Mathematical Methods in Physics

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October 13, 2014

Chapter 1

Vector Algebra

1.1 Linearly independent vectors

Given a set of m vectors, $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, ...,$\boldsymbol{a}_3$, a linear combination of these vectors is an expression:

$$c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + ... + c_m\boldsymbol{a}_m$$

Where c_1, c_2, ..., c_m are scalars.

Defintion: Linearly independent vectors satisfy the equation:

$$c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2 + ... + c_m\boldsymbol{a}_m = \boldsymbol{0}$$

if and only if c_1, c_2, ..., $c_m = 0$.

If the above equation holds when the scalars are not 0, we say that the set of vectors are *linearly dependent*. If they are linearly dependent, this means that at least one of them is a scalar multiple of the others.

Proof. If $c_1 \neq 0$:

$$\boldsymbol{a}_1 = \frac{1}{c_1}(c_2\boldsymbol{a}_2 + ... + c_m\boldsymbol{a}_m)$$

□

1.2 Linear vector space

A set of V elements $\boldsymbol{a}$, $\boldsymbol{b}$, ..., is called a linear vector space V. It's elements are all vectors if there are two operations called *addition* and *scalar multiplication*, and satisfy the following criteria:

- $\boldsymbol{a} + \boldsymbol{b} \in V$ for all $\boldsymbol{a}, \boldsymbol{b} \in V$ (closed under addition)
- $\lambda\boldsymbol{a} \in V$ for all $\lambda \in \mathbb{R}/\mathbb{C}$ and $\boldsymbol{a} \in V$ (closed under scalar multiplication)
- $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ for all $\boldsymbol{a}, \boldsymbol{b} \in V$ (commuitive)
- $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$ for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V$ (associative)

- $(\mu + \lambda)\boldsymbol{a} = \mu\boldsymbol{a} + \lambda\boldsymbol{a}$ for all $\boldsymbol{a} \in V$ and $\mu, \lambda \in \mathbb{R}/\mathbb{C}$ (distributitive)
- There exists a $\mathbf{0} \in V$ such that $\mathbf{0} + \boldsymbol{a} = \boldsymbol{a}$ for all $\boldsymbol{a} \in V$
- For all $\boldsymbol{a} in V$ there exists $-\boldsymbol{a}$ such that $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$

1.2.1 Dimension

The maximum number of linearly independent vectors in V is called the dimension of V ($\dim V$).

1.3 Basis

A basis is a set of linearly independent vectors that can be taken in scalar multiples to span the whole of the vector space (you can make any vector in the space with this set). The requirements are:

- The vectors are linearly independent
- $\dim(\text{Basis}) = \dim(V)$

One example of this is the *standard basis* of $\mathbb{R}^3$:

$$\hat{\imath} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \hat{\jmath} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

1.4 Orthonormal Vectors

A set of vectors are called orthonormal if they are all of unit length and mutually perpendicular, i.e.:

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0 \; \forall \; \boldsymbol{a}, \boldsymbol{b} \in V$$

1.5 Inner Products

Consider a vector space, V. The inner product of two vectors $\boldsymbol{a}$, $\boldsymbol{b} \in V$, is denoted by $(\boldsymbol{a}, \boldsymbol{b})$.

It is a scalar function which satisfies some properties:

- $(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{b}, \boldsymbol{a})^*$for all $\boldsymbol{a}, \boldsymbol{b} \in V$ (symmetry/commutitive)
- $(\boldsymbol{a}, \lambda\boldsymbol{b} + \mu\boldsymbol{c}) = \lambda(\boldsymbol{a}, \boldsymbol{b}) + \mu(\boldsymbol{a}, \boldsymbol{c})$ for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V$ and $\mu, \lambda \in \mathbb{R}/\mathbb{C}$. (distributive)

Two vectors in a general vector space are said to be orthogonal if $(\boldsymbol{a}, \boldsymbol{b}) = 0$.

1.5.1 Dot product

The dot product is the 'standard' inner product that we use on $\mathbb{R}^n$. It has the value:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta$$

We say that $|\boldsymbol{b}| \cos \theta$ is the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$. The value of the dot product can also be found using:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Complex inner product

Here we use a generalisation of the dot product:

$$< \boldsymbol{a}, \boldsymbol{b} >= \boldsymbol{a}^\dagger \cdot \boldsymbol{b} = a_x^* b_x + a_y^* b_x + \ldots$$

Where the † represents the complex-conjugate transpose of the original vector.

1.5.2 Vector product

The vector product, rather than returning a scalar, returns a vector. This vector is perpendicular to both 'input' vectors.

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \theta$$

If both vectors are in the same direction (i.e. $\theta = 0$) then the vector product returns 0. To calculate the vector returned:

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

Triple product

$$[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$$

1.6 Equation of a line

Imagine you have a point in space, which you know is on the line ($\boldsymbol{a}$). Then, imagine you have a vector that points in the direction of your line ($\boldsymbol{b}$). You can then find any point on the line ($\boldsymbol{r}$):

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$$

However, if you know two points ($\boldsymbol{c}$, $\boldsymbol{d}$), you can work out the direction vector:

$$\boldsymbol{r} = (\boldsymbol{c} - \boldsymbol{d})\lambda + \boldsymbol{c}$$

It is also possible to describe a line using a vector product. This is because you know that everything in the line is in the same direction as the direction vector $\boldsymbol{b}$:

$$(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{b} = 0$$

1.7 Equation of a plane

A plane goes through a point $\boldsymbol{a}$ with a normal vector $\hat{n}$. Any point in the plane can then be found by using a scalar product (a vector 'in' the plane dotted with the normal is always 0):

$$(\boldsymbol{r} - \boldsymbol{a}) \cdot \hat{\boldsymbol{n}} = 0$$

We can re-write this:

$$xn_x + yn_y + zn_z = \boldsymbol{a} \cdot \hat{\boldsymbol{n}} = d$$

Another way of getting a plane is to specify three points lying in the plane, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and specify vectors that lie in the plane using these:

$$\hat{\boldsymbol{n}} = (\boldsymbol{b} - \boldsymbol{a}) \times (\boldsymbol{b} - \boldsymbol{c})$$

1.8 Equation of a sphere

If we know the centre of the sphere, $\boldsymbol{c}$, and the radius, a, we know that the distance to any point on the surface ($\boldsymbol{r}$) from the centre is a:

$$|\boldsymbol{r} - \boldsymbol{c}| = a^2$$

This expands out to the familar equation:

$$(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 = a^2$$

Chapter 2

Matricies

2.1 Linear operators

An operator is an object that associates one vector space with another:

$$\boldsymbol{y} = A\boldsymbol{x}$$

An operator is said to be linear if:

- $A(\boldsymbol{a} + \boldsymbol{b}) = A\boldsymbol{a} + b\boldsymbol{b}$
- $A(\lambda\boldsymbol{a}) = \lambda(A\boldsymbol{a})$

We can write operators as matricies.

2.2 Matricies

A matrix is an array of elements written by enclosing them in parentheses. A vector is also a matrix, with only one column or row. By convention we write $\boldsymbol{x}$ as a column vector.

2.2.1 Matrix addition

For some element A_{ij}:

$$S = A + B \rightarrow S_{ij} = A_{ij} + B_{ij}$$

2.2.2 Multiplication by scalar

$$S = \lambda A \rightarrow S_{ij} = \lambda A_{ij}$$

2.2.3 Matrix multiplication

$$P = AB \rightarrow P_{ij} = \Sigma_n A_{in} B_{nj}$$

Where the matrices have dimensions $(x \times n)$ and $(n \times y)$ for matricies A and B respectively.

Matrix multiplication is *not* commutitive - i.e.:

$$AB \neq BA$$

2.2.4 Transpose

If a matrix is transposed, it returns the columns and rows switched:

$$A_{ij}^T = A_{ji}$$

The Hermition conjugate

This is the same as transpose, but also takes the complex conjugate:

$$A_{ij}^\dagger = A_{ji}^*$$

2.2.5 Trace

The trace of a matrix is the sum of the diagonal elements:

$$Tr(A) = \Sigma_n A_{nn}$$

The trace of a multiple product of matricies is invariant under a cyclic permutation of matricies:

$$Tr(ABC) = Tr(ACB) = Tr(BCA) = \ldots$$