Outline

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Common paradigms:

- Statistical frameworks: PCA, t-SNE
- Neural-based approaches: Autoencoder

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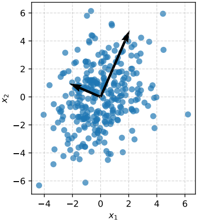

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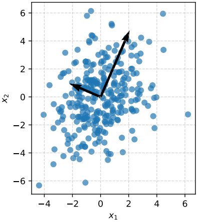

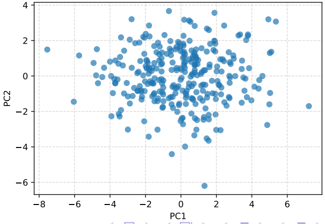

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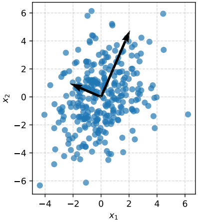

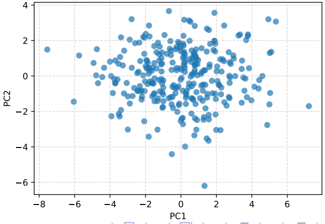

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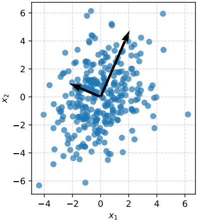

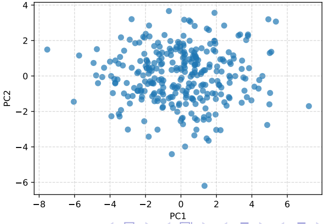

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  $\rightarrow$ Key condition: *eigenvector* must maximize $\mathrm{Var}(\mathbf{Dv})$

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- d'-th *eigenvector*: variance condition + orthogonal to $\{\mathbf{v}_i\}_{i=1}^{d'-1}$

$$\mathbf{v}_{d'} = \arg\max_{||\mathbf{v}||=1, \mathbf{v}\perp\{\mathbf{v}_i\}_{i=1}^{d'-1}} \mathbf{v}^T \Sigma \mathbf{v}$$

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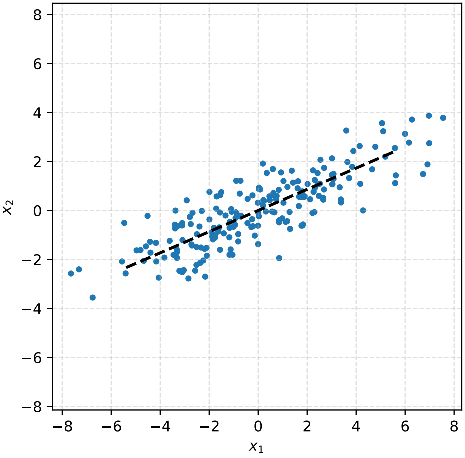
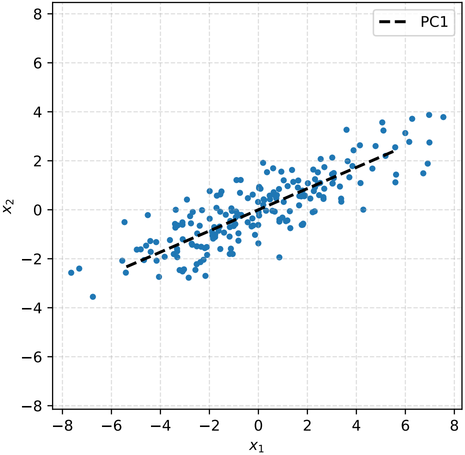
$$\mathrm{CumVar}(d') = \frac{\sum_{i=1}^{d'} \lambda_i}{\sum_{j=1}^{d} \lambda_j} \quad \text{where } \lambda_j = \mathbf{v}_j^T \Sigma \mathbf{v}_j$$

Example - Single component



Example - Two components



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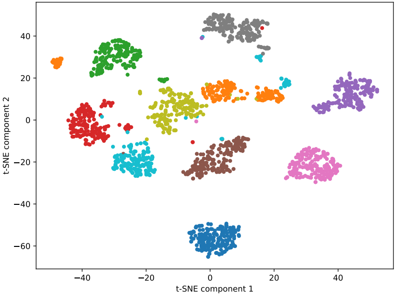

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$$\min \sum_{\mathbf{x} \in \mathcal{D}} P(\mathbf{x}) \log \frac{P(\mathbf{x})}{Q(\mathbf{x})} = \min D_{KL}(P||Q)$$

$D_{KL} \rightarrow$ Kullback-Leibler Divergence

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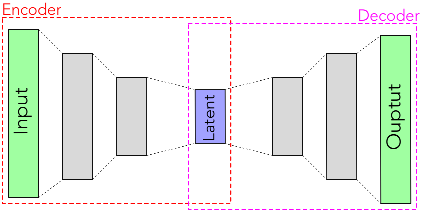

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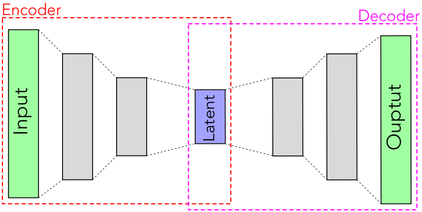

- Once trained, we are only interested in the Encoder
  - → Decoder in this case is used for training purposes

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Given an input datum $\mathbf{x}$:

- Embedded representation: $\mathbf{x}^e = E_{\phi_E}(\mathbf{x})$
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Optimization: minimize the difference between input and reconstruction

$$\underset{\phi_E,\phi_D}{\arg\min}\, \mathcal{L}\left(\mathbf{x}, \tilde{\mathbf{x}}\right) = \underset{\phi_E,\phi_D}{\arg\min}\, \mathcal{L}\left(\mathbf{x}, D_{\phi_D}\left(E_{\phi_E}(\mathbf{x})\right)\right)$$

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- Related to PCA:
  - → Single hidden layer with linear activation ⇒ Equivalent approaches

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   - **Strategies**: PCA error, Autoencoder, Isolation Forest

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3. Motif discovery
   - Mining patterns in time-series data collections

T6: Unsupervised learning

Fundamentos del Aprendizaje Automático

Curso 2025/2026