \\ 2 & 5 & 1 \end{vmatrix} \end{array} = -1 + 40 + 5\lambda - (-2 + 50 + 2\lambda) =$$

$$= -1 + 40 + 5\lambda + 2 - 50 - 2\lambda = 3\lambda - 9 = 0$$

$$3\lambda = 9 \longrightarrow \boxed{\lambda = \frac{9}{3} = 3}$$

- si $\lambda = 3$: $rg(A) = 2$ ✓
- si $\lambda \neq 3$: $rg(A) = 3$ ✓

Ejercicios

10 En caso de existir, calcular las matrices inversas de A y B mediante Gauss-Jordan:

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 3 & 7 & 9 \\ 1 & 5 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 4 & 1 & 2 & 0 \\ 2 & -1 & 3 & 1 \\ 4 & 2 & 1 & -5 \end{pmatrix}$$

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ 1 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & 3 & -3 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$F_2 \to F_2 - 3F_1$

$F_3 \to F_3 - F_1$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & -5 & 3 & -3 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 6 & 5 & 0 & -4 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -2 & -8 & 1 & 5 \end{array}\right)$$

$F_2 \longleftrightarrow F_3$

$F_1 \to F_1 - 4F_2$

$F_3 \to F_3 + 5F_2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -19 & 3 & 11 \\ 0 & 2 & 0 & 6 & -1 & -3 \\ 0 & 0 & -2 & -8 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -19 & 3 & 11 \\ 0 & 1 & 0 & 3 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 4 & -\frac{1}{2} & -\frac{5}{2} \end{array}\right)$$

$F_1 \to F_1 + 3F_3$

$F_2 \to 2F_2 - F_3$

$F_2 \to \frac{1}{2} F_2$

$F_3 \to -\frac{1}{2} F_3$

$I \qquad A^{-1}$

$$(B \mid I) \sim \left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & -1 & 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & -5 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$F_2 \to F_2 - 4F_1$

$F_3 \to F_3 - 2F_1$

$F_4 \to F_4 - 4F_1$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -6 & -12 & -4 & 1 & 0 & 0 \\ 0 & 1 & -1 & -5 & -2 & 0 & 1 & 0 \\ 0 & 6 & -7 & -17 & -4 & 0 & 0 & 1 \end{array}\right) \sim$$

$F_2 \leftrightarrow F_3$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -5 & -2 & 0 & 1 & 0 \\ 0 & 5 & -6 & -12 & -4 & 1 & 0 & 0 \\ 0 & 6 & -7 & -17 & -4 & 0 & 0 & 1 \end{array}\right) \sim$$

$F_1 \to F_1 + F_2$

$F_3 \to F_3 - 5F_2$

$F_4 \to F_4 - 6F_2$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -5 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 13 & 6 & 1 & -5 & 0 \\ \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{13} & 8 & 0 & -6 & 1 \end{array}\right) \to$$

Como no podemos obtener I de orden 4

$\Rightarrow B^{-1}$ no existe.

Ejercicios

⓫ Utilizando determinantes, calcular las inversas de las siguientes matrices:

a) $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & -3 & 3 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$

a) $|A| = \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} = 6 - (-2)\cdot(-1) = 6 - 2 = 4$

$$A^{-1} = \frac{1}{4} \cdot \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$$

b) $|A| = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & -3 & 3 \end{vmatrix} = 18 - 3 - (3 - 12) = 15 + 9 = 24$

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 15 & -1 & -6 \\ -3 & 5 & 6 \\ -3 & -3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

signos A_{ij}

$$A_{11} = + \begin{vmatrix} 3 & 2 \\ -3 & 3 \end{vmatrix} = 9 + 6 = 15$$

$$A_{12} = - \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -[3 - 2] = -1$$

$$A_{13} = + \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = -3 - 3 = -6$$

$$A_{21} = - \begin{vmatrix} 0 & 1 \\ -3 & 3 \end{vmatrix} = -[0+3] = -3$$

$$A_{22} = + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6-1 = 5$$

$$A_{23} = - \begin{vmatrix} 2 & 0 \\ 1 & -3 \end{vmatrix} = -[-6-0] = 6$$

$$A_{31} = + \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 0-3 = -3$$

$$A_{32} = - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -[4-1] = -3$$

$$A_{33} = + \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = 6$$

$$A^{-1} = \frac{1}{24} \cdot \begin{pmatrix} 15 & -3 & -3 \\ -1 & 5 & -3 \\ -6 & 6 & 6 \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{24} & \frac{5}{24} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

c) $$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{pmatrix}$$

Si A es diagonal

con $a_{ii} \neq 0$

Ejercicios

⓬ Hallar el valor de a para que la siguiente matriz sea invertible:

$$\begin{pmatrix} a+1 & 1 & 1 \\ 1 & a-1 & 1 \\ 0 & 1 & a+2 \end{pmatrix}$$

A será invertible si $|A| \neq 0$:

$$\begin{vmatrix} a+1 & 1 & 1 \\ 1 & a-1 & 1 \\ 0 & 1 & a+2 \end{vmatrix} = (a+1)\cdot(a-1)\cdot(a+2) + 1 - \overbrace{(a+1+a+2)}^{-2a-3} =$$

$$= (a+1)(a-1)(a+2) - 2a - 2 = (a+1)(a-1)(a+2) - 2(a+1) =$$

$$= (a+1)\cdot\big((a-1)(a+2)-2\big) = 0 \longrightarrow a+1 = 0 \longrightarrow \boxed{a=-1}$$

$$\downarrow$$

$$(a-1)(a+2) - 2 = 0 \longrightarrow a^2 + 2a - a - 2 - 2 = 0$$

$$a^2 + a - 4 = 0 \longrightarrow a = \frac{-1 \pm \sqrt{1-4\cdot 1\cdot(-4)}}{2\cdot 1} = \boxed{\frac{-1\pm\sqrt{17}}{2}}$$

A será invertible cuando $a \neq -1$ y $a \neq \frac{-1\pm\sqrt{17}}{2}$.

Ejercicios

⓭ Resolver la ecuación matricial:

$$B(2A + I) = AXA + B$$

Donde:

$$A = \begin{pmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$B(2A + I) = AXA + B \qquad 2BA + B\cdot\cancel{I}^{\,1} = AXA + B$$

$$A^{-1}\cdot 2BA + \cancel{B} = \overbrace{\cancel{A^{-1}\cdot A}}^{I}XA + \cancel{B} \qquad 2A^{-1}\cdot B\overbrace{\cancel{A\cdot A^{-1}}}^{I} = X\overbrace{\cancel{A\cdot A^{-1}}}^{I}$$

$$X = 2A^{-1}B \longrightarrow \qquad X = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 10 & 14 \\ -4 & -8 & -10 \end{pmatrix}\cdot\begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} =$$

$$A^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{pmatrix} \qquad = \underline{\underline{\begin{pmatrix} -2 & -8 & 14 \\ -6 & -18 & 32 \\ 4 & 14 & -26 \end{pmatrix}}}$$

Ejercicios

14 Resolver la ecuación matricial:

$$BAX + AX = C - I - 2DAX$$

Donde:

$$A = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \quad B = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \quad C = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad D = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

$$BAX + AX = C - I - 2DAX \qquad BAX + AX + 2DAX = C - I$$

¡a la derecha!

$$(B + I + 2D)\cdot AX = C - I$$

$$\overbrace{[(B+I+2D)\cdot A]^{-1}\cdot(B+I+2D)\cdot A}^{I}\cdot X = [(B+I+2D)A]^{-1}\cdot(C-I)$$

$$(B+I+2D)\cdot A = \left[\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\right]\cdot\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} =$$

$$= \begin{pmatrix}5 & 3\\ 3 & 5\end{pmatrix}\cdot\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}8 & 5\\ 8 & 3\end{pmatrix}$$

$$[(B+I+2D)\cdot A]^{-1} = \frac{1}{-16}\begin{pmatrix}3 & -5\\ -8 & 8\end{pmatrix}$$

$$X = -\frac{1}{16}\begin{pmatrix}3 & -5\\ -8 & 8\end{pmatrix}\cdot\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix} = -\frac{1}{16}\begin{pmatrix}-8 & 8\\ 16 & -16\end{pmatrix} = \underline{\underline{\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -1 & 1\end{pmatrix}}}$$

Ejercicios

15. Resolver el sistema matricial:

$$\left.\begin{aligned} 2X + 3Y &= A \\ 5X - 2Y &= B \end{aligned}\right\} \text{ donde: } A = \begin{pmatrix} 4 & 8 \\ 7 & 11 \end{pmatrix} \quad B = \begin{pmatrix} 10 & 1 \\ 8 & 18 \end{pmatrix}$$

$$\begin{aligned} 2X + 3Y = A &\xrightarrow{\cdot 2} 4X + 6Y = 2A \\ 5X - 2Y = B &\xrightarrow{\cdot 3} 15X - 6Y = 3B \end{aligned}$$

$+$

$$\underbrace{19}_{}X \quad \cancel{} = 2A + 3B \qquad X = \frac{1}{19}(2A + 3B)$$

¡es un nº!

$$3Y = A - 2X \longrightarrow Y = \frac{1}{3}(A - 2X)$$

$$X = \frac{1}{19}\left[\begin{pmatrix} 8 & 16 \\ 14 & 22 \end{pmatrix} + \begin{pmatrix} 30 & 3 \\ 24 & 54 \end{pmatrix}\right] = \frac{1}{19}\begin{pmatrix} 38 & 19 \\ 38 & 76 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}}}$$

$$Y = \frac{1}{3}\left[\begin{pmatrix} 4 & 8 \\ 7 & 11 \end{pmatrix} + \begin{pmatrix} -4 & -2 \\ -4 & -8 \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} 0 & 6 \\ 3 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}}}$$

⑯ Resolver el sistema matricial: $\left.\begin{aligned} X^t + AY &= B \\ X + Y^t C &= D \end{aligned}\right\}$ donde:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

$$X^t + AY = B \longrightarrow X^t + AY = B$$

$$X + Y^t \cdot C = D \xrightarrow{(\ )^t} X^t + (Y^t \cdot C)^t = D^t \rightarrow X^t + C^t \cdot Y = D^t$$

$$\ominus \qquad AY - C^t \cdot Y = B - D^t$$

¡derecha!

$$(A - C^t) \cdot Y = B - D^t$$

$$Y = (A - C^t)^{-1} \cdot (B - D^t)$$

$$Y = \left[\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} \right]^{-1} \cdot \left[\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \right] =$$

$$= \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{-1}{3} \begin{pmatrix} 1 & -1 \\ -3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} =$$

$$= -\frac{1}{3} \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}}}$$

$$X = D - Y^t \cdot C = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} \\ \frac{4}{3} & -\frac{1}{3} \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{11}{3} & \frac{7}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}}}$$