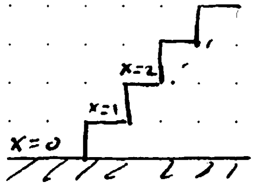
30 AUG, 2018

PROOF BY INDUCTION:

↳ PROVE THAT IT HOLDS ON GROUND (BASE CASE)

↳ PROVE THAT IT CAN GO UP BY 1



Ex: $\sum_{k=1}^{n}(2k-1)=n^2$ $\quad n=1,2,3\ldots$

1) BASE CASE: $n=1$ $\quad 2(1)-1=1^2$ ✓

2) ASSUME TRUE FOR $n-1$

$$\sum_{k=1}^{n-1}(2k-1)=(n-1)^2$$

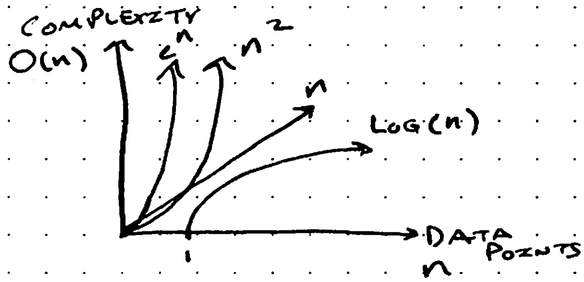
3) PROVE FOR n: $\sum_{k=1}^{n}(2k-1)=n^2$

↳ $$\sum_{k=1}^{n}(2k-1)=\underbrace{\sum_{k=1}^{n-1}(2k-1)}_{(n-1)^2}+(2n-1)$$

$$\sum_{k=1}^{n}(2k-1)=(n-1)^2+(2n-1)$$

$$= n^2\cancel{-2n+1}\cancel{+2n-1}$$

$$= n^2 \quad ✓$$



Ex: $n! > 2^n$; $n\geq 4$

BASE CASE: $n=4$ $\quad 4! > 2^4$

$24 > 16$ ✓

ASSUME $(n-1)! > 2^{(n-1)}$

PROVE FOR n: $n! = (n-1)!\,n > 2^{(n-1)}n > 2^{(n-1)}2 = 2^n$

B/C $n \geq 4 > 2$

$$n! > 2^n \; ; \quad n\geq 4$$

FOURIER TRANSFORM (HW1.1)

CONTINUOUS

↳ $F(u) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\underbrace{e^{-j2\pi ux}}_{\text{TRANSFORM KERNEL}}dx$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$$

$$= \cos\theta - j\sin\theta$$

$$e^{j\theta}+e^{-j\theta} = \cos\theta + j\sin\theta + \cos\theta - j\sin\theta$$

$$= 2\cos\theta = e^{j\theta}+e^{-j\theta}$$

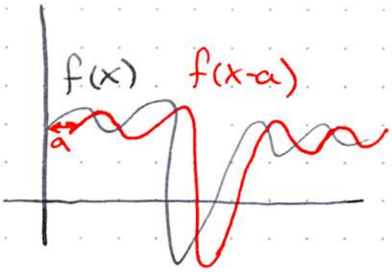
$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$e^{j\theta}-e^{-j\theta} = -2j\sin\theta$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$

$$F(n) = Re\{F(n)\} + Im\{F(n)\}$$

$$= |F(n)|e^{j\angle F(n)}$$



$$\mathcal{F}\{f(x-a)\} = e^{-j2\pi u a} F(u)$$

PROVE: SET S W/ n ELEMENTS HAS 2^n SUBSETS

INDUCTION:

BASE CASE $n=0$ # SUBSETS $2^0 = 1$, JUST $\emptyset$

$n=1$ # SUBSETS $2^1 = 2$, $\{\emptyset\}$ $\{a\}$

ASSUME: TRUE FOR $n-1$ ELEMENTS 2^{n-1} SUBSETS

WHEN S HAS n ELEMENTS, I.E. $(n-1)$ ELEMENTS + ONE MORE

↳ ALL SUBSETS OF $(n-1)$ ELEMENTS, 2^{n-1} ∴ ELEMENT b

PLUS ALL ABOVE W/ b ADDED, ANOTHER 2^{n-1}

TOTAL: $2(2^{n-1}) = 2^n$

FUNCTIONS & RELATIONS

~~RELATIONS~~ RELATIONS

↳ LET X & Y BE NONEMPTY SETS

↳ ANY SUBSET OF $X \times Y$ IS A RELATION FROM X TO Y

↳ ANY SUBSET OF $X \times X$ IS A RELATION IN X

↳ ~~RELATION~~ OR

GRAPHS

A GRAPH CONTAINS AN EVEN # OF ODD DEG. VERTICES

BY CONTRADICTION: ASSUME ∃ A GRAPH W/ AN ODD # OF ODD VERTICES

DUE TO HANDSHAKING THEOREM:

$$\sum_{1} \deg(v) = 2|E| = \text{EVEN}$$

Σ DEG. OF EVEN VER. = EVEN ODD # OF VERT.

Σ TOTAL = ΣODD + ΣEVEN ⇒ ODD CONTRADICTS HST

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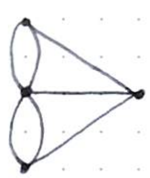
4 SEPT, 2018

7 BRIDGES OF KÖNIGSBERG:



CAN I START/END @ SAME VERTEX WHILE CROSSING EACH EDGE ONCE?

↳ NO, B/C 4 ~~EDGES~~ ODD EDGES

SIMPLE GRAPH:

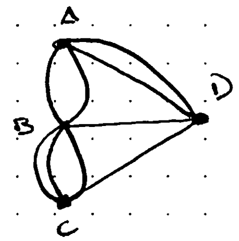
NO LOOPS

COMPLETE GRAPH:

EVERY NODE IS CONNECTED

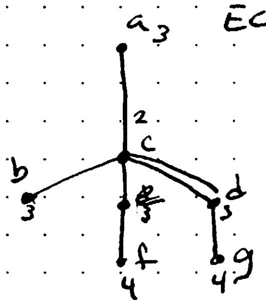
HANDSHAKING THM : DIRECTED GRAPH → ONLINE NOTES

EULERIAN PATHS & CYCLES
- ↳ VISITS EACH EDGE ONCE
- ↳ START/END SAME VERTEX



MODEL SO ALL EDGES ARE EVEN →
- ↳ AD, BC
- ↳ AB, CD

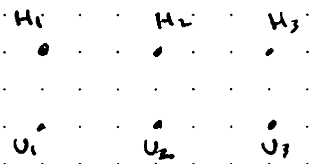
GRAPH ECCENTRICITY & CENTER
- ↳ DISTANCE MEASURED BY # OF EDGES BTWN 2 VERTICES (GEODESIC)
- ↳ ECCENTRICITY OF V IS MAX GEODESIC DISTANCE FROM V TO ANY VERTEX



ECC (a) : MAX(GEOD(a, ALL OTHERS)) = 3

CENTER OF GRAPH = LOWEST ECCENTRICITY

BIPARTITE GRAPH
- ↳ UNDIRECTED GRAPH WHICH CAN BE SPLIT INTO 2 SETS



CONNECT 3 HOUSES TO 3 UTILITIES W/O CROSSING LINES
- ↳ CAN'T BE DONE

K_3, K_4 - BOTH PLANAR
- ↳ NO LINES CROSSING
- ↳ CAN BE DRAWN W/ ONLY STRAIGHT LINES

6 Sep, 2018

GRAPH COMPLEMENT
↳ HAS ALL EDGES MISSING FROM ORIGINAL GRAPH

GRAPH COUNTING
↳ HOW MANY DIFFERENT SIMPLE GRAPHS HAVE n NODES ?

A GRAPH W/ n NODES CAN HAVE $\frac{n(n-1)}{2}$ EDGES
(n-1) EDGES PER NODE, n NODES
↳ DIVIDE BY 2 TO REMOVE DOUBLE COUNTS

THERE CAN BE AT LEAST $2^{\frac{n(n-1)}{2}}$ GRAPHS
↳ FORMULA ACCOUNTS FOR ISOMORPHIES

TREES
SPANNING TREE
↳ GRAPH THAT SPANS ALL VERTICES OF A GRAPH

EULER TOUR
↳ GO THROUGH ALL EDGES ~~ONCE~~ & RETURN HOME
↳ EULERIZE - ADD EDGES TO GRAPH TO MAKE ALL VERTICES EVEN

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11 SEPT. 2018
HW 2 DUE THURS
HW 3 DUE SEP 20 → NO CLASS, TURN HW IN TO MAIN OFFICE

MON SEP 24, 7-9 PM MAKE UP CLASS (MOSTLY REVIEW)
~~TEST EITHER ON TUES, SEP 25 OR THURS, SEP 27~~
TEST SEPT. 25

HW 2:

2) BIPARTITE FINITE GRAPH
IFF NO CYCLES OF ODD LENGTH

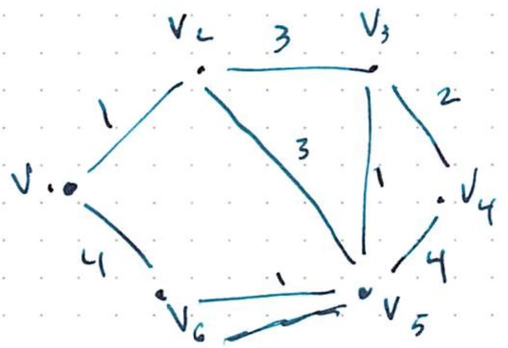
NEED TO PROVE THAT IF BP THEN NO ODD CYCLES
& IF NO ODD CYCLES THEN BP

GRAPH ALGORITHMS:
↳ DIJKSTRA
↳ SHORTEST DIST FROM START TO EVERY OTHER NODE

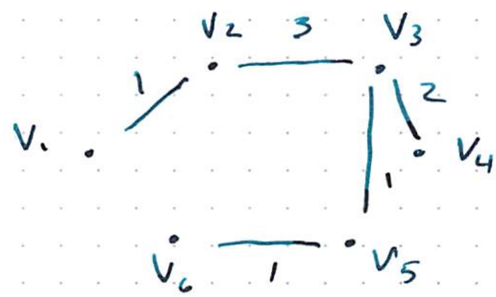
DYNAMIC PLANNING:
- REQUIRES BOOKKEEPING
- WORK BACKWARDS FROM END TO FIND MOST OPTIMAL PATH

PICK MIN
- COST(F) + 7
- COST(E) + 5 —
  - COST(C) + 3 + 5 —
  - COST(D) + 5 + 5
  - COST(G) + 7 + 5
- COST(G) + 6

TREES:
- NO CYCLES
  - ACYCLIC
- SPANNING TREE
  - FOR A GIVEN GRAPH THERE ARE USUALLY SEVERAL SPANNING TREES
  - A SPANNING TREE COVERS ALL VERTICES OF A GRAPH
- MINIMUM SPANNING TREE
  - SUM OF EDGE WEIGHT IS MINIMIZED
  - 2 ALGORITHMS (PRIM'S - GREEDY, KRUSKAL'S - NOT...?)
    - GREEDILY PICK ALL LOWEST-WEIGHT CONNECTED EDGE
      - IF IT WOULD MAKE A LOOP, SKIP IT
    - KRUSKAL'S
      - RANK ORDER ALL EDGES LOW → HIGH
      - ADD EDGES UNTIL TREE IS COMPLETE
        - SKIP EDGES WHICH WOULD MAKE LOOPS
    - ALGS MIGHT MAKE DIFF. TREES, BUT WILL ALWAYS GIVE THE SAME # RESULT



1, 1, 1, 2, 3, 3, 4, 4



KRUSKAL MINIMUM
(EITHER '3' WOULD WORK)

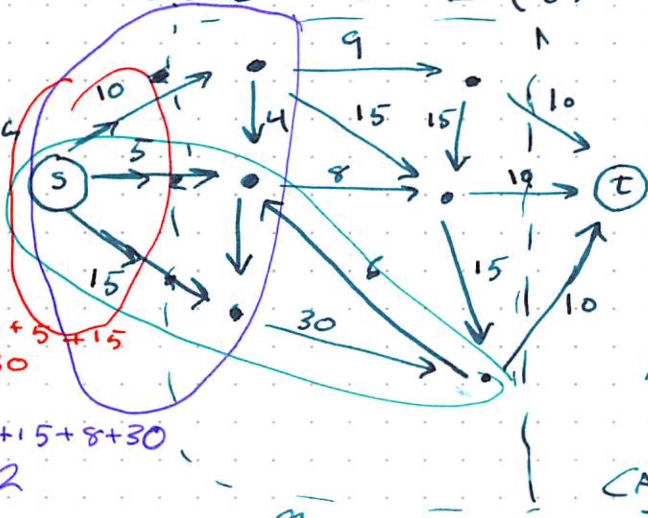
NETWORK MAX FLOW

↳ FLOW NETWORK

↳ START NODE = SOURCE (S)

↳ END NODE = SINK (t)

CAP GOING OUT FROM S



NETWORK

$\text{CAP} = 10 + 5 + 15 = 30$

$\text{CAP} = 9 + 15 + 8 + 30 = 62$

$\text{CAP} = 10 + 8 + 10 = 28$

MAX FLOW ⟺ MINIMUM CUT

MAX FLOW IS DETERMINED BY SLOWEST CONNECTION

AN S-t CUT IS A PARTITION (A, B) OF V w/ $s \in A$ & $t \in B$

CAPACITY OF A CUT(A, B) IS $\text{CAP}(A, B) = \sum_{E \text{ OUT OF } A} c(E)$

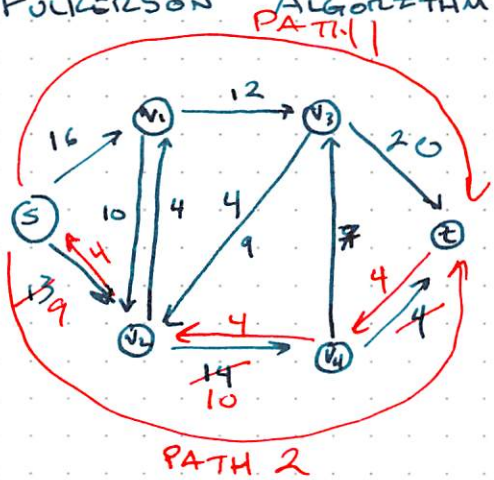
LOOK @ ALL PARTITIONS TO ~~FEN~~ FIND MIN CUT

ONE CUT MUST CONTAIN S & THE OTHER t

SEP. 13

FORD FULKERSON ALGORITHM

Ex:



PATH 1: S V_1 V_3 t <u>12</u>

PATH 2: S V_2 V_4 t <u>4</u>

CHECK PPT FOR ALGORITHM.

SEP. 18

TUES, SEPT. 25 - TEST 1
- 5 PROBLEMS
- HW 1, 2, 3
- ONE HANDWRITTEN NOTE PAGE

REVIEW: MON, SEPT. 24
7:00 - 8:15 PM

TRAVELLING SALESMAN SOLUTION METHODS

- BRUTE FORCE: $(n-1)!$
  - CRAZY BAD IDEA
  - COMPUTATIONALLY SLOW
- OPTIMIZING METHODS
  - PRETTY MUCH STILL BRUTE FORCE
- HEURISTICS
  - OBTAIN "GOOD" SOLUTIONS "QUICKLY" BY INTUITIVE METHODS
  - NOT GUARANTEED TO BE OPTIMAL
    - GREEDY?
    - EULER TOUR
    - DIJKSTRA
    - MST
    - BFS
    - DFS

GRAPH → MST → EULER TOUR
- SKIP RE-USED EDGES

MAPS:

REGION → NODE
BORDER → EDGE

FOUR COLOR THEOREM
- THE CHROMATIC # OF A PLANAR GRAPH IS NO GREATER THAN 4

SEP. 24

EXAM REVIEW

5 PROBLEMS - HW 1, 2, 3
1 DOUBLE SIDED PAGE OF NOTES

PROOFS:
↳ DIRECT
↳ CONTRAPOSITIVE (IF NOT Q → NOT P)
↳ CONTRADICTION
↳ CASES
↳ INDUCTION

GRAPHS:
↳ SIMPLE, COMPLETE, CONNECTED, BIPARTITE
↳ ECCENTRICITY, CENTER, DIAM, RADIUS
↳ ADJACENCY MATRIX
↳ INCIDENCE MATRIX
↳ PLANARITY, ISOMORPHISM
↳ HANDSHAKING THEOREM
↳ CYCLES, PATHS, EULERIAN PATHS
↳ ROUTE INSPECTION, DFS, BFS
↳ DIJKSTRA
↳ MIN SPANNING TREE
↳ MAX FLOW
↳ ASSIGNMENT HUNGARIAN ALG.
↳ TSP

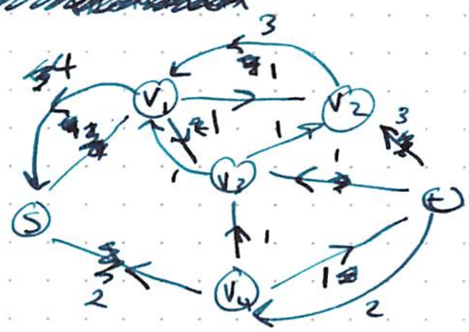
HIGH #s FOR MIN.

	A	B	C	D
F	29	30	18	~~[illegible]~~ 50
G	35	32	20	~~100~~ 50
U	25	25	14	~~100~~ 50
W	50	36	16	~~100~~ 50

ROW RED. → SUBTRACT MIN FROM EACH ROW
COL RED. → SUB MIN FROM EACH COL

	A	B	C	D
F	0	1	0	2
G	5	1	0	0
U	1	0	0	6
W	24	9	0	4

[redacted]



PATH 1: $S \to V_4 \to t$: 2

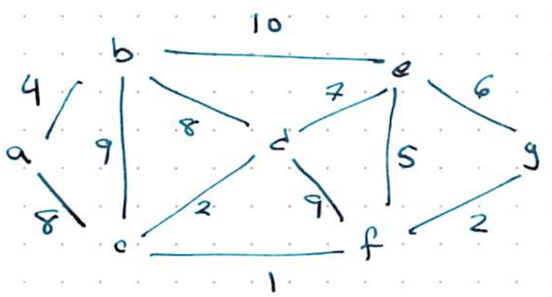
P2 : $S \to V_1 \to V_2 \to t$: 3

P_3 : $S \to V_1 \to V_3 \to t$: 1

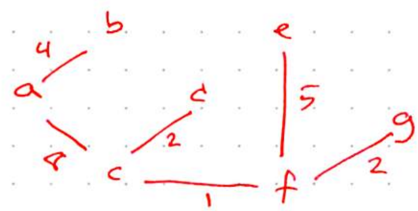
6

MIN CUT: (S), (t, V_4)

MST:



PRIM: START: a



TOTAL COST: 22

EULER TOUR - LOOK AT HW 3

PROOF - INDUCTION:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

BASE CASE: $n = 2$, $1+2 = 3$ ✓

$$\frac{2(2+1)}{2} = 3$$

ASSUME HOLDS FOR n-1:

$$\sum_{k=1}^{n-1} k = \frac{(n-1)n}{2} \to \text{PROVE FOR } n$$

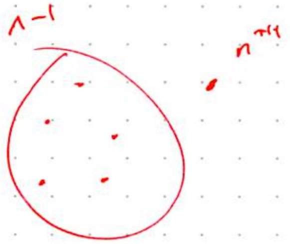
$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{n-1} k\right) + n = \frac{(n-1)n}{2} + n = \boxed{\frac{n(n+1)}{2}} \checkmark$$

G IS A SIMPLE, UNDIRECTED, CONNECTED GRAPH w/ n VERT-
MUST HAVE AT LEAST $(n-1)$ EDGES

BASE CASE: $n=2$

ASSUME TRUE FOR $(n-1)$:

FOR $(n-1)$ VERT. I NEED $(n-2)$ EDGES



FOR n VERT:

$(n-1)+1 = n$ VERT
$n-(2+1) = n-1$ EDGES

CASES:

PROVE: $n(n-1)$ IS EVEN

n IS ODD: $n = 2K+1$
$(2K+1)(2K) = 4K+2K = 6K$

n IS EVEN: $n = 2K$

$2K(2K-1) \quad 4K-2K = 2K$

$2K$ IS EVEN

SEPT. 27, 2018

PROBABILITY

BAYES RULE: $P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$

OCT [illegible], 2018

$$[x_1 \ \ x_2] \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + \tfrac{1}{2} x_2^2$$

$$[x_1 \ \ x_2] \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + \frac{1}{2}x_2 \end{bmatrix}} = x_1^2 + 2x_1x_2 + 2x_1x_2 + \tfrac{1}{2} x_2^2$$

OCT 4, 2018

$$\mathrm{VAR}(X) = E(X^2) - E(X)^2 = E((x-\mu)^2)$$

$$\mathrm{COV}(X,Y) = E\big((x-\mu_x)(y-\mu_y)\big)$$

$$\mathrm{CORRELATION}(X,Y) = \frac{\mathrm{COV}(X,Y)}{\sqrt{\mathrm{VAR}(X)}\sqrt{\mathrm{VAR}(Y)}}$$

UNCORRELATED RVs:

$\mathrm{COV}(X,Y) = 0$

$E\big((x-\mu_x)(y-\mu_y)\big) = 0$

$E\big(xy - \mu_x y - \mu_y x + \mu_x\mu_y\big) = 0$

$= E(xy) - \mu_x E(y) - \mu_y E(x) + \mu_x\mu_y = 0$

$E(xy) - \mu_x\mu_y - \mu_y\mu_x + \mu_x\mu_y = 0$

$E(xy) = E(x)E(y)$

JOINT RVs

FOR n RVs: $X_1, X_2, \ldots, X_n$

JOINT pdf IS $p(x_1, x_2, \ldots, x_n) = \mathrm{Prob}\begin{cases} X_1 = x_1 \\ \vdots \\ X_n = x_n \end{cases}$

Ex) P(DIE, COIN)

	1	2	3	4	5	6
H						
T						

* FOR INDEP RVs: $p(x_1 x_2) = p(x_1)p(x_2)$

FOR CONTINUOUS CASE:

JOINT pdf $f(x_1, \dots, x_n)$

$$\underset{x_1}{\int_{-\infty}^{\infty}} \underset{x_2}{\int_{-\infty}^{\infty}} \cdots \underset{x_n}{\int_{-\infty}^{\infty}} (f(x_1, \dots, x_n))\, dx_n \cdots dx_1 = 1$$

FOR INDEP. $f(x, y) = f(x)f(y)$

$f(x_1, \dots, x_n) = f(x_1) \cdots f(x_n)$

IID - INDEPENDANT, IDENTICALLY DISTRIBUTED VARIABLES

↳ $f(x_1 \dots x_n) = f(x_1) \cdots f(x_n)$

UNCORRELATED $\overset{?}{\Rightarrow}$ INDEPENDANT ✗ NOT ALWAYS

INDEPENDANT $\overset{?}{\Rightarrow}$ UNCORRELATED ✓

ASSUME INDEP RVs x, y : $f(x y) = f(x)f(y)$

CHECK CORRELATED:

$$E(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x y)\, f(x,y)\, dx\, dy \overset{?}{=} E(x)E(y)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x)\, y f(y)\, dx\, dy$$

$$= \underbrace{\int_{-\infty}^{\infty} x f(x)\, dx}_{E(x)} \underbrace{\int_{-\infty}^{\infty} y f(y)\, dy}_{E(y)}$$

$= E(x)E(y)$ ✓

MARGINAL PDF

GIVEN $f(x, y)$ JOINT PDF, FIND $f(x)$, $f(y)$

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

FIND $E(x)$, GIVEN $f(x, y)$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

1) FIND MARGINAL PDF $f_x(x)$

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \quad \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 2 \end{array}$$

$$f_x(x) = \int_{-\infty}^{\infty} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy = \frac{6}{7}\int_0^2 x^2\, dy + \frac{6}{7}\int_0^2 \frac{x}{2} y\, dy$$

↳ $E(x) = \frac{6}{7}\int_0^1 x\,(x + 2x^2)\, dx$

$= \frac{5}{7}$

Ex) $f(x,y) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 0 \le y \le 1 \end{cases}$ VALID?

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \overset{?}{=} 1$$

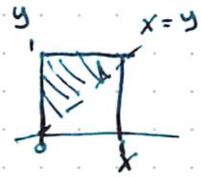
$Pr(x \le a \text{ AND } y \le b) \overset{?}{=} a \cdot b \cdot 1 = ab$ WHERE $0 < a < 1$, $0 < b < 1$

$Pr(x \le a) = 1 \cdot 1 \cdot a = a$

$Pr(y \le b) = 1 \cdot 1 \cdot b = b$

~~Pr(x≤y)~~

$P(x \le y) = 1/2$

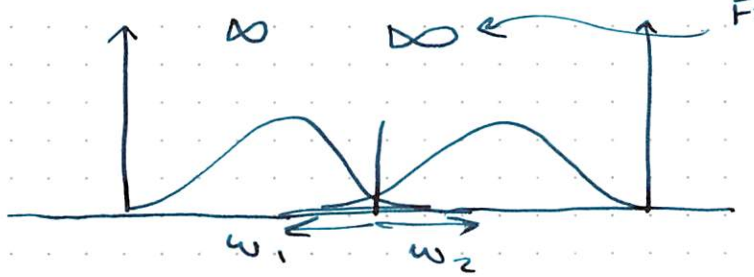


$$P(x \le y) = \int_0^1 \int_0^y 1\,dx\,dy = \int_x^1 \int_0^1 1\,dx\,dy$$

BAYESIAN DECISION

↳ MAKE DECISION ON A TWO CLASS PROBLEM

FISH, NOT INFINITY...



OCT 16, 2018

UNCORRELATED:

$Cov(x,y) = 0$

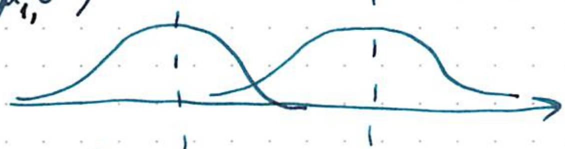
$E(xy) = E(x)E(y)$

INDEPENDANT:

$f(x,y) = f(x)f(y)$

BAYESIAN DECISIONS

$N(\mu_1, \sigma^2)$ $P(\omega_1)$ $P(\omega_2)$ $N(\mu_2, \sigma^2)$



MEASURE x

↳ DOES x BELONG TO ω_1 OR ω_2

$$P(\omega_1|x) \underset{\omega_2}{\overset{\omega_1}{\gtrless}} P(\omega_2|x)$$

BAYES RULE

~~$P(\omega_1|x)P(\omega_1) / P(x)$~~

$$p(\omega_1|x) = \frac{p(x|\omega_1)\,p(\omega_1)}{p(x)}$$

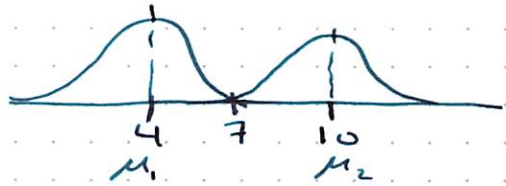
$$\frac{P(x|\omega_1)\,p(\omega_1)}{p(x)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{p(x|\omega_2)\,p(\omega_2)}{p(x)}$$

$$p(x|\omega_1)\,p(\omega_1) \underset{\omega_2}{\overset{\omega_1}{\gtrless}} p(x|\omega_2)\,p(\omega_2) \Rightarrow$$

BAYESIAN DECISION MAKING:

$$\boxed{\frac{p(x|\omega_1)}{p(x|\omega_2)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{p(\omega_2)}{p(\omega_1)}}$$

LIKELIHOOD RATIO TEST



Ex) $p(x|\omega_1) \sim N(4,1)$
$p(x|\omega_2) \sim N(10,1)$

$p(\omega_1) = p(\omega_2) = \frac{1}{2}$

GAUSSIAN: $\frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\mu)^2}{2}\right)$ $p(x|\omega_1)$

ω_1: $\frac{1}{\sqrt{2\pi(1)^2}} \exp\left(-\frac{(x-4)^2}{2}\right)$ $p(x/\omega_2)$

ω_2: $\frac{1}{\sqrt{2\pi}(1)^2} \exp\left(-\frac{(x-10)^2}{2}\right)$

LRT: $\Lambda(x) = \frac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-4)^2}{2}\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-10)^2}{2}\right)} \underset{\omega_2}{\overset{\omega_1}{\gtrless}} \frac{1/2}{1/2} = 1$

SOLVE FOR x

$$\exp\left(-\tfrac{1}{2}(x-4)^2 + \tfrac{1}{2}(x-10)^2\right) \underset{\omega_2}{\overset{\omega_1}{\gtrless}} 1$$

$$x \underset{\omega_2}{\overset{\omega_1}{\lessgtr}} 7$$

PROBABILITY OF ERROR

P_{ERR} (ACTUAL ω_1, DECISION ω_2) $= \frac{1}{2}\int_7^\infty p(\omega_1|x)\,dx$

P_{ERR} (ACTUAL ω_2, DECISION ω_1) $= \frac{1}{2}\int_{-\infty}^7 p(\omega_2|x)\,dx$

MAXIMUM LIKELIHOOD ESTIMATION (MLE)

ESTIMATE PARAMETERS FROM DATA (MEASUREMENTS)
CONSIDER MEASUREMENTS OF RV x: $\{x_1 \dots x_n\}$

THE PDF HAS A KNOWN FORM $f(x,\alpha)$ AND WE WANT TO ESTIMATE THE PARAM α
↳ USE MLE TO EST. α

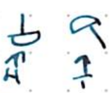
$$\underbrace{p(\theta|x)}_{\text{PROB. AFTER } x \text{ IS MEASURED, POSTERIOR}} = \frac{p(x|\theta)\,p(\theta)}{p(x)}$$

x: MEASUREMENT

LIKELIHOOD

PRIOR

Ex) FLIP A THUMB TACK



$p(\text{HEADS}) = \theta$
$p(\text{TAILS}) = 1-\theta$

5 TIMES! $\underbrace{\{H \; H \; T \; H \; T\}}_{\text{IID: INDEP. IDENT. DIST.}}$

$P(\text{DATA}|\theta) = P(H|\theta)P(H|\theta)P(T|\theta)P(H|\theta)P(T|\theta)$
$= \theta\cdot\theta(1-\theta)\cdot\theta\cdot(1-\theta)$

LIKELIHOOD $L(\theta) = \text{Prob}(\text{DATA}|\theta) = \theta^3(1-\theta)^2$

SELECT $\theta = \theta^*$ SUCH THAT THE LIKELIHOOD IS MAXIMIZED

$$\theta^* = \arg\max_\theta \{\theta^3(1-\theta)^2\} \Rightarrow \arg\max_\theta \{P(D|\theta)\}$$

MAX $\longrightarrow \frac{d}{d\theta}\{\theta^3(1-\theta)^2\} = 0$

SOLVE FOR θ

SOLVE USING LOG LIKELIHOOD

$$\theta^*_{MLE} = \arg\max_\theta \{\ln P(D|\theta)\}$$

$$\frac{d}{d\theta}\{\ln(\theta^3(1-\theta)^2)\} = 0$$

$$\frac{d}{d\theta}\{\ln\theta^3\} + \frac{d}{d\theta}\{\ln(1-\theta)^2\} = 0$$

$$\Rightarrow \boxed{\theta = \frac{3}{5}}$$

EX) LET $f(x,\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$ ESTIMATE μ

TAKE n MEASUREMENTS OF x: $x_1, \dots, x_n$; IID

FORM LIKELIHOOD FUNCTION:

$$L = \underbrace{f(x_1, x_2, \dots, x_n; \mu; \sigma^2)}_{\text{(JOINT pdf)}} = \prod_{i=1}^n f(x_i; \underbrace{\mu}_{\text{EST}}; \underbrace{\sigma^2}_{\text{KNOWN}})$$

$$= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x_i-\mu)^2}{2\sigma^2}\right)$$

LOG LIKELIHOOD:

$$\ln(L) = \ln\left(\prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x_i-\mu)^2}{2\sigma^2}\right)\right)$$

$$= \sum_{i=1}^n \ln\left(\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)\right)$$

$$= \sum_{i=1}^n \left(\ln\frac{1}{\sigma\sqrt{2\pi}}\right) + \sum_{i=1}^n \left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) = \operatorname{Log}(L)$$

$$\frac{\partial}{\partial\mu}\{\ln(L)\} = 0 \Rightarrow \underbrace{\frac{\partial}{\partial\mu}\left(\sum_{i=1}^n \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)\right)}_{0} + \sum_{i=1}^n \frac{-1}{2\sigma^2}\frac{d}{d\mu}\left((x-\mu)^2\right) = 0$$

$$= \sum_{i=1}^n 2(x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^n x_i = \sum_{i=1}^n \mu$$

$$= n \cdot \mu \Rightarrow \mu_{MLE} \frac{1}{n}\sum_{i=1}^n x_i$$

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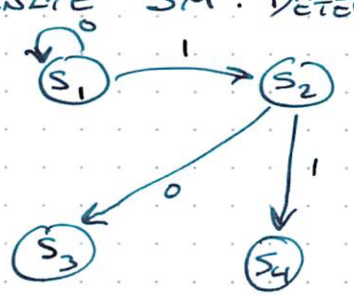
MARKOV MODEL:

FINITE SM: DETERMINISTIC



MARKOV CHAINS

SET OF STATES: $\{S_1 \ldots S_r\}$

MOVE FROM ONE ~~STEP~~ STATE TO ANOTHER (STEP)

W/ SOME PROBABILITY OF TRANSITION

P_{ij} IS PROB FROM STATE i TO STATE j

Ex) SIMPLE WEATHER MODEL

N: SUNNY → R OR S P =

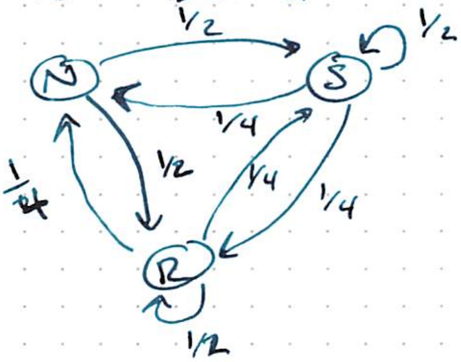
R: RAINY → R = 1/2 OR N OR S = 1/4

S: SNOW → S = 1/2, R OR N = 1/4

$$\begin{array}{c c} & \begin{array}{ccc} N & R & S \end{array} \\ \begin{array}{c} N \\ R \\ S \end{array} & \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \end{array}$$

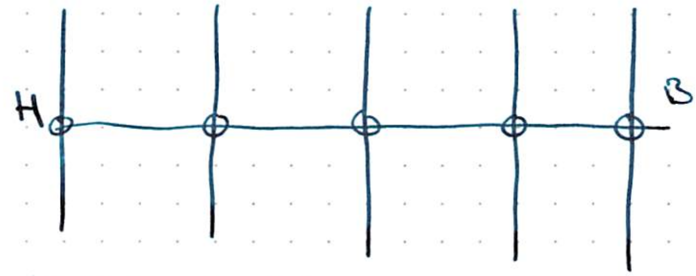
PROBABILITY TRANSITION MATRIX

STATE DIAGRAM:

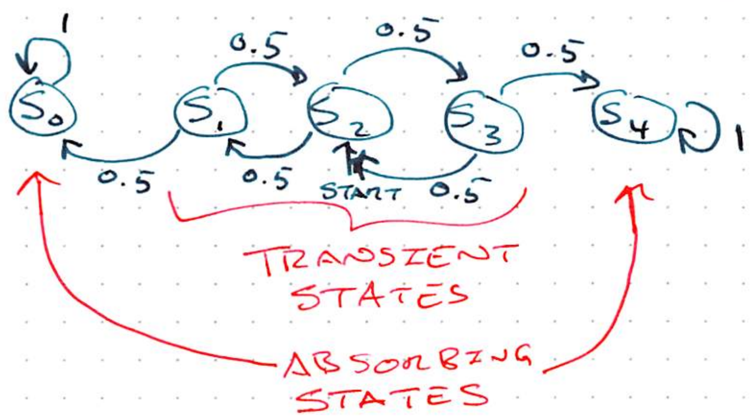


FOR A MARKOV MODEL, THE PROB. OF TRANSITION DEPENDS ON THE CURRENT STATE (OR A FINITE # OF PAST STATES)

DRUNK'S WALK:



H & B → ABSORBING STATES (STAY FOREVER)



TRANS MATRIX

$$P = \begin{array}{c|ccccc} & s_0 & s_1 & s_2 & s_3 & s_4 \\ \hline s_0 & 1 & 0 & 0 & 0 & 0 \\ s_1 & 0.5 & 0 & 0.5 & 0 & 0 \\ s_2 & 0 & 0.5 & 0 & 0.5 & 0 \\ s_3 & 0 & 0 & 0.5 & 0 & 0.5 \\ s_4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

CANONICAL FORM:

$$P = \begin{array}{cc|ccc|cc} & & \overbrace{s_1 \quad s_2 \quad s_3}^{\text{T}} & & & \overbrace{s_0 \quad s_4}^{\text{ABS}} & \\ \hline \text{TRANS} & s_1 & & & & & \\ & s_2 & & Q & & R & \\ & s_3 & & & & & \\ \hline \text{ABS.} & s_0 & 0 & 0 & 0 & 1 & 0 \\ & s_4 & 0 & 0 & 0 & 0 & 1 \end{array} \leftarrow \text{IDENTITY MATRIX}$$

FUNDAMENTAL MATRIX

THE MATRIX $N = (I - Q)^{-1}$
↳ FUND. MATRIX OF P

THE ENTRY n_{ij} OF N IS THE EXP. # OF TIMES WE GO THROUGH STATE S_j WHEN STARTING IN STATE S_i, BEFORE GETTING ABSORBED

$$P = \begin{array}{c|ccc|cc} & s_1 & s_2 & s_3 & s_0 & s_4 \\ \hline s_1 & 0 & 1/2 & 0 & 1/2 & 0 \\ s_2 & 1/2 & 0 & 1/2 & 0 & 0 \\ s_3 & 0 & 1/2 & 0 & 0 & 1/2 \\ \hline s_0 & 0 & 0 & 0 & 1 & 0 \\ s_4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$N = (I - Q)^{-1}$$

$$= \left\{ \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} \right\}^{-1}$$

$$= \begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & 3/2 & 1 & 1/2 \\ s_2 & 1 & 2 & 1 \\ s_3 & 1/2 & 1 & 3/2 \end{array}$$

TIME TO ABSORBTION

TOTAL # OF STEPS TO ABSORBTION WHEN STARTING @ STATE S_1

$$N = \begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix}$$

STARTING @ S_1, WE SPEND
$n_{11} = \frac{3}{2}$ IN S_1
$n_{12} = 1$ IN S_2
$n_{13} = 1/2$ IN S_3
$\Big\}$ TOTAL TIME $= \Sigma \boxed{= 3}$

IN GENERAL: TIME TO ABS. (STEPS)

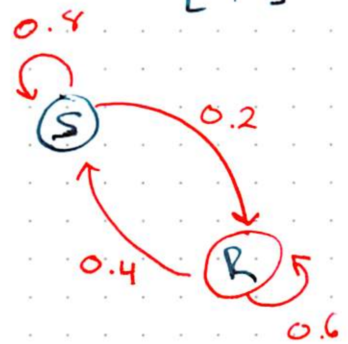
$$t = \begin{bmatrix} N \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{C} = N \cdot C$$

MAT OF 1's

$$= \begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

STEADY STATE BEHAVIOR OF MARKOV CHAINS

$$P = \begin{array}{c|cc} & S & R \\ \hline S & 0.8 & 0.2 \\ R & 0.4 & 0.6 \end{array}$$



AT STEADY STATE, HOW OFTEN DOES IT RAIN (R)?
↳ WHAT IS THE STEADY STATE PROBABILITY π_R AND π_S?

STATE VECTOR (START @ SUNNY)

$$\begin{array}{c} S \\ R \end{array}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\text{STATE}} \underbrace{\begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix}}_{\text{TRANS. MATRIX}} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0.8 & 0.2 \end{bmatrix}}_{\text{NEW STATE}} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} P \end{bmatrix}^2$$

NEXT ↳ $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} P \end{bmatrix}^3$

$$\Rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} P \end{bmatrix}^n$$

@ STEADY STATE P^n, n LARGE
π_1 & π_2 NO LONGER CHANGE

$$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} P \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$$

↳ STEADY STATE

SOLVE

$$\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$$

FOR STEADY STATE

$0.8\pi_1 + 0.4\pi_2 = \pi_1$

$0.2\pi_1 + 0.6\pi_2 = \pi_2$

$\pi_2 = \frac{\pi_1 - 0.8\pi_1}{0.4}$

$\pi_1 = 2\pi_2$

$\pi_1 + \pi_2 = 1$

$\pi_1 = 0.667 \rightarrow 2/3$

$\pi_2 = 0.333 \rightarrow 1/3$

CONFIRM π_1 & π_2 ARE STEADY STATE

↳ $\begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix} \begin{bmatrix} P \end{bmatrix}$

$$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \checkmark$$

OCT 23, 2018

EXAM 2 – THURSDAY NOV 1

↳ REVIEW TUES OCT 30, 7:00 PM

MARKOV:

TRANSITION FROM X_{t-1} TO X_t ONLY DEPENDS ON X_{t-1}

↳ CALLED MARKOV PROPERTY

$$P(\underset{\text{CURRENT}}{X_t} \mid \underset{\text{PREV.}}{X_{t-1}})$$

TRANSITION MATRIX

$$P = \begin{array}{c c} & \begin{array}{cc} \text{TRANS} & \text{ABS.} \end{array} \\ \begin{array}{c} \text{TRANS} \\ \text{ABS.} \end{array} & \left[\begin{array}{c|c} Q & R \\ \hline \emptyset & I \end{array}\right] \end{array}$$

CANONICAL FORM

FUNDAMENTAL MATRIX

$$N = (I - Q)^{-1}$$

$n_{ij} \equiv$ EXPECTED # OF TIMES IN S_j STARTING FROM S_i BEFORE ABSORBTION

GAMBLER'S RUIN PROBLEM

TIME TO ABSORPTION

$$N = s_i \begin{bmatrix} \bar{0} & \bar{0} & \bar{0} \\ & & \end{bmatrix} \qquad \sum_j n_{ij} = \text{TOTAL TIME BEFORE ABSORBTION}$$

PROBABILITY OF ABSORPTION

b_{ij} ELEMENT IN $\underset{t \times r}{B} = \underset{t \times t}{N} \cdot \underset{t \times r}{R}$

b_{ij} IS THE PROB OF ABSORPTION IN ABS. STATE S_j WHEN STARTING AT TRANS. STATE S_i

$$Q = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix}$$

$$B = N \cdot R = \begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 1/2 \end{bmatrix} \\ \begin{matrix} S_0 & S_4 \end{matrix} \end{array}$$

$$\Rightarrow B = \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

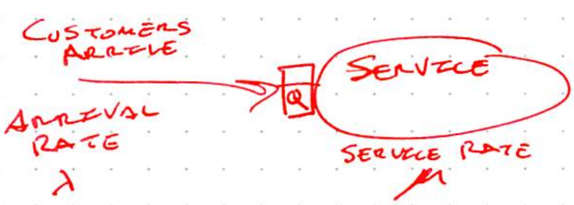
A REGULAR CHAIN IS ONE WHERE P^n HAS ALL NON-ZERO ELEMENTS

HIDDEN MARKOV MODELS

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{t-1} \arrow[r] \arrow[d] & X_t \arrow[d] \\ y_1 & y_2 & & y_{t-1} & y_t \end{tikzcd}$$

OUTPUT/MEASUREMENT TAKES SOME VALUES $\{O_1, O_2, \ldots, O_m\}$

QUEUEING SYSTEMS



ARRIVALS ARE RANDOM & FOLLOW A POISSON PROCESS

$\lambda = \frac{5 \text{ CUSTOMERS}}{\text{HOUR}}$

INTER ARRIVAL TIMES
↳ EXPONENTIAL

[redacted] OCT 25, 2018

QUEUEING EXAMPLE

λ = MEAN ARRIVAL RATE
μ = MEAN SERVICE RATE

$L = \frac{\lambda}{\mu - \lambda}$ $\quad$ $L_Q = \lambda W_Q$

$W = \frac{1}{\mu - \lambda}$ $\quad$ $W_Q = \frac{\lambda}{\mu(\mu - \lambda)}$

Ex)

$\lambda = 20/\text{hour}$ $\quad$ $W = \frac{1}{\mu - \lambda} = \frac{1}{27-20} = \frac{1}{7}$ HOUR = 8.6 MIN

$\mu = 27/\text{HOUR}$

$L = \lambda W = \frac{20}{7} \doteq 2.9$ AIRPLANES

$\mu = 22/\text{HR}$

$W = \frac{1}{\mu - \lambda} = \frac{1}{22-20} = \frac{1}{2} = 30$ MIN

LINEAR ALGEBRA

VECTOR SPACES

SET $\mathbb{X}$ OF VECTORS $x \in \mathbb{X}$
WITH VECTOR ADDITION + $\left(x+y \quad \begin{matrix} x \in \mathbb{X} \\ y \in \mathbb{X} \end{matrix}\right)$

SCALAR MULT $\cdot$ $\left(c \cdot x \quad \begin{matrix} c \in \mathbb{R} \\ x \in \mathbb{X} \end{matrix}\right)$

PROPERTIES

$x + y = y + x$ COMMUTATIVE
$x + (y + z) = (x + y) + z$ ASSOCIATIVE
$\left.\begin{matrix} (\alpha + \beta)x = \alpha x + \beta x \quad \alpha, \beta \in \mathbb{R} \\ (\alpha\beta)x = \alpha(\beta x) \end{matrix}\right\}$ DISTRIBUTIVE
$\vec{0} + x = x$

$0 \cdot x = \vec{0}$
$1 \cdot x = x$

CLOSURE:
$x + y \in \mathbb{X}$ ALL $x, y \in \mathbb{X}$

cx ALL $c \in \mathbb{R}$
$x \in \mathbb{R}$

LINEAR INDEP.

$\alpha x_1 + \alpha x_2 + \dots + \alpha x_n = 0$

IS POSSIBLE ONLY IF $\alpha_i = 0$
$i = 1 \dots n$

IF SOME α_i's ARE NOT ZERO
AND $\sum_{i=1}^{n} \alpha_i x_i = 0$ THEN WE
CAN SOLVE FOR ~~EVERY~~ ONE x_i
IN TERMS OF OTHERS
↳ LINEAR DEPENDANT

GIVEN VECTORS $x_1 \dots x_n$
THE SET OF ALL LINEAR
COMBINATIONS
$\left\{ \sum_{i=1}^{n} \alpha_i x_i \text{ For } \alpha_i \right\}$
↳ IS THE <u>SPAN</u> of $\{x_i\}$

BASES OF A VECTOR SPACE IS THE
SET $\{y_i\}$ SUCH THAT $\{y_i\}$ ARE
SPANNING THE VECTOR SPACE
AND $\{y_i\}$ ARE LINEARLY INDEPENDANT

BASES IS THE $\underbrace{\text{MINIMAL}}_{\text{LIN. INDEP}}$ SPANNING SET
OF THE V.S

THE DIMENSION OF A VECTOR SPACE IS THE # OF VECTORS IN
THE BASIS

BASES IS <u>NOT</u> UNIQUE

CONSIDER SUBSET M OF V.S. X

i.e. IF $x \in M$ THEN $x \in X$
THEN M INHERITS ALL PROPERTIES OF X EXCEPT
FOR CLOSURE (& NULL VECTOR)

EX) POLYNOMIAL OF DEGREE 4
$f(x) = a_0 x^0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

BASIS: $\left\{ x^0, x^1, x^2, x^3, x^4 \right\}$ DIM: 5

$f(x) = 3 - 5x + 2x^2 + x^3 - 9x^4$

$= 3(x^0) - 5(x^1) + 2(x^2) + 1(x^3) - 9(x^4)$

EX) 2x2 MATRICES $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ w/ $a, b, c, d \in \mathbb{R}$

VECTOR SPACE? ✓

BASIS: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

DIM = 4

NORMED VECTOR SPACE

A FUNCTION $\|\cdot\|$ $X \to \mathbb{R}^+$ IS CALLED A NORM (MAGNITUDE) AND SATISFIES

1) $\|X\| = 0$ IFF $X = \vec{0}$

2) $\|cX\| = |c|\,\|X\|$ $c \in \mathbb{R}$

3) $\|X+y\| \leq \|X\| + \|y\|$

EX) CONSIDER $\mathbb{R}^N$

$\|X\|_2 = \left[\sum_{i=0}^{N} (X_i) \right]^2$

$\|X\|_1 = \sum_{i=0}^{N} |X_i|$

$\|X\|_p \left(\sum_{i=0}^{N} |X_i|^p \right)^{1/p}$

OCT 30, 2018

NORM $\|\ \|$ $\mathbb{X} \to \mathbb{R}^+$

a) $\|\vec{x}\| = 0$ IFF $x = \vec{0}$

b) $\|c\vec{x}\| = |c|\,\|\vec{x}\|$ - HOMOGENEITY

c) $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ - SUBADDITIVITY

Ex1) p NORM $\|\vec{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

$\|\vec{x}\|_0 \equiv$ THE # OF NON-ZERO ELEMENTS x_i; WHICH ARE COMPONENTS IN $\vec{x}$

USED FOR SPARSITY; i.e. MOST x_i ARE ZERO & A FEW NON-ZERO USED FOR COMPRESSIVE SENSING

$\|\vec{x}\|_\infty = \text{MAX} \{ |x_1|, |x_2|, \dots, |x_n| \}$

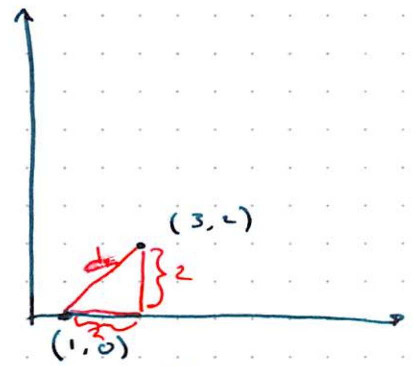
METRIC

$d : X \to [0, \ldots \infty)$

1) $d(x, y) = 0 \iff x = y$
2) $d(x, y) = d(y, x)$
3) $d(x, z) \leq d(x, y) + d(y + z)$ – TRIANGLE INEQ.

Ex) $d_2(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2}$

$d_1(x, y) = \left[\sum_{i=1}^{n} |x_i - y_i| \right]$



$d_1 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = |2| + |2| = 4$

INNER PRODUCT $\langle \cdot, \cdot \rangle$

INNER PRODUCT $\langle x, y \rangle$ OF $x, y \in X$ IS A MAPPING OF $X \times X \to \mathbb{R}$ (OR COMPLEX) THAT SATISFIES:

1) $\langle x, x \rangle \geq 0$
2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \to \alpha \in$ SCALAR
3) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
4) $\langle x, y \rangle = \langle y, x \rangle \;\forall\; \mathbb{R}$

$\langle x, y \rangle = \underbrace{\overline{\langle y, x \rangle}}_{\text{COMPLEX CONJ.}} \;\forall\; \mathbb{C}$

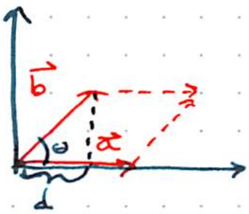
Ex) DOT PROD. $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, y \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots$$

$$= \sum_{i=1}^{n} x_i y_i$$

$$= \begin{bmatrix} x_1 \cdots x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x^T y$$

GEOMETRIC INTERPRETATION:



$a \cdot b \equiv$ SCALAR $= |a||b| \underbrace{\cos\theta}_{\text{PROJECTION}}$

$d = |b| \cos\theta$

Ex) VECTOR SPACE OF MATRICES OF DIMENSION $m \times n$

Σ OF DIAG ELEMENTS

$A_{m \times n}$, $B_{m \times n}$ $\langle A, B \rangle = ? = \text{TRACE}(A^T B)$

$\langle A, B \rangle = \text{TRACE}(A^T B) = a_{11} b_{11} + a_{21} b_{21} + a_{12} b_{12} + a_{22} b_{22}$

$$\underbrace{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \quad \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{matrix}}$$

$$A^T B = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} b_{11} + a_{21} b_{21} & \\ & a_{12} b_{12} + a_{22} b_{22} \end{bmatrix}$$

Ex) VECTOR SPACE OF FUNCTIONS $x(t)$ IN $[0, T]$

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t) y(t) dt$$

ORTHOGONALITY

TWO VECTORS ARE ARTHOGONAL $x \perp y$ IF $\langle x, y \rangle = 0$

RELATE NORM TO INNER PRODUCT

GIVEN INNER PRODUCT $\langle x, y \rangle$
↳ CAN WE INDUCE A NORM?

$$\langle a, a \rangle = a \cdot a$$
$$= |a||a| \cos(0) = |a|^2$$

$$\underbrace{\|x\| = \sqrt{\langle x, x \rangle}}_{\text{INDUCED NORM}}$$

GRAM SCHMIDT → ORTHONORMALIZATION

GIVEN A SET OF VECTORS $\{x_1, \dots, x_n\}$

FIND AN ORTHONORMAL SET OF VECTORS $\{u_1, \dots, u_m\}$ THAT HAVE THE SAME SPAN AS $\{x_1, \dots, x_n\}$

A SET OF VECTORS $\{u_1, \dots, u_m\}$ IS ORTHOGONAL IF $\langle u_i, u_j \rangle = \begin{cases} 0 & i \neq j \\ c & i = j \end{cases}$

ORTHONORMAL IF

$$\langle u_i, u_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

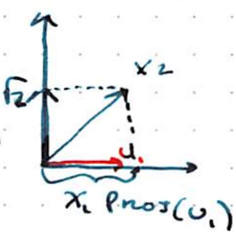
G.S. PROCESS:

1) NORMALIZE THE 1ST VECTOR $\frac{x_1}{\|x_1\|} = u_1$

2) $r_2 = x_2 - x_2 \text{ PROJ}(u_1)$

$r_2 = x_2 - \langle x_2, u_1 \rangle u_1$ $\quad r_2 \perp u_1$

$$u_2 = \frac{r_2}{\|r_2\|}$$



3) $r_3 = x_3 - \langle x_3, u_1 \rangle u_1 - \langle x_2, u_2 \rangle u_2$

$$u_3 = \frac{r_3}{\|r_3\|}$$

Ex) $X_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $X_3 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

1) $U_1 = \frac{X_1}{||X_1||} = \frac{1}{\sqrt{(1)^2 + (2)^2}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

2) $X_2 - \langle X_2, U_1 \rangle \cdot U_1 = r_2$

$\langle X_2, U_1 \rangle = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \sqrt{5}$

$r_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \frac{\sqrt{5}}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$U_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

ATopics Exam 2 Review

Topics: Bayes's Theorem
Total Law of Prob.

HW4, P2: A = Allergy
B = Test (+)

$$P(A|B) = \frac{P(B|A)P(A)}{\underbrace{P(B)}_{\text{Total Law}}}$$

$P(B) = P(A)t_p + P(\bar{A})f_p$

$P(A) = \frac{2}{10000} = 0.0002$

$t_p = 0.98$
$f_p = 0.01$

HW 4, P3: $E(X) = \mu_X$, $E(Y) = \mu_Y$, $Var(X) = \sigma_X^2$, $Var(Y) = \sigma_Y^2$

$Var(X) = E\{(X-\mu)^2\}$

$Var(\underbrace{aX+b}_{\alpha}) = E((\alpha - E(\alpha))^2) = E((aX+b - E(aX+b))^2)$

$= E\{(aX+b)^2\} - (E(aX+b))^2$

$E(aX+b) = aE(X) + \underset{\overset{\parallel}{b}}{E(b)} = aE(X) + b$

$Var(aX+b)$
$\parallel$
$a^2 Var(X)$

$Cov(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\} = E(XY) - E(X)E(Y)$

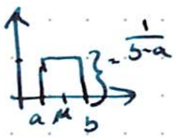
$Var(X-Y) = E\{[(X-Y) - E(X-Y)]^2\}$

$[X - E(X) - (Y - E(Y))]^2$

$= Var(X) + Var(Y) - 2Cov(X,Y)$

$$\mathrm{Cov}(aX+b,\ cY+d) = E\big((aX+b)(cY+d)\big) - E(aX+b)E(cY+d)$$
$$= ac\,\mathrm{Cov}(X,Y)$$

HW4 P4



HW4 P5 - MULTIVARIATE GAUSSIAN

$\cancel{f(\bar{x}) = \frac{1}{(2\pi)^{n/2}|K|^{1/2}} \exp(-\tfrac{1}{2}}$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad \mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix}$$

$\cancel{f(\bar{x}) = \frac{1}{(2\pi)^{n/2}|K|^{1/2}} \exp\left(-\tfrac{1}{2}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)^T\right.}$

Var(X_1)

$$K = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Cov(X_i, X_j)

Var(X_2)

Cov(X_j, X_i)

MUST BE SYMMETRIC

$$f(\bar{x}) = \frac{1}{(2\pi)^{2/2}|K|^{1/2}} \exp\left(-\tfrac{1}{2}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)^T K^{-1}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)\right)$$
$$= \frac{1}{2\pi\sqrt{15}} \exp\left(-\frac{1}{15}\left(2x_1^2 - X_1X_2 + 2x_2^2\right)\right)$$

HW5 P1

10 TRIALS $P(\text{SUCCESS}) = P(\text{DEF}) = 0.1$

$$P(X=k) = \binom{10}{k} p^k (1-p)^{10-k}$$

$$P(X=0) = \binom{10}{0} p^0 (1-p)^{10} = (1-p)^{10} = (0.9)^{10} = 0.3487$$

$$P(X>3) = 1 - (P(X=3) + P(X=2) + P(X=1) + P(X=0))$$

P2 POISSON (DISCRETE)

INTER-ARRIVAL TIMES ARE EXPONENTIALLY DISTRIBUTED (CONT.)

λ AVG. TIME BETWEEN ARRIVALS $= \frac{1}{4}$

$\lambda = 4/\text{HOUR}$

pdf: $f(x) = 4e^{-4x} \quad x \geq 0$

cdf: $\int_{-\infty}^{t} f(x)\,dx = 1 - e^{-4t} = P(X \leq t)$

b) $T_1 = 3$ HOURS

$\lambda = 3(4) = 12/3$ HOURS

$$P(X=0) = \frac{e^{-\lambda_1} \lambda_1^{k=0}}{k!}$$

P3 $\lambda = 1.5/\text{MIN}$

$$P(X=6) = \frac{3^6 e^{-3}}{6!} = 0.0504$$

$\lambda_1 = 3/2\text{MIN}$

Nov 6, 2018

Linear Combination

- $a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_n\vec{x}_n$
- $\sum_{i=1}^{n} a_i\vec{x}_i$
- $= a_1\begin{bmatrix}\vec{x}_1\end{bmatrix} + a_2\begin{bmatrix}x_2\end{bmatrix} + \dots + a_n\begin{bmatrix}x_3\end{bmatrix}$

$$= \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Systems of Equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\Downarrow$$
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Vectors

$$Ax = b \Rightarrow \underbrace{A^{-1}(Ax)}_{= I} = A^{-1}b$$

$$x = A^{-1}b$$

A^{-1} Exists if $\det(A) \neq 0$

~~Columns are linearly indep.~~

Change of Bases

- Vector Space w/ Basis $\{V_1, V_2, \dots, V_n\}$
- Any Vector $x \in X$ is represented as a linear combination of the basis vector

$$x = \sum_{i=1}^{n} a_i V_i$$

↳ In Matrix Notation: $x = \begin{bmatrix} V_1 & V_2 & \dots & V_n \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \Rightarrow x = V \cdot a$

another Basis $\{U_1, \dots, U_n\}$
then X is represented in terms of $\{U_1, \dots, U_n\}$ as

$$X = \sum_{i=1}^{n} b_i U_i \Rightarrow$$

$$X = \begin{bmatrix} U_1 & V_2 & \dots & U_n \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

GIVEN $x = V \cdot a$, $\{v_1, \dots, v_n\}$, a_i

FIND THE REPRESENTATION USING BASIS $\{u_1, \dots, u_n\}$
i.e. FIND b_i SUCH THAT $x = [U][b]$

$$\left.\begin{aligned} x &= Ub \\ x &= Va \end{aligned}\right\} \quad Va = Ub$$

$$b = U^{-1} V a$$

QR FACTORIZATION

$Ax = b$, DECOMPOSE $A = \underset{\text{ORTHOGONAL}}{Q}\ \underset{\text{UPPER TRIANGULAR}}{R}$

START W/ A & OBTAIN MATRIX Q

$$\underbrace{\begin{bmatrix} a_1 & \vdots & \dots & \vdots & a_n \end{bmatrix}}_{A} \qquad \begin{bmatrix} q_1 & \vdots & \dots & \vdots & q_n \end{bmatrix}$$

$$\{a_1 \dots a_n\} \xrightarrow[GS]{} \{q_1 \dots q_n\}$$

SOLVE FOR R

$A = QR$

$QRx = b$, $Q^{-1} = Q^T$

$\underbrace{Q^T Q}_{I} Rx = Q^T b$

Ex) $A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$ TO FIND Q → GS ON COLUMNS $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

1) $x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad u_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

2) $v_2 = x_2 - \langle x_2, u_1 \rangle u_1$

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{8}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$A = QR \qquad Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

SOLVE FOR R

~~Oct~~ NOV 8, 2018

QR DECOMPOSITION

$Ax = b$

$A = QR$

ORTHOGONAL (Q) UPPER TRIANGULAR (R)

$QRx = b \Rightarrow \underbrace{Q^T Q}_{= I} Rx = Q^T b$

$Rx = w = Q^T b$

Ex)

$A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$

$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$e_2 = x_2 - \langle x_2, u_1 \rangle u_1$

$= \frac{8}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow u_2 = \frac{e_2}{\|e_2\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A = QR \Rightarrow R = Q^T A = \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & -6 \\ 0 & 8 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$Rx = Q^T b$

$\frac{1}{5} \begin{bmatrix} 5 & -6 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$x_2 = 3/8$

$x_1 = 5/4$

$Ax = b \Rightarrow Ax = b$

$\Rightarrow (A^T A) x = A^T b$

A: $m \times n$, NOT SQUARE; x: $n \times 1$

$\underbrace{(A^T A)^{-1} (A^T A)}_{I} x = (A^T A)^{-1} A^T b$

$x = \underbrace{(A^T A)^{-1} A^T}_{\text{PSEUDO-INVERSE}} b$

MOORE PENROSE INV.

$Ax = b$

IF A IS $(m \times n)$ AND A^{-1} EXISTS

$x = (A^T A)^{-1} A^T b$

$= A^{-1} \underbrace{(A^T)^{-1} A^T}_{I} b$

$= A^{-1} b$

$\sqrt{\frac{1}{2} \left\langle \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\rangle}$

$\sqrt{\frac{1}{2} (1 + 4 + 1)} = 6 (1/2)$

$\sqrt{3}$ ✓

$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{-3}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$

$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \frac{-3}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$-1 - -3/2 \quad -1 + 3/2 = 0.5$

LEAST SQUARES ESTIMATE

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A^{T} = (A^TA)^{-1} A^T = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1} A^T b$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix}$$

B/C WE PROJECTED b (DATA) ON THE SPAN OF A (COLS, a_1, a_2)
TO OBTAIN THE ESTIMATE

ERROR $\perp$ ESTIMATE

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

LU DECOMPOSITION

$A = L \cdot U$

LOWER UPPER

$$Ax = b \qquad Ux = w$$

$$L \cdot Ux = b$$

$$L\underbrace{(Ux)}_{w} = b$$

$$L \cdot w = b$$

$$Ax = b \qquad \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} x = \begin{bmatrix} 27 \\ 0 \end{bmatrix}$$

AUGMENTED MATRIX:

$$\left[\begin{array}{cc:c} 1 & 1 & 27 \\ 2 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc:c} 1 & 0 & ? \\ 0 & 1 & ? \end{array}\right]$$

Nov 13

Eigenvalues & Eigenvectors

$Ax = y$

$\underbrace{Ax}_{\text{Matrix Transform}} = \lambda x$

Same Dir w/ Scaling Factor

When $Ax = \lambda x$

↳ x is eigenvector of A

↳ λ is eigenvalue of A

To find eigenvalues & eigenvectors:

$Ax = \lambda x$

$Ax - \lambda x = 0$

$(A - \lambda I)x = 0$

If $(A - \lambda I)^{-1}$ exists

then $x = (A - \lambda I)$

Eigenvector solution for $(A - \lambda I)^{-1}$ does not exist

↳ $\text{Det}(A - \lambda I) = 0$

Solve for λ to get eigenvalue

Given λ, $Ax = \lambda x \Rightarrow$ solve for x

Ex) $A = \begin{bmatrix} -1 & 1 \\ -12 & 6 \end{bmatrix}$

Char Eqn. $\text{Det}(A - \lambda I) = 0$

$\text{Det}(\lambda I - A) = 0$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ -12 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ -12 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+1 & -1 \\ 12 & \lambda-6 \end{bmatrix}$$

$$\text{Det}(\lambda I - A) = (\lambda+1)(\lambda-6) - (-1)(12) = 0$$

$$= \underbrace{\lambda^2 - 5\lambda + 6 = 0}_{\text{Characteristic Polynomial}}$$

Solve: $\left.\begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 2 \end{matrix}\right\}$ Two E-values

E-vector V_1 for λ_1

$A \cdot V_1 = \lambda_1 \cdot V_1$

$$\begin{bmatrix} -1 & 1 \\ -12 & 6 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 3 \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix}$$

$-V_{11} + V_{12} = 3V_{11} \Rightarrow V_{12} = 4V_{11}$

$-12V_{11} + 6V_{12} = 3V_{12} \Rightarrow 4V_{11} = V_{12}$

$V_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ etc.

Use unit vector:

$V_1 = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

SIMILARITY TRANSFORMATION

A, B SIMILAR IF THERE IS A MAPPING
$B = PAP^{-1}$

THEN $P^{-1}B = \underbrace{P^{-1}P}_{I}AP^{-1}$

$P^{-1}BP = APP^{-1}$
$P^{-1}BP = A$ ✓

SIMILAR MATRICES HAVE SAME DETERMINANT & SAME EIGENVALUES

$\det(P^{-1}AP - \lambda I) \stackrel{?}{=} \det(A - \lambda I)$

FOR SYMMETRIC MATRICES
↳ REAL E-VALUES
↳ ORTHOGONAL E-VECTORS
↳ SYM. MATRICES ARE POSITIVE DEFINITE

PRINCIPLE COMPONENT ANALYSIS

SET OF DATA VECTORS $\{x_1 \dots x_n\}$

FORM COVARIANCE MATRIX C (SYM)

FIND E-VALS & E-VECTS OF C

PCA EX) VECTORS = $\left\{ \begin{bmatrix} 0 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$

CENTER ~~DATA~~ $(x_i - \mu)$ $= \left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ $\mu = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

NOV 15, 2018
TEST 3 → NOV 29

MODAL MATRIX: EIGENVECTORS AS COLUMNS

$M = \left[v_1 \mid v_2 \mid \dots \mid v_n \right]$ IF v_i NORMALIZED THEN FOR SYM. MATRIX A
M IS ORTHOGONAL, $MM^T = I$

$A = P\Lambda P^{-1} = \underbrace{M}_{\text{MODAL}} \underbrace{\Lambda}_{\text{E-VALUE MAT}} M^{-1}$

$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

A FIND E-VALUES $\lambda_1 \cdots \lambda_n$

E-VECTORS $V_1 \cdots V_n$

$$\left.\begin{aligned} A V_1 &= \lambda V_1 \\ A V_2 &= \lambda V_2 \\ &\vdots \\ A V_n &= \lambda V_n \end{aligned}\right\} \begin{bmatrix} V_1 & V_2 & V_3 & \cdots & V_n \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

SET OF EQUS $\Rightarrow$ $AM = M\Lambda$

$$\boxed{AMM^{-1} = M\Lambda M^{-1}}$$

$$\underbrace{A^T A}_{\text{DIAGONAL}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix}$$

AA^T IS SYMMETRIC

$A^T A$ " "

COVARIANCE MATRIX OF N DATA SAMPLES: $\{x_1, x_2, \ldots, x_N\}$

SAMPLE MEAN:

$$m_N = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$C = E\{(x - \mu_x)(y - \mu_y)\}$$

$$\Downarrow$$

SAMPLE VARIANCE

$$\sigma_N^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - m_N)^2$$

$$C = \begin{array}{c|c} & x_1 \cdots x_n \\ \hline \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} & \\ & C_{ij} \\ & \end{bmatrix} \end{array} \qquad C_{ij} = E\{(x_i - \mu_i)(x_j - \mu_j)\}$$

$$C = \frac{1}{N} \sum_{i=1}^{N} x_i x_i^T$$

EIGENVECTORS OF C, E-VECTORS ARE ORTHOGONAL $V_i \perp V_j$ $i \neq j$

$\langle V_i, V_j \rangle = 0$

$$y_i = \Big[\overbrace{V_1 \,\vdots\, V_2 \,\vdots\, \cdots \,\vdots\, V_n}^{\text{ROTATES}}\Big]^T \overbrace{(x_i - m)}^{\text{CENTERS DATA}}$$

NEW COORDINATES ALIGNED W/ PRINCIPAL AXIS

i.e. DIRECTION OF MAX VAR IN DATA