5050 HW 5
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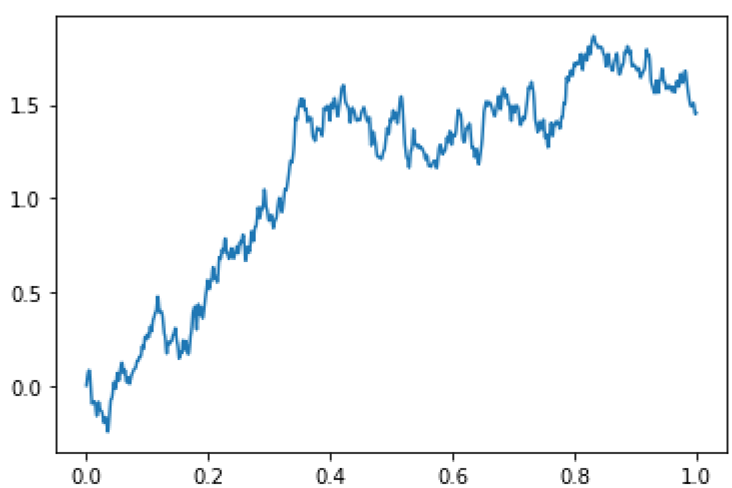
Problem 1
a)
Use Wiener's Fourier series defined on the interval $[0, 1]$

$$W(t) = X_0 + \frac{\sqrt{2}}{\pi} \sum_{j=1}^{\infty} \frac{X_j}{j} \sin(\pi j t)$$

In [1]:
```
# Code for part a
import numpy as np
import matplotlib.pyplot as plt
import math

# Return vector of times and values
def wiener(N,segments):
    # Construct time vector
    t = []
    for i in range(segments+1):
        t.append(i/segments)

    # Generate Xi vector
    Xi = []
    for i in range(0,N+1):
        Xi.append(np.random.normal(0,1))

    # Generate values of the BM
    Wt = []
    for time in t:
        value = Xi[0]*time
        for j in range(len(Xi)-1):
            value += (math.sqrt(2)/math.pi)*Xi[j+1]/(j+1)*math.sin(math.pi*(j+1)*
        Wt.append(value)

    # Return an array with [time,values]
    return [t,Wt]


w = wiener(500,1000)
plt.plot(w[0],w[1])
print("Plot of Wiener's Brownian Motion:")
plt.show()
```

```
Plot of Wiener's Brownian Motion:
```



b)

Use Donsker's theorem to simulate standard brownian motion on the time iterval $[0, 1]$ using a random walk

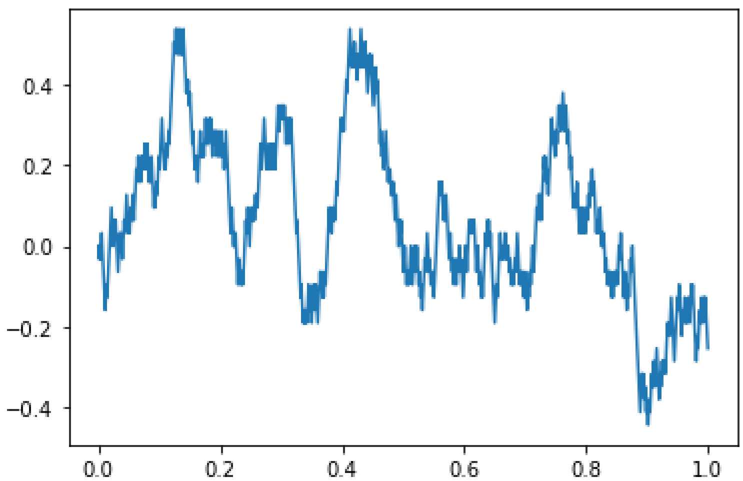
for the random walk increments use the following distributions

i) ± 1 equally likely
ii) Standard Normal
iii) Uniform(0,1)

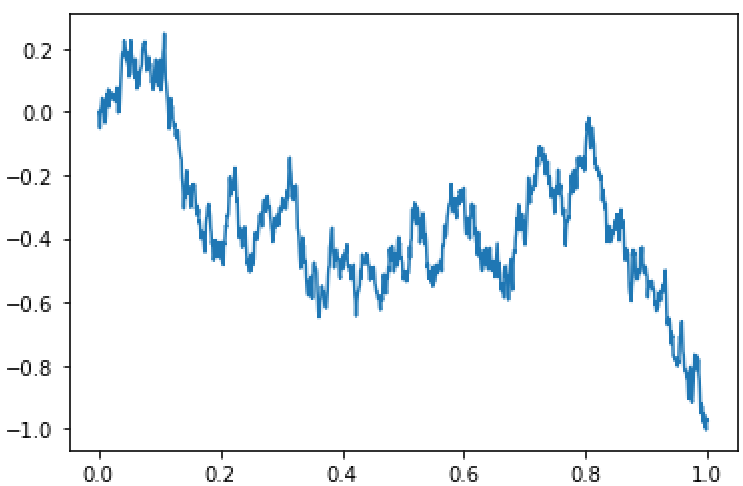
In [2]:
```
# Code for part b
def alternating_donsker(segments):
    # Construct time vector
    t = []
    for i in range(segments+1):
        t.append(i/segments)

    # Generate the random walk
    rw = [0]
    for i in range(segments):
        rw.append(2*np.random.binomial(1,.5)-1+rw[-1])

    # Scale the rw by 1 over the square root of N times the variance
    # Variance for this distribution is 1
    bmi = np.array(rw)
    bmi = bmi/(math.sqrt(segments*1))

    # Return an array with [time,values]
    return [t,bmi]


def normal_donsker(segments):
    # Construct time vector
    t = []
    for i in range(segments+1):
        t.append(i/segments)

    # Generate the random walk
    rw = [0]
    for i in range(segments):
        rw.append(np.random.normal(0,1)+rw[-1])

    # Scale the rw by 1 over the square root of N times the variance
    # Variance for this distribution is 1
    bmii = np.array(rw)
    bmii = bmii/(math.sqrt(segments*1))

    # Return an array with [time,values]
    return [t,bmii]


def uniform_donsker(segments):
    # Construct time vector
    t = []
    for i in range(segments+1):
        t.append(i/segments)

    # Generate the random walk
    rw = [0]
    for i in range(segments):
        rw.append(np.random.uniform(0,1)-.5+rw[-1])

    # Scale the rw by 1 over the square root of N times the variance
    # Variance for this distribution is 1/12
    bmiii = np.array(rw)
    bmiii = bmiii/(math.sqrt(segments*(1/12)))
```

```
    # Return an array with [time,values]
    return [t,bmiii]


i = alternating_donsker(1000)
plt.plot(i[0],i[1])
print("Brownian Motion from +-1 equally likely:")
plt.show()

ii = normal_donsker(1000)
plt.plot(ii[0],ii[1])
print("Brownian Motion from standard normal:")
plt.show()

iii = uniform_donsker(1000)
plt.plot(iii[0],iii[1])
print("Brownian Motion from uniform(0,1):")
plt.show()
```

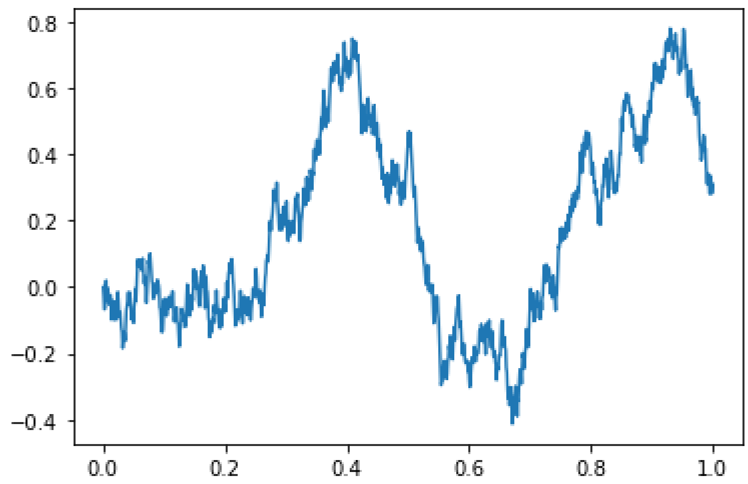
```
Brownian Motion from +-1 equally likely:
```




```
Brownian Motion from standard normal:
```




```
Brownian Motion from uniform(0,1):
```



c)

The Brownian motion from ± 1 equally likely looks the most different, especially with small values of n it looks much more spikey.
As the value for n gets larger all three start to look the same.

d)

$W(t)$ is distributed like a normal $(N(0, t))$ so the variance of $W(1/2) = 1/2$

In [3]:
```
# Code for d
A = []
I = []
II = []
III = []

for i in range(1000):
    A.append(wiener(25,500)[1][250])
    I.append(alternating_donsker(500)[1][250])
    II.append(normal_donsker(500)[1][250])
    III.append(uniform_donsker(500)[1][250])

print(f"Variance of Wiener Fourier series: {np.var(A)}")
print(f"Variance of Donsker's theorem using the distribution in b.i) : {np.var(I
print(f"Variance of Donsker's theorem using the distribution in b.ii) : {np.var(I
print(f"Variance of Donsker's theorem using the distribution in b.iii) : {np.var(
print("\nAll these variances are fairly close to the real value even with realati
```

```
Variance of Wiener Fourier series: 0.4554080640555858
Variance of Donsker's theorem using the distribution in b.i) : 0.51565368799999
99
Variance of Donsker's theorem using the distribution in b.ii) : 0.4742869181917
7937
Variance of Donsker's theorem using the distribution in b.iii) : 0.471700744372
0468

All these variances are fairly close to the real value even with realatively sm
all N's
```

e)

Brownian Motion has the property that any time step is independent of its past, so the correlation is

0

In [4]:
```
# Code for e
W12 = []
W1_12 = []
for i in range(1000):
    brown = alternating_donsker(500)
    W12.append(brown[1][250])
    W1_12.append(brown[1][500]-brown[1][250])
print(f"The correlation coefficient between W(1/2) and W(1)-W(1/2) is: {np.corrco
print("\nThe correlation is very close to zero.")
```

```
The correlation coefficient between W(1/2) and W(1)-W(1/2) is: 0.03719897494740
297

The correlation is very close to zero.
```

f) The probability that the maximum value of the Brownian motion is above 3 is given by:
$2P(B(1) > 3) = 2P(N(0,1) > 3) = 2(.001349898) = .0026998$

In [5]:
```
max_values = []
count = 0
for i in range(1000):
    motion = normal_donsker(500)
    m = np.amax(motion[1])
    max_values.append(m)
    if m > 3:
        count += 1
test_prob = count/1000
print(f"An estimate of the probability that the maximum is above 3 is:{test_prob}
print("\nThe estimate is usually very close to the theoretical value")
```

```
An estimate of the probability that the maximum is above 3 is:0.003

The estimate is usually very close to the theoretical value
```

Problem 2 Let $W(t)$ be standard Brownian motion.

a) $P(W(t) = 0 \text{ for some t with } 2 < t < 3)$

let $X(t) = W(2t)$ X is a brownian motion as shown on a previous homework, and
$P(W(t) = 0 \text{ for some t with } 2 < t < 3)$
becomes $P(X(s) = 0 \text{ for some s with } 1 < s < 3/2)$. Then we can use the formula for this and we get the probability
$1 - \frac{2}{\pi}\arctan(\frac{1}{\sqrt{1/2}}) \approx .3918$

b) $P(W(t) < 4 \text{ for all t with } 0 < t < 3)$

This asks what the probability is that a standard Brownian motion between 0 and 3 has maximum value smaller than 4,
The probability of a Brownian motion having maximum value of 4 or higher is given by:
$2P(W(3) > 4) = 2P(N(0,3) > 4)$ So, the probability of b) is
$1 - 2P(N(0,3) > 4) \approx 1 - .1824 \approx .81758$

c) $P(W(t) > 0 \text{ for all } t > 10)$

Brownian motion will cross any arbitrary point if it continues long enough,
This question asks what the probability is that a standard brownian motion will stay positive after time 10,
But the Brownian motion will eventually become negative so the probability of c) is 0.

In [ ]: