Classical Mechanics

Josh Borrow

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Chapter 1

Degrees of Freedom and Dynamical Variables

1.1 Dynamical variables

Dynamical variables are aset of variables that describe a system that changes over time. Equations of motion describe how these variables change. With Newtonian physics we relied on knowing about precise initial conditions before we had a clue what was going on.

1.2 Degrees of Freedom

Point masses have three degrees of freedom, in the x, y, z planes. A system of M point masses have:

$$N = 3M$$

degrees of freedom (3 for each particle). However, the existence of j independent constraints (rigid 'bars' between the masses) reduces this to:

$$N = 3M - j$$

This leads to the conclusion that any rigid body with more than two point masses connected with rigid rods has exactly 6 degrees of freedom - x, y, z of the centre of mass, along with three angles to describe the orientation. Another way of specifying this is simply to give two points within the plane.

1.2.1 Constraints and their types

The existance of these j constraints means that the masses, M, are no longer independent. To describe the system fully (as long as we know what the system looks like), we only need as many co-ordinates as there are degrees of freedom.

If the constraints are *holonomic* then it is possible to express the full position and orientation of the system, $\boldsymbol{r}$ as a function of these co-ordinates q_k:

$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, \ldots, q_N, t).$$

There are two types of holonomic constraints:

- Rheonomic - time dependent constraint
- Scleronomous - no *explicit* time dependence

If we can express the constraints as a function:

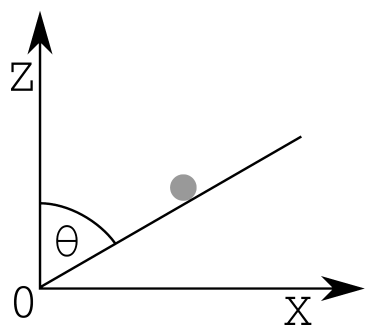
$$\boldsymbol{r} = \boldsymbol{r}(q_1, q_2, \ldots, q_N, t) = 0,$$

Then they are all holonomic.

Constraints are non-holonomic if the state depends on the path taken to achieve it - differential constraints or inequalities. These are usually velocities.

1.2.2 Examples

Example 1: point mass on an inclined plane. Here, we can find that all constraints are

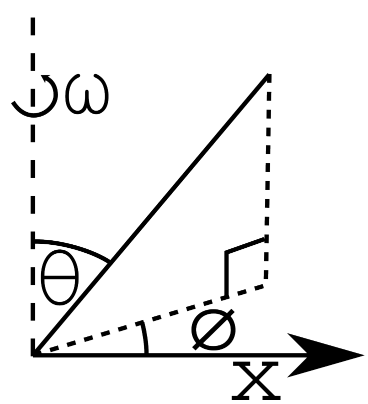


holonomic if we say:

$$\tan\theta = \frac{x}{z}; \;\rightarrow\; z\tan\theta - x = 0$$

This describes the system fully.

Example 2: bead on a rotating wire. For the rotating wire, we find two equations that



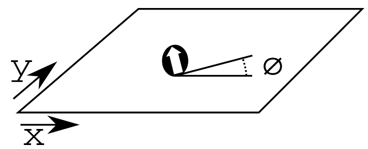
both have time dependence and hence are *rheonomic*:

$$x = z\tan\theta\cos\omega t$$

$$y = z\tan\theta\sin\omega t$$

These constraints mean that the particle always lies on the wire.

Example 3: coin, rolling on a plane. This is slightly more complicated, and is also non-



holonomic:

$$\mathrm{d}x = a\mathrm{d}\theta\cos\phi$$

$$\mathrm{d}x = a\mathrm{d}\theta\sin\phi$$

If we say that the angle that the coin has turned is θ. We can reach any point in this 4D space - i.e. we cannot separate the constraints from the dynamics.

1.3 Generalised co-ordinates

We always restrict the number of variables to a minimum because this means that we have fewer equations to solve (and mistakes to make). The minimum number of co-ordinates is equal to the number of degrees of freeom.

We regard co-ordinates and generalised velocities as independent. If we think simultaneously, then there is no reason that $\dot{x}$ is related to x. We have the relation:

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \Sigma_k \frac{\partial \boldsymbol{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}$$

Which we can reduce to the 'cancelling the dots' formula:

$$\frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_k} = \frac{\partial \boldsymbol{r}_i}{\partial q_k}$$

Chapter 2

The Lagrangian

When we use The Lagrangian, we assume that we have:

- Holonomic constraints
- The costraining forces do no work
- The applied forces are conservative

However, the potential function may change with time.

2.1 D'Alembert's Principle

From Newton's Second Law:

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i \;\rightarrow\; \Sigma_i(\boldsymbol{F}_i - \boldsymbol{p}_i) \cdot \delta\boldsymbol{r}_i = 0$$

where the $\boldsymbol{r}_i$ is a virtual displacement that is consistent with the constraints and is instantenous. We can also split the force up into the components that are from the constraint, and those that are applied:

$$\boldsymbol{F}_i = \boldsymbol{F}_i^c + \boldsymbol{F}_i^a \; .$$

As we know that the constraint forces do no work, we can simply replace the above with the applied forces:

$$\Sigma_i(\boldsymbol{F}_i^a - \boldsymbol{p}_i) \cdot \delta\boldsymbol{r}_i = 0$$

We have removed the constraint forces from this equation, but we need to rewrite the equation in terms of the generalised co-ordinates that are *independent* of each other. This is where the proof for *generalised equations of motion (EoM)* comes in - this is in the course notes or online. This leaves us with:

$$L = T - V \text{ where } \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \; .$$

We can use this to solve pretty much any classical mechanics problem. Apparently.

2.1.1 Examples of the Lagrangian Method

Example: a free particle in 3D space:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

There is no potential (as the particle is free):

$$L = T - V = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Putting this into the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0 \ .$$

For each q_k the answer is the same, so we will give a generalised one for x, y, z:

$$j = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{j}) - 0 = 0 \ \rightarrow \ m\ddot{j} = 0 \ ,$$

giving the equation of motion:

$$j(t) = \dot{j}(0)t + j(0) \ .$$

We also get a similar answer for constant potentials.

Ignoreable co-ordinates

If we have no explicit dependence on a co-ordinate, it can saftley be ignored in the final E-L calculation (i.e.

$$\frac{\partial L}{\partial q_k} = 0 \ ).$$

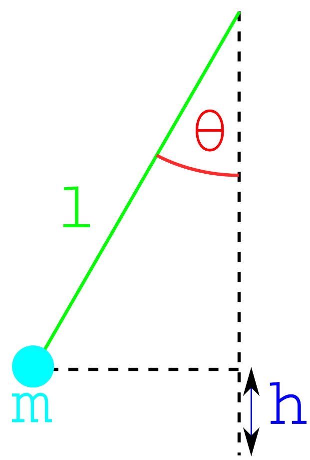
If a co-ordinate is not in the Lagrangian, it is a constant which returns a 'constant of the motion':

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

which is the conjugate momentum. For example, in the above:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p = \text{constant}$$

Example: a plane pendulum.



Here, we only need one generalised co-ordinate, θ, which describes how far away the pendulum is from it's lowest point. We assume the gravitational field is approximately constant.

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\ell\dot{\theta}\right)^2$$

$$V = mgh = mg\ell(1 - cos\theta)$$

Giving us:

$$L = \frac{1}{2}m\left(\ell\dot{\theta}\right)^2 - mg\ell(1 - cos\theta)$$

Using Euler-Lagrange for $q_1 = \theta$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\ell^2\dot{\theta}\right) + mg\ell\sin\theta = 0$$

Cancelling:

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

Now we use the small-angle approximation:

$$\ddot{\theta} = -\frac{g}{\ell}\theta$$

Which is SHM with $\omega = \sqrt{g/L}$.