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Graph Theory HW

1 2.1.11 (b)

Deduce that a digraph admits a topological sort if and only if it is acyclic.

($\Rightarrow$) Let digraph D have a topological sort. Assume for a contradiction that D has a cycle. This means that $\exists$ a cycle c. Let x and y be the first and last vertex in the topologocal sort, respectively, that are in this cycle. Since x and y are in a cycle $\exists$ $x-y$ walk, and $\exists$ $y-x$ walk. Since x comes before y in the topological sort, but $\exists$ $y-x$ walk, there must be an edge from a vertex from y to x meaning that the tail of y precedes its head $\lightning$. This contradicts the constraints of topological sort $\therefore$ D does not have a cycle.

($\Leftarrow$) Let digraph D be acyclic: As proved in class a acyclic digraph always has a source. Remove this source and place it in a linear ordering. We are, again, left with another acyclic digraph, which must have a source. Repeat the first step until there are no sources left (at this point it will be a set of verticies with no edges). Arbitrarily add these verticies to the linear ordering. As we add sources to the linear ordering we know that there cannot be a tail that precedes its head, because sources have no in edges. We are left with a topological sort. $\blacksquare$