Mathématiques du signal

$t \longrightarrow p$

$\delta(t) \longrightarrow 1$

$u(t) \longrightarrow 1/p$

$e^{-at}.u(t) \longrightarrow \frac{1}{p+a}$

$t.u(t) \longrightarrow \frac{1}{p^2}$

Transformation de Laplace inverse :

- $$X(p) = \frac{2p^2 + 12p + 6}{p(p+2)(p+3)}$$

$$X(p) = \frac{A}{p} + \frac{B}{p+2} + \frac{C}{p+3} = \frac{2p^2+12p+6}{p(p+2)(p+3)}$$

Décomposition en éléments simples.

$\rightarrow \times p \rightarrow p = 0 \Rightarrow A = 1$

$\rightarrow \times (p+2) \rightarrow p = -2 \Rightarrow B = 5$

$\rightarrow \times (p+3) \rightarrow p = -3 \Rightarrow C = -4.$

$$X(p) = \frac{1}{p} + \frac{5}{p+2} - \frac{4}{p+3} \Rightarrow x(t) = (1 + 5e^{-2t} - 4e^{-3t}).u(t)$$

- $$Y(p) = \frac{12 + 20p}{p^2(p^2+p-6)} = \frac{12+20p}{p^2(p+3)(p-2)}.$$

$$Y(p) = \frac{A}{p^2} + \frac{B}{p} + \frac{C}{p+3} + \frac{D}{p-2}.$$

$\rightarrow \times p^2 \rightarrow p = 0 \Rightarrow A = -2$

$\rightarrow \times (p+3) \rightarrow p = -3 \Rightarrow C = 16/15$

$\rightarrow \times (p-2) \rightarrow p = 2 \Rightarrow D = 13/5$

$\rightarrow \times p \rightarrow p \rightarrow \infty \Rightarrow B + C + D = 0 \Rightarrow B = -11/3.$

$$Y(p) = \frac{-2}{p^2} - \frac{11}{3p} + \frac{16}{15} \times \frac{1}{p+3} + \frac{13}{5} \times \frac{1}{p-2}$$

$$y(t) = (-2t \underbrace{- \frac{11}{3} + \frac{16}{15}e^{-3t} + \frac{13}{5}e^{2t}}_{\text{à vérifier avec les CI.}}).u(t)$$

Transformation de Laplace :

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad \text{(Formule d'Euler).}$$

$$\mathcal{L}[\sin(\omega t)\, u(t)] = \mathcal{L}\left[\frac{e^{j\theta} - e^{-j\theta}}{2j}\, u(t)\right]$$

$$= \frac{1}{2j}\left[\mathcal{L}[e^{j\theta} u(t)] - \mathcal{L}[e^{-j\theta} u(t)]\right]$$

$$= \frac{1}{2j}\left[\frac{1}{p-j\omega} - \frac{1}{p+j\omega}\right]$$

$$= \frac{\omega}{p^2+\omega^2}$$

Résolution d'une équation différentielle :

$$\ddot{y}(t) + \dot{y}(t) - 6\,y(t) = 12t + 20$$

$$(p^2 + p - 6)\,Y(p) = \mathcal{L}[(12t + 20)\,u(t)].$$

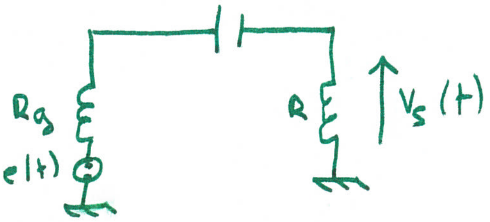
$$= 12\,\mathcal{L}(t\,u(t)) + 20\,\mathcal{L}(u(t))$$

$$= \frac{12}{p^2} + \frac{20}{p} = \frac{12 + 20p}{p^2}$$

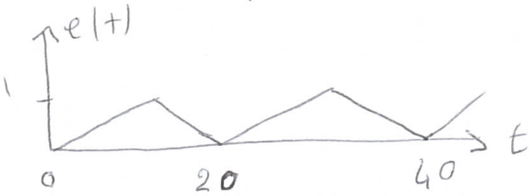
$$\Rightarrow \quad Y(p) = \frac{12 + 20p}{p^2(p^2+p-6)} \longrightarrow \text{cf pcdt}$$

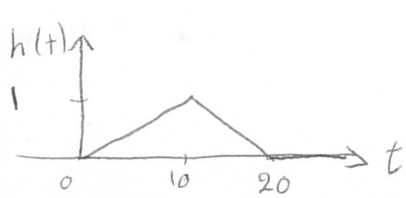
$$y(t) = -2t - \frac{11}{3} + \frac{13}{5}e^{2t} + \frac{16}{15}e^{-3t}$$

Réponse d'un circuit électronique :



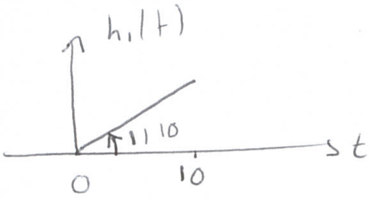
Calculer $E(p)$.



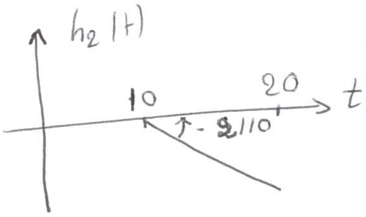


$$\mathcal{L}[h(t)] = H(p).$$

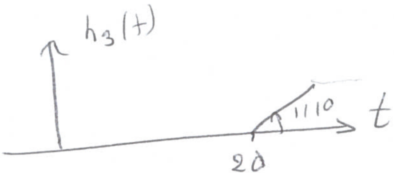
On décompose $h(t)$ sous la forme de 3 signaux :



$$H_1(p) = \frac{1}{10} \times \frac{1}{p^2}.$$



$$H_2(p) = e^{-10p}\left(\frac{-2}{10} \times \frac{1}{p^2}\right).$$



$$H_3(p) = e^{-20p}\left(\frac{1}{10} \times \frac{1}{p^2}\right).$$

$$\rightarrow H(p) = \frac{1}{10p^2}\left(1 - 2e^{-10p} + e^{-20p}\right).$$

$$\Rightarrow H(p) = \frac{(1 - e^{-10p})^2}{10\,p^2}$$

On $e(t) = h(t) + h(t-20) + h(t-40) \dots$

$$\mathcal{L}[e(t)] = \mathcal{L}[h(t)] + \mathcal{L}[h(t-20)] \dots$$

$$= H(p) + e^{-20p} H(p) \dots$$

$$= H(p)\left[1 + e^{-20p} + (e^{-20p})^2 \dots\right]$$

$\rightarrow$ série géométrique $\displaystyle\sum_{n \geq 0}^{\infty} x^n = \frac{1}{1-x}$

$$\Rightarrow E(p) = H(p) \times \frac{1}{1 - e^{-20p}} = H(p) \times \frac{1}{(1-e^{-10p})(1+e^{-10p})}$$

$$\Rightarrow E(p) = \frac{(1 - e^{-10p})}{10p^2(1 + e^{-10p})}$$

Calculer $V_s(p)$.

$$H(p) = \frac{V_s(p)}{E(p)} = \frac{R}{R_g + R + \frac{1}{Cp}} = \frac{RCp}{1 + (R + R_g)Cp}$$

$\Rightarrow V_s(p) = H(p) \times E(p)$

$$= \frac{RCp}{1+(R+R_g)Cp} \times \frac{1-e^{-10p}}{10p^2(1+e^{-10p})}$$

$$= \frac{RC}{10p} \times \frac{1-e^{-10p}}{(1+e^{-10p}) \times (1+\underbrace{(R+R_g)C}_{\text{suppose } = 1\mu s}p)}$$

$$= \frac{RC}{10} \times \frac{1-e^{-10p}}{(1+e^{-10p})(1+p)p} = \underbrace{\frac{RC}{10} \times \frac{1}{p(p+1)}}_{= X(p) \Rightarrow \frac{RC}{10}\left(\frac{1}{p} - \frac{1}{p+1}\right)} \times \underbrace{\frac{1-e^{-10p}}{1+e^{-10p}}}_{= \frac{1-2e^{-10p}+e^{-20p}}{1-e^{-20p}}}$$

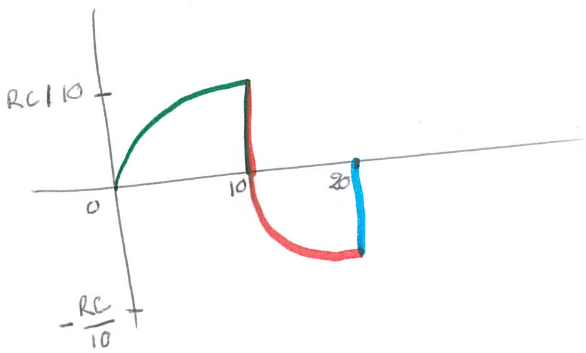
$$\rightarrow x(t) = \frac{RC}{10}(1-e^{-t})\,u(t)$$

$$\cancel{X}(t) = \frac{RC}{10}(1-e^{-t})\,u(t)$$

$$V_s(p) = \underbrace{X(p)\,(1-2e^{-10p}+e^{-20p})}_{N(p)} \times \frac{1}{1-e^{-20p}}$$

→ reproduction toutes les 20 µs du signal $n(t)$.

$$\rightarrow n(t) = x(t) - 2x(t-10) + x(t-20)$$



Théorème de Shannon :

Lorsque l'on échantillonne un signal continue à spectre fréquentiel borné [−N ; +N], on ne perd aucune information si la fréquence d'échantillonnage est supérieure au double de la plus haute fréquence N contenu dans le signal continu : $N < 1/2T$.

Transformée en Z :

M1 Méthode 1 : Calcul à partir de $f(t)$.

$$f^*(t) = \sum_{n=0}^{+\infty} f(nT)\, z^{-n}$$

M2 Méthode 2 : Calcul à partir de $F(p)$:

$$F(z) = \sum r_i \quad \text{(résidus)}.$$

$$F(p) = \frac{N(p)}{D(p)} \longrightarrow \text{pôles } p_i \text{ d'ordre } n_i \; . \longrightarrow r_i = \frac{N(p_i)}{D'(p_i)} \times \frac{1}{1 - e^{Tp_i} z^{-1}}$$

ou $r_i = \dfrac{1}{(n_i-1)!} \times \dfrac{d^{(n_i-1)}}{dp^{n_i-1}} \left[(p-p_i)^{n_i} \times \dfrac{F(p)}{1 - e^{Tp} z^{-1}} \right]$

- $\begin{cases} f(t) = u(t) \\ F(p) = 1/p \end{cases}$

M1 $\longrightarrow f(t) = u(t) \implies f(nT) = u(nT)$

$$\longrightarrow F(z) = \sum_{n=0}^{+\infty} f(nT)\, z^{-n}$$

$$= \sum_{n=0}^{+\infty} u(nT)\, z^{-n}$$

$$= \sum_{n=0}^{+\infty} z^{-n} \quad \text{(série géométrique)}$$

$$F(z) = \frac{1}{1 - z^{-1}}$$

$$F(z) = \frac{z}{z-1}$$

M2 $\rightarrow N(p) = 1 \; . \begin{cases} D(p) = p \rightarrow p_0 = 0 \text{ d'ordre } 1 \\ D'(p) = 1 \end{cases}$

$$r_i = \frac{1}{1} \times \frac{1}{1 - e^{T \times 0} z^{-1}} \implies F(z) = \frac{z}{z-1}$$

- $\begin{cases} f(t) = e^{-at}\, u(t) \\ F(p) = \dfrac{1}{p+a} \end{cases}$

$M1 \rightarrow f(nT) = e^{-anT}\, u(nT).$

$$F(z) = \sum_{n=0}^{+\infty} e^{-anT} z^{-n}$$

$$= \sum_{n=0}^{+\infty} \left(e^{-aT} z^{-1}\right)^n .$$

$$= \frac{1}{1 - e^{-aT} z^{-1}}$$

$M2 \rightarrow N(p) = 1.$ $\begin{cases} D(p) = p + a \rightarrow p_0 = -a \\ D'(p) = 1 \end{cases}$

$$r_i = \frac{1}{1} \times \frac{1}{1 - e^{-Ta} z^{-1}}$$

$$F(z) = \frac{1}{1 - e^{-aT} z^{-1}}$$

- $\begin{cases} f(t) = t\, u(t) \\ F(p) = 1/p^2 \end{cases}$

$M1 \rightarrow f(nT) = nT\, u(nT)$

$F(z) = \sum_{n=0}^{+\infty} nT z^{-n}$ ← compliqué.

$M2 \rightarrow N(p) = 1.$ $\begin{cases} D(p) = p^2 \leftarrow p_0 = 0 \text{ d'ordre } 2. \\ D'(p) = 2p \end{cases}$

$$r_1 = \frac{d}{dp}\left[\cancel{p^2} \times \frac{1/\cancel{p^2}}{1 - e^{Tp_i} z^{-1}} \right]$$

$$= \left(\frac{T e^{Tp_i} z^{-1}}{(1 - e^{Tp_i} z^{-1})^2} \right)_{p_i = 0} = \frac{T z^{-1}}{(1 - z^{-1})^2}$$

$$F(z) = \frac{Tz}{(z-1)^2}$$

- $F(p) = \dfrac{1}{(p+a)(p+b)}$

$M2 \rightarrow \quad N(p) = 1 \quad . \begin{cases} D(p) = (p+a)(p+b) \\ D'(p) = 2p + a + b \end{cases} \rightarrow \underbrace{p_1 = -a \text{ et } p_2 = -b}_{\text{ordre } 1.}$

$$\begin{cases} r_1 = \dfrac{1}{b-a} \times \dfrac{1}{1 - e^{-aT} z^{-1}} \\ r_2 = \dfrac{1}{a-b} \times \dfrac{1}{1 - e^{-bT} z^{-1}} \end{cases} \qquad F(z) = \frac{1}{b-a}\left[\frac{1}{1 - e^{-aT} z^{-1}} - \frac{1}{1 - e^{-bT} z^{-1}}\right].$$

- $f(t) = \sin(w_0 t)\, u(t)$

$f(nT) = \sin(n w_0 T)\, u(nT)$.

$$F(z) = \sum_{n=0}^{+\infty} \sin(n w_0 T) z^{-n} = \sum_{n=0}^{+\infty} \frac{e^{nj w_0 T} - e^{-nj w_0 T}}{2j} z^{-n}$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{+\infty} (e^{j w_0 T} z^{-1})^n - \sum_{n=0}^{+\infty} (e^{-j w_0 T} z^{-1})^n\right].$$

$$= \frac{1}{2j}\left[\frac{z}{z - e^{j w_0 T}} - \frac{z}{z - e^{-j w_0 T}}\right]$$

$$= \frac{z}{2j}\left(\frac{-e^{-j w_0 T} + e^{j w_0 T}}{(z^2 - (e^{j w_0 T} + e^{-j w_0 T}) z + 1)}\right)$$

$$F(z) = \frac{z \sin(w_0 T)}{z^2 - 2z\cos(w_0 T) + 1}$$

Transformée en z inverse :

$$F(z) = \frac{2z}{(z-1)(z-0,5)}$$

$$g(z) = \frac{F(z)}{z} = \frac{2}{(z-1)(z-0,5)} = \frac{4}{z-1} - \frac{4}{z-0,5}.$$

$$F(z) = \frac{4z}{z-1} - \frac{4z}{z-0,5}$$

$$f(nT) = 4\left(1 - (0,5)^n\right).$$