Approximation
Exercises with solutions

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1 Exercises

Exercise 1

Let $E = C^0([0,1])$ be the vector space of the continuous functions from $[0,1]$ to $\mathbb{R}$. We define on E the inner product $\langle , \rangle$ and the associated norm $\|.\|$ for any $(f,g) \in E^2$ by

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x \qquad \text{and} \qquad \|f\| = \sqrt{\langle f,f\rangle}$$

Let $F = \mathbb{R}_2[X]$ be the linear subspace of E made of the polynomial functions of degree at most 2. It is a linear subspace of dimension 3 : a basis of F is the family $(\varphi_0, \varphi_1, \varphi_2)$ with

$$\varphi_0(x) = 1, \qquad \varphi_1(x) = x, \qquad \varphi_2(x) = x^2$$

Let f be the function defined for on $[0,1]$ by $f(x) = e^x$.
Determine the best approximation of f by a polynomial of degree at most 2, in the sense of the norm $\|.\|$.

Exercise 2

Let $E = C^0([0,1])$ be the vector space of the continuous functions from $[0,1]$ to $\mathbb{R}$. We define the same inner product and norm as in exercise 1 : for any $(f,g) \in E^2$

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x \qquad \text{and} \qquad \|f\| = \sqrt{\langle f,f\rangle}$$

Let $(\varphi_0, \varphi_1, \varphi_2)$ be the family of functions defined for any $x \in [0,1]$ by

$$\varphi_0(x) = 1 \qquad \varphi_1(x) = \sqrt{3}(2x-1) \qquad \varphi_2(x) = \sqrt{5}(6x^2 - 6x + 1)$$

1. Prove that $(\varphi_0, \varphi_1, \varphi_2)$ is an orthonormal basis of $F = \mathbb{R}_2[X]$, the set of polynomial functions of degree at most 2.
2. Let f and g be the functions defined for any $x \in [0,1]$ by

$$f(x) = \cos(\pi x) \qquad g(x) = \sin(\pi x)$$

   (a) Calculate $\langle f,1\rangle$ and $\langle g,1\rangle$.
   (b) Using an integration by parts, deduce $\langle f,x\rangle$ and $\langle g,x\rangle$

(c) Using an integration by parts, deduce $\left\langle f, x^2 \right\rangle$ and $\left\langle g, x^2 \right\rangle$

(d) Deduce $\langle f, \varphi_i \rangle$ and $\langle g, \varphi_i \rangle$ for all $i \in \{0, 1, 2\}$.

3. Let P_f and P_g be the best approximations of f and g in F, in the sense of the norm $\|.\|$. Since P_f and P_g are in F, there exist $(a_0, a_1, a_2) \in \mathbb{R}^3$ and $(b_0, b_1, b_2) \in \mathbb{R}^3$ such that

$$P_f = a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 \qquad \text{and} \qquad P_f = b_0\varphi_0 + b_1\varphi_1 + b_2\varphi_2$$

(a) Write the system that (a_0, a_1, a_2) satisfies.

(b) Write the system that (b_0, b_1, b_2) satisfies.

(c) Deduce P_f and P_g.

2 Solutions

Exercise 1

The best approximation of f by a function of F is $P_0 = p_F(f)$, the orthogonal projection of f onto F. Indeed, for any $P \in F$, we have

$$\begin{aligned}
\|f-P\|^2 &= \left\|\big(f-p_F(f)\big)+\big(p_F(f)-P\big)\right\|^2 \\
&= \langle f-p_F(f), f-p_F(f)\rangle + 2\left\langle \underbrace{f-p_F(f)}_{\in F^\perp}, \underbrace{p_F(f)-P}_{\in F}\right\rangle + \langle p_F(f)-P, p_F(f)-P\rangle \\
&= \|f-p_F(f)\|^2 + \|p_F(f)-P\|^2 \\
&\geqslant \|f-p_F(f)\|^2
\end{aligned}$$

Now, we have

$$\begin{aligned}
P_0 = p_F(f) &\iff \begin{cases} P_0 \in F \\ f - P_0 \in F^\perp \end{cases} \\
&\iff \begin{cases} \exists (a_0,a_1,a_2) \in \mathbb{R}^3, P_0 = a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2 \\ \forall k \in \{0,1,2\}, \langle f - P_0, \varphi_k\rangle = 0 \end{cases}
\end{aligned}$$

We just need to calculate the coefficients $a_i, i \in \{0,1,2\}$. They are solutions of the system :

$$\left\{\begin{array}{rcl}
\langle f-a_0\varphi_0-a_1\varphi_1-a_2\varphi_2, \varphi_0\rangle &=& 0 \\
\langle f-a_0\varphi_0-a_1\varphi_1-a_2\varphi_2, \varphi_1\rangle &=& 0 \\
\langle f-a_0\varphi_0-a_1\varphi_1-a_2\varphi_2, \varphi_2\rangle &=& 0
\end{array}\right. \iff \left\{\begin{array}{rcl}
a_0\langle\varphi_0,\varphi_0\rangle + a_1\langle\varphi_1,\varphi_0\rangle + a_2\langle\varphi_2,\varphi_0\rangle &=& \langle f,\varphi_0\rangle \\
a_0\langle\varphi_0,\varphi_1\rangle + a_1\langle\varphi_1,\varphi_1\rangle + a_2\langle\varphi_2,\varphi_1\rangle &=& \langle f,\varphi_1\rangle \\
a_0\langle\varphi_0,\varphi_2\rangle + a_1\langle\varphi_1,\varphi_2\rangle + a_2\langle\varphi_2,\varphi_2\rangle &=& \langle f,\varphi_2\rangle
\end{array}\right.$$

But for any (i,j), we have

$$\langle\varphi_i,\varphi_j\rangle = \int_0^1 x^{i+j}\,\mathrm{d}x = \left[\frac{x^{i+j+1}}{i+j+1}\right]_0^1 = \frac{1}{i+j+1}$$

Furthermore,

$$\begin{aligned}
\langle f,\varphi_0\rangle &= \int_0^1 e^x\,\mathrm{d}x = [e^x]_0^1 = e-1 \\
\langle f,\varphi_1\rangle &= \int_0^1 xe^x\,\mathrm{d}x \\
&= [xe^x]_0^1 - \int_0^1 e^x\,\mathrm{d}x \\
&= e-(e-1) = 1 \\
\langle f,\varphi_2\rangle &= \int_0^1 x^2e^x\,\mathrm{d}x \\
&= \left[x^2e^x\right]_0^1 - \int_0^1 2xe^x\,\mathrm{d}x \\
&= e-2
\end{aligned}$$

So the system we must solve is

$$(S) \quad = \quad \begin{cases} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 &= e-1 \\ \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 &= 1 \\ \frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 &= e-2 \end{cases}$$

$$\iff \begin{cases} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 &= e-1 & \\ \frac{1}{6}a_1 + \frac{1}{6}a_2 &= -e+3 & 2\times \text{Eqn.}2 - \text{Eqn.}1 \\ \frac{1}{4}a_1 + \frac{4}{15}a_2 &= 2e-5 & 3\times \text{Eqn.}3 - \text{Eqn.}1 \end{cases}$$

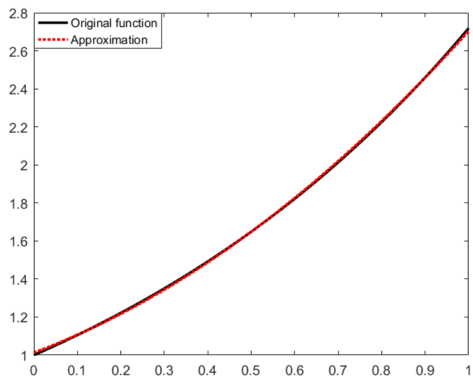
$$\iff \begin{cases} a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 &= e-1 & \\ \frac{1}{6}a_1 + \frac{1}{6}a_2 &= -e+3 & \\ \frac{1}{30}a_2 &= 7e-19 & 2\times \text{Eqn.}3 - 3\times\text{Eqn.}2 \end{cases}$$

$$\iff \begin{cases} a_2 &= 210e-570 \\ a_1 &= -216e+588 \\ a_0 &= 39e-105 \end{cases}$$

Thus, the best approximation of f with a function of $F = \mathbb{R}_2[X]$ is

$$P_0(x) = (210e-570)x^2 + (-216e+588)x + (39e-105)$$

The following figure shows the graphs of the function and of its approximation. We can hardly do the distinction between them.



Exercise 2

1. We have

$$\begin{aligned}
\langle \varphi_0, \varphi_1 \rangle &= \int_0^1 1 \times \sqrt{3}(2x-1)\,\mathrm{d}x \\
&= \left[\sqrt{3}(x^2-x)\right]_0^1 \\
&= 0 \\
\langle \varphi_0, \varphi_2 \rangle &= \int_0^1 1 \times \sqrt{5}(6x^2-6x+1)\,\mathrm{d}x \\
&= \left[\sqrt{5}(2x^3-3x^2+x)\right]_0^1 \\
&= 0 \\
\langle \varphi_1, \varphi_2 \rangle &= \int_0^1 \sqrt{3}(2x-1) \times \sqrt{5}(6x^2-6x+1)\,\mathrm{d}x \\
&= \int_0^1 (\sqrt{15}(12x^3-18x^2+8x-1)\,\mathrm{d}x \\
&= \left[\sqrt{15}(3x^4-6x^3+4x^2-x)\right]_0^1 \\
&= 0
\end{aligned}$$

and the family is orthogonal.

Furthermore,

$$\begin{aligned}
\|\varphi_0\|^2 = \langle \varphi_0, \varphi_0 \rangle &= \int_0^1 1 \times 1\,\mathrm{d}x \\
&= 1 \\
\|\varphi_1\|^2 = \langle \varphi_1, \varphi_1 \rangle &= \int_0^1 3(2x-1)^2\,\mathrm{d}x \\
&= \left[3 \times \tfrac{(2x-1)^3}{6}\right]_0^1 \\
&= 1 \\
\|\varphi_2\|^2 = \langle \varphi_2, \varphi_2 \rangle &= \int_0^1 5(6x^2-6x+1)^2\,\mathrm{d}x \\
&= 5\int_0^1 (36x^4-72x^3+48x^2-12x+1)\,\mathrm{d}x \\
&= 5\left[\tfrac{36}{5}x^5-18x^4+16x^3-6x^2+x\right]_0^1 \\
&= 5\left(\tfrac{36}{5}-18+16-6+1\right) \\
&= 5\left(\tfrac{36}{5}-7\right) \\
&= 1
\end{aligned}$$

and the family is orthonormal.
Since the family does not contain the zero function 0_E and is orthogonal, it is linearly independant. Since it is made of 3 independant vectors of F and since $\dim(F) = 3$, it is a basis of F. Finally, it is an orthonormal basis of F.

2. (a) We have

$$\begin{aligned}
\langle f, 1\rangle &= \int_0^1 \cos(\pi x)\,\mathrm{d}x \\
&= \left[\frac{\sin(\pi x)}{\pi}\right]_0^1 \\
&= 0 \\
\langle g, 1\rangle &= \int_0^1 \sin(\pi x)\,\mathrm{d}x \\
&= \left[\frac{-\cos(\pi x)}{\pi}\right]_0^1 \\
&= \frac{2}{\pi}
\end{aligned}$$

(b) Using integrations by parts, we find

$$\begin{aligned}
\langle f, x\rangle &= \int_0^1 x\cos(\pi x)\,\mathrm{d}x \\
&= \underbrace{\left[\frac{x\sin(\pi x)}{\pi}\right]_0^1}_{0} - \frac{1}{\pi}\underbrace{\int_0^1 \sin(\pi x)\,\mathrm{d}x}_{\frac{2}{\pi}} \\
&= -\frac{2}{\pi^2}
\end{aligned}$$

and

$$\begin{aligned}
\langle g, x\rangle &= \int_0^1 x\sin(\pi x)\,\mathrm{d}x \\
&= \underbrace{\left[\frac{-x\cos(\pi x)}{\pi}\right]_0^1}_{\frac{1}{\pi}} + \frac{1}{\pi}\underbrace{\int_0^1 \cos(\pi x)\,\mathrm{d}x}_{0} \\
&= \frac{1}{\pi}
\end{aligned}$$

(c) Using integrations by parts, we find

$$\begin{aligned}
\left\langle f, x^2\right\rangle &= \int_0^1 x^2\cos(\pi x)\,\mathrm{d}x \\
&= \underbrace{\left[\frac{x^2\sin(\pi x)}{\pi}\right]_0^1}_{0} - \frac{2}{\pi}\underbrace{\int_0^1 x\sin(\pi x)\,\mathrm{d}x}_{\frac{1}{\pi}} \\
&= -\frac{2}{\pi^2}
\end{aligned}$$

and

$$\begin{aligned}
\left\langle g, x^2\right\rangle &= \int_0^1 x^2\sin(\pi x)\,\mathrm{d}x \\
&= \underbrace{\left[\frac{-x^2\cos(\pi x)}{\pi}\right]_0^1}_{\frac{1}{\pi}} + \frac{2}{\pi}\underbrace{\int_0^1 x\cos(\pi x)\,\mathrm{d}x}_{-\frac{2}{\pi^2}} \\
&= \frac{1}{\pi} - \frac{4}{\pi^3}
\end{aligned}$$

(d) We have

$$\begin{aligned}
\langle f,\varphi_0\rangle &= \langle f,1\rangle \\
&= 0 \\
\langle f,\varphi_1\rangle &= \left\langle f,\sqrt{3}(2x-1)\right\rangle \\
&= \sqrt{3}\big(2\langle f,x\rangle - \langle f,1\rangle\big) \\
&= -\frac{4\sqrt{3}}{\pi^2} \\
\langle f,\varphi_2\rangle &= \left\langle f,\sqrt{5}(6x^2-6x+1)\right\rangle \\
&= \sqrt{5}\big(6\left\langle f,x^2\right\rangle - 6\langle f,x\rangle + \langle f,1\rangle\big) \\
&= \sqrt{5}\left[6\left(-\frac{2}{\pi^2}\right) - 6\left(-\frac{2}{\pi^2}\right) + 0\right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle g,\varphi_0\rangle &= \langle g,1\rangle \\
&= \frac{2}{\pi} \\
\langle g,\varphi_1\rangle &= \left\langle g,\sqrt{3}(2x-1)\right\rangle \\
&= \sqrt{3}\big(2\langle g,x\rangle - \langle g,1\rangle\big) \\
&= 0 \\
\langle g,\varphi_2\rangle &= \left\langle g,\sqrt{5}(6x^2-6x+1)\right\rangle \\
&= \sqrt{5}\big(6\left\langle g,x^2\right\rangle - 6\langle g,x\rangle + \langle g,1\rangle\big) \\
&= \sqrt{5}\left[6\left(\frac{1}{\pi} - \frac{4}{\pi^3}\right) - \frac{6}{\pi} + \frac{2}{\pi}\right] \\
&= \frac{2\sqrt{5}}{\pi}\left(1 - \frac{12}{\pi^2}\right)
\end{aligned}$$

3. (a) $P_f = a_0\varphi_0 + a_1\varphi_1 + a_2\varphi_2$ is the orthogonal projection of f onto $F = \mathrm{Span}(\{\varphi_0, \varphi_1 n \varphi_2\})$. Thus,

$$\begin{aligned}
P_f = p_F(f) &\iff f - P_f \in F^\perp \\
&\iff \left\{\begin{array}{rcl}
\langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_0\rangle &=& 0 \\
\langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_1\rangle &=& 0 \\
\langle f - a_0\varphi_0 - a_1\varphi_1 - a_2\varphi_2, \varphi_2\rangle &=& 0
\end{array}\right. \\
&\iff \left\{\begin{array}{rcl}
a_0\langle\varphi_0,\varphi_0\rangle + a_1\langle\varphi_1,\varphi_0\rangle + a_2\langle\varphi_2,\varphi_0\rangle &=& \langle f,\varphi_0\rangle \\
a_0\langle\varphi_0,\varphi_1\rangle + a_1\langle\varphi_1,\varphi_1\rangle + a_2\langle\varphi_2,\varphi_1\rangle &=& \langle f,\varphi_1\rangle \\
a_0\langle\varphi_0,\varphi_2\rangle + a_1\langle\varphi_1,\varphi_2\rangle + a_2\langle\varphi_2,\varphi_2\rangle &=& \langle f,\varphi_2\rangle
\end{array}\right.
\end{aligned}$$

But the family $(\varphi_0, \varphi_1, \varphi_2)$ is orthonormal, which means that

$$\langle\varphi_i,\varphi_j\rangle = \left|\begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{else}\end{array}\right.$$

So the latter system is

$$\left\{\begin{array}{rcl}
a_0 &=& \langle f,\varphi_0\rangle \\
a_1 &=& \langle f,\varphi_1\rangle \\
a_2 &=& \langle f,\varphi_2\rangle
\end{array}\right.$$

(b) In the same way, for $P_g = b_0\varphi_0 + b_1\varphi_1 + b_2\varphi_2$, we have

$$\left\{\begin{array}{rcl}
b_0 &=& \langle g,\varphi_0\rangle \\
b_1 &=& \langle g,\varphi_1\rangle \\
b_2 &=& \langle g,\varphi_2\rangle
\end{array}\right.$$

(c) According to the results of previous questions, the polynomials P_f and P_g are

$$\begin{aligned}
P_f(x) &= -\frac{4\sqrt{3}}{\pi^2}\varphi_1(x) \\
&= -\frac{12}{\pi^2}(2x-1) \\
P_g(x) &= \frac{2}{\pi}\varphi_0(x) + \frac{2\sqrt{5}}{\pi}\left(1-\frac{12}{\pi^2}\right)\varphi_2(x) \\
&= \frac{60}{\pi}\left(1-\frac{12}{\pi^2}\right)x(x-1) + \frac{12}{\pi}\left(1-\frac{10}{\pi^2}\right)
\end{aligned}$$

The following figure shows the graphs of the two functions and of their approximations.