Advanced Control Systems - Assignment 3

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1 Part 1

$$\dot{x}_1 = \mu(x_2)x_1 - x_1 u \quad (1)$$
$$\dot{x}_2 = -k_1\mu(x_2)x_1 - (\bar{p} - x_2)u \quad (2)$$
$$y = k_2\mu(x_2)x_1 \quad (3)$$

Q (1.1) & (1.2)

Let $\bar{x}$ be the steady state value of x and $\bar{x}^*$ be the optimal steady state value. There is one pair of equilibrium in the closed positive orthant that is $(x_1, x_2) = (\frac{1}{k_1}(\bar{p} - \frac{\bar{u}}{\bar{\eta}}), \frac{\bar{u}}{\bar{\eta}})$. There is one additional pair of equilibrium in closed positive orthant when $(x_1, x_2) = (0, \bar{p})$
At Equilibrium $\dot{x}_1 = \dot{x}_2 = 0$
$\therefore 0 = \bar{x_1}(\bar{\eta}x_2 - \bar{u}) \Rightarrow x_2 = \frac{\bar{u}}{\bar{\eta}}$ or $x_1 = 0$

Substitute $x_2 = \frac{\bar{u}}{\bar{\eta}}$ into equation (1)
$0 = -k_1\bar{\eta}\frac{\bar{u}}{\bar{\eta}}x_1 - (\bar{p} - \frac{\bar{u}}{\bar{\eta}})\bar{u}$
$\bar{x_1} = \frac{1}{k_1}(\bar{p} - \frac{\bar{u}}{\bar{\eta}})$

Substitute $x_1 = 0$ into equation (1)
$0 = (\bar{p} - \frac{\bar{u}}{\bar{\eta}})\bar{u}$
$x_2 = \bar{p}$

Q (1.3) Forward invarience

When calculating forward invarience $u = 0$. Equation (1) indicates that $\dot{x}_1 > 0$ when $x_1(0) = x_2(0) \geq 0$. Thus it can be confimed that x_1 satisfies forward invarience. For confirm x_2 we must examine the boundary of $x_2 = 0$ using linearisation.
$\dot{x}_2 = -k1\bar{\eta}x_1\delta x_2 - k1\bar{\eta}x_2\delta x_1$
As $t \Rightarrow \infty, x_2 \Rightarrow 0$ such that $\delta x_2 \Rightarrow 0$ and $\dot{x}_2 \Rightarrow 0$ and both values x_1 and x_2 are confined to the closed positive orthant.

Q (1.4) Asymptotic Stability

To prove global asymptotic or with a region D other methods must be used. Numerically, asymptotic stability of the equilibrium can be guaranteed by plotting a phase portrait, for a variety of initial conditions and input values U. To solve analytically, a far more robust and transferable method, you must use lyapunov. First we me must choose a candidate lyapunov system.
$V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ & $\dot{V}(x) = x_1\dot{x_1} + x_2\dot{x_2}$
Substituting in we find
$\dot{V}(x) = \bar{\eta}x_2x_1^2 - x_1^2u - k_1\bar{\eta}x_2^2x_1 - \bar{p}ux_2 + x_2^2u$
No simple cancellations are found and $\dot{V}(x)$ is not negative definite. This is expected as the origin is not a stable equilibrium point. We must shift the origin to the known equilibrium point. $x_1 \Rightarrow dx_1 - \bar{x_1}$ and $x_1 \Rightarrow dx_2 - \bar{x_2}$. Assuming asymptotic stability of the equilibrium for a given lyapunov candidate such as the one shown above cancellations will result in $\dot{V}(x)$ being negative definite.

Q (1.5) Outflow Equilibrium

Find equilibrium $\bar{y}$ in terms of U by substituting the equilibrium values $\bar{x_1}(u), \bar{x_2}(u)$

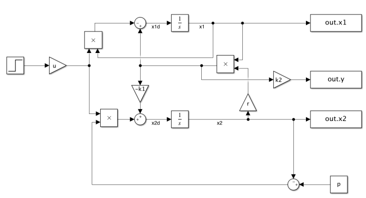
$$\begin{aligned}\bar{y}(\bar{u}) : \bar{y} &= k_1\bar{\eta}\bar{x_2}\bar{x_1} \\ &= k_2\bar{\eta}\frac{1}{k_1}(\bar{p} - \frac{\bar{u}}{\bar{\eta}})\frac{\bar{u}}{\bar{\eta}} \\ &= \frac{k_2}{k_1}(\bar{p} - \frac{\bar{u}}{\bar{\eta}})\bar{u} \\ &= \frac{k_2}{k_1}[\frac{-\bar{u}^2}{\bar{\eta}} + \bar{u}\bar{p}]\end{aligned}$$

Note that $\bar{y}$ is a negative parabola with a maximum, that can be found by determining where the gradient is zero.

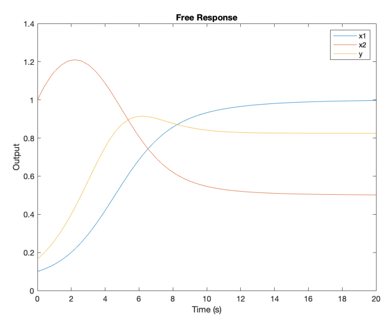
$$\frac{d\bar{y}}{d\bar{u}^*} = -2\frac{k_2}{k_1}[\frac{\bar{u}^*}{\bar{\eta}} + \bar{p}] = 0 \qquad \bar{u}^* = \frac{\bar{\eta}\bar{p}}{2}$$

$$\bar{y}^* = \frac{k_2}{k_1}[-\frac{\bar{\eta}^2\bar{p}^2}{4\bar{\eta}} + \frac{\bar{\eta}\bar{p}^2}{2}] = \frac{k_2}{k_1}[\frac{\bar{\eta}\bar{p}^2}{4}] = \frac{3}{1.5}[\frac{\bar{\eta}2^2}{4}] = 2\bar{\eta} = 1.1$$

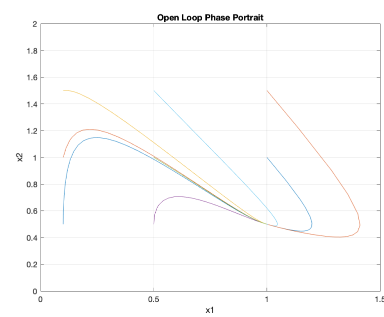
Q (1.6) MATLAB MODEL



(a) Simulink Plant Model



(b) Simulink Plant Model



(c) Steady State Output

Q (1.7) Confirming Maximum

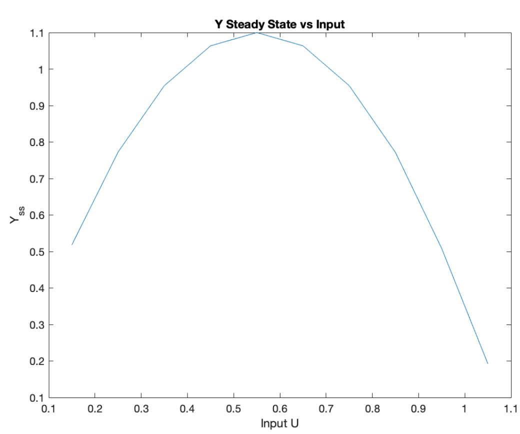


Figure 1: Steady state map illustrating $\bar{y}^*$ and $\bar{u}^*$

Analytically we can find $\bar{y}^*$:

$$\bar{u}^* = \frac{\bar{\eta}\bar{p}}{2} = \frac{2\bar{\eta}}{2} = \bar{\eta} = 0.55 \tag{4}$$

$$\bar{y}^*(\bar{u}^*) = \bar{y}^*(\bar{\eta}) = \frac{k_2}{k_1}[-\frac{\bar{\eta}^2}{\eta} + \bar{\eta}\bar{p}] \tag{5}$$

$$= -\frac{k_2}{k_1}[-\bar{\eta} + 2\bar{\eta}] = 2\bar{\eta} \tag{6}$$

As illustrated in the steady state vs input map of figure 1, input value 0.55 yields steady state output of 1.1 as expected.

2 Part 2

Q (2.1) Deriving l() & Q() & Parameter Estimation

From Q1.1-Q1.5 we can know that:

$$Q(\bar{u}) = \frac{k_2}{k_1}(\bar{p} - \frac{\bar{u}}{\bar{\eta}})\bar{u} \qquad \bar{x_1} = l1(\bar{u}) = \frac{1}{k_1}(\bar{p} - \frac{\bar{u}}{\bar{p}}) \qquad \bar{x_2} = l2(\bar{u}) = \frac{\bar{u}}{\bar{p}}$$

The following parameters will be manipulated in order to optimise the response.

- High Pass cut-off frequency - b
- Low Pass cut-off frequency - c
- Controller Gain - K
- Dither amplitude - a_{dith}
- Dither frequency - ω

Naive Estimation Of Parameters

A range of possible values for each parameter can be estimated before fine tuning during experimentation.
b: The high pass filter should be higher frequency that the dither signal as it must not filter out the plants response to the dither. Estimate $b = 2\omega$
c: The low pass cut-off frequency must be considerably faster than the dither and the high pass filter. Primarily to filter out high frequency noise in the controller.Estimate $c = 20\omega$
a_{dith}**:** The size of a_{dith} is proportional to the desired steady state error and the rate of convergence. Estimate $a_{dith} = 20\omega$
K: Optimal value varies wildly depending on other parameters. A lower gain is safer as it reduces the risk of wild oscillations that interfere with the dither. Estimate $K = 1$

Determining Dither Frequency

A faster dither frequency excites the system more radically and allows faster convergence to the optimum state. However if steady state cannot be achieved by the plant during these oscillations then the outcome will be erroneous.

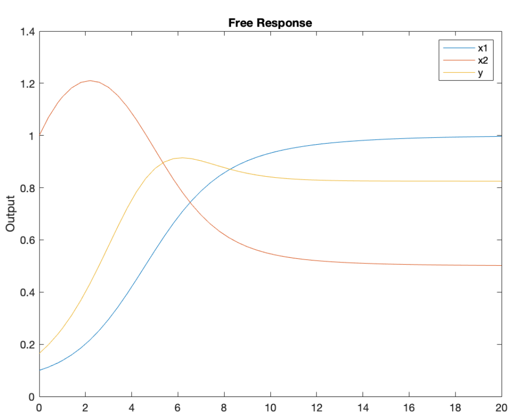


Figure 2: Free Response

The corresponding steady state values for $(x1, x2, y)$ are $(1, 0.5, 0.825)$. The time taken for each of the values to reach within 5% of the steady state is 10.7,11.4 and 8.6 seconds respectively. The is the result of an impulse or free response. This give a rough estimate of the upperbound timescales which the dither frequency should operate. Faster dither frequencies are likely to work because throughout the period of a dither the respective steady states will be close to each other and require less time to converge.

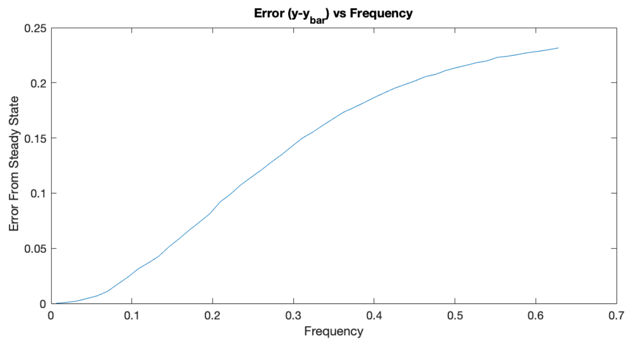


Figure 3: Deviation From Steady State

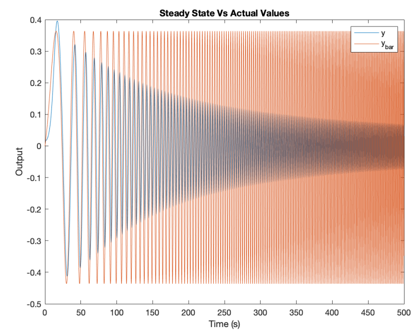


Figure 4: Chirp Signal and Steady State Error

The following graph illustrates the error between the analytical steady state and the actual y value achieved during the sinusoidal dither. The Chirp Signal illustrate how error with respect to true steady state varies with sinusoidal frequency. Lower frequencies allow for the plant to reach steady state, while For reference a error value of 0.26 occurs when a super-high frequency dither is used and the output y is a constant. This can be used as reference of "no information". No frequency exhibits a clear change in steady state error so, the dither frequency must be determined experimentally. Estimate $= 0.15 \approx \frac{\pi}{20}$

Q (2.2) Simulate Extremum Seeking

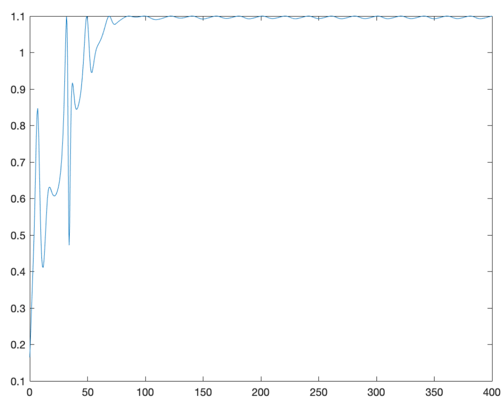


Figure 5: Extremum Seeking Simulation

Here, we can see that over time the system converges to the optimal value of 1.1. This matches the previously determined value for $\bar{y}(\bar{u}) = 2\bar{\eta} =$ 2*0.55=1.1
The oscillation about the optimum is due to the amplitude of the dither and results in an oscillation about the maximum with an amplitude of 0.007.

Q (2.3) Parameter Manipulation

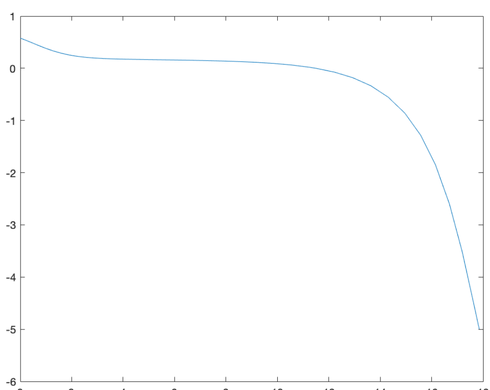
Results

Exp #	K	ω	a_{dith}	b	c	Settling Time	SSE
1	1	$2\pi/40$	0.05	2ω	π	63.834	0.007
2	0.4	$2\pi/40$	0.05	2ω	π	32.5	0.008
3	0.1	$2\pi/40$	0.05	2ω	π	158.873	0.081
4	0.4	$2\pi/40$	0.1	2ω	π	27.3	0.031
5	0.4	$2\pi/20$	0.05	2ω	π	57.127	0.007
6	0.4	$2\pi/60$	0.05	2ω	π	62.137	0.008
7	0.4	$2\pi/40$	0.05	2ω	$\pi/2$	34.498	0.008
8	0.4	$2\pi/40$	0.05	3ω	π	13.659	0.009
9	0.4	$2\pi/40$	0.05	4ω	π	115.519	0.017

From this an locally optimised result was determined in experiment 8. In general all variables suffer from trade-offs. From these experiments we can determine:

- Large gain improves response until large jerky movements are induced.
- Amplitude of a_{dith} improve settling time at the expense of steady state error
- Changing c for high pass filters made limited difference as the integrator already filters for lower frequencies, although improvements could be made.
- Varying frequency of dither ω K have a large impact on results compared to other variables.

Q (2.4) Initial Conditions & Stability



$x_1(0)$	$x_2(0)$	K	Stable?
0.9	1	0.1	Yes
0.9	1	1	No
0.1	2.1	1	Yes
0.1	2.1	0.1	No

Table 1: $w = 2 * pi/40, b = 3 * w, c = 20 * w, a_{dith} = .05$

Figure 6: Example of an unstable controller

With the conditions specified in the caption, adjusting K as shown can allow for stability when it otherwise wouldn't. Throughout the experimentation it was uncovered that the conditions for stability are highly dependant on all variables, including initial conditions. In no circumstance can you guaranteed to improve stability by naively adjusting a single variable in a given direction. The system is not open-loop stable for all initial conditions in the positive orthant. The system is not forward invarient for all $x_1(0), x_2(0) > 0$. In all cases where the controller was stable, the optimum was reached.

		x2															
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
x1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0.2	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
	0.3	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
	0.4	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	0.5	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	0.6	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
	0.7	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
	0.8	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
	0.9	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1.1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1.2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1.3	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1.4	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1.5	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0

Figure 7: Stability Of Stated Controller Given Initial Conditions

Q (2.5) Variations in $\bar{\eta}$

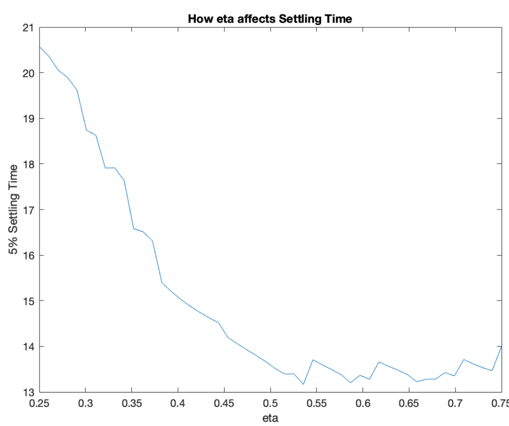


Figure 8: How a variation in η (eta) change the settling time

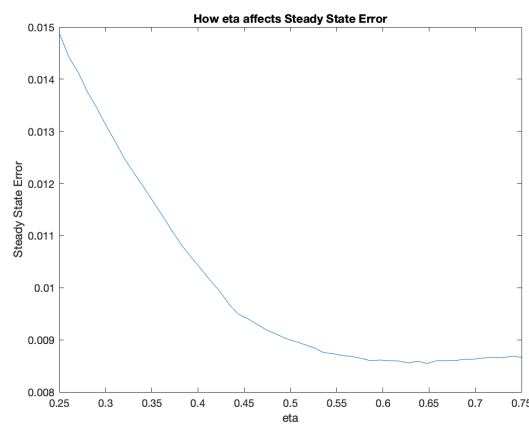
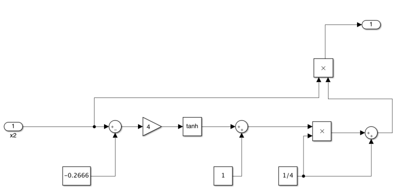
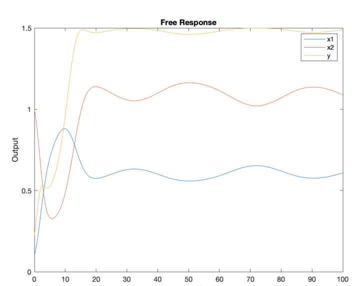


Figure 9: How a variation in η (eta) change the steady state error

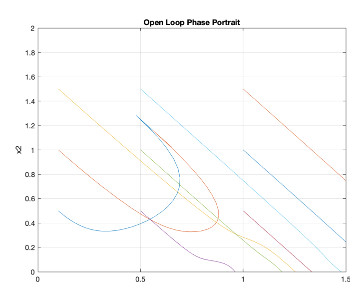
Q (2.5) Non linear $\bar{\eta}(x_2)$



(a) Simulink Non-linear $\bar{\eta}(x_2)$



(b) Free Response to standard initial conditions



(c) Open Loop Phase portrait

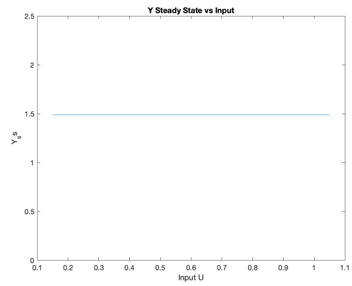


Figure 10: Steady State Optimum

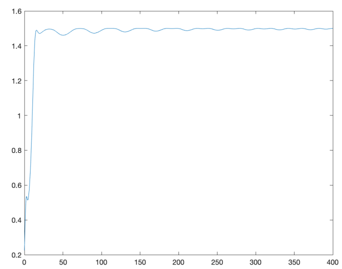


Figure 11: Extremum Seeking Simulation

Here, we can see that over time the system converges to the optimal value of 1.5. The oscillation about the optimum is due to the amplitude of the dither and results in an oscillation about the maximum with an amplitude of 0.009. The system has a 5% settling time of 14.002 seconds.

From the following diagrams we can make the following assumptions about the new plant.

- It is not open-loop stable for all initial conditions in the positive orthant.
- System is not forward invarient for all $x_1(0), x_2(0) > 0$
- The steady state optimum is always 1.5 no matter the input for the standard inital conditions and paramters.
- The controller allows the controller to be stable for a wide range of initial conditions, similar to the original system.

		x2															
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0
	0.2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
	0.3	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	0.4	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
	0.5	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
	0.6	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
x1	0.7	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	0.8	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	0.9	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1.1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1.2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	1.3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	1.4	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	1.5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 12: Stability Of Stated Controller With Non-linear $\bar{\eta}(x_2)$ Given Initial Conditions

Q (2.6) Summary, Non linear $\bar{\eta}(x_2)$

Referencing the results of (2.5) we can conclude that for the narrow range of parameters tested that an extremum seeking controller is stable. In all cases where the initial conditions do not result in an instability the optimum is reached. Due to the risk on instability based upon initial conditions, as with the original system steps must be taken to ensure that the intital conditions allow for stability to occur. Furthermore, due to the large and varying effect of changing the parameters it it difficult to know what effect varying parameters have on performance and stability without further analysis. In reality, because the ES controller is slow, adjustments could be made to the controller based upon additional information.

Matlab Instructions

Q1
"Main": Returns plots of free response, phase portrait and map of $\bar{u} \Rightarrow \bar{y}$
"Non linear Tank": Original Plant, Open Loop
"Non linear Tank Tanh": Second plant with non-linear $\mu(x_2)$, Open Loop
Q2:
"TestParamters": Runs Extrememum seeking simulation
"RobustController": Tests stability for initial conditions of intial system
"RobustNonLinearFunc": Tests stability for initial conditions of second system with non-linear $\mu(x_2)$
"Robust eta": Tests how stability of system and settling time related to $\bar{\eta}$
"Extremum Seeking": Extremum seeking controller simulation for original system
"Extremum Seeking Non linear": Extremum seeking controller simulation for second system with non-linear $\mu(x_2)$