Ultrafilter Aggregation of Δ_0-Definable Fragments and Induced Hilbert Structures

Jürgen Bigalke*

Abstract

We study bounded Δ_0-definability over the Jensen level $J_2 = \mathrm{rud}(\mathrm{HF})$ and analyse an ultrafilter-based aggregation mechanism transforming locally definable data into a canonical global comparison structure.

Countable families of locally Δ_0-definable fragments are aggregated via a fixed non-principal ultrafilter into second-order objects (Ultrasheaves). Δ_0-definable numerical comparison invariants induce, under aggregation, a well-defined global positive semidefinite kernel uniquely determined by the local data.

Under a finitary positivity assumption, the associated kernel construction yields a Hilbert structure canonical up to unitary equivalence. The resulting quadratic and factorisation properties are shown to arise purely from stabilisation phenomena governed by bounded definability over J_2, without additional analytic or topological assumptions.

Keywords: Δ_0–definability, Jensen hierarchy, ultrafilter aggregation, kernel methods, Hilbert–space representations

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1 Introduction

Initial segments of the Jensen hierarchy provide a canonical framework for the analysis of definability under severely bounded logical resources. Among these, the level

$$J_2 = \mathrm{rud}(\mathrm{HF})$$

occupies a distinguished position: it is a transitive structure closed under all rudimentary functions, yet sufficiently low to admit a detailed fine–structural analysis. Classical references include [Jen72, Dev84], with a focused systematic study of J_2 given in [Wea05].

Analytic representation spaces such as Banach or Hilbert spaces are traditionally introduced by enriching a set–theoretic framework with metric, topological, or measure–theoretic structure. From a fine–structural and definability–theoretic perspective, this raises a natural structural question: to what extent are such representations already implicit in the internal resources of set theory itself, without appeal to external analytic primitives?

The present article addresses this question at a purely structural level. We show that Hilbert–space representations arise canonically from bounded Δ_0–definable data once local definability is combined with a single global aggregation principle. The analysis is carried out concretely over J_2, which serves as a minimal environment in which nontrivial global structure can emerge from purely local definability constraints. While the construction is not intrinsically tied to J_2, this level is chosen for reasons of canonicity and minimality: it is the smallest natural domain supporting the required stability properties without additional axiomatic assumptions.

The central mechanism is an ultrafilter–based aggregation of locally definable data. We consider countable families of locally Δ_0–definable subsets of J_2, referred to as *local fragments*. Aggregating

*j.bigalke@juergen-bigalke.de

such families along a fixed non–principal ultrafilter yields global second–order objects, termed *Ultrasheaves*. The ultrafilter is treated throughout as an externally fixed parameter, characterised at the metatheoretic level by its standard axioms, and is not assumed to be definable over $(J_2, \in)$.

Locally Δ_0–definable numerical comparison invariants between fragments induce, under ultrafilter aggregation, a positive semidefinite kernel on the class of Ultrasheaves. Assuming only an explicit finitary positivity condition on this local comparison data, the associated global kernel is uniquely determined. By the standard kernel–to–Hilbert construction, such a kernel admits a Hilbert–space representation that is canonical in the representation–theoretic sense, that is, unique up to unitary equivalence. Numerical and geometric structure enter only at the level of representation; no topological, measure–theoretic, or probabilistic primitives are presupposed at the level of definability. All constructions are carried out within bounded Δ_0–definability over $(J_2, \in)$, rudimentary closure, and standard ultrafilter techniques. Positivity of the local comparison invariant is taken as explicit structural input: it is the minimal finitary condition ensuring that the aggregated comparison data determine a positive semidefinite kernel and hence admit a Hilbert–space representation.

The novelty of the present analysis therefore does not lie in the kernel–to–Hilbert construction itself, which is classical, but in the fine–structural localisation of this representation mechanism. The kernel–to–Hilbert passage emerges here as a canonical consequence of bounded definability over J_2 once a single aggregation principle is fixed. All objects entering the construction—local fragments, comparison data, kernels, and the resulting Hilbert–space representation—are induced canonically from this minimal input.

The subsequent sections analyse structural consequences of the resulting representation. These are not additional axioms or independent assumptions, but intrinsic constraints forced by bounded definability and ultrafilter aggregation. They serve to delimit the rigidity and expressive power of the induced Hilbert–space structure rather than to introduce new analytic content.

The paper is organised as follows. Section 2 recalls background material on the Jensen hierarchy and bounded definability. Section 3 introduces locally definable fragments and their aggregation into Ultrasheaves. Section 4 develops the induced comparison kernel and the associated Hilbert–space representation. Section 5 records several structural consequences of the construction. Section 6 concludes with brief remarks.

2 Definability in the Jensen Level J_2

We recall the background required for the subsequent construction and fix notation. The presentation is deliberately brief and restricted to standard facts from the fine structure of the Jensen hierarchy and bounded definability.

We work with the initial levels of Jensen's hierarchy. Let $J_0 = \varnothing$, let $J_1 = \mathrm{rud}(J_0) = \mathrm{HF}$ denote the set of hereditarily finite sets, and let

$$J_2 = \mathrm{rud}(J_1) = \mathrm{rud}(\mathrm{HF}).$$

The structure J_2 is a countable transitive set closed under all rudimentary functions. No further axiomatic assumptions are imposed.

Throughout the paper, definability over J_2 is understood exclusively in terms of bounded formulas. A subset $X \subseteq J_2$ is said to be Δ_0*–definable* if it is defined by a bounded formula in the language of set theory, allowing parameters from J_2. Bounded definability is stable under rudimentary operations and provides the basic notion of locality used in what follows.

In later sections we consider families of Δ_0–definable subsets of J_2 indexed by countable sets and aggregate them using ultrafilters. This aggregation separates local definability from global structure. While all local data are Δ_0–definable subsets of J_2, the resulting aggregated objects need not themselves belong to J_2. This distinction underlies the Ultrasheaf construction and the Hilbert–space representations developed below.

3 Locally Definable Fragments and Ultrasheaves

We introduce the local objects used in the subsequent aggregation and fix the corresponding equivalence notion. All notions of locality are understood exclusively in terms of bounded definability over the transitive Jensen level J_2.

Definition 3.1 (Locally definable fragment)**.** A *locally definable fragment* is a subset $X \subseteq J_2$ that is Δ_0–definable over the structure $(J_2, \in)$, allowing parameters from J_2.

Locally Δ_0–definable fragments form a class closed under the rudimentary operations of J_2 and exactly capture bounded definability.

Definition 3.2 ($\mathcal{U}$–equivalence)**.** Let I be a countable index set and let $(X_i)_{i\in I}$ be a family of locally definable fragments. Fix a non–principal ultrafilter $\mathcal{U}$ on I. Two families $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are said to be *$\mathcal{U}$–equivalent* if

$$\{\, i \in I : X_i = Y_i \,\} \in \mathcal{U}.$$

The ultrafilter $\mathcal{U}$ is fixed once and for all as a metatheoretic parameter. It is used solely to form equivalence classes of countable families of Δ_0–definable fragments; no internal presentation of $\mathcal{U}$ over $(J_2, \in)$ is part of the framework. Equivalently, $\mathcal{U}$ functions as an external aggregation scheme rather than as an object of the background structure.

Definition 3.3 (Ultrasheaf)**.** An *Ultrasheaf* is the $\mathcal{U}$–equivalence class of a family $(X_i)_{i\in I}$ of locally definable fragments. We denote such a class by $\langle X_i\rangle_{\mathcal{U}}$.

Ultrasheaves are obtained by ultrafilter aggregation of families of locally Δ_0–definable fragments. They are, in general, not elements of J_2, but second–order definable objects arising from the aggregation procedure.

Any property specified by a bounded Δ_0–formula and verified pointwise on a $\mathcal{U}$–large set of indices induces a well–defined attribute of the associated Ultrasheaf. Different choices of non–principal ultrafilters may yield non–isomorphic Ultrasheaves; no canonical choice of ultrafilter is assumed. Once a choice of $\mathcal{U}$ is fixed, the resulting class of Ultrasheaves admits canonical comparison constructions, which will be used in the subsequent sections.

4 Canonical Hilbert Structure

We show that the definability–theoretic structure introduced above admits a canonical Hilbert–space representation in a precise representation–theoretic sense. All constructions rely exclusively on bounded definability and ultrafilter aggregation.

Let $\mathcal{F}$ be the class of locally Δ_0–definable subsets of J_2. Assume a locally Δ_0–definable numerical comparison invariant

$$\langle\cdot,\cdot\rangle : \mathcal{F} \times \mathcal{F} \to \mathbb{D},$$

where $\mathbb{D}$ denotes the dyadic rationals coded in J_2. The choice of $\mathbb{D}$ is purely conventional and serves only to fix a concrete Δ_0–definable numerical codomain; any such domain would suffice for the construction.

The definition of $\langle\cdot,\cdot\rangle$ may involve bounded membership tests in the (possibly infinite) fragments $X, Y \subseteq J_2$ and is stable under rudimentary constructions. The invariant is assumed to be hermitian with respect to the involution of the numerical structure.

Comparison invariants satisfying these definability and stability conditions are standard in fine–structural analysis. In particular, Δ_0–definable numerical assignments over the transitive structure $(J_2, \in)$ are closed under rudimentary constructions; see, for example, [Dev84, Wea05]. Throughout this section, one such invariant is fixed as background data.

Definition 4.1 (Ultrasheaf comparison kernel)**.** Let $\langle\cdot,\cdot\rangle$ be fixed as above. For Ultrasheaves $\mathfrak{X} = \langle X_i\rangle_{\mathcal{U}}$ and $\mathfrak{Y} = \langle Y_i\rangle_{\mathcal{U}}$, define the *Ultrasheaf comparison kernel* K by

$$K(\mathfrak{X}, \mathfrak{Y}) \;:=\; \big[\, i \mapsto \langle X_i, Y_i\rangle \,\big]_{\mathcal{U}}.$$

The kernel K takes values in the space of $\mathcal{U}$–equivalence classes of $\mathbb{D}$–valued families indexed by I. It is independent of the choice of representatives by Δ_0–stability and invariance under $\mathcal{U}$–equivalence. If the local comparison invariant $\langle\cdot,\cdot\rangle$ is hermitian, then K inherits the corresponding hermitian symmetry with respect to the involution induced by pointwise conjugation on $\mathbb{D}$.

Lemma 4.2 (Finitary positivity condition)**.** *Assume that the local comparison invariant $\langle\cdot,\cdot\rangle : \mathcal{F}\times\mathcal{F}\to\mathbb{D}$ satisfies the following finitary positivity condition.*

For every $n<\omega$, every choice of locally Δ_0–definable families $(X_i^{(p)})_{i\in I}$ for $p<n$, and all coefficients $c_0,\dots,c_{n-1}\in\mathbb{D}$, the quadratic form

$$\sum_{p,q<n}\overline{c_p}\,\langle X_i^{(p)},X_i^{(q)}\rangle\,c_q$$

is non–negative for $\mathcal{U}$–many indices $i\in I$.

Equivalently, all finite Gram matrices associated with the local comparison invariant are positive semidefinite on $\mathcal{U}$–large sets.

Sketch of proof. Fix $n<\omega$, locally Δ_0–definable families $(X_i^{(p)})_{p<n}$, and coefficients $c_0,\dots,c_{n-1}\in\mathbb{D}$. By the finitary positivity condition, the associated quadratic form is non–negative on a $\mathcal{U}$–large set of indices.

For fixed n and fixed families, the expression above depends only on finitely many values of the local comparison invariant and is therefore determined by a finite system of algebraic inequalities. Since these inequalities are evaluated pointwise in i, their validity on a $\mathcal{U}$–large set is preserved under ultrafilter aggregation. □

The positivity of the local comparison invariant is a structural assumption. It is not derived from definability considerations, but isolates the minimal condition under which the aggregated comparison data admit a Hilbert–space representation.

Corollary 4.3 (Positive semidefiniteness of the comparison kernel)**.** *Under the assumptions of Lemma 4.2, the Ultrasheaf comparison kernel K is positive semidefinite.*

Proof. Fix Ultrasheaves $\mathfrak{X}_0,\dots,\mathfrak{X}_{n-1}$ with representatives $\mathfrak{X}_p=\langle X_i^{(p)}\rangle_{\mathcal{U}}$. Then for any fixed choice of coefficients $c_0,\dots,c_{n-1}\in\mathbb{D}$,

$$\sum_{p,q<n}\overline{c_p}\,K(\mathfrak{X}_p,\mathfrak{X}_q)\,c_q\geq 0,$$

which is exactly the defining condition for positive semidefiniteness of K. □

Given a positive semidefinite kernel K on Ultrasheaves, taking values in the $\mathcal{U}$–equivalence classes of numerical families, the standard kernel construction yields a pre–Hilbert space H_0 and a canonical mapping

$$\kappa\colon\mathfrak{X}\longmapsto\kappa(\mathfrak{X})\in H_0$$

satisfying

$$\langle\kappa(\mathfrak{X}),\kappa(\mathfrak{Y})\rangle_{H_0}\;=\;K(\mathfrak{X},\mathfrak{Y}),$$

as in the classical theory of positive definite kernels; see, for example, [Aro50].

Proposition 4.4 (Canonical Hilbert representation)**.** *The metric completion of H_0 yields a Hilbert space $\mathcal{H}$ over the numerical equivalence classes arising from ultrafilter aggregation, uniquely determined up to isometric isomorphism by the kernel K.*

Here "Hilbert space" refers to the completion of the induced pre–Hilbert space with respect to its inner product over the aggregated numerical quotient. Thus $\mathcal{H}$ carries precisely the structure forced by the kernel: an inner product and its metric completion.

The resulting Hilbert–space representation is canonical in the standard representation–theoretic sense, that is, uniquely determined by the kernel up to unitary equivalence. Canonically means: the representation is determined by K uniquely up to unitary equivalence. Accordingly, no further choices enter the construction beyond those already encoded by the kernel.

Identifying H_0 with its canonical image in $\mathcal{H}$, the map κ may be regarded as taking values in $\mathcal{H}$. The space $\mathcal{H}$ is thus canonically determined, relative to the chosen comparison invariant, as a representational completion of the definability–theoretic data encoded by Ultrasheaves.

5 Structural Consequences

This section records structural properties of the induced Hilbert–space representation that are not assumed a priori, but follow solely from bounded Δ_0–definability, ultrafilter aggregation, and the existence of a positive semidefinite comparison kernel. No additional definability or structural assumptions are introduced.

The statements collected here isolate consequences that are intrinsic to the representation mechanism itself. They are not independent features added to the formalism, but unavoidable structural constraints enforced by the kernel construction. Wherever possible, results are formulated entirely at the level of the comparison kernel; references to the associated Hilbert–space representation serve only as a secondary interpretative layer.

Quadratic structure. Let $\mathfrak{X} = \langle X_i \rangle_{\mathcal{U}}$ be an Ultrasheaf, and let $\kappa(\mathfrak{X}) \in \mathcal{H}$ denote its image under the canonical embedding induced by the comparison kernel. The inner product on $\mathcal{H}$ induces a canonical quadratic form

$$\|\kappa(\mathfrak{X})\|^2 \;:=\; \langle \kappa(\mathfrak{X}), \kappa(\mathfrak{X}) \rangle_{\mathcal{H}} \;=\; K(\mathfrak{X}, \mathfrak{X}).$$

This identity follows directly from the positive semidefiniteness of the kernel and requires no additional assumptions.

Canonical involution. The hermitian symmetry of the comparison kernel constructed in Section 4 induces a canonical conjugation on the associated Hilbert space $\mathcal{H}$. This involution is uniquely determined (up to unitary equivalence) by the kernel and reflects symmetry properties of the underlying local comparison invariants. At the definability level, it is intrinsic to the aggregation procedure and does not rely on any further structure.

Factorisation and stabilisation. Factorisation properties of the comparison kernel are governed by stabilisation phenomena at the level of bounded definability.

Let $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ be families of locally Δ_0–definable fragments giving rise to Ultrasheaves $\mathfrak{X} = \langle X_i \rangle_{\mathcal{U}}$ and $\mathfrak{Y} = \langle Y_i \rangle_{\mathcal{U}}$. Fix a Δ_0–definability equivalence relation $\equiv$ on the class $\mathcal{F}$ of locally definable fragments, identifying fragments that are indistinguishable with respect to the bounded definability data relevant for the comparison invariant.

We say that a family (X_i) stabilises modulo $\equiv$ if there exists a $\mathcal{U}$–large set $U \subseteq I$ such that all X_i with $i \in U$ lie in a single $\equiv$–class. Independent stabilisation of (X_i) and (Y_i) modulo $\equiv$ yields factorisation of the comparison kernel; failure of such independent stabilisation precludes factorisation. This dichotomy depends only on bounded definability and ultrafilter aggregation.

Kernel–level characterisation. Factorisation admits an intrinsic formulation entirely at the kernel level. Suppose that the relevant definability data decomposes into two independent components, labelled a and b. This induces corresponding kernel–level components of Ultrasheaves, written

$$\mathfrak{X} \longmapsto (\mathfrak{X}_a, \mathfrak{X}_b), \qquad \mathfrak{Y} \longmapsto (\mathfrak{Y}_a, \mathfrak{Y}_b),$$

defined up to $\mathcal{U}$–equivalence.

An Ultrasheaf $\mathfrak{X}$ is said to be *separable* with respect to this decomposition if, for all compatible Ultrasheaves $\mathfrak{Y}$,

$$K(\mathfrak{X}, \mathfrak{Y}) = K_a(\mathfrak{X}_a, \mathfrak{Y}_a)\, K_b(\mathfrak{X}_b, \mathfrak{Y}_b).$$

Failure of such a factorisation indicates irreducible global coherence at the kernel level. All statements in this section are therefore purely kernel–theoretic and do not presuppose any Hilbert–space representation.

Definition 5.1 (Definability equivalence)**.** Let $\mathcal{F}$ denote the class of locally Δ_0–definable subsets of J_2, allowing parameters from J_2. A binary relation $\equiv$ on $\mathcal{F}$ is called a Δ_0*–definability equivalence relation* if there exists a bounded Δ_0–formula $\varphi(X, Y, \vec{p})$ with parameters $\vec{p} \in J_2$ such that for all $X, Y \in \mathcal{F}$,

$$X \equiv Y \iff (J_2, \in) \models \varphi(X, Y, \vec{p}).$$

Lemma 5.2 (Independent stabilisation implies factorisation)**.** *Let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be families of locally Δ_0–definable fragments, and let $\mathfrak{X} = \langle X_i \rangle_{\mathcal{U}}$ and $\mathfrak{Y} = \langle Y_i \rangle_{\mathcal{U}}$ be the induced Ultrasheaves. Assume that the local comparison invariant $\langle \cdot, \cdot \rangle : \mathcal{F} \times \mathcal{F} \to \mathbb{D}$ is Δ_0–definable and hermitian.*

If there exist a Δ_0–definability equivalence relation $\equiv$ and a $\mathcal{U}$–large set $U \subseteq I$ such that both (X_i) and (Y_i) stabilise modulo $\equiv$ on U, and such that on U

$$\langle X_i, Y_i \rangle = \alpha([X_i]_\equiv) \cdot \beta([Y_i]_\equiv)$$

for functions α, β depending only on $\equiv$–classes, then the induced comparison kernel factorises.

Sketch. Stabilisation implies that the $\equiv$–classes $[X_i]_\equiv$ and $[Y_i]_\equiv$ are $\mathcal{U}$–eventually constant. Since α and β depend only on these classes, the associated functions are $\mathcal{U}$–eventually constant as well. Passing to ultrafilter equivalence classes yields

$$K(\mathfrak{X}, \mathfrak{Y}) = \alpha(\mathfrak{X})\, \beta(\mathfrak{Y}),$$

establishing factorisation. □

Corollary 5.3 (Factorisation versus definability dependence)**.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Ultrasheaves induced by locally Δ_0–definable families. The comparison kernel $K(\mathfrak{X}, \mathfrak{Y})$ factorises if and only if, on a $\mathcal{U}$–large set of indices, the local comparison invariant admits a multiplicative separation determined solely by the stabilised definability classes of the representing families.*

6 Conclusion

We have shown that Hilbert–space representations arise canonically from bounded definability alone. Starting from Δ_0–definable local fragments over the Jensen level $J_2 = \mathrm{rud}(\mathrm{HF})$, ultrafilter aggregation yields global second–order objects, termed Ultrasheaves. Under ultrafilter aggregation, locally Δ_0–definable numerical comparison invariants give rise to positive semidefinite kernels. The standard kernel construction then yields a canonical Hilbert–space representation.

These results isolate a minimal structural core underlying Hilbert–space theory. Quadratic structure, factorisation and non–factorisation phenomena, and the existence of a canonical conjugation arise as formal consequences of bounded definability, ultrafilter aggregation, and positivity of the induced kernel. From this perspective, Hilbert space appears as a representational completion of definability–theoretic data, uniquely determined (up to isometric isomorphism) by the chosen comparison invariant.

From a logical perspective, the significance of the construction lies in showing that bounded definability over a very low level of the Jensen hierarchy already suffices to enforce a rigid representation–theoretic geometry. The conclusions are formal and structural; questions of interpretation or dynamics lie outside the scope of the present note.

Several questions remain open. These include a systematic analysis of classes of locally definable comparison invariants, a finer investigation of the dependence of the resulting Hilbert–space representations on the choice of ultrafilter, and the identification of further structural features arising purely from definability–theoretic considerations. More broadly, the construction suggests a general representation–theoretic principle in which coherent global structure emerges from bounded definability via aggregation, inviting further study in other definability–controlled settings.

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