APIN- Correction of 2015 exam

Ex. 1 $\quad f(-2) = -19\ ;\ f(0) = -1\ ;\ f(1) = 5\ ;\ f(2) = 33$

1) $$L_0(x) = \frac{(x-0)(x-1)(x-2)}{(-2-0)(-2-1)(-2-2)} = \frac{x(x-1)(x-2)}{-24}$$

$$\Rightarrow L_0(3) = \frac{6}{-24} = -\frac{1}{4}$$

$$L_1(x) = \frac{(x+2)(x-1)(x-2)}{(0+2)(0-1)(0-2)} = \frac{(x+2)(x-1)(x-2)}{4}$$

$$\Rightarrow L_1(3) = \frac{10}{4}$$

$$L_2(x) = \frac{(x+2)(x-0)(x-2)}{(1+2)(1-0)(1-2)} = \frac{x(x+2)(x-2)}{-3}$$

$$\Rightarrow L_2(3) = -5$$

$$L_3(x) = \frac{(x+2)(x-0)(x-1)}{(2+2)(2-0)(2-1)} = \frac{x(x+2)(x-1)}{8}$$

$$\Rightarrow L_3(3) = \frac{15}{4}$$

2) $$f(3) \approx P_3(3) = -19\, L_0(3) - 1 . L_1(3) + 5\, L_2(3) + 33\, L_3(3)$$

$$= +\frac{19}{4} - \frac{10}{4} - 25 + \frac{15 \times 33}{4}$$

$$= \cancel{\frac{366}{4} = \frac{183}{2}}\ 101$$

3/ $\left| f(3) - P_3(3) \right| \leq \left| (3+2)(3-0)(3-1)(3-2) \right| \dfrac{\sup_{t \in [-2,3]} \left| f^{(4)}(t) \right|}{4!}$

$$\leq \frac{30}{4!} \sup_{t \in [-2,3]} \left| f^{(4)}(t) \right|$$

$$\leq \frac{5}{4} \sup_{[-2,3]} \left| f^{(4)}(t) \right|$$

<u>Ex. 2</u> $f(0) = 1$; $f(1) = 2$; $f(2) = 7$; $f(3) = 22$

1/

x_i	$f[x_i]$	$f[x_i x_{i+1}]$	$f[x_i x_{i+1} x_{i+2}]$	$f[x_0 x_1 x_2 x_3]$
0	1			
		1		
1	2		2	
		5		1
2	7		5	
		15		
3	22			

2/ $P_3(x) = 1 + 1(x-0) + 2(x-0)(x-1) + 1(x-0)(x-1)(x-2)$

$$= x^3 - x^2 + x + 1$$

3/ We can build the synthetic division table with $x - 4$:

	a_3	a_2	a_1	a_0	R
P_3	1	-1	1	1	0
P_2		1	3	13	53
P_1			1	7	41
P_0				1	11
					1

Then, we read the last column:

$$P_3(X) = (X-4)^3 + 11(X-4)^2 + 41(X-4) + 53$$

<u>Remark</u>: another redaction would consist in doing the divisions of P_k by $(X-4)$ one after one:

- $P_4 = X^3 - X^2 + X + 1 = (X-4)(X^2+3X+13) + 53$
- $X^2 + 3X + 13 = (X-4)(X+7) + 41$

$$\Rightarrow P_4 = (X-4)\left[(X-4)(X+7) + 41\right] + 53$$

$$= (X-4)^2 (X+7) + 41(X-4) + 53$$

- $X + 7 = (X-4) + 11$

$$\Rightarrow P_4 = (X-4)^2\left[(X-4) + 11\right] + 41(X-4) + 53$$

$$= (X-4)^3 + 11(X-4)^2 + 41(X-4) + 53$$

4) We take $f(4) \approx P_4(4)$

and $P_4(4) = 53$ so $f(4) \approx 53$

5) We have $P_4(x) = \underbrace{1}_{\frac{P_4'''(4)}{3!}} (x-4)^3 + \underbrace{11}_{\frac{P_4''(4)}{2!}} (x-4)^2 + \underbrace{41}_{\frac{P'(4)}{1!}}(x-4) + \underbrace{53}_{\frac{P(4)}{0!}}$

So $P_4'(4) = 41$

$P_4''(4) = 22$

$P_4'''(4) = 6$

<u>Ex. 3</u> For a function $f : [-1,1] \longrightarrow \mathbb{R}$, we denote $I(f)$

the integral $I(f) = \int_{-1}^{1} f(x)\,dx$

We have the nodes $x_0 = -1$; $x_1 = 0$; $x_2 = +1$

1) The interpolating polynomial of degree 2 using these nodes is

$$P_2(x) = f(-1)\frac{(x-0)(x-1)}{(-1-0)(-1-1)} + f(0)\frac{(x+1)(x-1)}{(0+1)(0-1)} + f(1)\frac{(x+1)(x-0)}{(1+1)(1-0)}$$

$$= f(-1)\frac{x(x-1)}{2} + f(0)\frac{(x-1)(x+1)}{-1} + f(1)\frac{x(x+1)}{2}$$

Indeed, this polynomial satisfies to

$$P_2(-1) = f(-1) \times \frac{(-1)(-1-1)}{2} - f(0)(-1-1)(-1+1) + f(1)\frac{(-1)(-1+1)}{2}$$

$$= f(-1)$$

$P_2(0) = f(0)$ and $P_2(1) = f(1)$

2/ $I(f)$ can be approximated with the rule R:

$$R(f) = I(P_2)$$

$$= f(-1)\int_{-1}^{1} \frac{1}{2} x(x-1)\,dx - f(0)\int_{-1}^{1}(x-1)(x+1)\,dx + f(1)\int_{-1}^{1}\frac{1}{2}x(x+1)\,dx$$

But $\int_{-1}^{1} \frac{1}{2} x(x-1)\,dx = \frac{1}{2}\int_{-1}^{1}(x^2-x)\,dx$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{3}$$

$$\int_{-1}^{1}(x-1)(x+1)\,dx = \int_{-1}^{1}(x^2-1)\,dx = \left[\frac{x^3}{3} - x\right]_{-1}^{1} = \frac{-4}{3}$$

$$\int_{-1}^{1}\frac{1}{2}x(x+1)\,dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{3}$$

So $R(f) = \frac{1}{3} f(-1) + \frac{4}{3} f(0) + \frac{1}{3} f(1)$

3/ * for $f(x)=1$, we have $I(f)=\int_{-1}^{1} 1\,dx=2$

$$R(f)=\frac{1}{3}f(-1)+\frac{4}{3}f(0)+\frac{1}{3}f(1)=2$$

So $I(f)=R(f)$

* for $f(x)=x$, we have

$$I(f)=\int_{-1}^{1} x\,dx=0$$

$$R(f)=\frac{1}{3}\times(-1)+\frac{4}{3}\times 0+\frac{1}{3}\times(+1)=0$$

So $R(f)=I(f)$

* for $f(x)=x^2$, we have

$$I(f)=\int_{-1}^{1} x^2 dx=2\int_{0}^{1} x^2 dx=2\left[\frac{x^3}{3}\right]_0^1=\frac{2}{3}$$

$$R(f)=\frac{1}{3}(-1)^2+\frac{4}{3}(0)^2+\frac{1}{3}(+1)^2=\frac{2}{3}$$

So $R(f)=I(f)$

* For $f(x)=x^3$, we have

$$I(f)=\int_{-1}^{1} x^3 dx=0$$

$$R(f)=\frac{1}{3}(-1)^3+\frac{4}{3}(0)^3+\frac{1}{3}(1)^3=0$$

So $R(f)=I(f)$

* For $f(x) = x^4$ we have

$$I(f) = \int_{-1}^{1} x^4\,dx = 2\int_0^1 x^4\,dx = 2\left[\frac{x^5}{5}\right]_0^1 = \frac{2}{5}$$

$$R(f) = \frac{(-1)^4}{3} + \frac{4(0)^4}{3} + \frac{(1)^4}{3} = \frac{2}{3}$$

So $R(f) \neq I(f)$

The degree of precision is hence $dop(R) = 3$

(Remark: we found $dop(R) = 3 \geq n = 2$, so we have the confirmation that the coefficients $\frac{1}{3}$, $\frac{4}{3}$, $+\frac{1}{3}$ of the rule are ok.

4/ * Since the degree of precision of the quadrature rule is 3, the Peano kernel is for any $t \in [-1,1]$:

$$K(t) = I\left(\frac{(x-t)_+^3}{3!}\right) - R\left(\frac{(x-t)_+^3}{3!}\right)$$

But $I\left((x-t)_+^3\right) = \int_{-1}^{1} (x-t)_+^3\,dx$

$(t \in [-1,1])$

$$= \int_{-1}^{t} \underbrace{(x-t)_+^3}_{=0}\,dx + \int_{t}^{+1} \underbrace{(x-t)_+^3}_{=(x-t)^3}\,dx$$

$$= \int_t^1 (x-t)^3 \, dx = \left[\frac{(x-t)^4}{4}\right]_t^1 = \frac{(1-t)^4}{4}$$

And $R\left((x-t)_+^3\right) = \frac{1}{3}\underbrace{(-1-t)_+^3}_{=0} + \frac{4}{3}\underbrace{(0-t)_+^3}_{=\begin{cases} -t^3 & \text{if } t<0 \\ 0 & \text{if } t \geq 0 \end{cases}} + \frac{1}{3}\underbrace{(1-t)_+^3}_{=(1-t)^3}$

So finally:

- If $t \geq 0$, $3!\,k(t) = \frac{(1-t)^4}{4} - \frac{(1-t)^3}{3}$

$$= (1-t)^3\left[\frac{1-t}{4} - \frac{1}{3}\right]$$

$$= (1-t)^3\left[-\frac{t}{4} - \frac{1}{12}\right]$$

- If $t \leq 0$, $3!\,k(t) = \frac{(1-t)^4}{4} - \frac{(1-t)^3}{3} + \frac{4}{3}t^3$

* Sign of k: if we use the binomial formula to develop $(1-t)^4$ and $(1-t)^3$, we obtain:

- $\forall t \in [-1,0]$, $3!\,k(t) = -\frac{1}{12} + \frac{t^2}{2} + \frac{2}{3}t^3 + \frac{t^4}{4}$
- $\forall t \in [0,1]$, $3!\,k(t) = -\frac{1}{12} + \frac{t^2}{2} - \frac{2}{3}t^3 + \frac{t^4}{4}$

Thus, k is an even function and we can study its sign on $[0,1]$ only.

But $\forall t \in [0,1]$, $3!\,k(t) = \underbrace{(1-t)^3}_{\geq 0} \underbrace{\left[\frac{-t}{4} - \frac{1}{12}\right]}_{\leq 0} \leq 0$

Thus, $k(t)$ has a constant sign on $[-1;1]$

* Therefore, there exists a Peano constant K_c, which satisfies

$\forall f \in C^4([-1,1])$, $\exists c \in [-1,1]$,

$$I(f) - R(f) = K_c \; f^{(4)}(c)$$

If we take for f the particular function $x \to x^4$, we have $I(f) = \frac{2}{5}$; $R(f) = \frac{2}{3}$; $f^{(4)}(c) = 4! = 24$

So $\frac{2}{5} - \frac{2}{3} = K_c \times 24$ and $\boxed{K_c = -\frac{1}{90}}$

5) We know that $\forall f \in C^4([-1,1])$, $\exists c \in [-1,1]$,

$$I(f) - R(f) = -\frac{1}{90} f^{(4)}(c)$$

Therefore, $\left| I(f) - R(f) \right| \leq \frac{1}{90} \sup_{[-1,1]} \left| f^{(4)}(x) \right|$