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Graph Theory HW

1 X4

1.1 (a)

Prove that every tournament has a hamilton directed path.

Take a tournament T. T has a directed path P of maximum length k, $v_0v_1v_2 \ldots v_k$. P starts at vertex $v_0 \in V(T)$ and ends at $v_k \in V(T)$. Assume for a contradiction that P is not a hamilton path. This means there exists a $u \in V(T)$ that is not in P. There is an edge going either to or from u and every $v_i \in P$. There are three cases that occur for u:

1. There is an edge going from u to v_0. If this is the case then we can simply add u to the beginning of P by adding the edge $u \to v_0$. This extends P so this is a contradiction.

2. There is an edge going from v_k to u. If this is the case we can add u to the end of P by adding the edge $v_k \to u$. This extends P so this is a contradiction.

3. The last case has an edge from v_0 to u and an edge from u to v_k. This means that we cannot extend this path at the beginning or end, and we must splice u somewhere inside P. In order to splice u in we need a situation where there exists an edge $v_{i-1} \to u$ and $u \to v_i$. As we go from v_0 to v_k for each v_i the edge between v_i and u must either be the same or different then that of v_{i-1} and u. Since $v_0 \to u$ and $u \to v_k$ there must be at least one v_i such that the edge between v_i and u is different then the edge between v_{i-1} and u, if we take the smallest i where this first occurs, we know that v_{i-1} is in the direction from $v_{i-1} \to u$ (This is because $v_0 \to u$ and the first time that an edge goes in a different direction, it must be the opposite of this). Let us splice u in using the edges $v_{i-1} \to u$ and $u \to v_i$. This extends P which is a contradiction.

Since we reach a contradiction in all three cases, P must be a hamilton path. ■

1.2 (b)

Show that every tournament of size three or more vertices is either strong or can be transformed into a strong tournement by the reorientation of just on arc.

Let us take a tournament. Based on part a, we know that there is a hamilton path beginning with a vertex $x \in V(T)$, and ending with a vertex $y \in V(T)$. There are two cases to consider:

1. If there is an edge going from $y \rightarrow x$ then there exists a hamilton cycle, and by following this cycle every vertex is reachable from every other vertex.

2. If there is an edge going from $x \rightarrow y$ then reverse this edge and we are left with Case 1 (a strong tournament by reorienting one edge). ■