Approximation and interpolation

Course

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This course focuses on the possible ways to «replace» a function $f : I \to \mathbb{R}$ by a polynomial. The term «replace» may have different meanings, depending on the context and the reasons why we're interested in such a «replacement».

The first part deals with function approximation. In this case, we know the exact function f and we wish to find a polynomial $P_n \in \mathbb{R}_n[X]$ such that a norm $\|f - P_n\|$ is minimal or, at least, not to high. This typically happens when the function f is complex, long to compute : then it may be useful to find a quicker way to estimate its values. The answer to this question depends on the norm that we choose.

The second part deals with function interpolation. In this case, we only know from f its values at $n+1$ «nodes» $x_0, x_1, \cdots, x_n$ and we want to find a polynomial $P_n \in \mathbb{R}_n[X]$ which takes the same values than f at these nodes. Furthermore, we wonder weither this interpolating polynomial has any chance to be a good approximation of f on the whole definition interval.

The third and last part is an application of function interpolation to numerical integration. We suppose that we need to compute an integral

$$I = \int_A^B f(x)\,\mathrm{d}x$$

and that we don't know any primitive of the function f. Then this integral may be approximated with numerical methods. These methods consist in spliting $[A, B]$ into small subintervals $[a_i, b_i]$ and in replacing f with an interpolating polynomial in each subinterval.

1 Approximation

We work with the following norms :

$$\left|\begin{array}{lcll} \|f\|_\infty & = & \sup\{|f(x)|, x \in I\} & \text{(uniform norm)} \\ \|f\|_2 & = & \sqrt{\displaystyle\int_I w(x) f^2(x)\,\mathrm{d}x} & \text{(quadratic norm)} \end{array}\right.$$

When I is a finite interval $I = [a, b]$, both norms are defined for any continuous function. If I is infinite, the function f should satisfy some conditions for $\|f\|$ to be defined. In any case, the set of adequate functions f is a vector space over $\mathbb{R}$, so we can add the functions, multiply them by a constant...

Finding a polynomial $P_n \in \mathbb{R}_n[x]$ which minimizes $\|f - P_n\|_\infty$ seems intuitively the best way to approximate f. Anyway, it is also the most difficult one and there are only poor results about this kind of approximation.

There are many more results about the $\|.\|_2$ approximation. This norm is derived from an inner product

$$\langle f, g\rangle = \int_I w(x)f(x)g(x)\,\mathrm{d}x$$

and all the concepts of linear algebra in inner product spaces, such as the orthogonal projection, can be used. We can also notice that, if we pick randomly x in I with a probability whose pdf is w, then $\|f - P_n\|_2^2$ is the expectation of the square error of the approximation.

1.1 Approximation with the uniform norm

Let $I = [a, b]$ and $f : I \to \mathbb{R}$. We denote by P_n^* the polynomial of $\mathbb{R}_n[X]$ such that

$$\forall P \in \mathbb{R}_n[X], \|f - P_n^*\|_\infty \leqslant \|f - P\|_\infty$$

The few results we have about the characterization of P_n^* are displayed in the two next sections :

1.1.1 Case of an arbitrary function

In this case, we only have a sufficient condition for P_n to be the best approximation of f in $\mathbb{R}_n[X]$ with the uniform norm on $[a, b]$.

Property : let's denote

$$M = \|f - P_n\|_\infty = \sup\{|f(x) - P_n(x)|, x \in [a, b]\}$$

If $f - P_n$ takes alternatively the values $+M, -M, +M, -M, \cdots$ in at least $n + 2$ points $x_0 < x_1 < \cdots < x_n < x_{n+1}$ in $[a, b]$, then $P_n = P_n^*$.

1.1.2 Case of the function $f(x) = x^{n+1}$

In this case, we can explicitly give P_n^*.

Chebyshev polynomials

- Definition : the Chebyshev sequence of polynomials, T_n, is defined by $T_0(X) = 1, T_1(X) = X$ and for all $n \in \mathbb{N}$,
$$T_{n+2}(X) = 2XT_{n+1}(X) - T_n(X)$$
For example, we have $T_0 = 1, \qquad T_1 = X, \qquad T_2 = 2X^2 - 1 \qquad$ and $\qquad T_3 = 4X^3 - 3X$
- Properties : this sequence T_n satisfies to the following properties :
  1. for any $n \in \mathbb{N}$, $\deg(T_n) = n$. Furthermore, when $n > 0$, the coefficient of the monomial of degree n is 2^{n-1},
  2. if $\theta = \arccos(x)$ that is, $x = \cos(\theta), \theta \in [0, \pi]$, then
  $$T_n(x) = T_n(\cos(\theta)) = \cos(n\theta)$$

3. for $x_k = \cos\left(\frac{k\pi}{n}\right)$, we have $T_n(x_k) = (-1)^k$

Approximation of $f(x) = x^{n+1}$ on $[-1,1]$

We have the following property : if we denote

$$T_{n+1} = 2^n X^{n+1} + a_n X^n + \cdots + a_1 X + a_0$$

then the best approximation of f by a polynomial of degree at most n over $[-1,1]$ is

$$P_n^*(X) = -\frac{1}{2^n}\big(a_n X^n + \cdots + a_1 X + a_0\big)$$

Furthermore, the error is

$$\|f - P_n^*\|_\infty = \frac{1}{2^n}$$

and this value is reached exactly $n+1$ times, at the points $x_k = \cos\left(\frac{k\pi}{n+1}\right)$, $0 \leqslant k \leqslant n$.

Remark : we then have $\lim\limits_{n\to+\infty} \|f - P_n^*\|_\infty = 0$, which is the definition of

1. $\lim\limits_{n\to+\infty} P_n^* = f$ in $\big(\mathscr{C}([-1,1],\mathbb{R}), \|.\|_\infty\big)$,
2. (P_n^*) converges uniformly to f on $[-1,1]$.

1.2 Approximation with the quadratic norm

1.2.1 Theoretical background

Let I be an interval of $\mathbb{R}$, w be a strictly positive function defined on I and let E be the set of continuous functions $f : I \to \mathbb{R}$ such that the integral

$$\int_I w(x) f^2(x)\,\mathrm{d}x$$

is defined.
If I is finite and if w is continuous over I, E is just the set of continuous functions over I. Anyway, it may happen that I is infinite or that w is not defined at the bounds of I, so the above integral is improper and E is restricted to the continuous functions f such that this integral is convergent.
We admit that in any case, E is a vector space over $\mathbb{R}$: the addition or substraction of functions of E is in E, as well as their product by a constant. We also admit that, for any f and g in E, then the integral

$$\int_I w(x) f(x) g(x)\,\mathrm{d}x$$

is defined.
Furthermore, we suppose that the set of polynomials $\mathbb{R}[X] \subset E$.

Then, we can define on E the inner product

$$\langle f, g\rangle = \int_I w(x) f(x) g(x)\,\mathrm{d}x$$

from which we derive the norm

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_I w(x) f^2(x)\,\mathrm{d}x}$$

The error of approximation of a function $f \in E$ by a polynomial P_n is then given by

$$\|f - P_n\|_2 = \sqrt{\int_I w(x)\big(f(x) - P_n(x)\big)^2\,\mathrm{d}x}$$

and we can refer to the theoretical results in inner product spaces.

In particular, we will refer to the orthgonal projection theorem.
Theorem : if $(E, \langle,\rangle)$ is a inner product space and if F is a *finite dimensional* subspace of E, then

$$F \oplus F^{\perp} = E$$

Consequence : we can define p_F, the orthogonal projection on F. For any $z \in E$, there exists a unique $(x, y) \in F \times F^{\perp}$ such that $z = x + y$. Then $p_F(z) = x$.

1.2.2 Application to function approximation

We use the notations introduced at the previous section and we remind that, for any $n \in \mathbb{N}$, $\mathbb{R}_n[X]$ is a vector subspace of E of dimension $n+1$. Its standard basis is $(1, X, X^2, \cdots, X^n)$.

Theorem : for any function $f \in E$, the best polynomial approximation of f of degree at most n is its orthogonal projection on $F = \mathbb{R}_n[X]$:

$$P_n^* = p_F(f)$$

Practical computation of P_n^* : let $(\phi_0, \phi_1, \cdots, \phi_n)$ be a basis of $F = \mathbb{R}_n[X]$ (typically, the standard basis but the process remains valid with other bases). Then we have

$$P_n^*(x) = \lambda_0\phi_0(x) + \cdots + \lambda_n\phi_n(x)$$

and we only needs to compute the coefficients λ_i. To do so, we write

$$\begin{aligned}
P_n^* = p_F(f) \quad &\iff \quad f - P_n^* \in F^{\perp} \\
&\iff \left\{\begin{array}{rcl} \langle f - P_n^*, \phi_0\rangle & = & 0 \\ \vdots & \vdots & \vdots \\ \langle f - P_n^*, \phi_n\rangle & = & 0 \end{array}\right. \\
&\iff \left\{\begin{array}{rcl} \langle f - \lambda_0\phi_0 - \cdots - \lambda_n\phi_n, \phi_0\rangle & = & 0 \\ \vdots & \vdots & \vdots \\ \langle f - \lambda_0\phi_0 - \cdots - \lambda_n\phi_n, \phi_n\rangle & = & 0 \end{array}\right. \\
&\iff \left\{\begin{array}{rcl} \lambda_0\langle\phi_0, \phi_0\rangle + \cdots + \lambda_n\langle\phi_n, \phi_0\rangle & = & \langle f, \phi_0\rangle \\ \vdots & \vdots & \vdots \\ \lambda_0\langle\phi_0, \phi_n\rangle + \cdots + \lambda_n\langle\phi_n, \phi_n\rangle & = & \langle f, \phi_n\rangle \end{array}\right.
\end{aligned}$$

The latter system has $n+1$ variables and $n+1$ equations. Furthermore, we can prove (admited here) that, since the family $(\phi_0, \cdots, \phi_n)$ is linearly independant, it has a unique solution, whatever its right hand side.
The function P_n^* is obtained in solving this system.

Example : let $I = [-1, 1]$ and $w(x) = 1$. The inner product and the quadratic norm are hence

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x \qquad \text{and} \qquad \|f - P_n\|_2 = \sqrt{\int_{-1}^{1} \big(f(x) - P_n(x)\big)^2\,\mathrm{d}x}$$

Let f be the function defined for any $x \in [-1, 1]$ by $f(x) = |x|$. What is P_2^*, the best polynomial approximation of f of degree at most 2 ?
Solution : we find $P_2^*(x) = \frac{15}{16}x^2 + \frac{3}{16}$.

1.2.3 Particular case of an orthonormal basis

Suppose that we have an orthonormal basis of $\mathbb{R}_n[X]$: the family $(\phi_0, \phi_1, \cdots, \phi_n)$ then satisfies

$$\langle \phi_i, \phi_j \rangle = \delta_{ij} = \left| \begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{array} \right.$$

In this case, the system that we must solve to obtain P_n^* is trivial :

$$\left\{ \begin{array}{ccc} \lambda_0 & = & \langle f, \phi_0 \rangle \\ \vdots & \vdots & \vdots \\ \lambda_n & = & \langle f, \phi_n \rangle \end{array} \right.$$

which leads to

$$P_n^* = \langle f, \phi_0 \rangle\, \phi_0 + \cdots + \langle f, \phi_n \rangle\, \phi_n$$

There is a process (Gram-Schmidt process) which enables to obtain, from the initial standard basis $(1, X, \cdots, X^n)$, an orthogonal one. Its result depends on the inner product $\langle , \rangle$ that is, on the interval I and the weight function w. Here are some famous examples :

1. The Legendre polynomials are orthogonal for the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$$

They are given by :

$$P_n(X) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (X-1)^{n-k}(X+1)^k$$

$$P_0 = 1, \qquad P_1 = X, \qquad P_2 = \frac{1}{2}(3X^2 - 1), \qquad P_3 = \frac{1}{2}(5X^3 - 3X)$$

2. The Laguerre polynomials are orthogonal for the inner product

$$\langle f, g \rangle = \int_{0}^{+\infty} f(x)g(x)e^{-x}\,\mathrm{d}x$$

They are defined by the recursive formula

$$L_0 = 1, L_1 = -X + 1, \forall k \in \mathbb{N}^*, L_{k+1} = \frac{(2k + 1 - X)L_k - kL_{k-1}}{k + 1}$$

and by a closed form

$$L_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} X^k$$

$$L_0 = 1, \qquad L_1 = -X + 1, \qquad L_2 = \frac{1}{2}(X^2 - 4X + 2)$$

3. The Hermitte polynomials are orthogonal for the inner product

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}\,\mathrm{d}x$$

They are defined by the recursive formula

$$H_0 = 1, \forall n \in \mathbb{N}, H_{n+1} = 2XH_n - H'_n$$

and we have

$$H_0 = 1, \qquad H_1 = 2X, \qquad H_2 = 4X^2 - 2, \qquad H_3 = 8X^3 - 12X$$

4. The Chebyshev polynomials are orthogonal for the inner product

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1 - x^2}}\,\mathrm{d}x$$

Recursive definition : $T_0 = 1, T_1 = X, \forall n \in \mathbb{N}, T_{n+2} = 2XT_{n+1} - T_n$

Closed definition : $T_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (X^2 - 1)^k X^{n-2k}$

First polynomials : $T_0 = 1, \qquad T_1 = X, \qquad T_2 = 2X^2 - 1, \qquad T_3 = 4X^3 - 3X$

Remark : the above families are orthogonal but not orthonormal. To derive orthonormal families, we just need to divide each polynomial by its norm.

2 Interpolation

Let f be a function, $\{x_0, x_1, \cdots, x_n\}$ be a set of $n + 1$ mutually different nodes, and suppose we only know from f its values $y_0, y_1, \cdots, y_n$ at these nodes.
The interpolation of f consists in finding a function g_n such that for any $i \in \{0, 1, \cdots, n\}, g_n(x_i) = y_i = f(x_i)$. We may see g_n as an «approximation» of f if we don't know the analytical form of f.

2.1 Existence, unicity and error of the interpolation polynomial of degree n

A way to interpolate f consists in searching g_n as a polynomial P_n of degree n. In theory, any polynomial $P \in \mathbb{R}_n[X]$ may be written

$$P = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$$

and we only need to find the coefficients $(a_i)_{0\leqslant i\leqslant n}$. To do so, we can solve the system

$$S : \left\{\begin{array}{rcl} a_0 + a_1x_0 + \cdots + a_nx_0^n & = & y_0 \\ a_0 + a_1x_1 + \cdots + a_nx_1^n & = & y_1 \\ \vdots & \vdots & \vdots \\ a_0 + a_1x_n + \cdots + a_nx_n^n & = & y_n \end{array}\right.$$

Property : if the $n+1$ nodes are mutually distincts, the above system S has a unique solution. It is called a «Vandermonde system». The square matrix of this system,

$$A = \begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & & \vdots & \\ 1 & x_n, & \cdots & x_n^n \end{pmatrix}$$

is regular and hence bijective.

Remark : if we had taken for P_n a polynomial of degree different from n, then the matrix A would not be square and hence would not be bijective.

Finally, as long as the $n+1$ nodes are all mutually different, then whatever the values of the y_i, there exists a unique polynomial P_n of degree at most n which interpolates f at these nodes.

The following theorem enables to evaluate the approximation error $\varepsilon_n = f - P_n$.

Theorem : if f is of class C^{n+1} over an interval $[a,b]$ and if all the nodes $x_0, x_1, \cdots, x_n$ are in $[a,b]$, then

$$\forall x \in [a,b], \exists c \in [a,b] \text{ such that } \varepsilon_n(x) = \prod_{i=0}^{n}(x - x_i) \times \frac{f^{(n+1)}(c)}{(n+1)!}$$

In particular, we have for any $x \in [a,b]$

$$|\varepsilon_n(x)| \leqslant \left|\prod_{i=0}^{n}(x - x_i)\right| \times \frac{\sup\limits_{t\in[a,b]} \left|f^{(n+1)}(t)\right|}{(n+1)!}$$

Remarks :

1. If the nodes are choosen arbitrarily, this error expression does not imply the uniform convergence of (P_n) to f when n tends to $+\infty$. Even the pointwise convergence is not sure.
2. If the nodes are given by
$$x_i = a + i\frac{b-a}{n}$$
then we have for any $x \in [a,b]$
$$|\varepsilon_n(x)| \leqslant \left(\frac{b-a}{n}\right)^n \sup_{t\in[a,b]} \left|f^{(n+1)}(t)\right|$$
and the convergence may be uniform. For example, it is if $\left|f^{(n+1)}\right|$ has an upper bound independant on n.

To calculate the interpolation polynomial of degree n, there are more convenient ways that just solving the system S. We now introduce the methods from Lagrange and from Newton.

2.2 Lagrange interpolation polynomial

2.2.1 Reminder about polynomials

Let $P \in \mathbb{R}[X]$ and let $x_i \in \mathbb{R}$. Then

1. x_i is a root of P iff $(X - x_i)$ divides P. In other words,

$$P(x_i) = 0 \Longleftrightarrow \exists K \in \mathbb{R}[X], P(X) = (X - x_i)K(X)$$

We say that x_i is a root of P with a multiplicity at least equal to 1,

2. x_i is a root of $P, P', \cdots, P^{m-1}$ iff $(X - x_i)^m$ divides P. In other words,

$$\left.\begin{array}{ccc} P(x_i) & = & 0 \\ P'(x_i) & = & 0 \\ \vdots & & \vdots \\ P^{(m-1)}(x_i) & = & 0 \end{array}\right\} \Longleftrightarrow \exists K \in \mathbb{R}[X], P(X) = (X - x_i)^m K(X)$$

We say that x_i is a root of P with a multiplicity at least equal to m,

3. x_i is a root of $P, P', \cdots, P^{m-1}$ and not of $P^{(m)}$ iff $(X - x_i)^m$ divides P but $(X - x_i)^{m+1}$ does not.

$$\left.\begin{array}{ccc} P(x_i) & = & 0 \\ P'(x_i) & = & 0 \\ \vdots & & \vdots \\ P^{(m-1)}(x_i) & = & 0 \\ p^{(m)}(x_i) & \neq & 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} (X - x_i)^m \mid P \\ (X - x_i)^{m+1} \nmid P \end{array}\right.$$

Then we say that x_i is a root of multiplicity exactly equal to m.

Exercise : let $\{x_0, x_1, x_2\}$ be a set of three mutually different nodes.

1. Find three polynomials L_0, L_1 and L_2 of degree 2 such that

$$\left\{\begin{array}{ccc} L_0(x_0) & = & 1 \\ L_0(x_1) & = & 0 \\ L_0(x_2) & = & 0 \end{array}\right. \qquad \left\{\begin{array}{ccc} L_1(x_0) & = & 0 \\ L_1(x_1) & = & 1 \\ L_1(x_2) & = & 0 \end{array}\right. \qquad \left\{\begin{array}{ccc} L_2(x_0) & = & 0 \\ L_2(x_1) & = & 0 \\ L_2(x_2) & = & 1 \end{array}\right.$$

2. Let $(y_0, y_1, y_2) \in \mathbb{R}^3$. Use the three above polynomials to determine the interpolation polynomial $P_2 \in \mathbb{R}_2[X]$, that is to say the unique polynomial in $\mathbb{R}_2[X]$ such that

$$\left\{\begin{array}{ccc} P_2(x_0) & = & y_0 \\ P_2(x_1) & = & y_1 \\ P_2(x_2) & = & y_2 \end{array}\right.$$

2.2.2 General method

Let $x_0, \cdots, x_n$ be $n+1$ mutually different nodes. We define for all $i \in \{0, 1 \cdots, n\}$ the Lagrange polynomials by

$$L_i(X) = \frac{\displaystyle\prod_{j \neq i}(X - x_j)}{\displaystyle\prod_{j \neq i}(x_i - x_j)}$$

These polynomials all have degree n and satisfy, for each $(i,j) \in \{0,1,\cdots,n\}^2$,

$$L_i(x_j) = \left| \begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{array} \right.$$

Therefore,

$$P_n(X) = \sum_{i=0}^{n} y_i L_i(X)$$

is the unique polynomial of degree at most n which interpolates f at the nodes.

Exercise 1 : let the function f satisfy

$$f(0) = 1 \qquad f(1) = 3 \qquad f(3) = 2 \qquad f(4) = 5$$

Find P_3, the interpolation polynomial of degree 3.
Answer : we find $P_3 = \frac{1}{2}X^3 - \frac{17}{6}X^2 + \frac{13}{3}X + 1$.

Exercise 2 : from the exam 2016. Let f be a continuous function whose values are given at the following nodes :

$$f(0) = 1 \qquad f(1) = 2 \qquad f(-1) = -4 \qquad f(3) = 16$$

1. Using the Lagrangian interpolation polynomial, compute an approximation of $f(2)$.
2. Assuming that f has continuous derivatives up to the order 4 over $[-1,3]$, provide an expression for the error of this approximation.

Exercise 3 : going further, interpolation of both f and f'. Let $x_0 = 1$ and $x_1 = 3$ be two nodes.

1. Find 4 polynomials A_0, A_1, B_0, B_1 in $\mathbb{R}_3[X]$ such that

$$\left\{ \begin{array}{lcl} A_0(x_0) & = & 1 \\ A_0'(x_0) & = & 0 \\ A_0(x_1) & = & 0 \\ A_0'(x_1) & = & 0 \end{array} \right. \qquad \left\{ \begin{array}{lcl} A_1(x_0) & = & 0 \\ A_1'(x_0) & = & 0 \\ A_1(x_1) & = & 1 \\ A_1'(x_1) & = & 0 \end{array} \right.$$

$$\left\{ \begin{array}{lcl} B_0(x_0) & = & 0 \\ B_0'(x_0) & = & 1 \\ B_0(x_1) & = & 0 \\ B_0'(x_1) & = & 0 \end{array} \right. \qquad \left\{ \begin{array}{lcl} B_1(x_0) & = & 0 \\ B_1'(x_0) & = & 0 \\ B_1(x_1) & = & 0 \\ B_1'(x_1) & = & 1 \end{array} \right.$$

2. Suppose that f is a differentiable function such that

$$f(x_0) = y_0 \qquad f'(x_0) = z_0 \qquad f(x_1) = y_1 \qquad f'(x_1) = z_1$$

Use the four above polynomials to determine $P \in \mathbb{R}_3[X]$ which interpolates both f and f' at the two nodes x_0 and x_1.

2.3 Newton interpolation polynomial

Definition : divided difference. For $k \in \{0, \cdots, n\}$, we define the divided differences of order k by :

$$\begin{array}{lll}
\text{Order } 0: \quad f[x_i] & = & f(x_i) \\
\text{Order } 1: \quad f[x_i, x_j] & = & \dfrac{f[x_j] - f[x_i]}{x_j - x_i} \\
\text{Order } 2: \quad f[x_i, x_j, x_k] & = & \dfrac{f[x_j, x_k] - f[x_i, x_j]}{x_k - x_i} \\
\text{Order } 3: \quad f[x_i, x_j, x_k, x_l] & = & \dfrac{f[x_j, x_k, x_l] - f[x_i, x_j, x_k]}{x_l - x_i} \\
\quad \vdots & \vdots & \\
\text{Order } n: \quad f[x_0, x_1, \cdots, x_n] & = & \dfrac{f[x_1, \cdots, x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0}
\end{array}$$

Theorem : for any set of mutually different nodes $\{x_0, \cdots, x_n\}$, the inpertolation polynomial is given by

$$\begin{aligned}
P_n(X) &= \sum_{k=0}^{n} \left(f[x_0, \cdots, x_k] \prod_{i=0}^{k-1} (X - x_i) \right) \\
&= f[x_0] + f[x_0, x_1](X - x_0) + f[x_0, x_1, x_2](X - x_0)(X - x_1) + \cdots \\
&\qquad \cdots + f[x_0, \cdots, x_n] \prod_{i=0}^{n-1} (X - x_i)
\end{aligned}$$

Example : let the function f satisfy

$$f(0) = 1 \qquad f(1) = 3 \qquad f(3) = 2 \qquad f(4) = 5$$

Find P_3, the interpolation polynomial of degree 3.

First, we build the divided difference table for these data :

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	1			
		2		
1	3		$-\frac{5}{6}$	
		$-\frac{1}{2}$		$\frac{1}{2}$
3	2		$\frac{7}{6}$	
		3		
4	5			

Then we use the terms from the upper diagonal of this table to get P_3 :

$$\begin{aligned}
P_3(X) &= 1 + 2(X - 0) - \tfrac{5}{6}(X - 0)(X - 1) + \tfrac{1}{2}(X - 0)(X - 1)(X - 3) \\
&= \tfrac{1}{2}X^3 - \tfrac{17}{6}X^2 + \tfrac{13}{3}X + 1
\end{aligned}$$

Exercise 1 : let f be a continuous function whose values are given at the following nodes :

$$f(0) = 1 \qquad f(1) = 2 \qquad f(-1) = -4 \qquad f(3) = 16$$

1. Construct the divided difference table for these data.
2. Use this result to derive the Newton interpolation polynomial for f.
3. Deduce an approximation of $f(2)$.
4. Assuming that f has continuous derivatives up to the order 4 over $[-1, 3]$, provide an expression for the error of this approximation.

Exercise 2 : let f be a continuous function whose values are given at the following nodes :

$$f(0) = 1 \qquad f(1) = 2 \qquad f(2) = 7 \qquad f(3) = 22$$

1. Construct the divided difference table for these data.
2. Use this result to derive the Newton interpolation polynomial for f.

2.4 Computing the value and derivatives of a polynomial : synthetic division

In this section, we consider an arbitrary polynomial Q_n of degree n, and a point $t \in \mathbb{R}$. The synthetic division (Horner's method) enables to compute $Q_n(t), Q'_n(t), \cdots, Q_n^{(k)}(t)$ efficiently, when $n = \deg(Q_n)$ is large.

Let

$$Q_n = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$$

To compute $Q_n(t)$, we do the euclidean division of Q_n by $X - t$:

$$Q_n(X) = (X - t) \underbrace{\left(b_{n-1} X^{n-1} + b_{n-2} X^{n-2} + \cdots + b_1 X + b_0\right)}_{Q_{n-1}(X)} + R_0$$

We can notice that we have $Q_n(t) = R_0$, since $(X - t)Q_{n-1}(X)$ cancels for $X = t$. Now, to compute the coefficients b_i and R_0, we can develop the latter expression and identify the monomial coefficients of each degree :

$$\left| \begin{array}{lllllll} a_n & = & b_{n-1} & \Longrightarrow & b_{n-1} & = & a_n \\ a_{n-1} & = & b_{n-2} - t b_{n-1} & \Longrightarrow & b_{n-2} & = & a_{n-1} + t b_{n-1} \\ & \vdots & & \vdots & & \vdots & \\ a_1 & = & b_0 - t b_1 & \Longrightarrow & b_0 & = & a_1 + t b_1 \\ a_0 & = & R_0 - t b_0 & \Longrightarrow & R_0 & = & a_0 + t b_0 \end{array} \right.$$

We hence obtain $R_0 = Q_n(t)$ in computing successively the b_i, starting from $b_{n-1} = a_n$ until $b_0 = a_1 + t b_1$ and $R_0 = a_0 + t b_0$. This requires n additions and n multiplications.
As a comparison, if we compute $Q_n(t)$ directly from the initial expression

$$Q_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

we need n additions and $2n-1$ multiplications.
In fact, the method just consists in writing

$$Q_n(t) = \Big(\big((a_n t + a_{n-1})t + a_{n-2}\big)t \cdots + a_1\Big)t + a_0$$

We admit that this is the most efficient way to compute $Q_n(t)$.

Now, we can iterate the process : if we do the euclidean division of Q_{n-1} by $X-t$, we obtain :

$$Q_{n-1}(X) = (X-t)\underbrace{\left(c_{n-2}X^{n-2} + \cdots + c_1 X + c_0\right)}_{Q_{n-2}(X)} + R_1$$

that we can inject in the expression of Q_n :

$$\begin{array}{rcl} Q_n(X) & = & (X-t)Q_{n-1}(X) + R_0 \\ & = & (X-t)\big((X-t)Q_{n-2}(X) + R_1\big) + R_0 \\ & = & (X-t)^2 Q_{n-2}(X) + R_1(X-t) + R_0 \end{array}$$

After n iterations, we get :

$$Q_n(X) = (X-t)^n Q_0(X) + R_{n-1}(X-t)^{n-1} + \cdots + R_1(X-t) + R_0$$

and since $\deg(Q_0) = 0$, the $(n+1)$-th iteration results in

$$Q_0 = (X-t)\times 0 + R_n$$

so finally,

$$Q_n(X) = R_n(X-t)^n + R_{n-1}(X-t)^{n-1} + \cdots + R_1(X-t) + R_0$$

Let's compare this expression with the Taylor Expansion of Q_n around t, at the order n. Since Q_n is a polynomial of degree n, its TE at the order n gives an exact expression :

$$Q_n(X) = Q_n(t) + (X-t)Q_n'(t) + \cdots + \frac{(X-t)^n}{n!}Q_n^{(n)}(t)$$

Thus, we have for each $k \in \{0, \cdots, n\}$

$$Q_n^{(k)}(t) = R_k\, k!$$

Example : let $Q_4(X) = 2X^4 - 5X^3 + X^2 - 7X + 6$. Compute the values of Q_4 and of its derivatives up to the order 4 in $t = 2$.
We construct the following table :

k	4	3	2	1	0	Remainder
Q_4 (coef. a_k)	2	-5	1	-7	6	
Q_3 (coef. b_k)		2	-1	-1	-9	$-12 = R_0$
Q_2 (coef. c_k)			2	3	5	$1 = R_1$
Q_1 (coef. d_k)				2	7	$19 = R_2$
Q_0 (coef. e_k)					2	$11 = R_3$
						$2 = R_4$

The line that corresponds to Q_i displays the result of the euclidean division of Q_{i+1} by $X-t$: it displays the quotient Q_i and the remainder. Thus, we have

$$\begin{array}{rclcl} Q_4(X) & = & 2X^4-5X^3+X^2-7X+6 & = & (X-2)(2X^3-X^2-X-9)-12 \\ Q_3(X) & = & 2X^3-X^2-X-9 & = & (X-2)(2X^2+3X+5)+1 \\ Q_2(X) & = & 2X^2+3X+5 & = & (X-2)(2X+7)+19 \\ Q_1(X) & = & 2X+7 & = & (X-2)2+11 \\ Q_0(X) & = & 2 & = & (X-2)0+2 \end{array}$$

and finally

$$Q_4(X) = 2(X-2)^4+11(X-2)^3+19(X-2)^2+(X-2)-12$$

So

$$\left| \begin{array}{l} Q_4(2) = -12 \\ Q_4'(2) = 1 \\ Q_4''(2) = 19\times 2! = 38 \\ Q_4^{(3)}(2) = 11\times 3! = 66 \\ Q_4^{(4)}(2) = 2\times 4! = 48 \end{array} \right.$$

Exercise : let P_3 be the interpolation polynomial for

$$f(0)=1 \qquad f(1)=3 \qquad f(3)=2 \qquad f(4)=5$$

In a previous exercise, we had found

$$P_3(X) = \frac{1}{2}X^3 - \frac{17}{6}X^2 + \frac{13}{3}X + 1$$

1. Write P_3 as a combination of terms $(X-2)^k$. Then deduce $P_3(2), P_3'(2), P_3''(2)$ and $P_3^{(3)}(2)$.
2. Same question in $t=-1$: write P_3 as a combination of terms $(X+1)^k$; then deduce $P_3(-1), P_3'(-1), P_3''(-1)$ and $P_3^{(3)}(-1)$.

Solution : we find

$$\begin{array}{rcl} P_3(X) & = & \frac{1}{2}(X-2)^3+\frac{1}{6}(X-2)^2-(X-2)+\frac{7}{3} \\ & = & \frac{1}{2}(X+1)^3-\frac{13}{3}(X+1)^2+\frac{23}{2}(X+1)-\frac{20}{3} \end{array}$$

3 Application of interpolation for numerical integration

Let $f:[\alpha,\beta]\to\mathbb{R}$ be a function. We suppose that we need to compute the integral

$$I(f) = \int_\alpha^\beta f(x)\,\mathrm{d}x$$

but that no exact method (antederivative, integration by parts, substitution) works. Then we can estimatate $I(f)$ with a «quadrature rule»

$$R(f) = \sum_{i=0}^{n} w_i f(x_i)$$

where the $n+1$ nodes $\{x_0,\cdots,x_n\}$ are in $[\alpha,\beta]$. The purpose of this section is to study how such a rule may be defined and to evaluate the integration error $I(f)-R(f)$.

3.1 Remark about the practical use of a quadrature rule

The following remark will not be asked in the subject of the final exam but it should help to understand the purpose of studying a quadrature rule : *the number of nodes is not intended to tend to* $+\infty$. Generally, no property like $R(f) - I(f) \underset{n\to+\infty}{\longrightarrow} 0$ has to be expected from this chapter because no such property exists. Anyway, we don't need such a property as we will see in the following example.

Example : practical use of the «left point rule» on $[0,1]$. If $f : [0,1] \longrightarrow \mathbb{R}$ is a function, we can approximate

$$I(f) = \int_0^1 f(x)\,\mathrm{d}x$$

with the quadrature rule $R(f) = f(0)$. This rule uses only one node, $x_0 = 0$. We will see further, in the next section, that if f has a continuous derivative on $[0,1]$ then

$$|I(f) - R(f)| \leqslant \frac{1}{2} \sup_{x\in[0,1]} |f'(x)| \qquad (*)$$

This inequality (*) does not prove that the integral approximation error is small, but it still enables to prove that this rule may help us.
Indeed, suppose that we need to approximate an arbitrary integral

$$J(g) = \int_a^b g(t)\,\mathrm{d}t$$

where the function g has a continuous derivative on $[a,b]$. First, we split $[a,b]$ into m sub-intervals $[a_i,b_i]$, which leads to split the integral $J(g)$ into m sub-integrals :

$$J(g) = \sum_{i=0}^{m-1} \underbrace{\int_{a_i}^{b_i} g(t)\,\mathrm{d}t}_{J_i(g)} \qquad \text{with} \qquad a_i = a + \frac{i}{m}(b-a) \quad \text{and} \quad b_i = a + \frac{i+1}{m}(b-a)$$

Second, we estimate each sub-integral $J_i(g)$ with the quadrature rule, by doing a change of variable in order to rescale the integration interval $[a_i,b_i]$ into $[0,1]$: we do the substitution $t = a_i + (b_i - a_i)x$ which results in

$$J_i(g) = (b_i - a_i) \int_0^1 \underbrace{g\Big(a_i + (b_i - a_i)x\Big)}_{f_i(x)}\,\mathrm{d}x = (b_i - a_i)\, I(f_i)$$

and the latter integral $I(f_i)$ is estimated by $R(f_i)$:

$$I(f_i) \approx R(f_i) = f_i(0) = g(a_i)$$

So finally,

$$J_i(g) \approx (b_i - a_i) g(a_i) \qquad \text{and} \qquad J(g) = \sum_{i=0}^{m-1} J_i(g) \approx \sum_{i=0}^{m-1} (b_i - a_i)\, g(a_i)$$

According to the inequality (*), the resulting error on $J_i(g)$ is bounded above by

$$\begin{aligned}
|J_i(g) - (b_i - a_i)g(a_i)| &= (b_i - a_i)\ |I(f_i) - R(f_i)| \\
&\leqslant (b_i - a_i) \times \frac{1}{2} \sup_{x \in [0,1]} |f_i'(x)| \\
&\leqslant \frac{1}{2}(b_i - a_i) \sup_{t \in [a_i,b_i]} |(b_i - a_i)g'(t)| \quad \text{since } f_i'(x) = (b_i - a_i)g'(t) \\
&\leqslant \frac{1}{2}(b_i - a_i)^2 \sup_{t \in [a,b]} |g'(t)|
\end{aligned}$$

If we replace $b_i - a_i$ with $\frac{b-a}{m}$, we obtain the following upper bound for the error on $J(g)$:

$$\begin{aligned}
\left| J(g) - \sum_{i=0}^{m-1} (b_i - a_i)g(a_i) \right| &\leqslant \sum_{i=0}^{m-1} |J_i(g) - (b_i - a_i)f(a_i)| \\
&\leqslant m \times \frac{1}{2} \left(\frac{b-a}{m} \right)^2 \sup_{t \in [a,b]} |g'(t)| \\
&\leqslant \frac{(b-a)^2}{2m} \sup_{t \in [a,b]} |g'(t)|
\end{aligned}$$

Thus, when the number m of sub-intervals tends to $+\infty$, the error on the integral estimation tends to 0 in $\frac{1}{m}$.

More generally, when we will study other quadrature rules $R(f)$ on an interval $[\alpha, \beta]$ (typically $[0,1]$ or $[-1,1]$), we will most of the time obtain an upper bound for the error in the form

$$|I(f) - R(f)| \leqslant K_c \sup_{x \in [\alpha,\beta]} \left| f^{(p+1)}(x) \right| \qquad (**)$$

where

1. p is the «degree of precision» of the quadrature rule. It depends on n, the number of nodes, as well as on the position of these nodes in $[\alpha, \beta]$;
2. K_c is the «Peano constant» of the rule.

Once we have these parameters, we use a similar approach to the one developped above : we split the integral $J(g)$ into the same sum of m sub-integrals and we approximate each sub-integral $J_i(g)$ by $(b_i - a_i)\,R(f_i)$ after a substitution which rescales the interval $[a_i, b_i]$ into $[\alpha, \beta]$. Using the inequality (**), we can prove that the approximation error on $J(g)$ is bounded by

$$K_c \, \frac{(b-a)^{p+2}}{(\beta - \alpha)m^{p+1}} \sup_{t \in [a,b]} \left| g^{(p+1)}(t) \right|$$

It hence tends to 0 in $\frac{1}{m^{p+1}}$.

The important point is that what will tend to $+\infty$ is not the number of nodes, but the number m of sub-intervals. When this number m tends to $+\infty$, the approximation error tends to 0 and the number of nodes, via the degree of precision p, controls the speed of convergence.

Thus, the purpose of the study of a quadrature rule is to be able to choose properly the weigths w_i, then to determine its degree or precision and its Peano constant (if it exists).

3.2 Definitions and properties about a quadrature rule

From now and until the end of the chapter, we will focus on integrals

$$I(f) = \int_\alpha^\beta f(x)\,\mathrm{d}x$$

that we try to estimate with a quadrature rule using $n+1$ distinct nodes $(x_0, \cdots, x_n) \in [\alpha, \beta]^{n+1}$:

$$R(f) = \sum_{i=0}^{n} w_i f(x_i)$$

To start with, let's see what happens if we replace $I(f)$ with $I(P_n)$, where P_n is the interpolating polynomial of degree n at the nodes $(x_0, \cdots, x_n)$: we have

$$P_n(X) = \sum_{i=0}^{n} f(x_i)L_i(X) \qquad \text{with} \qquad L_i(X) = \prod_{j \neq i} \frac{X - x_j}{x_i - x_j}$$

so

$$I(P_n) = \int_\alpha^\beta \left(\sum_{i=0}^{n} f(x_i)L_i(x) \right) \mathrm{d}x = \sum_{i=0}^{n} \left(f(x_i) \underbrace{\int_\alpha^\beta L_i(x)\,\mathrm{d}x}_{w_i} \right)$$

Definition : interpolation rule. A quadrature rule

$$R(f) = \sum_{i=0}^{n} w_i f(x_i)$$

is said to be an interpolation rule if it consists in writing

$$R(f) = I(P_n)$$

where P_n is the interpolating polynomial of degree n at the nodes. It is equivalent to

$$\forall i \in \{0, \cdots, n\}, w_i = \int_\alpha^\beta L_i(x)\,\mathrm{d}x = \int_\alpha^\beta \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}\,\mathrm{d}x$$

Examples : in the following examples, the integration interval is $[\alpha, \beta] = [0, 1]$.

1. Left point rule : it is the interpolation rule for a unique node $x_0 = 0$. Then we have $L_0 = 1$ and $w_0 = \int_0^1 L_(x)\,\mathrm{d}x = 1$. So
$$R_{\text{left}}(f) = f(0)$$
2. Mid-point rule : it is the interpolation rule for a unique node $x_0 = \frac{1}{2}$. As in the left point rule, we have $w_0 = 1$ and
$$R_{\text{mid}}(f) = f\left(\tfrac{1}{2}\right)$$

In both rules, we interpolate f by the degree 0 polynomial $P_0(X) = f(x_0)$. We deduce the rule $R(f) = \int_0^1 P_0(x)\,\mathrm{d}x = f(x_0)$.

Definition : degree of precision. A rule R is said to have the degree of precision p (we write $\mathrm{dop}(R) = p$) if

$$\begin{cases} \forall P \in \mathbb{R}_p[X], & R(P) = I(P) \\ \exists P \in \mathbb{R}_{p+1}[X], & R(P) \neq I(P) \end{cases}$$

Property : for any quadrature rule, we have

$$\mathrm{dop}(R) = p \Longleftrightarrow \begin{cases} R(x \mapsto 1) & = & I(x \mapsto 1) \\ R(x \mapsto x) & = & I(x \mapsto x) \\ \vdots & \vdots & \vdots \\ R(x \mapsto x^p) & = & I(x \mapsto x^p) \\ R(x \mapsto x^{p+1}) & \neq & I(x \mapsto x^{p+1}) \end{cases}$$

Examples : let's compute $\mathrm{dop}(R_{\text{left}})$ and $\mathrm{dop}(R_{\text{mid}})$.

1. Left point rule :

$$\begin{array}{llll} \text{For } p = 0: & f(x) = 1, & R_{\text{left}}(f) = f(0) = 1 \quad \text{and} & I(f) = \int_0^1 1\,\mathrm{d}x = 1 = R_{\text{left}}(f) \\ \text{For } p = 1: & f(x) = x, & R_{\text{left}}(f) = f(0) = 0 \quad \text{and} & I(f) = \int_0^1 x\,\mathrm{d}x = \frac{1}{2} \neq R_{\text{left}}(f) \end{array}$$

So $\mathrm{dop}(R_{\text{left}}) = 0$.

2. Mid-point rule :

$$\begin{array}{llll} \text{For } p = 0: & f(x) = 1, & R_{\text{mid}}(f) = f(\frac{1}{2}) = 1 \quad \text{and} & I(f) = \int_0^1 1\,\mathrm{d}x = 1 = R_{\text{mid}}(f) \\ \text{For } p = 1: & f(x) = x, & R_{\text{mid}}(f) = f(\frac{1}{2}) = \frac{1}{2} \quad \text{and} & I(f) = \int_0^1 x\,\mathrm{d}x = \frac{1}{2} = R_{\text{mid}}(f) \\ \text{For } p = 2: & f(x) = x^2, & R_{\text{mid}}(f) = f(\frac{1}{2}) = \frac{1}{4} \quad \text{and} & I(f) = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3} \neq R_{\text{mid}}(f) \end{array}$$

So $\mathrm{dop}(R_{\text{mid}}) = 1$.

Now, the following theorem connects the two previous definitions :
Theorem : let R be a quadrature rule using $n+1$ nodes $(x_0, \cdots, x_n)$. Then

$$R \text{ is an interpolation rule} \Longleftrightarrow \mathrm{dop}(R) \geqslant n$$

Because of this theorem, the only rules we will work with are interpolation rules. Such a rule is defined with no ambiguity by its $n+1$ nodes. In particular, the question weither its degree of precision is strictly larger than n or not depends only on these nodes.

Remark : this theorem is a good opportunity to check your computations. Indeed, in all the exercises, $R(f)$ will be an interpolation rule and you'll have to compute its coefficients w_i. Later, you'll be asked to determine its degree of precision. After this step, you can use

$$\mathrm{dop}(R) \geqslant n \Longleftrightarrow \text{the coefficients } w_i \text{ are all correct}$$

It may help you to detect a mistake in the w_i, or to confirm that they are right.

3.3 Expression of the integration error

In this section, we give an expression for the integration error

$$E(f) = I(f) - R(f)$$

First, for any $p \in \mathbb{N}$ and $t \in [\alpha, \beta]$, we define the particular function

$$\begin{array}{rccl} (x-t)_+^p : & [\alpha,\beta] & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \left| \begin{array}{ll} (x-t)^p & \text{if } x-t>0 \\ 0 & \text{if } x-t \leqslant 0 \end{array} \right. \end{array}$$

Then, the integration error is given by the following theorem.

Theorem (Peano). If $\operatorname{dop}(R) = p$ and if a function f has a continuous $(p+1)$-th derivative, then

$$E(f) = \int_\alpha^\beta K(t) f^{(p+1)}(t)\,\mathrm{d}t$$

where K is the «Peano kernel» of the rule, defined for any $t \in [\alpha, \beta]$ by

$$K(t) = E\left(\frac{(x-t)_+^p}{p!}\right)$$

Explanation : in this definition, we set $t \in [\alpha, \beta]$ and the function $(x-t)_+^p$ is seen as a function of x. We have

$$\begin{array}{rcl} E\left(\dfrac{(x-t)_+^p}{p!}\right) & = & I\left(\dfrac{(x-t)_+^p}{p!}\right) - R\left(\dfrac{(x-t)_+^p}{p!}\right) \\ & = & \displaystyle\int_\alpha^\beta \frac{(x-t)_+^p}{p!}\,\mathrm{d}x - \sum_{i=0}^n w_i \frac{(x_i-t)_+^p}{p!} \end{array}$$

which depends on t. The Peano kernel $K(t)$ is this integration error.

Particular case when K has a constant sign : if the sign of $K(t)$ does not depend on t, we define the «Peano constant» of the rule to be

$$K_c = \int_\alpha^\beta K(t)\,\mathrm{d}t$$

Then we have the following property : for any function $f \in C^{p+1}([\alpha, \beta])$,

$$\exists c \in [\alpha, \beta],\ E(f) = K_c\, f^{(p+1)}(c)$$

Thus,

$$|E(f)| \leqslant |K_c| \times \sup_{t \in [\alpha,\beta]} \left| f^{(p+1)}(t) \right|$$

In practice, once we have proven that $K(t)$ has a constant sign, we compute K_c using the integration error on the particular function $f_0 : x \mapsto x^{p+1}$. Indeed, its $(p+1)$-th derivative is constant and we have evaluated previously $I(f_0)$ and $R(f_0)$, when computing $\operatorname{dop}(R)$. We hence have easily $E(f_0)$ and $f_0^{(p+1)}(c) = (p+1)!$, from which we can deduce K_c.

Example 1 : left point rule on $[0,1]$**.** The rule is $R_{\text{left}}(f) = f(0)$ and we have seen previously that its degree of precision is $p = 0$. Therefore, its Peano kernel is defined for any $t \in [0,1]$ by

$$K(t) = I\left(\frac{(x-t)^0_+}{0!}\right) - R_{\text{left}}\left(\frac{(x-t)^0_+}{0!}\right)$$

The integral is

$$\begin{aligned} I\Big((x-t)^0_+\Big) &= \int_0^1 (x-t)^0_+ \,\mathrm{d}x \\ &= \int_0^t \underbrace{(x-t)^0_+}_{0} \,\mathrm{d}x + \int_t^1 \underbrace{(x-t)^0_+}_{1} \,\mathrm{d}x \\ &= \int_t^1 1 \,\mathrm{d}t \\ &= 1-t \end{aligned}$$

and the quadrature rule is

$$\begin{aligned} R_{\text{left}}\Big((x-t)^0_+\Big) &= (0-t)^0_+ \\ &= 0 \qquad \text{since } (0-t) \leqslant 0 \end{aligned}$$

Therefore,

$$K(t) = 1-t$$

It has a constant sign over $[0,1]$, so there exists a Peano constant K_c. If we take for f the particular function $f_0 : x \mapsto x^{p+1} = x$, we have

$$\exists c \in [0,1],\ I(f_0) - R_{\text{left}}(f_0) = K_c f_0'(c)$$

But $f_0' = 1$ and we have seen, when computing the degree of precision, that $I(f_0) - R_{\text{left}}(f_0) = \frac{1}{2} - 0 = \frac{1}{2}$. Thus, the Peano constant is

$$K_c = \frac{1}{2} \qquad \text{and} \qquad \forall f \in C^1([0,1]),\quad |I(f) - R_{\text{left}}(f)| \leqslant \frac{1}{2} \sup_{[0,1]} |f'|$$

Example 2 : mid-point rule on $[0,1]$**.** The quadrature rule is $R_{\text{mid}}(f) = f(\frac{1}{2})$ and we have seen that its degree of precision is $p = 1$. The Peano kernel is hence defined for any $t \in [0,1]$ by

$$K(t) = I\left(\frac{(x-t)^1_+}{1!}\right) - R_{\text{mid}}\left(\frac{(x-t)^1_+}{1!}\right)$$

The integral is

$$\begin{aligned} I\Big((x-t)^1_+\Big) &= \int_0^1 (x-t)^1_+ \,\mathrm{d}x \\ &= \int_0^t \underbrace{(x-t)^1_+}_{0} \,\mathrm{d}x + \int_t^1 \underbrace{(x-t)^1_+}_{x-t} \,\mathrm{d}x \\ &= \int_t^1 (x-t) \,\mathrm{d}t \\ &= \left[\tfrac{(x-t)^2}{2}\right]_t^1 \\ &= \frac{(1-t)^2}{2} \end{aligned}$$

and the quadrature rule is

$$R_{\text{mid}}\Big((x-t)_+^1\Big) = (\tfrac{1}{2}-t)_+^1 = \left| \begin{array}{ll} 0 & \text{if } t \geqslant \frac{1}{2} \\ \frac{1}{2}-t & \text{if } t < \frac{1}{2} \end{array} \right.$$

Thus, when $t \geqslant \frac{1}{2}$,

$$K(t) = \frac{(1-t)^2}{2} - 0 = \frac{(1-t)^2}{2}$$

and when $t < \frac{1}{2}$,

$$\begin{aligned} K(t) &= \frac{(1-t)^2}{2} - \left(\frac{1}{2}-t\right) \\ &= \frac{1-2t+t^2}{2} - \frac{1}{2} + t \\ &= \frac{t^2}{2} \end{aligned}$$

Observation : in this rule, the node(s) have a symmetry relative to the center of the integration interval $[0,1]$: if x_i is a node, $1-x_i$ is also a node. The Peano kernel follows this symmetry : $K(t) = K(1-t)$.

We can also observe that the Peano kernel has a constant positive sign. There hence exists a Peano constant K_c. If we take for f the particular function $f_0 : x \mapsto x^{p+1} = x^2$, then

$$\exists c \in [0,1], I(f_0) - R_{\text{mid}}(f_0) = K_c\, f''(c)$$

But $f_0'' = 2$ and the integration error in f_0 is $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Finally, the Peano constant is

$$K_c = \frac{1}{24} \qquad \text{and} \qquad \forall f \in C^2([0,1]),\ |I(f) - R_{\text{mid}}(f)| \leqslant \frac{1}{24} \sup_{[0,1]} |f''|$$

Exercise : trapezoidal rule on $[-1,1]$**.** We estimate $I(f)$ with the quadrature rule using the nodes $x_0 = -1$ and $x_1 = 1$:

$$I(f) = \int_{-1}^{1} f(x)\,\mathrm{d}x \qquad \text{and} \qquad R(f) = w_0 f(-1) + w_1 f(1)$$

1. Using the Lagrange's method, determine the interpolating polynomial of f at the nodes and deduce the coefficients w_0 and w_1.
   Solution : we find $w_0 = w_1 = 1$, so $R(f) = f(-1) + f(1)$
2. Determine the degree of precision of the rule.
   Solution : we find $\text{dop}(R) = 1$
3. Determine the Peano kernel and study its sign.
   Solution : $K(t) = \frac{t^2-1}{2} \leqslant 0$
4. Deduce an expression for the integration error, assuming that $f \in C^2([-1,1])$.
   Solution : $K_c = -\frac{2}{3}$, so $|I(f) - R(f)| \leqslant \frac{2}{3} \sup\limits_{[-1,1]} |f''|$

Observation : in this rule, the nodes have a symmetry relative to the center of the integration interval $[-1,1]$: if x_i is a node, $-x_i$ is also a node. The Peano kernel K conserves this symmetry : $\forall t \in [-1,1], K(t) = K(-t)$.
This symmetry conservation is satisfied by any quadrature rule whose nodes are symmetric relatively to the center of the integration interval $[\alpha,\beta]$. If for any node x_i, $\alpha+\beta-x_i$ is also a node, then

$$\forall t \in [\alpha,\beta], K(t) = K(\alpha+\beta-t)$$