5050 HW 6
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**Problem 1**
Consider the heat equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + 3\frac{\partial u}{\partial t}(t,x)$$

with initial heat profile $u(0,x) = e^{-x^2}$ where we can solve this equation using a standard Brownian motion

$$u(t,x) = E[e^{-(x+\sigma B(t)+\mu t)^2}]$$

**(a)** Compute $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(az+b)^2} e^{-z^2/2} dz$

First we expand the squares in the exponent

$$e^{-(az+b)^2} e^{-z^2/2} = e^{-(\frac{z^2}{2}+a^2z^2+2abz+b^2)}$$

Then completing the square

$$e^{-(\sqrt{a^2+1/2}x+\frac{ab}{\sqrt{a^2+1/2}})} e^{-(\frac{-a^2b^2}{a^2+1/2}+b^2)}$$

where the first exponential is in the form of a normal variable and the second exponential is a constant
Therefore the integral is

$$\frac{e^{-\frac{b^2}{2a^2+1}}}{\sqrt{2a^2+1}}$$

In [1]:
```
# Code for Brownian Motion
import numpy as np
import matplotlib.pyplot as plt
import math

def brownian(time, segments):
    # Construct time vector
    t = []
    for i in range(segments+1):
        t.append(time*i/segments)

    # Generate the random walk
    rw = [0]
    for i in range(segments):
        rw.append(np.random.normal(0,1)+rw[-1])

    # Scale the rw by 1 over the square root of N times the variance
    bm = np.array(rw)
    bm = bm/(math.sqrt(segments))*math.sqrt(time)

    # Return an array with [time,values]
    return [t,bm]
```

**Problem 2**

Let $B(t)$ be standard Brownian motion and let $Y(t)$ be 1 for $0 \leq t < 2$
and $3B(2)$ for $t \geq 2$ prove that:

$$M(t) = \int_0^t Y(s)dB(s)$$

is a martingale

We can start by seperating the stochastic integrals into two parts, before $t = 2$ and, at or after $t = 2$.
Then for both scenarios we need to show that the integral is a "fair game"
for $0 \leq t < 2$,

$$M(t) = \int_0^t 1dB(s) = B(t) - B(0) = B(t)$$

and for $t \geq 2$

$$M(t) = \int_0^2 1dB(s) + \int_2^t 3B(2)dB(s) = B(2) + 3B(2)(B(t) - B(2))$$

Then we need to show $E[M(t)|\mathcal{F}_s] = M(s)$ for all $s < t$ which gives 3 cases: $s < t < 2$, $2 \leq s < t$ and, $s < 2 \leq t$
For the first case $E[M(t)|\mathcal{F}_s] = E[B(t)|\mathcal{F}_s] = E[B(t-s)] + B(s) = B(s)$
For the second $E[M(t)|\mathcal{F}_s] = 3B(2)E[B(t-s)] + B(s) = B(s)$
For the third case $E[M(t)|\mathcal{F}_s] = E[B(2) + 3B(2)(B(t) - B(2))|\mathcal{F}_s] = B(s)$
Therefore we can say $M(t)$ is a martingale.

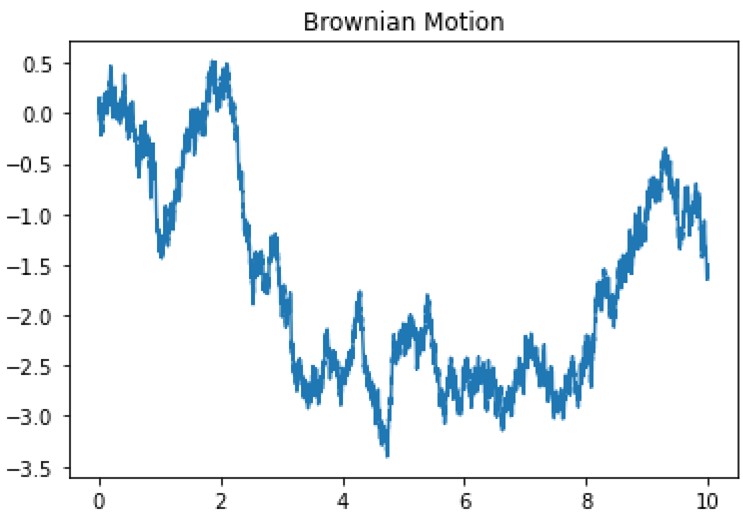
**Problem 3**

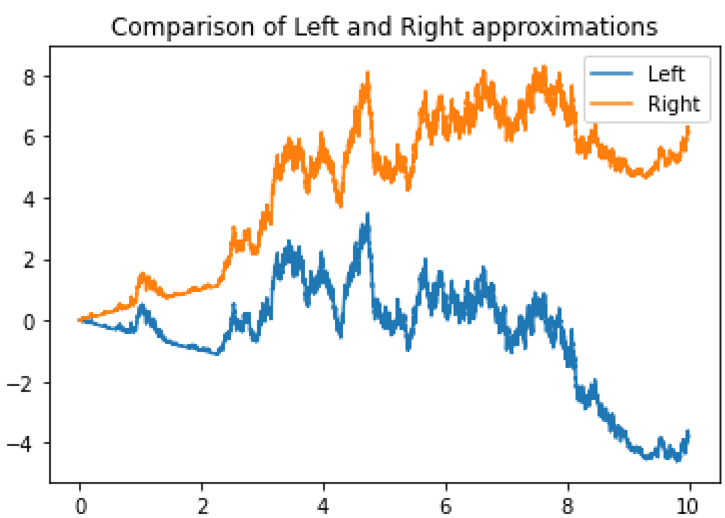
Let B(t) be standard Brownian Motion and consider the stochastic integral for $0 \leq t \leq 10$

$$\int_0^t B(s)dB(s)$$

In [6]:
```
# Code for part a
b = brownian(10,10000)
# b outputs a pair of arrays [time][values]
plt.plot(b[0],b[1])
plt.title("Brownian Motion")
plt.show()

# Numerically compute \B(s)dB(s)
left = [0]
right = [0]
for i in range(len(b[0])-1):
    left.append(b[1][i]*(b[1][i+1]-b[1][i])+left[-1])
    right.append(b[1][i+1]*(b[1][i+1]-b[1][i])+right[-1])
plt.plot(b[0],left)
plt.plot(b[0],right)
plt.title("Comparison of Left and Right approximations")
plt.legend(["Left","Right"])
plt.show()
```




**(b)**

Which approximation is correct

The left approximation is correct because it is a martingale (stays near zero)
wheras the right approximation tends to drift off.
This is a stochastic integral of progressivly measurable function which should be a martingale

**(c)**
Calculate the integral analytically

We start by applying the Ito formula
$f(B(t)) = f(B(0)) + \int_0^t f'(B(s))dB(s) + 1/2 \int_0^t f''(B(s))ds$
with $f = x^2$

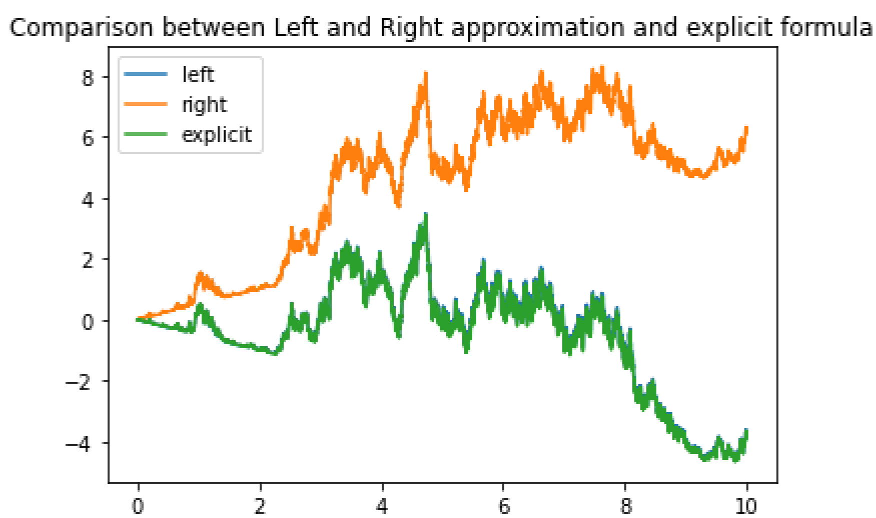
$$B(t)^2 = B(0)^2 + 2\int_0^t B(s)dB(s) + \int_0^t ds$$

then by rearrangement we get

$$\int_0^t B(s)dB(s) = \frac{B(t)^2 - t}{2}$$

In [7]:

```
# Code for part d
explicit =[]
for i in range(len(b[0])):
    explicit.append((b[1][i]**2-b[0][i])/2)
plt.plot(b[0],left)
plt.plot(b[0],right)
plt.plot(b[0],explicit)
plt.legend(["left","right","explicit"])
plt.title("Comparison between Left and Right approximation and explicit formula")
plt.show()
print("The explicit formula matches up almost perfectly with the left approximati
```

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The explicit formula matches up almost perfectly with the left approximation as
suggested by the answer to b
```

In [ ]: