Ejercicio: Consideremos el e.v euclídeo de $\mathbb{R}^3$ con el producto escalar usual, y en él, la base:

$$B = \{ \underset{\vec{v}_1}{(1,1,1)}, \underset{\vec{v}_2}{(1,-1,0)}, \underset{\vec{v}_3}{(1,0,-1)} \}$$

Obtener una base ortonormal para dicho espacio.

Aplicamos el proceso de Gram - Schmidt:

$$\vec{w}_1 = \vec{v}_1 = \underline{\underline{(1,1,1)}}$$

$$\vec{w}_2 = \vec{v}_2 - \cancel{\frac{\vec{v}_2 \bullet \vec{w}_1}{|\vec{w}_1|^2}}^{0} \cdot \vec{w}_1 = \underline{\underline{(1,-1,0)}}$$

$$\frac{\vec{v}_2 \bullet \vec{w}_1}{\underset{\vec{w}_1 \bullet \vec{w}_1}{\overset{\shortparallel}{|\vec{w}_1|^2}}} = \frac{1 \cdot 1 + 1(-1) + 1 \cdot 0}{1^2 + 1^2 + 1^2} = \frac{0}{3} = 0$$

$$|\vec{w}_1|^2 = \left(\sqrt{\vec{w}_1 \bullet \vec{w}_1}\right)^2 = \vec{w}_1 \bullet \vec{w}_1$$

$$\vec{w}_3 = \vec{v}_3 - \cancel{\frac{\vec{v}_3 \bullet \vec{w}_1}{|\vec{w}_1|^2}}^{0} \cdot \vec{w}_1 - \overset{\frac{1}{2}}{\overset{\shortparallel}{\frac{\vec{v}_3 \bullet \vec{w}_2}{|\vec{w}_2|^2}}} \cdot \vec{w}_2 = (1,0,-1) - \frac{1}{2}(1,-1,0) =$$

$$= \underline{\underline{\left(\frac{1}{2}, \frac{1}{2}, -1\right)}}$$

$$\frac{\vec{v}_3 \bullet \vec{w}_1}{|\vec{w}_1|^2} = \frac{1 \cdot 1 + 1 \cdot 0 + 1(-1)}{3} = 0$$

$$\frac{\vec{v}_3 \bullet \vec{w}_2}{|\vec{w}_2|^2} = \frac{1 \cdot 1 + (-1) \cdot 0 + 0(-1)}{1^2 + (-1)^2 + 0^2} = \frac{1}{2}$$

$$|\vec{w}_2|^2 = \vec{w}_2 \bullet \vec{w}_2$$

Base ortogonal : $B' = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \{(1,1,1), (1,-1,0), (\frac{1}{2}, \frac{1}{2}, -1)\}$

Base ortonormal : $B'' = \left\{ \frac{\vec{w}_1}{|\vec{w}_1|}, \frac{\vec{w}_2}{|\vec{w}_2|}, \frac{\vec{w}_3}{|\vec{w}_3|} \right\} =$ $\quad |\vec{w}_3| = \frac{\sqrt{6}}{2}$

$$= \underline{\underline{\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right) \right\}}}$$

$$|\vec{w}_3| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$$

Una matriz ortogonal sería : $A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$ $\quad A \cdot A^t = I$

(base ortonormal en col.)

Ejercicio: En $\mathbb{R}^3$ con el producto escalar estándar, se considera el subespacio:

$$U \equiv \begin{cases} 2x + y - z = 0 \\ x - y + 3z = 0 \end{cases}$$

Calcular unas ecuaciones paramétricas de $U^\perp$.

- **Base de U**: ec. imp $\longrightarrow$ ec. par $\longrightarrow$ s. gen $\longrightarrow$ base

$$\left.\begin{array}{l} 2x + y - z = 0 \\ x - y + 3z = 0 \end{array}\right\} \begin{array}{l} \longrightarrow 2x + x + 3z - z = 0 \rightarrow 3x + 2z = 0 \rightarrow x = -\frac{2}{3}z \\ \longrightarrow y = x + 3z \end{array}$$

3 inc − 2 ec = 1 par (α)

$$y = -\frac{2}{3}z + 3z = \frac{7}{3}z \qquad (z \in \mathbb{R})$$

$$\begin{cases} x = -\frac{2}{3}\alpha \\ y = \frac{7}{3}\alpha \\ z = \alpha \end{cases} \longrightarrow (x, y, z) = \alpha\left(-\frac{2}{3}, \frac{7}{3}, 1\right)$$

S. gen $\rightarrow$ L. I $\rightarrow$ base

$(\alpha \in \mathbb{R})$

$$B_U = \{\underbrace{\left(-\frac{2}{3}, \frac{7}{3}, 1\right)}_{\vec{n}}\}$$

- <u>Cálculo de $U^\perp$</u> : $U^\perp = \{ \vec{v} \in \mathbb{R}^3 / \vec{v} \perp \vec{u} \ \ \forall \vec{u} \in U \}$

$\vec{v} = (x, y, z) \longrightarrow \vec{v} \cdot \vec{u} = 0 \longrightarrow (x, y, z) \cdot \left(-\frac{2}{3}, \frac{7}{3}, 1 \right) = 0$

$-\frac{2}{3}x + \frac{7}{3}y + z = 0 \xrightarrow{2 \text{ par}}$ $\boxed{U^\perp \equiv \begin{cases} x = \alpha \\ y = \beta \qquad (\alpha, \beta \in \mathbb{R}) \\ z = \frac{2}{3}\alpha - \frac{7}{3}\beta \end{cases}}$

ec. imp. de $U^\perp$

Ejercicio: En $\mathbb{R}^3$ con el producto escalar usual, calcular la proyección del vector $(1,0,0)$ sobre el subespacio $U \equiv x + y = 0$.

$$\left.\begin{array}{l} x = -y \\ z \in \mathbb{R} \end{array}\right\} \longrightarrow \left.\begin{array}{l} x = -\alpha \\ y = \alpha \\ z = \beta \end{array}\right\} \longrightarrow (x,y,z) = \alpha(-1,1,0) + \beta(0,0,1)$$

3 inc $-$ 1 ec $=$ 2 par

$(\alpha, \beta \in \mathbb{R})$

$$B_U = \{\underbrace{(-1,1,0)}_{\vec{u}_1}, \underbrace{(0,0,1)}_{\vec{u}_2}\}$$

$*$ Si B_U no fuera ortogonal $\longrightarrow$ Calculamos base ortogonal con Gram-Schmidt.

$$\underbrace{\vec{u}_1 \bullet \vec{u}_2}_{\perp} = -1\cdot 0 + 1\cdot 0 + 0\cdot 1 = 0 \rightarrow B_U \text{ es ortogonal} \checkmark$$

$$\vec{v} = (1,0,0)$$

$$\text{proy}_U(\vec{v}) = \frac{\vec{v} \bullet \vec{u}_1}{\underbrace{|\vec{u}_1|^2}_{\vec{u}_1 \bullet \vec{u}_1}} \cdot \vec{u}_1 + \frac{\vec{v} \bullet \vec{u}_2}{\underbrace{|\vec{u}_2|^2}_{\vec{u}_2 \bullet \vec{u}_2}} \cdot \vec{u}_2 =$$

$$= \frac{-1}{2}(-1,1,0) + \underbrace{\cancel{\frac{0}{1}(0,0,1)}}_{\vec{0}} = \boxed{\left(\frac{1}{2}, -\frac{1}{2}, 0\right)}$$

Ejercicio: Hallar la solución aproximada por el método de los mínimos cuadrados del sistema sobredeterminado:

↳ + ecuaciones que incógnitas

$$\left.\begin{aligned} x + y &= 1 \\ x + 2y &= 0 \\ -x + y &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} &\to 0 + 0 \neq 1 \quad \text{S.I} \\ &\to x + 2x = 0 \to 3x = 0 \to x = 0 \\ &\to y = x \to y = 0 \end{aligned}\right\} \quad A = \begin{array}{cc} x & y \end{array} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A^t \cdot A = \underset{2 \times 3}{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}} \cdot \underset{3 \times 2}{\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}} = \underset{2 \times 2}{\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}}$$

$$(A^t \cdot A)^{-1} = \frac{1}{14} \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$$

$$\hat{X} = (A^t \cdot A)^{-1} \cdot A^t \cdot B = \frac{1}{14} \underset{2 \times 2}{\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}} \cdot \underset{2 \times 3}{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$= \frac{1}{14} \underset{2 \times 3}{\begin{pmatrix} 4 & 2 & -8 \\ 1 & 4 & 5 \end{pmatrix}} \cdot \underset{3 \times 1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} = \frac{1}{14} \underset{2 \times 1}{\begin{pmatrix} 4 \\ 1 \end{pmatrix}} = \begin{pmatrix} \frac{2}{7} \\ \frac{1}{14} \end{pmatrix}$$

$$\boxed{\hat{x} = \frac{2}{7} \qquad \hat{y} = \frac{1}{14}}$$

Ejercicio: Ajustar los datos $(\underset{x_1}{0},\underset{y_1}{1})$, $(\underset{x_2}{1},\underset{y_2}{3})$, $(\underset{x_3}{2},\underset{y_3}{4})$, $(\underset{x_4}{3},\underset{y_4}{4})$ y $(\underset{x_5}{4},\underset{y_5}{5})$ mediante una función lineal utilizando el método de mínimos cuadrados.

$$A = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_4 & 1 \\ x_5 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \qquad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \\ 5 \end{pmatrix}$$

Entonces:

$$A^t \cdot A = \underbrace{\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}}_{2\times 5} \cdot \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}}_{5\times 2} = \underbrace{\begin{pmatrix} 30 & 10 \\ 10 & 5 \end{pmatrix}}_{2\times 2}$$

$$(A^t \cdot A)^{-1} = \frac{1}{50}\begin{pmatrix} 5 & -10 \\ -10 & 30 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

$$X = (A^t \cdot A)^{-1} \cdot A^t \cdot Y =$$

$$= \begin{pmatrix} \frac{1}{10} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \\ 5 \end{pmatrix} =$$

2×2 $\quad$ 2×5

$$= \begin{pmatrix} -\frac{1}{5} & -\frac{1}{10} & 0 & \frac{1}{10} & \frac{1}{5} \\ \frac{3}{5} & \frac{2}{5} & \frac{1}{5} & 0 & -\frac{1}{5} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 4 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} \frac{9}{10} \\ \frac{8}{5} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

2×5 $\quad$ 5×1 $\quad$ 2×1

La recta de mínimos cuadrados que mejor ajusta los puntos es:

$$\boxed{y = ax + b = \frac{9}{10}x + \frac{8}{5}}$$