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Advances in quantum metrology

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The statistical error in any estimation can be reduced by repeating the measurement and averaging the results. The central limit theorem implies that the reduction is proportional to the square root of the number of repetitions. Quantum metrology is the use of quantum techniques such as entanglement to yield higher statistical precision than purely classical approaches. In this Review, we analyse some of the most promising recent developments of this research field and point out some of the new experiments. We then look at one of the major new trends of the field: analyses of the effects of noise and experimental imperfections.

Any measurement process can be divided into three distinct sections: the preparation of a probe, its interaction with the system to be measured and the probe readout. This process is often plagued by statistical or systematic errors. The source of statistical errors can be accidental (for example, resulting from insufficient control of the probes or the measured system) or fundamental (for example, resulting from Heisenberg uncertainty relations). Whatever their origin, we can reduce the effect of statistical errors by repeating the measurement and averaging the outcomes. This is a consequence of the central limit theorem, which states that the average of a large number n of independent measurements (each having a standard deviation $\Delta\sigma$) will converge to a Gaussian distribution with standard deviation $\Delta\sigma/\sqrt{n}$, so that the error on average scales as $n^{-1/2}$. Although many different definitions of n are used in the literature, the safest is to consider n as the number of interactions between the probes and the system to be measured; namely n is the number of times that the system is sampled. The statistical scaling of errors with $n^{-1/2}$ is referred to as the 'standard quantum limit' (SQL) or 'shot noise' in quantum optics, and assumes no more than classical statistical correlations between different probes, which are typically uncorrelated. It therefore fails to fully exploit the quantum nature of the system and probes. A more favourable statistical scaling of errors can be achieved if quantum effects such as entanglement are used to correlate the probes before letting them interact with the system to be measured[1–6], allowing the SQL boundary to be surpassed through non-classical strategies. Nevertheless, quantum mechanics still sets ultimate limits in precision through Heisenberg-like uncertainty relations, which are typically referred to as 'Heisenberg bounds' and scale as n^{-1}. Quantum metrology — part of the emerging field of quantum technology[7] — aims to study these bounds and the quantum strategies that allow us to attain them. More generally, quantum metrology deals with the measurement and discrimination procedures that receive some kind of enhancement (in precision, efficiency or simplicity of implementation, for example) through the use of quantum effects.

This Review aims to summarize some of the most recent developments in the field. A more historical and didactic perspective on quantum metrology can be found in ref. 1. We start by introducing some important results in quantum estimation theory that focus on optimization of the probe readout. We then report some recent findings obtained in the context of parameter estimation for channels, which also enables optimization of the probe preparation. It is only at this stage that Heisenberg-like scaling is obtained. Schemes based on filtering protocols and nonlinear effects will also be introduced. Finally we will deal with how the fragile quantum metrology protocols can be analysed in the presence of noise.

Optimized quantum measurements

In its simplest version, a typical quantum estimation problem[8–14] consists of recovering the value of a continuous parameter x (say the phase φ of Fig. 1) encoded in a fixed set of states ρ_x of a quantum system S. The quantum estimation problem can be described as a two-step process, in which we first perform a measurement on S and then extrapolate the value of x with some data-processing of the measurement results. To identify the most general connection between quantum states and measurement results, it is useful to express the measurement in terms of a set of Hilbert space operators $\{E_y^{(n)}\}$ that form a positive operator valued measure (POVM) $\varepsilon^{(n)}$, where n is the number of copies of ρ_x used to obtain a single result y. The POVM defines the conditional probability of obtaining the outcome y when the initial state is ρ_x, as $p_n(y|x) = \mathrm{Tr}(E_y^{(n)}\rho_x^{\otimes n})$. Data-processing the result y will yield our estimate z of the value of x and, in the most general case, will be characterized by assigning some conditional probability $p_{\mathrm{est}}^{(n)}(z|y)$. Ideally, we would like z to be as close as possible to x; unfortunately, however, depending on the physics of the problem and possibly on the selected estimation strategy, there will often be some residual uncertainty. This uncertainty is fully characterized by the probability

$$P_n(z|x) := \sum_y p_{\mathrm{est}}^{(n)}(z|y)\, p_n(y|x)$$

which describes the statistical dependence of z on the true value x. The root-mean-square error (RMSE) for this uncertainty is therefore

$$\delta X_n := \sqrt{\sum_z [z-x]^2 P_n(z|x)}$$

which provides a good measure of the estimation accuracy[8] (different measures are possible[11] but will not be treated here). Assuming that the estimation function is asymptotically locally unbiased, δX_n can be shown to obey the Cramér–Rao bound[15], given by

$$\delta X_n \geqslant 1/\sqrt{F_n(x)} \qquad (1)$$

where

$$F_n(x) := \sum_y \left(\frac{\partial p_n(y|x)}{\partial x}\right)^2 \Big/ p_n(y|x)$$

is the Fisher information associated with the selected POVM

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measurement $\varepsilon^{(n)}$. The asymptotically locally unbiased assumption for which equation (1) holds requires that as $n \to \infty$ there exists at least one value of x for which the estimator tends to the correct value, and at this point the derivative of the estimator is unity[8]. All reasonable estimators satisfy this condition, whereas pathological ones do not (for example, trivial estimation procedures that produce the correct value of x by chance, as is the case of a stopped clock that is correct twice per day). A generalization of equation (1) that applies to all estimation strategies without the assumption of being locally unbiased was obtained[9,10] by replacing δX_n on the left-hand side of equation (1) with its regularized version

$$\delta\widetilde{X}_n := \sqrt{\sum_z [\widetilde{z}(x) - x]^2 P_n(z|x)}$$

where $\widetilde{z}(x)$ is the quantity z divided by

$$\frac{\partial}{\partial x}\sum_z z P_n(z|x)$$

Even though the inequality for $\delta\widetilde{X}_n$ refers to a larger number of estimation strategies than for $\delta\widetilde{X}_n$, it is weaker because $\delta\widetilde{X}_n \geq \delta X_n$. Furthermore, the above regularization implies that $\delta\widetilde{X}_n$ will diverge for pathological estimators that would be excluded by the restrictions used to derive equation (1).

Equation (1) provides a lower bound for the accuracy achievable when using a chosen measurement scheme formally represented by the POVM $\varepsilon^{(n)}$, which enters in the definition of the conditional probability $p_n(y|x)$. To get an accuracy bound that is independent of such a choice, one should minimize the resulting expression with respect to all conceivable measurements. In quantum mechanics this is equivalent to optimizing the right-hand side of equation (1) with respect to all possible POVMs[8–14], giving

$$\delta X_n \geqslant \frac{1}{\sqrt{\max_{\varepsilon^{(n)}}[F_n(x)]}} \geqslant \frac{1}{\sqrt{nJ(\rho_x)}} \qquad (2)$$

The term on the right is known as the quantum Cramér–Rao (q-CR) bound, which is obtained by exploiting the upper bound of $\max_{\varepsilon^{(n)}}[F_n(x)]$ in terms of the quantum Fisher information $J(\rho_x) := \mathrm{Tr}[R_{\rho_x}^{-1}(\rho_x')\rho R_{\rho_x}^{-1}(\rho_x')]$, where $\rho_x' = \partial\rho_x/\partial x$ and

$$R_{\rho_x}^{-1}(O) := \sum_{j,k:\,\lambda_j+\lambda_k\neq 0}\left(\frac{2O_{jk}|j\rangle\langle k|}{(\lambda_j+\lambda_k)}\right)$$

is the symmetric logarithmic derivative[8] written in the basis that diagonalizes

$$\rho_x = \sum_j \lambda_j |j\rangle\langle j|$$

For instance, consider the case in which ρ_x are pure states of the form $|\Psi_x\rangle = e^{-iHx}|0\rangle$, with H being a Hermitian operator and $|0\rangle$ being a reference vector. The q-CR bound then takes the simple form of an uncertainty relation[8–11]:

$$\delta X_n \geqslant \frac{1}{2\Delta H\sqrt{n}} \qquad (3)$$

where

$$\Delta H := \sqrt{\langle (H - \langle H\rangle)^2\rangle}$$

is the spread of H on $|0\rangle$.

To fully appreciate the significance of the q-CR bound given above, it is important to consider three properties. First, the bound in equation (2) holds for all possible POVMs, including those that operate jointly on the n copies while exploiting entanglement resources. Second, the SQL scaling $n^{-1/2}$ on the right-hand side of equation (2) is a direct consequence of the additivity of the quantum Fisher information when applied to tensor states $\rho_x^{\otimes n}$ (that is, $J(\rho_x^{\otimes n}) = nJ(\rho_x)$). Third, in the asymptotic limit of large n, the q-CR bound is always achievable and the estimation strategy that attains it can be constructed through local measurements and adaptive estimators — a strategy that uses only local operations and classical communication (LOCC)[12,13,16–20]. This implies that entangled resources at the measurement stage are not necessary to achieve the q-CR bound. Local measurements and some clever classical data-processing are sufficient. It also shows that the quantity $(nJ(x))^{-1/2}$ has a clear operational meaning and can be used to quantify the difficulty of the estimation problem. For the case in which the parameter to be estimated depends on time (waveform estimation), an extension of the theory presented here is given in ref. 21.

Finally, it is also useful to briefly consider the multiparameter case, in which x is a vector of random variables[8,11–14]. Although in this case one can construct an inequality in terms of the associated quantum Fisher information, in general this will not coincide with the ultimate achievable bound. This inequality applies only to all n-body separable measurements[16] — POVMs that may act globally on $\rho_x^{\otimes n}$ but whose elements can be expressed as convex combinations of tensor products of positive operators (a larger class than LOCC[22]). If we instead allow generic joint measurements that exploit entangled resources in a non-trivial fashion, then the bound may not hold and better performance can be expected, even though the $n^{-1/2}$ scaling still holds.

Optimized input states and measurements

Analysis in the previous section focused on scenarios in which the set of states $\rho_x^{\otimes n}$ is fixed by the estimation problem, and can therefore only be used to describe the last stage (probe readout) of a general measurement. Because equations (2) and (3) obey a SQL scaling law of $n^{-1/2}$ to obtain a quantum-metrology-type enhancement, we also need to consider the probe preparation and interaction stages. For this purpose it is useful to describe the correspondence $x \to \rho_x$ in terms of a quantum channel Φ_x that produces ρ_x when acting on the initial input state ρ_0 of the probe, through the mapping $\Phi_x(\rho_0) = \rho_x$. The aim of this section is to characterize the best estimate of x that can be obtained for a fixed number n of applications of Φ_x, while optimizing the measurements, the estimation functions and choice of the initial state[23–40]. This captures the basic aspects of most quantum metrology applications[1].

Figures 2 and 3 show that we can employ different strategies depending on the initial state of the probes, the measurement scheme and the operation to be performed on the system (either parallel or sequential)[34]. In full analogy with quantum channel communication theory, we can also consider entanglement-assisted schemes — the most general method for performing measurements — in which the incoming probes are entangled with an external ancillary system A that is not affected by the channel Φ_x. Enhancements in accuracy through entanglement-assisted schemes for parallel configurations have been derived for special classes of quantum channels Φ_x (refs 23,25,26,36,37).

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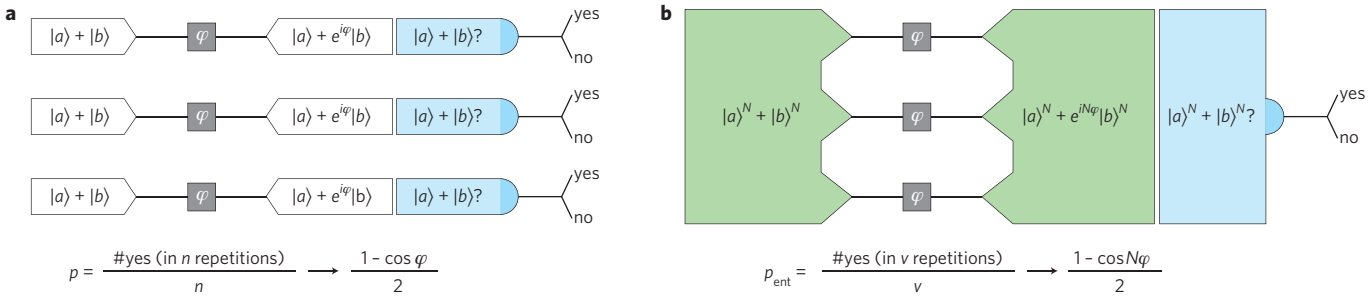

Figure 1 | Ramsey interferometry. The aim of Ramsey interferometry is to measure an unknown relative phase φ picked up by two orthogonal states ($|a\rangle$, $|b\rangle$) of an atomic probing system. This procedure can be generalized to other interferometric measurements such as frequency-standards, magnetometry and optical phase[2]. **a**, In a conventional set-up, 'probe preparation' consists of producing each atom in the superposition $|\Psi_{in}\rangle = (|a\rangle + |b\rangle)/\sqrt{2}$, which yields the output state $|\Psi_{\varphi}\rangle = (|a\rangle + e^{i\varphi}|b\rangle)/\sqrt{2}$ after the probing stage (shown as grey boxes). 'Readout' consists of checking whether $|\Psi_{\varphi}\rangle$ is still in the initial state $|\Psi_{in}\rangle$, which occurs with probability $p = |\langle\Psi_{in}|\Psi_{\varphi}\rangle|^2 = (1 - \cos\varphi)/2$. Thus, by taking the ratio between the number of successes and the total number of readouts, we can recover the phase φ. If we repeat this measurement n times, the associated error on our estimation of φ can then be evaluated using the standard deviation on the determination of p and by error propagation theory to obtain an SQL scaling of $n^{1/2}$. **b**, The quantum-enhanced case. A simple quantum strategy consists of dividing the n probes into groups of N, prepared in an entangled state $(|a\rangle^{\otimes N} + |b\rangle^{\otimes N})/\sqrt{2}$. Because each of the N vectors $|b\rangle$ acquires a relative phase φ, the final state is $(|a\rangle^{\otimes N} + e^{iN\varphi}|b\rangle^{\otimes N})/\sqrt{2}$. The probability that this state is equal to the initial one is now $p_{ent} = (1 - \cos N\varphi)/2$. Because we have $\nu = n/N$ groups of probes, we can repeat this procedure ν times with the same resources to obtain an error of $\delta\varphi_n = 1/\sqrt{(nN)}$, which is an $N^{1/2}$ increase in precision over the previous case, namely the Heisenberg bound, which scales as $1/(N\sqrt{\nu})$ (refs 3,4).

Optimizing with respect to all available resources can be a difficult problem. For simplicity we will focus on the parallel strategies in Fig. 2 while also briefly discussing sequential strategies in the caption of Fig. 3. In this context the conventional approach is to split the optimization into two stages[23,25,26,36,37]: first fixing a (possibly entangled) state $\rho_0^{(n)}$ of the n probes and optimizing with respect to the measurements, and then minimizing the overall error with respect to $\rho_0^{(n)}$. From the results of the previous section it follows that the accuracy of the first stage will not be larger than the q-CR bound associated with the selected state. Thus, the second optimization yields

$$\delta X_n \geqslant \min_{\rho_0^{(n)}} \left[\frac{1}{\sqrt{J(\rho_x^{(n)})}} \right] \tag{4}$$

where $J(\rho_x^{(n)})$ is the quantum Fisher information of the output state $\rho_x^{(n)}$. Because the function J is convex[23], the minimum is always achieved by choosing pure states. In addition, owing to the additivity properties of J, it is clear that by restricting the minimization in equation (4) to input states that are separable with respect to n probes, as in the classical input, classical output (CC) or classical input, quantum output (CQ) strategies of Fig. 2, one can achieve a maximum scaling of $n^{-1/2}$ (SQL)[34]. Achieving a better scaling requires the use of entangled states, as in the quantum input classical output (QC) or quantum input quantum output (QQ) strategies of Fig. 2. To exemplify this, we focus on the special case in which the channel Φ_x performs random unitary rotations of e^{-iHx}. This is a paradigmatic model, which, among others, includes the Ramsey set-up of Fig. 1 and the Mach–Zehnder interferometer[1,2,41,42]. Assume that the total number $n = \nu N$ of probes is split into $\nu \gg 1$ groups of N elements, where each group is prepared in the same entangled input state $|\Psi_0^{(N)}\rangle$. The output state of all n probes after passing through the black-boxes is $|\Psi_x^{(N)}\rangle^{\otimes\nu} := (e^{-iHx})^{\otimes\nu}|\Psi_0^{(N)}\rangle^{\otimes\nu}$. Equation (4) can then be evaluated using equation (3), yielding[34]

$$\delta X_n \geqslant \frac{c}{N\sqrt{\nu}} \tag{5}$$

where c is the inverse of the largest gap in the spectrum of the generator H of the transformation Φ_x. For $N > 1$, equation (5) presents a $N^{1/2}$ enhancement with respect to the SQL. The term on the right, known as the Heisenberg bound, can be attained by preparing the probes of each group in the entangled state $|\Psi_0^{(N)}\rangle = (|\lambda_{max}\rangle^{\otimes N} + |\lambda_{min}\rangle^{\otimes N})/\sqrt{2}$, where $|\lambda_{max}\rangle$ and $|\lambda_{min}\rangle$ are the eigenvectors associated with the maximum and minimum eigenvalues of H, respectively. For the Ramsey set-up, $|\Psi_0^{(N)}\rangle$ corresponds to the maximally entangled state $(|a\rangle^{\otimes N} + |b\rangle^{\otimes N})/\sqrt{2}$ of Fig. 1. In the literature of atomic physics, where $|a\rangle$ and $|b\rangle$ refer to different atomic levels, $|\Psi_0^{(N)}\rangle$ has characteristics analogous to 'spin-squeezed states'[43,44] or 'Schrödinger's cat states'[45–47], which have been experimentally demonstrated using trapped ions[45–48], Bose–Einstein condensates[49], macroscopic atomic ensembles[50] and NMR[51]. From the previous section it follows that equation (5) is asymptotically attainable through a maximum-likelihood approach for $\nu \gg 1$ by exploiting POVMs that act locally on the blocks. Alternatively[34], equation (5) is also achievable through adaptive strategies based on a POVM that acts locally on each probe, so that the yield of QC strategies is similar to that of QQ strategies.

In contrast, the situation is more complicated for finite ν — specifically for $\nu = 1$ — where equation (5) would yield the impressive n^{-1} scaling for the RMSE of the Heisenberg bound. This problem was analysed in refs 29,38 for the special case in which Φ_x induces unitary transformations, where a scaling of the order of $n^{-1}\log(n)$ was achieved through local adaptive strategies. In ref. 38 it was also observed that if the Φ_x channels are programmable, then a sub-shot-noise scaling of the accuracy is not allowed (Φ_x are 'programmable' if they can be expressed in terms of a constant interaction with an external ancillary system B whose initial state encodes a dependence on the parameter x; that is, $\Phi_x(\bullet) = \mathrm{Tr}_B\{V[\sigma_x \otimes (\bullet)]V^+\}$, where V is a unitary transformation acting on B and S, σ_x is the state of B, and Tr_B is the partial trace of B; ref. 52). Examples for which the no-go theorem of ref. 38 hold are provided by classical channels (quantum channels that cannot carry quantum information) and depolarizing channels[37]; for these families of maps the best estimation accuracy of the parameter x will scale at most as $n^{-1/2}$. It was also recently pointed out that for finite ν, the two-step optimization approach adopted in the derivation of equation (4) in general fails to provide the achievable bound[24,53]. Instead, one must adopt a min–max optimization scheme that minimizes the maximum RMSE, which always yields attainable bounds. This method gives slightly worse performance than the two-step approach, but,

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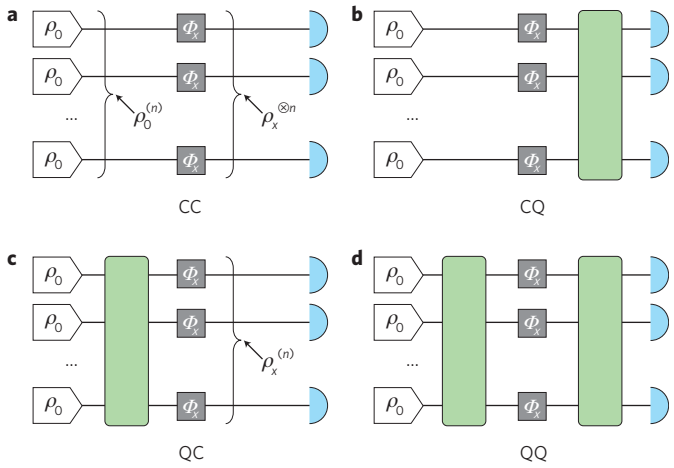

Figure 2 | Schematic representation of parallel estimation strategies. The white wedges represent the input probes entering the apparatus in a separable joint state $\rho_0^{(n)}$, the grey boxes represents channels, the blue semicircles describe local measurements on the probe and green boxes represent an entangling operation among the probes. **a**, In CC strategies, the probes are prepared in a separable state $\rho_0^{(n)}$ (although not necessarily all in the same state, as depicted here), and LOCC measurements are made at the output. In this case, averaging the local results yields a decrease in the result's precision that scales (at most) as the SQL — that is, as $n^{-1/2}$. **b**, In CQ strategies, entanglement among the probes is generated just before detection. **c**, In QC strategies, entanglement among the probes is generated before they are fed into the channel, with no entanglement resource employed at the detection stage. **d**, The most general is the QQ strategy, in which entanglement can be used both at the probe preparation and detection stages. By construction, the QQ strategy provides the best performance, and the CC strategy provides the worst. When estimating unitary channels, CQ has the same $n^{-1/2}$ yield as CC, whereas QC and QQ can both achieve the Heisenberg bound n^{-1} (ref. 34). Non-unitary channels have equal or worse performance (as any non-unitary map can be purified into a unitary), depending on their action on the probes[34,38].

apart from numerical prefactors, maintains the same n^{-1} scaling, at least when the quantum channels Φ_x induce unitary transformations. It is also worth mentioning that analogous n^{-1} scalings for specific problems of quantum channel estimation have been obtained using error measures that are substantially different from the RMSE benchmark adopted here. From such results one would not be able to infer the exact n^{-1} scaling for the RMSE, which follows directly from the q-CR bound. References 28,30–33 in particular consider the problem of estimating unitary rotations in finite-dimension systems using a metric introduced by Holevo[11].

Applications in quantum interferometry

The prototypical example of the quantum estimation procedure in an interferometric application is provided by the Mach–Zehnder interferometer. In this set-up, two input optical modes merge at a 50/50 beamsplitter, propagate along two paths of different lengths to accumulate an unknown relative phase shift φ, and then merge at a second 50/50 beamsplitter. The goal is to recover the value of φ by measuring the signals emerging from the interferometer, while employing a limited amount of resources by setting an upper limit N on either the maximum number or the average number of photons entering the interferometer during each experimental run. Using a and b to indicate the annihilation operators associated with the two internal paths of the interferometer, the problem of recovering φ reduces to estimating a channel $\Phi_{x=\varphi}$ that induces a unitary rotation e^{-iHx}, with $H = (a^{+}a - b^{+}b)/2$ being the effective system Hamiltonian[2,41,54]. Through this technique, the two-step optimization strategy that brought us to equation (5) can be used to set a lower bound on the RMSE. First, consider the situation in which the generic POVM measurements are performed on ν independent preparations of the interferometer. In this case, equation (3) yields the following bound:

$$\delta\varphi_\nu \geqslant \min_{|\Psi\rangle}\left[\frac{1}{2\Delta H\sqrt{\nu}}\right] \tag{6}$$

where the minimization is performed over the set of input states $|\Psi\rangle$ satisfying the selected photon number constraint (either the maximum number or the average number) and ΔH is the associated energy spread. Under both constraints, an optimal input $|\Psi\rangle$ is provided by a state that, at the level of the internal modes of the interferometer, can be expressed as a 'NOON' state[2,55] — a superposition of the form $(|N,0\rangle + |0,N\rangle)/\sqrt{2}$, in which N photons are propagating along the first or the second optical path[56]. NOON states are the formal analogue of the highly entangled superposition states that achieve the Heisenberg bound in a Ramsey configuration (Fig. 1). In fact, NOON states exhibit a special sensitivity with respect to the transformation that encodes the random variable φ, and get transformed into output states $(e^{-i\varphi N/2}|N,0\rangle + e^{i\varphi N/2}|0,N\rangle)/\sqrt{2}$, where the phase φ is effectively multiplied by a factor N. Equation (6) then yields a lower bound of the form

$$\delta\varphi_\nu \geqslant \frac{1}{N\sqrt{\nu}} \tag{7}$$

which, for any given N, is achievable in the limit of large ν, for example, through maximum-likelihood estimation based on photo-counting statistics at the output ports of the interferometer. Equation (7) shows a $N^{-1/2}$ enhancement over standard estimation approaches in which, for instance, the input ports of the interferometer are fed with coherent states of average photon number N. Such procedures show a SQL scaling of $\delta\varphi_\nu = 1/\sqrt{(\nu N)}$, in which all the νN photons contribute independently to the estimation process. For this reason equation (7) can be seen as the quantum optical counterpart of the Heisenberg bound of equation (5).

The attainability of equation (7) requires certain extra considerations. First, it assumes the ability to create NOON states. This is possible through rather complicated optical schemes[57–61] that, although are highly refined, rely on post-selection[60,62–66]. States that possess a high fidelity with NOON states for large N can be obtained simply by mixing a squeezed vacuum state and a coherent state at a beamsplitter[67–69]. Introducing such states at the input of a Mach–Zehnder interferometer allows a scaling of N^{-1} (ref. 5). Achieving this, however, requires a proper estimation process[70] — a point that may not be sufficiently clear in parts of the literature, where sub-optimal performance can often be traced back to poor processing of the measurement outcomes. Analogous N^{-1} performance can also be achieved by employing different sources and/or by using estimator functions that are simpler than the maximum-likelihood approach. These schemes are typically based on adaptive strategies, in which the parameter φ is pushed towards an optimal working point that guarantees high performance. Two recent proposals of this are given in refs 71,72, whereas older demonstrations are provided in refs 1,2. Alternative schemes are based on sequential strategies (Fig. 3), in which a single photon pulse recursively probes the phase shift φ by passing through the delay line multiple times[29,34,38–40].

It is also non-trivial to achieve equation (7) for finite values of

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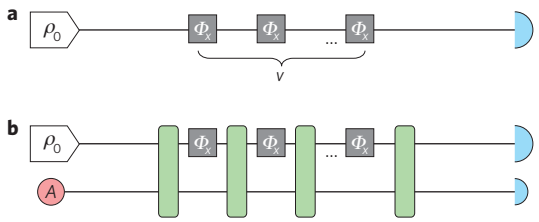

Figure 3 | Sequential strategies. a, In a sequential (or 'multi-round') scheme[29,34,38–40], a single probe state sequentially samples the n grey boxes. **b**, In an entanglement-assisted sequential scheme, an external ancillary system A aids the estimation. These schemes have some advantages over parallel schemes because in principle they do not require the use of multipartite entanglement among n different probes to achieve sub-shot-noise scaling. Furthermore, they can be used to simulate other schemes by proper tuning of the intermediate controls. The most general estimation scheme is a combination of the sequential and parallel schemes of Fig. 2. An optimization over the possible dispositions is presented in ref. 33.

ν (refs 4,42,53,54,56,70,73–76). Several numerical analyses support the evidence that Heisenberg-like N^{-1} scaling should be achievable in the limit of large N for $\nu = 1$, where equation (7) would yield a scaling analogous to the strong Heisenberg bound. In particular, refs 54,70,74,75 study the asymptotic behaviour of the error confidence by adopting a Bayesian estimation strategy[42]. Recent work by Hayashi[53], however, seems to settle this problem by showing that, although for $\nu = 1$ the bound of equation (7) is not exactly achievable, one could still reach an asymptotic N^{-1} scaling for the RMSE by adopting a min–max optimization approach[24].

Filtering protocols

Creating the complex and fragile quantum states necessary for achieving quantum-enhanced metrology is a troublesome process. Alternative protocols create robust classical states and then filter the high-resolution states at the measurement stage. This idea is based on post-selection[77,78]: once a high-resolution quantum state has been detected at the output, one can interpret the whole experiment as having employed such a state since the input.

The filtering process means that some of the resources available at the input are wasted; the system is sampled with many more resources than those actually used for the parameter estimation. Moreover, because the system is sampled with classical states, which have no quantum correlations, it is clear that the SQL cannot be beaten — there is no increase in resolution over the optimal classical strategy that could employ all the resources, without filtering. For this reason, filtering protocols cannot be considered proper quantum metrology protocols, according to the definition given above. Nevertheless, they can be extremely useful in the common case when it is impractical to use optimal classical strategies that employ all the resources used to sample the system. Moreover, in practical situations, considerations of efficiency rarely play a role, but robustness to noise is paramount. Classical states are, by definition, more robust than quantum states. In addition, there are situations in which filtering methods can achieve tasks that would be impossible using purely classical strategies, and post-selecting high-resolution quantum states is often very simple.

Reference 78 provides the theory of filtering protocols for phase estimation, along with the conditions necessary to distinguish super-sensitivity (the error in the estimation being lower than allowed when using classical resources) from super-resolution (the error in the estimation being lower than allowed when using a classical procedure that exploits only the resources retained by the filtering process). Filtering protocols can achieve super-resolution if an appropriate measuring strategy is used, but can never properly achieve super-sensitivity. A super-resolution experiment is presented in ref. 78, where it is pointed out that no experiments so far have achieved super-sensitivity in optical phase estimation, as they have all exploited some form of post-selection.

Nonlinear estimation strategies

Several authors[79–96] have recently considered the possibility of using nonlinear effects to go beyond the N^{-1} Heisenberg-like scaling in phase estimation problems. These new regimes are referred to as 'super-Heisenberg' in ref. 84, but proper accounting of the resources shows that they are fully compatible with the analysis presented in the previous sections[97]. In fact, it is important to emphasize that in our treatment n and N are connected to the number of times the system is sampled. To perform significant comparisons among different quantum and classical strategies, it is important to specify what the relevant resources are for each strategy, such as the number of probes, required energy and measurement duration time. Ultimately, the idea of nonlinearity-based proposals is to consider settings in which the unitary transformation that 'writes' the unknown parameter x into the probing signals is characterized by many-body Hamiltonian generators that are no longer extensive functions of the number of probes employed in the estimation[82–84,86–88,90,91,94–96], or, for the optical implementations that yielded equation (6), in the photon number operator of the input signals[79–81,85,86,89,92,93]. Consequently, in these set-ups, the mapping $(e^{-iHx})^{\otimes n}$ acting on the input states $\rho_0^{(n)}$ is replaced by a transformation of the form $e^{-iH^{(n)}x}$ that couples the probes non-trivially. In this context, the minimization of equation (6) is no longer forced to adhere to equation (7). For instance, RMSE with scalings of the order of N^{-k} can be obtained when using Hamiltonians that involve k-system interactions between the probes[82], whereas a 2^{-N} scaling can be achieved by introducing an exponentially large number of coupling terms[86]. Proposed implementations include scattering in Bose condensates[84,88], Duffing nonlinearity in nanomechanical resonators[85], two-pass effective nonlinearity with an atomic ensemble[87], Kerr-like nonlinearities[79–81,89,93] and nonlinear quantum atom–light interfaces[94,95].

Quantum metrology in the presence of noise

The study of noisy quantum metrology is a special case of parameter estimation in which the map Φ_x describing the encoding of the unknown parameter x also contains a description of the noise. Even though very few general results are known about the use of quantum metrology in the presence of noise, many of the findings detailed in the previous sections can be used to characterize noise effects. Several examples in which Heisenberg scaling can be retained even in the presence of noise have been discussed in ref. 38. However, application of the programmable criterion introduced in ref. 38 shows that even a small amount of depolarizing noise is sufficient to ruin any sub-shot-noise performance. Typical quantum metrology protocols indeed seem to be extremely sensitive to noise. Does this then imply that quantum advantages cannot be attained in any practical metrology application[68,98–100]?

As a paradigmatic example, consider the case of phase estimation in optical systems. Here, the incoherent loss of a single photon transforms a NOON state into a statistical mixture of $(|N{-}1,0\rangle\langle N{-}1,0| + |0,N{-}1\rangle\langle 0,N{-}1|)/2$, which is useless for phase-sensing. Is it, therefore, ever possible to outperform classical strategies in practice[100]? Surprisingly, it has been shown that it is asymptotically possible to do so, but only by a constant factor in parallel strategies[101,102]; for any nonzero loss, for a sufficiently high number of photons, the scaling of the optimal phase sensing is proportional to the scaling of the shot noise ($N^{-1/2}$). Although this means that quantum approaches are useful in highly controlled environments[102] (such as for gravitational wave detection[5]), they only allow for very small enhancements in free-space target

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acquisition[102]. The shot-noise limit can be beaten[103], however, and the optimal states to achieve this for parallel strategies in the presence of loss have been calculated numerically using various optimization techniques for a fixed number of input photons[104,105] and for photon-number detection[106]. A post-selected proof-of-principle experiment that employs some of these optimal states was recently performed[66]. For very low loss levels, NOON states retain their optimality[104,106] and can be approximated by states that are easy to generate[67,68]. Researchers recently proposed a very simple proposal based on parametric downconversion that can be realized without post-selection[72], achieving the Heisenberg bound for low loss and degrading slowly for increasing noise levels. In the case of significant loss at the detection stage, achieving the shot-noise limit might be difficult because most of the photons that interacted with the phase shifter and contain phase information are lost. A simple strategy that amplifies the signal before detection and can asymptotically achieve the shot-noise limit was experimentally tested in ref. 107 for single-photon probe states. In contrast, the optimal states for sequential (or 'multi-round') interferometry[39,40] have not been obtained so far in the presence of noise[100]. However, more robust single-mode states have also been analysed as an alternative to fragile two-mode states, including pure Gaussian states in the presence of phase diffusion[108], mixed Gaussian states in the presence of loss[109] and single-mode variants of two-mode states[110].

We now briefly present the quantum protocols for estimating phase and frequency in the presence of dephasing, as it highlights some of the subtleties encountered by quantum metrology in the presence of noise. Phase φ and frequency ω are related simply by $\varphi = \omega t$. Phase estimation is robust, giving it a quantum advantage even in the presence of dephasing. However, frequency estimation[3] is much less robust[111], and for any nonzero value of dephasing, the maximally entangled state ceases to present any advantage over a classically correlated state[111]. An analogous result also exists for magnetometry[112].

Phase can be estimated using the Ramsey set-up of Fig. 1. For the case of separable probes (Fig. 1a), the probability of finding the probe in its initial state in the presence of dephasing becomes $p = (1 + e^{-\gamma t}\cos\varphi)/2$, where $\gamma \geq 0$ quantifies the dephasing rate and t is the time elapsed from preparation of the state until measurement readout. Analogously, for the case of N entangled probes (Fig. 1b) we find $p_{\mathrm{ent}} = (1 + e^{-N\gamma t}\cos N\varphi)/2$, where the factor N in the exponent results from the entangled state's exponential sensitivity to dephasing. For phase estimates, it is possible to assume that the time between preparation and measurement is arbitrarily short such that no dephasing takes place, allowing the typical $N^{1/2}$ enhancement of quantum metrology to be recovered in the presence of dephasing. In contrast, for frequency estimates[3,111], the error diverges for $t \to 0$ because $\Delta\omega = \Delta\varphi/t$. For $\gamma = 0$, we have $\Delta\omega = 1/(t\sqrt{n})$ for the separable strategy and $\Delta\omega_n = 1/(tN\sqrt{\nu})$ for the entangled strategy. Optimization over t says that the measurement time must be as long as possible. The entangled protocol is $\sqrt{N}$ times more precise than the separated protocol for the same t, thereby allowing the quantum enhancement to persist. This enhancement disappears suddenly for $\gamma > 0$ as the optimal measurement times are different for the separable and the entangled case; the entangled state decays N times faster (as emphasized by the time dependence of p_{ent}) so the entangled strategy requires a measurement time that is N times shorter, thus negating the N-fold quantum advantage. However, the N-fold speed advantage of the entangled procedure allows us to repeat it N times and then average the results to recover an error of $1/\sqrt{N}$, matching that of the separable procedure and thus removing all quantum advantage. In refs 111,113 it is shown that non-maximally entangled states can be used to beat the separable procedure by ~40%. Note, however, that if the observation time t in atomic clocks is constrained by experimental issues (typically by fluctuations in the local oscillator), then entanglement allows a sub-shot-noise scaling[114,115] of $N^{-2/3}$. A generalization of the frequency measurement theory is given in ref. 116, which also covers many noisy estimation measurements that consider the measurement duration time and production rate of the probes to be relevant resources.

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Additional information

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