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Graph Theory HW

1 X4

1.1 (a)

Prove that every tournement has a hamilton directed path.

Take a tournement T, T has a maximal directed path p of length k starting at vertex $v_0 \in V(T)$ and ending at $v_k \in V(T)$. Assume for a contradiction that p is not a hamilton path. This means there exists a $u \in V(T)$ that is not in the p. There is an edge going either to or from u and $v_i \in p$. There are three cases that occur for u:

1. All edges are going from u to each $v_i \in p$. If this is the case then we can simply add u to p by adding the edge $u \to v_0$. This extends p $\lightning$. We can continue this for each vertex not in p, leaving with the maximal path, being a hamilton path.

2. All edges are going from each $v_i \in p$ to u. If this is the case we can add u to p by adding the edge $v_k \to u$. This extends p $\lightning$. By the same logic above, we have a hamilton path.

3. The edges are not uniformally in one direction. This means that at some point there are two edges in opposite directions (one from $u \to v_i$ and on from $v_{i+1} \to u$. This means that we can splice u into path p, this can be done by instead of following the edge $v_i v_{i+1}$ the path goes along $v_i u$ then $u v_{i+1}$ and then continues along on its ususal way. In this way we have extended p $\lightning$. By the same logic above we have a hamilton path.$\blacksquare$

1.2 (b)

Show that every tournement of size three or more vertices is either strong or can be transformed into a strong tournement by the reorientation of just on arc.

Based on part a, we know that there is a hamiltonion path begining with x, and ending with y. There are two cases to consider:

1. If there is an edge going from $y \to x$ then there exists a hamiltonian cycle, by following this cycle every vertex is reachable of every toher vertex.

2. If there is an edge going from $x \to y$ then reverse this edge and we are left with Case 1 (a strong tournement by reorienting one edge). $\blacksquare$