Laboratorium 11

June 19, 2024

1 Zadanie 1

1.1 $f(x, y) = x^2 - 4xy + y^2$

Pochodne cząstkowe tej funkcji wynoszą

$$\frac{\partial f}{\partial x} = 2x - 4y$$
$$\frac{\partial f}{\partial y} = 2y - 4x$$

Punkty krytyczne można znaleźć rozwiązując liniowy układ równań

$$\begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[1]:
```
import numpy as np
```

[2]:
```
A = np.array([[2, -4], [-4, 2]])
b = np.array([0, 0])
x, y = np.linalg.solve(A, b)
print(f"x = {x}, y = {y}")
```

```
x = -0.0, y = -0.0
```

W wyniku czego otrzymujemy

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Macierz Hessego H wygląda następująco

$$H = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix}$$

Minory tej macierzy mają następujące wyznaczniki

$$\begin{cases} H_1 = 2 \\ H_2 = 4 - 16 = -12 \end{cases}$$

Macierz jest nieokreślona, zatem punkt $(x, y) = (0, 0)$ jest punktem siodłowym.

Funkcja jest nieograniczona, zatem nie posiada minimum ani maksimum globalnego.

1.2 $f(x, y) = x^4 - 4xy + y^4$

Pochodne cząstkowe

$$\frac{\partial f}{\partial x} = 4x^3 - 4y$$
$$\frac{\partial f}{\partial y} = 4y^3 - 4x$$

Aby znaleźć punkty krytyczne należy rozwiązać nieliniowy układ równań

$$\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases}$$

[3]:
```
import scipy.optimize as spo
```

[4]:
```
def fprime(args):
    x = args[0]
    y = args[1]
    return [
        4*x**3 - 4*y,
        4*y**3 - 4*x
    ]
```

Potencjalne rozwiązania układu można łatwo oszacować analitycznie.

[5]:
```
x1, y1 = spo.fsolve(fprime, (0, 0))
x2, y2 = spo.fsolve(fprime, (1, 1))
x3, y3 = spo.fsolve(fprime, (-1, -1))
```

[6]:
```
print(f"(x1, y1) = ({x1}, {y1}), (x2, y2) = ({x2}, {y2}), (x3, y3) = ({x3},␣
 ↪{y3})")
```

```
(x1, y1) = (0.0, 0.0), (x2, y2) = (1.0, 1.0), (x3, y3) = (-1.0, -1.0)
```

Otrzymaliśmy 3 punkty krytyczne

$$\begin{cases} P_1 = (0, 0) \\ P_2 = (1, 1) \\ P_3 = (-1, -1) \end{cases}$$

Macierz Hessego

$$H(x, y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

[7]:
```
def make_hessian(P):
    x = P[0]
    y = P[1]
    return np.array([
        [12*x**2, -4],
        [-4, 12*y**2]
```

```
    ])
```

[8]:
```
P1 = (x1, y1)
P2 = (x2, y2)
P3 = (x3, y3)
```

[9]:
```
H1 = make_hessian(P1)
try:
    np.linalg.cholesky(H1)
except np.linalg.LinAlgError:
    print("H1 is NOT positive definite!")
else:
    print("H1 is positive definite!")
```

```
H1 is NOT positive definite!
```

Macierz nie jest dodatnio określona, niestety trzeba teraz ręcznie sprawdzić, czy jest ujemnie określona

$$H(0,0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}$$

$$\begin{cases} H_1 = 0 \\ H_2 = -16 \end{cases}$$

Macierz jest nieokreślona, zatem punkt $(0,0)$ jest punktem siodłowym.

[10]:
```
H2 = make_hessian(P2)
try:
    np.linalg.cholesky(H2)
except np.linalg.LinAlgError:
    print("H1 is NOT positive definite!")
else:
    print("H1 is positive definite!")
```

```
H1 is positive definite!
```

Macierz $H(1,1)$ jest dodatnio określona, zatem w punkcie $(1,1)$ funkcja przyjmuje minimum.

[11]:
```
H3 = make_hessian(P3)
try:
    np.linalg.cholesky(H3)
except np.linalg.LinAlgError:
    print("H1 is NOT positive definite!")
else:
    print("H1 is positive definite!")
```

```
H1 is positive definite!
```

Macierz $H(1,1)$ również jest dodatnio określona, zatem w punkcie $(1,1)$ funkcja przyjmuje minimum.

[12]:
```
f = lambda x, y: x**4 - 4*x*y + y**4
val2 = f(x2, y2)
val3 = f(x3, y3)
print(val2, val3)
```

```
-2.0 -2.0
```

Funkcja przyjmuje takie same wartości w punktach P_1 i P_2, punkty te są minimami globalnymi

1.3 $f(x,y) = 2x^3 - 3x^2 - 6xy(x - y - 1)$

Pochodne cząstkowe wynoszą

$$\frac{\partial f}{\partial x} = 6(x^2 - 2xy - x + y^2 + y)$$

$$\frac{\partial f}{\partial y} = -6x(x - 2y - 1)$$

$$\begin{cases} x^2 - 2xy - x + y^2 + y = 0 \\ x^2 - 2yx - x = 0 \end{cases}$$

[13]:
```
def fprime(P):
    x = P[0]
    y = P[1]
    return [
        x**2 - 2*x*y - x + y**2 + y,
        x**2 - 2*x*y - x
    ]
```

[14]:
```
solution_set = set()
for x in range(-1, 2):
    for y in range(-1, 2):
        sol_x, sol_y = spo.fsolve(fprime, (x, y))
        solution_set.add((round(sol_x, 5), round(sol_y, 5)))
print(*solution_set, sep="\n")
```

```
(1.0, 0.0)
(0.0, -1.0)
(-1.0, -1.0)
(0.0, 0.0)
```

Funkcja ma 4 punkty krytyczne

$$\begin{cases} P_1 = (-1,-1) \\ P_2 = (0,0) \\ P_3 = (1,0) \\ P_4 = (0,-1) \end{cases}$$

Macierz Hessego

$$H(x,y) = \begin{bmatrix} 12x - 12y - 6 & -6(x-y-1) - 6x + 6y \\ -6(x-y-1) - 6x + 6y & 12x \end{bmatrix}$$

[15]:
```
def make_hessian(P):
    x = P[0]
    y = P[1]
    return np.array([
        [12*x-12*y-6, -6*(x-y-1) - 6*x + 6*y],
        [-6*(x-y-1) - 6*x + 6*y, 12*x]
    ])
```

[16]:
```
# returns 1 if negative definite, -1 if semi definite and 0 if indefinite
def check_negative_definite(H):
    flag = True
    if H[0][0] > 0:
        return 0
    elif H[0][0] == 0:
        flag = False

    det = np.linalg.det(H)
    if det < 0:
        return 0
    elif det == 0:
        flag = False

    return 1 if flag else -1
```

[17]:
```
H1 = make_hessian((-1, -1))
try:
    np.linalg.cholesky(H1)
except np.linalg.LinAlgError:
    print("H1 is NOT positive definite!")
else:
    print("H1 is positive definite!")
```

```
H1 is NOT positive definite!
```

Macierz nie jest dodatnio określona, trzeba ją sprawdzić ręcznie.

[18]:
```
check_negative_definite(H1)
```

[18]:
```
1
```

Funkcja zwróciła 1, co oznacza, że macierz jest ujemnie określona, zatem w punkcie $(-1, -1)$ występuje maksimum.

[19]:
```
H2 = make_hessian((0, 0))
try:
    np.linalg.cholesky(H2)
except np.linalg.LinAlgError:
    print("H1 is NOT positive definite!")
else:
    print("H1 is positive definite!")
```

```
H1 is NOT positive definite!
```

[20]:
```
check_negative_definite(H2)
```

[20]: 0

Macierz jest nieokreślona, co oznacza, że punkt $(0,0)$ jest punktem siodłowym.

[21]:
```
H3 = make_hessian((1, 0))
try:
    np.linalg.cholesky(H3)
except np.linalg.LinAlgError:
    print("H1 is NOT positive definite!")
else:
    print("H1 is positive definite!")
```

```
H1 is positive definite!
```

Macierz jest dodatnio określona, w punkcie $(1,0)$ jest minimum.

[22]:
```
H4 = make_hessian((0, -1))
try:
    np.linalg.cholesky(H4)
except np.linalg.LinAlgError:
    print("H1 is NOT positive definite!")
else:
    print("H1 is positive definite!")
```

```
H1 is NOT positive definite!
```

[23]:
```
check_negative_definite(H4)
```

[23]: 0

Macierz jest nieokreślona, punkt $(0,-1)$ jest punktem siodłowym

Funkcja nie posiada maksimum ani minimum globalnego.

1.4 $f(x,y) = (x-y)^4 + x^2 - y^2 - 2x + 2y + 1$

$$\frac{\partial f}{\partial x} = 4(x-y)^3 + 2x - 1$$

$$\frac{\partial f}{\partial y} = -4(x-y)^3 - 2y + 2$$

[24]:
```
def fprime(P):
    x = P[0]
    y = P[1]
    return [
        4*(x-y)**3 + 2*x - 1,
        -4*(x-y)**3 - 2*y + 2
    ]
```

[25]:
```
solution_set = set()
for x in range(-2, 3):
    for y in range(-2, 3):
        sol_x, sol_y = spo.fsolve(fprime, (x, y))
        solution_set.add((round(sol_x, 5), round(sol_y, 5)))
print(*solution_set, sep="\n")
```

```
(0.75, 1.25)
```

Funkcja ma tylko 1 punkt krytyczny $P = (0.75, 1.25)$

Macierz Hessego dla punktu P

$$H = \begin{bmatrix} 5 & -3 \\ -3 & 1 \end{bmatrix}$$

$$\begin{cases} H_1 = 5 \\ H_2 = 5 - 9 = -4 \end{cases}$$

Macierz jest nieokreślona, punkt $(0.75, 1.25)$ jest punktem siodłowym, funkcja nie ma minimum ani maksimum globalnego.

2 Zadanie 2

$$F(\mathbf{x}) = \lambda_1 \sum_{i=0}^{n} \sum_{j=0}^{k-1} \frac{1}{\epsilon + ||x_i - r_j||_2^2} + \lambda_2 \sum_{i=0}^{n-1} ||x_{i+1} - x_i||_2^2$$

$$F : \mathbb{R}^{n \times 2} \to \mathbb{R}$$

Funkcja największego spadku dla $\mathbf{x}$.

$$\nabla F(\mathbf{x}) = \frac{\partial F}{\partial \mathbf{x}} = -\lambda_1 \sum_{j=0}^{k-1} \left[\frac{2(x_i - r_j)}{(\epsilon + ||x_i - r_j||_2^2)^2} \right] + \lambda_2(-2(x_{i+1} - x_i) + 2(x_i - x_{i-1}))$$

$$\nabla F : \mathbb{R}^{n \times 2} \to \mathbb{R}^{n \times 2}$$

Algorytm najmniejszego spadku wykorzystuje początkowe ustawienie punktów $\mathbf{x}_0$, następnie na ich podstawie wyliczane są następne ustawienia, dla których funkcja osiąga coraz mniejsze wartości.

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha_k \nabla F(\mathbf{x}_k)$$

Gdzie α szukane jest poprzez minimalizację

$$\min_{\alpha} F(\mathbf{x}_k - \alpha \nabla F(\mathbf{x}_k))$$

Minimalizacja jest minimalizacją jednej zmiennej α, więc można zastosować algorytm *golden section search*.
Do minimalizacji *golden section search* jako przedział podstawowy został wybrany przedział $[0, 441]$, ze względu na wymiary planszy robota 21×21

[26]:
```
import random
from copy import deepcopy
import matplotlib.pyplot as plt
```

[27]:
```
def F(x, r):
    lambda1 = lambda2 = 1
    n = len(x)
    k = len(r)
    epsilon = 1e-13

    res1 = 0
    for i in range(n):
        for j in range(k):
            res1 += 1 / (epsilon + np.linalg.norm(x[i] - r[j])**2)

    res1 *= lambda1

    res2 = 0
    for i in range(n-1):
        res2 += np.linalg.norm(x[i+1] - x[i])**2

    res2 *= lambda2

    return res1 + res2


def gradient(x, r):
    lambda1 = lambda2 = 1
    n = len(x)
    k = len(r)
    epsilon = 1e-13

    res = [None for _ in range(n)]
    for i in range(1, n-1):
        res[i] = -lambda1 * np.sum([(2 * (x[i] - r[j])) / (epsilon + np.linalg.
 ↪norm(x[i] - r[j])**2)**2 for j in range(k)], axis=0) + lambda2*(-2*(x[i+1] -␣
 ↪x[i]) + 2*(x[i] - x[i] - 1))

    res[0] = np.array([0, 0])
```

```
    res[-1] = np.array([0, 0])
    return np.array(res)
```

[28]:
```
def golden_search(f, x, r):
    a = 0
    b = 441
    t = (np.sqrt(5)-1) / 2
    x1 = a + (1 - t) * (b - a)
    f1 = f(x1, x, r)
    x2 = a + t*(b-a)
    f2 = f(x2, x, r)
    tol = 1e-5
    while (b-a) > tol:
        if f1 > f2:
            a = x1
            x1 = x2
            f1 = f2
            x2 = a + t*(b-a)
            f2 = f(x2, x, r)
        else:
            b = x2
            x2 = x1
            f2 = f1
            x1 = a + (1 - t) * (b - a)
            f1 = f(x1, x, r)

    return (a+b)/2

def F_min(alfa, x0, r):
    return F(x0 - alfa * gradient(x0, r), r)

def minimize(x, r):
    x0 = deepcopy(x)
    for _ in range(400):
        alfa = golden_search(F_min, x0, r)
        x0 = x0 - alfa * gradient(x0, r)
    return x0
```

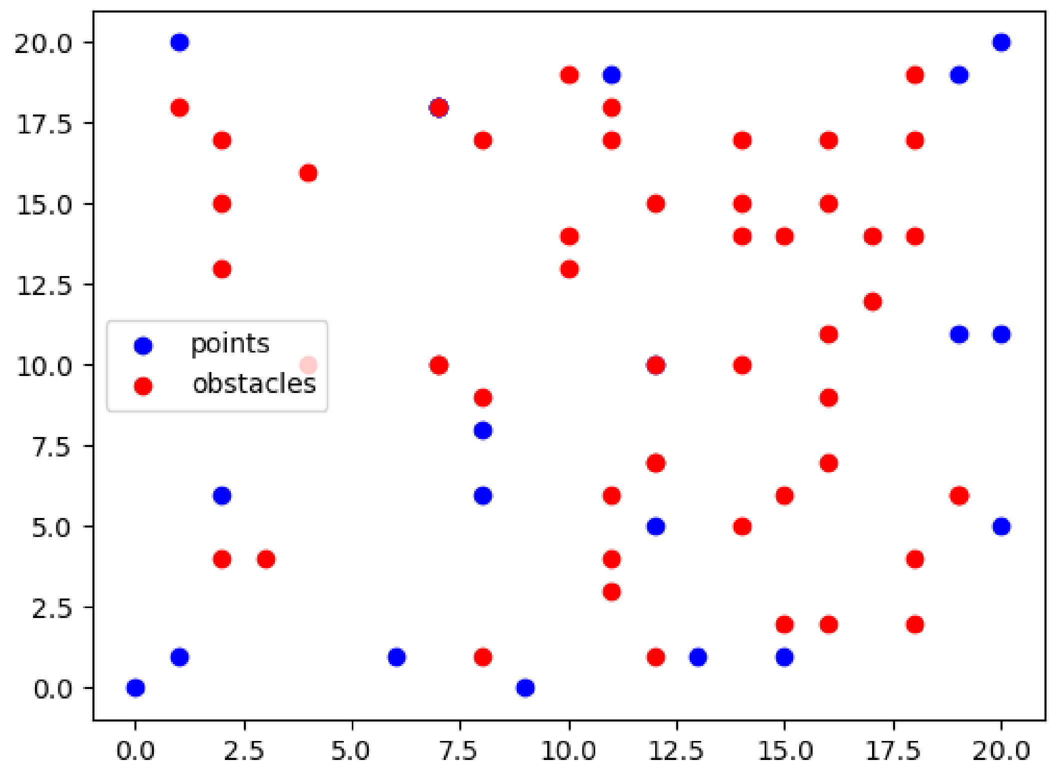
[29]:
```
rng = random.Random(2137)
```

2.1 Próba 1

[30]:
```
points = [np.array([rng.randrange(0, 21), rng.randrange(0,21)]) for _ in␣
 ↪range(21)]
points[0] = np.array([0, 0])
points[-1] = np.array([20, 20])
```

```
obstacles = [np.array([rng.randrange(1, 20), rng.randrange(1, 20)]) for _ in
  range(50)]
points_xs = [points[i][0] for i in range(21)]
points_ys = [points[i][1] for i in range(21)]
obstacles_xs = [obstacles[i][0] for i in range(50)]
obstacles_ys = [obstacles[i][1] for i in range(50)]
```

[31]:
```
plt.scatter(points_xs, points_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



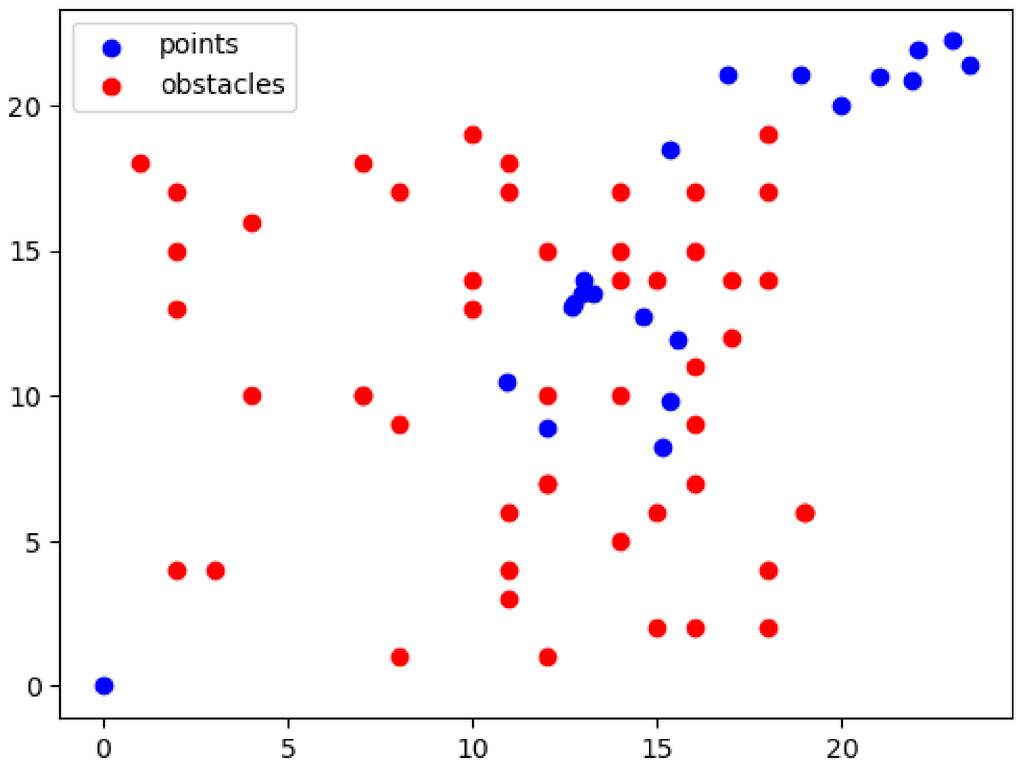
[32]:
```
shortest = minimize(points, obstacles)
```

[33]:
```
print(f"Function result: {F(points, obstacles)}")
```

```
Function result: 40000000004863.68
```

[34]:
```
shortest_xs = [shortest[i][0] for i in range(21)]
shortest_ys = [shortest[i][1] for i in range(21)]
```

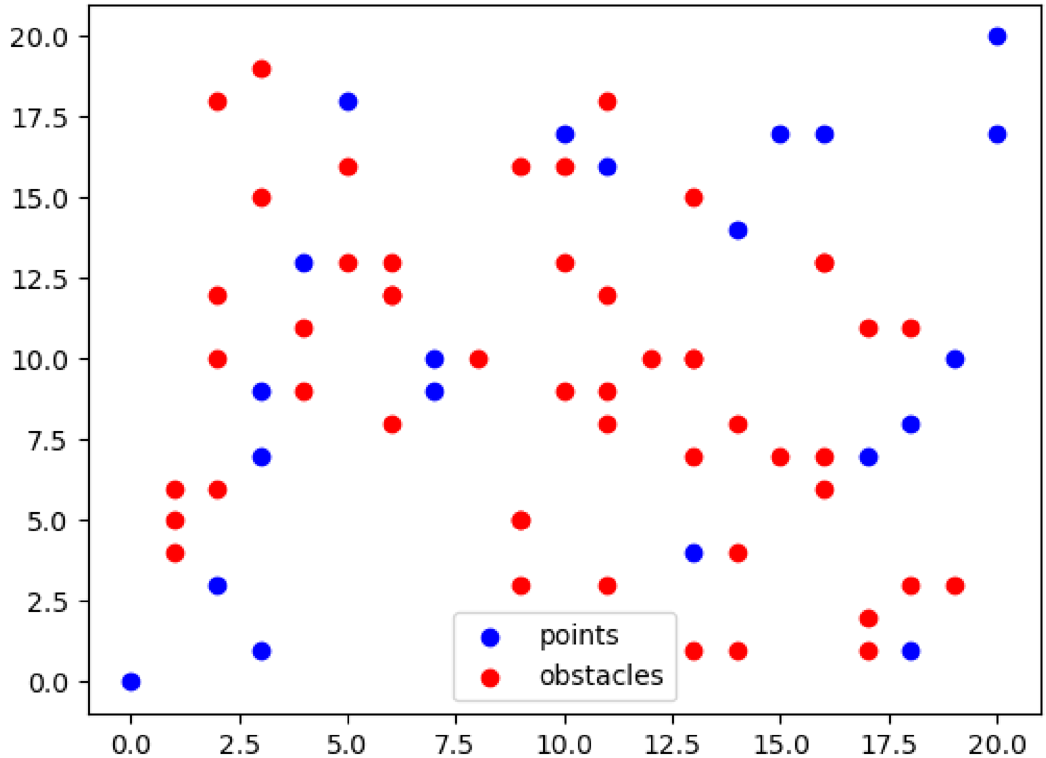
[35]:
```
plt.scatter(shortest_xs, shortest_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



2.2 Próba 2

[36]:
```
points = [np.array([rng.randrange(0, 21), rng.randrange(0,21)]) for _ in␣
  ↪range(21)]
points[0] = np.array([0, 0])
points[-1] = np.array([20, 20])
obstacles = [np.array([rng.randrange(1, 20), rng.randrange(1, 20)]) for _ in␣
  ↪range(50)]
points_xs = [points[i][0] for i in range(21)]
points_ys = [points[i][1] for i in range(21)]
obstacles_xs = [obstacles[i][0] for i in range(50)]
obstacles_ys = [obstacles[i][1] for i in range(50)]
```

[37]:
```
plt.scatter(points_xs, points_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



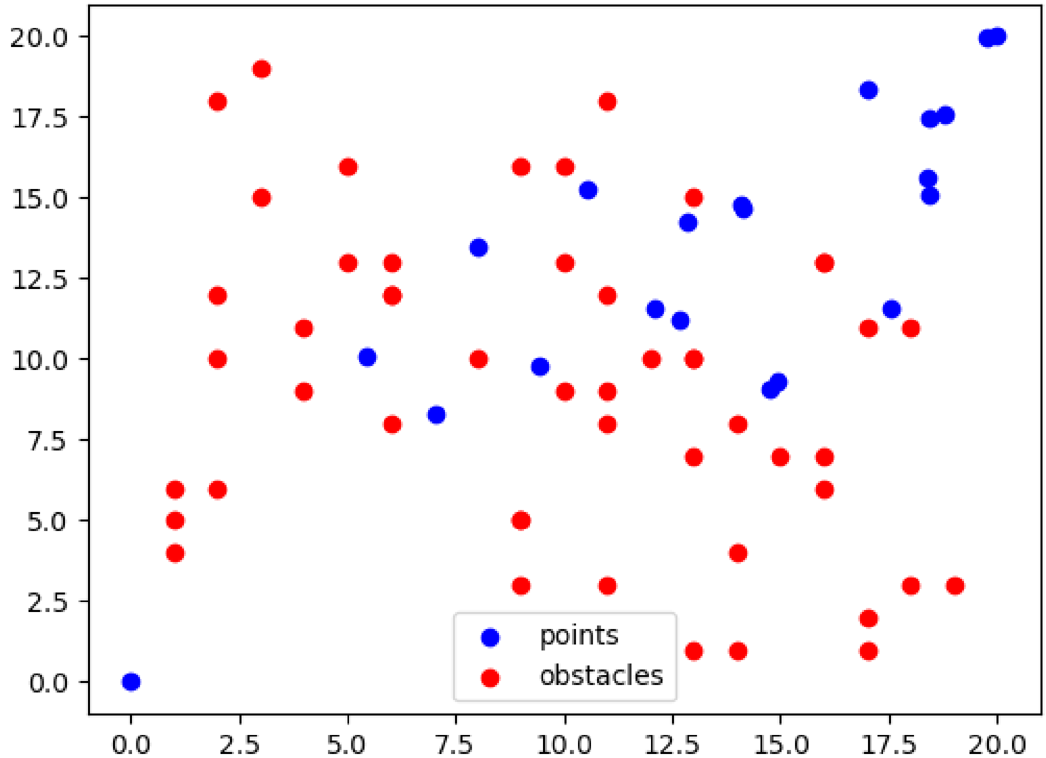
[38]:
```
shortest = minimize(points, obstacles)
```

[39]:
```
print(f"Function result: {F(points, obstacles)}")
```

Function result: 2540.1306437524804

[40]:
```
shortest_xs = [shortest[i][0] for i in range(21)]
shortest_ys = [shortest[i][1] for i in range(21)]
```

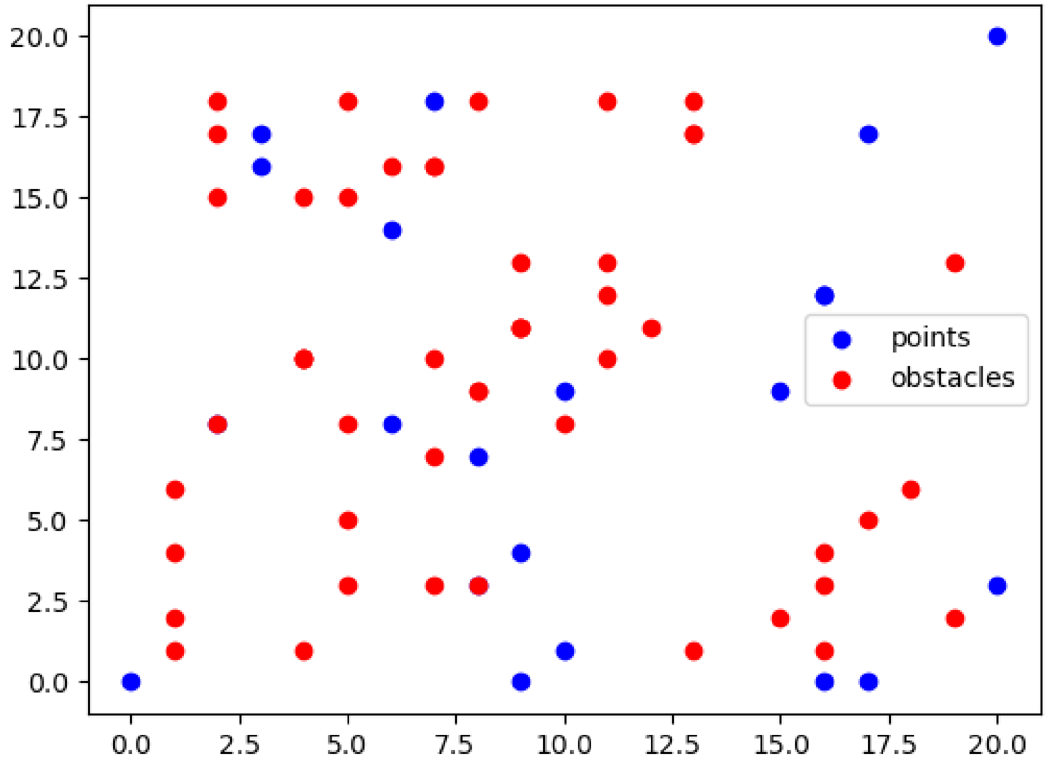
[41]:
```
plt.scatter(shortest_xs, shortest_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



2.3 Próba 3

[42]:
```
points = [np.array([rng.randrange(0, 21), rng.randrange(0,21)]) for _ in␣
  ↪range(21)]
points[0] = np.array([0, 0])
points[-1] = np.array([20, 20])
obstacles = [np.array([rng.randrange(1, 20), rng.randrange(1, 20)]) for _ in␣
  ↪range(50)]
points_xs = [points[i][0] for i in range(21)]
points_ys = [points[i][1] for i in range(21)]
obstacles_xs = [obstacles[i][0] for i in range(50)]
obstacles_ys = [obstacles[i][1] for i in range(50)]
```

[43]:
```
plt.scatter(points_xs, points_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



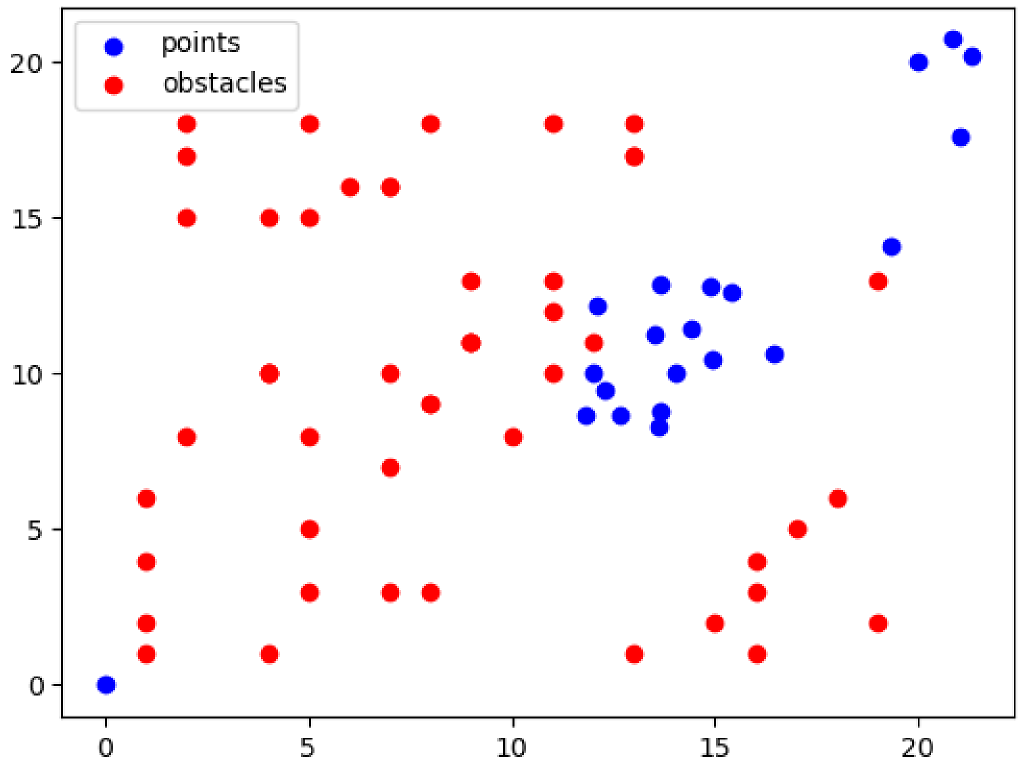
[44]:
```
shortest = minimize(points, obstacles)
```

[45]:
```
print(f"Function result: {F(points, obstacles)}")
```

```
Function result: 20000000003079.777
```

[46]:
```
shortest_xs = [shortest[i][0] for i in range(21)]
shortest_ys = [shortest[i][1] for i in range(21)]
```

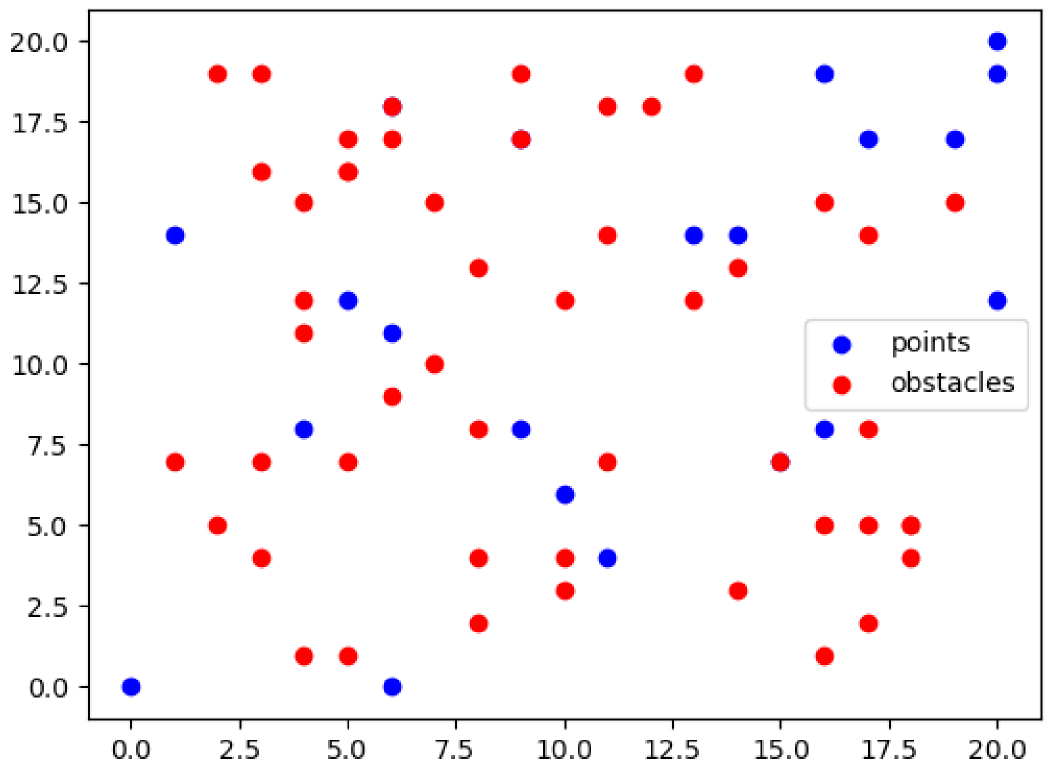
[47]:
```
plt.scatter(shortest_xs, shortest_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



2.4 Próba 4

[48]:
```
points = [np.array([rng.randrange(0, 21), rng.randrange(0,21)]) for _ in␣
↪range(21)]
points[0] = np.array([0, 0])
points[-1] = np.array([20, 20])
obstacles = [np.array([rng.randrange(1, 20), rng.randrange(1, 20)]) for _ in␣
↪range(50)]
points_xs = [points[i][0] for i in range(21)]
points_ys = [points[i][1] for i in range(21)]
obstacles_xs = [obstacles[i][0] for i in range(50)]
obstacles_ys = [obstacles[i][1] for i in range(50)]
```

[49]:
```
plt.scatter(points_xs, points_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



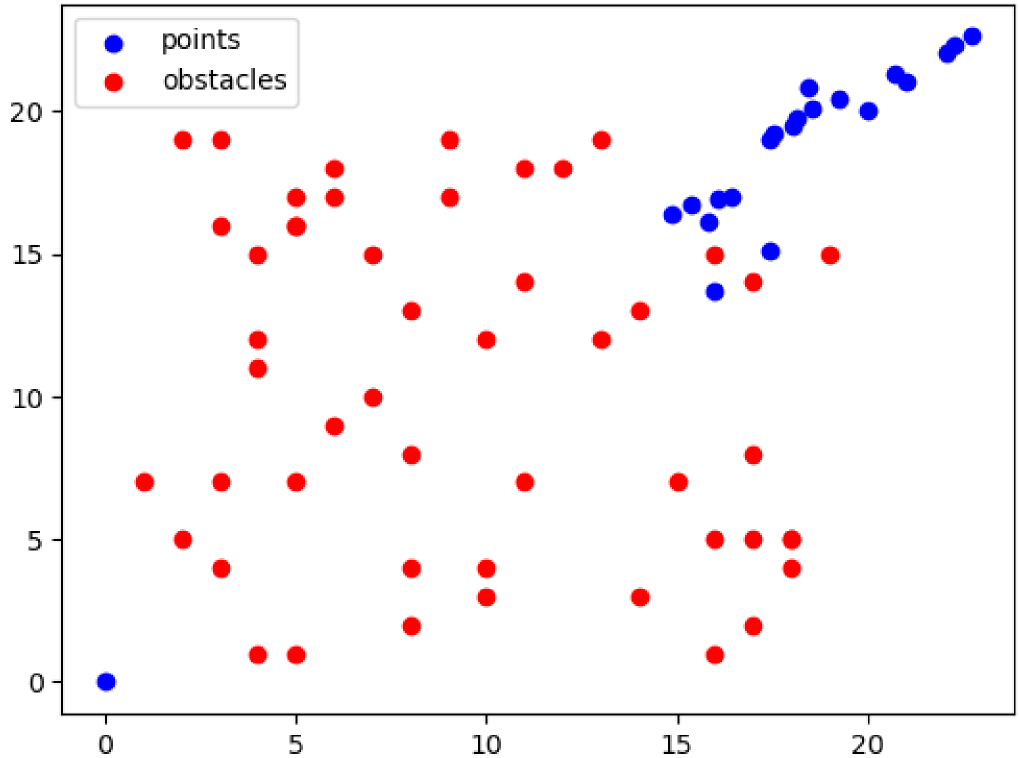
[50]:
```
shortest = minimize(points, obstacles)
```

[51]:
```
print(f"Function result: {F(points, obstacles)}")
```

```
Function result: 30000000003386.863
```

[52]:
```
shortest_xs = [shortest[i][0] for i in range(21)]
shortest_ys = [shortest[i][1] for i in range(21)]
```

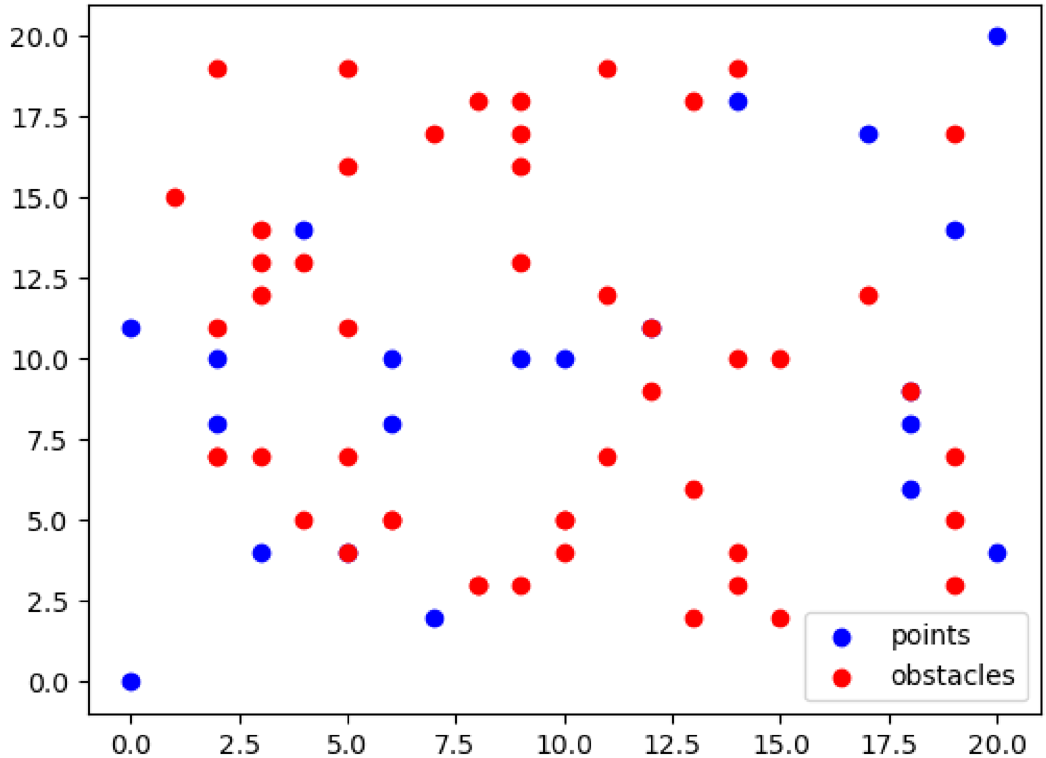
[53]:
```
plt.scatter(shortest_xs, shortest_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



2.5 Próba 5

[54]:
```
points = [np.array([rng.randrange(0, 21), rng.randrange(0,21)]) for _ in␣
↪range(21)]
points[0] = np.array([0, 0])
points[-1] = np.array([20, 20])
obstacles = [np.array([rng.randrange(1, 20), rng.randrange(1, 20)]) for _ in␣
↪range(50)]
points_xs = [points[i][0] for i in range(21)]
points_ys = [points[i][1] for i in range(21)]
obstacles_xs = [obstacles[i][0] for i in range(50)]
obstacles_ys = [obstacles[i][1] for i in range(50)]
```

[55]:
```
plt.scatter(points_xs, points_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



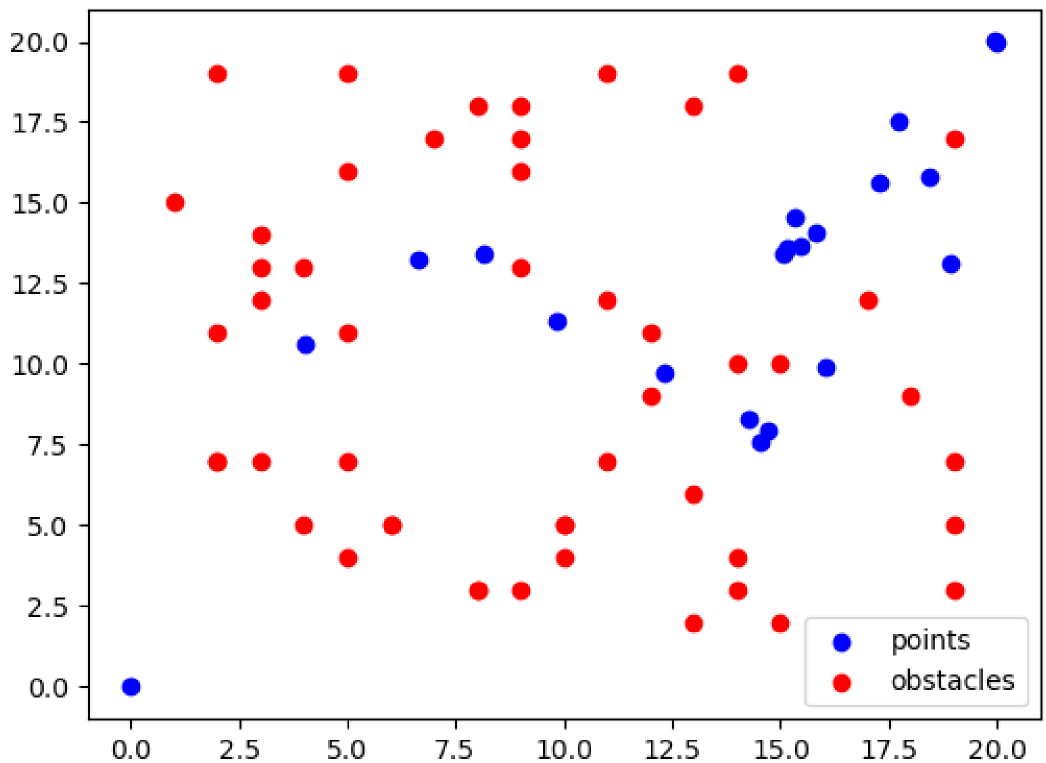
[56]:
```
shortest = minimize(points, obstacles)
```

[57]:
```
print(f"Function result: {F(points, obstacles)}")
```

```
Function result: 30000000002305.12
```

[58]:
```
shortest_xs = [shortest[i][0] for i in range(21)]
shortest_ys = [shortest[i][1] for i in range(21)]
```

[59]:
```
plt.scatter(shortest_xs, shortest_ys, color="blue", label="points")
plt.scatter(obstacles_xs, obstacles_ys, color="red", label="obstacles")
plt.legend()
plt.show()
```



2.6 Wnioski

Można zauważyć pewną prawidłowość w wyliczaniu ścieżki robota. Pierwszy przystanek jest w znacznej odległości od pola startowego. Wszystkie pozostałe przystanki są znacznie bliżej siebie.