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Conditional risk

Expected loss associated to each action $\alpha_i \in \mathcal{A}$ given $\mathbf{x}$:

$$R(\alpha_i|\mathbf{x}) = \sum_{j=1}^{|\mathcal{W}|} \lambda(\alpha_i|w_j) \cdot P(w_j|\mathbf{x})$$

If I see feature vector $\mathbf{x}$*, what is my expected loss if I decide* α_i*?*
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Risk (or loss)

Average over all possible observations of conditional risk

$$R = \int_{\mathbb{R}^d} R\left(\gamma(\mathbf{x})|\mathbf{x}\right) \cdot p(\mathbf{x}) d\mathbf{x}$$

where decision rule $\gamma : \mathbb{R}^d \rightarrow \mathcal{A}$ selects the appropriate action to take

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- Bayes risk: Best performance that can be achieved
  $\rightarrow$ Usually denoted as R^*

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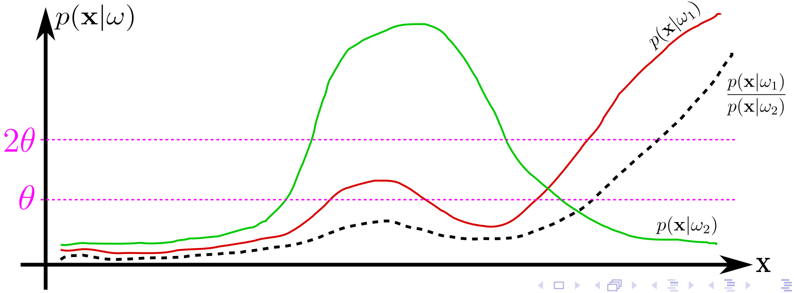

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$\rightarrow$ Bayes error:

$\Rightarrow$ Bayes risk for classification problems with zero-one loss

$\Rightarrow$ Irreducible misclassification probability

Summary

- **Loss** (λ): cost of one action (α_i) VS a state of nature (ω_j)
- **Conditional risk** ($R(\alpha_i|\mathbf{x})$): expected loss for observation $\mathbf{x}$ and action (α_i)
- **Risk** (R): also *expected loss*; overall expected loss for a decision rule (α_i) across the whole distribution of data
- **Bayes risk**: the minimum possible risk (the theoretical lower bound)
- **Bayes rule**: Action selection function (γ) that selects the optimal action that minimizes the risk (Bayes risk)
- **Bayes error**: Bayes risk for classification problems with zero-one loss

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- When $c = 2$ (binary classification), only one function is considered:

$$g(\mathbf{x}) = g_1(\mathbf{x}) - g_2(\mathbf{x})$$

Outline

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  - → **Parametric Pattern Recognition**: Particular case in which $p(\mathbf{x}|\omega_i)$ can be characterized by few parameters
  - → Simplifies the problem of *estimating an unknown function* $p(\mathbf{x}|\omega_i)$ to one of estimating the parameters of a distribution

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$\Rightarrow$ What about **prior** $P(\omega)$?

Parameter estimation

Likelihood

The function that measures how likely is to obtain precisely this dataset, given the parameters is the so-called **likelihood** of $\boldsymbol{\theta}$ with regard to $\mathcal{D}$.

$$p(\mathcal{D}|\boldsymbol{\theta}) = \prod_{n=1}^{|\mathcal{D}|} p(\mathbf{x}_n|\boldsymbol{\theta})$$

Recall: Samples are drawn i.i.d. from the original data distribution

Parameter estimation

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  - $\rightarrow$ In practise *log-likelihood* function: $l(\boldsymbol{\theta}) \equiv \ln p(\mathcal{D}|\boldsymbol{\theta})$

Maximum Likelihood Estimation

- The MLE problem may be formally described as:

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} l(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\theta}} \ln p(\mathcal{D}|\boldsymbol{\theta})$$

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- The set of necessary conditions for the MLE of $\boldsymbol{\theta}$ can be obtained from the set of $|\boldsymbol{\theta}|$ equations:

$$\boldsymbol{\nabla}_{\boldsymbol{\theta}}\, l(\boldsymbol{\theta}) = 0 \ \text{ where } \ \boldsymbol{\nabla}_{\boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial}{\partial \theta_1} \\ \dots \\ \frac{\partial}{\partial \theta_{|\boldsymbol{\theta}|}} \end{bmatrix}$$

Gaussian case

Univariate case

Gaussian distribution with $\boldsymbol{\theta} = (\mu, \sigma^2)$ where $\mathbf{x} \in \mathbb{R} \equiv x$:

$$\hat{\mu} = \frac{1}{|\mathcal{D}|} \sum_{n=1}^{|\mathcal{D}|} x_n \qquad \hat{\sigma^2} = \frac{1}{|\mathcal{D}|} \sum_{n=1}^{|\mathcal{D}|} (x_n - \hat{\mu})^2$$

Multivariate case

Gaussian distribution with $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\mathbf{x} \in \mathbb{R}^d$:

$$\hat{\boldsymbol{\mu}} = \frac{1}{|\mathcal{D}|} \sum_{n=1}^{|\mathcal{D}|} \mathbf{x}_n \qquad \hat{\boldsymbol{\Sigma}} = \frac{1}{|\mathcal{D}|} \sum_{n=1}^{|\mathcal{D}|} (\mathbf{x}_n - \hat{\boldsymbol{\mu}})\,(\mathbf{x}_n - \hat{\boldsymbol{\mu}})^t$$