5 Dada la matriz:

$$A = \begin{pmatrix} b-1 & 0 & 0 \\ -1 & b & 2-b \\ 0 & 0 & 2 \end{pmatrix} \text{ con } b \in \mathbb{R}$$

Comprobar que $\lambda = 2$ es un valor propio de A para todo b y estudiar en qué casos $\lambda = 3$ es un valor propio.

$\lambda = 2$ será un valor propio de A si:

$$|A - \lambda I| = 0 \longrightarrow |A - 2I| = 0$$

$$\begin{vmatrix} b-3 & 0 & 0 \\ -1 & b-2 & 2-b \\ 0 & 0 & 0 \end{vmatrix} = 0 \rightarrow \boxed{\text{Para todo } b \in \mathbb{R} \ \lambda = 2 \text{ es valor propio de } A.}$$

Fila nula

Analizamos cuándo $\lambda = 3$ será un valor propio de A: $|A - 3I| = 0$

$$\begin{vmatrix} b-4 & 0 & 0 \\ -1 & b-3 & 2-b \\ 0 & 0 & -1 \end{vmatrix} = -(b-4)(b-3) = 0$$

$b - 4 = 0$ $\quad$ $b - 3 = 0$

$\underline{\underline{b = 4}}$ $\quad$ $\underline{\underline{b = 3}}$

$\boxed{\lambda = 3 \text{ será un valor propio de } A \text{ cuando } b = 3 \text{ o } b = 4.}$

6 Determinar los autovalores reales y subespacios propios asociados a las matrices:

a) $A = \begin{pmatrix} -2 & 3 \\ -4 & 5 \end{pmatrix}$ $\rightarrow$ **b)** $B = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix}$ $\rightarrow$ **c)** $C = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$

b) $B = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} \rightarrow |B - \lambda I| = 0$

$$\begin{vmatrix} -4-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix} = (-4-\lambda)\cdot(2-\lambda) + 9 = 0$$

$$-8 + 4\lambda - 2\lambda + \lambda^2 + 9 = 0 \rightarrow \lambda^2 + 2\lambda + 1 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4\cdot 1\cdot 1}}{2\cdot 1} = \frac{-2 \pm 0}{2} = \begin{cases} \lambda = -1 \\ \lambda = -1 \end{cases}$$

$$\boxed{\lambda = -1 \quad m = 2}$$

- <u>Para $\lambda = -1$:</u> $(B + I)\cdot\vec{v} = \vec{0}$

$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{matrix} -\cancel{3}x + \cancel{3}y = 0 \\ \text{-------------} \end{matrix}\right\} \rightarrow y = x, \quad \underset{\alpha}{\overset{\parallel}{x}} \in \mathbb{R}$$

$$\boxed{V_\lambda = \{(\alpha, \alpha) \in \mathbb{R}^2 / \alpha \in \mathbb{R}\}} \qquad \dim(V_{\lambda_1}) = 1$$

c) $C = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ $\quad |C - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & -5 \\ 2 & 3-\lambda \end{vmatrix} = 0 \rightarrow \underbrace{(-3-\lambda)}_{}(3-\lambda) + 10 = 0$$

$-(3+\lambda)$

$$-(9-\lambda^2) + 10 = 0 \rightarrow \lambda^2 + 1 = 0 \rightarrow \lambda^2 = -1$$

$\lambda = \pm\sqrt{-1} \notin \mathbb{R}$ $\quad$ $\boxed{\text{La matriz } C \text{ no tiene autovalores reales}}$

9 Estudiar si la siguiente matriz es diagonalizable:

$$A = \begin{pmatrix} -11 & -8 & -8 \\ 8 & 5 & 8 \\ 4 & 4 & 1 \end{pmatrix}$$

En caso afirmativo, encontrar una matriz diagonal D y una matriz de paso P que verifiquen que $AP = PD$.

$$|A - \lambda I| = 0 \longrightarrow \begin{vmatrix} -11-\lambda & -8 & -8 \\ 8 & 5-\lambda & 8 \\ 4 & 4 & 1-\lambda \end{vmatrix} \underset{F_1 = F_1 + F_2}{=}$$

$$= \begin{vmatrix} -3-\lambda & -3-\lambda & 0 \\ 8 & 5-\lambda & 8 \\ 4 & 4 & 1-\lambda \end{vmatrix} = (-3-\lambda) \cdot \begin{vmatrix} 1 & 1 & 0 \\ 8 & 5-\lambda & 8 \\ 4 & 4 & 1-\lambda \end{vmatrix} \underset{C_2 = C_2 - C_1}{=}$$

$$= (-3-\lambda) \begin{vmatrix} 1 & 0 & 0 \\ 8 & -3-\lambda & 8 \\ 4 & 0 & 1-\lambda \end{vmatrix} = (-3-\lambda) \cdot (-3-\lambda) \cdot (1-\lambda) = 0$$

$$(-3-\lambda)^2 \cdot (1-\lambda) = 0$$

$$-3-\lambda = 0 \qquad 1-\lambda = 0$$

$\lambda_1 = -3$	$\lambda_2 = 1$
$m_1 = 2$	$m_2 = 1$

• $\underline{\text{Para } \lambda_1 = -3}$: $(A+3I)\cdot\vec{v} = \vec{0}$

$$\begin{pmatrix} -8 & -8 & -8 \\ 8 & 8 & 8 \\ 4 & 4 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{array}{l} \text{--------------------} \\ \text{--------------------} \\ \cancel{4}x + \cancel{4}y + \cancel{4}z = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} x = -y-z \\ \underset{\alpha}{y}, \underset{\beta}{z} \in \mathbb{R} \end{array}\right\} \qquad V_{\lambda_1} = \{(-\alpha-\beta, \alpha, \beta) \in \mathbb{R}^3 / \alpha, \beta \in \mathbb{R}\}$$

$$B_{\lambda_1} = \{\underbrace{(-1,1,0),(-1,0,1)}_{L.I}\}$$

$$\dim(V_{\lambda_1}) = 2 \qquad \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\boxed{\text{La matriz } A \text{ es diagonalizable}}$

1. $m_1 + m_2 = 2 + 1 = 3 = n$ ✓

2. $m_1 = \underbrace{\dim(V_{\lambda_1})}_{2 \text{ autovec.}} = 2$ y $m_2 = \underbrace{\dim(V_{\lambda_2})}_{1 \text{ autovec.}} = 1$ ✓

• <u>Para $\lambda_2 = 1$:</u> $(A - I) \cdot \vec{v} = \vec{0}$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{pmatrix} -12 & -8 & -8 \\ 8 & 4 & 8 \\ 4 & 4 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{array}{l} ① = -(② + ③) \\ \hline 8x + 4y + 8z = 0 \\ \cancel{4}x + \cancel{4}y \quad = 0 \end{array}\right\}$$

$$\downarrow$$

$$② \xrightarrow{:4} 2x + y + 2z = 0$$

$$\underbrace{-2y + y}_{-y} + 2z = 0 \longrightarrow \left.\begin{array}{l} x = -y \\ \\ z = \frac{1}{2} y \end{array}\right\} \begin{array}{c} y \in \mathbb{R} \\ \shortparallel \\ \alpha \end{array}$$

$$V_{\lambda_2} = \{ (-\alpha, \alpha, \tfrac{1}{2}\alpha) \in \mathbb{R}^3 / \alpha \in \mathbb{R} \}$$

$$\downarrow \alpha = 2$$

$$B_{\lambda_2} = \{ (-2, 2, 1) \} \longrightarrow \vec{v}_3 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

$$\boxed{D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} -1 & -1 & -2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}}$$

13 Estudiar en función de los parámetros $a, b \in \mathbb{R}$ si la matriz $A = \begin{pmatrix} a & 1 & b \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ es diagonalizable.

$$|A - \lambda I| = 0 \longrightarrow \begin{vmatrix} a-\lambda & 1 & b \\ 0 & 2-\lambda & 0 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)\cdot(2-\lambda)\cdot(-1-\lambda) = 0 \quad \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 2 \\ \lambda_3 = a \end{cases}$$

- $\underline{\text{Si } a \neq -1, 2 \text{ y } b \in \mathbb{R}}$: $m_1 = m_2 = m_3 = 1 \rightarrow \boxed{A \text{ es diagonaliz.}}$

- $\underline{\text{Si } a = -1}$: $m_1 = 2 \quad m_2 = 1$

  Para $\lambda_1 = -1$: $(A + I)\cdot \vec{v} = \vec{0}$

$a = -1$

$$\begin{pmatrix} 0 & 1 & b \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \left. \begin{array}{r} y + bz = 0 \\ y = 0 \end{array} \right\}$$

$$\cancel{0} + bz = 0 \longrightarrow \underline{\underline{z = \frac{0}{b}}} \; ?$$

$$\longrightarrow \boxed{\text{si } b \neq 0} : \quad \left.\begin{array}{l} \overset{\alpha}{\overset{\shortparallel}{x}} \in \mathbb{R} \\ y = 0 \\ z = 0 \end{array}\right\} \to \text{1 autovector !}$$

(1 par.)

$\boxed{\text{A no es diagonaliz.}}$

$$\longrightarrow \boxed{\text{si } b = 0} \quad \left.\begin{array}{l} y = 0 \\ \cancel{y = 0} \end{array}\right\} \to \text{2 autovect.}$$

$\boxed{\text{A es diagonaliz.}}$

$$\underset{\alpha}{\underset{\shortparallel}{x}}, \underset{\beta}{\underset{\shortparallel}{z}} \in \mathbb{R}$$

• Si $a = 2$: $m_1 = 1$ $m_2 = 2$

Para $\lambda_2 = 2$: $(A - 2I) \cdot \vec{v} = \vec{0}$

$a = 2$

$$\begin{pmatrix} 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \left.\begin{array}{l} y + bz = 0 \\ \\ y - 3z = 0 \end{array}\right\}$$

$$\downarrow$$

$$y = 3z$$

$$\to 3z + bz = 0$$

$$z(b+3) = 0$$

$$z = \frac{0}{b+3} = ?$$

$\longrightarrow$ $\boxed{\text{si } b+3 \neq 0 : b \neq -3}$

$$\left.\begin{array}{l} y = 3z = 0 \\ z = 0 \end{array}\right\} \quad \underset{\alpha}{\underset{\parallel}{x}} \in \mathbb{R} \quad \rightarrow \quad 1 \text{ autovec.}$$

$\boxed{\text{A no es diag.}}$

$\longrightarrow$ $\boxed{\text{si } b+3 = 0 : b = -3}$

$$\left.\begin{array}{l} y = -3z \\ \hline \cancel{y = -3z} \end{array}\right\} \quad \underset{\alpha}{\underset{\parallel}{x}},\ \underset{\beta}{\underset{\parallel}{z}} \in \mathbb{R} \quad \rightarrow \quad 2 \text{ autovec.}$$

$\boxed{\text{A sí es diagonaliz}}$

14 Mediante diagonalización, calcular:

$$\underbrace{A^{k+1}=\begin{pmatrix}1&1\\1&1\end{pmatrix}}_{A}^{k+1},\ k\in\mathbb{N}$$

Si A es diagonalizable: $A^n = P\cdot D^n\cdot P^{-1} \Rightarrow A^{K+1} = P\cdot D^{K+1}\cdot P^{-1}$

$|A-\lambda I| = 0 \longrightarrow \begin{vmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix} = 0$

$(1-\lambda)^2-1 = 0 \longrightarrow \cancel{1}-2\lambda+\lambda^2-\cancel{1} = 0 \longrightarrow \lambda\cdot(\lambda-2) = 0$

$$\boxed{\begin{array}{ll}\lambda_1 = 0 & m_1 = 1\\ \lambda_2 = 2 & m_2 = 1\end{array}}$$

$\longrightarrow$ A es diagonalizable ya que $m_1 = m_2 = 1$.

- Para $\lambda_1 = 0$: $(A-0\cdot I)\cdot\vec{v} = \vec{0}$

$\begin{pmatrix}1&1\\1&1\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \longrightarrow \left.\begin{array}{l}\text{------}\\ x+y = 0\end{array}\right\} \to \begin{array}{l}\overset{\alpha}{\underset{\shortparallel}{}}\\ y\in\mathbb{R}\\ x=-y\end{array}$

$V_{\lambda_1} = \{(-\alpha,\alpha)\in\mathbb{R}^2 / \alpha\in\mathbb{R}\}$

$B_{\lambda_1} = \{(-1,1)\} \longrightarrow \vec{v}_1 = \begin{pmatrix}-1\\1\end{pmatrix}$

• <u>Para $\lambda_2 = 2$</u> : $(A - 2I)\cdot\vec{v} = \vec{0}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \left.\begin{matrix} \text{- - - - - - - - -} \\ x - y = 0 \end{matrix}\right\} \rightarrow x = y \qquad \overset{\alpha}{\overset{\shortparallel}{y}} \in \mathbb{R}$$

$$V_{\lambda_2} = \{(\alpha, \alpha) \in \mathbb{R}^2 / \alpha \in \mathbb{R}\}$$

$$B_{\lambda_2} = \{(1,1)\} \longrightarrow \underline{\underline{\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$$

Entonces :

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad y \quad P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Calculamos P^{-1} : $$P^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Entonces :

$$A^{K+1} = \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}_{P} \cdot \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}^{K+1}}_{A} \cdot \underbrace{\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}_{P^{-1}} =$$

$$= \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 2^{K+1} \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2^{K+1} \\ 0 & 2^{K+1} \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2^{K+1} & 2^{K+1} \\ 2^{K+1} & 2^{K+1} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2^K & 2^K \\ 2^K & 2^K \end{pmatrix}}}$$