Approximation and Interpolation Examination

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Duration: 1 hour and 30 minutes.

Use of reference notes and textbooks of any form is strictly prohibited.
Use of a calculator is permitted.
The 3 questions in this examination are mutually independent.

Question 1:

Consider the real–valued function $f : \mathbb{R} \to \mathbb{R}$ whose values are given only at a finite number of nodes $x_0 = -2, \quad x_1 = 0, \quad x_2 = 1, \quad \text{and} \quad x_3 = 2$:

$$f(-2) = -19, \quad f(0) = -1, \quad f(1) = 5, \quad \text{and} \quad f(2) = 33.$$

Recall that the *Lagrange formula* for the interpolating polynomial of this data is given by

$$p_3(x) = \sum_{j=0}^{3} f(x_j)\, L_j(x)$$

where the functions $L_j(x)$ are the *fundamental Lagrange polynomials* associated with each node, $j = 0, 1, 2, 3$.

1): Calculate the values $L_j(3)$ for each $j = 0, 1, 2, 3$.

2): Estimate the value of $f(3)$ by interpolation.

3): Assuming that the function $f(x)$ has continuous derives up to and including order 4 in the interval $(-2, 2) \subset \mathbb{R}$, give the expression for the error in the approximation of $f(x)$ when using this interpolating polynomial.

Question 2:

Our task is to use the Newton divided difference method to obtain the interpolating polynomial.
Suppose we are given a real–valued function $f : \mathbb{R} \to \mathbb{R}$ whose values are known only at the points $x_0 = 0, \quad x_1 = 1, \quad x_2 = 2, \quad \text{and} \quad x_3 = 3$:

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 7, \quad \text{and} \quad f(3) = 22.$$

1): Construct the divided difference table for this data.

2): Use this result to derive the interpolating polynomial for $f(x)$.

3): Show that this interpolating polynomial can be expressed in the form

$$p_3(x) = R_3\,(x-4)^3 + R_2\,(x-4)^2 + R_1\,(x-4) + R_0$$

where $R_0,\ R_1,\ R_2,$ and R_3 are constants to be determined.

4): Provide an approximation for $f(4)$.

5): Approximate the derivatives $f'(4),\ f''(4),$ and $f'''(4)$.

Question 3:

In this question we shall construct a numerical method of integration using the Lagrange formula for an interpolating polynomial.
We assume that our real–valued function of interest $f(x)$ is continuous over the interval $[-1,1] \subset \mathbb{R}$, and that our nodes of interpolation are $x_0 = -1,\quad x_1 = 0,\quad$ and $\quad x_2 = 1$.
If $p_2(x)$ is the interpolating polynomial of degree 2 for $f(x)$ using these nodes,

1): show that

$$p_2(x) = \sum_{j=0}^{2} f(x_j)\,L_j(x)$$

where $L_0(x) = \frac{1}{2}x(x-1),\quad L_1(x) = -(x+1)(x-1),\quad$ and $\quad L_2(x) = \frac{1}{2}x(x+1)$

2): and deduce the following approximation formula:

$$\int_{-1}^{1} f(x)\,dx \approx \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1).$$

3): Determine the order (the *degree of precision*) of this quadrature rule.

4): Assuming that the derivatives $f'(x), f''(x), \ldots, f^{(m+1)}(x)$ are continuous on the interval $(-1,1)$, where m is the degree of precision, evaluate the *Peano constant.*

5): Provide an upper bound for the absolute value of the approximation error for this quadrature rule.