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Graph Theory HW

1 X4

1.1 (a)

Prove that every tournement has a hamilton directed path.

Take a tournament T. T has a directed path P of maximal length k, $v_0v_1v_2\dots v_k$. P starts at vertex $v_0 \in V(T)$ and ends at $v_k \in V(T)$. Assume for a contradiction that P is not a hamilton path. This means there exists a $u \in V(T)$ that is not in the P. There is an edge going either to or from u and $\forall v_i \in P$. There are three cases that occur for u:

1. All edges are going from u to each $v_i \in P$. If this is the case then we can simply add u to the beginning of P by adding the edge $u \to v_0$. This extends P this is a contradiction. We can continue this for each vertex not in P, leaving with the maximal path, being a hamilton path.

2. All edges are going from each $v_i \in P$ to u. If this is the case we can add u to the end of P by adding the edge $v_k \to u$. This extends P this is a contradiction. By the same logic above, we have a hamilton path.

3. The edges between u and $\forall v_i \in P$ are not uniformally in one direction. This means that at some point there are two edges in opposite directions (one from $v_i \to u$ and on from $u \to v_{i+1}$). This means that we can splice u into path P, this can be done by instead of following the edge v_iv_{i+1} the path goes along v_iu then uv_{i+1} and then continues along on its ususal way. In this way we have extended P this is a contradiction. By the same logic above we have a hamilton path.■

1.2 (b)

Show that every tournament of size three or more vertices is either strong or can be transformed into a strong tournement by the reorientation of just on arc.

Let us take a tournament Based on part a, we know that there is a hamiltonion path beginning with a vertex $x \in V(T)$, and ending with a vertex $y \in V(T)$. There are two cases to consider:

1. If there is an edge going from $y \to x$ then there exists a hamiltonian cycle, and by following this cycle every vertex is reachable from every other vertex.

2. If there is an edge going from $x \to y$ then reverse this edge and we are left with Case 1 (a strong tournement by reorienting one edge). ■